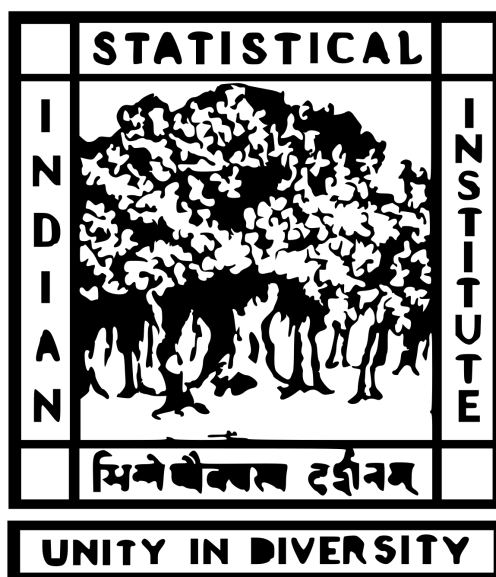# On arc minimal directed graph of diameter two

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A thesis presented for the completion of
M Tech. in Computer Science

Dedication

To my parents and all fun lovers.

Abstract

The *weak distance* $\overrightarrow{d}_w(u, v)$ between any two vertices u and v $\overrightarrow{G}$ is the shortest directed path connecting u and v. The *weak diameter* of an oriented graph $\overrightarrow{G}$ $diam_w(\overrightarrow{G}) = \max\{d(u, v) | u, v \in V(\overrightarrow{G})\}$. We denote $f_w(n)$ as the minimum arcs that an n vertex oriented graph required to be in a weak diameter two. And oclique is an oriented graph with weak diameter at most 2.

In this thesis, we have settled an unknown question about $f_w(n)$ being piecewise increasing which was previously known to be asymptotically increasing, improved the previous best upper bound of $f_w(n)$. In the domain of the planar ocliques we have characterised the minimal planar ocliques showing the possible effects of removing arcs from it, and have shown some possible applications of these effects.

Acknowledgement

This journey at ISI Kolkata has been more memorable because of this master dissertation work. I came across some wonderful persons because of this. The people at Room No. -614, 6th floor ACMU, PJA Building are simply amazing, knowledgeable, helpful and warm. And not only that because of this I tasted and enjoyed freedom. Such freedom was never given to me when it came to academics. I certainly would like to thank my friends who provided the necessary distractions at the time when I needed the most.

Prof. Sandip Das my supervisor, not only allowed me to choose my own comfortable topic for my dissertation but also allowed me to use any relevant or irrelevant methods to attack the problem at hand. A heartily special thanks to Dr. Sagnik Sen and Swathy Prabhu Maharaj , who shared their thoughts and time with me for my problems. And were immensely patient with a simpleton guy like me. Without them it would have been impossible for me to make any progress. And finally thanks to my parents because of their love and selfless support.

Contents

Chapter 1

Introduction

The set of vertices and edges of a graph G is denoted by $V(G)$ and $E(G)$, respectively. *Order* of a graph means the number of vertices in that graph while *size* of a graph means the number of edges (or arcs) in it. The *distance* $d(u,v)$ between any two pair of vertices u,v is the length (number of edges) of a shortest path connecting u and v. The *diameter* of a graph G is $diam(G)= \max\{d(u,v)|u,v \in V(G)\}$.

An *oriented graph* $\overrightarrow{G}$ is a directed graph having no directed cycle of length one or two with the set of vertices $V(\overrightarrow{G})$ and set of arcs $A(\overrightarrow{G})$. By replacing each edge uv of simple graph G with an arc $\overrightarrow{uv}$ we obtain an oriented graph $\overrightarrow{G}$; $\overrightarrow{G}$ is the orientation of G, and G is the underlying graph of $\overrightarrow{G}$. The *length* of a directed path/cycle in a directed graph is the number of distinct arcs in it. The *weak distance* $\overrightarrow{d}_w(u,v)$ between any two vertices u and v $\overrightarrow{G}$ is the shortest directed path connecting u and v. Note that the connecting directed path between u and v can be either from u to v or from v to u. The *weak diameter* of an oriented graph $\overrightarrow{G}$ $diam_w(\overrightarrow{G})= \max\{d(u,v)|u,v \in V(\overrightarrow{G})\}$. Two arcs $\overrightarrow{uw}$ and $\overrightarrow{wv}$ in an oriented graph together are called a *directed 2-path* or a *2-dipath* where u and v are the terminal vertices and w is the internal vertex.

The journey of our problem began holding hands with Erdős, Rényi and Sõs [3]. They started by defining the function

$$f_d(n) = \min\{|E(G)| \text{ such that } diam(G) \leq d \text{ and } |V(G)| = n\}.$$

They gave some asymptotic bounds for general diameter d and bettered that bound for some constrained graphs, leaving us with the analogous questions for oriented graphs. These questions were again asked by Dawes and Meijer [2] and also by Známi [15]. It turns out that its not very easy to answer those questions even for diameter 2. For answering those analogous questions Erdős, Rényi and Sõs [3] defined

$$f_w(n) = \min\{|E(G)| \text{ such that } diam_w(\overrightarrow{G}) \leq d \text{ and } |V(G)| = n\}.$$

There are few results relating to this $f_w(n)$ lets look at them in a chronological order. The first result regarding this is due to Katona and Szemerédi [7].

Theorem 1.1 (Katona and Szemerédi 1967 [7])**.**

$$\frac{n}{2}log_2\left(\frac{n}{2}\right) \leq f_w(n) \leq n\lceil log_2 n\rceil$$

Later independent studies were made by Kostochka, Luczak, Simonyi and Sopena [10] and Furedi, Horak, Pareek and Zhu [4] on $f_w(n)$ both proved that they are asymptotically increasing and gave their own bounds.

Theorem 1.2 (Furedi, Horak, Pareek and Zhu 1997 [4])**.**

$$(1 - o(1))nlog_2 n \leq f_w(n) \leq nlog_2 n - \frac{3}{2}n$$

A little weaker bounds were given by Kostochka, Luczak, Simonyi and Sopena [10].

Theorem 1.3 (Kostochka, Luczak, Simonyi and Sopena 1999 [10])**.**

$$n \left(log_2(n) - 4log_2 log_2(n) - 5 \right) \le f_w(n) \le \lceil log_2 n \rceil \left(n - \lceil log_2 n \rceil \right)$$

An asymptotic bound for $f_w(n)$ was given by Furedi, Horak, Pareek and Zhu [4] again by Kostochka, Luczak, Simonyi and Sopena [10], thus it was asymptotically increasing.

Theorem 1.4 (Furedi, Horak, Pareek and Zhu 1997 [4] and Kostochka, Luczak, Simonyi and Sopena 1999 [10])**.**

$$\lim_{n \to \infty} \frac{f_w(n)}{nlog_2 n} \to 1$$

If a graph G has an orientation $\overrightarrow{G}$ that makes it have weak diameter at most d then we call such an orientation as d-weak orientation. The following result shows that the computational feasibility is not there when it comes to finding out an weak orientation of a graph of a given diameter.

Theorem 1.5 (Kirgizov, Duvignan and Bensmail 2016 [8])**.** *Determining whether a graph G has d-weak orientation is NP-complete, for $d \ge 2$.*

One of the natural questions we address in this thesis is the following:

Question 1.1. *Can we improve the existing upper and lower bounds of $f_w(n)$?*

We manage to improve the upper bound. Moreover, we consider another question asked by Eric Sopena in a personal communication.

Question 1.2. *Is $f_w(n)$ increasing?*

We show that the function is, indeed, increasing.

$$* * *$$

In this thesis, we are also interested in studying the structure of oriented graphs $\overrightarrow{G}$ with $diam_w(\overrightarrow{G}) \le 2$ along with the function $f_w(n)$. This problem is also well motivated as it turns out that the graphs with weak diameter at most 2 plays a key role in the theory of oriented graph coloring as well and are interesting objects of study in that domain.

Courcelle [1] introduced the concept of oriented coloring while studying monadic second order logic. An *oriented k-coloring* of an oriented graph $\overrightarrow{G}$ is a mapping ϕ from the vertex $V(\overrightarrow{G})$ to the set $\{1, 2, ...k\}$ such that

(i) $\phi(u) \ne \phi(v)$ whenever v and u are adjacent.

(ii) for two arcs $\overrightarrow{ub}$ and $\overrightarrow{wx}$ of $\overrightarrow{G}$, $\phi(u) = \phi(x)$ implies $\phi(v) \ne \phi(w)$.

The *oriented chromatic number* $\chi_o(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$ is the minimum number k for which the graph has an oriented k-coloring.

That the above definition is indeed an extension of the theory of undirected graph coloring was noted by Hell and Nešetřil [6] using graph homomorphism. We will omit the part involving graph homomorphism as it is not inside the scope of this thesis. A similar extension of the concept of cliques for undirected graphs was done by Klostermeyer and MacGillivray [].

An *oriented absolute clique* or *oclique* is an oriented graph $\overrightarrow{G}$ for which $\chi_o(\overrightarrow{G}) = |V(\overrightarrow{G})|$. The *oriented absolute clique number* $\omega_{ao}(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$ is the maximum order of an oclique contained in $\overrightarrow{G}$ as a subgraph. The following characterization of ocliques due to Klostermeyer and MacGillivray [9] connects our initial problem of interest to the theory of oriented coloring.

Proposition 1.1 (Klostermeyer and MacGillivary 2004 [9])**.** *An oriented graph $\overrightarrow{G}$ is an oclique if and only if $diam_w(\overrightarrow{G}) = 2$.*
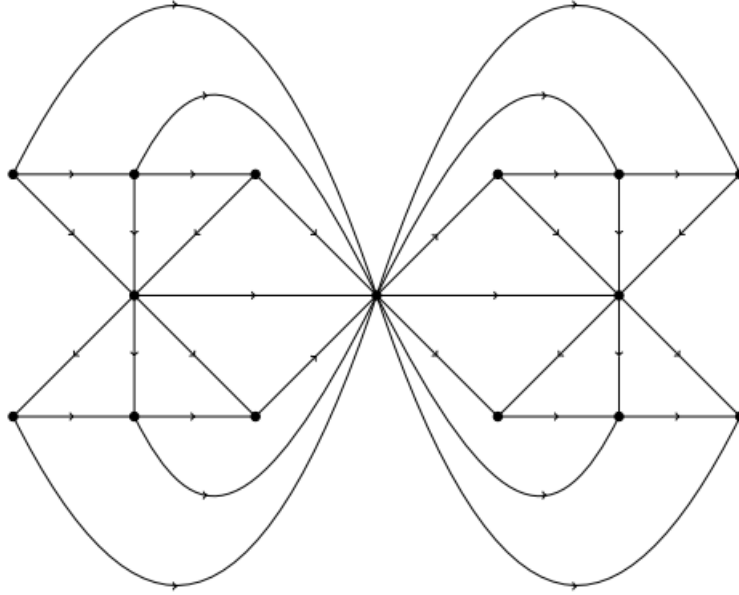
Figure 1.1: The minimal planar oclique on 15 vertices. [13]

Thus note that an oclique and a weak 2 diameter graph is the same thing. Hence the above result connects our object of interest to the theory of oriented graph coloring.

Thus we return to the same question as asked by Erdős, Rényi and Sõs [3]. So in the proofs of the $f_w(n)$ we use the properties of oriented coloring. Theorem 1.2 was the best known bound till date, we have improved that in chapter 1. Also we have added further information about the $f_w(n)$'s property by proving that it is piecewise increasing, which was asked by Eric Sopena in a personal communication. $f_w(n)$ was previously known to be asymptotically increasing.

Now lets peep into the domain of planar graphs. We know that planar graphs has a maximum clique size of 4. That is in case of K_5 and higher higher cliques we do not have any planar representation of it. This has been proved in the famous and celebrated Kurtawoski theorem [14]. So does there exist any upper bound for planar ocliques as well? The answer is yes.

Theorem 1.6 (Nandy, Sen and Sopena 2015 [13]). *A planar oclique has order at most 15. Furthermore, any planar oclique of order 15 contains the planar oclique $\overrightarrow{P}$ depicted in figure 1.1 as a spanning subgraph.*

Thus by Theorem 1.6 we know that the highest order of a planar oclique is 15. Not only that, there is only one minimal oclique of order 15. By minimal we mean that no proper subgraph H of the underlying graph G of the oclique $\overrightarrow{G}$, has an orientation $\overrightarrow{H}$ that makes it an oclique of the same order as $\overrightarrow{G}$. This immediately raises some questions. For example, what about the planar ocliques of lower order? How many minimal planar ocliques are there of order n for $n \leq 14$? In particular, Nandy, Sen and Sopena [13] asked the following question:

Question 1.3. *Characterize the set L of the graphs such that a planar graph can be oriented as an oclique if and only if it contains one of the graphs from L as a sub graph?*

We partially answer this question.

* * *

In Chapter 2 of this thesis we have proved $f_w(n)$ is strictly increasing. We also worked out an upper bound of $f_w(n)$ and finally have shown why this beats the previously best known upper bound.

Later in Chapter 3 we have shown some properties and of the minimal planar ocliques for lower order graphs and have tried to give a general theorem of the planar graphs as a whole. While Chapter 4 lists out all the source codes that we have used in this process.

Finally we conclude our thesis in Chapter 5.

Chapter 2

Size of an oclique of order n

2.1 The increasing property

We here to show that the number of edges required for increasing order of oclique is also strictly increasing. We will give a method of obtaining a lesser order oclique from the higher order oclique. This will produce an oclique of lower order with lesser number of arcs.

Let us denote $f_w(n)$ to be the minimum number of edges required by a graph $\overrightarrow{G}$ with n vertices to be an oclique of order n.

So we take $f_w(n) = m$, now we will show later that $f_w(n-1) \leq m-1$. Before that let us make a small observation.

Observation 2.1. *Merging two vertices with same oriented color, will not create any new cycle of two or less in the resultant graph.*

Definition 2.1. *Let $\overrightarrow{G}(V,A)$ be a directed graph and let $v \in V(G)$. Let $N^+(v)$ and $N^-(v)$ be the out-neighbours and in neighbours of v respectively. Interchanging the in-neighbours with the out-neighbours of v is called $\boldsymbol{push(\overrightarrow{G}, v)}$. The new di-graph that is created is named as $\overrightarrow{G_{new}}$.*

Lemma 2.1. *Let $\overrightarrow{G}(V,A)$ be an oclique of order $n+1$. Applying the operation $push(\overrightarrow{G}, v)$ for any vertex $v \in V(G)$, the oriented colouring decreases by at most 1. That is $\chi_o(\overrightarrow{G_{new}}) \geq n$.*

Proof. Consider any two pairs (x,y) with $x, y \neq v$. Now they were either adjacent or connected via a 2-path. If they were adjacent then they remained adjacent after $push(\overrightarrow{G}, v)$. Else if they had v in there 2-path as an internal vertex then even after push they still disagree upon v so they still are connected. Then also (x,y) remain connected as those edges are not affected. So colouring of those n vertices remain same. Thus requiring atleast n colors to color $\overrightarrow{G_{new}}$. □

Theorem 2.1. *$f_w(n)$ is a strictly increasing.*

Proof. We begin with a oclique of order $n+1$ where $f_w(n+1) = m$. Now we pick any vertex $v_1 \in V(\overrightarrow{G})$ and apply $push(\overrightarrow{G}, v_1)$. So by lemma 2.1 the new graph say $\overrightarrow{G_1}$ has colouring $\chi_o(\overrightarrow{G_1}) \geq n$. Now if $\chi_o(\overrightarrow{G_1}) = n$ then we are done as we can merge two vertices say (v_1, x) of the same color. Thereby reducing the number of edges by number of their common neighbours. Notice that, we must have atleast one common neighbours as they were in an oclique to begin with and also they cannot be adjacent as they have the same color. After merging we have an oclique of order n. So $f_w(n) \leq m-1$. If $\chi_o(\overrightarrow{G_1}) = n+1$, then we pick another vertex $v_2 \neq v_1$, and apply $push(\overrightarrow{G_1}, v_2)$ creating $\overrightarrow{G_2}$. Even then also if $\chi_o(\overrightarrow{G_2} = n+1$. Then we pick $v_3 \neq v_2 \neq v_1$ and so on until we find $\overrightarrow{G_k}$ such that $\chi_o(\overrightarrow{G_k}) = n$, then we perform the merge. If no such $\overrightarrow{G_k}$ exits then we adopt the following procedure.

We fix a vertex $v \in V(\overrightarrow{G})$ and take $v_1 \in N^+(v)$ and then apply $push(\overrightarrow{G}, v_1)$ creating $\overrightarrow{G_1}$. Then we take another vertex $v_2 \in N^+(v)$ in the graph $\overrightarrow{G_1}$ again apply $push(\overrightarrow{G_1}, v_2)$ creating $\overrightarrow{G_2}$. This way we continue until we make $N^+(v) = \phi$, say we achive this in the graph $\overrightarrow{G_r}$. This makes v a sink vertex. So v does not lie in the two dipath of any other vertices. Hence removing v from $\overrightarrow{G_r}$ will create an

oclique $\overrightarrow{H}$ of order n. And as $deg(v) > 0$ so $f_w(n) \leq m - 1$.

So we showed for any n, $f_w(n + 1) > f_w(n)$. Hence $f_w(n)$ is strictly increasing. □

Let us denote $F_w(n, r)$ as the minimum number of edges required for a relative oriented clique $\overrightarrow{R}$ which has r good vertices and $n - r$ helper vertices. n is the total number of vertices. Thus $n \geq r$. We will prove some properties of relative oriented clique also. Notice that, $F_w(n, n) = f_w(n)$.

Corollary 2.1. $F_w(n, r) \geq f_w(r)$

Proof. We color with oriented colors the whole of the relative oriented clique including the helper vertices also. If any helper vertices can be colored as the same color as that of any other helper or good vertex then we may merge them to a single vertex. And keep on repeating this step until there is no helper vertices thus proving our claim, as we do not increase the edges.

If such a situation is not there, that is there exits no helper vertex that can be colored as the same color as any other helper or good vertex. Then we basically have an oclique of order r+number of helper vertices left, and this requires more edges than $f_w(r)$ by theorem 2.1. □

Proposition 2.1. $F_w(n, r)$ *is strictly increasing with respect to* r.

Proof. Consider the relative oriented clique *overrigharrowR*. And consider the edge between an helper vertex h and a good vertex g. Now if we delete that edge then the oriented color decreases by atmost one. This is because only the connection of g with other good vertices is hampered, and that's the only thing that affects the oriented coloring that is the connection between good vertices. And here only one good vertex is affected. So coloring decreases by one. Thus $F_w(n, r) > F_w(n, r - 1)$. □

2.2 Finding out the pattern

As mentioned earlier that the problem of determining the function $f_w(n)$, was first posed by Erdős,Rényi and Sõs [3]. It was again asked by Dawes and Meijer [2] and also by Znám [15].

For unoriented graphs the answer to this question is trivial. Such a graph has $n - 1$ edges and star is the only extremal graph.

The best known upper bound for $f_w(n)$ was given by Furedi, Horak, Pareek and Zhu [4]. And is given as :

$$f_w(n) \leq n\log_2 n - \frac{3}{2}n$$

for $n \geq 9$.

This bound beats the previously known bound given by Katona and Szemeredi [7] as :

$$f_w(n) \leq n\lceil \log_2 n \rceil$$

. In this section we provide a way of generating ocliques, following a certain method. Now the number of arcs that these ocliques require surely forms an upper bound to the $f_w(n)$. We derive certain properties that the graphs, derived from our said procedure, posses. And finally we show that the upperbound this method gives is tighter than the previously known upper bound.

Definition 2.2. *We say that an oriented graph* $\overrightarrow{G}$ *is* **cut-structured** *if* $\overrightarrow{G}$ *is an oclique with one dominator vertex* v. *Such that* v *is an cut vertex.*

From the definition it can be observed that the positive neighbours $(N^+(v))$ as well as the negative neighbours $(N^-(v))$ of the one dominator vertex v are ocliques on there own. As $N^+(v)$ and $N^-(v)$ have no connection. We denote the minimum edge requirement for a cut vertexed graph of order n, to be an oclique as $g_w(n)$. Thus one thing immediately follows.

$$f_w(n + 1) \leq n + g_w(k) + g_w(n - k) \text{ where } 0 \leq k \leq n \text{ and } k \text{ minimizes the RHS.} \quad (2.1)$$

Equation 2.1 gives us an hint as what to our procedure will be. But we state it formally.

Notice that we have initialized $g_w(5) = 5$ which is not a cut-structured graph but it reduces the error

Result: returns $g_w(n)$
initialize : $g_w(1) = 0, g_w(2) = 1, g_w(3) = 2, g_w(4) = 4, g_w(5) = 5$;
$i = 6$;
while $i \leq n$ **do**
> $k = \lceil \frac{n}{2} \rceil$;
> $g_w(i) = min_{\forall j, k \leq j \leq i-2}(g_w(j) + g_w(i - 1 - j))$;

end
return $g_w(n)$;

in the upper bound of $f_w(n)$. As $g_w(5)$ is actually 6 but a lower sized oclique is available so $g_w(5)$ is initialized that way.

Now for $n \geq 11$ we have observed some interesting patterns. Although the pattern starts from $n \geq 6$ but we prefer to start from 11 as it helps our analysis. Which are stated below.

Feature 2.1. *The difference between the $g_w(n)$ and $g_w(n+1)$ varies alternately between x and $x+1$, where $n \geq 11$ and $x \in \mathbb{N}$. For a length of 12.2^i for some value i. Then it varies alternately by $x + 1$ and $x + 2$, for the next 12.2^{i+1}.*

Definition 2.3. *The point when the alternate change in difference (of $g_w(n)$) increases in value, we call that **change point**. In other words the alternating change of $g_w(n) - g_w(n-1)$ as x and $x+1$, changes to $x + 1$ and $x + 2$ and the value of n at which it such a change occurs is called the change point.*

Feature 2.2. *At change points the value of $g_w(n+1) - g_w(n) - (g_w(n) - g_w(n-1))$ is 2. Where n is the change point.*

One easy observation that can be made is that the alternating difference ends at lower value and starts from the higher value. This is evident from the feature 2.1 and 2.2.

We have observed these features 2.1 and 2.1 for some finite i then we claim that it holds for all i and hence for all $n \geq 11$.

Theorem 2.2. *If feature 2.1 and 2.2 holds for some finite i then it also holds for any other value of i.*

Proof. As we have feature 2.1 true for some finite i thus for some n, so we use the strong law of induction. We assume that $\{2n, 2n+1, 2n+2\}$ in such an interval of 12.2^r so that induction holds till $r - 1$. Notice that $2n < 11 + 12 \times 2^0 + 12 \times 2^1 ... + 12 \times 2^r$.

$$2n < 11 + 12(2^{r+1} - 1)$$
$$n < \frac{11}{2} + 6(2^{r+1} - 1)$$
$$n < 11 + 12(2^r - 1)$$

Thus n lies with in the range of our induction.
For some $2n + 1$ we need some k such that $g_w(k) + g_w(2n - k)$ is minimized and $k \geq 2n - k$. So lets begin with $k = n$ and keep on increasing k until the sum is minimized. Now there are 4 cases here that are as follows.
Case 1: k is equidistant from the right and the left change points.
Case 2: k is closer to the right change point, but not on change point.
Case 3: $2n + 1 - k$ is closer to the left change point but on the change point.
Case 4: k is at the change point.

Case 1: As mentioned above that we start from $k = n$. Now we increase the value of k to $n + 1$ so the value of $2n - k$ becomes $n - 1$. By feature 2.1 and induction we have $g_w(n + 1) - g_w(n) = x$ and $g_w(n) - g_w(n - 1) = x + 1$ or vice-versa.
If the first case has occurred that is $g_w(n + 1) - g_w(n) = x$ and $g_w(n) - g_w(n - 1) = x + 1$ then we have decrement of edges by choosing $2n - k = n - 1$ over $2n - k = n$, but we also have increment of edges by choosing $k = n + 1$ over $k = n$ so ultimately we decrease the edge requirement by one

edge $(x + 1 - x = 1)$. But if we again increase the value of k from $n + 1$ to $n + 2$ then by feature 2.1 $g_w(n + 2) - g_w(n + 1) = x + 1$ and $g_w(n - 1) - g_w(n - 2) = x$. So we increase the edge requirement by 1. In the next increment of k we again decrease it by one only from the case $g_w(n - 1) + g_w(n + 1)$. We thus move on and keep increasing the value of k until the change point is reached. In that case as k was initially equally close to both of the change points so during crossing the change point there is increase in the value of $g_w(n - k) + g_w(k)$. As $g_w(k + 1) - g_w(k) = x + 2$ so is $g_w(n - k) - g_w(n - k - 1) = x - 1$ (by feature 2.2) then further changes will only lead to the increment as $x + 2$ or $x + 1$ while the decrement will only be $x - 1$ or x. Leading to net rise in the sum $g_w(2n - k) + g_w(k)$. And this keeps on rising. Thus we have for $k = n + 1$ attained minimum $g_w(n - k) + g_w(k)$, for the case $g_w(n + 1) - g_w(n) = x$ and $g_w(n) - g_w(n - 1) = x + 1$.

Now consider $2n + 2$ then we start $k' = n + 1$ and thus $2n + 1 - k' = n$. And also note $g_w(n + 2) - g_w(n + 1) = x + 1$ and $g_w(n + 1) - g_w(n) = x$ so increasing the value of k' does not affect the value of $g_w(2n + 1 - k') + g_w(k')$. As k' is greater than n so k' is closer to the right change point than $2n + 1 - k'$ is from the left change point. So increasing the value of k' when k' is at the change point means that the sum $g_w(2n + 1 - k') + g_w(k')$ increases by the value of atleast 1 and further changes will only increase the sum $g_w(2n + 1 - k') + g_w(k')$ in accordance with feature 2.1. So $k' = n + 1$ minimizes the sum $g_w(2n + 1 - k') + g_w(k')$. Now :

$$g_w(2n + 2) - g_w(2n + 1) = 2n + 1 + g_w(n + 1) + g_w(n) - 2n - g_w(n + 1) - g_w(n - 1)$$
$$g_w(2n + 2) - g_w(2n + 1) = 1 + g_w(n) - g_w(n - 1)$$
$$g_w(2n + 2) - g_w(2n + 1) = 2 + x$$

It can be shown that in the case of $g_w(n + 1) - g_w(n) = x$, $g_w(2n + 2) = 2g_w(n + 1) + 1$. So $g_w(2n + 2) - g_w(2n + 1) = x + 1$. Thus we have an alternating sequences. Similarly we can show for the case where $g_w(n + 1) - g_w(n) = x + 1$, that the alternating sequences prevail. Thus we prove our claim in this case 1.

Note : There cannot exist any such k equidistant from both left change point and right change point. As the number of elements between two change points are even. Although this proof gives the idea about the following cases to come.

Case 2: From the proof of the Case 1 its clear that when no change point is not there in between k and $2n - k$ then increasing the value of k increases or decreases the value of $g_w(2n - k) + g_w(k)$ by 1 alternately. Similarly when k crosses reaches the change point. The value of $g_w(k) - g_w(k - 1)$ becomes $x + 2$ while $g_w(2n - k) - g_w(2n - k - 1)$ becomes $x + 1$ or x. Thus increasing the value of $g_w(2n - k) + g_w(k)$ by atleast 1. And the value of $g_w(2n - k) + g_w(k)$ keeps on increasing when k is increased when there is one or more change point in between them as seen in the proof of case 1. It can be shown similar to case 1 $g_w(2n + 1) = 2g_w(n) + 2n$ for $g_w(n + 1) - g_w(n) = x + 1$. Then :

$$g_w(2n + 2) = g_w(n + 1) + g_w(n) + 2n + 1 \text{ thus}$$
$$g_w(2n + 2) - g_w(2n + 1) = x + 2 \text{ and}$$
$$g_w(2n) = g_w(n) + g_w(n - 1) + 2n - 1 \text{ making}$$
$$g_w(2n + 1) - g_w(2n) = x + 1$$

Similar calculations can be shown for $g_w(n + 1) - g_w(n) = x$.

Case 3: Same as case 2 by reversing the roles of $2n + 1 - k$ and k.

Case 4: When k is an change point then by feature 2.2 we have $g_w(k) - g_w(k - 1) = x$ while $g_w(k + 1) - g_w(k) = x + 2$ and we know that $g_w(2n - k) - g_w(2n - k - 1) = x + 1$ or x. So $g_w(2n - k) + g_w(k)$ increases by 1 with the increment of k. And as noted above that when a change point is there between k and $2n - k$ with the increase in value of k, $g_w(2n - k)$ also increases. Hence we prove our claim.

$\square$

2.3 Formulating the upper bound

As we have proven the existence of pattern in $g_w(n)$. We will here encode this pattern in algebraic sense and thus have an upper bound for $f_w(n)$. Now we have seen by feature 2.1 that for a length of 12×2^i, difference between consecutive $g_w(n)$ varies alternately between x and $x+1$. Taking advantage of this situation we will find an explicit formula for $g_w(n)$ that forms the upper bound of $f_w(n)$. We have

$$n = 11 + 12 \times (2^0) + 12 \times (2^1)... + 12 \times (2^i) + \delta$$
$$n - 11 = 12 \times (2^{i+1} - 1) + \delta$$
$$\frac{n-11}{12} = 2^{i+1} - 1 + \frac{\delta}{12}$$
$$2^{i+1} \leq \frac{n-11}{12} + 1$$
$$i \leq log_2\left(\frac{n+1}{12}\right) - 1$$
$$i = \left\lfloor log_2\left(\frac{n+1}{12}\right) \right\rfloor - 1$$

δ represents the further addition that is needed to LHS to make it equal to n. Surely $0 \leq \delta < 12 \times 2^{i+1}$. Now lets see the explicit formula for $g_w(n)$.

$$g_w(n) = 20 + S + k \tag{2.2}$$

For $i > 0$,

$$S = \sum_{r=0}^{r=i}(7 + 2r) \times 6 \times 2^r = 42(2^{i+1} - 1) + 12((i-1)2^{i+1} + 2)$$

.

For $i = 0$

$$S = 42$$

.

And for $i < 0$ we have,

$$S = 0$$

.

If δ is even.

$$k = (7 + 2(i+1)) \times \frac{\delta}{2}$$

If δ is odd.

$$k = (7 + 2(i+1)) \times \frac{\delta - 1}{2} + (4 + i + 1)$$

.

δ can be calculated as

$$\delta = n - 11 - (12(2^{i+1} - 1))$$

2.4 Beating the previous upper bound

In this section we try to prove why this upper bound is better than the previously known upper bound. As mentioned earlier that Furedi,Horak,Pareek and Zhu [4] gave the best known upper bound, using this same cut structure ocliques in a bit different way. They have assumed that :

$$f_w(n) \leq h\left(\left\lceil\frac{n-1}{2}\right\rceil\right) + h\left(\left\lfloor\frac{n-1}{2}\right\rfloor\right) + n - 1$$

,n is the number of vertices.

Where $h(n)$ follows the same cut structured oclique as $g_w(n)$, but it follows the above mentioned equation in its computation. That is $h(n) = h\left(\left\lceil\frac{n-1}{2}\right\rceil\right) + h\left(\left\lfloor\frac{n-1}{2}\right\rfloor\right) + n - 1$.

But as seen in the previous section that we have not restricted ourselves as in $h(n)$ but rather we found out the structure that minimizes the number of edges. By finding out the j which minimizes the value of $g_w(n-1-j) + g_w(j)$ thus minimizing $g_w(n)$, making $g_w(n)$ the function which gives the minimum edge requirement for cut structured ocliques. Hence $g_w(n) \le h(n)$.

This fact trivially shows that why our upper bound beats the previous upper bound. But still we do a more rigorous analysis and find out that our upper bound is strictly less than the previously known upper bound.

From previous analysis we have $2^{i+1} = \frac{n+1}{12} + \frac{\delta}{12}$. Let $\log_2(n+2) = p$. Using the fact that $\log_2\left(\frac{n+1}{12}\right) < p - 3.5$. But as i is always an integer so we have i as $p - 4.5 - b$. Where $0 < b < 2$ is the value subtracted from $p - 3.5$ to make it have an integer value. Note that here we do the analysis for $i \ge 0$ that is for $n > 23$

Now we assume that $\delta = \frac{n}{a}$ for any real positive number a. Clearly $a > 2$. Now consider the following thing:

$$\log_2\left(\frac{n - \delta + 1}{12}\right) = p - 3.5 - b$$
$$\log_2\left(\frac{n - \frac{n}{a} + 1}{12}\right) = p - 3.5 - b$$
$$12 \times \frac{a}{a-1} \le 2^{3.5} \times 2^b$$
$$\frac{a}{a-1} \le 2^b$$
$$\frac{1}{a} \le \frac{2^b - 1}{2^b}$$
$$\frac{n}{a} \le \frac{2^b - 1}{2^b} \times n$$

Thus we have..

$$\delta \le \frac{2^b - 1}{2^b} \times n \tag{2.3}$$

Now having the equalities in place lets begin the analysis. We know that $g_w(n) = 20 + S + k$. Thus for S we have

$$S = 42\left(\frac{n+1}{12} - 1 - \frac{\delta}{12}\right) + 12\left((p - 5.5 - b)\left(\frac{n+1}{12} - \frac{\delta}{12}\right) + 2\right)$$
$$= 3.5(n+1) - 3.5\delta - 42 + (p - 5.5 - b)(n + 1 - \delta) + 24$$
$$\therefore S = (2 + b)\delta + p + np - (2 + b)n - p\delta - 20 - b$$

Similarly we have for k as

$$k = (7 + 2(p - 3.5 - b)) \times \frac{\delta}{2}$$
$$\therefore k = p\delta - b\delta$$

Hence,

$$g_w(n) = 20 + S + k$$
$$g_w(n) = 2\delta + np + p - (2+b)n - b$$
$$g_w(n) \leq 2 \times \frac{2^b - 1}{2^b} \times n - (2+b)n + np + p - b \text{ using } 2.3$$
$$\leq n\left(\frac{2(2^b - 1)}{2^b} - 2 - b\right) + np + p - b$$
$$\leq -n\left(\frac{2}{2^b} + b\right) + np + p - b$$

The term $\left(\frac{2}{2^b} + b\right)$ attains minimum value for $b = \log_2(2 \ln 2)$, which is 1.914. Thus we have :

$$f_w(n) \leq np - 1.914n + p - b \text{ as } f_w(n) \leq g_w(n) \tag{2.4}$$

And as mentioned earlier, the previously known best known upper bound as given by Furedi,Horak,Pareek and Zhu [4] is

$$f_w(n) \leq n\log_2(n) - 1.5n \tag{2.5}$$

Subtracting the RHS of equation of 2.5 from the RHS equation of 2.4 (let that difference be D), we have

$$D = np - n\log_2(n) - 0.414n + p - b$$
$$D = n\left(\log_2(\frac{n+2}{n})\right) - 0.414n + \log_2(n+2) - b$$

Notice that $\log_2(\frac{n+2}{n}) < 0.115$ for $n \geq 24$. Thus we have

$$D < -0.299n + \log_2(n+2) - b \text{ thus,}$$
$$D < 0 \text{ for } n \geq 24$$

But D tells us more than just the fact that our bound is tighter than the previously known bound. It addresses the problem of linear error in the previous upper bound as mentioned by Furedi,Horak,Pareek and Zhu [4]. As we see in D the term $-0.299n$ dominates every other term thus we see an almost (because of the log term) linear improvement in the upper bound for our upper bound compared with the previous upper bound. Thus our bound is better than the existing one.

Chapter 3

Minimal Planar Ocliques

An advantage of dealing with planar graphs is that we have to deal with only two types of dominator cases. Henning and Goddard [5] showed that any planar graph with diameter two is dominated by atmost two vertices except for a graph on 9 vertices as illustrated in MacGillivary and Seyffarth [11]. That graph has domination number 3. But that graph is not an oclique. So we deal with graphs having domination number atmost 2.

We have mainly three results, which are as follows :

1. Affect in the oriented coloring of the planar graphs upon the removal of edges.

2. Affect of edge removal on some planar ocliques minimality.

3. The minimum edge requirements for two dominator cases in the planar graph.

3.1 Minimum edge requirement

In this section we give minimum edge requirements of planar graphs of order 4 to 7 for two dominator case.

Observation 3.1. *Dominator vertices of the ocliques $\overrightarrow{G}(V, E)$ of order n has atleast $n - 1$ edges adjacent to it.*

Proof. For one dominator case ,the dominator vertex has to be adjacent to exactly $n - 1$ other vertices.Thus requiring $n - 1$ edges.
For two dominator case we require atleast $m - 2$ edges to dominate other vertices in the graph except for d_1 and d_2. For the case of the no common neighbour between the two dominator vertex the underlying graph G has distance of d_1 to d_2 more than two so we have an edge between d_1 and d_2. Thus making the sum total $n - 1$.
For the case of one or more common neighbour. We require atleast $n - 1$ edges. □

Proposition 3.1. *Planar oclique of order 4 with 2 dominator requires atleast 4 edges.*

Proof. A connected graph of 4 vertices and 3 edges is either a path of length 3 or a claw. Both are which are not ocliques. □

Proposition 3.2. *Planar oclique of order 5 with 2 dominator requires atleast 5 edges.*

Proof. All connected graph of 5 vertices and four edges are trees only the 4 star has diameter 2 but has no orientation that makes it an oclique. □

Proposition 3.3. *Planar oclique of order 6 with 2 dominator requires atleast 8 edges.*

Proof. let $\{x, y, a, b, c, d\}$ be the vertices and x, y be the two dominators. The dominators cannot have three common neighbours as one vertex then becomes a pendant vertex and it can be in 2-dipath with at most 3 vertices. If x, y has two common neighbours a, b, then c, d cannot form a triangle with one dominator as both will miss the other dominator. Hence x, c, d, y has to be a 4-dipath. This forces (x, c, d, y, a) and (x, c, d, y, b) to be directed 5 cycles forcing a to miss b. If x, y has d as the common

neighbour, then arcs xb, xc, and ya are needed to preserve the 2-dominator property. Without loss of generality ac is an arc that gives rise to the directed 5-cycle (x, d, y, a, c). As b is a 2-degree vertex it can be in 2-dipath with at most 2 vertices in the directed 5-cycle thereby missing one of them. If x, y has no common neighbour then they have to be adjacent to avoid missing each other. The degree sum of x, y is exactly 6. The degree sum of a, b, c, d is exactly 8 as there are only 7 arcs. Therefore, ignoring the arc xy all vertices must have degree 2 making the graph a 6-cycle with a chord which is not an oclique. $\square$

For an underlying oclique G, its induced subgraph H is the subgraph we get after removing the dominators from G.

Proposition 3.4. *A 7 vertex planar oclique of domination number two requires atleast 10 vertices.*

Proof. We will prove this by contradiction .Say oclique of size 7 can be done using 9 edges. Let the two dominator be d_1 and d_2.
Case 1: With one or no common neighbours, between two dominators d_1, d_2.
Then H have 5 vertices with 3 edges. So we have atleast 3 vertices with one or less degree in H. It can be seen that there are more than or equal to, three 2 degree vertices in G(which are isolated vertices or 1 degree vertices in H). Notice that when we put back the dominators u is adjacent to atleast one of them let that be d_1, and the other one be v. Without loss of generality let x_1 be the other 2 degree
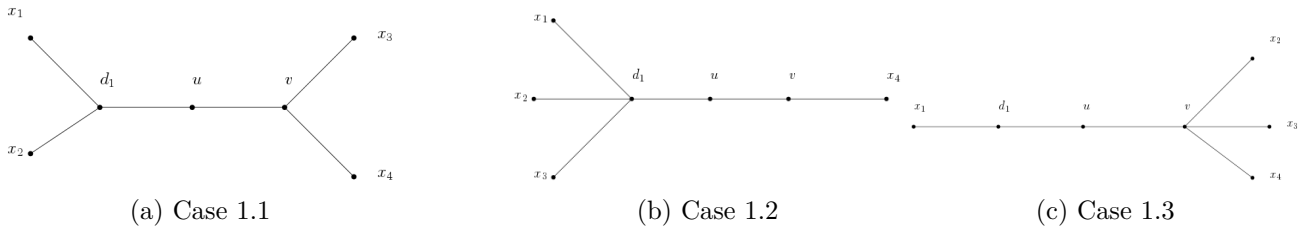

(a) Case 1.1   (b) Case 1.2   (c) Case 1.3

Figure 3.1: Cases for 7

vertex. With respect to figure 3.1a, notice that x_1 agrees with x_2 on d_1 and x_3 agrees with x_4 on v to see u. Now x_1 cannot be adjacent to v. If x_1 was adjacent to v then we could remove x_1 or u which does not hamper connectivity of d_1 and v thus we have a oclique of size 6 but with $9 - 2 = 7$ edges which is not possible as oclique of 6 requires atleast 8 edges by proposition 3.3.
If x_1 be adjacent to x_2. Then we can delete x_1 and have an oclique of 6 vertices but with 7 edges, as deleting x_1 in this case will not affect connectivity. But this contradicts proposition 3.3.
Let x_1 be adjecent to x_3 so x_3 has to dominate x_2. Now for x_1 to see x_4 it uses either d_1 or x_3 this uses up all 9 edges. If $d_1$is adjacent to x_4 then x_3 misses x_4.Else if x_3 is adjacent to x_4 then d_1 misses x_4.
Now lets consider the next case as shown above figure 3.1b. With respect to figure 3.1b notice that x_1 agrees with x_2 and x_3 on d_1 to see u. Again lets consider x_1 to be the two degree vertex, the above arguments hold for x_1 being adjacent to v and x_2, x_3.So x_1 is adjacent to x_4. So x_4 is adjacent to x_2 and x_3. Thus using up all 9 edges but x_2 still misses x_3, as they agree upon all their common neighbours.
With regard to figure 3.1c x_1 is adjacent to x_2, with out loss of generality. For x_1 to see x_3, x_4 via d_1 or x_2 it requires atleast 2 edges. So we end up using all 9 edges but x_3 misses x_4 as they agree upon all their common neighbours.
Case 2: With more than one common neighbours between d_1 and d_2.
As the number of common neighbours increases the edges in H decreases. Thus it can be seen that there are atleast three 2 degree vertices in G. And thus all above arguments hold.
Now if there were any oclique of smaller size then those planar ocliques can be extended to higher sized ocliques (this can always be done as the planar graph is not triangulated as number of edges is less $3n - 6$). This completes the proof. $\square$

3.2 A property of minimal planar ocliques

In this section we try to answer a property of the planar graphs. If a planar graph of domination number two has no planar oclique of order $n \leq 14$ requires atleast m arcs. And suppose k is the least arc size of that order n planar oclique such that it does not have any minimal oclique of size k and $k \geq m$. Then do we have a minimal oclique of size $k' > k$? Well the answer to this is not known but our intuition says its no. We have charted put some method to prove it for graphs of order 4,5,6 which are shown in later section.

Swathy Prabhu Maharaj have prepared a set of minimal ocliques using the help of nauty library [12]. And have generated minimal planar ocliques of order from 4 to 14 with domination number 2. For each order he have found out minimal planar ocliques upto the size k. Where k is the smallest arc number greater than the minimum arcs required for which $k + 1$ has no minimal planar oclique. We denote that set as $\mathcal{F}$

Theorem 3.1. *For a minimal planar ocliques $\overrightarrow{G}$ there exists two edges e_1 and e_2 in $E(G)$ of the underlying graph G whose removal creates $G(V, E\backslash\{e_1, e_2\})$, which have no $n - 1$ order oclique in it. Assuming conjecture 1 to be true*

We will prove this theorem by finding contradiction. We will first assume that the Theorem 3.1 is false and then find some characters in such a graph $\overrightarrow{H}$. So by contrary of Theorem 3.1, no matter which two edges we remove from H we have a orientation that makes $\overrightarrow{H}(V, E\backslash\{e_1, e_2\})$ as $n - 1$ colorable.

Conjecture 1. *For any two pair of vertices $u, v \in V(H)$, there exists three disjoint path of length 2 or less between u, v in H.*

We believe that this conjecture 1 is true as suggested by our computer analysis on all minimal planar graphs of diameter two. In our computer analysis we have taken all available minimal ocliques of planar graphs in $\mathcal{F}$. We need not need this theorem 3.1 for planar ocliques of order 4. But for higher order oclique we have used this theorem. So here we assume that this 1 is true and begin proving the theorem.

Let $\delta(G)$ be the smallest degree in a simple graph G.

Observation 3.2. $\delta(H) \geq 3$

As we have made some observations about the minimal graph H. Now we are ready to prove Theorem 3.1.

Proof. As we have seen from observation 3.2 that $\delta(H) \geq 3$ and for planar graphs we know that there exists a vertex of degree less than or equal to 5. Therefore $3 \leq \delta(H) \leq 5$. So here we do a case wise analysis of all the degree and show contradiction for each case. Notice that we also begin from $n > 4$ as $\delta(H) \geq 3$ this creates a K_4 for $n = 4$, which is not a minimal oclique. We will fix the minimum degree vertex as x and its neighbours are named as $v_1, v_2, v_3....$

Case $\delta(H)=3$:
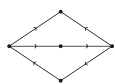
(a) 5 oclique   (b) 6 oclique   (c) 7 oclique

Figure 3.2: Counter Examples

For **n=5**: Let y be another vertex not adjacent to x. So it connected to x via v_1, v_2 and v_3. That is shown in figure 3.2a. And it has an orientation that makes it 5 oriented colorable. And clearly there are vertices whose degree is less than 3. And adding further edges will contradict the minimality .Thus creating an contradiction as it does not satisfy observation 3.2.

For **n > 5**: For this case we have at least two vertices that are non-adjacent to x so they must see x

via v_1, v_2 and v_3. This creates a $K_{3,3}$.

Case $\delta(H)=4$:

For **n=5**: For this case the only possible configuration for H is K_5, which is neither planar nor minimal oclique. So for n=5 we have a contradiction.

For **n=6**: Let y be another vertex not adjacent to x. So it connected to x via v_1, v_2 and v_3. But its also adjacent to v_4 as minimum degree is four and it cannot be adjacent to x. This is shown in figure 3.2b. And it has orientation that makes it 6 oriented colorable. Thus it contradicts observation 3.2, and adding further edges will contradict the minimality of H.

For **n=7**: Just like the previous case we have a vertex y non adjacent to x. So its connected via v_1, v_2, v_3. This divides the plane into three faces .$\mathcal{F}_1$ the outer face, $\mathcal{F}_2$ the face contained by x, v_1, y, v_2, x and $\mathcal{F}_3$ the face contained by x, v_2, y, v_3, x. As minimum degree is four therefore we have another vertex v_4 which is adjacent to x. Assume v_4 lie in $\mathcal{F}_2$. Let another non-adjacent vertex be z. Now z must lie in the same face as v_4 is. This is because every face has only 2 vertices among v_1, v_2, v_3, but z requires atleast 3 vertices from the neighbour hood x by conjecture 1. Thus z lies in $\mathcal{F}_2$. So z becomes adjacent to v_1, v_2 and v_4 to see x in accordance with conjecture 1. But this divides the face into three faces C_1, C_2 and C_3. C_1 is bounded by x, v_1, z, v_4, x, C_2 is bounded by x, v_2, z, v_4, x and C_3 is bounded by y, v_1, z, v_2, y. This shows that y and v_4 does not share any common face. Thus they cannot be adjacent. But as minimum degree is four so this implies that y and z are adjacent as y cannot be adjacent to x by our construction. The resultant graph is shown in the figure 3.2c. And it also has an orientation that makes it 7 oriented coloring. Thus contradicting observation 3.2 and adding edges will contradict the minimality of H . Hence a contradiction. Similar arguments hold for v_4 being in $\mathcal{F}_1$ or $\mathcal{F}_3$.

For **n > 7**: In this case we have another vertex named u which is non adjacent to x apart from z and y. So by conjecture 1 we have seen that u need to be adjacent to atleast three vertices among v_1, v_2, v_3, and v_4.So we connect u to any three vertices among them. Now consider the graph minor H_m of H which we get by merging the vertices y and z to create y'. Now this y' is adjacent to all of the neighbours of x. Now no matter what three vertices among the neighbours of x, u is adjacent to we have an $K_{3,3}$ as minor of H, thus contradicting the planarity of H.

Case $\delta(H)=5$:

For **n=6**: Only possibility is K_6 which is neither planar nor minimal oclique.

For **n=7**: In this case x has only one non adjacent vertex y. Ans as the minimum degree is so y is also adjacent to all of the neighbourhood of x. Now as the degree of v_1 is two so it must be adjacent to atleast three more vertices among v_2, v_3, v_4 and v_5. But this creates an $K_{3,3}$.

For **n=8**: In this case apart from y we will have another vertex z which is not adjacent to x. So just like y, the vertex z is adjacent to three of the neighbours of x. If y is adjacent to all neighbours of x this creates an $K_{3,3}$. So y is adjacent to only four neighbours (say v_1, v_2, v_3, v_4) of x and z to maintain the minimum five degree condition. Now z must lie in the same face as the v_5 to maintain the conditions imposed by conjecture 1 and to maintain planarity. Assume that v_5 lie in the face enclosed by x, v_1, y, v_2, x.But this means that z got enclosed in a face where it can be adjacent to only four vertices namely y, v_1, v_2 and v_5 (it cannot be adjacent to x by our construction). But minimum degree is five so its an contradiction.

For **n=9**: In this case apart from y we will have another two vertices z, u which are not adjacent to x. One case is that u and z are not adjacent. So both u and z have atleast four common neighbours with x. This creates an $K_{3,3}$.

So the remaining case is that u and z are adjacent. Notice that u and z both are adjacent to y. Else both y and u (or z the one which is not adjacent to y) have to be adjacent to atleast four neighbours of x. Thus creating $K_{3,3}$. Again we assume that y is adjacent to v_1, v_2, v_3. This divides the plane into three faces. As u and z are adjacent we have them in the same face. Without loss of generality let them be in the face bounded by x, v_1, y, v_2, x. Observe that vertices u and z must be adjacent to atleast one of the vertices among v_1 or v_2, to maintain its minimum five degree condition. If u is adjacent to v_1 (or v_2) then z cannot be adjacent to v_1 (or v_2) as it encloses u in a face bounded by y, z, v_1, y which will limit the degree of u to only three. We assume with out loss of generality that u is adjacent to v_1, so z is adjacent to v_2 Thus both v_4 and v_5 are present in the same face as u and z so

that u and z have degree atleast five. So u is adjacent to v_1, v_4, v_5, z, y this encloses v_4 (say, can be v_5 also depending on which comes after $v1$ in the arrangement around x, here we assume v_4 comes next to v_1) in a face bounded by v_1, x, v_5, u, v_1. So v_4 cannot be adjacent to z this limits the degree of z to four. Thus a contradiction.

For $\mathbf{n > 9}$: Here also we first fix an non-adjacent vertex y. Apart from y we have atleast 3 vertices that are not adjacent to x. Now if we merge y with any two of those non-adjacent vertices thus creating a new vertex y'. This y' is adjacent to all the five neighbours of x. Thus the third non-adjacent vertex together with y', x and x's neighbour forms an $K_{3,3}$. So we have an planarity contradiction.

Thus we prove theorem 3.1 by contradicting its negation scenario.

$\square$

3.3 Maximality of minimality of planar ocliques

Here we demonstrate some application of theorem 3.1 of the previous section. As stated earlier that we here show that if there exists no minimal planar oclique of some order n and size k whose domination number more than 1, then there exists no minimal oclique of size k_1 and order n, where $k_1 > k$.
As a matter of fact we show it for $n = 4, 5, 6$. We will show this one by one and finally use this to create the final result that we mentioned above.
But first we need to look into some facts about oclique.

Observation 3.3. *Ocliques with domination number greater than one has no pendant vertex*

Proposition 3.5. *Removal of a single edge in an oclique $\overrightarrow{G}$ of size n with domination number greater than one, does not disconnect the graph.*

Proof. We prove this by contradiction. Say we have an edge (u, v) in G such that removal of which decomposes it into 2 components C_1 and C_2. And let $u \in C_1$ and $v \in C_2$. So in this case $N(v)$ in C_2 is not visible to $N(u)$ in C_1 thus one of these sets are empty. With out loss of generality say $N(u) = \phi$, this makes u an pendant vertex. This is a contradiction as pendent vertices are not present in ocliques with domination number greater than one. $\square$

Corollary 3.1. *Removal of two edges from the an oclique decomposes it into atmost 2 components.*

Proposition 3.6. *Let $\overrightarrow{G}$ be an oclique with dominator number greater than one. And if removal of two edges makes the oclique $\overrightarrow{G}$ disconnected . Then one of two component has atmost two vertices in it.*

Proof. By corollary 3.1 we have atmost two components say C_1 and C_2. With out loss of generality say $|C_2|$ is the smaller of the two. Let the two edges to be deleted be (x, v) and (y, u) where $u, v \in C_2$ and $x, y \in C_1$ (u and v may be same). So either $N(u)\backslash\{v\}$ and $N(v)\backslash\{u\}$ are empty or $N(x)\backslash\{y\}$ and $N(y)\backslash\{x\}$ are empty else they will not be visible to another vertices in different component. Now as C_2 was assumed to be the smaller of the two so $N(v)\backslash\{u\}$ and $N(u)\backslash\{v\} = \phi$. Thus it proves our claim. $\square$

Let us denote $D(G)$ to be the maximum degree in a graph G. We will refer all minimal ocliques as $\overrightarrow{G}$ an the underlying graph as G. And in such graphs we fix the max degree vertex as v. And its neighbours as $v_1, v_2, ...$

Lemma 3.1. *There are no minimal planar oclique of size 5 or more and order 4.*

Proof. Before we begin the proof we must notice that a four cycle and a triangle with a pendant attached to it, has an orientation that makes it four oriented colorable. Suppose we have a minimal graph $\overrightarrow{G}$. Then such a graph must posses a max degree vertex v as degree 3. Therefore we have 2 edges that has end points in $v1, v2$ and v_3. Thus we have an minimal oclique with as proper subgraph in $\overrightarrow{G}$. This contradicts the minimality of $\overrightarrow{G}$. $\square$

Lemma 3.2. *There are no minimal planar oclique of size 7 or more and order 5.*

Proof. If there were any minimal oclique of size 7 and order 5 then we must have a three or less oriented colorable subgraph with 5 edges and 5 vertices by theorem 3.1. We will show that under every possible subgraphs of 5 edges and 5 vertices that can be formed from an oclique we have an oriented four (or more) coloring.

There are 2 cases one is that removal of edges break the graphs into two components (by corollary 3.1) and the other one is that only one component is there. So lets first check the cases when we have 2 component. When one of the component has two vertices then the other one has three vertices. This means that there can be atmost 4 edges in that graph. Three in one component and one in the other. So one component must have one vertex and other must have 4 vertices with 5 edges in it. And lemma 3.1 proof shows that we always have an four coloring in such cases.

Now we deal with the case where we have only one component. If we have a four degree vertex say v with neighbours v_1, v_2, v_3 and v_4. Now as there are five edges so one edge has endpoints in v_1, v_2, v_3 and v_4, thus making a four coloring. If we have a three degree vertex say v with neighbours v_1, v_2 and v_3. Then we have another vertex y left. Notice that there cannot be any edge between v_1, v_2 and v_3 else we would have an oriented four coloring. Hence y has the left over 2 edges incident to it, y can not be adjacent to v as its degree is 3. So y is adjacent to two vertices among v_1, v_2 and v_3, say v_1, v_2. Then there is an four cycle namely y, v_1, v, v_2, thus having a four coloring. So only possibility is max degree is 2. But as there are 5 edges it means that we have a five cycle. Which is itself five colorable. □

Lemma 3.3. *There are no minimal planar oclique with domination number two, of size 10 or more and order 6.*

Proof. By theorem 3.1, we must have an four or less oriented coloring. We will show that under every possible subgraphs of 8 edges and 6 vertices that can be formed from an oclique we have an oriented five (or more) coloring.

There are 2 cases one is that removal of edges break the graphs into two components (by corollary 3.1) and the other one is that only one component is there. Lets see the case of two components. Now by proposition 3.6 we have one component having atmost two vertices.

When a component have only two vertices that means other one has 4 vertices. This means it can have atmost $6 + 1 = 7$ edges but we have 8 edges. So we cannot have this case.

So one component must have one vertex. This leaves the other vertex with 5 vertices and 8 edges. As there are 8 edges we must have a degree four vertex v. Let the neighbours of v be v_1, v_2, v_3 and v_4. Thus we have $8 - 4 = 4$ edges for four vertices v_1, v_2, v_3 and v_4. This means that v_1, v_2, v_3 and v_4 has four oriented coloring and as v is adjacent to all v_1, v_2, v_3 and v_4 this means that we have five oriented coloring. So for two component cases we are done.

So the remaining case is that all the vertices must be in one component. As the number of edges are 8 so there must be a vertex with degree 3 or more.

Case 1:$D(G) = 3$

Let such a vertex with degree 3 be v. So in the induced graph where we remove v we have 5 vertices with 5 edges. By lemma 3.2 we have that for all configuration of 5 vertex and 5 edges we have a four oriented coloring or more. Now we put back v in a face which is either bounded by a triangle, or a four cycle or a five cycle and connect it with 3 vertices so that we have five coloring.

Case 2:$D(G) = 4$

Let such a vertex with degree 4 be v. Neighbours of v are v_1, v_2, v_3 and v_4 and the remaining vertex be u. Notice that $K_{3,2}$ has oriented 5 coloring. So u cannot be adjacent to more than 2 vertices among v_1, v_2, v_3 and v_4. Say its adjacent to v_1 and v_2. So we have $8 - 4 - 2 = 2$ edges more left with end points in v_1, v_2, v_3 and v_4. If these 2 edges makes a path (say v_1, v_2, v_3) then we have 5 coloring with v as one dominator, oclique of size 3 as positive neighbours and oclique of size one as negetive neighbour. So two edges cannot form a path so one edge is (v_1, v_2) and other is (v_4, v_3). This again is a 5 oclique with oclique of size 2 on the positive side and negetive side of v.

Similar arguments follow when u is adjacent to only one vertex among v_1, v_2, v_3 and v_4. Case 3:$D(G) = 5$

Then u becomes one dominator which we don't allow by construction. □

Chapter 4

Source Codes

4.1 Code for finding Upper Bound

We have used the following code for finding out the $g(n)$ in accordance with the algorithm mentioned in section 2.2. The language is C.

```c
#include <stdio.h>
#include <stdlib.h>

int MAX;

void find(int *ar,int last)
{
  int inc,dec,min,sum,c1,c2;
  c2=inc=last/2;
  c1=dec=last-inc;
  min=ar[dec-1]+ar[inc-1];
  sum=min;
  while(1)
    {
      if(sum<min)
        {
          min=sum;
          c1=dec;
          c2=inc;
        }
      if(dec>0 && inc <=last && ar[inc-1]<=min)
        {
          dec--;
          inc++;
          sum=ar[dec-1]+ar[inc-1];
        }
      else
        {
          ar[last]=min+last;
          //  printf("Chosen :: %d \t %d \t Value : %d",c1,c2,ar[last]);
          return;
        }
    }
}

int* initialize()
{
```

```c
  int *ar;
  ar=(int *)malloc(MAX*sizeof(int));
  ar[0]=0;
  ar[1]=1;
  ar[2]=2;
  ar[3]=4;
  ar[4]=5;
  return ar;
}

int main()
{
  int *ar=initialize();
  int i,j,diff=0,d1,d2;
  scanf("%d",&MAX);
  for(i=5;i<MAX;i++)
    {
      // printf("For vertex number %d\t",i+1);
      find(ar,i);
      //printf("\n");
    }
  printf("Difference :\n");
  for(i=5;i<MAX-1;i++)
    {
      j=i+2;
      d1=ar[i+1]-ar[i];
      diff=d1-d2;
      if(i>5 && diff==2)
        printf("the diff of 2 at %d and %d\n",i+1,j);
      d2=d1;
    }
    return 0;
}
```

4.2 Code for conformation

The following code written in C was created to test Theorem 3.1 on minimal planar ocliques. This code removes 2 edges then computes the square of the adjacent matrix, after that it checks whether more than a pair of vertices are at a distance greater than two by counting the number of off diagonal zeros. If the number of such zeros are more than 3 then its an "Successful" else "Unsuccessful".

```c
#include <stdio.h>
#include <stdlib.h>

#ifndef DEBUG
#define DEBUG
#define dprintf //
#endif


struct adj{
  int a;
  int mark;
};
typedef struct adj* Adj;
```

```c
Adj* allocate(int n){
    int i,j;
    Adj *s;
    s=(Adj*)malloc(n*sizeof(Adj));
    for(i=0;i<n;i++)
        {
            s[i]=(Adj)malloc(n*sizeof(struct adj));
            for(j=0;j<n;j++)
                s[i][j].mark=0;
        }

    return s;
}

void freeing(Adj* x,int n)
{
    int i;
    for(i=0;i<n;i++)
        free(x[i]);
    free(x);
}

Adj* multiplying(Adj *g,int n)
{//finding a square
    int i,j,k,sum;
    Adj* x=allocate(n);
    for(i=0;i<n;i++)
        {
            for(j=0;j<n;j++)
                {
                    sum=0;
                    for(k=0;k<n;k++)
                        {
                            sum+=g[i][k].a*g[k][j].a;
                        }
                    x[i][j].a=sum;
                    x[i][j].mark=0;
                }
        }
    for(i=0;i<n;i++)
        {
            for(j=0;j<n;j++)
                {
                    x[i][j].a+=g[i][j].a;
                }
        }
    return x;
}

int check_clique(Adj *x,int n)
{// checking wether its a clique or not
    int i,j,count=0;
    for(i=0;i<n;i++)
        {
```

```c
        for ( j =0; j <n ; j++)
          {
            if ( i !=j && x [ i ] [ j ] . a==0)
              count++;
          }
      }
    if ( count >=4)
      return 0;
    return 1;
}

void taking_queue (Adj *g , int n){
  int i , j , count =0;
  Adj* x;
  int **ar ;
  ar=(int **) malloc (2* sizeof (int *) );
  for ( i =0; i <2; i++)
    ar [ i]=(int *) malloc (3*n* sizeof (int ) );
  for ( i =0; i <n ; i++)
    {
      for ( j =0; j <n ; j++)
        {
          if ( g [ i ] [ j ] . mark==0 && g [ i ] [ j ] . a==1)
            {
              g [ i ] [ j ] . mark=1;
              ar [ 0 ] [ count]=i ;
              ar [ 1 ] [ count]=j ;
              g [ j ] [ i ] . mark=1;
              count++;
            }
        }
    }

  for ( i =0; i <count ; i++)
    {
      for ( j=i +1; j <count ; j++)
        {
          g [ ar [ 0 ] [ i ] ] [ ar [ 1 ] [ i ] ] . a=0;
          g [ ar [ 1 ] [ i ] ] [ ar [ 0 ] [ i ] ] . a=0;
          g [ ar [ 1 ] [ j ] ] [ ar [ 0 ] [ j ] ] . a=0;
          g [ ar [ 0 ] [ j ] ] [ ar [ 1 ] [ j ] ] . a=0;
          x=multiplying (g , n ) ;
          g [ ar [ 0 ] [ i ] ] [ ar [ 1 ] [ i ] ] . a=1;
          g [ ar [ 1 ] [ i ] ] [ ar [ 0 ] [ i ] ] . a=1;
          g [ ar [ 1 ] [ j ] ] [ ar [ 0 ] [ j ] ] . a=1;
          g [ ar [ 0 ] [ j ] ] [ ar [ 1 ] [ j ] ] . a=1;
          if ( check_clique (x , n)==0)
            {
              printf (" Successful \n" );
              free ( ar [ 0 ] ) ;
              free ( ar [ 1 ] ) ;
              free ( ar ) ;
              freeing (x , n ) ;
              return ;
            }
```

```c
            freeing(x,n);
        }
    }
    printf("Usuccessful\n");
    free(ar[0]);
    free(ar[1]);
    free(ar);

    return;
}


int main()
{
    int n,times,i,j,k;
    Adj *g;
    scanf("%d",&n);
    scanf("%d",&times);
    times/=(n+1);
    dprintf("times=%d\n",times);
    for(k=0;k<times;k++)
        {
            scanf("%d",&n);
            g=allocate(n);
            for(i=0;i<n;i++)
                {
                    for(j=0;j<n;j++)
                        {
                            scanf("%d",&g[i][j].a);
                        }
                }
            dprintf("going in for i=%d\n",i);
            taking_queue(g,n);
            freeing(g,n);
        }
    return 0;
}
```

Chapter 5

Conclusion

In Chapter 2 we have found a tighter upper bound for $f_w(n)$. We believe that this upper bound is the actual explicit formula for $f_w(n)$. The minimum cut structured ocliques that we have defined in the same chapter are the ocliques with minimum edge requirment. In other words if $g_w(n)$ is the minimum arc requirement of a cut structured oclique, we believe that $g_w(n) = f_w(n)$.

Conjecture 2. $g_w(n) = f_w(n)$

We reach to this intuition by our computer simulations which shows that ocliques of higher order requires more arcs than the ocliques with domination number one. And we believe that cut structured ocliques minimize that for one dominator vertices.

In case of planar graphs as we see that our theorem 3.1 is not yet fully proved. We may try by proving conjecture 1 by showing that there exists an orientation in the contradicting graph $\overrightarrow{H}$ such that the vertices which are not at a distance three are ocliques.

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