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Social Choice and Welfare

Social welfare and equality

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Abstract. The main purpose of the paper is to provide a unified framework within which normatively significant equality indices can be derived from social welfare orderings. The paper contains a functional representation of the class of social evaluation functions generating relative equality indices.

1. Introduction

The ethical approach to the measurement of income equality tries to derive indices of equality from (ordinal) social evaluation functions.[1] Ethical indices of equality have obvious normative significance in the sense that an increase in equality with per capita income constant represents an increase in social welfare. Thus, the implicit idea is that "social welfare" is a function of "size" and "distribution", increasing in size as well as in the level of equality associated with the distribution.

The intuitive basis of the ethical approach would be on firm ground if it was the case that every social welfare ordering implied a unique equality ordering. The social welfare ordering embodies all our ethical values about social welfare. If equality is also an ethical concept, then it seems reasonable to demand that the ethical values incorporated in the social welfare ordering should be sufficient to single out a unique equality index. Unfortunately, we show in Theorem 1, that it is possible to obtain ordinally different equality indices which are all normatively significant with respect to a given social welfare ordering.

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[1] See, for instance, Atkinson [1], Dalton [7], Desgupta, Sen and Starrett [8], Ebert [9], Esteban [10], Kats [11], Kolm [12], Sen [14], Sheshinski [15].

This suggests that additional assumptions about the nature of equality (additional to those incorporated in the social welfare ordering) must be introduced to single out equality indices which are normatively significant with given social evaluation functions. The main purpose of this paper is to provide a unified framework within which this exercise can be carried out. More specifically, we impose the restriction of mean-invariance equivalence on admissible equality indices. Consider, for instance, the two classes of equality indices which have received the most attention in the literature. Relative equality indices are homogeneous of degree zero in incomes, so that an equal proportional change in incomes leaves the level of equality unchanged. Absolute equality indices, on the other hand, are invariant to equal absolute changes in individual incomes. However, one can think of other patterns of distributions of additional incomes so as to leave equality unchanged.[2] We define two equality indices to be mean-invariant equivalent if they require additional incomes to be distributed in the same way to maintain the same level of equality. Thus, the relative equality indices are mean-invariant equivalent to one another. However, if E^1 is a relative index and E^2 is an absolute index, then E^1 and E^2 are not mean-invariant equivalent to one another.

In Theorem 2, we show that any social welfare ordering implies a unique normatively significant equality ordering within any prespecified mean-invariance equivalence class. Thus, any form of mean invariance equivalence, together with the ethical values underlying the social welfare ordering, is sufficient to single out a unique equality ordering. Although similar results have been derived for relative equality indices by Blackorby and Donaldson [2, 4] and Ebert [9], and for absolute indices by Blackorby and Donaldson [3], the present result shows that a case-by-case treatment is not necessary.[3]

Of course, if the equality index E is restricted to a prespecified mean-invariance class, then this imposes a corresponding restriction on the form of the social welfare ordering which generates it. We derive the specific nature of the restriction in Theorem 3. A corollary of Theorem 3 gives Ebert's [9] characterisation of the class of social evaluation functions generating relative equality indices. This class is considerably wider than the class of homothetic social evaluation functions. In the final section, we provide a functional representation of this class. Of course, analogous representation theorems can be derived for other classes of mean-invariant equality indices.

2. Notation and definitions

We assume that society is composed of n individuals and that there is one single good: income. Let x_i be the amount of income assigned to individual i, $i = 1, \ldots, n$. We shall denote by X^n the set of feasible income allocations,

$$X^n = \{x \mid x \in \mathfrak{R}^n_+ \setminus \{0\}\} ,$$

[2] See Kolm [12], Bossert and Pfingsten [5] and Chakravarty [6].
[3] See Remark 4 for further discussion of this issue.

and by S^n the set of feasible income allocations with a total income equal to unity, i.e.

$$S^n = \left\{ x \mid x \in X^n, \sum_{i=1}^{n} x_i = 1 \right\} .$$

μ_x denotes the mean income corresponding to the income allocation x.

Let e represent the n-dimensional vector with every component equal to unity and e^i the n-dimensional vector with zeroes in every component but for the i-th which is unity. The mean income can thus be written as $\mu_x = \frac{1}{n} x \cdot e$.

The ordering $\succeq$ ranks any pair of vectors $x, y \in X^n$, and $x \succeq y$ means that x is ranked at least as high as y. We shall write $x \sim y$ for $x \succeq y$ and $y \succeq x$, and $x \succ y$ for $x \succeq y$ and not $y \succeq x$. We shall examine the relationship between two orderings, $\succeq_w$ and $\succeq_E$, one in terms of social welfare and one in terms of equality. Specifically, $x \succeq_w Y$ means that x is at least as socially desirable as y and $x \succeq_E y$ means that there is at least as much equality under x as under y.

Our main aim is to explore whether equality orderings can be inferred from welfare orderings. This exercise can be made meaningful only by placing some restrictions on the properties that an ordering has to exhibit to be identified as a welfare ordering or as an equality ordering. A natural condition is that the two orderings should satisfy some kind of distributional sensitivity as a positive response to Daltonian progressive transfers. It is wellknown that this property corresponds to imposing S-concavity on both orderings.

Definition 1. An ordering $\succeq$ is *Semi-Strictly S-concave*[4] when:

(i) $Qx \succeq x$ for all $x \in X^n$ and all bistochastic matrices Q, and
(ii) $\mu_x \cdot e \succ n\mu_x e^n$ for all $x \in X^n$.

Implicit in our analysis is the presumption that social welfare is a function of the "size" of total income and distributional equality. This decomposition can have some substance only if we can completely separate changes in total income from distributional changes. In other words, we need to require equality orderings to be invariant to changes in total income. An assumption frequently made in the literature (see, for instance, Blackorby and Donaldson [3] and Ebert [9]) is that the equality index is homogeneous of degree zero in incomes (i.e. is a relative index). This assumption implies that the equality ordering satisfies $x \sim_E \lambda x$ for all $x \in X^n$ and $\lambda > 0$. But mean invariance has also been specified in a number of alternative ways. Blackorby and Donaldson [3] and Kolm [12] have investigated the properties of *absolute* equality indices showing invariance with respect to additions of equal absolute amounts of income, i.e. $x \sim_E (x + \lambda e)$ for all $x \in X^n$ and for all $\lambda \in \Re$ such that $(x + \lambda e) \in X^n$. Further, Kolm [12], Bossert and Pfingsten [5] and Chakravarty [6] have explored compromise indices showing invariance to combinations of the other two extreme cases.

The common feature of relative, absolute and compromise indices of equality is that there always exists a way of distributing additional income in nonnegative

[4] A bistochastic matrix is a matrix whose elements are non-negative and such that each row and each column adds up to unity. Condition (i) is standard. Condition (ii) simply rules out strict linearity, i.e. $Qx \sim x$ for all $x \in X^n$ and all Q and is weaker than strict S-concavity which requires (i) to hold with strict inequality for all bistochastic matrices Q such that Qx is not a permutation of x.

amounts so as to leave equality unchanged. In other words, given any two positive numbers μ and μ' $(\mu > \mu')$ there exist distributions x and y such that $\mu_x = \mu, \mu_y = \mu', x > y$,[5] and $x \sim_E y$. This general notion of mean-invariance is formalised in the next definition.

Definition 2. The equality ordering $\succeq_E$ is *mean-invariant* if and only if for every $x \in X^n$ and every $\mu > \mu_x$, there exists x_μ^E with $x_\mu^E \cdot e = n\mu, x_\mu^E > x$ and $x_\mu^E \sim_E x$.

Of course, two equality rankings $\succeq_E$ and $\succeq_{E'}$ may well require different schemes for distributing additional incomes in order to leave equality rankings unchanged. This will be the case, for instance, if one is a relative index while the other is an absolute index. So, it makes sense to say that two equality rankings are "similar" or of the same kind if they both require additional incomes to be distributed in the same way to maintain equality rankings.

Definition 3. The equality orderings $\succeq_E$ and $\succeq_{E'}$, are *mean-invariant equivalent* if and only if for every $x \in X^n$ and for every $\mu > \mu_x$ and every x_μ^E we have that $x_\mu^E \sim_{E'} x$.

Observe that all relative equality indices are mean-invariant equivalent and that the same can be said of absolute as well as of compromise indices.

3. From welfare to equality

In this section we explore the possibilities of constructing equality rankings which are consistent with a given welfare ranking.

Our primitive concept now is a welfare ranking $\succeq_w$ defined over X^n. We assume that $\succeq_w$ is continuous, semi strictly S-concave and *increasing along rays.*[6] We take $W: X^n \to \Re$ to be a particular numerical representation of that ordering. Since welfare is an ordinal concept, any increasing monotonic transformation of W also represents the same ranking $\succeq_w$. The question we ask is whether we can construct an equality ranking being "related" to a welfare ranking $\succeq_w$. This provides the motivation for the next definition.

Definition 4. The equality ordering $\succeq_E$ is *consistent* with the welfare ordering $\succeq_w$ if and only if for all $x, y \in X^n$ with $\mu_x = \mu_y$,

$$x \succeq_E y \leftrightarrow x \succeq_w y \ .$$

Definition 4 formalises our intuitive notion that welfare depends on "size" and "distribution", with the welfare ranking showing a preference for greater equality.[7] Clearly, once we agree on this principle, for pairs of income distributions with the same total income, the welfare ranking must coincide with the equality ranking.

It is also obvious that the ethical foundation of equality rankings would be on firmer grounds if it is the case that a given welfare ranking $\succeq_w$ implies a unique equality ranking $\succeq_E$. (Of course, the converse question of whether a

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[5] Given any $x, y \in X^n$, (i) $x \geqq y$ if $x_i \geqq y_i \forall i = 1, \ldots, n$ (ii) $x > y$ if $x \geqq y$ and $x \neq y$.

[6] $\succeq_w$ is increasing along rays if $\forall x \in X^n$, $\alpha x \succ_w x$, $\forall \alpha > 1$.

[7] This notion of consistency has already been used by Esteban [10]. Blackorby and Donaldson [4] call this condition when applied to relative equality indices Relative Inequality Aggregation Property (RIAP).

particular $\succeq_E$ implies a unique $\succeq_w$ ranking is also important. On this see Ebert [19]). Unfortunately, we show in the next theorem that unless more restrictions are imposed on the nature of the welfare and equality rankings $\succeq_w$ and $\succeq_E$, no $\succeq_w$ can imply a unique $\succeq_E$.

*Assumption W**. The welfare ranking $\succeq_w$ is continuous, semi strictly *S*-concave and increasing along rays.

Theorem 1. *For any $\succeq_w$ satisfying Assumption W* there exist at least two different consistent equality rankings, both satisfying continuity, mean invvariance and semi strict S-concavity.*

Proof. Let $W: X^n \to \Re$ be a particular continuous numerical representation of $\succeq_w$. Choose $a, b > 0$. Let us construct equality indices E^1, E^2, and $E^3: X^n \to \Re$ satisfying $\forall x \in X^n$

$$E^1(x) = \frac{W(x) - W(n\mu_x e^n)}{E(\mu_x e) - W(n\mu_x e^n)}, \tag{3.1}$$

$$E^2(x) = \frac{a + b[W(x) - W(n\mu_x e^n)]}{a + b[W(\mu_x e) - W(n\mu_x e^n)]}, \quad \text{and} \tag{3.2}$$

$$E^3(x) = \frac{W(x)}{W(\mu_x e)}. \tag{3.3}$$

All three indices are clearly consistent with W. Moreover, the fact that $E^1(\mu e) = E^2(\mu e) = E^3(\mu e) = 1$, together with the continuity and *S*-concavity of W imply that E^1, E^2 and E^3 also possess these properties and are mean invariant.

Let us first assume that W is such that there exist μ_1 and $\mu_2, \mu_2 > \mu_1 > 0$, such that $\Omega(\mu_1) \equiv W(\mu_1 e) - W(n\mu_1 e^n) \neq W(\mu_2 e) - W(n\mu_2 e^n) \equiv \Omega(\mu_2)$. Let us then choose $x, y \in X^n$ such that $E^1(x) = E^1(y) = \bar{E} < 1$ and $\mu_x = \mu_1$ and $\mu_y = \mu_2$. In that case,

$$\begin{aligned} E^2(x) - E^2(y) &= \frac{a + b[W(x) - W(n\mu_x e^n)]}{a + b[W(\mu_x e) - W(n\mu_x e^n)]} \\ &\quad - \frac{a + b[W(y) - W(n\mu_y e^n)]}{a + b[W(\mu_y e) - W(n\mu_y e^n)]} \\ &= \frac{ab(1 - \bar{E})[\Omega(\mu_2) - \Omega(\mu_1)]}{[a + b\Omega(\mu_1)][a + b\Omega(\mu_2)]} \end{aligned}$$

Hence, $E^2(x) \neq E^2(y)$, so that E^1 and E^2 cannot represent the same equality ordering.

Assume now that W is such that $\Omega(\mu) = K$ for all $\mu > 0$. Then choose $x, y \in X^n$ such that $E^3(x) = E^3(y) = \tilde{E} < 1$ and $\mu_y > \mu_x > 0$. Now,

$$E^1(x)-E^1(y)=\frac{\tilde{E}W(\mu_x e)-W(n\mu_x e^n)}{K}$$
$$-\frac{\tilde{E}W(\mu_y e)-W(n\mu_y e^n)}{K}$$
$$=\frac{\tilde{E}-1}{K}(W(\mu_x e)-W(\mu_y e))<0\ .$$

Hence, in that case E^1 and E^3 cannot represent the same quality ordering. This establishes Theorem 1.

Theorem 1 shows that the mere property of consistency alone does not have sufficient bite to single out a unique equality index. If two equality rankings are consistent with a given welfare ranking then they must rank distributions with the same mean income in an identical manner. However, equality rankings over distributions with different mean incomes need not coincide. Hence, any welfare index can have multiple associated equality indices having vastly different properties. Presumably, our ethical judgements about income distributions, encompassing notions of equality and the size-distribution trade-off are all incorporated into the welfare ranking $\succeq_w$. Despite this, the implication of Theorem 1 is that a welfare ranking is still not a perfect mechanism for singling out a unique equality ranking.

This suggests that we should have a priori restrictions on the class of "permissible" equality rankings, these a priori restrictions being independent of the welfare ranking itself. Of course, the two sets of restrictions cannot be completely independent of one another, because a specific family of welfare rankings will generally impose restrictions on the class of associated (consistent) equality indices and conversely.

We now explore the avenue of imposing additional restrictions on the class of permissible equality rankings. In particular, we will be concerned with restricting equality rankings to belong to a (any) specific class of mean-invariant equivalent ordering, as defined in Definition 4.

Theorem 2. Let $\succeq_E$ and $\succeq_{E'}$ be mean invariant equivalent to each other. If $\succeq_E$ and $\succeq_{E'}$ are consistent with a given welfare ranking $\succeq_w$, then $\succeq_E$ and $\succeq_{E'}$ are identical.

Proof. Suppose $\succeq_E$ and $\succeq_{E'}$ are mean invariant equivalent, consistent with $\succeq_w$ but not identical. Then, there exist distributions x and y such that $x \succ_E y$ and $y \succeq_{E'} x$.

Then $\mu_x \neq \mu_y$. For suppose $\mu_x=\mu_y$. Since $\succeq_{E'}$ is consistent wih $\succeq_w$, $y \succeq_{E'} x \Rightarrow y \succeq_w x$. This contradicts $x \succ_E y$.

So, without loss of generality, assume $\mu_x > \mu_y$. Since $\succeq_E$ is mean-invariant, there exists z with $\mu_z=\mu_x$ such that $z \sim_E y$, and hence $x \succ_E z$. Since $\succeq_E$ and $\succeq_{E'}$ are mean-invariant equivalent, it must be that $z \sim_{E'} y$, and hence $z \succeq_{E'} x$. But, ($\mu_z=\mu_x$ and $x \succ_E z$) and ($z \succeq_{E'} x$) imply that $\succeq_E$ and $\succeq_{E'}$ cannot both be consistent with $\succeq_w$. This contradiction proves the theorem.

The implication of Theorem 1 is that there is no hope of deriving a unique equality ranking from a given welfare ranking unless restrictions in addition to consistency are imposed on equality indices. Theorem 2 provides a way out of the nonuniqueness problem. Suppose there is agreement on the specific way of distributing additional incomes in order to leave equality unchanged. In other words, attention is being restricted to a *particular* class of equality rankings, the members of the class being mean-invariant equivalent to one another. Then, Theorem 2 states that there can be at most one consistent equality ranking within this class for every $\succeq_w$. Theorem 2 therefore generalises the corresponding results of Blackorby and Donaldson [2] and Ebert [9] on relative equality indices, and Blackorby and Donaldson [3] on absolute equality indices.

Suppose $\succeq_E$ is a mean-invariant equivalent ranking. Then there is a correspondence $H(\succeq_E)$ from $S^n \times \Re_+$, to subsets of X^n such that for all $(x, \mu) \in S^n \times \Re_+$, $H(x, \mu, \succeq_E) = \{y \mid \mu_y = \mu \text{ and } y \sim_E x\}$. For instance, if $\succeq_E$ is a relative equality index, then $n\mu x \in H(x, \mu, \succeq_E)$.

Theorem 3. *A mean-invariant equality ranking $\succeq_E$ is consistent with a given welfare ranking $\succeq_w$ only if for all $x, y \in S^n$, for all $\mu \in \mathbb{R}_+$, for all $x' \in H(x, \mu, \succeq_E)$, for all $y' \in H(y, \mu, \succeq_E)$, $x \succeq_w y \Leftrightarrow x' \succeq_w y'$.*

Conversely, if $\succeq_w$ is a welfare ranking with the property that for some correspondence F from $S^n \times \Re_+$ to subsets of X^n, for all $x, y \in S^n, x \succeq_w y \Leftrightarrow x' \succeq_w y'$ for all $x' \in F(x, \mu)$, $y' \in F(y, \mu)$, then there exists an equality ranking $\succeq_E$ consistent with $\succeq_w$ and $H(x, \mu, \succeq_E) = F(x, \mu)$.

Proof. Suppose $x, y \in S^n$, so that $\mu_x = \mu_y$. Suppose $\succeq_E$ is consistent with $\succeq_w$. Then $x \succeq_w y \Leftrightarrow x \succeq_E y$. If $x' \in H(x, \mu, \succeq_E)$ and $y' \in H(y, \mu, \succeq_E)$, then $x' \sim_E x$ and $y' \sim_E y$. So, by transitivity of $\succeq_E$, $x \succeq_w y \Leftrightarrow x' \succeq_E y'$. But, $\mu_{x'} = \mu_{y'} = \mu$. So, consistency of $\succeq_E$ implies that $x' \succeq_w y'$. Then, $x' \succeq_E y' \Rightarrow x \succeq_E y \Rightarrow x \succeq_w y$. This proves the first part of the theorem.

Conversely, suppose W has the stated property with respect to a correspondence F. Define an equality index $\succeq_E$ such that for all $x, y \in X^n$, if $\mu_x = \mu_y$, then $x \succeq_w y \Leftrightarrow x \succeq_E y$. It is obvious that $\succeq_E$ is consistent with $\succeq_w$. Also, it is easy to check that $H(x, \mu, \succeq_E) = F(x, \mu)$ for all $(x, \mu) \in S^n \times \Re_+$.

Corollary 1. *There exists a relative equality index consistent with $\succeq_w$ iff for all $x, y \in X^n$ such that $\mu_x = \mu_y$,*

$$x \succeq_w y \Leftrightarrow \alpha x \succeq_w \alpha y \ , \quad \forall \alpha > 0 \tag{3.4}$$

Remark 1. Corollary 1 was proved by Ebert [9].

Corollary 2. *There exists an absolute equality ranking consistent with $\succeq_w$ iff for all $x, y \in X^n$, such that $\mu_x = \mu_y$,*

$$x \succeq_w y \Leftrightarrow (x + \lambda e) \succeq_w (y + \lambda e) \ , \quad \forall \lambda \text{ such that}$$

$$(x + \lambda e), \ (y + \lambda e) \in \mathrm{X}^n \ . \tag{3.5}$$

Remark 2. Condition (3.5) is more general than the *translatability* of social welfare functions used in Blackorby and Donaldson [3]. Thus, Corollary 2 *generalizes* the results in [3] and shows that this weakening of translatability characterizes the class of welfare rankings generating consistent absolute equality measures.

Remark 3. Theorem 3 establishes necessary and sufficient conditions on $\succeq_w$ to be consistent with some mean-invariant equality ordering $\succeq_E$. The correspondence H can be used to establish a one-to-one relationship between the welfare ordering and a member of a particular of *mean-invariant equivalent* equality orderings. Note that it is possible to have *distinct* equality orderings $\succeq_E$ and

$\succeq_{E'}$ with $H(x,\mu,\succeq_E)=H(x,\mu\succeq_{E'})$ for all $(x,\mu)\in S^n\times\Re_+$.[8] However, $\succeq_E$ and $\succeq_{E'}$ cannot be mean-invariant equivalent to one another. This follows from Theorems 2 and 3. For, from Theorem 3, we can construct a welfare ordering $\succeq_w$ such that $\succeq_E$ and $\succeq_{E'}$ are consistent with $\succeq_w$. Theorem 2 then implies that $\succeq_E$ and $\succeq_{E'}$ must be in different mean invariant equivalent classes.

Remark 4. We have mentioned earlier that the main purpose of this paper is to show that the derivation of normatively significant equality orderings within any prespecified mean-invariance class from a given welfare ordering can be conducted in a unified framework. This statement, however requires a qualification. Suppose $\succeq_E$ is a relative index. Pick any $x\in X^n$. Let $y=\frac{1}{\mu_x}x$. Then, $y\in S^n$ and $x\sim_E y$. Hence, for every $x\in X^n$, there exists $y\in S^n$ such that $x\sim_E y$. This suggests that for relative equality indices, S^n can be used as a *reference set* in the following sense. Take any $x,y\in X^n$. 'Invert' $H(\cdot,\succeq_E)$ to find out x',y' such that $x\sim_E x'$, $y\sim_E y'$ and compare x' and y' according to $\succeq_E$. Thus, knowledge of $\succeq_E$ on the set S^n enables us to compare the extent of equality within *any* pair of distributions in X^n. However, S^n *cannot* be used as the reference set if $\succeq_E$ is not a relative index. If $\succeq_E$ is not a relative index, then given an arbitrary $x\in X^n$, there may not be any $y\in S^N$ such that $x\sim_E y$. Indeed, there cannot be any one reference set which will work for all classes of mean invariant equality indices, *unless* the domain of the welfare and equality rankings is enlarged to allow for negative individual incomes.

4. A representation theorem

In the literature on the measurement of inequality, relative measures have received the most attention. In view of this, Ebert's characterisation of the class of social evaluation functions implying relative equality indices (see Corollary 1) is particularly interesting. In this section we offer a functional representation of this class.

Definition 5. A social welfare ordering $\succeq_w$ is *weakly homothetic* if and only if for all $x,y\in X^n$ such that $\mu_x=\mu_y$, $x\succeq_w y\Leftrightarrow\alpha x\succeq_w\alpha y\,\forall\alpha>0$.

Theorem 4. *$\succeq_w$ is weakly homothetic if and only if for any representation $W:\Re^n_+\to\Re$ of $\succeq_w$ there exist functions $V:\Re\times\Re_{++}\to\Re$ strictly increasing in its first argument and $g:S^n\to\Re$ such that*

$$W(x)=V\left(g\left(\frac{x}{\mu_x}\right),\mu_x\right). \tag{4.1}$$

Proof[9]. (Necessity) In light of (4.1), weak homotheticity requires

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[8] Of course, this can only occur if $\succeq_E$ is not a relative index.

[9] This proof has been suggested by an anonymous referee.

$$W\left(\frac{x}{\mu_x}\right) \geqq W\left(\frac{y}{\mu_y}\right) \Leftrightarrow W\left(\frac{\lambda x}{\mu_x}\right) \geqq W\left(\frac{\lambda y}{\mu_y}\right)$$

for all $x, y \in X^n$ and all $\lambda \in \mathfrak{R}_{++}$.

This is equivalent to requiring that $W\left(\frac{\lambda x}{\mu_x}\right)$ be an increasing transformation of $W\left(\frac{x}{\mu_x}\right)$ for each $\lambda \in \mathfrak{R}_{++}$, that is, $W\left(\frac{\lambda_x}{\mu_x}\right) = V\left(W\left(\frac{x}{\mu_x}\right), \lambda\right)$, where $V: \mathfrak{R} \times \mathfrak{R}_{++} \to \mathfrak{R}$ is strictly increasing in its first argument. Choosing $\lambda = \mu_x$, we obtain

$$W(x) = V\left(W\left(\frac{x}{\mu_x}\right), \mu_x\right) \quad \text{for all } x \in X^n ,$$

which clearly implies that there exists a function $g: S^n \to \mathfrak{R}$ such that

$$W(x) = V\left(g\left(\frac{x}{\mu_x}\right), \mu_x\right) \text{ for all } x \in X^n .$$

The proof of sufficiency is straightforward.

Remark 5. If we assume that $\geqq_w$ is increasing along rays, then the function V has to be increasing in its second argument as well.

Remark 6. If $\geqq_w$ is increasing along rays, then a trivial implication of Theorem 4 is an alternative representation according to which W is weakly homothetic if and only if there exist $f: \mathfrak{R}^2_{++} \to \mathfrak{R}$ and $\xi: X^n \to \mathfrak{R}_{++}$, with ξ positively linearly homogeneous, such that $W(x) = f(\mu_x, \xi(x))$.

Of course, Theorem 4 suggests that the natural candidate for the relative index (unique up to monotonic transformations) implied by any (weakly homothetic) social evaluation function W is simply $W\left(\frac{x}{\mu_x}\right)$. Remark 4 clarifies why there may not be any such 'natural candidate' for nonrelative indices.

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