Econometrica, Vol. 52, No. 5 (September, 1984)

EFFECTIVITY FUNCTIONS AND ACCEPTABLE GAME FORMS

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A game form is *acceptable* if for every preference profile, a Nash equilibrium exists and the outcomes corresponding to Nash equilibria are Pareto efficient. A game form is *strongly consistent* if the set of strong Nash equilibria is always nonempty. The paper shows that no game form can be both acceptable and strongly consistent. The set of game forms which are both acceptable and dominance-solvable is also characterized in terms of the effectivity functions of game forms.

1. INTRODUCTION

THE MAIN FOCUS of the literature on implementation is on rules by which a group of individuals arrives at a choice among available alternatives. Such a rule, called a game form, specifies the permissible strategies for each individual and associates with each combination of permissible strategies a particular element out of the set of alternatives.

The result of Gibbard [2] and Satterthwaite [13] demonstrates that when there are more than two alternatives, there is no nondictatorial game form with dominant strategies for all individual preference profiles. In a noncooperative framework, if agents do not have dominant strategies, then it is no longer possible to determine unambiguously the outcome selected by the agents. In order to predict the outcome(s) selected by the agents, one has to first describe the strategic behavior of agents, or in other words, to select an equilibrium concept for games in strategic form.

In the noncooperative framework, the main equilibrium concepts analyzed have been Nash equilibrium and sophisticated behavior (see [1]), the latter being closely related to the notion of perfect equilibrium. Moulin [6, 7] presents several interesting results on dominance solvable game forms, which are game forms under which sophisticated behavior of the agents leads to a single outcome for all preference profiles. Hurwicz and Schmeidler [3] show that there are several game forms with the property that all Nash equilibria are Pareto-efficient and the set of Nash equilibria is always nonempty. (Such game forms are called acceptable.)

In the cooperative context, Maskin [5] shows that when there are more than two alternatives, only dictatorial game forms possess the property that the outcome set corresponding to strong equilibria is a singleton. However, the class of strongly consistent game forms, i.e., game forms under which the set of strong equilibria is always nonempty, has been characterized by Moulin and Peleg [9] in terms of effectivity functions. (An effectivity function is a representation of the distribution of power in the game form.)

Clearly, these results dispel to some extent the pessimism created by the Gibbard–Satterthwaite result. If the planner *knows* which concept of equilibrium

[1] I am most grateful to two anonymous referees for remarkably detailed comments on an earlier draft of the paper.

is the best description of the individual's strategic behavior, then he can construct a suitable game form which performs reasonably under that notion of equilibrium. However, suppose the planner believes that equilibrium of type α is the relevant concept and based on this belief he specifies the game form g^*. His belief may be wrong in the sense that the informational framework and other structural characteristics lead agents to select outcomes which correspond to equilibria of type β. Since g^* may be an "undesirable" mechanism under equilibrium of type β, the planner should choose mechanisms which perform well under *several* equilibrium concepts.

The main focus of this paper is on the possibility of constructing acceptable game forms which are also strongly consistent and/or dominance-solvable. The effectivity function of a strongly consistent or dominance-solvable game form is *maximal.* We show that if the effectivity function of a game form is maximal, then the game form is acceptable only if its effectivity function is that of a strong, proper simple game. This immediately implies that no game form can be both acceptable and strongly consistent. However, we characterize the class of acceptable and dominance-solvable game forms. It turns out that for every strong, proper, simple game, there is an acceptable and dominance-solvable game form whose effectivity function coincides with that of the given simple game.

2. THE FRAMEWORK

Let $N = \{1, 2, \ldots, n\}$ be a set of voters or players, with $n \geq 3$. A is a set of alternatives with $|A| \geq 3$. We denote by L the set of all *linear orderings*[2] over A. For every $i \in N$, $R_i \in L$ is the preference relation of i on the outcome space A. A *special choice correspondence* (SCC) summarizes the ethical norms guiding society's choice out of A.

DEFINITION 2.1: An SCC[3] is a function $H: L^N \to 2^A$.

We note that in our notation for any set B, 2^B represents the set of all *non-empty* subsets of B. So, for any preference profile $R^N \in L^N$, $H(R^N)$ represents the set of "best" outcomes for society, given the ethical considerations underlying H. In this paper, we will restrict attention to those SCC's satisfying the well-known Pareto criterion. For any $R^N \in L^N$, let $Q(R^N) = \{a \in A \mid$ for no $b \in A$, bR_ia for all $i \in N\}$.

DEFINITION 2.2: An SCC H is *Paretian* iff for all $R^N \in L^N$, $H(R^N) \subseteq Q(R^N)$.

[2] A binary relation B defined over a set Y is a linear ordering iff it satisfies: (i) *Connectedness*: For all distinct $x, y \in Y$, (xBy or yBx). (ii) *Transitivity*: For all $x, y, z \in Y$, $[(xBy$ and $yBz) \to (xBz)]$. (iii) *Antisymmetry*: For all distinct $x, y \in Y$, $(xBy \to \sim yBx)$.

[3] If for all preference profiles, the SCC is single-valued, then it is called a social choice function (SCF).

In view of the fact that SCC's do not induce truthful revelation of preferences, SCC's cannot in general be used as the actual aggregation mechanism. In order to "implement" a given SCC, the planner may have to take recourse to more abstract aggregation mechanisms.

DEFINITION 2.3: A *game form* g is a $(n+1)$-tuple $g=(X_i,\ i \in N; \pi)$ where (i) X_i is the strategy set (or message space) of individual i, (ii) π is a single valued mapping from $X_N = \times_{i \in N} X_i$ *onto* A.

The mapping π describes the decision rule, if for all i, agent i chooses strategy x_i, the resulting strategy n-tuple is denoted $\underline{x}=(x_i)_{i \in N}$ and the decision rule π forces the outcome $\pi(\underline{x}) \in A$.

For any $T \in 2^N - \{N\}$, $x_T=(x_i)_{i \in T}$; $x_{-T}=(x_i)_{i \in N-T}$. Thus, a strategy n-tuple $\underline{x}$ will often be represented as (x_T, x_{-T}). When T is a singleton, we will simply write x_{-i} instead of $x_{-\{i\}}$, etc. A game form g is *nondictatorial* iff there does not exist $i \in N$ such that for all $a \in A$, there is $x_i \in X_i$ with $\pi\ (x_i, x_{-i}) = a$ for all $x_{-i} \in X_{-i}$. Since dictatorial game forms represent an extremely arbitrary distribution of power, we only consider nondictatorial game forms.

Suppose the planner specifies a particular game form g to the set of players. In order to "predict" the outcome(s) selected by the agents, the planner must first know the strategic behavior of the agents. This amounts to selecting an equilibrium concept for games in strategic form. Each equilibrium reflects the information that each individual has about the others' preferences and strategic behavior, etc. In this paper, we will be concerned with three concepts of equilibrium behavior.

DEFINITION 2.4: Given a game form $g=(X_i,\ i \in N; \pi)$, and a profile $R^N \in L^N$, $\underline{x} \in X_N$ is a *Nash equilibrium* of g at R^N iff:

$$\text{for all } i \in N, \text{ for all } y_i \in X_i, \quad \pi(y_i, x_{-i}) \neq \pi(\underline{x}) \neq \pi(\underline{x}) R_i \pi(y_i, x_{-i}).$$

We denote by $\mathrm{NE}(g, R^N)$ the possibly empty set of Nash equilibria of g at R^N.

In order to introduce our next notion of equilibrium-sophisticated behavior, we need some further notation. Let the game form g be given, and R^N be a fixed profile. For any subsets $Y_i \subseteq X_i$, and any agent $j \in N$, we denote by $D_j(R_j;\ Y_i,\ i \in N)$ the set of agent j's undominated strategies when the strategy spaces are restricted to Y_i, $i \in N$. Hence x_j belongs to $D_j(R_j;\ Y_i,\ i \in N)$ iff $x_j \in Y_j$ and there is no $y_j \in Y_j$ such that:

$$\text{for all } x_{-j} \in Y_{-j}: \quad \pi(x_j, x_{-j}) \neq \pi(y_j, x_{-j}) \rightarrow \pi(y_j, x_{-j}) R_j \pi(x_j, x_{-j})$$

$$\text{and there is } x_{-j} \in Y_{-j}: \quad \pi(x_j, x_{-j}) \neq \pi(y_j, x_{-j}).$$

The successive elimination of dominated strategies is the following N-tuple of decreasing sequences: $X_j^t, j \in N, t \in N$:

$$X_j = X_j^0, \qquad X_j^{t+1} = D_j\big(R_j; X_i^t, i \in N\big) \subset X_j^t.$$

Let for all integers t, $X^t = \times_{i \in N} X_i^t$.

DEFINITION 2.5: g is *dominance solvable* iff for all $R^N \in L^N$, there is an integer t such that $\pi(X^t)$ is a singleton.

In this case, X_i^t is called the set of *sophisticated strategies* of player i, and will be denoted by $D^S(g, R_i)$. Let $D^S(g, R^N) = \times_{i \in N} D^S(g, R_i)$.

The behavioral assumptions underlying the concept of sophisticated behavior are complete information and noncooperation. Each agent is aware of the whole preference profile, and eliminates his dominated strategies on the assumption that others are also doing so. A game form is dominance-solvable if the successive elimination of dominated strategies reduces the strategy space to X^t and π is constant over X^t.

While both Nash equilibrium and sophisticated behavior do not require collusion amongst agents, the concept of strong equilibrium assumes the possibility of collusion among agents.

DEFINITION 2.6: Let $g = (X_i, i \in N; \pi)$ be a given game form. $x \in X_N$ is a *strong equilibrium* of g at R^N iff there is no $T \in 2^N$ and $x'_T \in X_T$ such that $\pi(x'_T, x_{-T}) R_i \pi(\underline{x})$ for all $i \in T$.

Let $\mathrm{SE}(g, R^N)$ denote the set of strong equilibria of g at R^N. So, if $\underline{x} \in \mathrm{SE}(g, R^N)$, then no coalition T can deviate from $\underline{x}$ and make everyone in T better off. Clearly, $\mathrm{SE}(g, R^N) \subseteq \mathrm{NE}(g, R^N)$.

Armed with these notions of equilibrium, we can now make precise the concept of implementation. An SCC H is implementable via equilibrium of type α iff there is a game form g such that for each profile, the set of outcomes associated with equilibria of type α under g coincides with the set of best outcomes under H. The underlying motivation for this definition is that whatever the individuals' preferences might be, and provided their strategic behavior is adequately described by equilibria of type α, the individuals will eventually select the outcomes recommended by H.

DEFINITION 2.7: (i) An SCC H is *Nash implementable* iff there is a game form g such that for all $R^N \in L^N$, $H(R^N) = \pi(\mathrm{NE}(g, R^N))$.

(ii) An SCC H is *strongly implementable* iff there is a game form g such that for all $R^N \in L^N$, $H(R^N) = \pi(\mathrm{SE}(g, R^N))$.

(iii) An SCC H is *implementable via sophisticated equilibrium* iff there is a dominance-solvable game form g such that for all $R^N \in L^N$, $H(R^N) = \pi(D^S(g, R^N))$.

Note that if H is implementable via sophisticated equilibrium, then $H(R^N)$ is always a singleton, i.e., H is a *social choice function* (SCF).

Suppose H is Nash implementable by the game form g, and moreover H is Paretian. Then, g has the property that $\pi(\mathrm{NE}(g, R^N)) \subseteq Q(R^N)$ and $\mathrm{NE}(g, R^N) \neq \emptyset$ for all $R^N \in L^N$. Following Hurwicz and Schmeidler **[3]**, such game forms will be called *acceptable*. Similarly, if an SCC h is strongly implementable by a game form g, then for all profiles $R^N \in L^N$, $\mathrm{SE}(g, R^N) \neq \emptyset$. (Of course, outcomes associated with strong equilibria must necessarily be Pareto efficient.) Such game forms will be called *strongly consistent*, a terminology due to Moulin and Peleg **[9]**.

3. THE INCOMPATIBILITY OF STRONG CONSISTENCY AND ACCEPTABILITY

The main result of this section is to show that no game form can be both acceptable as well as strongly consistent. As we have remarked earlier, the literature on implementation[4] has provided several interesting positive results. For instance, the results of Maskin **[4, 5]**, Moulin and Peleg **[9]**, and Moulin **[6, 7]** show that a large class of SCC's is implementable under all three notions of equilibrium mentioned above. So, if the planner *knows* which concept of equilibrium is the best description of agents' strategic behavior, then he can construct a suitable game form implementing a reasonable SCC. However, suppose the planner believes that strong equilibrium is the relevant concept, and based on this belief, he specifies the strongly consistent game form g^* which strongly implements the desired SCC H^*. However, it may not be possible for agents to collude, and there may be many Nash equilibria of g^* whose outcomes are not elements of H^*. In particular, some Nash equilibria may not even be Pareto efficient. Faced with this possibility, the planner may well want to search for game forms which are both strongly consistent as well as acceptable. Our result shows that this is not possible.

This result may seem somewhat counter-intuitive since there are game forms which are both strongly consistent as well as dominance-solvable. Maskin **[4]** had shown that when $|A| \geq |N|$,[5] a strongly consistent game form must endow at least some agent with "veto power," i.e., the power to eliminate alternatives unilaterally. Following this insightful result, Moulin and Peleg **[9]** identified the set of strongly consistent game forms; it turns out that Mueller's **[10]** "voting by veto" is the principal element of this set.[6] Voting by veto game forms are of course also dominance-solvable.

To understand the nature of the voting by veto game forms, it is simplest to consider the case where $|A| = p > n = |N|$. Choose nonnegative integers p_1,

[4] Moulin **[8]** and Peleg **[12]** give very lucid surveys of the recent results in this area.

[5] Actually, Maskin **[4]** overlooked the restriction on the relative cardinalities of A and N. This error was later corrected by Moulin and Peleg **[9]**.

[6] Moulin **[7]** contains an extensive discussion on the merits of the voting by veto mechanism.

$p_2, \ldots, p_n$ such that

$$p_1 + p_2 + \cdots + p_n = p - 1. \tag{3.1}$$

Given the vector $(p_1, p_2, \ldots, p_n)$, let

player 1 veto the first p_1 outcomes among A, say A_1,

player 2 veto the first p_2 outcomes among $A - A_1$, say A_2,

⋮

player n veto finally p_n outcomes among $A = \bigcup_{i=1}^{n-1} A_i$, say A_n.

From (3.1), only one element will remain not vetoed by any player, and this will be the outcome of the voting by veto game form.

Unfortunately, while this game form has great strategic appeal particularly in the cooperative context, it is not acceptable. The following example illustrates.

EXAMPLE 3.1: Let $N = \{1,2\}$, $A = \{x, y, z\}$, and $p_1 = p_2 = 1$. Suppose the two agents unanimously prefer x to y to z. Consider the following pair of strategies in the voting by veto game form:

agent 1 eliminates x,

agent 2 eliminates z when 1 eliminates x; otherwise 2 eliminates x.

Clearly, the outcome will be y; the specified pair of strategies will be a Nash equilibrium, although x is the only Pareto optimal element.

In a sense, our result showing the incompatibility of strong consistency and acceptability "generalizes" this example. In a heuristic sense, while strong consistency requires veto power, thus giving even "losing" coalitions the power to veto some alternatives, acceptable game forms entail a much sharper distribution of power in which "winning" coalitions usurp all the power.

In order to prove the result, we provide a partial characterization of acceptable game forms in terms of their effectivity functions. An effectivity function is a representation of the distribution of power in the given game form.

Let $T \in 2^N$, $B \in 2^A$, and $g = (X_N, \pi)$ be a given game form. Then, T is *effective for B in g* iff:

$$\text{there is } x_T \in X_T \text{ such that } \pi(x_T, x_{N-T}) \in B \text{ for all } x_{N-T} \in X_{N-T}. \tag{3.2}$$

If T is effective for B in g, this will be denoted as $T \operatorname{eff}_g B$.

DEFINITION 3.2.: The *effectivity function* of g is a function $E(g)$ from 2^N to 2^A such that for all $T \in 2^N$, $E(g, T) = \{B \in 2^A / T \operatorname{eff}_g B\}$.

An effectivity function is *maximal* iff

(3.3) for all $B \in 2^A$, for all $T \in 2N$,

$$[B \notin E(g,T)] \rightarrow [(A-B) \in E(g,N-T)].$$

The following results due respectively to Moulin [7] and Moulin and Peleg [9] will be used in the sequel.

LEMMA 3.3: (i) *If g is dominance-solvable, then $E(g)$ is maximal.* (ii) *If g is strongly consistent, then $E(g)$ is maximal.*

Given a game form g, a coalition T is *winning in g* iff

(3.4) for all $a \in A$, $T\,\mathrm{eff}_g\{a\}$.

In other words, a coalition T is winning if it can enforce the choice of any element in A.

A *simple game* is an ordered pair $G = (N, W)$ where W is the set of winning coalitions satisfying the monotonicity property:

(3.5) $T \in W$ and $T \subset T' \rightarrow T' \in W$.

DEFINITION 3.4: Let $G = (N, W)$ be a simple game. G is a *strong, proper* simple game iff for all $T \in 2^N$, $T \in W \leftrightarrow (N-T) \notin W$.

Thus, in a strong, proper simple game, a coalition is not winning iff its complement in N is winning. Simple games embody a particularly sharp distribution of power since only winning coalitions have any power. If G is a simple game, then its effectivity function $E(G)$ is:

(3.6) for all $B \in 2^A$ and all $T \in 2^N$,

$$[B \in E(G,T)] \text{ iff } [T \in W \text{ or } B = A].$$

We are now ready for our first result.

THEOREM 3.5: *Let g be an acceptable game form. If $E(g)$ is maximal, then $E(g) = E(G)$, where G is a strong, proper simple game.*

PROOF: Let g be any game form such that $E(g)$ is maximal. Suppose $E(g)$ is not the associated effectivity function of any strong, proper simple game. Then, there is a coalition T such that neither T nor its complement is winning under g.

Hence, there is $T \in 2^N$, and $a^*, b^* \in A$ such that

(3.7) $\quad \sim T\,\text{eff}_g\{a^*\}$,

(3.8) $\quad \sim(N-T)\text{eff}_g\{b^*\}$.

Since $E(g)$ is maximal, (3.3), (3.7), and (3.8) imply

(3.9) $\quad (N-T)\text{eff}_g(A-\{a^*\})$,

(3.10) $\quad T\,\text{eff}_g(A-\{b^*\})$.

There are two possibilities:

(i) $\quad T\,\text{eff}_g(A-\{a^*\})$,

(ii) $\quad \sim T\,\text{eff}_g(A-\{a^*\})$.

CASE (i): Suppose $T\,\text{eff}_g(A-\{a^*\})$. Then, there is x_T^* such that

(3.11) $\quad$ for all x_{N-T}: $\quad \pi(x_T^*, X_{N-T}) \in (A-\{a^*\})$.

Similarly, (3.9) implies that there is x_{N-T}^* such that

(3.12) $\quad$ for all x_T, $\quad \pi(x_T, x_{N-T}^*) \in A-\{a^*\}$.

Let $\underline{x}^* = (x_T^*, x_{N-T}^*)$, and suppose $\pi(x^*) = b \in (A-\{a^*\})$. Consider the profile R^N such that:

(3.13) $\quad$ for all $i \in N$: $\quad a^* R_i b R_i y \quad$ for all $y \in A-\{a^*, b\}$.

We show that $\underline{x}^* \in \text{NE}(g, R^N)$. Suppose that $\underline{x}^*$ is not a Nash equilibrium. Then, there is $i \in N$ and $x_i \in X_i$ such that $\pi(x_i, x_{-i}^*) R_i b$. Given the specification of R^N, we must have $\pi(x_i, x_{-i}^*) = a^*$. However, (3.11) and (3.12) together imply that for all $i \in N$, $\pi(x_i, x_{-i}^*) \neq a^*$ for any $x_i \in X_i$. Hence, $\underline{x}^* \in \text{NE}(g, R^N)$. Since $\pi(\underline{x}^*) = b$ and $a^* R_i(b)$ for all $i \in N$, g is not acceptable.

CASE (ii): Suppose $\sim T\,\text{eff}_g(A-\{a^*\})$. Since $E(g)$ is maximal,

(3.14) $\quad (N-T)\text{eff}_g\{a^*\}$.

So, there is x_{N-T}^* such that

(3.15) $\quad$ for all $x_T \in X_T$: $\quad \pi(x_T, x_{N-T}^*) = a^*$.

Also, (3.10) implies that there is x_T^* such that

(3.16) $\quad$ for all $x_{N-T} \in X_{N-T}$: $\quad \pi(x_T^*, x_{N-T}) \in A-\{b^*\}$.

Let $\underline{x}^* = (x_T^*, x_{N-T}^*)$. Clearly $\pi(\underline{x}^*) = a^*$. Consider R^N such that

(3.17) $\quad$ for all $i \in N$, $\quad b^* R_i a^* R_i y \quad$ for all $y \in A-\{a^*, b^*\}$.

Suppose $\underline{x}^*$ is not a Nash equilibrium of g at R^N. Then, there is $i \in N$ and $x_i \in X_i$ such that $\pi(x_i, x^*_{-i}) R_i a^*$. From (3.17), we must have $\pi(x_i, x^*_{-i}) = b^*$. However, (3.15) and (3.16) together imply that for all $i \in N$, for all $x_i \in X_i$, $\pi(x_i, x^*_{-1}) \neq b^*$.

Hence, $\underline{x}^* \in \mathrm{NE}(g, R^N)$. Since $\pi(\underline{x}^*) = a^*$ and $b^* R_i a^*$ for all $i \in N$, g is not acceptable. This completes the proof of the theorem.

The main result of this section now follows from Lemma 3.3 and Theorem 3.5.

THEOREM 3.6: *Let g be any nondictatorial and acceptable game form. Then, g is not strongly consistent.*

PROOF: If g is strongly consistent, then by Lemma 3.3, $E(g)$ is maximal. If g is also acceptable, then $E(g) = E(G)$, where $G = (N, W)$ is a strong, proper simple game.

Let T^* be a *minimal* winning coalition, i.e. $T \in W$ and for all $i \in T^*$, $(T^* - \{i\}) \notin W$. Since G is strong, for all $i \in T^*$, $((N - T^*) U \{i\}) \in W$. Without loss of generality, let $T^* = \{1, 2, \ldots, k\}$, where $k \geq 2$ since g (and hence G) is nondictatorial. Let $T_1 = \{1, k+1, k+2, \ldots, n\}$, and $T_2 = \{2, k+1, \ldots, n\}$. Then, $\{T^*, T_1, T_2\} \subset W$. Clearly, since $E(g) = E(G)$, T^*, T_1 and T_2 are also winning coalitions in g.

Since $|A| \geq 3$, let $B = \{a, b, c\} \subset A$. Consider the following R^N:

for all $i = 2, 3, \ldots, k$: $a R_i b R_i c R_i d$ for all $d \in A - B$,

for all $i = k+1, k+2, \ldots, n$: $b R_i c R_i a R_i d$ for all $d \in A - B$,

$c R_1 a R_1 b R_1 d$ for all $d \in A - B$.

Obviously, if $\underline{x} \in \mathrm{SE}(g, R^N)$, then $\pi(\underline{x}) \in B$. Suppose $\pi(\underline{x}') = a$. Then individuals in T_1 (a winning coalition) can switch their strategies to enforce the outcome c, which they all prefer to a. Hence, $\underline{x}'$ is not a strong equilibrium of g at R^N. For exactly analogous reasons, if $\pi(\underline{x}) = b$ or $\pi(\underline{x}) = c$, then $\underline{x} \in \mathrm{SE}(g, R^N)$.

Hence, $\mathrm{SE}(g, R^N) = \emptyset$, and g is not strongly consistent.

REMARK 3.6: Theorem 3.5 could also be proved with the help of results from Moulin and Peleg [**9**] and Nakamura [**11**]. The "direct" proof economizes on additional definitions.

4. STRONG PROPER SIMPLE GAMES AND ACCEPTABLE GAME FORMS

Theorem 3.5 and Lemma 3.3 also suggest that the search for dominance-solvable game forms which are also acceptable must be restricted to those whose effectivity functions correspond to that of some strong proper simple game. It is trivial to show that such game forms exist, the *direct kingmaker* game form (see [**3**]) being an obvious example. However, in this section, we prove that the converse is also true; if G is a strong, proper simple game, then it is always possible to construct a game form g which is both dominance-solvable and acceptable with $E(G) = E(g)$. So, this provides a complete characterization in terms of effectivity functions of game forms which are dominance-solvable and acceptable.

4.1. *Tree-based Game Forms*

We make a brief digression on game forms derived from *finite game trees.* A finite tree is a pair $\Gamma = (M, \sigma)$ where M is a *finite* set of nodes, and σ associates to each node its nearby predecessor. We require that σ satisfy the following properties: (i) There is a unique node m_1, the origin of Γ, with $\sigma(m_1) = m_1$. (ii) There is an integer k such that $\sigma^k(m) = m_1$ for all $m \in M$. A node m such that $\sigma^{-1}(m) = \emptyset$ is called a *terminal* node of Γ, and their set is denoted Z. For a nonterminal node m, $\sigma^{-1}(m)$ is the set of successor nodes of m. A *path* is a sequence of nodes from m_1 to a terminal node such that if $p = \{m_1, m_2, \ldots, m_k\}$ is a path, then m_k is a terminal node and $m_1 = \sigma(m_{i+1})$ for all $i = 1, 2, \ldots, k - 1$. If $p = \{m_1, m_2, \ldots, m_k\}$ is a path, then $p(m_i, +)$ is the subset of p from m_i to m_k, and $p(m_i, -)$ is the subset of p from m_1 to m_i. Hence, for any path p, $p(m_i, +) \cap p(m_i, -) = \{m_i\}$ and $p(m_i, +) \cup p(m_i, -) = p$. In what follows, it will sometimes be useful to consider the following partition of M. Let $M_1 = \{m_1\}$, and for $k > 1$, $M_k = \bigcup_{m \in M_{k-1}} \sigma^{-1}(m)$. For all $k \geq 1$, M_k will be called nodes of order k.

Given a set B of outcomes, we construct a game form on B by assigning to each terminal node an element of B, and to each nonterminal node a game form bearing on its successor nodes. Formally, let θ be a mapping from Z *onto* B, and for all $m \in M - Z$, let $g(m) = (X_i(m), i \in N; \pi(m))$ be a game form on $\sigma^{-1}(m)$. Then, the game form associated to $(\Gamma; \theta; g_m, m \in M - Z)$ is: (i) A strategy x_i of player i associates to each nonterminal node $m \in M - Z$ an element $x_i(m) \in X_i(m)$. Their set is denoted X_i. (ii) For each strategy n-tuple $\underline{x}$, we define $\pi(\underline{x}) = \theta(m_T)$ where m_T is the first terminal node of the sequence

$$m_1, \ldots, m_{t+1} = \pi(m_t)(\underline{x}(m_t)) \qquad \text{where} \quad \underline{x}(m) = (x_i(m), i \in N).$$

Voting by veto is a tree-based game form. Another example of a tree-based game form is the kingmaker game form. In this game form, a player (say agent 1) selects the "king for a day," dictatorially out of the set $\{2, 3, \ldots, n\}$. The person so selected dictates the final choice out of A. Note that at each nonterminal node m, $\pi(m)$ is dictatorial—agent 1 being the dictator in $\pi(m_1)$, and each of the potential "kings for a day" being dictators at corresponding nodes of order 2. Hence, Moulin's **[6]** result implies that the game form is dominance-solvable; Hurwicz and Schmeidler **[3]** have shown that direct kingmaker game forms are acceptable.

Also, note that the effectivity function of the direct kingmaker game form is indeed that of a strong proper simple game. Agent 1 together with any other agent i forms a winning coalition since 1 can enforce the choice of i as the king for a day, and i once selected can choose any element out of A. Conversely,

$(N - \{1\})$, the set of potential kings for a day, is also a winning coalition. Thus, when N contains more than 3 members, the distribution of power is very asymmetric, with agent 1 being all-powerful. Fortunately, we show in Section 4.2, that indirect kingmaker procedures can be constructed whose effectivity functions coincide with that of any prespecified strong proper simple game.

4.2. *Constructing Kingmaker Procedures with Prespecified Effectivity Function*

Let $G = (N, W)$ be any strong proper simple game which is fixed for the rest of this section. In this section, we will show that an indirect kingmaker game form g^* can be constructed such that $E(g^*) = E(G)$.

In a general or indirect kingmaker game form, the king for a day is chosen by an elective process which is not necessarily dictatorial. Thus, for $n \geq 3$, let $N = N_1 \cup N_2$, with $|N_1| \geq 1$, and $|N_2| \geq 2$. Each member of N_1 "votes" for his choice of king for a day from among members N_2, and if $i^* \in N_2$ is elected, then i^* chooses an element from A dictatorially. So, let $\hat{g} = (S_i,\ i \in N_1; \hat{\pi})$ be an *auxiliary* game form with the set of players N_1 and the set of outcomes N_2; hence $\hat{\pi} : \times_{i \in N_1} S_i \to N_2$. Then the kingmaker game form $g = (X_i,\ i \in N; \pi)$ on A is defined as follows:

$$\begin{aligned} &\text{for all} \quad i \in N_1 - N_2: && X_i = S_i, \\ &\text{for all} \quad i \in N_2 - N_1: && X_i = A, \\ &\text{for all} \quad i \in N_1 \cap N_2: && X_i = S_i X A. \end{aligned}$$

For an arbitrary list of strategies

$$\underline{x} = ((y_i), i \in N_1 - N_2; (z_i), i \in N_2 - N_1; (y_i, z_i), i \in N_1 \cap N_2)$$

we have $\pi(\underline{x}) = z_j$, where $j = \hat{\pi}((y_i),\ i \in N_1)$ is the elected king for a day.

In the particular kingmaker game that we will construct, the auxiliary game form itself will be derived from a finite tree. First, construct a "universal" tree $\Gamma^* = (M^*, \sigma)$ in the following manner. For each $k = 1, 2, \ldots, n-2$, for all $m \in M_k^*$, $|\sigma^{-1}(m)| = n - k$, and $M_{n-1}^* = Z$ is the set of terminal nodes. Note that in Γ^*, each path will consist of exactly $(n-1)$ nodes. Also, $|M_2^*| = n - 1$, $|M_3^*| = (n-1)(n-2)$, and so on. Now define a function μ from M^* to N as follows: (i) $\mu(m_1) = 1$; and for all k, $1 < k \leq n - 1$, (ii) for all $m \in M_k^*$, $\mu(m) \in [N - \mu(\bigcup_{i=1}^{k-1} \sigma^i(m))]$; (iii) for all $m, m' \in M_k^*$, $\sigma(m) = \sigma(m') \to \mu(m) \neq \mu(m')$. Note that (ii) and (iii) imply that for any $m \in M_k^*$, $k \geq 1$,

$$\mu\left[\sigma^{-1}(m)\right] = \left(N - \mu\left[\bigcup_{j=1}^{k-1} \sigma^j(m) \cup \{m\}\right]\right).$$

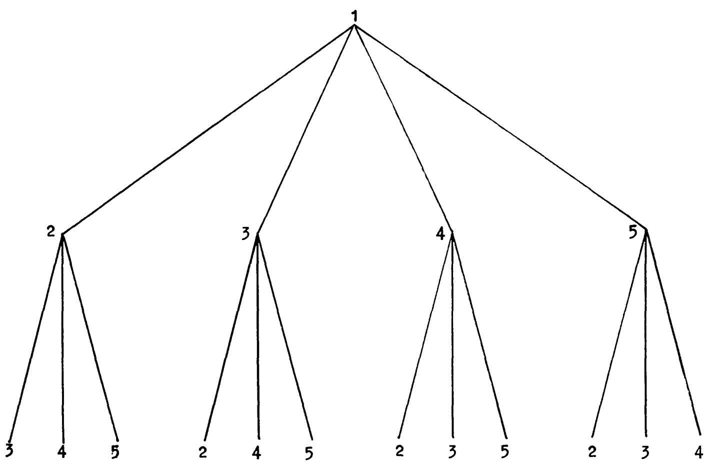
This in turn implies that for all $k > 1$, $\mu(M_k^*) = (N - \{1\})$.

Then, the *tree based on* G, denoted $\Gamma(G)$, is constructed out of Γ^* in the following way. In $\Gamma(G)$, each path p in Γ^* is "terminated" at the *first* node m where $\mu(p(m, -)) \in W$. So, if m is a terminal node in $\Gamma(G)$, then m satisfies two conditions: (i) $\mu[\bigcup_{j=1}^{k-1}\sigma^j(m) \cup \{m\}] \in W$; (ii) for all $m' \neq m$; if $m' \in p(m, -)$ then $\mu(p(m', -)) \notin W$. Clearly, some nonterminal nodes of Γ^* will become terminal nodes in $\Gamma(G)$,[7] so that the set of nodes in $\Gamma(G)$, $M(G) \subset M^*$. Henceforth, all references to a tree will pertain to $\Gamma(G)$.

Our aim is to construct a kingmaker game form g^* whose effectivity function will coincide with that of G. We construct g^* such that the auxiliary game form $\hat{g}$ of g^* is derived from $(\Gamma(G), \mu)$. At each nonterminal node m of $\Gamma(G)$, $\mu(m)$ is the dictator of $\pi(m)$, and his strategy set is simply $\sigma^{-1}(m)$, i.e., a strategy consists of selecting a successor node. Given our earlier discussion of game forms derived from trees and kingmaker game forms, this completes the description of g^*. Note that N_2, the set of potential "kings for a day," is $\mu[Z(G)] \subset (N - \{1\})$. The reader can check that if every individual $i \in (N - \{1\})$ belongs to some minimal winning coalition, then N_2 will in fact coincide with $(N - \{1\})$.

We want to show that the set of winning coalitions in g^* is indeed W. Before proving this proposition formally, we give an example in order to make the nature of our construction clearer.

EXAMPLE 4.1: Let $n = 5$. Let $\overline{G} = (N, \overline{W})$ be the majority game in which every 3-member coalition is winning. Then, in the corresponding "universal" tree Γ^*, the set of terminal nodes will be nodes of order 4. $\Gamma(\overline{G})$, from which the auxiliary game is derived, will have as terminal nodes all nodes of order 3. $\Gamma(\overline{G})$ is shown in Figure 1.



The integers are the values of $\mu(m)$ at each node m.

FIGURE 1.

[7]Thus, given n, the universal tree is uniquely defined. (G), on the other hand, will depend on the specification of G.

Figure 1 represents an indirect kingmaker procedure. Agent 1 is the indirect kingmaker, and can select a direct kingmaker out of the set $\{2,3,4,5\}$. If agent 1 selects i as the direct kingmaker, then the latter can select an element of $(N - \{1,i\})$ as the king for a day. It is easy to see why agent 1 together with any two other agents forms a winning coalition; agent 1 selects i as the direct kingmaker, and i in turn selects j as the king for a day. Moreover, any 3 individuals out of $\{2,3,4,5\}$ also form a winning coalition; they can ensure that one of them is finally elected as king for a day. Hence, in this simple case, $\overline{W}$ would indeed be the set of winning coalitions in this procedure.

The formal proof for the arbitrary game G runs along similar lines.

THEOREM 4.2: *Let g^* be the kingmaker game form in which the auxiliary game form $\hat{g}$ is derived from $\Gamma(G)$. Then, $E(g^*) = E(G)$.*

PROOF: Since G is strong, so that only winning coalitions have any "power," it is enough to show that the set of winning coalitions in g^* is W.

STEP 1: Let $T \in W$. If $1 \in T$, then from the construction of $\Gamma(G)$, it is clear that there is a path $p \subset \mu^{-1}(T)$.

Suppose $1 \notin T$. Then, since $(N - T) \notin W$ and $1 \in (N - T)$, there is no path p which is wholly contained in $\mu^{-1}(N - T)$. Hence for *any path* p^*, $p^* \cap \mu^{-1}(T) \neq \emptyset$. Let m be the first element of p^* such that $\mu(m) \in T$, i.e. $\mu(p^*(m, -)) \subset (N - T) \cup \mu(m)$.

Choose $\hat{T} \subset T$ such that:

$$S = \mu(p^*(m,-)) \cup \hat{T} \quad \text{and} \quad S \in W, \tag{4.1}$$

$$\text{for no} \quad i \in \hat{T}, \qquad (S - \{i\}) \in W. \tag{4.2}$$

Such $\hat{T}$ exists since $T \subset \mu(p^*(m, -)) \cup T$ and $T \in W$. Since $S \in W$ and $1 \in S$, there is a path $\hat{p}$ such that

$$p^* \cap \hat{p} = p^*(m,-) \tag{4.3}$$

and

$$\hat{p}(m,+) \subset \mu^{-1}(\hat{T}) \subset \mu^{-1}(T). \tag{4.4}$$

Hence, if $T \in W$, for every path p, the following is true:

(4.5) there is $m \in p$ and a path $\hat{p}$ such that

$$m \in \hat{p} \cap p \text{ and } \hat{p}(m,+) \subset \mu^{-1}(T).$$

STEP 2: Pick any $a^* \in A$. For all $i \in T$, for all $m \in M(G)$, if $i = \mu(m)$, let

$$\hat{x}_i(m) = a^* \qquad \text{if} \quad m \in Z(G), \tag{4.6}$$

$$\hat{x}_i(m) = \sigma^{-1}(m) \cap \mu^{-1}(T) \qquad \text{if} \quad m \in M(G) - Z(G). \tag{4.7}$$

Equation (4.6) means that all individuals in T pick a^* if elected king for a day.

(4.7) implies that at all nonterminal nodes of $\Gamma(G)$ where individuals in T are dictators, the successor node will also be in $\mu^{-1}(T)$. In view of (4.5), it is obvious that if individuals use strategies according to (4.7) the elected king for a day will be a member of T. From (4.6), $T\,\mathrm{eff}_{g^*}\{a^*\}$. Since this is true for all $a \in A$, T is winning in g^*.

STEP 3: Now, suppose $T \notin W$. Since G is strong, $(N - T) \in W$. From Steps 1 and 2, $(N - T)$ is winning in g^*, and hence T is not winning in g^*.

Hence, the set of winning coalitions in g^* is W, so that $E(g^*) = E(G)$.

4.3. *Acceptability of* g^*

Notice that g^* is derived from a finite tree such that at each node m, $\pi(m)$ is dictatorial. Hence, from Moulin [**6**], g^* is dominance-solvable. We now show that g^* is also acceptable.

THEOREM 4.3: *g^* is acceptable.*

PROOF: Suppose $\underline{x} \in \mathrm{NE}(g^*, R^N)$, and $\pi(\underline{x}) = b$. Then, b must be the R_i-maximal element of the king for a day, and hence $b \in Q(R^N)$.

Hence, g^* is acceptable if $\mathrm{NE}(g^*, R^N) \neq \emptyset$ for all $R^N \in L^N$. Let $\overline{m} \in Z(G)$, and $\hat{p}$ be the path in $\Gamma(G)$ containing $\overline{m}$. Let $i = \mu(\overline{m})$, and a^* be the R_i-maximal element in A. Consider any $\underline{x}$ satisfying:

(4.8) $\quad$ for all $\quad m \in Z(G), \quad$ for all $\quad j \in N,$

$\quad$ if $\quad j = \mu(m), \quad$ then $\quad x_j(m) = a^*;$

(4.9) $\quad$ for all $\quad m \in \hat{p} - Z(G), \quad$ for all $\quad j \in N,$

$\quad$ if $\quad j = \mu(m), \quad$ then $\quad x_j(m) = \sigma^{-1}(m) \cap \hat{p}.$

Note that (4.8) implies that all potential kings for a day nominate a^*, while (4.9) implies that at all nonterminal nodes in $\hat{p}$, the dictator chooses a successor node belonging to $\hat{p}$.

We show that any $\underline{x}$ satisfying (4.8) and (4.9) is a Nash equilibrium by showing that:

(4.10) $\quad$ for all $\quad j \neq i, \quad$ for all $\quad \hat{x}_j \in X_j, \qquad \pi(\hat{x}_j, x_{-j}) = a^*.$

Since a^* is R_i-maximal in A, i has no interest in switching his strategy, and hence (4.10) ensures that $\underline{x}$ is a Nash equilibrium.

Clearly, (4.11) is true for all $j \notin \mu(\hat{p})$. Suppose $j = \mu(m)$ and $m \in \hat{p}$. Given the specification of μ and $\Gamma(G)$, individual j by changing his strategy can at most select as king for a day some individual $k \in [N - \mu(p(m, -))]$. In particular, j cannot get himself elected. Thus, (4.8) implies that (4.10) is true.

So, g^* is acceptable.

Gathering our previous results, we are now able to state the main result of this section.

THEOREM 4.4: *Let* eff* *be an effectivity function. Then, there is a game form g which is acceptable and dominance-solvable with* $E(g) = \text{eff}^*$ *iff* eff* *is the effectivity function of a strong, proper simple game.*

The following result is also interesting.

THEOREM 4.5: *Let* eff* *be a maximal effectivity function. Then, there is an acceptable game form g with* $E(g) = \text{eff}^*$ *iff* eff* *is the effectivity function of a strong, proper simple game.*

REMARK 4.6: Theorem 4.5 is *not* a complete characterization of acceptable game forms in terms of effectivity functions since there are acceptable game forms whose effectivity functions are not maximal. See Moulin [7] for an example.

REMARK 4.7: Let eff* be the effectivity function of a strong, proper simple game, and $E(g) = \text{eff}^*$. Theorem 4.5 does *not* assert that g is acceptable.

5. CONCLUDING REMARKS

The existing literature on implementation analyzes the strategic properties of game forms under several concepts of equilibrium. As these equilibrium concepts are quite different, the present paper looks for game forms which could possibly combine acceptability, dominance solvability, and strong consistency. Our first result shows that no nondictatorial game form is both strongly consistent and acceptable. The second result shows that kingmaker procedures play a central role in the family of game forms which are together acceptable and dominance-solvable; the effectivity function of such game forms is that of a strong proper simple game, and an explicit kingmaker procedure can be constructed of which the effectivity function is that of any given strong proper simple game.

We conclude with a comment on the differences between Nash implementability and implementability via sophisticated equilibrium. Suppose the kingmaker game form g^* is used as the aggregation mechanism. Then, the SCC $\hat{H}$ implemented via sophisticated equilibrium will in fact be single-valued, i.e., $\hat{H}$ will be a SCF. Moreover, the effectivity function of H will coincide with that of g^*. On the other hand, the SCC H' which is Nash-implemented by g^* will be the following:

$$H'(R^N) = \{a \mid a \text{ is } R_i\text{-maximal for some } i \in N_2\}.$$

The validity of this assertion will be clear from the proof given to show that $\text{NE}(g^*, R^N) \neq \emptyset$ for all $R^N \in L^N$ in Theorem 4.3. Notice that not only is H^* not single-valued, but also the effectivity function of H' will not coincide with that of g^*. For instance, in H', individuals in $(N - N_2)$ are "dummies" since their preferences do not count. Of course, individuals in $(N - N_2)$ are not dummies in g^*. However, it is easy to show that $\hat{H}(R^N) \subset H'(R^N)$ for all $R^N \in L^N$. So, the planner who specifies a kingmaker game form g^* will ensure that if individuals cannot collude, then at least the "minimal" requirement of Pareto efficiency will be satisfied; if in addition, the agents behave sophisticatedly, then the planner can also "predict" the outcome that will be selected by the players. We view this as a powerful justification for the use of kingmaker game forms.

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Manuscript received March, 1983; final revision received January, 1984.

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