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GENERAL BALANCED DESIGNS THROUGH REINFORCEMENT*

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SUMMARY. General efficiency balanced designs, introduced by Das and Ghosh (1985), constructed by the reinforcement of balanced incomplete block designs are characterized to unify the concept of reinforced designs and supplemented designs.

1. Introduction

In available literature on incomplete block designs, the concept of balanced designs has been put forth differently in different contexts. There are, for example, balanced incomplete block (BIB) designs, variance-balanced (VB) designs, efficiency-balanced (EB) designs and so on (cf. Caliński (1971, 1977), Caliński and Ceranka (1974), Puri and Nigam (1975), Saha (1976), Williams (1975)). For some discussion on the practical utility of such designs, see Clarke and Ricketts (1982). Recently, Das and Ghosh (1985) introduced the concept of "general efficiency balanced (GEB) designs" to unify the above-mentioned balanced designs and constructed several series of VB designs and EB designs as subclasses of GEB designs. These constructions are based on a technique of reinforcement, which was first defined by Das (1958) with stress on the importance of this approach in applied field.

Consider a block design in v treatments and b blocks with replication numbers $r_1, \ldots, r_v$ and block sizes $k_1, \ldots, k_b$. Let $\boldsymbol{N}$ be the incidence matrix of the design, $\boldsymbol{r} = (r_1, \ldots, r_v)'$, $\boldsymbol{R} = \text{Diag}(r_1, \ldots, r_v)$, $\boldsymbol{K} = \text{Diag}(k_1, \ldots, k_b)$. Let $\boldsymbol{C} = \boldsymbol{R} - \boldsymbol{N}\boldsymbol{K}^{-1}\boldsymbol{N}'$ be the usual $\boldsymbol{C}$-matrix of the design. The design will be called general efficiency balanced (GEB) in the sense of Das and Ghosh (1985) if for some $a, s_1, \ldots, s_v$ (> 0), $\boldsymbol{C}$ is of the form

$$\boldsymbol{C} = a(\boldsymbol{S} - g^{-1}\boldsymbol{s}\boldsymbol{s}'), \qquad \ldots \quad (1.1)$$

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where $\boldsymbol{S} = \text{Diag}(s_1, \ldots, s_v)$, $\boldsymbol{s} = (s_1, \ldots, s_v)'$ and $g = \sum_{i=1}^{v} s_i$. In particular, if in (1.1) $\boldsymbol{s} = \boldsymbol{r}$, then the design will be called efficiency-balanced (EB) and if $\boldsymbol{s} = \boldsymbol{1}_v^{(v\times 1)} = (1, \ldots, 1)'$, then the design will be called variance-balanced (VB).

Let d_0 be a BIB design with parameters v, b, r, k, λ, and having incidence matrix $\boldsymbol{N}_0$. To avoid trivialities, we shall assume $\lambda > 0$. Suppose t ($\geqslant 1$) treatments are added, each p ($\geqslant 0$) times, to each block of d_0. Then n ($\geqslant 0$) more blocks are taken in each of which each of the original v treatments occurs u_1 ($\geqslant 0$) times and each of the t new treatments occurs u_2 ($\geqslant 0$) times. The resulting design d with parameters (t, p, n, u_1, u_2) (in addition to the original parameters v, b, r, k, λ) will be said to have been obtained through a reinforcement of d_0. Note that if $n = 0$ (in this case the resulting design is called a supplemented design, see e.g. Caliński and Ceranka (1974), Puri, Nigam and Narain (1977)), then automatically $u_1 = u_2 = 0$; otherwise, at least one of u_1 and u_2 is positive. Also $(p, n) \neq (0, 0)$, for otherwise d becomes the same as d_0.

In this paper, we consider the construction of VB, EB and GEB designs based on reinforcement of a BIB design. It is attempted to develop a unified theory considering all possible choices of t, p, n, u_1 and u_2. Among other things it has been analytically shown that starting from a BIB design in v treatments one cannot construct a VB or an EB design in $(v+2)$ or more treatments through reinforcement. This provides a theoretical justification to a statement in Das and Ghosh (1985, p. 71, lines 25–28). It is, however, seen that GEB designs with $(v+2)$ or more treatments can be constructed in abundance. Special attention has been given to the construction of binary designs.

2. The case ($n = 0$)

Consider first the situation when $n = 0$. Then $p > 0$, and the $\boldsymbol{C}$-matrix of the design d, obtained through reinforcement, is given by

$$\boldsymbol{C} = \begin{pmatrix} r\boldsymbol{I}_v & 0 \\ 0 & pb\boldsymbol{I}_t \end{pmatrix} - (k+pt)^{-1} \begin{pmatrix} \boldsymbol{N}_0\boldsymbol{N}_0' & rp\boldsymbol{J}_{vt} \\ rp\boldsymbol{J}_{tv} & p^2b\boldsymbol{J}_{tt} \end{pmatrix} \quad \ldots \quad (2.1)$$

where $\boldsymbol{I}_h$ is the identity matrix of order h and $\boldsymbol{J}_{hh'}$ is the $h\times h'$ matrix with all unity.

Theorem 2.1 : *If $n = 0$, then d cannot be a VB or an EB design.*

Proof : If d is a VB design then by (2.1), $\lambda/(k+pt) = rp/(k+pt)$, which is impossible as $r > \lambda$ and $p \geqslant 1$. If d is an EB design then by (2.1), for some $a > 0$,

$$\lambda/(k+pt) = ar^2/r_0, \; rp/(k+pt) = arpb/r_0,$$

where $r_0 = rv+pbt$. But the above yields $r^2 = \lambda b$, which is impossible as

$$r^2-\lambda b = r(b-r)/(v-1) > 0. \qquad \ldots \quad (2.2)$$

Theorem 2.2 : (i) *If* $n = 0$ *and* $t \geqslant 2$, *then* d *cannot be a GEB design.* (ii) *If* $n = 0$ *and* $t = 1$, *then* d *is always a GEB design.*

Proof : (i) If d is a GEB design, then from (1.1) a little reflexion shows that C must be of the form

$$\boldsymbol{C} = a\left[\begin{pmatrix} s\boldsymbol{I}_v & 0 \\ 0 & z\boldsymbol{I}_t \end{pmatrix} - g^{-1}\begin{pmatrix} s\mathbf{1}_v \\ z\mathbf{1}_t \end{pmatrix}(s\mathbf{1}_v', z\mathbf{1}_t')\right], \qquad \ldots \quad (2.3)$$

where $g = vs+tz$, and $\mathbf{1}_h = \boldsymbol{J}_{h1}$. Comparing the off-diagonal elements of (2.1) with those of (2.3)

$$\lambda/(k+pt) = as^2/g,$$

$$rp/(k+pt) = asz/g,$$

$$p^2b/(k+pt) = az^2/g, \qquad \ldots \quad (2.4)$$

whence it follows that $r^2 = \lambda b$, which is impossible by (2.2). This proves (i). Note that the last relation in (2.4) can arise only when $t \geqslant 2$.

(ii) With $t = 1$, by (2.1),

$$\boldsymbol{C} = \begin{pmatrix} r\boldsymbol{I}_v & 0 \\ 0 & pb \end{pmatrix} - (k+p)^{-1}\begin{pmatrix} \boldsymbol{N}_0\boldsymbol{N}_0' & rp\mathbf{1}_v \\ rp\mathbf{1}_v' & p^2b \end{pmatrix}$$

$$= a\left[\begin{pmatrix} s\boldsymbol{I}_v & 0 \\ 0 & z \end{pmatrix} - g^{-1}\begin{pmatrix} s\mathbf{1}_v \\ z \end{pmatrix}(s\mathbf{1}_v' \quad z)\right], \qquad \ldots \quad (2.5)$$

where $s = \lambda/(k+p)$, $z = rp/(k+p)$, $g = vs+z$, $a = (vs+z)/s$. The detailed verification of (2.5) follows from elementary considerations. The proof of (ii) follows from (1.1) and (2.5).

3. Construction of VB and EB designs ($n > 0$)

In the rest of the paper we shall consider the situation $n > 0$. Then as noted earlier $(p, u_2) \neq (0, 0)$ and $(u_1, u_2) \neq (0, 0)$. The $\boldsymbol{C}$-matrix of the design d obtained through reinforcement is now given by

$$\boldsymbol{C} = \begin{pmatrix} (r+u_1n)\boldsymbol{I}_v & 0 \\ 0 & (pb+u_2n)\boldsymbol{I}_t \end{pmatrix} - (k+pt)^{-1}\begin{pmatrix} \boldsymbol{N}_0\boldsymbol{N}_0' & rp\boldsymbol{J}_{vt} \\ rp\boldsymbol{J}_{tv} & p^2b\boldsymbol{J}_{tt} \end{pmatrix}$$

$$-(u_1v+u_2t)^{-1}\begin{pmatrix} u_1^2n\boldsymbol{J}_{vv} & u_1u_2n\boldsymbol{J}_{vt} \\ u_1u_2n\boldsymbol{J}_{tv} & u_2^2n\boldsymbol{J}_{tt} \end{pmatrix}. \qquad \ldots \quad (3.1)$$

This section will deal with the construction problems for VB and EB designs. The same problems for GEB designs will be taken up in the next section.

Theorem 3.1 : (i) *If* $n > 0$, $t \geqslant 2$, *then* d *cannot be a VB design.* (ii) *If* $n > 0$, $t = 1$, *then* d *is a VB design if and only if*

$$(rp-\lambda)/(k+p) = u_1 n(u_1-u_2)/(u_1 v+u_2).$$

Proof : (i) Let $t \geqslant 2$. If d is VB then by (3.1),

$$\begin{aligned}\lambda/(k+pt)+u_1^2 n/(u_1 v+u_2 t) &= rp/(k+pt)+u_1 u_2 n/(u_1 v+u_2 t)\\ &= p^2 b/(k+pt)+u_2^2 n/(u_1 v+u_2 t). \qquad \ldots \quad (3.2)\end{aligned}$$

The above follows by considering the off-diagonal elements in (3.1). Note that the third term in (3.2) can arise only when $t \geqslant 2$.

If $p = 0$, then from the second equality in (3.2), $u_2 n(u_1-u_2) = 0$. Since $(p, u_2) \neq (0, 0)$, this yields $u_1 = u_2$. Now, with $p = 0$, the first equality in (3.2) reduces to $\lambda = 0$, which is impossible.

Suppose now $p \geqslant 1$. Then $p^2 b > rp > \lambda$, and by (3.2),

$$u_1 > u_2 > 0. \qquad \ldots \quad (3.3)$$

Since by (3.2), $(rp-\lambda)/(p^2 b-rp) = u_1/u_2$, it follows from (3.3) that $p^2 b-2rp+\lambda < 0$, which implies that $p < [r+(r^2-b\lambda)^{1/2}]/b$. This is, however, impossible as $[r+(r^2-b\lambda)^{1/2}]/b \leqslant 1$ and $p \geqslant 1$. This completes the proof of (i).

(ii) Follows trivially from the first equality in (3.2) noting that the third term in (3.2) cannot arise when $t = 1$.

Remark 3.1 : The designs in Sections 5.1 and 5.2 of Das and Ghosh (1985) follow from Theorem 3.1(ii) taking $p = 0$, $u_1 = 1$, $u_2 = 1+\lambda(v+1)/(nk-\lambda)$ and $p \geqslant 1$, $u_1 = v(rp-\lambda)/[n(k+p)]$, $u_2 = 0$ respectively.

Remark 3.2 : It is interesting to examine whether Theorem 3.1(ii) may be employed to generate binary VB designs. In order that d is binary each of p, u_1, u_2 must be 0 or 1. Since $(p, u_2) \neq (0, 0)$, $(u_1, u_2) \neq (0, 0)$, one has the following possibilities : (A) $p = u_1 = 0$, $u_2 = 1$, (B) $p = 0$, $u_1 = u_2 = 1$, (C) $p = 1$, $u_1 = 0$, $u_2 = 1$, (D) $p = u_1 = 1$, $u_2 = 0$, (E) $p = u_1 = u_2 = 1$. Among these under (A), (B), (C) and (E), the condition stated in Theorem 3.1(ii) cannot hold as one can easily verify. In the situation (D) the condition reduces to $n = v(r-\lambda)/(k+1)$ and this has been considered in Section 5.4.2 of Das and Ghosh (1985).

Theorem 3.2 : (i) *If* $n > 0$, $t \geqslant 2$, *then* d *cannot be an EB design.* (ii) *If* $n > 0$, $t = 1$, *then* d *is an EB design if and only if*

$$u_1 n(pbu_1 - ru_2)/(u_1 v + u_2) = [rp(r + u_1 n) - \lambda(pb + u_2 n)]/(k + p).$$

Proof : Let $t \geqslant 2$ and if possible suppose d is an EB design. Then considering the off-diagonal elements in (3.1) one has

$$\lambda/(k+pt) + u_1^2 n/(u_1 v + u_2 t) = a(r + u_1 n)^2/r_0, \qquad \ldots \quad (3.4a)$$

$$rp/(k+pt) + u_1 u_2 n/(u_1 v + u_2 t) = a(r + u_1 n)(pb + u_2 n)/r_0, \qquad \ldots \quad (3.4b)$$

$$p^2 b/(k+pt) + u_2^2 n/(u_1 v + u_2 t) = a(pb + u_2 n)^2/r_0, \qquad \ldots \quad (3.4c)$$

where $r_0 = v(r + u_1 n) + t(pb + u_2 n)$. Observe that (3.4c) can arise only when $t \geqslant 2$. By (3.4a, b, c),

$$pn(pbu_1 - ru_2)/(k+pt) = u_2 n(pbu_1 - ru_2)/(u_1 v + u_2 t), \qquad \ldots \quad (3.5a)$$

$$[rp(r + u_1 n) - \lambda(pb + u_2 n)]/(k+pt) = u_1 n(pbu_1 - ru_2)/(u_1 v + u_2 t). \qquad \ldots \quad (3.5b)$$

First suppose $pbu_1 \neq ru_2$. Then by (3.5a),

$$p/(k+pt) = u_2/(u_1 v + u_2 t),$$

which, on simplification, yields $pbu_1 = ru_2$. Thus $pbu_1 \neq ru_2$ is impossible and one must have

$$pbu_1 = ru_2. \qquad \ldots \quad (3.6)$$

Under (3.6), $u_1, u_2 > 0$, for otherwise $(u_1, u_2) = (0, 0)$ which is impossible. By (3.5b), (3.6), $rp(r + u_1 n) = \lambda(pb + u_2 n) = (\lambda u_2/u_1)(r + u_1 n)$, and hence $rp = \lambda u_2/u_1$, which, together with (3.6), yields $r^2 = \lambda b$; but this is impossible by (2.2). This completes the proof of (i). The proof of (ii) is straightforward and follows along the line of the proof of Theorem 2.2(ii).

Remark 3.3 : The designs in Sections 3.2 and 3.3 of Das and Ghosh (1985) follow from Theorem 3.2(ii) taking $p = 0$, $u_1 = 1$, $u_2 = (r - \lambda)/\lambda$ and $p \geqslant 1$, $u_1 = (r - \lambda)/(np)$, $u_2 = 0$ respectively.

Remark 3.4 : It is interesting to investigate the derivation of binary EB designs from Theorem 3.2(ii). As in Remark 3.2, the situations (A)–(E) listed there have to be considered. It is easy to see that under (A) the condition stated in Theorem 3.2(ii) cannot hold. Under (B) the condition becomes $r = 2\lambda$, and there are many choices of d_0 satisfying this relation. In particular, the series of BIB designs with $v = b = 4l + 3$, $r = k = 2l + 2$, $\lambda = l + 1$, where $4l + 3$ is a prime or prime power, satisfies $r = 2\lambda$. Under (C) the condition reduces to $n = (r^2 - b\lambda)/\lambda$, and construction is possible if $(r^2 - b\lambda)/\lambda$ is a positive integer. This integrality condition is seen to hold for many choices of d_0 (in particular it holds for every d_0 with $\lambda = 1$). Similarly, under (D) the condition in Theorem 3.2(ii) becomes $n = r-\lambda$, and clearly construction is possible for every choice of d_0. Finally, under (E) one obtains a simplified form of the condition in Theorem 3.2(ii) as $n = r(v+1)/(v-2k-1)$, and construction is possible provided $r(v+1)/(v-2k-1)$ is a positive integer. There are again many choices of d_0 satisfying this integrality condition (e.g. some possible choices of d_0 are $v = 6$, $b = 15$, $r = 5$, $k = 2$, $\lambda = 1$; $v = 7$, $b = 21$, $r = 6$, $k = 2$, $\lambda = 1$; $v = 15$, $b = 35$, $r = 7$, $k = 3$, $\lambda = 1$; and so on). Observe that if the integrality condition under (C) or (E) is satisfied by some BIB design d_0 then the same condition is satisfied by BIB design obtained by repeating d_0 several times. Das and Ghosh (1985) considered some aspects of the situations (B) and (D) above, while the findings under (C) and (E) appear to be new.

Theorems 2.1, 3.1(i), 3.2(i) show that it is impossible to construct a VB or an EB design in $(v+2)$ or more treatments through the reinforcement of a BIB design in v treatments. These theorems also imply that the result is fairly strong in the sense that the impossibility holds even when non-binary designs are allowed (with values of p, u_1, u_2 possibly greater than unity). This provides a theoretical justification to a statement in Das and Ghosh (1985, p. 71, lines 25–28).

4. Construction of GEB designs ($n > 0$)

With $n > 0$, we shall explore in this section the situations under which d can be a GEB design in the sense of Das and Ghosh (1985).

Theorem 4.1 : (i) *If* $n > 0$, $t = 1$, *then* d *is always a GEB design.* (ii) *If* $n > 0$, $t \geqslant 2$, *then* d *is a GEB design if and only if*

$$[\lambda/(k+pt)+u_1^2n/(u_1v+u_2t)][p^2b/(k+pt)+u_2^2n/(u_1v+u_2t)]$$
$$= [rp/(k+pt)+u_1u_2n/(u_1v+u_2t)]^2.$$

Proof : The proof of (i) follows along the line of proof of Theorem 2.2(ii). The validity of (ii) is an immediate consequence of (3.1) and the following lemma :

Lemma : *Suppose a design* d *in* $v+t$ *treatments* $(v, t \geqslant 2)$ *has a* $\boldsymbol{C}$*-matrix of the form*

$$\boldsymbol{C} = \begin{pmatrix} f_1I_v-f_2J_{vv} & -f_3J_{vt} \\ -f_3J_{tv} & f_4I_t-f_5J_{tt} \end{pmatrix}$$

where $f_1, f_2, f_4, f_5 > 0$, *and obviously* $f_1 = f_2v+f_3t$, $f_4 = f_3v+f_5t$. *Then* d *is a GEB design if and only if* $f_2f_5 = f_3^2$.

The proof of the lemma follows from (1.1) and the details are omitted here.

Remark 4.1 : Theorems 2.2(ii) and 4.1(i) show that if a single new treatment is added then the reinforcement of a BIB design always leads to a GEB design. If further the condition in Theorem 3.1(ii) or that in Theorem 3.2(ii) holds then this GEB design is actually a VB or an EB design.

Remark 4.2 : For $n > 0$, $t \geqslant 2$, as before we shall give special attention to the construction of binary GEB designs considering the situations (A)–(E) listed in Remark 3.2. Under (A) or (B) it may be seen that the condition stated in Theorem 4.1(ii) can never be realized. Under (C), (D) or (E) this condition reduces to

$$n = (r^2 - b\lambda)t/[\lambda(k+t)], \qquad \ldots \quad (4.1)$$

$$n = (r-\lambda)/(k+t), \qquad \ldots \quad (4.2)$$

or

$$n = (r^2 - b\lambda)(v+t)/[(b-2r+\lambda)(k+t)], \qquad \ldots \quad (4.3)$$

respectively and for a fixed t and a fixed d_0 construction is possible whenever the right-hand number in one of the above is a positive integer. These integrality conditions are seen to hold in a large number of cases. For example, the right-hand member of (4.1) is a positive integer if $t = s$, $v = s^2$, $b = s^2+s$, $r = s+1$, $k = s$, $\lambda = 1$, s being an odd prime or prime power. Similarly, the integrality condition (4.2) holds if, in particular, $t = 3(l-1)$, $v = 6l+3$, $b = (3l+1)(2l+1)$, $r = 3l+1$, $k = 3$, $\lambda = 1$, where $l\,(\geqslant 2)$ is an integer. Some examples where the right-hand member of (4.3) is a positive integer are : $t = 4$, $v = 4$, $b = 6$, $r = 3$, $k = 2$, $\lambda = 1$; $t = 21$, $v = 9$, $b = 12$, $r = 4$, $k = 3$, $\lambda = 1$; $t = 10$, $v = 8$, $b = 14$, $r = 7$, $k = 4$, $\lambda = 3$, and so on. Numerous further examples may be given (see e.g. Raghavarao (1971, Ch. 5)) where the right-hand members of (4.1)–(4.3) are positve integers. Also if for some t and for some BIB design d_0 the right-hand member of (4.1), (4.2) or (4.3) is a positive integer, then the same holds for the same t and a BIB design obtained by repeating d_0 several times. Thus starting from a BIB design with v treatments although one cannot obtain through reinforcement a VB or an EB design in $(v+2)$ or more treatments, GEB designs in $(v+2)$ or more treatments can be constructed in abundance and this is true even if one restricts only to binary GEB designs. This shows that reinforcement is a very powerful tool in the construction of GEB designs.

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