Approximate Isometries on Euclidean Spaces

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1. INTRODUCTION. Let E and F be Banach spaces. An *isometry* from E to F is a map $f: E \to F$ such that

$$\|f(x) - f(y)\| = \|x - y\| \quad \text{for all } x, y \in E. \tag{1}$$

Every isometry is continuous and injective. Among the earliest theorems for Banach spaces is the Mazur-Ulam Theorem [**13**]. This says that if f is a surjective isometry between real Banach spaces E and F, and if $f(0) = 0$, then f is linear. The conclusion is not valid for complex Banach spaces (just consider the complex conjugation on $\mathbb{C}$). The hypothesis of surjectivity is essential in general, but can be dropped for a large class of Banach spaces that includes real Hilbert spaces. The condition $f(0) = 0$ is necessary for f to be linear. If f is any isometry then $f - f(0)$ is also an isometry, so this condition is no serious restriction.

If distances are known imprecisely one may not be able to say whether f is an isometry. Then the concept of an approximate isometry is useful. Given $\varepsilon > 0$, a map $f: E \to F$ is called an *ε-isometry* if

$$\big|\|f(x) - f(y)\| - \|x - y\|\big| < \varepsilon \quad \text{for all } x, y \in E. \tag{2}$$

Note that if f is an ε-isometry then so is $f - f(0)$. The following problem was posed by Hyers and Ulam [**9**]. If f is a surjective ε-isometry between real Banach spaces E and F such that $f(0) = 0$, then does there exist a surjective linear isometry $g: E \to F$ such that

$$\|f(x) - g(x)\| \le K\varepsilon \quad \text{for all } x \in E, \tag{3}$$

where the constant K is independent of f, but can depend on the spaces E and F? Hyers and Ulam [**9**] showed that if $E = F$ is a real Hilbert space then the answer is in the affirmative with $K \le 10$.

The Hyers-Ulam problem has been solved over the years. It is only recently that it was shown that the sharp value of K is 2 for all Banach spaces [**14**].

The aim of this note is to discuss some of these matters, to explain a part of the original Hyers-Ulam ideas, and to show how to extract some more results from them. One major issue of concern through the article is how essential the assumption of surjectivity of f is for the conclusions.

2. ISOMETRIES. Let E be any Banach space and let x, y be any two points of E. The *algebraic midpoint* of x and y is the vector $m(x, y) = (x + y)/2$. A *metric midpoint* of x and y is any point z of E that satisfies

$$\|z - x\| = \|z - y\| = \frac{1}{2}\|x - y\|. \tag{4}$$

The algebraic midpoint is always a metric midpoint. It is easy to see that if E is a Hilbert space there are no other metric midpoints for any pair of vectors x, y. This is not always so in all Banach spaces. Here is an easy example:

Let E be the space $\mathbb{R}^2$ with the l_1-norm; i.e., if $x = (x_1, x_2)$ then $\|x\| = |x_1| + |x_2|$. Let $x = (1,0)$ and $y = (0,1)$. The algebraic midpoint of x and y is $(\frac{1}{2}, \frac{1}{2})$. This is at distance 1 from x and y. So are all points z of the form (t, t), where $0 \le t \le 1$. All these points are metric midpoints of x and y. A pictorial representation of this phenomenon might be helpful; see Figure 1. The unit ball of E is a diamond centered at the origin. Shift this diamond's center to $(1,0)$ and then to $(0,1)$. The intersection of the boundaries of these two diamonds is precisely the set of metric midpoints of x and y.

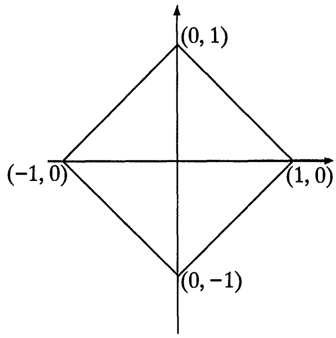

Figure 1

Let $M_0(x, y)$ be the set of all metric midpoints of x and y. It is easy to see that $M_0(x, y)$ is a closed, convex, and bounded subset of E.

There is a class of Banach spaces in which the norm is chosen so as to ensure that for all pairs x, y the set $M_0(x, y)$ is just the singleton $\{m(x, y)\}$. These are the *strictly convex* Banach spaces. The space E is called strictly convex, if whenever $\|x\| = \|y\| = 1$ and $\|(x+y)/2\| = 1$, then $x = y$ (that is, every point of the unit ball of E is an extreme point). For $1 < p < \infty$, l_p is strictly convex. A simple calculation with norms shows that if E is strictly convex then $M_0(x, y) = \{m(x, y)\}$ for all $x, y \in E$.

The importance of this observation is the following. The relation (4) that defines metric midpoints is unchanged under isometries, so if f: $E \to F$ is an isometry and if F is strictly convex then

$$f\left(\frac{x+y}{2}\right) = f(m(x,y)) = m(f(x), f(y)) = \frac{f(x)+f(y)}{2}.$$

Thus *every* isometry f from a Banach space E into a strictly convex Banach space F satisfies the equation

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}, \quad \text{for all } x, y \in E. \tag{5}$$

Now if $f(0) = 0$, this says that $f(x/2) = f(x)/2$ for all x. It follows, again from (5), that f is *additive*:

$$f(x+y) = f(x) + f(y), \quad \text{for all } x, y \in E. \tag{6}$$

It is clear from this equation that $f(nx) = nf(x)$, for every positive integer n. Also, choosing $y = -x$ in (6) we see that $f(x) = -f(x)$ for all x. Hence, $f(nx) = nf(x)$ for every integer n. Now it is easy to see that $f(rx) = rf(x)$ for every rational number r. Since f is continuous, for all *real* α we have

$$f(\alpha x) = \alpha f(x) \quad \text{for all } x \in E.$$

Thus f is real linear even if the spaces E and F are complex; if they are real then f is linear. This proves the Mazur-Ulam Theorem in the special case when the space F is strictly convex. Note that in this case we did not require that f be surjective.

When the set $M_0(x, y)$ contains points other than $m(x, y)$, the preceding argument does not work. However, it is possible to give a metric characterization of the algebraic midpoint and then use a modified version of the above argument. Here is an outline of the argument.

Let x, y be any two points of a Banach space E. Starting with the set $M_0(x, y)$, define, inductively, for $n = 1, 2, \ldots,$

$$M_n(x, y) = \left\{u \in M_{n-1}(x, y) : \|u - v\| \leq \frac{d_{n-1}}{2} \quad \text{for all } v \in M_{n-1}(x, y)\right\}.$$

Here, $d_n = \operatorname{diam} M_n$. Then we have a nested sequence of closed sets $M_0(x, y) \supset M_1(x, y) \supset M_2(x, y) \supset \cdots$, with $\operatorname{diam} M_n \leq d_0/2^n$. It is not difficult to prove that the point $m(x, y)$ is in $M_n(x, y)$ for all n. Hence,

$$\bigcap_{n=0}^{\infty} M_n(x, y) = \{m(x, y)\}. \tag{7}$$

This gives a metric characterization of the algebraic midpoint $m(x, y)$.

Now note that if f is a surjective isometry from a Banach space E onto a Banach space F, then

$$M_n(f(x), f(y)) = f(M_n(x, y)) \quad \text{for all } n.$$

At this step of the proof we do need to assume that f is surjective. If f were not surjective, we could have in F two points $f(x)$ and $f(y)$ whose metric midpoint is outside the range of f. So, from (7) we have

$$\left\{\frac{f(x) + f(y)}{2}\right\} = \bigcap_{n=0}^{\infty} M_n(f(x), f(y)) = \bigcap_{n=0}^{\infty} f(M_n(x, y)).$$

Since f is injective,

$$f\left(\bigcap_{n=0}^{\infty} M_n(x, y)\right) = \bigcap_{n=0}^{\infty} f(M_n(x, y)).$$

Now appealing to (7) again, we have

$$\frac{f(x) + f(y)}{2} = f\left(\frac{x + y}{2}\right).$$

As before, from this we can conclude that f is real linear. This proves the Mazur-Ulam Theorem.

Let us now give some simple examples to illustrate the necessity of the surjectivity assumption for general Banach spaces. Let $E = \mathbb{R}$ and let $F = \mathbb{R}^2$ with the ℓ_∞-norm; i.e., if $x = (x_1, x_2)$ then $\|x\| = \max(|x_1|, |x_2|)$. Let $f: E \to F$ be the map $f(t) = (t, \sin t)$. Since $|\sin t - \sin s| \leq |t - s|$ for all t and s, it follows that f is isometric. Clearly f is not linear. To see another example, let $E = \mathbb{R}$ and let

$F = \mathbb{R}^2$ with the l_1-norm. Define the map $f\colon E \to F$ as

$$f(t) = \begin{cases} (t,0) & \text{if} \quad -1 \le t \le 1 \\ (-1, t+1) & \text{if} \quad t \le -1 \\ (1, t-1) & \text{if} \quad t \ge 1 \end{cases}.$$

This is the piecewise linear curve illustrated in Figure 2. It is easy to verify that

$$\|f(t) - f(s)\| = |t - s| \quad \text{for all } t, s \in \mathbb{R}.$$

Thus f is isometric, but not linear.

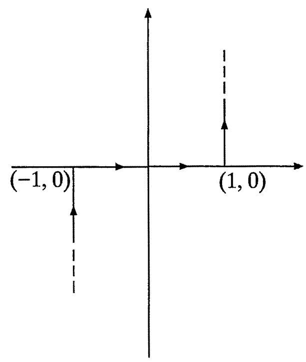

Figure 2

Can this phenomenon occur if $\dim E = \dim F$? The answer is no for finite-dimensional spaces. It was shown by Charzyński [**5**], [**6**, p. 143] that if E, F are n-dimensional real normed spaces then every isometry $f\colon E \to F$ satisfying $f(0) = 0$, is linear. Note that this implies that f is surjective.

Here is a simple proof of this theorem. Obviously, f maps S_E^r, the sphere of radius r centered at the origin of E into the sphere of the same type in F. Assume that there exists $r > 0$ such that $f(S_E^r)$ is a proper subset of S_F^r. Take any point $y \in S_F^r \setminus f(S_E^r)$. Then the restriction of f to S_E^r is an embedding of S_E^r into $S_F^r \setminus \{y\}$. If we have two different norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{R}^n$ then every sphere with a positive radius r with respect to the norm $\|\cdot\|_1$ centered at 0 is homeomorphic to the unit sphere with respect to $\|\cdot\|_2$ (the homeomorphism can be defined by $x \mapsto x/\|x\|_2$ for every x with $\|x\|_1 = r$). Hence, the restriction of f to S_E^r can be considered as an embedding of the standard sphere S^{n-1} into the punctured sphere S^{n-1}, which is homeomorphic to $\mathbb{R}^{n-1}$. It is well-known that such embeddings do not exist. So, f must be surjective, and therefore, by the Mazur-Ulam theorem, it is linear. It is interesting to note that the name of Ulam is associated also with the theorem from topology used here. This is the Borsuk-Ulam Theorem; see [**12**, p. 170].

There is a more general version of the Mazur-Ulam Theorem that goes beyond Banach spaces to locally convex topological vector spaces. See [**6**, Chapter VII]. The idea of the proof is essentially the same, but now the algebraic midpoint is characterized in terms of prenorms.

3. APPROXIMATE ISOMETRIES. We have defined ε-isometries in Section 1 and explained the Hyers-Ulam problem. Since surjectivity of f is a necessary requirement in the Mazur-Ulam Theorem, it is natural to impose that condition here too. However, there is a significant difference between the two problems in this respect. In Section 2 we explained how for a large class of Banach spaces (including Euclidean spaces) the Mazur-Ulam Theorem can be proved without this assumption. Hyers and Ulam gave an example of an ε-approximate isometry f from $\mathbb{R}$ into the Euclidean space $\mathbb{R}^2$, with $f(0) = 0$, that cannot be uniformly approximated by any linear isometry $g\colon \mathbb{R} \to \mathbb{R}^2$. They defined $f\colon \mathbb{R} \to \mathbb{R}^2$ by

$$f(t) = \begin{cases} (t, 0) & \text{if} \quad t \leq 1 \\ (t, c \log t) & \text{if} \quad t > 1 \end{cases}.$$

Then for each ε, we can choose c such that f is an ε-isometry. To see this note that $\log t$ is a concave function, and hence for $1 < s < t$,

$$\frac{\log t - \log s}{t - s} \leq \frac{\log t}{t - 1}.$$

Since $(\log t)/t \to 0$ as $t \to \infty$, this means that $\|f(t) - f(s)\|$ is asymptotically like $|t - s|$. More formally, it is an easy exercise to show that f is an ε-isometry whenever

$$\varepsilon > c^2 \max_{t>1} \left\{ \frac{(\log t)^2}{2t - 2} \right\}.$$

However, the set $\{\|f(t) - g(t)\| : t \in \mathbb{R}\}$ is unbounded for every linear isometry $g\colon \mathbb{R} \to \mathbb{R}^2$.

After the Hyers-Ulam solution of the problem for Hilbert spaces, there were several papers giving partial solutions for special Banach spaces. A breakthrough was made by Gruber **[8]**, who proved that if a constant K satisfying (3) can be found (for a given pair of real Banach spaces E and F) then this inequality remains true if we choose $K = 5$. Further, he proved that this can always be done if E and F are finite-dimensional. In the general case of all real Banach spaces this was proved by Gevirtz **[7]**. Finally, it was shown by Omladič and Šemrl that the choice $K = 2$ works in (3) for all real Banach spaces E and F **[14]**. Here is a simple example that shows the inequality (3) with $K = 2$ is sharp. Define $f\colon \mathbb{R} \to \mathbb{R}$ by

$$f(t) = \begin{cases} t - 1 & \text{if} \quad t \notin [0, 1/2] \\ -3t & \text{if} \quad t \in [0, 1/2] \end{cases}.$$

One can easily check that f is a surjective 1-isometry satisfying $f(0) = 0$. The only linear isometries $g\colon \mathbb{R} \to \mathbb{R}$ are $g(t) = t$ and $g(t) = -t$. Obviously, the second one does not approximate f uniformly, while $\max|f(t) - t| = |f(\frac{1}{2}) - \frac{1}{2}| = 2$.

4. EUCLIDEAN SPACES. The Hyers-Ulam example explained in Section 3 can be modified to show that if E and F are real Hilbert spaces with either $\dim E < \dim F$, or $\dim E = \dim F = \infty$, then there exists an ε-isometry $f\colon E \to F$, $f(0) = 0$, that is not uniformly close to any linear isometry. Of course, such an f is not surjective.

What happens in the remaining case, $\dim E = \dim F < \infty$? The following theorem gives the answer.

Theorem 1. *Let E_n be an n-dimensional Euclidean space and let $f\colon E_n \to E_n$ be an ε-isometry satisfying $f(0) = 0$. Then there exists a unique bijective linear isometry*

$g\colon E_n \to E_n$ *such that*

$$\|f(x) - g(x)\| \le 2\varepsilon$$

for all $x \in E_n$.

Our proof has two steps. First we use two theorems from the Hyers-Ulam paper to find an isometry g and a constant K (depending on n) such that the inequality (3) is true. Then we use the special inner product structure of E_n to show that K can be replaced by 2. This argument is simpler than the one in **[14]** for arbitrary Banach spaces, and requires no assumption of surjectivity on f.

The inner product between two vectors x and y will be denoted by $\langle x, y\rangle$. Let $f\colon E_n \to E_n$ be an ε-isometry satisfying $f(0) = 0$. Assume for a moment that f can be uniformly approximated by a linear isometry $g\colon E_n \to E_n$, that is, there exists a positive real constant M such that

$$\|f(x) - g(x)\| \le M \quad \text{for all } x \in E_n.$$

Let m be an arbitrary positive integer. Replacing x in this inequality by $2^m x$, dividing the obtained inequality by 2^m, and using linearity of g we get

$$\left\|\frac{f(2^m x)}{2^m} - g(x)\right\| \le \frac{M}{2^m} \quad \text{for all } x \in E_n$$

and for all positive integers m. This shows that if

$$\lim_{m \to \infty} \frac{f(2^m x)}{2^m} \tag{8}$$

exists, then a linear isometry g can approximate f uniformly if and only if $g(x)$ is equal to this limit for every x. The sequence in (8) is now called the *Hyers-Ulam sequence*.

The first result in the Hyers-Ulam paper **[9]** states that this sequence does converge for every x.

Lemma 2. *Let E_n be an n-dimensional Euclidean space. Suppose that $\varepsilon > 0$ and that $f\colon E_n \to E_n$ is an ε-isometry satisfying $f(0) = 0$. Then*

$$g(x) = \lim_{m \to \infty} \frac{f(2^m x)}{2^m}$$

exists for every $x \in E_n$. The mapping g is a linear bijective isometry.

After this, Hyers and Ulam prove that an ε-isometry (not necessarily surjective) "approximately preserves" orthogonality, in the following sense:

Lemma 3. *Let f and g be as in Lemma 2, and let $u \in E_n$ be a unit vector. Then for every $x \in E_n$ orthogonal to u we have $|\langle f(x), g(u)\rangle| \le 3\varepsilon$.*

Proof of Theorem 1: Let $g\colon E_n \to E_n$ be as in Lemma 2. Since g^{-1} is an isometry, $g^{-1} \circ f$ is an ε-isometry. Note that $g^{-1} \circ f$ sends zero to zero and

$$\lim_{m \to \infty} \frac{(g^{-1} \circ f)(2^m x)}{2^m} = x$$

for all x. As it is enough to prove the conclusion for $g^{-1} \circ f$, we can assume with no loss of generality that $g(x) = x$ for every x.

First we show, using induction, the existence of a constant K (depending on n) such that $\|f(x) - x\| \le K\varepsilon$ for all x. Let f be an ε-isometry on E_1. Since $f(0) = 0$ we have $||f(x)| - |x|| < \varepsilon$ for all x. So, either $|f(x) - x| < \varepsilon$ or $|f(x) + x| < \varepsilon$.

For all x outside a large neighborhood of 0, only one of these can be true. It is now easy to find a constant K such that $|f(x) - x| \leq K\varepsilon$ for all x.

Assume now that we have already proved the assertion for $n - 1$ dimensional Euclidean spaces. Let x be any vector in E_n and let u be any unit vector orthogonal to x. By Lemma 3 with $g(y) \equiv y$ we have $|\langle f(x), u\rangle| \leq 3\varepsilon$. Let P be the orthoprojector onto $[u]^{\perp}$. For any $w \in [u]^{\perp}$ we define $f_1(w) = Pf(w)$. We claim that f_1 is a 7ε-isometry on $[u]^{\perp}$ satisfying $f_1(0) = 0$ and

$$\lim_{m\to\infty} \frac{f_1(2^m w)}{2^m} = w$$

for all w. Obviously, $f_1(0) = 0$. Next note that

$$\begin{aligned}
&\big|\|f_1(w) - f_1(w')\| - \|w - w'\|\big| \\
&\quad = \big|\|f(w) - \langle f(w), u\rangle u - f(w') + \langle f(w'), u\rangle u\| - \|w - w'\|\big| \\
&\quad \leq \big|\|f(w) - f(w')\| - \|w - w'\|\big| + 6\varepsilon \leq 7\varepsilon.
\end{aligned}$$

Finally,

$$\lim_{m\to\infty} \frac{f_1(2^m w)}{2^m} = \lim_{m\to\infty} \frac{Pf(2^m w)}{2^m} = Pw = w.$$

By the induction hypothesis, there exists a positive constant K_{n-1} such that

$$\|f_1(w) - w\| \leq 7K_{n-1}\varepsilon$$

for all $w \in [u]^{\perp}$. It follows that

$$\|f(x) - x\| = \|f_1(x) + \langle f(x), u\rangle u - x\| \leq 7K_{n-1}\varepsilon + 3\varepsilon.$$

Since x was an arbitrary vector, the induction step is over.

Now we will show how to replace K by 2. Take any $x \in E_n$ and set $\|f(x) - x\| = a$. Assume that $a \neq 0$. Denote by y the unit vector satisfying $f(x) - x = ay$. The vector x can be written as $x = x_0 + by$, $b \in \mathbb{R}$, where x_0 and y are orthogonal. For every positive integer m we have $f(x + my) = x + my + v_m$, where $\|v_m\| \leq K\varepsilon$ because of what we have shown in the first step. Write $v_m = b_m y + u_m$, $b_m \in \mathbb{R}$, where u_m and y are orthogonal. Consequently, $\|u_m\| \leq K\varepsilon$ and $|b_m| \leq K\varepsilon$. Using the fact that f is an ε-isometry with $f(0) = 0$ we have

$$\big|\|f(x + my)\| - \|x + my\|\big| < \varepsilon.$$

This can be rewritten as

$$\big|\|(m + b + b_m)y + (x_0 + u_m)\| - \|(m + b)y + x_0\|\big| < \varepsilon.$$

Since u_m is bounded and x_0 and u_m are orthogonal to y, this shows that for every $\mu > 0$ we have $|b_m| < \varepsilon + \mu$ if m is large enough.

Since f is an ε-isometry, we have

$$m - \varepsilon < \|f(x + my) - f(x)\| < m + \varepsilon,$$

or equivalently,

$$m - \varepsilon < \|(m - a + b_m)y + u_m\| < m + \varepsilon.$$

For large m this norm can be brought as close to $m - a + b_m$ as we wish. Since for large m we have $|b_m| < \varepsilon + \mu$ with μ being arbitrarily small, this is possible only if $a \leq 2\varepsilon$. ■

We should remark that in the second part of our proof no reference was made to the finite dimensionality of the spaces involved. Thus, the factor 10 obtained by Hyers and Ulam (for the case of surjective isometries between infinite-dimensional Hilbert spaces) can be reduced to 2 using this argument.

It would be nice to extend Theorem 1 to ε-isometries $f\colon E \to F$ where E and F are arbitrary n-dimensional real normed spaces. In this case we have the following substitute for Lemma 2: There exists an increasing sequence (m_k) of positive integers such that

$$g(x) = \lim_{k\to\infty} \frac{f(m_k x)}{m_k} \tag{9}$$

exists for every $x \in E$. The mapping g is a linear bijective isometry. To prove this we first observe that the definition of ε-isometry implies that the sequence $(n^{-1}f(nx))$ is bounded for every $x \in E$. We choose a dense subset $\{z_1, z_2, \dots\}$ in E. Applying the Cantor diagonal procedure we can find an increasing sequence (m_k) of positive integers such that

$$g(z_p) = \lim_{k\to\infty} \frac{f(m_k z_p)}{m_k}$$

exists for every positive integer p. Using the definition of ε-isometry once again we see that (9) exists for every $x \in E$. Clearly, $g(0) = 0$. To prove that g is an isometry we replace x and y in (2) by $m_k x$ and $m_k y$, respectively. Dividing the obtained inequality by m_k and sending k to infinity we conclude that g is an isometry. We have already proved that g must be linear.

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