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Binary designs are not always the best

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ABSTRACT

We give an example of a nonbinary block design which is better than any binary design with respect to the E-optimality criterion. This shows that the class of binary designs is not essentially complete, at least with respect to E-optimality.

RÉSUMÉ

On présente un exemple d'un schéma non binaire qui, relativement au critère de E-optimalité, est meilleur que tout schéma binaire. Cela démontre que la classe des schémas binaires n'est pas essentiellement complète, du moins relativement à la E-optimalité.

1. INTRODUCTION

When the size of the block is less than the number of treatments, it is customary to use a binary design, i.e., a design in which every treatment appears at most once in any particular block. It is generally felt that the class of binary designs is essentially complete in the sense that for any nonbinary design there exists a binary design which is at least as good. One of the outstanding unresolved problems in the theory of optimal block designs is the essential completeness of the class of binary designs (John and Mitchell 1977, Shah and Sinha 1989). In a private communication to one of the authors, the late Professor J.C. Kiefer stated that binary designs are the best, but the result is not yet established. In this note we examine this conjecture with respect to E-optimality and show that in fact the conjecture is false.

Consider designs for comparing v treatments using b blocks each of size k. Let $\mathbf{r} = (r_1, \ldots, r_v)^{\mathsf{T}}$ denote the vector giving replication numbers for the v treatments, and let $\mathbf{r}^{\delta}$ denote the $v \times v$ diagonal matrix of replication numbers. Under the usual fixed-effects additive model, the information matrix for treatment effects is given by

$$\mathbf{C} = \mathbf{r}^{\delta} - \frac{\mathbf{N}\mathbf{N}^{\mathsf{T}}}{k}, \tag{1.1}$$

where $\mathbf{N} = ((n_{ij}))$ is the $v \times b$ incidence matrix of the design (n_{ij} = number of times ith treatment is applied to experimental units in block j). Let $\lambda_{ii'}$ denote the elements of the concurrence matrix $\mathbf{N}\mathbf{N}^{\mathsf{T}}$. A design is said to be connected if rank of $\mathbf{C}$ is $v - 1$. The smallest nonzero eigenvalue of $\mathbf{C}$ will be denoted by $x_{(1)}$. The class of connected designs for comparing v treatments using b blocks each of size k will be denoted by $D(v, b, k)$. A design in $D(v, b, k)$ which maximizes $x_{(1)}$ is said to be E-optimal.

2. MAIN RESULT

We start with a balanced incomplete block design with seven treatments arranged in seven blocks, each of size three. We then replace treatment 7 by treatment 6, so that we have a nonbinary design for six treatments arranged in seven blocks each of size three. It has been shown Bagchi (1988) that this design is E-optimal in $D(6,7,3)$. It is easy to show that for this design $kx_{(1)} = 7$. We shall now show that for any binary design in this class $D(6,7,3)$ one has $kx_{(1)} < 7$, so that no binary design is E-optimal.

The search among the competing binary designs is narrowed down by using the following bounds for $kx_{(1)}$ given in Constantine (1981):

$$kx_{(1)} \leq \frac{vr_i(k-1)}{v-1} \qquad \text{for all } i, \tag{2.1}$$

$$kx_{(1)} \leq \frac{k-1}{2}(r_i + r_{i'}) + \lambda_{ii'} \qquad \text{for all } i \neq i', \tag{2.2}$$

We shall also use the following bound obtained by Jacroux (1983):

$$kx_{(1)} \leq \frac{\{\bar{r}(k-1) - (m-1)z\}v}{v-m}, \tag{2.3}$$

where $\bar{r}$ is the replication number realized by a set of m (say) treatments in a design, and $z = \min_{i<i'} \lambda_{ii'}$ over this set of m treatments.

From (2.1) it follows that if $r_1 \geq \cdots \geq r_6$, the only possibilities for a competing binary design to have $kx_{(1)} \geq 7$ are

$$\mathbf{r} = \mathbf{r}_{\mathrm{I}} = (6,\ 3,\ 3,\ 3,\ 3,\ 3)^{\mathsf{T}},$$

$$\mathbf{r} = \mathbf{r}_{\mathrm{II}} = (5,\ 4,\ 3,\ 3,\ 3,\ 3)^{\mathsf{T}},$$

$$\mathbf{r} = \mathbf{r}_{\mathrm{III}} = (4,\ 4,\ 4,\ 3,\ 3,\ 3)^{\mathsf{T}}.$$

We shall now deal with these three cases.

2.1. *Case I:* $\mathbf{r} = \mathbf{r}_{\mathrm{I}} = (6,3,3,3,3,3)^{\mathsf{T}}$.

From (2.2) we see that if $\lambda_{ii'} = 0$ for any pair involving treatments 2 to 6, then $kx_{(1)} < 7$. Thus for $kx_{(1)} \geq 7$, each of these $\binom{5}{2} = 10$ pairs must come at least once in a block. Since $r_1 = 6$, the design can have at most nine pairs (in a block) involving treatments 2 to 6. Thus we must have $\lambda_{ii'} = 0$ for at least one pair, and hence for any binary design with the above $\mathbf{r}$, we have $kx_{(1)} < 7$.

2.2. *Case II:* $\mathbf{r} = \mathbf{r}_{\mathrm{II}} = (5,4,3,3,3,3)^{\mathsf{T}}$.

Again (2.2) requires that each pair from treatments 3 to 6 must come at least once in a block. It is also easy to see that the pair (1, 2) must come in either three or four blocks of the design. Each of these two situations can be seen to give a unique design (up to isomorphism). These designs together with the respective value of $kx_{(1)}$ are given below:

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 4 & 5 & 3 & 4 \\ 3 & 4 & 5 & 6 & 6 & 6 & 5 \end{array} \qquad \begin{array}{ccccccc} 1 & 1 & 1 & 1 & 1 & 3 & 3 \\ 2 & 2 & 2 & 2 & 5 & 4 & 4 \\ 3 & 4 & 5 & 6 & 6 & 6 & 5 \end{array}$$

$$kx_{(1)} = 6.68 \qquad\qquad kx_{(1)} = 6.00.$$

2.3. *Case III:* $\mathbf{r} = \mathbf{r}_{\mathrm{III}} = (4,4,4,3,3,3)^{\mathsf{T}}$.

The relations (2.2) and (2.3) imply that the following are necessary conditions for a binary design to have $kx_{(1)} \geq 7$:

(1) $\lambda_{ii'} > 0$ for all pairs from treatments 4, 5, and 6.
(2) $\lambda_{ii'} < 2$ for at least one pair from treatments 4, 5, and 6.
(3) $\lambda_{ii'} < 3$ for at least one pair from treatments 1, 2, and 4.

In view of the above, the block $(1,2,3)$ can occur 0, 1, or 2 times. We deal with these possibilities below.

III(a): $(1,2,3)$ *not occurring as a block.*

It can be easily seen that there are only three possible designs. These designs are shown below:

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 2 & 2 & 4 \\ 2 & 2 & 3 & 3 & 3 & 3 & 5 \\ 4 & 4 & 5 & 5 & 6 & 6 & 6 \end{array} \qquad \begin{array}{ccccccc} 1 & 1 & 1 & 1 & 2 & 2 & 4 \\ 2 & 2 & 3 & 3 & 3 & 3 & 5 \\ 4 & 6 & 4 & 6 & 5 & 5 & 6 \end{array}$$

$$kx_{(1)} = 6.00 \qquad kx_{(1)} = 6.00$$

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 2 & 2 & 4 \\ 2 & 2 & 3 & 3 & 3 & 3 & 5 \\ 4 & 5 & 4 & 6 & 5 & 6 & 6 \end{array}$$

$$kx_{(1)} = 6.70.$$

III(b): $(1,2,3)$ *occurring once as a block.*

We look at the composition of the blocks other than the block $(1,2,3)$. The following four possible mutually exclusive cases arise:

(1) Out of the pairs $(1,2)$, $(1,3)$, $(2,3)$, one pair, say $(1,2)$, occurs three times in a block of the design, whereas the other two pairs do not occur. This gives us only one design, which is given below:

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 4 & 4 & 5 \\ 3 & 4 & 5 & 6 & 5 & 6 & 6 \end{array}$$

$$kx_{(1)} = 6.00.$$

(2) The pair $(1,2)$ occurs twice, and the pair $(1,3)$ occurs once, whereas the pair $(2,3)$ does not occur in a block. This gives the two designs shown below:

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 2 & 3 & 4 & 4 & 5 \\ 3 & 4 & 6 & 5 & 5 & 6 & 6 \end{array} \qquad \begin{array}{ccccccc} 1 & 1 & 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 2 & 3 & 4 & 4 & 5 \\ 3 & 4 & 5 & 6 & 5 & 6 & 6 \end{array}$$

$$kx_{(1)} = 6.39 \qquad kx_{(1)} = 5.40.$$

(3) The pairs $(1,2)$, $(1,3)$, $(2,3)$ each occur exactly once in a block. This leads to the following three designs:

$$\begin{array}{ccccccc} 1 & 1 & 1 & 2 & 1 & 2 & 3 \\ 2 & 2 & 3 & 3 & 4 & 4 & 5 \\ 3 & 4 & 5 & 6 & 5 & 6 & 6 \end{array} \qquad \begin{array}{ccccccc} 1 & 1 & 1 & 2 & 1 & 2 & 3 \\ 2 & 2 & 3 & 3 & 4 & 4 & 5 \\ 3 & 4 & 6 & 5 & 5 & 6 & 6 \end{array}$$

$$kx_{(1)} = 6.00 \qquad kx_{(1)} = 6.00$$

$$\begin{array}{ccccccc} 1 & 1 & 1 & 2 & 1 & 2 & 3 \\ 2 & 2 & 3 & 3 & 4 & 4 & 5 \\ 3 & 6 & 4 & 5 & 5 & 6 & 6 \end{array}$$

$$kx_{(1)} = 6.70.$$

(4) The pair $(1,2)$ occurs two times, whereas the pairs $(1,3)$ and $(2,3)$ occur only once in a block. This gives the following three designs:

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 3 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 & 6 & 5 & 6 \end{array} \qquad \begin{array}{ccccccc} 1 & 1 & 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 3 & 3 & 4 & 5 \\ 3 & 5 & 6 & 4 & 6 & 5 & 6 \end{array}$$
$$kx_{(1)} = 6.00 \qquad\qquad kx_{(1)} = 5.64$$

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 3 & 3 & 4 & 5 \\ 3 & 6 & 6 & 4 & 5 & 5 & 6 \end{array}$$
$$kx_{(1)} = 4.92.$$

III(c): $(1,2,3)$ *occurring two times as a block.*

This is again subdivided into three possible mutually exclusive cases as follows:

(1) In the blocks other than $(1,2,3)$, the pairs $(1,2)$ and $(1,3)$ occur in a block once each, and the pair $(2,3)$ does not occur in a block. This gives the following three designs:

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 3 & 5 & 5 & 5 \\ 3 & 3 & 4 & 4 & 6 & 6 & 6 \end{array} \qquad \begin{array}{ccccccc} 1 & 1 & 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 3 & 4 & 5 & 5 \\ 3 & 3 & 4 & 5 & 6 & 6 & 6 \end{array}$$
$$kx_{(1)} = 4.17 \qquad\qquad kx_{(1)} = 5.09$$

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 3 & 5 & 4 & 5 \\ 3 & 3 & 4 & 5 & 6 & 6 & 6 \end{array}$$
$$kx_{(1)} = 5.09.$$

(2) The pair $(1,2)$ comes two times in a block, whereas the pairs $(1,3)$ and $(2,3)$ do not occur in a block. The following two designs arise:

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 3 & 3 & 4 \\ 2 & 2 & 2 & 2 & 5 & 5 & 5 \\ 3 & 3 & 4 & 4 & 6 & 6 & 6 \end{array} \qquad \begin{array}{ccccccc} 1 & 1 & 1 & 1 & 3 & 3 & 4 \\ 2 & 2 & 2 & 2 & 4 & 5 & 5 \\ 3 & 3 & 4 & 5 & 6 & 6 & 6 \end{array}$$
$$kx_{(1)} = 3.20 \qquad\qquad kx_{(1)} = 4.63.$$

(3) Only one pair, say $(1,2)$, comes precisely once in a block, and the pairs $(1,3)$ and $(2,3)$ do not occur in a block. This gives the following three designs:

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 2 & 4 & 4 & 4 & 5 \\ 3 & 3 & 6 & 5 & 5 & 6 & 6 \end{array} \qquad \begin{array}{ccccccc} 1 & 1 & 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 2 & 4 & 4 & 4 & 5 \\ 3 & 3 & 6 & 5 & 6 & 5 & 6 \end{array}$$
$$kx_{(1)} = 6.00 \qquad\qquad kx_{(1)} = 5.61$$

$$\begin{array}{ccccccc} 1 & 1 & 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 2 & 4 & 5 & 4 & 4 \\ 3 & 3 & 6 & 6 & 6 & 5 & 5 \end{array}$$
$$kx_{(1)} = 4.92.$$

We thus see that there are only 22 possible nonisomorphic binary designs which satisfy the necessary conditions for $kx_{(1)} \geq 7$, and for each of these designs we find that $kx_{(1)}$ is strictly less than 7. Thus, it follows that no binary design in $D(6,7,3)$ is E-optimal.

3. REMARKS

The above result concerns E-optimality only. Many of the binary designs, especially the nearly balanced ones (Cheng and Wu 1981), are superior to the E-optimal (nonbinary) design with respect to the D- or A-optimality criteria. The conjecture of essential completeness of the class of binary designs with respect to A- or D-optimality still stands, and it appears to us that proving (or showing the falsity of) this conjecture is an extremely difficult task.

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