ON THE EXISTENCE OF TREES WITH GIVEN DEGREES

By V. V. MENON

Indian Statistical Institute

SUMMARY. A necessary and sufficient condition for the existence of an oriented tree with given demi-degrees has been given. As a corollary, the problem of existence of an unoriented tree with given degrees has been solved. Finally, algorithms for the construction of such trees are obtained.

1. Introduction

Preliminary notions : In the following, by a graph we mean a set X of elements called vertices, and a set Γ of pairs of elements from X.

An *oriented* graph is a graph in which the members of Γ are ordered pairs of vertices. If (x_i, x_j) is such a pair, we say that it is an *arc directed from* x_i to x_j, and call x_i and x_j as the initial and terminal vertices, respectively, of the arc. They together constitute the end-vertices of the arc.

Definition 1 : The *demi-degrees* of the vertex x_i is the pair (d_i^+, d_i^-), where d_i^+ is the number of arcs whose initial vertex is x_i, and d_i^- is the number of arcs whose terminal vertex is x_i.

The pairs (d_i^+, d_i^-), $i = 1, 2, \ldots, n$, constitute the demi-degrees of the graph with n vertices x_i, $i = 1, \ldots, n$, where (d_i^+, d_i^-) stands for the demi-degrees of x_i.

On the other hand, an *unoriented* graph is a graph where the members of Γ, called *edges*, are unordered pairs of vertices. If (x_i, x_j) is an edge, x_i and x_j are called its end-vertices.

Definition 2 : The *degree* of the vertex x_i is the number of edges which have x_i as an end-vertex.

The numbers d_i, $i = 1, 2, \ldots, n$, where d_i is the degree of the vertex x_i $(i = 1, \ldots, n)$ constitute the degrees of the graph.

The arcs and edges are represented respectively as directed and undirected lines joining the pertinent vertices. For example, in Figure 1(a) is given the arc (x_i, x_j) and in Figure 1(b), the edge (x_i, x_j).

Fig. 1(a)

Fig. 1(b)

In a graph (oriented or unoriented), a *chain* between x_i and x_j is defined as a sequence of arcs (edges) $u_1, \ldots, u_k$ such that

(i) x_i is an end vertex of u_1, and x_j is an end-vertex of u_k,

(ii) u_i has one end-vertex in common with u_{i-1} and the other with u_{i+1}, for $i = 2, 3, \ldots, k-1$.

The graph is said to be connected if there is a chain between each pair of vertices.

Definition 3 : An *oriented* (*unoriented*) *tree* is an oriented (unoriented) connected graph with n vertices where $n \geqslant 2$, and $n-1$ arcs (edges).

The problem : The problem we are concerned with is that of existence of an oriented tree with given demi-degrees, that is, of a tree whose demi-degrees are some given pairs of integers. The soluton is given as Theorem 1 below.

As a corollary, one solves a related problem : that of the existence of an unoriented tree with given degrees; in other words, of finding the conditions on a set of n integers such that they can be the degrees of a tree with n vertices (Ore, 1962). The conditions appear as Theorem 2.

The main results are stated below.

Theorem 1 : *Let* (a_i, b_i), $i = 1, ..., n$, *be given pairs of non-negative integers. A necessary and sufficient condition for the existence of an oriented tree with n vertices and with demi-degrees* (a_i, b_i), $i = 1, ..., n$, *is that*

$$a_i+b_i \geqslant 1 \quad \textit{for all} \quad i,$$

and

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = n-1.$$

Theorem 2 : *Let* ρ_i, $i = 1, ..., n$, *be given non-negative integers. A necessary and sufficient condition for the existence of an unoriented tree with n vertices and degrees* ρ_i, $i = 1, ..., n$, *is that*

$$\rho_i \geqslant 1 \quad \textit{for all} \quad i,$$

and

$$\sum_{i=1}^{n} \rho_i = 2(n-1).$$

2. Proofs of theorems 1 and 2

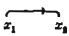
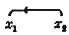
The necessity of the conditions in both the theorems follows immediately from the Definitions 1, 2 and 3 if we note that (1) each arc has one initial and one terminal vertex while each edge has two end-vertices, and (2) in any tree, at least one chain starts from a vertex x_i.

It remains to prove the sufficiency of the conditions in both the theorems.

(1) *Sufficiency of the conditions in Theorem* 1: We shall prove the result by induction on n, the number of pairs.

When $n = 2$, the conditions imply that

either (a) $\begin{cases} a_1 = 1 = b_2 \\ b_1 = 0 = a_2, \end{cases}$ or (b) $\begin{cases} a_1 = 0 = b_2 \\ b_1 = 1 = a_2 \end{cases}$

Consider the trees (a)  and (b)  , respectively in those cases. We see that the demi-degrees of the tree are (1, 0), (0, 1), (the same as (a_1, b_1), (a_2, b_2)) in the case (a), and (0, 1), (1, 0) (the same as (a_1, b_1), (a_2, b_2)) in the case (b). Hence the theorem is true for $n = 2$. We now assume the truth of the theorem for $n-1$ pairs.

Consider the $n(n > 2)$ pairs (a_i, b_i), which satisfy the conditions. Since $a_i+b_i \geqslant 1$ for all i, it follows from $\sum_{i=1}^{n} (a_i+b_i) = \sum_1^n a_i+\sum_1^n b_i = 2(n-1)$ that $a_i+b_i \geqslant 2$ for at most $n-2$ pairs.

In other words, $a_i+b_i < 2$ i.e., $a_i+b_i = 1$, for at least two pairs. Without loss of generality we can assume (by renumbering the pairs, if necessary) that

$$a_n \leqslant a_{n-1} \text{ and } a_{n-1}+b_{n-1} = a_n+b_n = 1. \qquad \ldots \quad (1)$$

In the following we also make use of the fact that $a_j+b_j \geqslant 2$ for at least one j, (because otherwise each $a_i+b_i = 1$ and $\sum_1^n (a_i+b_i) = n < 2(n-1)$).

Consider n points $x_1, \ldots, x_n$. We shall construct an oriented tree with these vertices and with demi-degrees

$$\left.\begin{array}{l} d_i^+ = a_i \\ d_i^- = b_i \end{array}\right\} \quad i = 1, 2, \ldots, n.$$

In view of (1), there are three possible cases.

Case 1 : $(a_n, b_n) = (a_{n-1}, b_{n-1}) = (1, 0)$. Since $\sum_1^n b_i = n-1$, it follows that there is a j_0 such that $b_{j_0} \geqslant 2$. Consider the $n-1$ vertices $x_1, \ldots, x_{j_0}, \ldots x_{n-1}$ corresponding to $(n-1)$ pairs $(a_1, b_1), \ldots, (a_{j_0}, b_{j_0}-1), \ldots, (a_{n-1}, b_{n-1})$. These pairs satisfy the conditions of the theorem, as is easy to verify. Hence a tree T with these vertices and demi-degrees exists by hypothesis of induction. Now consider the graph obtained by taking the tree T augmented by the vertex x_n and the arc (x_n, x_{j_0}) [see Fig. 2(a)]. This graph is obviously a tree with the required demi-degrees.

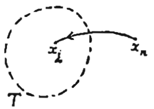


Fig. 2(a)

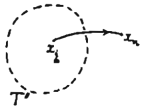


Fig. 2(b)

Case 2 : $(a_n, b_n) = (a_{n-1}, b_{n-1}) = (0, 1)$. Since $\sum_1^n a_i = n-1$, there is a j_0 such that $a_{j_0} \geqslant 2$. Consider the vertices $x_1, \ldots, x_{j_0} \ldots x_{n-1}$ with the series of corresponding $n-1$ pairs $(a_1, b_1), \ldots, (a_{j_0}-1, b_{j_0}), \ldots, (a_{n-1}, b_{n-1})$, these pairs satisfy the conditions of the theorem whence a tree T with these vertices and demi-degrees exists. If we augment T as in Figure 2(b) we verify that the graph obtained is indeed a tree with the required demi-degrees.

Case 3 : $(a_n, b_n) = (0, 1)$; $(a_{n-1}, b_{n-1}) = (1, 0)$. There exists a j_0 such that $a_{j_0}+b_{j_0} \geqslant 2$. If $a_{j_0} \geqslant 1$, consider the $n-1$ vertices $x_1, ..., x_{j_0}, ..., x_{n-1}$, with the pairs $(a_1, b_1), ..., (a_{j_0}-1, b_{j_0}), ..., (a_{n-1}, b_{n-1})$. These pairs satisfy the conditions of the theorem. Therefore a tree T_1 exists with these vertices and corresponding demi-degrees. One verifies that the graph in Figure 3(a) is a tree with required demi-degrees.

If $b_{j_0} \geqslant 1$ we consider the $n-1$ vertices $x_1, ..., x_{j_0}, ..., x_{n-2}, x_n$ and pairs $(a_1, b_1), ..., (a_{j_0}, b_{j_0}-1), ..., (a_{n-2}, b_{n-2}), (a_n, b_n)$ and build up the tree T_2 with these vertices and demi-degrees, as in other cases. The graph in Fig. 3(a) is then the required tree.

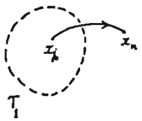


Fig. 3(a)

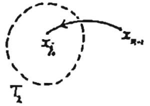


Fig. 3(b)

This proves the theorem.

(2) *Sufficiency of the conditions for Theorem* 2 : We are given numbers $\rho_i \geqslant 1$, $1 \leqslant i \leqslant n$, such that $\sum_1^n \rho_i = 2(n-1)$. Consider the n pairs (a_i, b_i), $1 \leqslant i \leqslant n$, where

$$a_i = \begin{cases} 1 & \text{for } 1 \leqslant i \leqslant n-1 \\ 0 & \text{for } i = n \end{cases}$$

and $$b_i = \rho_i - a_i, \quad 1 \leqslant i \leqslant n.$$

Then the pairs (a_i, b_i) satisfy the conditions of Theorem 1. Hence an oriented tree with these demi-degrees exists. Consider the unoriented graph G obtained from this tree by disregarding the orientation. It is easily verified that G is the required tree.

3. Algorithms for constructing the trees

The method of proof for Theorem 1 suggests a simple algorithm for constructing trees with given demi-degrees. For the case of trees with given degrees, a simpler algorithm exists.

Algorithm 1(*tree with given demi-degrees*). Let the given demi-degrees (satisfying the conditions of the Theorem 1) (a_i, b_i), $1 \leqslant i \leqslant n$, be arranged in the descending order of magnitude of a_i+b_i. Then necessarily $a_{n-1}+b_{n-1} = a_n+b_n = 1$. We take the vertices as $x_1, ..., x_n$.

(i) If $(a_n, b_n) = (0, 1) = (a_{n-1}, b_{n-1})$, find the largest j_0 with $a_{j_0} \geqslant 2$. Join x_{j_0} to x_n by the arc (x_{j_0}, x_n) and for the next stage consider vertices $x_1, ..., x_{n-1}$ with the corresponding pairs $(a_1, b_1), ..., (a_{j_0}-1, b_{j_0}), ..., (a_{n-1}, b_{n-1})$.

(ii) if $(a_n, b_n) = (1, 0) = (a_{n-1}, b_{n-1})$, find the largest j_0 with $b_{j_0} \geqslant 2$. Join x_n to x_{j_0} by an arc (x_n, x_{j_0}) and for the next stage consider vertices $x_1, \ldots, x_{n-1}$ with the pairs $(a_1, b_1), \ldots, (a_{j_0}, b_{j_0}-1), \ldots, (a_{n-1}, b_{n-1})$.

(iii) If $(a_n, b_n) = (0, 1)$ and $(a_{n-1}, b_{n-1}) = (1, 0)$, find the largest j_0 with $a_{j_0}+b_{j_0} \geqslant 2$. If $a_{j_0} \geqslant 1$, draw the arc (x_{j_0}, x_n) and consider for the next stage the vertices $x_1, \ldots, x_{j_0}, \ldots, x_{n-1}$ together with the corresponding pairs $(a_1, b_1), \ldots, (a_{j_0}-1, b_{j_0}), \ldots, (a_{n-1}, b_{n-1})$.

If $b_{j_0} \geqslant 1$, draw the arc (x_{n-1}, x_{j_0}) and consider next the vertices $x_1, \ldots, x_{j_0}, \ldots, x_{n-2}, x_n$ with the pairs $(a_1, b_1), \ldots, (a_{j_0}, b_{j_0}-1), \ldots, (a_{n-2}, b_{n-2}), (a_n, b_n)$.

(iv) If $(a_n, b_n) = (1, 0)$ and $(a_{n-1}, b_{n-1}) = (0, 1)$ apply (iii) with n and $n-1$ interchanged.

At each stage, in this manner, we go on reducing the number of pairs and Σa_i, Σb_i, each by unity.

We go on doing this for $n-1$ stages, in each of which an arc in the tree is determined, and we will obtain the required tree.

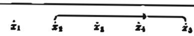
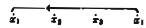
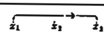
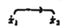
Algorithm 2 (*tree with given degrees*). Let the required degrees be ρ_i, $1 \leqslant i \leqslant n$, (which satisfy the conditions in Theorem 2). We renumber them so that $\rho_1, \ldots, \rho_n$ is a non-ascending sequence. Consider vertices $x_1, \ldots, x_n$. We correspond x_i to ρ_i, $1 \leqslant i \leqslant n$. We must have $\rho_{n-1} = \rho_n = 1$. Take the largest j such that $\rho_j \neq 1$ and join x_n to x_j by an edge. Consider next the $n-1$ vertices $x_1, \ldots, x_j, \ldots, x_{n-1}$ and the corresponding numbers $\rho_1, \ldots, \rho_j-1, \ldots, \rho_{n-1}$. Apply the above procedure to these in the next stage.

In $n-1$ stages we obtain a tree in this manner.

4. Examples

Example 1 : Suppose we are given the pairs (1, 0), (0, 1), (1, 1), (2, 1), (0. 1), satisfying the conditions of the Theorem 1, and we want to construct a tree with these demi-degrees. First we rearrange them and obtain (2, 1), (1, 1), (0, 1), (1, 0), (0, 1). Assume the corresponding vertices to be x_1, x_2, x_3, x_4, x_5. The following table shows the application of the algorithm.

TABLE 1. ALGORITHM 1 APPLIED TO EXAMPLE 1

stage	pairs available at stage i					arc added during stage i and the available vertices
i	1	2	3	4	5	
1	(2,1)	(1,1)*	(0,1)	(1,0)*	(0,1)*	 x_1 x_2 x_3 x_4 x_5
2	(2,1)*	(0,1)	(0,1)*	(1,0)*		 x_1 x_2 x_3 x_1
3	(2,0)*	(0,1)*	(0,1)*			 x_1 x_2 x_3
4	(1,0)*	(0,1)*				 x_1 x_2

Note : The asterisks show the pairs actually relevant to corresponding stage of the algorithm.

The arcs of the tree are, therefore, (x_1, x_2), (x_1, x_3), (x_2, x_5) and (x_4, x_1), as in Figure 4.

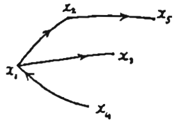


Fig. 4

Example 2 : Let the required degrees be 1, 2, 1, 1, 3.

We first rearrange them as 3, 2, 1, 1, 1. Take vertices x_1, x_2, x_3, x_4, x_5. The following table shows the relevant procedure.

TABLE 2. ALGORITHM 2 APPLIED TO EXAMPLE 2

stage	pairs available at stage i					edge added during stage i and the available vertices				
i	1	2	3	4	5					
1	3	2*	1	1	1*	$\dot{x}_1$	$\dot{x}_2$	$\dot{x}_3$	$\dot{x}_4$	$\dot{x}_5$
2	3*	1	1	1*		$\dot{x}_1$	$\dot{x}_2$	$\dot{x}_3$	$\dot{x}_4$	
3	2*	1	1*			$\dot{x}_1$	$\dot{x}_2$	$\dot{x}_3$		
4	1*	1*				$\dot{x}_1$	$\dot{x}_2$			

Note : Asterisks show the numbers relevant to the corresponding stage of the algorithm.

Thus the tree has the edges (x_1, x_2), (x_1, x_3), (x_1, x_4) and (x_2, x_5) as in Figure 5.

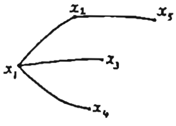


Fig. 5

REFERENCE

Ore, O. (1962) : *Theory of Graphs*, American Mathematical Society Colloquium Publications, 38, Chapter 4, Section 4.1., Problem 3.

Paper received : October, 1963.