# A REPRESENTATION THEOREM FOR $G_\delta$-VALUED MULTIFUNCTIONS

By S. M. Srivastava

**1. Introduction.** In this paper we prove the following representation theorem for $G_\delta$-valued multifunctions:

THEOREM 1.1 *Let $T$, $X$ be Polish spaces, $\mathfrak{I}$ a countably generated sub $\sigma$-field of the Borel $\sigma$-field $\mathcal{B}_T$ and $F: T \to X$ a multifunction. Then the following are equivalent*:

(A) *$F$ is $\mathfrak{I}$-measurable, $Gr(F) \in \mathfrak{I} \otimes \mathcal{B}_X$ and $F(t)$ is a $G_\delta$ in $X$ for each $t \in T$.*

(B) *There is a function $f: T \times \Sigma \to X$ such that for $t \in T, f(t, .)$ is a continuous, closed map from $\Sigma$ onto $F(t)$ and for $\sigma \in \Sigma, f(., \sigma)$ is $\mathfrak{I}$-measurable, where $\Sigma$ is the space of irrationals.*

The necessary definitions and notation are given in Section 2 where we also state some known results for easy reference. In Section 3 we prove the implication $(A) \Rightarrow (B)$ when $X$ is, moreover, zero-dimensional; this implication for an arbitrary Polish space $X$ is proved in Section 4. The implication $(B) \Rightarrow (A)$ is proved in Section 5.

The author [10] had earlier established the existence of a $\mathfrak{I}$-measurable selector for a multifunction $F: T \to X$ satisfying condition $(A)$. Various representation theorems for such multifunctions are also proved in [9]. Similar results for multifunctions taking closed values in a Polish space can be found in [5], [11].

Our result can be viewed as a sectionwise version of the following well known characterization of Polish spaces: a *second countable, metrizable space is completely metrizable if and only if it is the image of irrationals under a closed continuous function*. The 'if' part of this result was proved by Vaĭnšteĭn [14] and we carry over this proof for each $F(t)$, $t \in T$, uniformly to prove the implication $(B) \Rightarrow (A)$. Engelking [4] proved the 'only if' part of the above result.

I received invaluable help and suggestions from Drs. A. Maitra and H. Sarbadhikari while working on the problems considered in this paper. I express my deep indebtedness to them.

Manuscript received October 25, 1978



0002-9327/80/1021-0165 $01.50

**2. Definitions and Notation.** The set of positive integers will be denoted by $N$. $S$ will denote the set of all finite sequences of positive integers, including the empty sequence $e$. For each non-negative integer $k$, we denote by $S_k$ the set of elements of $S$ of length $k$. For $s \in S$, $|s|$ will denote the length of $s$ and if $i \leq |s|$ is a positive integer, $s_i$ will denote the $i$-th coordinate of $s$. If $s \in S$ and $n \in N$, $sn$ will denote the catenation of $s$ and $n$. We put $\Sigma = N^N$. Endowed with the product of discrete topologies on $N$, $\Sigma$ becomes a homeomorph of the irrationals. For $\sigma \in \Sigma$ and $k \in N$, $\sigma_k$ will denote the $k$-th coordinate of $\sigma$ and $\sigma|k = (\sigma_1, \ldots, \sigma_k)$. If $k = 0$, $\sigma|k = e$. If $s \in S_k$, $\Sigma_s$ will denote the set $\{\sigma \in \Sigma : \sigma|k = s\}$.

Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. We denote by $\mathcal{A} \otimes \mathcal{B}$ the product of the $\sigma$-fields $\mathcal{A}$ and $\mathcal{B}$. We say that the $\sigma$-field $\mathcal{A}$ is *countably generated* if there exist subsets $A_n$, $n \in N$, of $X$ such that $\mathcal{A}$ is generated by $\{A_n : n \in N\}$. A non-empty set $A \in \mathcal{A}$ is called an A-*atom* if $A \supseteq B \in \mathcal{A} \Rightarrow B = A$ or $B = \emptyset$. If $Z \subseteq X$, $\mathcal{A}|Z$ will denote the trace of the $\sigma$-field $\mathcal{A}$ on $Z$. So, $\mathcal{A}|Z = \{A \cap Z : A \in \mathcal{A}\}$. If $X$ is a metric space, $\mathcal{B}_X$ will denote the *Borel $\sigma$-field* of $X$. If $E \subset X \times Y$ and $x \in X$, $E^x$ will denote the set $\{y \in Y : (x, y) \in E\}$ and will be called the *section* of $E$ at $x$. We use $\Pi_X$ to denote the projection from $X \times Y$ to $X$.

A *multifunction* $F : T \to X$ is a function whose domain is $T$ and whose values are non-empty subsets of $X$. A function $f : T \to X$ is called a *selector* for $F$ if $f(t) \in F(t)$ for each $t \in T$. The set $\{(t, x) \in T \times X : x \in F(t)\}$ is denoted by $Gr(F)$ and is called the *graph* of $F$. If $X$ is a metric space and $\mathfrak{I}$ is a $\sigma$-field on $T$, we say that $F$ is $\mathfrak{I}$-*measurable* if the set $\{t \in T : F(t) \cap V \neq \emptyset\} \in \mathfrak{I}$ for every open set $V$ in $X$.

Let $X$, $Y$ be topological spaces and $A \subset X$. We say that $A$ is a *retract* of $X$ if there is a continuous function $f : X \to A$ such that $f(x) = x$ for each $x \in A$. The map $f$ is called a *retraction* of $X$ onto $A$. A continuous function $g : X \to Y$ is called *closed* if for every closed set $C$ in $X$ $g(C)$ is relatively closed in the range of $g$.

The rest of our terminology is from [6].

Now we state two results which will be useful in the sequel.

Lemma 2.1. *Let $T$, $X$ be Polish spaces and $\mathfrak{I}$ a countably generated sub $\sigma$-field of $\mathcal{B}_T$. Let $B \in \mathfrak{I} \otimes \mathcal{B}_X$ and let the sections of $B$ be $\sigma$-compact. Then $\Pi_T(B) \in \mathfrak{I}$.*

Proof: By a result of Arsenin and Kunugui [1] (See also [13]) it follows that $\Pi_T(B)$ is Borel in $T$. Further, $\Pi_T(B)$ is a union of $\mathfrak{I}$-atoms. As $\mathfrak{I}$ is countably generated, by a result of Blackwell [2], $\Pi_T(B) \in \mathfrak{I}$.

The next is a very useful result for $G_\delta$-valued multifunctions. A proof of this is given in [10].

LEMMA 2.2 *Let $T$, $X$ be Polish spaces and $\mathfrak{J}$ a countably generated sub σ-field of $\mathcal{B}_T$. Let $G \in \mathfrak{J} \otimes \mathcal{B}_X$ and $G^t$ be a $G_\delta$ in $X$ for each $t \in T$. Then there exist sets $G_n \in \mathfrak{J} \otimes \mathcal{B}_X$ such that $G_n{}^t$ is open in $X$ for $t \in T$ and $n \in N$ and $G = \bigcap_{n=1}^{\infty} G_n$.*

**3. The zero-dimensional case.** Our first result is on closed valued multifunctions. This result is itself interesting and it is very easy to deduce (under a weaker measurability condition) Ioffe's representation theorem for closed valued multifunctions [5] from this

PROPOSITION 3.1 *Let $(T, \mathfrak{J})$ be a measurable space and $F: T \to \Sigma$ be a $\mathfrak{J}$-measurable multifunction such that $F(t)$ is closed in $\Sigma$ for each $t \in T$. Then there is a map $g: T \times \Sigma \to \Sigma$ such that*

(i) *for each $t \in T$, $g(t, .)$ is a closed retraction of $\Sigma$ onto $F(t)$, and*

(ii) *for $\sigma \in \Sigma$, $g(., \sigma)$ is $\mathfrak{J}$-measurable.*

*Proof.* Let $s \in S$. Let $T_s = \{t \in T: F(t) \cap \Sigma_s \neq \emptyset\}$. As $F$ is $\mathfrak{J}$-measurable, $T_s \in \mathfrak{J}$. Define a closed valued multifunction $F_s: T_s \to \Sigma$ by

$$F_s(t) = F(t) \cap \Sigma_s, \qquad t \in T_s.$$

$F_s$ is $\mathfrak{J}|T_s$-measurable. By the selection theorem of Kuratowski and Ryll-Nardzewski [8], we get a $\mathfrak{J}|T_s$-measurable selector $f_s: T_s \to \Sigma$ for $F_s$. Now, define $g: T \times \Sigma \to \Sigma$ by

$$\begin{aligned} g(t, \sigma) &= \sigma && \text{if } \sigma \in F(t) \\ &= f_{\sigma|n-1}(t) && \text{if } \sigma \notin F(t) \text{ and } n \text{ is the first positive integer } m \text{ such that } F(t) \cap \Sigma_{\sigma|m} = \emptyset. \end{aligned}$$

As $F$ is closed valued, $g$ is defined on all of $T \times \Sigma$. (i) is easily checked. To check (ii), fix a $\sigma \in \Sigma$, and define

$$T^n = \Big(\bigcap_{m<n} T_{\sigma|m}\Big) \setminus T_{\sigma|n}, \qquad n \in N$$

The sets $T^n$, $n \in N$, belong to $\mathfrak{J}$ and are pairwise disjoint.

Further,

$$g(t, \sigma) = f_{\sigma|n-1}(t) \qquad \text{if } t \in T^n$$

$$= \sigma \qquad \text{if } t \in T \setminus \Big(\bigcup_{n=1}^{\infty} T^n\Big).$$

It follows that $g(\,.\,, \sigma)$ is $\mathfrak{J}$-measurable.

*From now on, in this and in the next section, $T$, $X$ will denote arbitrary Polish spaces and $\mathfrak{J}$ a countably generated sub $\sigma$-field of $\mathcal{B}_T \cdot X$ will be given a complete metric such that diam $(X) < 1$. We fix a base $\{V_n : n \in N\}$ for the topology of $X$ such that it is closed under finite intersections and finite unions, $V_1 = \emptyset$ and $V_2 = X$. In this section $X$ will be, moreover, zero-dimensional and basic open sets will be closed as well. Finally, in both these sections $F: T \to X$ will denote a multifunction satisfying condition (A).* $G_n$, $n \in N$, will be a sequence of sets in $\mathfrak{J} \otimes \mathcal{B}_X$ *such that $G_n{}^t$ is open for $t \in T$ and $n \in N$ and $G = \bigcap_{n=1}^{\infty} G_n$, where $G$ denotes the graph of $F$.* The existence of such a sequence of sets is ensured by Lemma 2.2.

LEMMA 3.2 *Let $X$ be compact. Then for each $t \in T$ there is a system $\{n_s{}^t : s \in S\}$ of positive integers and a system $\{F_s{}^{(t)} : s \in S\}$ of clopen subsets of $X$ such that for $s \in S_k$, $k$ is a non-negative integer, and $t \in T$*

(i) *$t' \to n_s{}^{t'}$ is a $\mathfrak{J}$-measurable map defined on $T$,*

(ii) *diam $(F_s{}^{(t)}) < 2^{-k}$,*

(iii) *$G^t \subseteq F_e{}^{(t)}$ and $G^t \cap F_s{}^{(t)} \subseteq \bigcup_{\lambda=1}^{\infty} F_s{}^{(t)}$,*

(iv) *$F_{sm}{}^{(t)} \subseteq G_{k+1}{}^t \cap F_s{}^{(t)}$, $\quad m \in N$,*

(v) *$F_s{}^{(t)} = V_{n_s{}^t}$ if $k = 0$, or $k \in N$ and $s_k = 1$.*
*$= V_{n_s{}^t} \setminus \bigcup_{i < s_k} V_{n^t{}_{s|k-1,i}}$ if $k \in N$ and $s_k > 1$.*

*In particular, it follows that if $s, s' \in S_k$ and $s \neq s'$ then $F_s{}^{(t)} \cap F_{s'}{}^{(t)} = \emptyset$.*

*Proof.* We define these by induction on $|s|$.

Define $n_e{}^t = 2$ and $F_e{}^{(t)} = V_{n_e{}^t}$, $t \in T$. (i)-(v) are satisfied for $s = e$ and $t \in T$. Suppose $n_s{}^t$ and $F_s{}^{(t)}$ are defined for $t \in T$ and $s \in S$ of length $\leq k$ satisfying (i)-(v). Fix an $s \in S_k$. We observe that the set $\{t \in T : U \subseteq G^t{}_{k+1} \cap F_s{}^{(t)}\} \in \mathfrak{J}$ for every open set $U$ in $X$. To see this let $t \in T$. We have:

If $k = 0$, or $k \in N$ and $s_k = 1$, then

$$U \subseteq G^t{}_{k+1} \cap F_s{}^{(t)} \Leftrightarrow (\exists l \in N)\,(n_s{}^t = l \text{ and } U \subseteq G^t{}_{k+1} \cap V_l)$$

whereas if $k \in N$ and $s_k > 1$, then

$$U \subseteq G^t{}_{k+1} \cap F_s{}^{(t)} \Leftrightarrow (\exists (l_1, \ldots, l_{s_k}) \in N^{s_k} ((\forall i \leq s_k) (n^t{}_{s|k-1,i} = l_i)$$

and

$$U \subset G^t{}_{k+1} \cap (V_{l_{s_k}} \setminus \bigcup_{i<s_k} V_{l_i}))$$

By the induction hypothesis and Lemma 2.1, the assertion is now easy to check. For each $t \in T$, we now define $n^t{}_{sp}$, $p \in N$, by induction on $p$. For $m \in N$, let

$$\begin{aligned} T_m{}^0 = \emptyset \qquad & \text{if } \operatorname{diam}(V_m) \geq 2^{-(k+1)} \text{ or } m = 1 \\ = \{t \in T: V_m \subset G^t{}_{k+1} \cap F_s{}^{(t)} \\ & \text{and} \\ (\forall l < m)(\operatorname{diam}(V_l) < 2^{-(k+1)} \Rightarrow V_l \not\subset G^t{}_{k+1} \cap \\ F_s{}^{(t)})\}, \qquad & \text{if } \operatorname{diam}(V_m) < 2^{-(k+1)} \text{ and } m > 1 \end{aligned}$$

By the above observation, the sets $T_m{}^0$, $m \in N$, belong to $\mathfrak{I}$ and are pairwise disjoint. Define

$$\begin{aligned} n^t{}_{s1} &= m \qquad \text{if } t \in T_m{}^0 \\ &= 1 \qquad \text{if } t \in T \setminus \bigcup_{m=1}^{\infty} T_m{}^0 \end{aligned}$$

Clearly, the map $t \to n^t{}_{s1}$ is $\mathfrak{I}$-measurable. Suppose for some $p \in N$, maps $t \to n^t{}_{si}$ are defined for every $i \leq p$ and are $\mathfrak{I}$-measurable. For $m \in N$, let

$$T_m{}^p = \emptyset \qquad \text{if } \operatorname{diam}(V_m) \geq 2^{-(k+1)},$$

$$= \{t \in T: n^t{}_{sp} < m, V_m \subseteq G^t{}_{k+1} \cap F_s{}^{(t)}$$

and

$$(\forall l < m)(\operatorname{diam}(V_l) < 2^{-(k+1)} \Rightarrow (n^t{}_{sp} \geq m \text{ or}$$

$$V_l \not\subset G^t{}_{k+1} \cap F_s{}^{(t)}))\}, \qquad \text{if } \operatorname{diam}(V_m) < 2^{-(k+1)}.$$

The sets $T_m{}^p$, $m \geq 1$, belong to $\mathfrak{J}$ and are pairwise disjoint. Define

$$n^t{}_{s,p+1} = m \qquad \text{if } t \in T_m{}^p,$$

$$= 1 \qquad \text{if } t \in T \setminus \bigcup_{m=1}^{\infty} T_m{}^p.$$

As $s \in S_k$ and $p \in N$ were arbitrary, this completes the definition of $\{n^t{}_{s'} : s' \in S_{k+1}\}$. We define $\{F_{s'}{}^{(t)} : s' \in S_{k+1}\}$ satisfying ($v$), $t \in T$. It is easy to verify that the systems $\{n_s{}^t : s \in S\}$ and $\{F_s{}^{(t)} : s \in S\}$ thus defined satisfy the required conditions for each $t \in T$.

*Proof of (A) ⇒ (B) when X is a zero-dimensional, Polish space.* Since each zero-dimensional Polish space can be embedded in a zero-dimensional compact metric space in which it will automatically be a $G_\delta$, we see that it is sufficient to prove the result when $X$ is, moreover, compact. So, we assume that $X$ is a compact, zero-dimensional, metric space. We get a system $\{n_s{}^t : s \in S\}$ of positive integers and a system $\{F_s{}^{(t)} : s \in S\}$ of clopen sets in $X$ satisfying (i)-(v) of Lemma 3.2. We define a multifunction $H : T \to \Sigma$ by

$$H(t) = \{\sigma \in \Sigma : F_{\sigma|k}{}^{(t)} \neq \emptyset \text{ for all } k \in N\}, \qquad t \in T.$$

Using standard arguments, we show that $H(t)$ is closed in $\Sigma$ for each $t \in T$. Further, $H$ is $\mathfrak{J}$-measurable. To see this, let $t \in T$ and $s \in S_k$. Then

$$H(t) \cap \Sigma_s \neq \emptyset \Leftrightarrow G^t \cap F_s{}^{(t)} \neq \emptyset,$$

and if $k = 0$, or $k \in N$ and $s_k = 1$, then

$$F_s{}^{(t)} \cap G^t \neq \emptyset \Leftrightarrow F(t) \cap V_{n_s{}^t} \neq \emptyset$$

$$\Leftrightarrow (\exists l \in N)(n_s{}^t = l \quad \text{and} \quad F(t) \cap V_l \neq \emptyset)$$

whereas if $k \in N$ and $s_k > 1$, then

$$F_s{}^{(t)} \cap G^t \neq \emptyset \Leftrightarrow F(t) \cap (V_{n_s{}^t} \setminus \bigcup_{i < s_k} V_{n^t{}_{s|k-1,i}}) \neq \emptyset$$

$$\Leftrightarrow (\exists (l_1 \cdots l_{s_k}) \in N^{s_k})((\forall i \leq s_k)(n^t{}_{s|k-1,i} = l_i)$$

and

$$F(t) \cap (V_{l_{s_k}} \setminus \bigcup_{i < s_k} V_{l_i}) \neq \emptyset)$$

By $\mathfrak{I}$-measurability of $F$ and the condition (i) of Lemma 3.2, it follows that $\{t \in T : H(t) \cap \Sigma_s \neq \emptyset\} \in \mathfrak{I}$. Thus, $H$ is $\mathfrak{I}$-measurable. By Proposition 3.1, let $h : T \times \Sigma \to \Sigma$ be a map such that for each $t \in T$, $h(t, .)$ is a closed retraction of $\Sigma$ onto $H(t)$ and for each $\sigma \in \Sigma$, $h(., \sigma)$ is $\mathfrak{I}$-measurable.

Now, define a map $g : Gr(H) \to X$ by taking $g(t, \sigma)$ to be the unique point in $\bigcap_{k=1}^{\infty} F_{\sigma|k}^{(t)}$, $(t, \sigma) \in Gr(H)$. By standard arguments, we show that for each $t \in T$, $g(t, .)$ is a homeomorphism from $H(t)$ onto $G^t = F(t)$. Let $U \subseteq X$ be open and $(t, \sigma) \in Gr(H)$. Then

$$\begin{aligned} g(t, \sigma) \in U &\Leftrightarrow \bigcap_k F_{\sigma|k}^{(t)} \subseteq U \\ &\Leftrightarrow (\exists k)\,(F_{\sigma|k}^{(t)} \subseteq U) \\ &\Leftrightarrow (\exists s \in S)\,(\sigma \in \Sigma_s \text{ and } F_s^{(t)} \subseteq U). \end{aligned}$$

Thus,

$$g^{-1}(U) = Gr(H) \cap \bigcup_{s \in S} (\{t \in T : F_s^{(t)} \subset U\} \times \Sigma_s)$$

We argue as before and show that for every $s \in S$, $\{t \in T : F_s^{(t)} \subseteq U\} \in \mathfrak{I}$. It follows that $g$ is $\mathfrak{I} \otimes \mathcal{B}_\Sigma | Gr(H)$-measurable.

Finally, define $f : T \times \Sigma \to X$ by

$$f(t, \sigma) = g(t, h(t, \sigma)),\ t \in T,\ \sigma \in \Sigma.$$

It is easily checked that $f$ has the desired properties.

**4. The General Case.** The main idea contained in this part of the proof is contained in Ponomarev [12].

LEMMA 4.1. *Let $X$ be compact. Then for $t \in T$ and $i, j \in N$ there exist positive integers $n_{ij}^t$ and $n_i^t$ such that*

(i) *the maps $t \to n_i^t$ and $t \to n_{ij}^t$ are $\mathfrak{I}$-measurable,*

(ii) $diam\,(V_{n_{ij}^t}) \leq 2^{-i}$,
(iii) $\overline{F(t)} \subseteq \bigcup_{m=1}^{\infty} V_{n_{im}^t}$,
(iv) $m > n_i^t \Rightarrow n_{im}^t = 1$.

*Proof.* Let $\tilde{G} = \{(t, x) \in T \times X : x \in \overline{F(t)}\}$. For every open set $U$ in $X$, $\{t \in T : \tilde{G}^t \cap U \neq \emptyset\} = \{t \in T : F(t) \cap U \neq \emptyset\} \in \mathfrak{J}$. Fix $i \in N$. We shall define maps $t \to n_{ij}^t$, $j \in N$, by induction on $j$. For $m \in N$, let

$$\begin{aligned} T_m^0 &= \emptyset \qquad \text{if diam}\,(V_m) \geq 2^{-i} \\ &= \{t \in T : \tilde{G}^t \cap V_m \neq \emptyset \text{ and} \\ &\qquad (\forall l < m)(\text{diam}\,(V_l) < 2^{-i} \Rightarrow \tilde{G}^t \cap V_l = \emptyset)\}, \\ &\qquad \text{if diam}\,(V_m) < 2^{-i}. \end{aligned}$$

By the above observation, the sets $T_m^0$, $m \in N$, belong to $\mathfrak{J}$ and are pairwise disjoint. Also, $T = \bigcup_{m=1}^{\infty} T_m^0$. We define

$$n_{i1}^t = m \qquad \text{if } t \in T_m^0.$$

The map $t \to n_{i1}^t$ is clearly $\mathfrak{J}$-measurable. Now, suppose for some $p \in N$, $n_{ij}^t$ is defined for all $j \leq p$ and $t \in T$ and the maps $t \to n_{ij}^t$, $j \leq p$, are $\mathfrak{J}$-measurable. We observe that for every open set $U$ in $X$,

$$\{t \in T : (\tilde{G}^t \setminus \bigcup_{j \leq p} V_{n_{ij}^t}) \cap U \neq \emptyset\} \in \mathfrak{J}.$$

To see this, first observe that if $t \in T$ and $x \in X$, then

$$\begin{aligned} (t, x) \notin \tilde{G} &\Leftrightarrow x \notin \overline{F(t)} \\ &\Leftrightarrow (\exists n \in N)(x \in V_n \text{ and } V_n \cap F(t) = \emptyset). \end{aligned}$$

So that

$$T \times X \setminus \tilde{G} = \bigcup_{n=1}^{\infty} (\{t \in T : F(t) \cap V_n = \emptyset\} \times V_n) \in \mathfrak{J} \otimes \mathcal{B}_X.$$

The above assertion now follows from the induction hypothesis, Lemma 2.1 and the following equivalence for every $t \in T$:

$$(\tilde{G}^t \setminus \bigcup_{j \leq p} V_{n_{ij}^t}) \cap U \neq \emptyset \Leftrightarrow (\exists (l_1 \cdots l_p) \in N^p)((\forall_j \leq p)(n_{ij}^t = l_j)$$

and

$$(\tilde{G}^t \setminus \bigcup_{j \le p} V_{l_j}) \cap U \neq \emptyset).$$

For $m \in N$, define

$$T_m{}^p = \emptyset \qquad \text{if diam}\,(V_m) \ge 2^{-i}$$

$$\{t \in T : n_{ip}{}^t < m, (\tilde{G}^t \setminus \bigcup_{j \le p} V_{n_{ij}{}^t}) \cap V_m \neq \emptyset$$

and

$$(\forall l < m)\,(\text{diam}(V_l) < 2^{-i} \Rightarrow (l \le n_{ip}{}^t \text{ or}$$

$$(\tilde{G}^t \setminus \bigcup_{j \le p} V_{n_{ij}{}^t}) \cap V_l = \emptyset))\}, \qquad \text{if diam}\,(V_m) < 2^{-i}.$$

By the observation made above, it follows that the sets $T_m{}^p$, $m \in N$, belong to $\mathfrak{I}$ and are pairwise disjoint. We define

$$n^t{}_{i,p+1} = m \qquad \text{if } t \in T_m{}^p$$

$$= 1 \qquad \text{if } t \in T \setminus \bigcup_{m=1}^{\infty} T_m{}^p.$$

As $p \in N$ was arbitrary, this completes the definition of the maps $t \to n_{ij}{}^t$, $j \in N$. To define $n_i{}^t$, $t \in T$, notice that $\tilde{G}^t$ is compact and so, $(\exists m \in N)$ $(\forall l > m)\,(n_{il}{}^t = 1)$. We define $n_i{}^t$ to be the first such positive integer $m$, $t \in T$. It is an easy matter to verify that conditions (i)-(iv) are satisfied.

LEMMA 4.2 *Let $X$ be compact. Then there is a set $B \subseteq T \times \Sigma$ and a map $g : B \to X$ such that for $t \in T$*

(i) $B \in \mathfrak{I} \otimes \mathcal{B}_\Sigma$,
(ii) $B^t$ *is non-empty and compact.*
(iii) $g(t, .)$ *is a continuous map from* $B^t$ onto $\overline{F(t)}$,
(iv) *$D$ is a dense subset of* $\overline{F(t)} \Rightarrow \{\sigma \in \Sigma : g(t, \sigma) \in D\}$ *is dense in* $B^t$,
(v) *$g$ is* $(\mathfrak{I} \otimes \mathcal{B}_\Sigma)|B$*-measurable.*

*Proof.* For $t \in T$ and $i, j \in N$ we get positive integers $n_i{}^t$ and $n_{ij}{}^t$ satisfying condition (i)-(iv) of Lemma 4.1. Let $\tilde{G} = \{(t, x) \in T \times X : x \in \overline{F(t)}\}$ and let

$$U_{ij}{}^{(t)} = V_{n_{ij}{}^t} \cap \tilde{G}^t \qquad \text{if } j = 1$$

$$(V_{n_{ij}{}^t} \cap \tilde{G}^t) \setminus \bigcup_{l<j} (\overline{V_{n_{il}{}^t} \cap \tilde{G}^t}) \qquad \text{if } j > 1.$$

We have

(1) $U_{ij}{}^{(t)}$ is relatively open in $\tilde{G}^t$,

(2) $\operatorname{diam}(U_{ij}{}^{(t)}) < 2^{-i}$,

(3) $m \neq n \Rightarrow U_{im}{}^{(t)} \cap U_{in}{}^{(t)} = \emptyset$,

(4) $m > n_i{}^t \Rightarrow U_{im}{}^{(t)} = \emptyset$

(5) $\tilde{G}^t = \bigcup_{k=1}^{\infty} \overline{U_{ik}{}^{(t)}}$

(6) for every open set $U$ in $X$, $\{t \in T : \tilde{G}^t \cap U \subseteq U_{ij}{}^{(t)}\} \in \mathfrak{J}$,

(7) if $P$ is a finite subset of $N \times N$ and if $U \subseteq X$ is open then

$$\{t \in T : \bigcap_{(m,n) \in P} U_{mn}{}^{(t)} \cap U \neq \emptyset\} \in \mathfrak{J}.$$

Properties (1)-(5) are clear. To see (6), notice that if $j = 1$

$$\tilde{G}^t \cap U \subseteq U_{ij}{}^{(t)} \Leftrightarrow (\exists l \in N)\,(n_{ij}{}^t = l \text{ and } \tilde{G}^t \cap U \subseteq V_l)$$

while if $j > 1$

$$\tilde{G}^t \cap U \subset U_{ij}{}^{(t)} \Leftrightarrow \tilde{G}^t \cap U \subset V_{n_{ij}{}^t} \setminus \bigcup_{k<j} (\overline{V_{n_{ik}{}^t} \cap \tilde{G}^t})$$

$$\Leftrightarrow \tilde{G}^t \cap U \subseteq V_{n_{ij}{}^t} \text{ and } (\forall k < j)\,(\tilde{G}^t \cap U \cap V_{n_{ik}{}^t} = \emptyset)$$

$$\Leftrightarrow (\exists (l_1 \cdots l_j) \in N^j)\,((\forall k \leq j)\,(n_{ik}{}^t = l_k),\ \tilde{G}^t \cap U \subseteq V_{l_j} \text{ and } (\forall k < j)\,(\tilde{G}^t \cap U \cap V_{l_k} = \emptyset))$$

Now, (6) follows from (i) of Lemma 4.1 and Lemma 2.1. (Note that $\tilde{G} \in \mathfrak{J} \otimes \mathcal{B}_X$). To prove (7), first notice that

$$\bigcap_{(m,n)\in P} U_{mn}^{(t)} \cap U \neq \emptyset \Leftrightarrow (\exists k \in N)(V_k \subseteq U \text{ and}$$

$$(\forall (m,n) \in P)(\tilde{G}^t \cap V_k \subseteq U_{mn}^{(t)})).$$

Now, (7) follows from (6).

For $t \in T$ and $i, j, \in N$, we define the following by induction on $i$:

$$\begin{aligned} m_i^t &= n_i^t && \text{if } i = 1 \\ &= m_{i-1}^t \cdot n_i^t && \text{if } i > 1, \\ \text{and} \qquad W_{ij}^{(t)} &= U_{ij}^{(t)} && \text{if } i = 1 \\ &= W_{i-1,k}^{(t)} \cap U_{il}^{(t)} && \text{if } i > 1,\ 1 \leq k \leq m_{i-1}^t, \\ & && 1 \leq l \leq n_i^t \text{ and } j = (k-1)n_i^t + l \\ &= \emptyset && \text{if } j > m_i^t. \end{aligned}$$

We have

(a) the map $t \to m_i^t$ is $\mathfrak{J}$-measurable,
(b) $W_{ij}^{(t)}$ is relatively open in $\tilde{G}^t$,
(c) $\operatorname{diam}(W_{ij}^{(t)}) < 2^{-i}$,
(d) $m \neq n \Rightarrow W_{im}^{(t)} \cap W_{in}^{(t)} = \emptyset$,
(e) $k > m_i^t \Rightarrow W_{ik}^{(t)} = \emptyset$,
(f) $\tilde{G}^t = \bigcup_{l=1}^{\infty} \overline{W_{il}^{(t)}}$,
(g) $\underline{(\forall (i,j) \in N \times N)}\,(\exists k \in N)\,(W_{i+1,j}^{(t)} \subseteq W_{ik}^{(t)})$,
(h) $\overline{W_{i+1,j}^{(t)}} \subseteq \overline{W_{ik}^{(t)}} \Rightarrow W_{i+1,j}^{(t)} \subseteq W_{ik}^{(t)},\ k \in N$,
(i) $\{t \in T : W_{ij}^{(t)} \neq \emptyset\} \in \mathfrak{J}$,
(j) $U$ is open in $X \Rightarrow \{t \in T : \overline{W_{ij}^{(t)}} \subseteq U\} \in \mathfrak{J}$,
(k) $\{t \in T : W_{k+1,m}^{(t)} \subseteq W_{kn}^{(t)}\} \in \mathfrak{J}$.

(a)-(h) are easily verified. (i) follows from (7). Also, from (7), we get that the closed set-valued function $t \to \overline{W_{ij}^{(t)}}$ is $\mathfrak{J}$-measurable. Hence, by [15, Theorem 4.2], its graph is in $\mathfrak{J} \otimes \mathcal{B}_X$. Now, (j) follows from Lemma 2.1. To verify (k), notice

$$\begin{aligned} \{t \in T : W_{k+1,m}^{(t)} \subseteq W_{kn}^{(t)}\} &= \{t \in T : W_{k+1,m}^{(t)} = \emptyset\}\ U\ \{t \in T : \emptyset \\ &\qquad \neq W_{k+1,m}^{(t)} \subseteq W_{kn}^{(t)}\} \\ &= \{t \in T : W_{k+1,m}^{(t)} = \emptyset\}\ U \cup \{t \in T : j_n \\ &\qquad \leq m_k^t,\ p = n_{k+1}^t\}, \end{aligned}$$

where the last union is taken over all $(p, q) \in N \times N$ such that $q \leq p$ and $j_m = (j_n - 1).p + q$. Now, (k) follows from (i), (a) and (i) of Lemma 4.1.

We define

$$B = \{(t, \sigma) \in T \times \Sigma : (\forall k)(W_{k\sigma_k}{}^{(t)} \neq \emptyset \text{ and } \overline{W^{(t)}{}_{k+1,\sigma_{k+1}}} \subseteq \overline{W_{k\sigma_k}{}^{(t)}})\}$$

$$= \{(t, \sigma) \in T \times \Sigma : (\forall k)(W_{k\sigma_k}{}^{(t)} \neq \emptyset \text{ and } W^{(t)}{}_{k+1,\sigma_{k+1}} \subseteq W_{k\sigma_k}{}^{(t)})\}$$

$$= \{(t, \sigma) \in T \times \Sigma : (\forall k)(W_{1\sigma_1}{}^{(t)} \supseteq \cdots \supseteq W_{k\sigma_k}{}^{(t)} \neq \emptyset)\}$$

$$= \bigcap_{k=1}^{\infty} \bigcup_{s \in S_k} (\{t \in T : W_{1s_1}{}^{(t)} \supseteq \cdots \supseteq W_{ks_k}{}^{(t)} \neq \emptyset\} \times \Sigma_s)$$

From (i) and (k), it follows that $B \in \mathfrak{I} \otimes \mathcal{B}_\Sigma$. By König's infinity lemma [7, pp. 326] we get that $B^t \neq \emptyset$, for each $t \in T$. It is easy to check that for $t \in T$, $B^t$ is closed in $\Sigma$ and $B^t \subseteq \times_{i=1}^{\infty}(\{1, \ldots, m_i{}^t\})$. Thus, $B^t$ is a non-empty, compact subset of $\Sigma$, $t \in T$. We define $g : B \to X$ by taking $g(t, \sigma)$ to be the unique point in $\bigcap_{k=1}^{\infty} \overline{W_{k\sigma_k}{}^{(t)}}$, $(t, \sigma) \in B$. Using König's infinity lemma, we check that $g(t, .)$ is a continuous map from $B^t$ onto $\tilde{G}^t$, $t \in T$.

For a proof of (iv) the reader is referred to Ponomarev [12]. Finally, if $(t, \sigma) \in B$ and $U$ is open in $X$, then

$$g(t, \sigma) \in U \Leftrightarrow (\exists k \in N)(\exists m \in N)(\overline{W_{km}{}^{(t)}} \subseteq U \text{ and } \sigma_k = m)$$

From (j), it follows that $g$ is $\mathfrak{I} \otimes \mathcal{B}_\Sigma | B$-measurable.

*Proof of* $(A) \Rightarrow (B)$. Since each Polish space can be embedded in a compact metric space in which it will automatically be a $G_\sigma$, it is sufficient to prove the result for a compact metric $X$. So we assume that $X$ is a compact metric space. We get a set $B \subseteq T \times \Sigma$ and a map $g : B \to X$ satisfying conditions (i)–(v) of Lemma 4.2. We define a multifunction $H : T \to \Sigma$ by

$$H(t) = \{\sigma \in \Sigma : g(t, \sigma) \in F(t)\}, \qquad t \in T.$$

$H(t)$ is a non-empty, $G_\delta$ set in $\Sigma$ and by (iv) of Lemma 4.2, $H(t)$ is dense in $B^t$, $t \in T$. Thus by (i) and (ii) of Lemma 4.2 and Lemma 2.1, it follows that $H$ is $\mathfrak{I}$-measurable. By (i) and (v) of Lemma 4.2 and the fact that $Gr(F) \in \mathfrak{I} \otimes \mathcal{B}_X$, we get that $Gr(H) \in T \otimes B_\Sigma$. By $(A) \Rightarrow (B)$ for zero-dimensional Polish spaces proved in section 3, we get a map $h: T \times \Sigma \to \Sigma$ such that for each $t \in T$, $h(t, .)$ is continuous, closed and onto $H(t)$ and for each $\sigma \in \Sigma$, $h(., \sigma)$ is $\mathfrak{J}$-measurable. Define $f: T \times \Sigma \to X$ by

$$f(t, \sigma) = g(t, h(t, \sigma)),\ t \in T,\ \sigma \in \Sigma.$$

It is easily checked that $f$ satisfies $(B)$.

**5. Proof of** $(B) \Rightarrow (A)$. We first check that $F$ is $\mathfrak{J}$-measurable. Let $\{\sigma^n : n \in N\}$ be a dense sequence in $\Sigma$. Then $\{f(t, \sigma^n): n \in N\}$ is dense in $F(t)$, $t \in T$. Therefore, for $U \subseteq X$ open,

$$\{t \in T: F(t) \cap U \neq \emptyset\} = \bigcup_{n=1}^{\infty} (f(., \sigma^n)^{-1}(U)) \in \mathfrak{J}.$$

Now, let $\{U_n : n \in N\}$ and $\{V_n : n \in N\}$ be bases for $\Sigma$ and $X$ respectively. We define a set $B \subseteq T \times \Sigma$ as follows:

> $(t, \sigma) \in B \Leftrightarrow (\exists x \in X)$ (either $f(t, .,)^{-1}(x)$ is not open and $\sigma$ is a boundary point of it, or $f(t, .)^{-1}(x)$ is open and $\sigma = \sigma^n$, where $n$ is the first positive integer $m$ such that $f(t, \sigma^m) = x$).

It is easily checked that for $t \in T$, $B^t$ is closed in $\Sigma$ and $f(t, B^t) = f(t, \Sigma) = F(t)$. It follows from a result of Vaĭnšteĭn [14] (see also [3, p. 204]) that the restriction of $f(t, .)$ on $B^t$ is perfect. Vaĭnšteĭn [14] proved that if a separable metric space $Z$ is the image of a Polish space under a perfect map, $Z$ is Polish. From this it follows that $F(t)$ is a $G_\delta$ in $X$ for each $t \in T$. Finally, observe that

$$\begin{aligned}(t,\sigma)\in B \Leftrightarrow \text{Either } & [(\forall m)\{\sigma \in U_m \Rightarrow (\exists k)(\sigma^k \in U_m \text{ and } f(t, \sigma^k) \neq f(t,\sigma))\}] \\ \text{or } & [(\exists n)\{\sigma = \sigma^n, (\forall l < n)(f(t, \sigma^l) \neq f(t, \sigma^n)) \\ & \quad \text{and } (\exists p)(f(t, \sigma^n) \in V_p \text{ and} \\ & \qquad (\forall l)(f(t, \sigma^l) \in V_p \Rightarrow f(t, \sigma^l) = f(t, \sigma^n)))\}]\end{aligned}$$

Thus, $B \in \mathfrak{J} \otimes \mathcal{B}_\Sigma$. Now,

$$(t, x) \in Gr(F) \Leftrightarrow (\exists \sigma \in \Sigma)((t, \sigma) \in B \text{ and } f(t, \sigma) = x).$$

Therefore

$$Gr(F) = \Pi_{T \times X} (\{(t, \sigma, x) \in T \times \Sigma \times X : (t, \sigma) \in B \text{ and } f(t, \sigma) = x\})$$

By Lemma 2.1, $Gr(F) \in \mathfrak{J} \otimes \mathcal{B}_X$.

INDIAN STATISTICAL INSTITUTE, CALCUTTA.

## REFERENCES

[1] W. Arsenin and A. Ljapunov, "Theory of A sets," (Russian). *Uspekhi* **5** (1950), pp. 45-108.

[2] D. Blackwell, "On a class of probability spaces," *Proc. 3rd. Berkeley Sympos. Math. Statist. and Prob.* 2 (1956), pp. 1-6.

[3] R. Engelking, *Outline of general topology*, North-Holland, Amsterdam, 1968.

[4] ______, "On closed images of the space of irrationals," *Proc. Amer. Math. Soc.* **21** (1969), pp. 583-586.

[5] A. D. Ioffe, "Representation theorems for multifunctions and analytic sets," *Bull. Amer. Math. Soc.* **84** (1978), pp. 142-144.

[6] K. Kuratowski, *Topology*, Vol. 1, Academic Press, New York and London, PWN, Warsaw, 1966.

[7] ______, and A. Mostowski, *Set Theory*, North-Holland Publishing Company, Amsterdam, New York, Oxford, PWN, 1976.

[8] ______, and C. Ryll-Nardzewski, "A general theorem on selectors," *Bull. Acad. Polon. Sci., Ser. Sci. Math. Astron. Phys.* **13** (1965), pp. 397-403.

[9] H. Sarbadhikari and S. M. Srivastava, "Parametrizations of $G_\delta$-valued multifunctions," *Trans. Amer. Math. Soc.*, to appear.

[10] S. M. Srivastava, "Selection Theorems for $G_\delta$-valued multifunctions," *Trans. Amer. Math. Soc.*, 254(1979), pp. 283-294.

[11] ______, "A representation theorem for closed valued multifunctions," *Bull. Acad. Polon. Sci.*, to appear.

[12] V. Ponomarev, "Normal spaces as images of zero-dimensional ones," *Soviet Math. Doklady* **1** (1960), pp. 774-777.

[13] J. Saint-Raymond, "Boreliens A Coupes $k_\sigma$," *Bull. Soc. Math. France.* **104** (1976), pp. 389-400.

[14] I. A. Vaĭnšteĭn, "On closed mappings of metric spaces," *Doklady Akad. Nauk SSSR* (NS) **57** (1947), pp. 319-321.

[15] D. H. Wagner, "Survey of Measurable Selection Theorems," *Siam J. Control and Optimization*, **15** (1977), pp. 859-903.