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SOME RELATIONS AMONG INEQUALITY MEASURES

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SUMMARY. This paper develops a number of inequalities among some standard inequality measures popular in econometrics. In most cases the inequalities obtained cannot be made more stringent.

1. Introduction

Over the last few years substantial work has been done on inequality measures. The social welfare functions leading to different inequality measures have been studied extensively (see e.g. Chakravarty, 1981, 1982; Sheshinski, 1972). This paper derives some inequalities relating to such measures. The results are expected to be helpful in exploring possible inter-relationships among the measures.

With non-negative observations $x_1, x_2, \ldots, x_n$, having positive arithmetic mean $\bar{x}$, the following inequality measures (cf. Bhandari, 1985; Marshall and Olkin, 1979; Ord *et al.*, 1983; Sen, 1973 among others) have been considered in this paper. Note that these measures are all normalized i.e., equal zero when $x_1, x_2, \ldots, x_n$ are all equal.

(i) Gini's coefficient :

$$G = G(x_1, \ldots, x_n) = \sum_{i<j=1}^{n}\sum |x_i - x_j| / (n^2\bar{x}),$$

(ii) Coefficient of variation :

$$C = C(x_1, \ldots, x_n) = \left[\sum_{i=1}^{n} (x_i - \bar{x})^2 / (n\bar{x}^2) \right]^{1/2},$$

(iii) Measure derived from Mellin transformation :

$$H_\lambda = H_\lambda(x_1, \ldots, x_n) = \left\{ \sum_{i=1}^{n} (x_i/\bar{x})^\lambda / n \right\} - 1 \; (\lambda > 1),$$

(iv) Theil's entropy measure :

$$T = T(x_1, \ldots, x_n) = \left[\sum_{i=1}^{n} (x_i/\bar{x}) \log(x_i/\bar{x}) \right] / n.$$

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For the sake of simplicity in presentation, we have taken in (iii) above a form of H_λ which is a strictly increasing function of the conventional form (Ord *et al.*, 1983).

2. Some lemmas

The distributions under consideration are non-degenerate.

Lemma 2.1 :

$$(n-1) \sum_{i<j=1}^{n}\sum (x_i-x_j)^2 \leqslant \Big(\sum_{i<j=1}^{n}\sum |x_i-x_j| \Big)^2$$

$$\leqslant \frac{1}{3} (n^2-1) \sum_{i<j=1}^{n}\sum (x_i-x_j)^2.$$

Proof : Without loss of generality, let $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$. Let $t = n-1$, $\alpha_i = x_{i+1}-x_i$ $(1 \leqslant i \leqslant t)$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_t)'$. Then it can be seen that

$$\sum_{i<j=1}^{n}\sum (x_i-x_j)^2 = \boldsymbol{\alpha}' M \boldsymbol{\alpha}, \quad \sum_{i<j=1}^{n}\sum |x_i-x_j| = \boldsymbol{l}'\boldsymbol{\alpha}, \qquad \ldots \quad (2.1)$$

where $\boldsymbol{l} = (l_1, \ldots, l_t)'$, $l_i = i(n-i)$ $(1 \leqslant i \leqslant t)$ and $M^{(t\times t)} = ((m_{ij}))$ is a symmetric matrix with $m_{ij} = i(n-j)$ $(1 \leqslant i \leqslant j \leqslant t)$. One may check that M is positive definite (p.d.) and M^{-1} has entries $\frac{2}{n}$ along the principal diagonal, $-\frac{1}{n}$ just below and above the principal diagonal and 0 elsewhere (cf. Mukerjee and Saharay, 1985).

Since M is p.d. by (2.1) and Cauchy-Schwarz inequality (Rao, 1973, p. 60),

$$\Big(\sum_{i<j=1}^{n}\sum |x_i-x_j| \Big)^2 = (\boldsymbol{l}'\boldsymbol{\alpha})^2 \leqslant (\boldsymbol{l}'M^{-1}\boldsymbol{l})(\boldsymbol{\alpha}'M\boldsymbol{\alpha})$$

$$= (\boldsymbol{l}'M^{-1}\boldsymbol{l}) \sum_{i<j=1}^{n}\sum (x_i-x_j)^2.$$

After considerable algebra, $\boldsymbol{l}'M^{-1}\boldsymbol{l} = \frac{1}{3}(n^2-1)$ and the right-hand inequality follows. Also noting that

$$l_i l_j \geqslant (n-1) m_{ij} \; (1 \leqslant i, j \leqslant t),$$

the left-hand inequality follows immediately from (2.1) and the fact that $\boldsymbol{\alpha} \geqslant \mathbf{0}$.

Remark : The right-hand inequality in Lemma 2.1 attains equality iff $\alpha_1 = \alpha_2 = \ldots = \alpha_t$ i.e., if $x_1, x_2, \ldots, x_n$ are equispaced which means

$$x_i = x_1+(i-1)(x_n-x_1)/(n-1) \;\; (i = 1, 2, \ldots, n).$$

The left-hand inequality attains equality iff $x_1 = x_2 = \ldots = x_{n-1} \leqslant x_n$.

Lemma 2.2 : *Let* $S = \{(x_1, \ldots, x_n) : x_i \geqslant 0 \; (1 \leqslant i \leqslant n), \; \bar{x} > 0\}$. *Then*

$$1 \leqslant \left(\sum_{i=1}^{n} (x_i/\bar{x})^{\mu} \right) \Big/ \left(\sum_{i=1}^{n} (x_i/\bar{x})^{\lambda} \right) \leqslant n^{\mu-\lambda},$$

provided $(x_1, \ldots, x_n) \in S$ *and* $1 < \lambda < \mu$.

Proof : The right-hand inequality follows trivially as

$$(x_i/\bar{x})^{\mu} \leqslant n^{\mu-\lambda}(x_i/\bar{x})^{\lambda} \;\; (1 \leqslant i \leqslant n).$$

To prove the left-hand inequality, let $f(\xi) = \Sigma_{i=1}^{n}(x_i/\bar{x})^{\xi}$. Observe that for $(x_1, \ldots, x_n) \in S$, $f'(1) = \Sigma_{i=1}^{n}(x_i/\bar{x})\log(x_i/\bar{x}) \geqslant 0$ (by Jensen's inequality) and $f''(\xi) \geqslant 0$ for $\xi \geqslant 1$. Hence $f(\xi)$ is non-decreasing in ξ for $\xi \geqslant 1$, so that $f(\mu) \geqslant f(\lambda)$, completing the proof.

Remark : The right-hand inequality in Lemma 2.2 attains equality iff among $x_1, \ldots, x_n$ exactly one is positive while the rest equal 0. The left-hand inequality attains equality iff $x_1, \ldots, x_n$ are all equal.

3. Main results

The above lemmas will be applied in this section to derive inequalities among the measures considered in Section 1. The notation is as before and $x_1, \ldots, x_n$ are non-negative observations with $\bar{x} > 0$.

Theorem 3.1 : $(n-1)^{1/2}n^{-1}C \leqslant G \leqslant ((n^2-1)/3)^{1/2}n^{-1}C$.

Theorem 3.2 : *For* $1 < \lambda < \mu$, $H_{\lambda} \leqslant H_{\mu} \leqslant n^{\mu-\lambda}(H_{\lambda}+1)-1$.

Since $H_2 = C^2$, the following corollary holds :

Corollary 3.1 : (i) *For* $\lambda > 2$, $n^{2-\lambda}(H_{\lambda}+1)-1 \leqslant C^2 \leqslant H_{\lambda}$;

(ii) *For* $1 < \lambda < 2$, $H_{\lambda} \leqslant C^2 \leqslant n^{2-\lambda}(H_{\lambda}+1)-1$.

Theorems 3.1 and 3.2 are immediate consequences of Lemmas 2.1 and 2.2. In particular, the proof of Theorem 3.1 utilizes the fact that

$$\sum_{i=1}^{n} (x_i-\bar{x})^2 = n^{-1} \sum_{i<j=1}^{n} (x_i-x_j)^2.$$

The case of equality in these theorems may be obtained from the remarks following the respective lemmas. As for Theil's measure, one has the following result.

Theorem 3.3 : $T \leqslant e^{-1}(H_2+1) = e^{-1}(C^2+1)$.

Proof : Note that $\max_{y>0} y^{-1} \log y = e^{-1}$ and hence $y \log y \leqslant e^{-1}y^2$ for $y \geqslant 0$. Therefore, defining $y_i = x_i/\bar{x}$ $(1 \leqslant i \leqslant n)$,

$$T/(H_2+1) = (\sum_{i=1}^{n} y_i \log y_i)/(\sum_{i=1}^{n} y_i^2) \leqslant e^{-1},$$

completing the proof.

Remark : In Theorem 3.3, equality holds iff every non-zero x_i equals $\bar{x}e$ which is, however, impossible since e is irrational. However, it may be checked that $T/(H_2+1)$ can be made arbitrarily close to e^{-1} for sufficiently large n provided $x_1, \ldots, x_n$ are suitably chosen.

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