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A ONE-ONE SELECTION THEOREM

H. SARBADHIKARI

ABSTRACT. Let X, Y be Polish spaces without isolated points and $B \subseteq X \times Y$ a Borel set such that $\{x: B_x \text{ is nonmeager}\}$ is comeager in X and $\{y: B^y \text{ is nonmeager}\}$ is comeager in Y. There is a comeager Borel $E \subseteq X$, a comeager Borel $F \subseteq Y$ and a Borel isomorphism f from E onto F such that graph of $f \subseteq B$.

1. Introduction. In [3] Mauldin proved that if $B \subset [0,1] \times [0,1]$ is a Borel set such that $\lambda\{x: \lambda(B_x) > 0\} = 1$ and $\lambda\{y: \lambda(B^y) > 0\} = 1$, where λ is the Lebesgue measure, then there exist Borel sets E and F of full measure and a Borel isomorphism f from E onto F such that the graph of $f \subseteq B$. Our main theorem is a category analogue of this. Throughout this paper, X, Y are taken to be Polish spaces without isolated points.

2. The main result. Our main theorem reads

Let $B \subset X \times Y$ be a Borel set such that $\{x: B_x$ is nonmeager$\}$ is comeager in X and $\{y: B^y$ is nonmeager$\}$ is comeager in Y. Then there is a comeager Borel $E \subseteq X$, a comeager Borel $F \subseteq Y$ and a Borel isomorphism f from E onto F such that the graph of $f = \{(x, y): y = f(x)\} \subseteq B$.

Our proof is analogous to that in [3] where several subsidiary results are proved, leading to the main theorem.

THEOREM 1. *Let $X = Y = [0,1]$, $B \subseteq X \times Y$ be a Borel set such that $\{x: B_x$ is comeager$\}$ is comeager in X. There is a comeager Borel $E \subseteq X$, a meager Borel $F \subseteq Y$ and a Borel isomorphism f from E onto F with graph $f \subseteq B$.*

PROOF. Fix an open base $\{U_n: n = 1, 2, \dots\}$ for Y consisting of nonempty intervals.

Note that B is comeager in $X \times Y$. Hence there exist dense open sets $V_1 \supseteq V_2 \supseteq \cdots$ in $X \times Y$ with $\bigcap_n V_n \subseteq B$.

By induction on n, we define a sequence of Borel sets $\{H_n: n = 1, 2, \dots\}$ such that for all n

(1) $H_{n+1} \subseteq H_n \subseteq V_n, \dots$.

(2) There exist a sequence $\{B_{ni}: i = 1, 2, \dots\}$ of pairwise disjoint nonmeager G_δ sets in X with $B_{ni} \subseteq (k_i/2^n, (k_i + 1)/2^n)$ for some integer k_i and $\bigcup_i B_{ni}$ comeager

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in X; a sequence $\{[a_{ni}, b_{ni}]: i = 1, 2, \dots\}$ of pairwise disjoint closed intervals of length > 0 and $\leqslant 1/2^n$ and a nonempty open $W_n \subseteq U_n$ with $W_n \cap \bigcup_i [a_{ni}, b_{ni}] = \varnothing$ such that $H_n = \bigcup_i B_{ni} X(a_{ni}, b_{ni})$.

(3) For all x, $\overline{H_{n+1x}} \subseteq H_{nx}$.

Now $\bigcap_n H_n$ is the graph of the required function f.

Construction of H_n. Suppose H_m has been defined and equals $\bigcup_i B_{mi} X(a_{mi}, b_{mi})$ as in condition (2). For each i, we define H^i_{m+1} so that $\bigcup_i H^i_{m+1} = H_{m+1}$.

Fix i. Note that $V_{m+1} \cap (B_{mi} \times (a_{mi}, b_{mi}))$ is comeager in $B_{mi} X(a_{mi}, b_{mi})$. Hence by the Kuratowski-Ulam theorem, $\{y \in (a_{mi}, b_{mi}): V^y_{m+1} \cap B_{mi}$ is comeager in $B_{mi}\}$ is comeager in (a_{mi}, b_{mi}). Pick y_1, y_2 from this set with $y_1 < y_2$. Let $M >$ maximum$\{1/(y_2 - y_1), 1/l, 2^m\}$, where $l = \max(\text{length}\, U_{m+1} \cap (a_{mi}, b_{mi}), 1)$ and

$$A^i_n = \left\{x \in B_{mi}: \left[y_1 - \frac{1}{4n}, y_1 + \frac{1}{4n}\right] \cup \left[y_2 - \frac{1}{4n}, y_2 + \frac{1}{4n}\right] \subseteq V_{m+1x} \cap (a_{mi}, b_{mi})\right\}, \qquad n \geqslant M.$$

Then A^i_n is coanalytic and $\bigcup_{n \geqslant M} A^i_n = V^{y_1}_{m+1} \cap V^{y_2}_{m+1} \cap B_{mi}$ is a comeager Borel set in B_{mi}.

Find pairwise disjoint Borel sets $B^i_n \subset A^i_n$, $n \geqslant M$, with $\bigcup_n B^i_n = \bigcup_n A^i_n$. Put $C^i_n = B^i_n \cap (k_i/2^n, (2k_i + 1)/2^{n+1})$, $D^i_n = B^i_n \cap ((2k_i + 1)/2^{n+1}, (k_i + 1)/2^n)$. Note that by possibly ignoring a meager set, we can suppose C^i_n and D^i_n to be nonmeager G_δ sets in X. Put

$$H^i_{m+1} = \bigcup_{n \geqslant M} \left(C^i_n X\left(y_1 - \frac{1}{4n}, y_1 - \frac{1}{4n+1}\right) \cup D^i_n X\left(y_2 - \frac{1}{4n}, y_2 - \frac{1}{4n+1}\right)\right).$$

To construct H_1, use $V_1 \cap X \times (0,1)$ as a comeager open set in $X \times Y$ and proceed as above.

COROLLARY. *The previous theorem is true even when X and Y are arbitrary Polish spaces without isolated points.*

PROOF. Since the irrationals are homeomorphic to a comeager G_δ subset of $[0,1]$, the result is true if $X = Y =$ irrationals.

Now any Polish space without isolated points contains a comeager G_δ set homeomorphic to irrationals. Thus the result is true for X, Y such spaces.

THEOREM 2. *Let $B \subseteq X \times Y$ be such that $\{x: B_x$ is nonmeager$\}$ is comeager. Then there is a comeager Borel $E \subseteq X$, a meager Borel $F \subseteq Y$ and a Borel isomorphism f on E onto F such that graph $f \subseteq B$.*

PROOF. Let $U_1, U_2, \dots$ be a countable open base for Y. Let $B^*_n = \{x: B_x \cap U_n$ is comeager in $U_n\}$ and $A_n = B^*_n - \bigcup_{m<n} B^*_m$. A_n is Borel for all n and $\bigcup_n A_n$ is comeager in X.

By ignoring a meager set if necessary, we can suppose that each A_n is a nonmeager G_δ. By induction on n, we define f_n on $E_n \subseteq A_n$. We then define $f(x) = f_n(x)$ for $x \in E_n$.

Suppose f_k, $k \leqslant m$, has been defined and range $f_k \subseteq$ a meager F_σ set, say $F_k \subseteq U_k$. Put

$$B_{m+1} = A_{m+1} X \left(U_{m+1} - \bigcup_{i=1}^{m} F_i \right) \cap B.$$

B_{m+1} is a Borel subset of $A_{m+1}X(U_{m+1} - \bigcup_{i=1}^{m} F_i)$ and $\{x\colon B_{m+1x}$ is comeager (in $U_{m+1} - \bigcup_{i=1}^{m} F_i)\} = A_{m+1}$. By applying the previous result, get a comeager $G_\delta E_{m+1}$ in A_{m+1} and a Borel isomorphism f_{m+1} on E_{m+1} into $U_{m+1} - \bigcup_{i=1}^{m} F_i$ such that range f_{m+1} is meager.

If $f(x) = f_n(x)$ for $x \in E_n$, f is a Borel isomorphism on $\bigcup_n E_n$ into $\bigcup_n F_n$. Thus domain f is comeager and the range is meager.

PROOF OF THE MAIN THEOREM. Find Borel sets $E_1 \subseteq X$, $F_1 \subseteq Y$ such that E_1 is comeager, F_1 is meager and there is a Borel isomorphism h from E_1 onto F_1 satisfying graph $h \subseteq B$.

Find Borel sets $G \subseteq Y$, $H \subseteq X$ such that G is comeager, H is meager and there is a Borel isomorphism g from G onto H satisfying $\{(x, y)\colon x = g(y)\} \subseteq B - X \times F_1$.

Define f on $E_1 \cup H$ by

$$\begin{aligned} f(x) &= g^{-1}(x) \quad \text{if } x \in H, \\ &= h(x) \qquad \text{if } x \in E_1 - H. \end{aligned}$$

Putting $E = E_1 \cup H$, $F = $ range f, we get the result.

REMARKS. In [3] Mauldin raises some interesting questions of which the following are still open to our knowledge.

1. If $B \subseteq [0,1] \times [0,1]$ is a Borel set with B_x, B^y of positive Lebesgue measure for all x and y, is there a Borel isomorphism of $[0,1]$ onto $[0,1]$ whose graph is a subset of B?

2. Is the category analog of the above true?

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INDIAN STATISTICAL INSTITUTE, DIVISION OF THEORETICAL STATISTICS AND MATHEMATICS, 203 BARRACKPORE TRUNK ROAD, CALCUTTA 700 035, INDIA