STRATEGIC CANDIDACY AND VOTING PROCEDURES

By

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We study the incentives of candidates to strategically affect the outcome of a voting procedure. We show that the outcomes of *every* nondictatorial voting procedure that satisfies unanimity will be affected by the incentives of noncontending candidates (i.e., who cannot win the election) to influence the outcome by entering or exiting the election.

Keywords: Strategic candidacy, voting procedures, candidates, voting rules.

1. INTRODUCTION

THE DECISION OF A CANDIDATE to enter an election can affect its outcome even in situations where the candidate is not the winner of the election. For instance, consider a scenario in which three national parties A, B, and C can contest an election in which the winner is decided by plurality rule. Although party A may have the highest number of first-preference votes, it may still fail to win the election if, for instance, B drops out of the race in order to let C win.[2] If the voting process is viewed as a mapping from preferences to outcomes, then strategic behavior in the first stage can be just as important as strategic voting in the second stage. As we shall show, this phenomenon is important to *all* voting procedures, and thus spans applications ranging from political elections to committee decisions.

To be precise, we consider a framework in which there is a finite set of voters and potential candidates. We allow for the possibility that some or all of the candidates may also be voters, and consider situations where each individual (including candidates) has preferences over the set of all candidates. We examine a two-stage procedure where in a first stage candidates decide on

[2] An example of such strategic exit decisions arose in the recent election in Israel. On the last day before the election, Bishara (the Arab party candidate), Mordecai (the center party candidate), and Begin (the ultra-nationalist party candidate) each exited. Most critical was Mordecai's exit which gave Barak enough votes to win in the first round of the election, as most of the voters who supported Mordecai had Barak as a second choice. Without Mordecai's exit, it is likely that none of the candidates would have won in the first round and that there would have been a run-off between Barak and Netanyahu two weeks later. While the only affect of this was to avoid having a run-off, it was still an openly strategic action by Mordecai as he made it clear that he preferred Barak to win given that he would not. As a more direct example, without Perot's entry into the 1992 U.S. Presidential election Bush might have defeated Clinton. For an empirical look at such "what-if" questions in the 1987 British election, see Alvarez, Nagler, and Bowler (2000).

whether or not they will enter the election, and then in a second stage a voting procedure is implemented to select from the candidates who enter.

Before outlining our analysis, let us describe in more detail the way in which we model voting procedures. We model a voting procedure as specifying the winning candidate as a function of the set of entering candidates and voters' preferences over the entering candidates. The only restriction that we place on such a voting procedure is that it satisfy *unanimity*. Unanimity requires that if all voters find the same candidate most preferred out of the entering candidates, then that candidate is selected.

We focus on the following issue. Which voting procedures are not influenced by candidates' incentives to exit an election? More precisely, for which voting procedures is it always a Nash equilibrium for all candidates to enter the election? We call this condition "candidate stability." We show that if the sets of voters and candidates are distinct, then the only voting procedures satisfying candidate stability are dictatorial procedures. When the set of candidates and voters overlap, then there exist nondictatorial voting procedures that satisfy candidate stability and unanimity. However, we show that none of these voting procedures satisfy an appealing "almost"-unanimity condition together with a very weak monotonicity condition that is satisfied by most standard voting procedures (e.g., tree implementable procedures, Condorcet consistent procedures, scoring rules, etc.). This implies that most standard voting procedures fail to satisfy candidate stability regardless of the overlap between candidates and voters.

We should mention that these results are not simple extensions of an Arrow-type impossibility theorem, even though we invoke Arrow's theorem at one point in the proof of the first theorem. The bulk of the proof develops the joint implications of candidate stability and unanimity. We discuss this in more detail in what follows.

Why should we care whether a voting procedure is candidate stable? Regardless of how one feels about candidate stability as a normative property, the results here must be taken seriously if we are at all interested in evaluating and comparing voting procedures. Much of what we know about voting rules is based on comparisons of the properties of different voting procedures when the set of candidates is taken as given.[3] The results here show that the outcome of *all* nondictatorial and unanimous voting procedures will be influenced by the entry decisions of candidates. This implies that it is not valid to treat the set of candidates as fixed for any nondictatorial voting procedure. As most of what we know about voting rules treats the set of candidates as fixed, our results suggest that these need to be revisited accounting for strategic candidacy. For instance, example 5 below shows that the Pareto property of voting by successive elimination is upset when one allows for strategic candidacy. So, the results here show

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[3] There are some exceptions, such as Tideman (1987) who shows that entry by "clone" candidates can upset various scoring rules.

that in order to make meaningful comparisons across voting rules, strategic entry and exit effects must be understood.

The Related Literature

Before proceeding to the body of our analysis, let us briefly discuss the relationship of this work to the most closely related literature.

The most closely related papers to this one are papers by Dutta and Pattanaik (1978), Osborne and Slivinski (1996), and Besley and Coate (1997).[4] Dutta and Pattanaik (1978) analyze a setting where in a first stage (before voting) individuals sponsor or propose alternatives out of a set. Next, in a second stage, voting takes place over the set of proposed alternatives. Their main result is to show that there are circumstances under which sponsors indulge in strategic behavior by not proposing their most preferred alternatives. More recently, Besley and Coate (1997) and Osborne and Slivinski (1996) analyzed strategic candidacy in the context of representative democracy. In their models citizen-candidates can contest an election in which the winner is decided by plurality rule. The main focus of these papers is to determine the number of candidates who will enter the election as well as the pattern of entry. They exhibit situations where candidates who have no chance of winning may enter the fray simply in order to influence the outcome of the election, thus noting strategic candidacy.

Although each of the three above-mentioned papers analyzes issues related to strategic candidacy, the focus of our paper is quite different. We are interested in understanding whether strategic candidacy is an issue that is pervasive to all voting procedures. We conclude that it is.

Also related is a companion paper, Dutta, Jackson, and Le Breton (2000), where we analyze the effects of strategic candidacy on a particular voting procedure, voting by successive elimination. As argued above, given the result that all nondictatorial and unanimous voting procedures will be influenced by candidate entry decisions, it is important to re-examine previous characterizations of voting procedures to see how the analysis changes when we account for strategic candidacy. In the companion paper we conclude that strategic candidacy enlarges the set of outcomes of a voting procedure, and for voting by successive elimination this expansion is characterizable. It turns out that strategic candidacy allows for the election of Pareto dominated candidates at some profiles.

There is also a rich theoretical literature on the strategic effects of agenda manipulation, which is not as closely related to our work, but still should be mentioned.[5] The agenda manipulation literature takes seriously the strategic

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[4] See also Majumdar (1956) and Mueller (1978). The effects of candidacy have been considered in many other settings too, such as models of elections where the positioning of candidates can affect entry decisions of other candidates (e.g., Palfrey (1984)).

[5] A nice discussion of some of the main contributions to the agenda manipulation literature appears in Ordeshook (1986).

ordering of alternatives, or more generally the effects of varying the game form used for voting. However, the agenda manipulation literature considers the set of alternatives as fixed, and instead focuses on effects of changes in the voting procedure.

Finally, the underlying issue that we are considering here is of importance to the general literature on mechanism design and implementation. This literature usually takes the set of feasible alternatives to be exogenous and focuses on the difficulties raised by the presence of incentives of players regarding information that they might privately hold. There are notable exceptions. Papers by Postlewaite (1979), Hurwicz, Maskin, and Postlewaite (1995), and Hong (1996, 1998) have considered strategic withholding of endowments in exchange settings,[6] and thus are similarly motivated to understand the strategic effects of control of the feasible set of alternatives. As we examine a very different setting, our work is complementary to these papers with regard to the broader goal of developing an understanding of collective decision making when the set of alternatives is endogenous.

The remainder of the paper is structured as follows. In Section 2, we outline the setting and provide definitions. In Section 3, we discuss the implications of candidate stability when there is no overlap between candidates and voters. In Section 4, we follow up on this question when there is overlap between voters and candidates. In Section 5, we provide a preliminary examination of some important issues for further analysis.

2. DEFINITIONS

Candidates and Voters

Let $\mathcal{N} = \{1, \ldots, n\}$ be a finite set of individuals.

$\mathscr{C} \subset \mathcal{N}$ is the set of potential *candidates*. Generic candidates are denoted a, b, c, d, e. We consider the case where $\#\mathscr{C} \geq 3$, as the case with just two potential candidates is easily handled with a majority vote and the possibilities for strategic candidacy are trivial.

$\mathscr{V} \subset \mathcal{N}$ is the set of *voters*. Let $m = \#\mathscr{V}$. Generic voters are denoted i, j, k.

Without loss of generality, assume that $\mathcal{N} = \mathscr{C} \cup \mathscr{V}$. This allows for the case where $\mathscr{C} \cap \mathscr{V} \neq \varnothing$ (e.g., $\mathscr{C} = \mathscr{V}$ or $\mathscr{C} \subset \mathscr{V}$) and the case where $\mathscr{C} \cap \mathscr{V} = \varnothing$. We will discuss how the overlap between candidates and voters matters, as it applies. Unless otherwise stated, no assumption is made about the overlap.

Preferences

Individuals have strict preferences over the set of candidates represented by a complete, transitive, and asymmetric binary relation. The preference relation of i is denoted by P_i. Let $\mathscr{P}$ denote the set of all profiles of such preference relations and P denotes a generic profile in $\mathscr{P}$.

[6] Hong (1998) addresses private information of an agent about the feasible set of alternatives.

Say that aR_ib if either aP_ib or $a = b$.

Given any $A \subset \mathscr{C}$, let $P_i|_A$ denote the binary relation on A induced by P_i, and $P|_A$ the profile of induced relations.

Given any nonempty $B \subset \mathscr{C}$, let $top(B, P_i)$ denote the candidate $a \in B$ such that aP_ib for all $b \in B$, $b \neq a$.

In many situations it is natural to assume that a candidate finds him or herself most preferred.[7] The restricted domain of preferences

$$\mathscr{P}^r = \{P \in \mathscr{P} \mid a \in \mathscr{C} \Rightarrow top(\mathscr{C}, P_a) = a\},$$

captures this assumption.

Without such a domain restriction a basic problem can arise. For instance, in the extreme case where candidates find themselves least preferred[8] any elected candidate prefers to exit. So to give candidate stability its best shot at being satisfied, we work with the restricted domain. We shall see that even on the restricted domain candidate stability is an impossible condition to satisfy.

Voting Procedures

A *voting procedure* is a function $V : 2^{\mathscr{C}} \setminus \{\varnothing\} \times \mathscr{P}^r \to \mathscr{C}$ such that for all $A \in 2^{\mathscr{C}} \setminus \{\varnothing\}$ and $P \in \mathscr{P}^r$:

  (i) $V(A, P) \in A$,

  (ii) $V(A, P) = V(A, P')$ for all $P' \in \mathscr{P}^r$ such that $P_i = P_i'$ for all $i \in \mathscr{V}$, and

  (iii) $V(A, P) = V(A, P')$ for all $P' \in \mathscr{P}^r$ such that $P|_A = P'|_A$.

Item (i) says that a voting procedure chooses from the set of available candidates A. As stated, a voting procedure is a function and thus is single-valued. So any ties are already broken by the procedure and in some deterministic manner.[9]

Item (ii) says that a voting procedure is determined only by voters' preferences. In our setting the profile P includes a specification of candidates' preferences, and the candidates in some cases may not be voters. Restriction (ii) is essentially without loss of generality, as we could simply define $\mathscr{V}$ to be the set of individuals whose preferences matter for V.

Item (iii) says that the voting procedure depends only on preferences over the set of feasible (i.e., entering) candidates. This condition is similar to Arrow's independence of irrelevant alternatives condition, except defined over voting procedures instead of social welfare orderings.[10] We emphasize that this inde-

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[7] For instance, this will be true in the framework of Besley and Coate (1997), where each candidate is identified with her most preferred alternative in some policy space.

[8] This might arise if the election is for a position that nobody wants to fill, e.g., chairman of a department.

[9] This leaves open the case of random tie-breaking. For example, imagine a plurality rule with ties broken by the flip of a fair (multi-faced) coin. We conjecture that with appropriate restrictions on von Neumann-Morgenstern preferences over lotteries on candidates, the results presented here extend essentially intact.

[10] This condition is sometimes called "independence of infeasible alternatives" in the choice setting.

pendence condition is very weak in the context of voting procedures, as there are many voting procedures that are nondictatorial and satisfy unanimity, as discussed in more detail below.

If the set of entering candidates is fixed, then the voting procedure becomes a social choice function. It is because a voting procedure is more general than either choice rules or social choice functions that we use the new name voting procedure.[11]

Unanimity

V satisfies *unanimity* if $V(B, P) = b$ for any $B \subset \mathscr{C}$, $P \in \mathscr{P}^r$, and $b \in B$ such that $\text{top}(B, P_i) = b$ for each $i \in \mathscr{V}$.[12]

If for each A, $V(A, \cdot)$ is a social choice function that is implementable via some standard solution concept such as Nash equilibrium, subgame perfect equilibrium, undominated Nash equilibrium, or the iterative elimination of weakly dominated strategies, via a game form (not necessarily finite or of perfect information) with range A, then V is a voting procedure that satisfies unanimity. Thus, if for each A the voting rule that society uses can be modeled (implemented) by a game form, then that will result in a unanimous voting procedure. We emphasize, however, that we can be completely agnostic on the derivation of the voting procedure V and still satisfy these properties. For instance, solving a knock-out voting tree (e.g., see Moulin (1988)) with sincere voting will also lead to a voting procedure V that satisfies unanimity, as will plurality rule under sincere voting with a deterministic tie-breaking rule.

Candidate Stability

The candidate stability condition used in the main result in the paper is the following.

A voting procedure V is *candidate stable* if $V(\mathscr{C}, P) R_a V(\mathscr{C} \setminus \{a\}, P)$ for all $a \in \mathscr{C}$ and $P \in \mathscr{P}^r$.

In thinking about a game where candidates are deciding whether or not to enter, candidate stability requires that it be a Nash equilibrium for all candidates to enter.[13] Candidate stability is a weak way of capturing the idea that no

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[11] The term "social choice function" has been used in the nonbinary choice literature to describe the same functions that we are calling "voting procedures." We chose not to use the name social choice function, as that term is now commonly used to indicate a function for which the set of candidates is fixed, and we want to explicitly focus on the importance of candidate entry.

[12] Given the self-preference of candidates, unanimity on $\mathscr{C}$ is always satisfied when there are at least two candidates who are voters. In the Appendix we discuss a strengthening of unanimity for the case where $\mathscr{C} \cap \mathscr{V} \neq \varnothing$.

[13] In the proof of Theorem 1 in the Appendix, we use as a tool a variation on the notion of candidate stability that addresses issues associated with nonstrategic entry/exit decisions. Such a condition can also be of interest.

candidate can strategically affect the outcome of a voting procedure by with-drawing from a contest. It is weak in that it does not require anything like it to be a dominant strategy for candidates to enter.

Two features of this concept are worth emphasizing. First, the condition only applies to candidates who are not being elected. Given candidates' self-prefer-ence, they could only conceivably benefit from exiting if they were not being elected to begin with. Thus, candidate stability only applies to situations where the candidate entering or exiting only affects the outcome by swinging the vote in terms of which other candidate is elected. Second, candidate stability only requires that this be true when all candidates enter. That is, the condition only compares $V(\mathscr{C}, P)$ and $V(\mathscr{C} \setminus \{a\}, P)$, but makes no statement about the rela-tionship between $V(A, P)$ and $V(A \setminus \{a\}, P)$ for other $A \subset \mathscr{C}$. We detail the relationship between candidate stability and the set of entry equilibrium out-comes in the concluding remarks.

3. THE IMPLICATIONS OF CANDIDATE STABILITY

There are many nondictatorial voting procedures that satisfy unanimity (and even Pareto optimality). This follows from our earlier claim. However, as we show below, only dictatorial voting procedures satisfy candidate stability. Before stating the theorem, let us examine some examples to get a feeling for how voting procedures fail to satisfy candidate stability.

First, we examine the example of plurality rule under sincere voting for which there are many examples of strategic candidacy, as mentioned in the introduc-tion.[14]

EXAMPLE 1: Let $\mathscr{C} = \{a, b, c\}$, $m = 7$. For any $A \subset \mathscr{C}$, $V(A, P)$ is determined by plurality rule (under sincere voting) with ties broken in any deterministic manner.

Consider the profile of voters' preferences where

$$aP_i bP_i c \quad \text{for} \quad i = 1, 2, 3;$$

$$bP_i aP_i c \quad \text{for} \quad i = 4, 5;$$

$$cP_i bP_i a \quad \text{for} \quad i = 6, 7.$$

In this case, a wins if all candidates enter, while b wins if candidate c exits. If candidate c has 6's preferences (or is 6), then candidate stability is violated. Note that under Borda voting we end up with exactly the same outcomes, so this is not simply an artifact or plurality rule.[15]

[14] See Osborne and Slivinski (1996) and Besley and Coate (1997) for detailed analyses of candidacy decisions under plurality rule.

[15] This example also demonstrates that failures of candidate stability can arise at preference profiles where the majority relationship is transitive. Here, b is a Condorcet winner and c is a Condorcet loser.

Failures to satisfy candidate stability occur not only with sincere voting, but also under strategic voting procedures. Let us consider an example of one of the most extensively studied strategic voting procedures. Voting by successive elimination[16] defines a unanimous and nondictatorial voting procedure (actually under either strategic or sincere voting), but fails to satisfy candidate stability.

EXAMPLE 2: Let $\mathscr{C} = \{a, b, c\}$, $\mathscr{V} = \{1, 2, 3\}$, and consider $A \subset \mathscr{C}$.
If $\#A = 1$, then $V(A, P) \in A$.
If $\#A = 2$, then $V(A, P)$ is determined by majority rule.
If $\#A = 3$, then $V(A, P)$ is determined by first holding a majority vote over a versus b, then matching the winner in a majority vote against c.

When $V(A, P)$ is defined using sophisticated voting, V is clearly a voting procedure, satisfies unanimity, and is nondictatorial. However, V is not candidate stable. This is seen, for instance, when voters have the following preferences: aP_1bP_1c, bP_2cP_2a, and cP_3aP_3b. At this preference profile, b will be the outcome with sophisticated voting (1 and 2 vote for b in the first stage, since 1 knows that a would lose in the second stage). However, if c were to exit, then the outcome would be a since a majority prefers a to b. This is a violation of candidate stability, since if c had the preferences of voter 3, then c would prefer to exit. This example easily extends to more voters.

The following theorem shows that such violations of candidate stability exist for every nondictatorial and unanimous voting procedure.

A voting procedure V is *dictatorial* if there exists a voter $i \in \mathscr{V}$ such that $V(\mathscr{C}, P) = \text{top}(\mathscr{C}, P_i)$ and $V(\mathscr{C} \setminus \{a\}, P) = \text{top}(\mathscr{C} \setminus \{a\}, P_i)$ for all $P \in \mathscr{P}^r$ and $a \in \mathscr{C}$.[17]

THEOREM 1: *If* $\mathscr{C} \cap \mathscr{V} = \varnothing$, *and a voting procedure* V *is candidate stable and unanimous, then* V *is dictatorial.*[18]

The proof of Theorem 1 appears in the Appendix.
Theorem 1 says that if candidates cannot vote, then any voting procedure that satisfies unanimity and candidate stability must be dictatorial. The implication is

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[16] For formal definitions of voting by successive elimination and sophisticated voting, see Banks (1985) or Shepsle and Weingast (1984). Dutta, Jackson, and Le Breton (2000) characterize the strategic affects of candidate entry/exit decisions on the outcome of voting by successive elimination.

[17] The definition of dictatorial is unusual in applying only to certain sets of candidates. This is due to Theorem 1 on a voting procedure being dictatorial only holding on the sets $\mathscr{C}$ and $\mathscr{C} \setminus \{a\}$, as those are the only sets that we consider under candidate stability. It is easy to show that if candidate stability is applied on all sets, or under the much stronger condition that it is a dominant strategy for each candidate to enter, then V is dictatorial on all sets. However, those stronger assumptions are not in line with our question of whether strategic candidacy changes the outcomes of a voting procedure away from those with a fixed candidate set.

[18] The converse holds if unanimity is only required to hold when at least $\#\mathscr{C} - 1$ candidates enter.

that *every* nondictatorial and unanimous method by which a society elects a candidate will be open to strategic manipulation on the part of the candidates.

To see some of the logic behind the proof of Theorem 1, note that if such a voting procedure were rationalizable by some social welfare ordering (i.e., represented choices consistent with some social welfare ordering), then one could apply Arrow's (1951) impossibility theorem to deduce the result. However, there are voting procedures that are unanimous and candidate stable, but are *not* rationalizable. Thus, the logic of Arrow's theorem cannot be directly applied. The way in which we prove Theorem 1 is to show that candidate stability and unanimity imply that the voting procedure is rationalizable on very restricted domains. In particular, the voting procedure is rationalizable on a domain of preferences where all voters find the same three alternatives most preferred, and agree with some fixed preference profile on other alternatives.[19] We thus conclude that for each three alternatives that appear at the top of voters' preferences the voting procedure is dictatorial on such a restricted domain. We then tie such domains together through repeated application of candidate stability to show that the same voter must dictate on each such restricted domain where all voters have the same three candidates at the top of their preferences. Finally, we use candidate stability and unanimity again to argue that the same voter must dictate on the rest of the domain.

4. OVERLAP BETWEEN CANDIDATES AND VOTERS

Theorem 1 only covers the case where candidates are not voters. This covers some settings of interest, but there are many settings where candidates are voters and so it is important to understand that case. When there is overlap in candidates and voters, there are nondictatorial voting procedures that are candidate stable and unanimous. However, this class of voting procedures is quite restricted as we shall show. We begin with an example of a nondictatorial voting procedure that is candidate stable and unanimous when candidates can vote.

EXAMPLE 3: Let $\mathscr{C} = \{a, b, c\}$ and $\mathscr{V} = \{a, b, c, 1\}$. Consider a voting procedure described as follows. If only *two* candidates enter, then Voter 1 chooses the winner. If all the candidates enter, then Voter 1 determines an ordering of the

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[19] There are other methods of proof. On the same special restricted domain, we could show that the voting procedure satisfies strong positive association (a.k.a. monotonicity) and then invoke the Muller and Satterthwaite (1977) theorem. John Weymark has shown us another approach: Using Hansson's Theorem (1968) it follows that a candidate stable and unanimous voting procedure satisfies Pareto optimality when at most one candidate stays out. Then using Pareto optimality on this domain, one can invoke a theorem of Grether and Plott (1982) from the nonbinary choice literature to prove Theorem 1. Although the Grether and Plott theorem is not concerned with candidate stability, it uses a revealed preference axiom that is mathematically similar. See Weymark (2001). Phil Reny has pointed out that our conditions suffice to produce a direct proof that mimics the proof of Arrow's theorem (and the Gibbard-Satterthwaite theorem) in Reny (1999).

three candidates $\{a, b, c\}$. Then, the candidates are voted on by successive elimination (as in Example 1), but according to the ordering of the candidates suggested by voter 1. In particular, only the candidates a, b, and c get to vote in the successive elimination procedure. Thus, voter 1 only sets the ordering of the agenda. It is easily checked that if, for instance, candidate a is second most preferred by candidates b and c, then a is a Condorcet winner (considering only candidate preferences) and so regardless of voter 1's ordering of the agenda candidate a will be selected. Thus, a can be selected even when it is 1's worst alternative. Moreover, each candidate's least preferred alternative can be the outcome for some preference profile. It can be checked by direct calculation that this procedure is candidate stable and strongly unanimous (as defined in the Appendix).

Although Example 3 shows that when there is an overlap between voters and candidates it is possible to satisfy candidate stability, it remains an extremely demanding requirement. In the example, the outcome is voter 1's most preferred candidate except when the other three voters all have the same most preferred candidate (subject to their self-preference). More generally, the voting procedures that are candidate stable must exhibit a similar strong imbalance of power among voters. The rest of this section is devoted to making this imbalance of power precise.

Procedures with characteristics similar to the one in Example 3 exist for any number of candidates. For example, consider a procedure where:

(i) if some candidates do not enter, then there is a voter (say i) who is not a candidate who dictates, and

(ii) if all candidates enter, then i selects an ordering over candidates, say $a, b, c, \ldots, k$, and a vote is then taken between a and b. If one voter (other than a) prefers a to b, then a is elected. Otherwise a vote is taken between b and c. If one voter (other than b) prefers b to c, then b is elected.

This procedure is candidate stable.[20] In this sort of voting procedure, although it is not dictatorial, i effectively dictates as long as there is at least one other voter who does not find i's most preferred candidate least preferred. So, such a procedure makes some choices against the unanimous wishes of $m - 2$ voters.

As we now show, all candidate stable voting procedures that are tree implementable must violate such an $m - 2$ unanimity condition. In fact, candidate stability and $m - 2$ unanimity are in conflict for a class of voting procedures that is much larger than tree implementable rules. We now provide formal definitions to state these results.

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[20] This procedure is defined under sincere voting. Under sophisticated voting more complicated examples with similar features can be constructed.

Tree Implementability

For each set of candidates $A \subset \mathscr{C}$ consider any finite extensive game form of perfect information Γ_A with range A. A function $V : 2^{\mathscr{C}} \setminus \{\varnothing\} \times \mathscr{P}^r \to \mathscr{C}$ is *tree implementable* if $V(A, P)$ is the subgame perfect equilibrium[21] outcome of Γ_A given the preference profile P, for every $A \in 2^{\mathscr{C}} \setminus \{\varnothing\}$ and $P \in \mathscr{P}^r$. (See the Appendix for more detailed definitions.)

Note that sophisticated voting on binary trees results in tree implementable voting procedures (see Dutta and Sen (1993)).

Restricted Top Sets

$B \subset \mathscr{C}$ is a *restricted top set* at P_i relative to $A \subset \mathscr{C}$ for $i \in \mathscr{V}$ if $bP_i a$ for each $b \in B$ and $a \in A \setminus (B \cup \{i\})$.

$B \subset \mathscr{C}$ is a *restricted top set* at $P \in \mathscr{P}^r$ relative to $A \subset \mathscr{C}$, if B is a restricted top set at A for every $i \in \mathscr{V}$.

A restricted top set for i is a set B of candidates whom voter i finds preferred to each candidate not in B, excepting the self-preference of i if i is a candidate.

m-2 Unanimity

Let us first state the $m - 2$ unanimity condition informally. The condition requires that if all but at most 2 voters find candidate a most preferred, the remaining 2 voters find a at least third most preferred, and all voters' preferences agree over all remaining candidates (all subject to self-preference restrictions), then a must be elected. This is an appealing condition when there are at least 5 voters, as it then says that when there is an extreme amount of agreement in voters' preferences favoring a given candidate, then that candidate should be chosen.

More formally, $m - 2$ unanimity is defined as follows.

A profile $\bar{P} \in \mathscr{P}^r$ is *uniform* if $a\bar{P}_i b$ implies $a\bar{P}_j b$ for all $\{a, b\} \subset \mathscr{C}$, $i \notin \{a, b\}$, and $j \notin \{a, b\}$.

Let $\mathscr{P}^r_B(\bar{P})$ denote the set of all $P \in \mathscr{P}^r$ such that B is a restricted top set of P relative to $\mathscr{C}$ for every $i \in \mathscr{V}$ and $P|_{\mathscr{C} \setminus B} = \bar{P}|_{\mathscr{C} \setminus B}$, where $\bar{P}$ is uniform.

So, $\mathscr{P}^r_B(\bar{P})$ is the set of preference profiles such that all voters have B as a restricted top set and have identical preferences over the remaining candidates, subject to self-preference.

A voting procedure V is $m - 2$ *unanimous* if $V(\mathscr{C}, P) = a$ whenever $\{a\}$ is a restricted top set of $P \in \mathscr{P}^r_{\{a,b,c\}}(\bar{P})$ relative to $\mathscr{C}$ except for at most 2 voters, where $\bar{P}$ is a uniform profile and $\{a, b, c\} \subset \mathscr{C}$.

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[21] Given the strict preference domain, for such game forms subgame perfect equilibrium, backwards induction, and the iterative elimination of weakly dominated strategies coincide.

THEOREM 2: *No voting procedure is tree implementable, candidate stable, and* $m - 2$ *unanimous.*[22]

Another way to state the above theorem, is to say if V is tree implementable and $m - 2$ unanimous, then it must fail to be candidate stable.

We prove Theorem 2 by proving the following stronger theorem that uses a necessary condition for tree implementability. This weaker condition is satisfied by some voting procedures, such as Borda rule, that are not tree implementable.

Top Pair Monotonicity

A voting procedure is *top pair monotonic* if for any uniform $\bar{P} \in \mathcal{P}^r$, $A \subset \mathscr{C}$ with $\#A \geq \#\mathscr{C} - 1$, $\{a, b\} \subset A$, and $P \in \mathscr{P}^r_{\{a, b\}}(\bar{P})$:

(i) $V(A, P) \in \{a, b\}$, and

(ii) if $P' \in \mathscr{P}^r_{\{a, b\}}(\bar{P})$ is such that $aP_i b \Rightarrow aP_i' b$, then $V(A, P) = a \Rightarrow V(A, P') = a$.

Top pair monotonicity requires that if there exists a restricted top set of two candidates, and preferences are uniform outside of this restricted top set, then a voting procedure must choose from that restricted top set. It also requires that if we start at a preference profile as described above and only change preferences so that we improve the standing of the elected candidate (and make no other changes in the orderings), then the same candidate is elected.

Top pair monotonicity is a necessary condition for tree implementability, as we show in the Appendix. It is also satisfied by some prominent voting procedures that are not tree implementable such as Borda rule and plurality rule.

THEOREM 3: *No voting procedure is top pair monotonic, candidate stable, and* $m - 2$ *unanimous.*

As most prominent nondictatorial voting procedures are top pair monotonic and $m - 2$ unanimous,[23] the implication of Theorem 3 is that there is a large class of reasonable voting procedures that fail to satisfy candidate stability, regardless of the overlap between candidates and voters.

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[22] Somewhat surprisingly, if the definition of $m - 2$ unanimity is weakened to apply only with $P \in \mathscr{P}^r_{\{a, b\}}(\bar{P})$ rather than when $P \in P^r_{\{a, b, c\}}(\bar{P})$, then Theorems 2 and 3 no longer hold. However, the counterexamples are quite specific and the theorems hold when $m - 2$ is replaced by requiring that $m - 3$ unanimity hold for $P \in P^r_{\{a, b\}}(\bar{P})$.

[23] The reason that we say most procedures, rather than all, is that some scoring rules fail to satisfy $m - 2$ unanimity for certain cases of overlap between candidates and voters. As a simple example, if all candidates are voters and there are two additional voters who are not candidates, then sincere plurality voting fails $m - 2$ unanimity. In that case, candidates vote for themselves and the remaining two voters decide the election (with ties broken in some fixed manner). Even if some candidate is the restricted favorite of all candidates and second most preferred by the remaining two voters, another candidate who is the favorite of the two remaining voters can be elected. With a small number of voters ($8 \geq m$, but not for more) one can find examples where Borda scoring also fails to satisfy $m - 2$ unanimity. However, in these cases one can directly check that candidate stability is violated.

5. CONCLUDING REMARKS

Theorems 1, 2, and 3 suggest that strategic candidacy will be important in determining the outcomes of reasonable voting procedures. Our analysis here is thus a step in a larger analysis. We conclude this paper with a comment on the results contained here and observations that look forward to a more detailed analysis of strategic candidacy.

Entry Equilibria and Candidate Stability

We have emphasized that a main implication of the results here is that considering strategic candidacy will change the characteristics of all (nondictatorial) voting procedures, because all such procedures are not candidate stable.

There is one potential loophole in this argument that we now close. One could imagine that a voting procedure could fail to be candidate stable, and yet have all the Nash equilibria in the associated candidate entry game still result in the same candidate being elected as if all candidates entered.

Say that $A \subset \mathscr{C}$ is an *entry equilibrium* relative to V and $P \in \mathscr{P}^r$ if $V(A, P) R_a V(A \setminus \{a\}, P)$ for all $a \in A$ and $V(A, P) R_a V(A \cup \{a\}, P)$ for all $a \in \mathscr{C} \setminus A$.

An entry equilibrium is a Nash equilibrium of the game where candidates decide simultaneously whether to enter the election.

V is *entry equilibrium stable* if for all $P \in \mathscr{P}^r$ (i) there exists an entry equilibrium A such that $V(\mathscr{C}, P) = V(A, P)$ and (ii) $V(\mathscr{C}, P) = V(A, P)$ for every entry equilibrium A.

Entry equilibrium stability says that the predictions of the outcomes of a voting procedure are the same if one simply assumes that all candidates enter as if one carefully analyzes the set of candidates who choose to enter.

As we now show for the case of $\mathscr{C} \cap \mathscr{V} = \varnothing$, every nondictatorial and unanimous voting procedure must violate entry equilibrium stability.

LEMMA 1: *If $\mathscr{C} \cap \mathscr{V} = \varnothing$, then if a voting procedure V is entry equilibrium stable, it is candidate stable.*

The proof of Lemma 1 is a slight variation on the proof of Lemma 2 in the Appendix, and is therefore omitted.

Lemma 1 and our Theorem 1 imply that if $\mathscr{C} \cap \mathscr{V} = \varnothing$, and V is entry equilibrium stable and unanimous, then V is dictatorial.[24] So, if candidate stability is violated, then entry equilibrium stability will be violated and any

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[24] In fact, one can prove that entry equilibrium stability implies that $V(\mathscr{C}, P) = V(A, P)$ for all $P \in \mathscr{P}^r$, $A \subset \mathscr{C}$ such that $V(\mathscr{C}, P) \in A$. This involves an induction proof that builds off of ideas in our proof of Lemma 2 in the Appendix. From this one can use entry equilibrium stability to prove that choice and Pareto axioms are satisfied. Appealing to Arrow's theorem, one can deduce that the same voter must be a dictator over all sets of candidates, instead of just on sets of the form $\mathscr{C}$ and $\mathscr{C} \setminus \{a\}$.

nondictatorial (and unanimous) voting procedure will fail to be entry equilibrium stable. This validates our interpretation that a failure of candidate stability means that the strategic entry equilibrium outcomes will differ from the predictions made without accounting for strategic candidacy.

We remark that both (i) and (ii) are critical to the definition of entry equilibrium stability and the validity of Lemma 1.

EXAMPLE 4: Let $\mathscr{V} = \{1, 2, 3\}$, $\mathscr{C} = \{a, b, c\}$ where the two sets are disjoint. $V(\mathscr{C}, P)$ is the Condorcet winner at P over $\mathscr{C}$ if one exists, and $V(\mathscr{C}, P) = a$ otherwise. For any $A \neq \mathscr{C}$, $V(A, P)$ is the Condorcet winner at P over A.

If some candidate is a Condorcet winner, then it is clear that $\mathscr{C}$ will be an entry equilibrium. If we are at a profile where there is no Condorcet winner and instead there is a cycle, then $\{a, b\}$ will be an entry equilibrium. So (i) of entry equilibrium is satisfied. However, (ii) is not satisfied, as when there is a cycle in the majority relation, $\{a, c\}$ is also an entry equilibrium when $cP_b a$.

In any nondictatorial and unanimous example where (i) is satisfied but (ii) is violated, such as Example 4 above, there are some entry equilibria that result in elected candidates different from that predicted by $V(\mathscr{C}, P)$. So, if a voting procedure violates either (i) or (ii), then an analysis of the properties of the voting procedure that treats $\mathscr{C}$ as fixed will not give a complete understanding of the properties of the voting procedure. This echoes the arguments made in the introduction about the implications of violations of candidate stability. Knowing that a voting procedure satisfies (i) but not (ii) is not reassuring: the voting procedure will be a correspondence that will include outcomes that are not consistent with an analysis that takes $\mathscr{C}$ as fixed, and so an analysis of its properties that ignore the other possible equilibrium outcomes will be incomplete and misleading.

For the case of overlap between candidates and voters, entry equilibrium stability does not imply candidate stability.[25] So, it is an open question what the characterizations of entry equilibrium stable voting procedures look like with an overlap of candidates and voters.[26] In other words, we do not know whether Theorems 2 and 3 have analogs for entry equilibrium stability.

Restricted Domains of Preferences

Strategic candidacy is only an issue at certain preference profiles for any given voting procedure. For instance, if voters unanimously prefer a given candidate a, then the exit of some other candidate b will not affect the outcome of the voting. In fact there are much larger restricted domains of preferences where

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[25] Entry equilibrium stability implies candidate stability when there are three candidates, but not with four or more.

[26] As a first observation on this, Example 5, letting candidate a fill the role of voter 1, shows that voting by successive elimination will not be entry equilibrium stable with overlap.

certain voting procedures are candidate stable. We note one such domain.

Let $\mathscr{P}^C \subset \mathscr{P}^r$ denote the domain of preferences such that for every preference profile there exists a Condorcet winner. That is,

$$\mathscr{P}^C = \{P \in \mathscr{P}^r \mid \exists a \in \mathscr{C} \text{ such that}$$

$$\#\{i \in \mathscr{V} \mid aP_ib\} > \#\mathscr{V}/2 \ \forall b \in \mathscr{C}, \ b \neq a\}.$$

If there is an odd number of voters, then single-peaked preferences are a subset of $\mathscr{P}^C$.

A voting procedure V is *Condorcet consistent* if it selects the Condorcet winner at each $P \in \mathscr{P}^C$.

The following proposition follows directly from noting that if a is a Condorcet winner at P relative to $\mathscr{C}$, then a is a Condorcet winner at P relative to $\mathscr{C}\backslash\{b\}$ for any $b \neq a$.

PROPOSITION 1: *Any Condorcet consistent voting procedure is unanimous and candidate stable on* $\mathscr{P}^C$.

For instance, Proposition 1 implies that with an odd number of voters on a domain of single-peaked preferences, median voting will be candidate stable.

Multi-Valued Voting Procedures

In our analysis, we have only considered voting procedures that are single valued. There are contexts of interest, however, where it is natural to consider voting procedures that have multiple outcomes. For example, plurality rule can result in multiple equilibria (e.g., see Besley and Coate (1997)). To fully study strategic candidacy with set-valued voting procedures, one needs to be careful in dealing with the source of the multiplicity.[27] What does it mean for a voting procedure to be set-valued? Are several candidates being elected? Are there multiple equilibria arising from strategic voting with each candidate in the set corresponding to at least one equilibrium? Are there ties in the voting with the outcome set representing the tied candidates? Each of these views of multiplicity can lead to a different way in which a candidate views the impact of entering or exiting an election.

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[27] Hülya Eraslan pointed out to us that Theorem 1 extends to multi-valued voting procedures with minor changes to the proof under the following multi-valued variation of candidate stability. For all $a \in \mathscr{C}$ and $P \in \mathscr{P}^r$: (1) If $a \notin V(\mathscr{C}, P)$, then $V(\mathscr{C}, P) = V(\mathscr{C}\backslash\{a\}, P)$, and (2) $V(\mathscr{C}, P) \subset V(\mathscr{C}\backslash\{a\}, P)$ $\cup \{a\}$ for each $a \in \mathscr{C}$. (1) is a multi-valued analog of the strong candidate stability condition that we discuss in the Appendix, and arguments for it can be made in some circumstances. However, (2) is less clearly an implication of considering candidate entry or exit decisions. This is an important question for future exploration, as there are some standard voting procedures (such as the top cycle correspondence) that satisfy (1) but not (2). See Eraslan and McLemman (2001) and also Rodriguez-Alvarez (2001).

Inefficiency due to Strategic Candidacy

The following example from Dutta, Jackson, and Le Breton (2000) shows that there are voting procedures for which strategic candidacy can result in inefficiency. More specifically, the candidate elected under sophisticated voting by successive elimination once strategic candidacy is accounted for is Pareto dominated by the candidate elected if all candidates are forced to enter, considering the preferences of all voters and candidates (except, of course, the preference of the strategic equilibrium candidate himself!).[28] As voting by successive elimination (under sophisticated voting with a fixed set of candidates) has been extolled for its efficiency properties, this is a disturbing aspect of strategic candidacy. This suggests that in future research we ask whether there exist (nondictatorial) voting procedures for which all entry equilibria are Pareto efficient.

EXAMPLE 5:[29] Let $\mathscr{V} = \{1, 2, 3\}$, $\mathscr{C} = \{a, b, c, d, e\}$ where the two sets are disjoint. The voting procedure is again voting by successive elimination. A vote is first taken to eliminate e or d. The winner (according to majority rule) then "fights" against c, the winner here against b, and finally the survivor against a. The voting equilibrium is *sophisticated voting*, due to Farquharson (1969), which in this case coincides with the iterated elimination of weakly dominated strategies.

Voters' preferences are:

$$aP_1cP_1eP_1bP_1d,$$

$$bP_2dP_2aP_2cP_2e,$$

$$eP_3bP_3dP_3cP_3a.$$

Let $xM(P)y$ denote that at least two voters prefer x to y. Then,

$$aM(P)c, aM(P)e,$$

$$bM(P)a, bM(P)c, bM(P)d,$$

$$cM(P)e,$$

$$dM(P)a, dM(P)c,$$

$$eM(P)b, eM(P)d.$$

The Shepsle-Weingast (1984) algorithm can be used to derive the outcome for each set of entering candidates. When all candidates enter, the outcome is b. When only $\{c, d\}$, $\{c, d, e\}$, or $\{a, c, d, e\}$ enter, the outcome is d. If $\{d, e\}$ or $\{b, c, d, e\}$ enter, the outcome is e.

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[28] This example is a strengthening of a previous example, and we thank Eric Maskin for suggesting that we look for an example where the strategic outcome was not only Pareto dominated, but was Pareto dominated by the outcome that would have occurred if all candidates entered.

[29] There are simpler examples with similar features. However, those examples are ad hoc while voting by successive elimination is extensively used and well-studied.

Let us consider what happens if we account for strategic candidacy. Consider the following preferences of candidates. Each candidate ranks herself first. All candidates other than d rank b higher than d. Also, candidates c and b prefer d to e. Given these preferences, it is an equilibrium (anticipating the subsequent voting) for exactly the set $\{c, d, e\}$ to enter. This results in the outcome of d. However, all voters and candidates (other than d) prefer b to d. Thus, the outcome under this entry equilibrium is Pareto dominated (except for d's self-preference) by the outcome that would occur if all candidates entered. It is not an equilibrium for all candidates to enter if $eP_a b$.

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APPENDIX

We prove Theorem 1, by proving a stronger result that includes the case of any overlap between $\mathscr{V}$ and $\mathscr{C}$. This result requires a stronger candidate stability and unanimity condition to handle the overlap, but these conditions simplify to candidate stability and unanimity in the case of no overlap, as shown in Lemma 2, below. The stronger statement is useful in the proof.

V satisfies *strong unanimity* if for all $B \subset \mathscr{C}$ and $P \in \mathscr{P}^r$, if $\{b\}$ is a restricted top set of B, then $V(B, P) = b$.

This condition coincides with unanimity when $\mathscr{C} \cap \mathscr{V} = \varnothing$. More generally, it is a unanimity condition that ignores candidates' preferences for themselves.

V is *strongly candidate stable* if for each $a \in \mathscr{C}$ and $P \in \mathscr{P}^r$ such that $a \ne V(\mathscr{C}, P)$, $V(\mathscr{C}, P) = V(\mathscr{C} \setminus \{a\}, P)$.

This is a strengthening of candidate stability to require that the outcome not change at all if a candidate exits. However, this condition is implied by candidate stability in the case where $\mathscr{C} \cap \mathscr{V} = \varnothing$, and thus under the premise of Theorem 1.

LEMMA 2: *If $\mathscr{C} \cap \mathscr{V} = \varnothing$ and a voting procedure V is candidate stable and unanimous, then V is strongly candidate stable and strongly unanimous.*

PROOF OF LEMMA 2: The satisfaction of strong unanimity is obvious, so we show that strong candidate stability is satisfied. Suppose V is candidate stable. Consider $a \in \mathscr{C}$ and $P \in \mathscr{P}^r$ such that $a \ne V(\mathscr{C}, P)$. We need to show that $V(\mathscr{C}, P) = V(\mathscr{C} \setminus \{a\}, P)$. Suppose to the contrary that $V(\mathscr{C}, P) \ne V(\mathscr{C} \setminus \{a\}, P)$. Since under candidate stability $V(\mathscr{C}, P) R_a V(\mathscr{C} \setminus \{a\}, P)$, it must be that $V(\mathscr{C}, P) P_a V(\mathscr{C} \setminus \{a\}, P)$. Since $V(\mathscr{C}, P) \ne a$, we can find P_a' such that $(P_{-a}, P_a') \in \mathscr{P}^r$ and $V(\mathscr{C} \setminus \{a\}, P) P_a' V(\mathscr{C}, P)$. Since V depends only on voters' preferences, it follows that $V(\mathscr{C} \setminus \{a\}, P_{-a}, P_a') P_a' V(\mathscr{C}, P_{-a}, P_a')$. This contradicts the fact that V is candidate stable, and so our supposition was incorrect.   Q.E.D.

Theorem 1 then follows from the following theorem.

THEOREM 4: *On the domain* $\mathscr{P}^r$, *if a voting procedure is strongly candidate stable and satisfies strong unanimity, then it is dictatorial and the dictator is in* $\mathscr{V} \setminus \mathscr{C}$.

Note that Theorem 4 explicitly states that the dictator must be in $\mathscr{V} \setminus \mathscr{C}$. There is an obvious explanation for this requirement. If an individual $i \in \mathscr{V} \cap \mathscr{C}$ was the dictator, then i would always be the chosen outcome given that i prefers herself to all other candidates. But this would violate strong unanimity. Hence, one implication of the theorem is that if $\mathscr{V} = \mathscr{C}$, then there is no voting procedure that can satisfy the stated conditions.

The following definitions are useful in the proof of Theorem 4.

V satisfies the *choice axiom* if for any $P \in \mathscr{P}^r$ and $A, B \subset \mathscr{C}$, if $B \subset A$ and $V(A, P) \in B$, then $V(A, P) = V(B, P)$.

This is a single-valued version of choice axioms in Chernoff (1954) and Arrow (1959) and is also equivalent to what Nash (1950) called Independence of Irrelevant Alternatives.

Consider any three distinct candidates $a, b, c \in \mathscr{C}$ and any preference profile $\bar{P} \in \mathscr{P}^r$. Let $\mathscr{P}_{\{a,b,c\}}(\bar{P})$ be the set of $P \in \mathscr{P}^r$ such that for each $i \in \mathscr{V}$:

  (i) $P_i|_{\mathscr{C} \setminus \{a,b,c\}} = \bar{P}_i|_{\mathscr{C} \setminus \{a,b,c\}}$, and
  (ii) $\{a, b, c\}$ is a restricted top set for P.

PROOF OF THEOREM 4: We establish the theorem from the following sequence of lemmas.

LEMMA 3: *Consider any three distinct candidates* $a, b, c \in \mathscr{C}$ *and* $\bar{P} \in \mathscr{P}^r$. *If* V *satisfies strong candidate stability and strong unanimity, then* $V(\mathscr{C}, P) \in \{a, b, c\}$ *for any* $P \in \mathscr{P}_{\{a,b,c\}}(\bar{P})$.

PROOF OF LEMMA 3: Pick any $\{a, b, c\} \subset \mathscr{C}$ and $\bar{P} \in \mathscr{P}^r$. Consider $P' \in \mathscr{P}_{\{a,b,c\}}(\bar{P})$ such that $\{a, b\}$ is a restricted top set. Either $V(\mathscr{C}, P') \neq b$ or $V(\mathscr{C}, P') \neq a$. Without loss of generality, suppose that $V(\mathscr{C}, P') \neq b$. By strong candidate stability $V(\mathscr{C} \setminus \{b\}, P') = V(\mathscr{C}, P')$. By strong unanimity $V(\mathscr{C} \setminus \{b\}, P') = a$. Combining these two equalities implies that $V(\mathscr{C}, P') = a$. Since the choice of a and b was arbitrary, it follows that if $\{d, e\} \subset \{a, b, c\}$ is a restricted top set of $P' \in \mathscr{P}_{\{a,b,c\}}(\bar{P})$, then $V(\mathscr{C}, P') \in \{d, e\}$.

So, consider any $P \in \mathscr{P}_{\{a,b,c\}}(\bar{P})$. Suppose to the contrary of the lemma that $V(\mathscr{C}, P) = f \notin \{a, b, c\}$. By strong candidate stability it follows that $V(\mathscr{C} \setminus \{c\}, P) = f$. Consider $P' \in \mathscr{P}_{\{a,b,c\}}(\bar{P})$ such that $P'|_{\mathscr{C} \setminus \{c\}} = P|_{\mathscr{C} \setminus \{c\}}$, and $\{a, b\}$ is a restricted top set for P' (so we have found P' from P by moving c to third position in each preference ranking and leaving the other relative rankings unchanged). It follows from (iii) in the definition of voting procedure that $V(\mathscr{C} \setminus \{c\}, P') = f$. From our previous argument we also know that $V(\mathscr{C}, P') \in \{a, b\}$. This contradicts strong candidate stability, since $f \notin \{a, b\}$. Thus, our supposition was wrong.   Q.E.D.

Let $\mathscr{P}^{r\mathscr{V}}_{\{a,b,c\}}$ denote the set of profiles of preferences of voters restricted to the set $\{a, b, c\}$, where preference profiles are required to be in $\mathscr{P}^r$.

LEMMA 4: *Fix any* $\{a, b, c\} \subset \mathscr{C}$. *Suppose* $\hat{V}: 2^{\{a,b,c\}} \setminus \{\varnothing\} \times \mathscr{P}^{r\mathscr{V}}_{\{a,b,c\}} \to \{a, b, c\}$ *is a voting procedure as a function of preference profiles in* $\mathscr{P}^{r\mathscr{V}}_{\{a,b,c\}}$ *and selecting from* $\{a, b, c\}$. *If* $\hat{V}$ *satisfies strong unanimity and is such that* $\hat{V}(\cdot, P)$ *is rationalizable by a linear order on the set* $\{a, b, c\}$ *for each* $P \in \mathscr{P}^{r\mathscr{V}}_{\{a,b,c\}}$, *then* $\hat{V}$ *is dictatorial on* $\mathscr{P}^{r\mathscr{V}}_{\{a,b,c\}}$ *with dictator* $i \notin \{a, b, c\}$.

PROOF OF LEMMA 4: Consider $P' \in \mathscr{P}^{r\mathscr{V}}_{\{a,b,c\}}$ such that $aP'_a bP'_a c$, $bP'_b cP'_b a$, and $cP'_c aP'_c b$. It follows from strong unanimity that if $aP_i bP_i c$ for all $i \notin \{a, b, c\}$, then $\hat{V}(\{a, b\}, P_{-\{a,b,c\}}, P'_{\{a,b,c\}}) = a$ and $\hat{V}(\{b, c\}, P_{-\{a,b,c\}}, P'_{\{a,b,c\}}) = b$. So, by rationalizability, $\hat{V}(\{a, b, c\}, P_{-\{a,b,c\}}, P'_{\{a,b,c\}}) = a$. Similarly, setting $bP_i cP_i a$ for all $i \notin \{a, b, c\}$, leads to $\hat{V}(\{a, b, c\}, P_{-\{a,b,c\}}, P'_{\{a,b,c\}}) = b$, and setting $cP_i aP_i b$ for all $i \notin \{a, b, c\}$, leads to $\hat{V}(\{a, b, c\}, P_{-\{a,b,c\}}, P'_{\{a,b,c\}}) = c$.

Thus, viewing $\hat{V}(\{a,b,c\},\cdot,P'_{\{a,b,c\}})$ as a function of $P_{-\{a,b,c\}}$, it is nonimposed, rationalizable, and satisfies Arrow's independence axiom by (iii) in the definition of voting procedure. So, by Wilson's (1972) Theorem,[30] there exists $i \notin \{a,b,c\}$ such that either $\hat{V}(A,P_{-\{a,b,c\}},P'_{\{a,b,c\}}) = \text{top}(A,P_i)$ for each $A \subset \{a,b,c\}$ and $P_{-\{a,b,c\}}$, or $\hat{V}(A,P_{-\{a,b,c\}},P'_{\{a,b,c\}}) = \text{bottom}(A,P_i)$ for each $A \subset \{a,b,c\}$ and $P_{-\{a,b,c\}}$ (where bottom(A,P_i) is the lowest ranked candidate in A under P_i). Since, as we argued above, it follows from strong unanimity that if aP_ibP_ic for all $i \notin \{a,b,c\}$, then $\hat{V}(\{a,b\},P_{-\{a,b,c\}},P'_{\{a,b,c\}}) = a$, it must be that $\hat{V}(A,P_{-\{a,b,c\}},P'_{\{a,b,c\}}) = \text{top}(A,P_i)$ for each $A \subset \{a,b,c\}$ and $P_{-\{a,b,c\}}$.

Thus, i is a dictator at profiles $P \in \mathscr{P}^{r\mathscr{V}}_{\{a,b,c\}}$ where $P_{\{a,b,c\}} = P'_{\{a,b,c\}}$. Next, we show that i is a dictator for every $P \in \mathscr{P}^{r\mathscr{V}}_{\{a,b,c\}}$. We do this by showing that if i is a dictator on profiles $P \in \mathscr{P}^{r\mathscr{V}}_{\{a,b,c\}}$ such that $P_{\{a,b,c\}} = P''_{\{a,b,c\}}$ (for some arbitrary P''), and $P'''_{\{a,b,c\}}$ differs from $P''_{\{a,b,c\}}$ for only one member of $\{a,b,c\}$, then i is a dictator on profiles $P \in \mathscr{P}^{r\mathscr{V}}_{\{a,b,c\}}$ such that $P_{\{a,b,c\}} = P'''_{\{a,b,c\}}$. Since we may pass from $P'_{\{a,b,c\}}$, as defined above, to any other restricted profile for $\{a,b,c\}$ through a sequence of such one-at-a-time changes, this suffices to prove the result.

By the symmetry of the problem there is no loss of generality in assuming that $P''_b = P'''_b$ and $P''_c = P'''_c$, while $aP''_a bP''_a c$ and $aP'''_a cP'''_a b$. By (iii) in the definition of voting procedure, $\hat{V}(\{a,b\},P) = \text{top}(\{a,b\},P_i)$ and $\hat{V}(\{a,c\},P) = \text{top}(\{a,c\},P_i)$ for every $P \in \mathscr{P}^{r\mathscr{V}}_{\{a,b,c\}}$ such that $P_{\{a,b,c\}} = P'''_{\{a,b,c\}}$. Rationalizability then implies that $\hat{V}(\{b,c\},P) = \text{top}(\{b,c\},P_i)$ for every $P \in \mathscr{P}^{r\mathscr{V}}_{\{a,b,c\}}$ such that $P_{\{a,b,c\}} = P'''_{\{a,b,c\}}$ and either bP_iaP_ic or cP_iaP_ib. Then, (iii) in the definition of voting procedure implies that $\hat{V}(\{b,c\},P) = \text{top}(\{b,c\},P_i)$ for every $P \in \mathscr{P}^{r\mathscr{V}}_{\{a,b,c\}}$ such that $P_{\{a,b,c\}} = P'''_{\{a,b,c\}}$, as i's ranking of a is irrelevant. Since i dictates on all pairs of candidates, rationalizability implies that also $\hat{V}(\{a,b,c\},P) = \text{top}(\{a,b,c\},P_i)$ for every $P \in \mathscr{P}^{r\mathscr{V}}_{\{a,b,c\}}$ such that $P_{\{a,b,c\}} = P'''_{\{a,b,c\}}$.   Q.E.D.

LEMMA 5: *Consider any three distinct candidates $\{a,b,c\} \subset \mathscr{C}$ and $\bar{P} \in \mathscr{P}^r$. If V satisfies strong candidate stability and strong unanimity, then V is dictatorial on $\mathscr{P}_{\{a,b,c\}}(\bar{P})$.*[31]

PROOF OF LEMMA 5: Consider any three distinct candidates $a,b,c \in \mathscr{C}$ and $\bar{P} \in \mathscr{P}$.

First, note that strong unanimity implies that $V(\mathscr{C}\backslash\{a,b\},P) = c$, $V(\mathscr{C}\backslash\{a,b\},P) = b$, and $V(\mathscr{C}\backslash\{b,c\},P) = a$, for any $P \in \mathscr{P}_{\{a,b,c\}}(\bar{P})$. Next, it follows from Lemma 3 that $V(\mathscr{C},P) \in \{a,b,c\}$ for all $P \in \mathscr{P}_{\{a,b,c\}}(\bar{P})$. Strong candidate stability implies that $V(\mathscr{C},P) = a \Rightarrow V(\mathscr{C}\backslash\{b\},P) = V(\mathscr{C}\backslash\{c\},P) = a$, and likewise that $V(\mathscr{C},P) = b \Rightarrow V(\mathscr{C}\backslash\{a\},P) = V(\mathscr{C}\backslash\{c\},P) = b$, and $V(\mathscr{C},P) = c \Rightarrow V(\mathscr{C}\backslash\{a\},P) = V(\mathscr{C}\backslash\{b\},P) = c$. Thus, it follows that $V(\mathscr{C}\backslash\{d\},P) \in \{a,b,c\}$ for all $d \in \{a,b,c\}$ and $P \in \mathscr{P}_{\{a,b,c\}}(\bar{P})$.

Define $\hat{V}: 2^{\{a,b,c\}}\backslash\{\varnothing\} \times \mathscr{P}^{r\mathscr{V}}_{\{a,b,c\}} \to \{a,b,c\}$, by

$$\hat{V}(B,P|^{\mathscr{V}}_{\{a,b,c\}}) = V(B \cup (\mathscr{C}\backslash\{a,b,c\}),P)$$

for each $B \subseteq \{a,b,c\}$ ($B \neq \varnothing$), and $P \in \mathscr{P}_{\{a,b,c\}}(\bar{P})$, where $P|^{\mathscr{V}}_{\{a,b,c\}}$ denotes the profile of preferences for $i \in \mathscr{V}$ restricted to $\{a,b,c\}$, induced by P. This is well defined, since $V(\mathscr{C},P) \in \{a,b,c\}$, $V(\mathscr{C}\backslash\{d\},P) \in \{a,b,c\}$, and $V(\mathscr{C}\backslash\{d,e\},P) \in \{a,b,c\}$ for all $\{d,e\} \subset \{a,b,c\}$ and $P \in \mathscr{P}_{\{a,b,c\}}(\bar{P})$. Moreover, by the arguments above, the choice axiom is satisfied by $\hat{V}$ relative to $\{a,b,c\}$ on $\mathscr{P}^{r\mathscr{V}}_{\{a,b,c\}}$. This implies, by a theorem of Sen (1971), that $\hat{V}(\cdot,P)$ is rationalizable by a linear order (see Moulin (1988, p. 308)) for any P. It then follows from Lemma 4 that $\hat{V}$ is dictatorial on $\mathscr{P}^{r\mathscr{V}}_{\{a,b,c\}}$ with a dictator in $\mathscr{V}\backslash\{a,b,c\}$. Thus, from the definition of $\hat{V}$ it follows that there exists $i \in \mathscr{V}$ such that $V(\mathscr{C},P) = \text{top}(\mathscr{C},P_i)$ and $V(\mathscr{C}\backslash\{d\},P) = \text{top}(\mathscr{C}\backslash\{d\},P_i)$ for all $P \in \mathscr{P}_{\{a,b,c\}}(\bar{P})$ and $d \in \{a,b,c\}$. Then from strong candidate stability it follows that $V(\mathscr{C}\backslash\{d\},P) = \text{top}(\mathscr{C}\backslash\{d\},P_i)$ for all $P \in \mathscr{P}_{\{a,b,c\}}(\bar{P})$ and $d \notin \{a,b,c\}$.   Q.E.D.

[30] Wilson's theorem states that if a social welfare ordering satisfies Arrow's independence axiom, then it is either a dictatorship, anti-dictatorship (choosing the worst candidate of the anti-dictating voter's preference), or is constant. Being nonimposed rules out the possibility of a constant social welfare ordering.

[31] There exists a voter $i \in \mathscr{V}$ such that $V(\mathscr{C},P) = \text{top}(\mathscr{C},P_i)$, $V(\mathscr{C}\backslash\{d\},P) = \text{top}(\mathscr{C}\backslash\{d\},P_i)$, for all $P \in \mathscr{P}_{\{a,b,c\}}(\bar{P})$ and $d \in \mathscr{C}$.

Let

$$\mathscr{P}_{\{a,b,c\}} = \{P \in \mathscr{P}^r \mid \{a,b,c\} \text{ is a restricted top set for } P\}.$$

LEMMA 6: *Consider any three distinct candidates $a, b, c \in \mathscr{C}$. If V satisfies strong candidate stability and strong unanimity, then V is dictatorial on $\mathscr{P}_{\{a,b,c\}}$.*

Lemma 6 is stronger than Lemma 5 in that it applies to a larger set of preferences ($\mathscr{P}_{\{a,b,c\}}$ instead of $\mathscr{P}_{\{a,b,c\}}(\bar{P})$).

PROOF OF LEMMA 6: Consider $\{a,b,c\}$, $\bar{P}$ and $d \notin \{a,b,c\}$. Let $\tilde{P}$ be such that $\bar{P}|_{\mathscr{C}\setminus\{d\}} = \tilde{P}|_{\mathscr{C}\setminus\{d\}}$. By Lemma 5 there exists a dictator $i \in \mathscr{V}\setminus\{a,b,c\}$ on $\mathscr{P}_{\{a,b,c\}}(\bar{P})$. Similarly, there exists a dictator $j \in \mathscr{V}\setminus\{a,b,c\}$ on $\mathscr{P}_{\{a,b,c\}}(\tilde{P})$. To establish the Lemma, we need only show that $i = j$, since $\bar{P}$ and d are arbitrary (and this logic can be applied iteratively).

Suppose to the contrary that $i \neq j$. Consider $P \in \mathscr{P}_{\{a,b,c\}}(\bar{P})$ with $a = \text{top}(\mathscr{C}\setminus\{i\}, P_i)$ and $b = \text{top}(\mathscr{C}\setminus\{j\}, P_j)$, and take any $P' \in \mathscr{P}_{\{a,b,c\}}(\tilde{P})$ such that $P|_{\mathscr{C}\setminus\{d\}} = P'|_{\mathscr{C}\setminus\{d\}}$. It follows that $V(\mathscr{C}, P) = a$ and $V(\mathscr{C}, P') = b$, and so strong candidate stability implies that $V(\mathscr{C}\setminus\{d\}, P) = a$ and $V(\mathscr{C}\setminus\{d\}, P') = b$. However, this contradicts the fact that V satisfies (iii) in the definition of voting procedure, since $P|_{\mathscr{C}\setminus\{d\}} = P'|_{\mathscr{C}\setminus\{d\}}$.   Q.E.D.

LEMMA 7: *If V satisfies strong candidate stability and strong unanimity, then V is dictatorial on $\mathscr{P}_{\{a,b,c\}}$ for every $\{a,b,c\} \subset \mathscr{C}$ (with the same dictator $i \in \mathscr{V}\setminus\mathscr{C}$ on each of these domains).*

Lemma 7 is stronger than Lemma 6, since it implies that the same i is dictator on $\mathscr{P}_{\{a,b,c\}}$ for every $\{a,b,c\} \subset \mathscr{C}$.

PROOF OF LEMMA 7: This follows directly if $\#\mathscr{C} = 3$. So suppose that $\#\mathscr{C} \geq 4$. It is enough to consider any $\{a,b,d\}$ distinct from $\{a,b,c\}$, and show that the same voter dictates on $\mathscr{P}_{\{a,b,c\}}$ and $\mathscr{P}_{\{a,b,d\}}$ (which in turn implies that $i \notin \{c,d\}$ for arbitrary c and d). By Lemma 6, there exists a dictator $j \in \mathscr{V}\setminus\{a,b,d\}$ on $\mathscr{P}_{\{a,b,d\}}$ and a dictator $i \in \mathscr{V}\setminus\{a,b,c\}$ on $\mathscr{P}_{\{a,b,c\}}$. Suppose to the contrary that $i \neq j$. Consider $P \in \mathscr{P}_{\{a,b,c\}}$ with $\{a,b,c,d\}$ a restricted top set for P and with $aP_ibP_icP_id$, $bP_jaP_jcP_jd$, and $aP_kbP_kcP_kd$ for any $k \in \mathscr{V}\setminus\{i,j\}$ (with each of these specifications subject to self-preference). Consider $P' \in \mathscr{P}_{\{a,b,d\}}$ such that $P|_{\mathscr{C}\setminus d} = P'|_{\mathscr{C}\setminus d}$, and $P|_{\mathscr{C}\setminus c} = P'|_{\mathscr{C}\setminus c}$. Thus, we have only reversed the place of c and d in the rankings. It follows that $V(\mathscr{C}, P) = a$ and $V(\mathscr{C}, P') = b$, and so strong candidate stability implies that $V(\mathscr{C}\setminus\{d\}, P) = a$ and $V(\mathscr{C}\setminus\{d\}, P') = b$. However, this contradicts the fact that V satisfies (iii) in the definition of voting procedure, since $P|_{\mathscr{C}\setminus\{d\}} = P'|_{\mathscr{C}\setminus\{d\}}$.   Q.E.D.

We now complete the proof of Theorem 4. Find i from Lemma 7 (noting that $i \notin \mathscr{C}$), and consider any $P \in \mathscr{P}^r$. Without loss of generality suppose that $\{a,b,c\}$ is top relative to P_i and that $\text{top}(\mathscr{C}, P_i) = a$. We need to show that $V(\mathscr{C}, P) = \text{top}(\mathscr{C}, P_i) = a$, and $V(\mathscr{C}\setminus\{d\}, P) = \text{top}(\mathscr{C}\setminus\{d\}, P_i)$, for any $d \in \mathscr{C}$.

Consider P'_{-i} such that $P'_{-i}|_{\{a,b,c\}} = P_{-i}|_{\{a,b,c\}}$, $P'_{-i}|_{\mathscr{C}\setminus\{a,b,c\}} = P_{-i}|_{\mathscr{C}\setminus\{a,b,c\}}$, and $P_i, P'_{-i} \in \mathscr{P}_{\{a,b,c\}}$ (so $\{a,b,c\}$ is a restricted top set for P'_{-i}). It follows from Lemma 7 that $V(\mathscr{C}, P_i, P'_{-i}) = \text{top}(\mathscr{C}, P_i)$ $= a$, and $V(\mathscr{C}\setminus\{d\}, P_i, P'_{-i}) = \text{top}(\mathscr{C}\setminus\{d\}, P_i)$, for any $d \in \mathscr{C}$. Find a voter j and alternatives $f = \text{bottom}(\{a,b,c\}, P_j)$ and $e = \text{top}(\mathscr{C}\setminus\{a,b,c,j\}, P_j)$ such that eP_jf and fP'_je. (Clearly $j \neq i$.) Consider P''_j which agrees with P'_j on $\mathscr{C}\setminus\{e\}$ and $\mathscr{C}\setminus\{f\}$, and agrees with P_j on $\{e,f\}$. (Thus, we have only switched e and f in the ranking of P_j.) Here $V(\mathscr{C}, P_i, P'_{-i}) = \text{top}(\mathscr{C}, P_i) = a = V(\mathscr{C}\setminus\{e\}, P_i, P'_{-i})$. Since P''_j agrees with P'_j on $\mathscr{C}\setminus\{e\}$ it follows from (iii) in the definition of voting procedure that $a = V(\mathscr{C}\setminus\{e\}, P_i, P'_{-i,j}, P''_j)$. Thus, from strong candidate stability it follows that $V(\mathscr{C}, P_i, P'_{-i,j}, P''_j)$ $\in \{a,e\}$. Consider two cases:

Case 1—$f = a$. Since P''_j agrees with P'_j on $\mathscr{C}\setminus\{f\}$, (iii) in the definition of voting procedure implies that $V(\mathscr{C}\setminus\{f\}, P_i, P'_{-i,j}, P''_j) = V(\mathscr{C}\setminus\{f\}, P_i, P'_{-i}) = b$. Then, since V is candidate stable it must be that $a = V(\mathscr{C}, P_i, P'_{-i,j}, P''_j)$. In this case, strong candidate stability also implies that

$a = V(\mathscr{C} \setminus \{d\}, P_i, P'_{-i,j}, P''_j)$, for any $d \neq a$, and since we know that $f = a$, it follows that $V(\mathscr{C} \setminus \{a\}, P_i, P'_{-i,j}, P''_j) = b$. The last two sentences imply that in this case i dictates at $P_i, P'_{-i,j}, P''_j$.

Case 2—$f \neq a$. Since P''_j agrees with P'_j on $\mathscr{C} \setminus \{f\}$, (iii) in the definition of voting procedure implies that $a = V(\mathscr{C} \setminus \{f\}, P_i, P'_{-i,j}, P''_j)$. So, from strong candidate stability it follows that $V(\mathscr{C}, P_i, P'_{-i,j}, P''_j) \in \{a, f\}$. Thus, since we also know that $V(\mathscr{C}, P_i, P'_{-i,j}, P''_j) \in \{a, e\}$, it follows that $V(\mathscr{C}, P_i, P'_{-i,j}, P''_j) = a$. Strong candidate stability then implies that $a = V(\mathscr{C} \setminus \{d\}, P_i, P'_{-i,j}, P''_j)$, for any $d \neq a$. To show that in this case i dictates at $P_i, P'_{-i,j}, P''_j$, it is only left to show that $b = V(\mathscr{C} \setminus \{a\}, P_i, P'_{-i,j}, P''_j)$. Notice that by the same reasoning that we have applied up to this point, we can conclude that $b = V(\mathscr{C} \setminus \{a\}, P'''_i, P'_{-i,j}, P''_j)$, where P'''_i differs from P_i only in switching the ranking of a and b. Thus, it follows from (iii) of the definition of voting procedure that $b = V(\mathscr{C} \setminus \{a\}, P_i, P'_{-i,j}, P''_j)$.

This argument can be repeated, with one such change at each stage for some j between an $f \in \{a, b, c\}$ and $e \notin \{a, b, c\}$, to complete the transition from P'_{-i} to P_{-i}. Q.E.D.

As Theorem 2 is proven using Theorem 3, we provide a proof of Theorem 3 first.

PROOF OF THEOREM 3: The proof used the following lemma.

LEMMA 8: *Consider any three distinct candidates $\{a, b, c\} \subset \mathscr{C}$ and uniform $\bar{P} \in \mathscr{P}^r$. If V is candidate stable and top pair monotonic, then either (i) there exists $\{x, y\} \subset \{a, b, c\}$ such that $V(\mathscr{C} \setminus \{x\}, P) = y$ for all $P \in \mathscr{P}^r_{\{a,b,c\}}(\bar{P})$ or (ii) there exist $i \notin \{a, b, c\}$ such that $V(A, P) = \text{top}(A, P_i)$ for all $P \in \mathscr{P}^r_{\{a,b,c\}}(\bar{P})$ and all $A \subset \mathscr{C}$ with $\#A = \#\mathscr{C} - 1$ and $\mathscr{C} \setminus A \subset \{a, b, c\}$.*

PROOF OF LEMMA 8: Consider any three distinct candidates $a, b, c \in \mathscr{C}$ and uniform $\bar{P} \in \mathscr{P}^r$. We define a binary relation $R(P)$ on $\{a, b, c\}$ for a given $P \in \mathscr{P}^r_{\{a,b,c\}}(\bar{P})$ as follows: $xR(P)y$ if $x = V(\mathscr{C} \setminus \{z\}, P)$ where x, y, and z are generic elements of $\{a, b, c\}$. By (i) of top pair monotonicity, $R(P)$ is complete.

Let us show that $R(P)$ is transitive. Assume to the contrary that there exists $P \in \mathscr{P}^r_{\{a,b,c\}}(\bar{P})$ such that (without loss of generality)

$$a = V(\mathscr{C} \setminus \{c\}, P), \qquad b = V(\mathscr{C} \setminus \{a\}, P), \qquad \text{and}$$

$$c = V(\mathscr{C} \setminus \{b\}, P).$$

Let P' be defined by

$$P'_i = P_i \quad \text{for all} \quad \notin \{a, b, c\},$$

$$aP'_a bP'_a c,$$

$$bP'_b cP'_b a,$$

$$cP'_c aP'_c b,$$

and $P'_j | \mathscr{C} \setminus \{a, b, c\} = P_j | \mathscr{C} \setminus \{a, b, c\}$ for $j \in \{a, b, c\}$. Since V is top pair monotonic it follows that

$$a = V(\mathscr{C} \setminus \{c\}, P'), \quad b = V(\mathscr{C} \setminus \{a\}, P'), \quad \text{and}$$

$$c = V(\mathscr{C} \setminus \{b\}, P').$$

By candidate stability it follows that $V(\mathscr{C}, P') \in \{a, b, c\}$. If $V(\mathscr{C}, P') = a$, then since $V(\mathscr{C} \setminus \{b\}, P') = c$ and $cP'_b a$ we contradict that V is candidate stable. Analogous arguments contradict that $V(\mathscr{C}, P') = b$ or c. Thus, our supposition was incorrect and $R(P)$ is transitive.

Assume that (i) of Lemma 8 does not hold. Then by top pair monotonicity it follows that $R(\cdot)$ is strongly unanimous on $\mathscr{P}^r_{\{a,b,c\}}(\bar{P})$ ($xR(P)y$ whenever $xP_i y$ for all $i \neq y$ and $\{x, y\} \subset \{a, b, c\}$).

By (iii) of the definition of voting procedure, we deduce that $R(P)$ also satisfies Arrow's independence of irrelevant alternatives. Thus, $R(P)$ is transitive, strongly unanimous, and satisfies the independence of irrelevant alternatives. By arguments analogous to those in Lemma 4, we

conclude that $R(\cdot)$ is dictatorial with a dictator $i \notin \{a,b,c\}$. By the definition of $R(P)$, this implies (ii) of Lemma 8.   Q.E.D.

We now complete the proof of Theorem 3. Let $\{a,b,c\} \subset \mathscr{C}$ and consider a uniform $\overline{P} \in \mathscr{P}^r$. From Lemma 8, either (i) there exists $\{x,y\} \subset \{a,b,c\}$ such that $V(\mathscr{C} \setminus \{x\}, P) = y$ for all $P \in \mathscr{P}^r_{\{a,b,c\}}(\overline{P})$ or (ii) there exist $i \notin \{a,b,c\}$ such that $V(A, P) = \mathrm{top}(A, P_i)$ for all $P \in \mathscr{P}^r_{\{a,b,c\}}(\overline{P})$ and all $A \subset \mathscr{C}$ with $\#A = \#\mathscr{C} - 1$ and $\mathscr{C} \setminus A \subset \{a,b,c\}$.

Suppose first that (i) holds and say without loss of generality that $V(\mathscr{C} \setminus \{c\}, P) = a$ for all $P \in \mathscr{P}^r_{\{a,b,c\}}(\overline{P})$. Let $P \in \mathscr{P}^r_{\{a,b,c\}}(\overline{P})$ be as follows:

$$aP_a c P_a b,$$

$$bP_b a P_b c,$$

$$cP_i b P_i a \quad \text{for all} \quad i \in \mathscr{V} \setminus \{a,b\}.$$

Since $aP_b c$ and $V(\mathscr{C} \setminus \{c\}, P) = a$, we deduce from candidate stability that $V(\mathscr{C}, P) \neq c$; however $\{c\}$ is a restricted top set in $\mathscr{C}$ for the coalition $S = \mathscr{V} \setminus \{b\}$, and so we violate $m - 2$ unanimity.

So, suppose instead that (ii) holds and let $P \in P_{\{a,b,c\}}(\overline{P})$ be as follows:

$$aP_a b P_a c,$$

$$aP_i b P_b c,$$

$$cP_j b P_j a \quad \text{for all} \quad j \in \mathscr{V} \setminus \{a,i\} \quad \text{subject to self-preference}.$$

Since $bP_a c$ and $V(\mathscr{C} \setminus \{a\}, P) = b$ we deduce from candidate stability that $V(\mathscr{C}, P) \neq c$; however $\{c\}$ is a restricted top set in $\mathscr{C}$ for the coalition $S = \mathscr{V} \setminus \{a,i\}$, which contradicts $m - 2$ unanimity.

                                                                 Q.E.D.

Before proving Theorem 2 we provide the formal definitions underlying tree implementability.

A *tree* (or finite extensive game form of perfect information) $\Gamma(B) = (\Lambda, <, \Phi, \rho)^{32}$ on $B \subseteq \mathscr{C}$ satisfies the following:

(i) Λ is a finite set of nodes.

(ii) $<$ is a partial order of Λ that has a unique least element λ^0, called the initial node, and a set of maximal elements Λ^T called terminal nodes.

(iii) There is at most one chain between $\lambda, \lambda' \in \Lambda$, where a chain between λ and λ' is a sequence $\lambda < \lambda_1 \cdots < \lambda_k < \lambda'$.

(iv) Φ is a function from Λ^T onto B specifying the candidate who is elected at every terminal node.

(v) ρ is a function from $\Lambda \setminus \Lambda^T$ onto $\mathscr{V}$ specifying the voter who moves at every nonterminal node.

A node λ' is called a *successor* of λ if $\lambda < \lambda'$ and there is no λ'' such that $\lambda < \lambda'' < \lambda'$. The set of successors of λ is denoted $S(\lambda, <)$.

The backwards induction solution of $\Gamma(B)$ at P, denoted by $\pi(\Gamma(B), P)$ is derived as follows.

(i) For each $\lambda \in \Lambda^T$, define

$$z(\lambda, B, P) = \Phi(\lambda).$$

(ii) For each $\lambda \in \Lambda \setminus \Lambda^T$, let $z(\lambda, B, P)$ be the maximal element in the set $\{z(\lambda', B, P) \mid \lambda' \in S(\lambda, <)\}$ according to the ordering $P_{\rho(\lambda)}$.

The backwards induction solution to $\Gamma(B)$ at (B, P) is $\pi(\Gamma(B), P) = z(\lambda_0, B, P)$.

A voting procedure $V(\cdot, \cdot)$ is *tree implementable* if there exists a set of trees $\{\Gamma(B) \mid B \subseteq \mathscr{C}\}$ such that for all $P \in \mathscr{P}^r$, for all $B \subseteq \mathscr{C}$, $V(B, P) = \pi(\Gamma(B), P)$.

The following useful Lemma follows directly from a result on adjacency in Dutta and Sen (1993).

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[32] When there is no opportunity for confusion, we omit the notation indicating the dependence of Λ, Φ, etc. on the set B.

LEMMA 9: *Consider V that is tree implementable and $B \subset A \subset \mathscr{C}$. If $V(A, P) \in B$ for some $P \in \mathscr{P}_B^r(\bar{P})$, then $V(A, P) \in B$ for all $P \in \mathscr{P}_B^r(\bar{P})$.*

PROOF OF THEOREM 2: We show that if V is tree implementable, candidate stable, and $m - 2$ unanimous, then it is top pair monotonic. The theorem then follows from Theorem 3. Lemma 10 establishes (i) of top pair monotonicity and Lemma 11 establishes (ii) of top pair monotonicity.

LEMMA 10: *Suppose a voting procedure $V(\cdot, \cdot)$ is tree implementable, $(m - 2)$ unanimous and candidate stable. Consider any three distinct candidates $\{a, b, c\} \subset \mathscr{C}$ and uniform $\bar{P} \in \mathscr{P}^r$. Then $V(\mathscr{C} \setminus \{a\}, P) \in \{b, c\}$ for all $P \in \mathscr{P}_{\{a,b,c\}}^r(\bar{P})$.*

PROOF OF LEMMA 10: Label the candidates $a_1, a_2, a_3, \ldots$, where $a = a_1$, $b = a_2$, $c = a_3$, and $a_4, a_5, \ldots$ are in the order of rankings under the uniform preference $\bar{P}$ (so $a_k \bar{P}_i a_{k+1}$ for all $i \neq a_{k+1}$ and $k > 3$).

CLAIM: *Let $V(\mathscr{C} \setminus \{a\}, \cdot)$ be tree implementable. If $V(\mathscr{C} \setminus \{a\}, P) \notin \{b, c\}$ for some $P \in \mathscr{P}_{\{a,b,c\}}^r(\bar{P})$, then there exists $k > 3$ such that $V(\mathscr{C} \setminus a, P) = a_k$ for all $P \in \mathscr{P}_{\{a_1, \ldots, a_k\}}^r(\bar{P})$.*

PROOF OF CLAIM: The proof is by induction on the length of the tree.

First, consider V implemented by a tree of length one; that is, a tree with only one nonterminal node λ^0. Let i be the voter choosing at λ^0, i.e., $\rho(\lambda^0)$. If i is a candidate and i is elected at some terminal node (i.e., $i = \Phi(\lambda)$ for some $\lambda \in \Lambda^T$), then by self preference the outcome will be i for all $P \in \mathscr{P}^r$ and so the claim is satisfied. If i is not elected at any terminal node, and either b or c is elected at some terminal node, then it follows that i will select an outcome in $\{b, c\}$ for all $P \in \mathscr{P}_{\{a,b,c\}}^r(\bar{P})$, and so the claim is satisfied. If none of $\{i, b, c\}$ are elected at any terminal node, then let k be the lowest label such that a_k is available at some terminal node. It follows that i will choose a_k for all $P \in \mathscr{P}_{\{a_1, \ldots, a_k\}}^r(\bar{P})$, and so the claim is satisfied.

Next, let us consider the induction step. Suppose that the claim is true for all trees of length $t - 1$ and consider a tree of length t. Let $i = \rho(\lambda^0)$. Each choice that i can make leads to some tree of length no more than $t - 1$. We know that the claim applies for each such tree. Suppose that $i \notin \{b, c\}$ and there is at least one tree in i's choice set such that $a_k = i$ is the outcome for all $P \in \mathscr{P}_{\{a_1, \ldots, a_k\}}^r(\bar{P})$. Then it follows the outcome of the overall tree will be $a_k = i$ for all $P \in \mathscr{P}_{\{a_1, \ldots, a_k\}}^r(\bar{P})$, and so the claim is satisfied. Consider next the case where $i \notin \{b, c\}$ and there is no tree in i's choice set such that $a_k = i$ is the outcome for all $P \in \mathscr{P}_{\{a_1, \ldots, a_k\}}^r(\bar{P})$. If there is a tree in i's choice set such that the outcome is in $\{b, c\}$ for all $P \in \mathscr{P}_{\{a,b,c\}}^r(\bar{P})$, then it follows that i's choice will result in an outcome in $\{b, c\}$ for all $P \in \mathscr{P}_{\{a,b,c\}}^r(\bar{P})$. Consider next the case where $i \notin \{b, c\}$ and neither of the above cases apply. Then for each tree in i's choice set there is some a_k such that a_k is the outcome for all $P \in \mathscr{P}_{\{a_1, \ldots, a_k\}}^r(\bar{P})$. Let k' be the lowest indexed such k. Then it follows that the outcome will be $a_{k'}$ for all $P \in \mathscr{P}_{\{a_1, \ldots, a_{k'}\}}^r(\bar{P})$, and so the claim is satisfied. Finally, consider the case where $i \in \{b, c\}$. If there is a tree in i's choice set such that the outcome is in $\{b, c\}$ for all $P \in \mathscr{P}_{\{a,b,c\}}^r(\bar{P})$, then it follows that i's choice will result in an outcome in $\{b, c\}$ for all $P \in \mathscr{P}_{\{a,b,c\}}^r(\bar{P})$. Otherwise, by the same argument given above there is some $a_{k'}$ such that $a_{k'}$ for all $P \in \mathscr{P}_{\{a_1, \ldots, a_{k'}\}}^r(\bar{P})$, and so the claim is satisfied in this case as well. This completes the proof of the claim. $Q.E.D.$

Now let us complete the proof of Lemma 10. Suppose the contrary of Lemma 10. Then by the claim, there exists $k > 3$ such that $V(\mathscr{C} \setminus a, P) = a_k$ for all $P \in \mathscr{P}_{\{a_1, \ldots, a_k\}}^r(\bar{P})$. Consider a profile $P \in \mathscr{P}_{\{b, a_k\}}^r(\bar{P})$ such that all voters have b ranked first, and a_k ranked second and a ranked third (subject to self-preference), and agree with $\bar{P}$ elsewhere, except for a who has a_k ranked above b instead. By $m - 2$ unanimity it follows that $V(\mathscr{C}, P) = b$. From our supposition we know that $V(\mathscr{C} \setminus a, P) = a_k$. This contradicts candidate stability since $a_k P_a b$. $Q.E.D.$

LEMMA 11: *Let V be a tree implementable, $A \subset \mathscr{C}$ and $P, P' \in \mathscr{P}_{\{a, b\}}^r(\bar{P})$ be such that $aP_i b \Rightarrow aP_i' b$ for all $i \in \mathscr{V}$. If $V(A, P) = a$, then $V(A, P') = a$.*

PROOF OF LEMMA 11: *Let Γ_A denote the tree implementing $V(A, \cdot)$. The proof is by induction on the length of the tree Γ_A. Let $\Gamma_A^1, \Gamma_A^2, \ldots, \Gamma_A^K$ denote the subtrees following each of the possible K moves to the player, say i, attached to the root node.*

Assume that $V(A, P) = a$. Then there exists at least one $k \in \{1, 2, \ldots, K\}$ such that $V_{\Gamma_A^k}(A, P) = a$. Then from the induction hypothesis, we deduce that $V_{\Gamma_A^k}(A, P') = a$. If $V_{\Gamma_A^l}(A, P) = b$, then from Lemma 9, $V_{\Gamma_A^l}(A, P) \in \{a, b\}$. Finally if $V_{\Gamma_A^l}(A, P) \notin \{a, b\}$, then from Lemma 9 we deduce that $V_{\Gamma_A^l}(A, P') = V_{\Gamma_A^l}(A, P)$. This implies that the node player i can still choose a and the menu of choices has not expanded. Then it follows that $V(A, P') = V(A, P) = a$. Q.E.D.

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