Soc Choice Welf (2009) 32:253–273
DOI 10.1007/s00355-008-0322-z

ORIGINAL PAPER

Deprivation, welfare and inequality

Brice Magdalou · Patrick Moyes

Received: 22 July 2006 / Accepted: 4 July 2008 / Published online: 2 August 2008

Abstract We provide a characterization of the generalised satisfaction—in our terminology non-deprivation—quasi-ordering introduced by S.R. Chakravarty (Keio Econ Stud 34:17–32, (1997)) for making welfare comparisons. The non-deprivation quasi-ordering obeys a weaker version of the principle of transfers: welfare improves only for *specific* combinations of progressive transfers, which impose that the same amount be taken from richer individuals and allocated to one arbitrary poorer individual. We identify the extended Gini social welfare functions that are consistent with this principle and we show that the unanimity of value judgements among this class is identical to the ranking of distributions implied by the non-deprivation quasi-ordering. We extend the approach to the measurement of inequality by considering the corresponding relative and absolute ethical inequality indices.

1 Introduction and motivation

There is a widespread agreement in the literature to appeal to the generalised Lorenz dominance criterion for making welfare comparisons across societies starting with

This is a shortened version of Magdalou and Moyes (2008), which contains the details of the proofs as well as an empirical illustration.

B. Magdalou
LAMETA (UMR CNRS 5474), Université Montpellier 1, CNRS,
Avenue de la Mer, Site Richter, C.S. 79606, 34960 Montpellier Cédex 2, France
e-mail: bricemagdalou@lameta.univ-montp1.fr

P. Moyes (✉)
GREThA (UMR CNRS 5113), Université Montesquieu Bordeaux 4, CNRS,
Avenue Léon Duguit, 33608 Pessac, France
e-mail: patrick.moyes@u-bordeaux4.fr

their distributions of income (see Kolm 1969; Shorrocks 1983). Although the generalised Lorenz quasi-ordering does only provide a partial ranking of the distributions under comparison, it nevertheless constitutes a first step in the appraisal of the distribution of well-being that can later on be supplemented by the choice of particular indices in order to resolve the cases of inconclusiveness. Much of the attractiveness of the generalised Lorenz criterion—beyond its simplicity and elegance—stems from its association with the Pigou-Dalton condition. According to the latter any transfer from a richer individual to a poorer one that does not modify their respective positions on the income scale—a so-called progressive transfer—reduces inequality, and as long as total income is left unchanged also increases welfare. The generalised Lorenz quasi-ordering and its variants have been extensively used in practice for making welfare and inequality comparisons with a reasonable degree of success.

Notwithstanding its wide application in theoretical and empirical work, the generalised Lorenz criterion is not the only possibility for passing welfare judgements. There is evidence that the magnitude of the income received by an individual constitutes only part of the relevant information for the assessment of her well-being. Individuals are not living in complete isolation and they may be inclined to evaluate their life circumstances by comparison with the situations of particular reference groups of individuals. The social status of an individual, which we can assimilate in a first round with her position in the social hierarchy, is considered an important dimension of one person's self-assessment of her well-being (see, e.g. Weiss and Fershtman 1998). Similarly feelings such as envy, resentment and satisfaction may contribute to explain the way in which trends in well-being derived from subjective measures differ from those based on standard indicators focusing exclusively on the material dimensions of well-being.[1] In this respect, the notion of individual deprivation may be particularly useful for understanding the way a person's self-evaluation of her well-being departs from standard monetary measures. Indeed, according to Runciman (1966), what matters for the individual's own appraisal of well-being in a given situation is not what she gets, but rather how much she feels deprived as compared to those individuals who, she considers, are treated more favourably than she is.

The concept of deprivation has been echoed in the inequality and welfare literature even though this has been done at the cost of an oversimplification. In the economic literature the level of deprivation experienced by any individual in a given situation is associated with the average difference between her income and the incomes of the individuals richer than her (see, e.g. Yitzhaki 1979; Berrebi and Silber 1985; Hey and Lambert 1980; Ebert and Moyes 2000 among others). If one believes that it is desirable to reduce deprivation, then it is natural to declare that a distribution is better than another distribution if the amount of deprivation experienced by each individual is less in the first situation than in the second. Credit has to be given to Kakwani (1984) for having made this idea precise through the introduction of his [relative] deprivation quasi-ordering. According to the latter, deprivation in

[1] It is an important question from an ethical point of view to decide whether sentiments like envy and self-contentment have to be counted when comparing different social states. It is not the purpose of this paper to address this issue, which raises important and difficult philosophical questions.

a society decreases as the deprivation curve, which indicates the levels of deprivation attained by all the individuals in the population, goes down. Kakwani (1984)'s suggestion has given rise to a considerable literature aimed at refining and extending the original deprivation quasi-ordering. Particularly important for our purpose in this strand of research are the contributions of Chakravarty (1997) and Chateauneuf and Moyes (2006).

One of the aims of Chateauneuf and Moyes (2006) is to uncover the implicit inequality value judgements embedded in Kakwani (1984)'s deprivation quasi-ordering. Because their focus is on inequality rather than on welfare, they restrict attention to income distributions with the same mean. They identify the class of extended Gini social welfare functions that are consistent with the absolute version of Kakwani (1984)'s deprivation quasi-ordering. In addition they propose an alternative to the notion of a progressive transfer, which possesses the property that, if a distribution is ranked above another by the deprivation quasi-ordering, then the former can be derived from the latter by means of such transfers.[2] A serious limitation of this work from a practical point of view is the fixed mean restriction, which prevents welfare comparisons from being made in real world situations where the income distributions typically differ in size. Inspection of Proposition 5.3 in Chateauneuf and Moyes (2006) suggests however, a simple method for deciding whether one distribution is welfare-superior to another distribution. The test would consist in comparing the means and the deprivation curves of the distributions under consideration: the distribution with the higher mean and the lowest deprivation curve will be ranked above the other distribution by all the extended Gini welfare functions they considered. The problem is that this procedure does not permit one to identify *all* the cases where the application of unanimity among the relevant class of extended Gini welfare functions is decisive. A criterion is missing, which would allow one to detect the distributions which are ranked in the same way by all the extended Gini welfare functions.

On the contrary Chakravarty (1997) suggested a modification of the deprivation curve that permits welfare judgements to be made when the distributions under comparison have unequal means. Building on a suggestion by Yitzhaki (1979) in a slightly different framework, Chakravarty (1997)'s proposal consists in taking the complement of a person's deprivation to mean income as a measure of her well-being. The so-called generalised satisfaction of an individual constitutes the relevant information for making welfare comparisons. A distribution is considered better than another distribution if the generalised satisfaction curve, which indicates the levels of satisfaction attained by all the individuals in the population, moves upwards. Chakravarty (1997) considers a class of welfare functions, which has as special cases the extended Gini welfare functions Chateauneuf and Moyes (2006) focused on. Therefore the extent to which both approaches fit together is not clear and this is an issue that needs clarification. Furthermore, the characterisation of the generalised satisfaction quasi-ordering provided by Chakravarty (1997) is not really illuminating.

[2] They also investigate the properties of two alternative inequality quasi-orderings related to the class of extended Gini social welfare functions. These criteria are not relevant for the present paper and we refer the interested reader to the original contribution.

Admittedly it is shown that, if one distribution is ranked above another by the generalised satisfaction quasi-ordering, then the former can be obtained from the latter by means of a so-called *fair transformation*, and conversely. However, it is immediately clear that both statements are just two different ways of saying the same thing, which does not add a lot to our comprehension of the complex equalising process leading to generalised satisfaction dominance.

Our paper aims at integrating the contributions of Chakravarty (1997) and Chateauneuf and Moyes (2006) in a comprehensive and consistent model. We introduce the non-deprivation quasi-ordering and we show that, if one distribution is ranked above another by this criterion, then the dominating distribution can be obtained from the dominated one by successive applications of the equalising transformations considered by Chateauneuf and Moyes (2006) and/or increments, and conversely. Because our non-deprivation quasi-ordering proves to be formally identical to Chakravarty (1997)'s generalised satisfaction quasi-ordering, this result sheds light upon the precise nature of the value judgements contained implicitly in the latter criterion. We also demonstrate that the partial ordering implied by the non-deprivation quasi-ordering coincides with the way all extended Gini social welfare functions—whose weighting functions are non-decreasing and star-shaped—rank the distributions. This result extends to all monotone social welfare functions that attach a positive value to decreases in deprivation, which is actually the case considered by Chakravarty (1997). The non-deprivation—equivalently the generalised satisfaction–quasi-ordering is operationally more efficient than the two-step procedure based on comparisons of mean incomes and deprivation curves, and it has to be substituted for the latter. Previous work does not make clear the relationship between the generalised satisfaction curve and the quantile function. We show that the non-deprivation curve is a modified version of the quantile curve where successive first differences in incomes are given decreasing weights.

We present in Sect. 2 our conceptual framework and we introduce the notions of social welfare functions and inequality indices that will be subsequently used. Section 3 is concerned with the standard approach to welfare and inequality measurement, which builds on the principle of transfers. We recall the standard criteria of rank order dominance, generalised Lorenz dominance, relative and absolute Lorenz dominance and we present without proofs the well-known equivalences between these quasi-orderings, the underlying transformations of the distributions and the corresponding classes of welfare and inequality measures. Our main contribution is contained in Sect. 4, where we introduce the welfare and inequality dominance criteria that build upon the extent to which every individual feels deprived. We weaken the principle of transfers by imposing restrictions on the way the progressive transfers are combined. Next we introduce the non-deprivation quasi-ordering and we show that it is equivalent to the unanimous ranking generated by all the extended Gini social welfare functions that are consistent with our restricted principle of transfers. Appropriate normalisations of the distributions allow us to derive the corresponding relative and absolute inequality quasi-orderings. We therefore devote Sect. 5 to a discussion and clarification of the relationships between our contribution and related work in the literature. Finally we summarise our results in Sect. 6, which also hints at some directions for future research.

2 Notation and preliminary definitions

We assume throughout that incomes are drawn from an interval D of $\mathbb{R}$. An *income distribution*—or equivalently, a *situation*—for a population consisting of n identical individuals ($n \geq 2$) is a list $\mathbf{x} := (x_1, x_2, \ldots, x_n)$ where $x_i \in D$ is the income of individual i. We suppose that incomes are arranged non-decreasingly and we use $\mathscr{Y}_n(D)$ to represent the *set of income distributions*. The arithmetic *mean* of distribution $\mathbf{x} \in \mathscr{Y}_n(D)$ is indicated by $\mu(\mathbf{x}) := \sum_{i=1}^{n} x_i/n$. We denote as $F(\cdot\,; \mathbf{x})$ the *cumulative distribution function* of $\mathbf{x} \in \mathscr{Y}_n(D)$ and we let $F^{-1}(\cdot\,; \mathbf{x})$ represent the *quantile function* of $\mathbf{x}$ (see Gastwirth 1971). A *social welfare function* is a continuous mapping $W: \mathscr{Y}_n(D) \longrightarrow \mathbb{R}$ such that $W(\mathbf{x})$ measures the welfare of society in situation $\mathbf{x} \in \mathscr{Y}_n(D)$ and we indicate by $\mathscr{W}$ the set of such functions. Similarly, an *inequality index* is a continuous mapping $I: \mathscr{Y}_n(D) \longrightarrow \mathbb{R}$ such that $I(\mathbf{x})$ represents the degree of inequality in situation $\mathbf{x} \in \mathscr{Y}_n(D)$ and we denote as $\mathscr{I}$ the set of inequality indices. We restrict attention to ethical inequality indices, namely the indices that are derived from a social welfare function (see Blackorby et al. 1999 for a survey of the literature). The *relative inequality index* is defined by $I^R(\mathbf{x}) := 1 - \Xi(\mathbf{x})/\mu(\mathbf{x})$ and the *absolute inequality index* by $I^A(\mathbf{x}) := \mu(\mathbf{x}) - \Xi(\mathbf{x})$, where $\Xi(\mathbf{x})$ represents the *equally distributed equivalent income* of situation $\mathbf{x}$.

The literature has mainly focused on two general families of social welfare functions up to now. The *utilitarian approach* assumes that social welfare is simply the sum of the utilities achieved by the individuals and it is defined by

$$W_u(\mathbf{x}) := \frac{1}{n}\sum_{i=1}^{n} u\,(x_i)\,, \quad \forall\, \mathbf{x} \in \mathscr{Y}_n(D), \tag{2.1}$$

where $u \in \mathscr{U} := \{u : D \longrightarrow \mathbb{R} \mid u \text{ is continuous}\}$ is the *utility function* defined up to an increasing and affine transformation. According to the second approach—the *extended Gini* model—social welfare is given by

$$W_f(\mathbf{x}) := \sum_{i=1}^{n}\left[f\left(\frac{n-i+1}{n}\right) - f\left(\frac{n-i}{n}\right)\right] x_i, \quad \forall\, \mathbf{x} \in \mathscr{Y}_n(D), \tag{2.2}$$

where $f \in \mathscr{F} := \{f : [0,1] \longrightarrow [0,1] \mid f \text{ is continuous}, f(0) = 0 \text{ and } f(1) = 1\}$ is the *weighting function* (see Weymark 1981; Yaari 1987, 1988; Ebert 1988). The utility function and the weighting function capture the preferences of the ethical observer within the utilitarian and extended Gini models, respectively. The two former models are actually special cases of the *rank-dependent expected utility* model popularised by Quiggin (1993).[3] We indicate respectively by I_u^R and I_u^A the relative and absolute inequality indices derived from the utilitarian social welfare function (2.1) when the ethical observer's preferences are captured by the utility function u. Similarly, the relative and absolute inequality indices corresponding to the extended Gini social welfare function (2.2) are denoted by I_f^R and I_f^A, respectively.

[3] Actually a characterisation of the so-called rank-dependent expected utility model is provided in Ebert (1988), where emphasis is on inequality measurement rather than on choice under uncertainty.

3 Lorenz consistent welfare and inequality measures

We briefly expose the standard theory of welfare and inequality measurement that constitutes the benchmark of our approach. Given two income distributions $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$, we will say that $\mathbf{x}$ is obtained from $\mathbf{y}$ by means of an *increment* if there exist an income amount $\Delta > 0$ and an individual i such that $x_i = y_i + \Delta$ and $x_h = y_h$, for all $h \neq i$. It is usually considered that an increase in someone's income other things equal results in a social welfare improvement, hence the following condition:

MONOTONICITY [MON]. For all $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$, we have $W(\mathbf{x}) \geq W(\mathbf{y})$ whenever $\mathbf{x}$ is obtained from $\mathbf{y}$ by means of an increment.

The non-decreasingness of u and f guarantees that the utilitarian and the extended Gini social welfare functions satisfy MON. Given two income distributions $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$, we will say that $\mathbf{x}$ *rank order dominates* $\mathbf{y}$, which we write $\mathbf{x} \geq_{RO} \mathbf{y}$, if and only if $RO(k/n; \mathbf{x}) \geq RO(k/n; \mathbf{y})$, for all $k = 1, 2, \ldots, n$, where $RO(k/n; \mathbf{x}) := F^{-1}(k/n; \mathbf{x})$, for all $k = 1, 2, \ldots, n$. A distribution $\mathbf{x}$ will be ranked above another distribution $\mathbf{y}$ by all monotone social welfare functions W—and in particular by all utilitarian and extended Gini social welfare functions with respectively u and f non-decreasing—if and only if $\mathbf{x}$ rank order dominates $\mathbf{y}$. Given two distributions $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$, we will say that $\mathbf{x}$ is obtained from $\mathbf{y}$ by means of a *progressive transfer*, if there exists an income amount $\Delta > 0$ and two individuals i, j such that

$$x_h = y_h, \quad \forall\, h \neq i, j; \tag{3.1a}$$

$$x_i = y_i + \Delta; \;\; x_j = y_j - \Delta; \;\; \text{and} \tag{3.1b}$$

$$\Delta \leq \left(y_j - y_i\right)/\,2. \tag{3.1c}$$

It is typically assumed in normative economics that inequality is reduced and welfare increased by a transfer of income from a richer individual to a poorer individual.

PRINCIPLE OF TRANSFERS [PT]. For all $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$, we have $W(\mathbf{x}) \geq W(\mathbf{y})$ and $I(\mathbf{x}) \leq I(\mathbf{y})$ whenever $\mathbf{x}$ is obtained from $\mathbf{y}$ by means of a progressive transfer.

We let $\mathscr{W}^2$ represent the set of social welfare functions that satisfy MON and PT. It is a straightforward exercise to check that the concavity of u and the convexity of f respectively guarantee that the utilitarian and the extended Gini social welfare functions obey PT. We denote as $\mathscr{U}^2$ the class of utility functions that are non-decreasing and concave and as $\mathscr{F}^2$ the class of weighting functions that are non-decreasing and convex. The *generalised Lorenz curve* of distribution $\mathbf{x} \in \mathscr{Y}_n(D)$ is defined by $GL(p; \mathbf{x}) := p\,x_1$, for all $0 \leq p \leq 1/n$, and

$$GL(p; \mathbf{x}) := p\,x_1 + \sum_{j=2}^{k} \left(\frac{n\,p - j + 1}{n}\right)\left[x_j - x_{j-1}\right], \quad \forall\ \frac{k-1}{n} < p \leq \frac{k}{n}, \tag{3.2}$$

where $k = 2, 3, \ldots, n$ (see Shorrocks 1983; Moyes 1999). Given two income distributions $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$, we will say that $\mathbf{x}$ *generalised Lorenz dominates* $\mathbf{y}$, which we write $\mathbf{x} \geq_{GL} \mathbf{y}$, if and only if $GL(k/n; \mathbf{x}) \geq GL(k/n; \mathbf{y})$, for all $k = 1, 2, \ldots, n$.

It has long been recognised that the notion of progressive transfer is closely associated with the generalised Lorenz quasi-ordering as the following result demonstrates (see Kolm 1969; Marshall and Olkin 1979; Sen 1973; Shorrocks 1983; Foster 1985 among others):

Theorem 3.1 *Let* $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$. *The following five statements are equivalent:*

(a) $\mathbf{x}$ *is obtained from* $\mathbf{y}$ *by means of a finite sequence of increments and/or progressive transfers.*
(b) $W(\mathbf{x}) \geq W(\mathbf{y})$, *for all* $W \in \mathscr{W}^2$.
(b1) $W_u(\mathbf{x}) \geq W_u(\mathbf{y})$, *for all* $u \in \mathscr{U}^2$.
(b2) $W_f(\mathbf{x}) \geq W_f(\mathbf{y})$, *for all* $f \in \mathscr{F}^2$.
(c) $\mathbf{x} \geq_{GL} \mathbf{y}$.

Theorem 3.1 indicates that both the utilitarian and the extended Gini social welfare functions are consistent with the generalised Lorenz quasi-ordering provided that appropriate restrictions be placed on the utility and the weighting functions. Furthermore, the rankings of distributions generated by the utilitarian and the extended Gini models under the constraint of unanimity among the classes of utility and weighting functions prove to be identical.

The preceding approach generalises in a straightforward way to inequality measurement by appropriate normalisation of the distributions under comparisons (see Moyes 1999). Given two income distributions $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$ with $D \subseteq \mathbb{R}_{++}$, we will say that $\mathbf{x}$ is obtained from $\mathbf{y}$ by means of a *scale transformation* if there exists $\lambda > 0$ such that $x_i = \lambda y_i$, for all $i = 1, 2, \ldots, n$.

SCALE INVARIANCE [SI]. For all $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$ with $D \subseteq \mathbb{R}_{++}$, we have $I(\mathbf{x}) = I(\mathbf{y})$ whenever $\mathbf{x}$ is obtained from $\mathbf{y}$ by means of a scale transformation.

We let $\mathscr{I}^2$ and $\mathscr{I}^R$ represent the set of inequality indices that satisfy conditions PT and SI, respectively. The *relative Lorenz curve* of distribution $\mathbf{x} \in \mathscr{Y}_n(D)$ with $D \subseteq \mathbb{R}_{++}$ is defined by $RL(p; \mathbf{x}) := GL(p; \hat{\mathbf{x}})$, for all $p \in [0, 1]$, where $\hat{\mathbf{x}} := (\hat{x}_1, \ldots, \hat{x}_n)$ is the *reduced distribution* corresponding to $\mathbf{x}$ with $\hat{x}_i := x_i / \mu(\mathbf{x})$, for all $i = 1, 2, \ldots, n$. Given two income distributions $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$ with $D \subseteq \mathbb{R}_{++}$, we will then say that $\mathbf{x}$ *relative Lorenz dominates* $\mathbf{y}$, which we write $\mathbf{x} \geq_{RL} \mathbf{y}$, if and only if $RL(k/n; \mathbf{x}) \geq RL(k/n; \mathbf{y})$, for all $k = 1, 2, \ldots, (n-1)$. The following result establishes the connections between relative Lorenz dominance, progressive transfers and scale transformations, and the requirement of unanimity among all extended Gini ethical observers who subscribe to the principle of transfers and scale invariance.

Theorem 3.2 *Let* $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$ *with* $D \subseteq \mathbb{R}_{++}$. *The following four statements are equivalent:*

(a) $\mathbf{x}$ *is obtained from* $\mathbf{y}$ *by means of a scale transformation and/or a finite sequence of progressive transfers.*
(b) $I(\mathbf{x}) \leq I(\mathbf{y})$, *for all* $I \in \mathscr{I}^2 \cap \mathscr{I}^R$.
(b2) $I_f^R(\mathbf{x}) \leq I_f^R(\mathbf{y})$, *for all* $f \in \mathscr{F}^2$.
(d) $\mathbf{x} \geq_{RL} \mathbf{y}$.

Given two income distributions $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$, we will say that $\mathbf{x}$ is obtained from $\mathbf{y}$ by means of a *translation transformation* if there exists $\gamma \in \mathbb{R}$ such that $x_i = y_i + \gamma$, for all $i = 1, 2, \ldots, n$.

TRANSLATION INVARIANCE [TI]. For all $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$, we have $I(\mathbf{x}) = I(\mathbf{y})$ whenever $\mathbf{x}$ is obtained from $\mathbf{y}$ by means of a translation transformation.

We let $\mathscr{I}^A$ represent the set of inequality indices that satisfy condition TI. The *absolute Lorenz curve* of distribution $\mathbf{x} \in \mathscr{Y}_n(D)$ is defined by $AL(p; \mathbf{x}) := GL(p; \tilde{\mathbf{x}})$, for all $p \in [0, 1]$, where $\tilde{\mathbf{x}} := (\tilde{x}_1, \ldots, \tilde{x}_n)$ is the *centred distribution* corresponding to $\mathbf{x}$ with $\tilde{x}_i := x_i - \mu(\mathbf{x})$, for all $i = 1, 2, \ldots, n$ (see Moyes 1987). Given two income distributions $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$, we will say that $\mathbf{x}$ *absolute Lorenz dominates* $\mathbf{y}$, which we write $\mathbf{x} \geq_{AL} \mathbf{y}$, if and only if $AL(k/n; \mathbf{x}) \geq AL(k/n; \mathbf{y})$, for all $k = 1, 2, \ldots, (n-1)$. The following result constitutes the analogue of Theorem 3.2 in the case of absolute inequality:

Theorem 3.3 *Let* $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$. *The following four statements are equivalent:*

(a) $\mathbf{x}$ *is obtained from* $\mathbf{y}$ *by means of a translation transformation and/or a finite sequence of progressive transfers.*
(b) $I(\mathbf{x}) \leq I(\mathbf{y})$, *for all* $I \in \mathscr{I}^2 \cap \mathscr{I}^A$.
(b2) $I_f^A(\mathbf{x}) \leq I_f^A(\mathbf{y})$, *for all* $f \in \mathscr{F}^2$.
(d) $\mathbf{x} \geq_{AL} \mathbf{y}$.

4 Social welfare and inequality as the absence of deprivation

4.1 The measurement of social welfare

Building upon Runciman (1966) and assimilating an individual's social status with her income, Kakwani (1984) has proposed to use the deprivation curve in order to compare income distributions. For reasons that will become obvious later on, we depart slightly from Kakwani's suggestion and we introduce the *non-deprivation curve* of distribution $\mathbf{x} \in \mathscr{Y}_n(D)$ defined by $ND(p; \mathbf{x}) := x_1$, for all $0 \leq p \leq 1/n$, and

$$ND(p; \mathbf{x}) := x_1 + \sum_{j=2}^{k} \left(\frac{n-j+1}{n}\right) \left[x_j - x_{j-1}\right], \quad \forall \, \frac{k-1}{n} < p \leq \frac{k}{n}, \tag{4.1}$$

where $k = 2, 3, \ldots, n$. We interpret $ND(k/n; \mathbf{x})$ as a measure of the feeling of non-deprivation of individual with rank $p = k/n$ in situation $\mathbf{x}$. In order to make the meaning of $ND(p; \mathbf{x})$ more transparent, we can develop and rearrange (4.1) when $k \geq 2$ to get

$$ND\left(\frac{k}{n}; \mathbf{x}\right) = GL\left(\frac{k}{n}; \mathbf{x}\right) + \frac{n-k}{n} x_k = \mu\left(\mathbf{x}^k\right), \tag{4.2}$$

where $\mathbf{x}^k := \left(x_1^k, \ldots, x_n^k\right)$ is the *censored distribution* obtained from $\mathbf{x}$ by letting $x_i^k := x_i$, for all $i = 1, 2, \ldots, k$, and $x_i^k := x_k$, for all $i = k+1, k+2, \ldots, n$.

Manipulating (4.2) one step further we finally obtain

$$ND\left(\frac{k}{n};\mathbf{x}\right) = \mu(\mathbf{x}) - ADP\left(\frac{k}{n};\mathbf{x}\right), \tag{4.3}$$

where ADP $(k/n;\mathbf{x}) := \sum_{j=k}^{n}(x_j - x_k)/n$, for all $k = 1, 2, \ldots, n$, is a measure of the *absolute deprivation* of individual k in situation $\mathbf{x}$ (see Ebert and Moyes 2000; Chateauneuf and Moyes 2006). Therefore, the non-deprivation of person k is identical to the difference between the mean income and her absolute deprivation: this is actually nothing else than what Chakravarty (1997) called person's k *generalised satisfaction.*[4]

Definition (4.1) makes clear that the non-deprivation curve is an increasing step function with jumps occurring possibly at $p = k/n$, with $k = 1, 2, \ldots, n-1$. Like the quantile curve, the non-deprivation curve attains its minimum value equal to $F^{-1}(0;\mathbf{x}) = x_1$ when $p = 0$. Like the generalised Lorenz curve, it reaches its maximum value equal to mean income $\mu(\mathbf{x})$ when $p = 1$. Notice that, in our framework where distributions are discrete, the quantile function can be equivalently rewritten as $F^{-1}(p;\mathbf{x}) := x_1$, for all $0 \le p \le 1/n$, and

$$F^{-1}(p;\mathbf{x}) := x_1 + \sum_{j=2}^{k}\left[x_j - x_{j-1}\right],\ \forall\ \frac{k-1}{n} < p \le \frac{k}{n}, \tag{4.4}$$

where $k = 2, 3, \ldots, n$. Comparing (4.1) and (4.4) makes clear that the quantile and the non-deprivation curves only differ in the way the adjacent pairwise income differences are weighted. In the case of the quantile curve, the first differences in incomes are all given the same weight equal to unity, while, in the case of the non-deprivation curve, the weights are decreasing at a constant rate.

We follow Chakravarty (1997) and compare income distributions on the basis of their non-deprivation curves. Given two income distributions $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$, we will say that $\mathbf{x}$ *non-deprivation dominates* $\mathbf{y}$, which we write $\mathbf{x} \geq_{ND} \mathbf{y}$, if and only if $ND(k/n;\mathbf{x}) \geq ND(k/n;\mathbf{y})$, for all $k = 1, 2, \ldots, n$. A distribution is considered better than another distribution if the non-deprivation experienced by any individual is not smaller in the first distribution than in the second distribution. An immediate question is to know how the non-deprivation quasi-ordering behaves in comparison with the rank order and generalised Lorenz quasi-orderings. Chakravarty et al. (1995, Theorem 1) have shown that non-deprivation dominance implies generalised Lorenz dominance when the distributions under comparison have equal means. The following result dispenses with the equal mean restriction:

Proposition 4.1 (a) *If* $n = 2$, *then* $\geq_{RO} \subset \geq_{ND} = \geq_{GL}$. *(b) If* $n > 2$, *then* $\geq_{RO} \subset \geq_{ND} \subset \geq_{GL}$.

Proof Because statement (a) is obvious, we only prove statement (b). It follows immediately from (4.2) that, if $x_k \geq y_k$, for all $k = 1, 2, \ldots, n$, then $\mathbf{x} \geq_{ND} \mathbf{y}$. To establish

4 We avoid here the term of *satisfaction* because it has been already used in Chateauneuf and Moyes (2006) with a different meaning.

that $\mathbf{x} \geq_{ND} \mathbf{y}$ implies $\mathbf{x} \geq_{GL} \mathbf{y}$, it suffices to show that $ND(k/n;\mathbf{x}) \geq ND(k/n;\mathbf{y})$ implies $GL(k/n;\mathbf{x}) \geq GL(k/n;\mathbf{y})$, for all $k = 1, 2, \ldots, n$. Clearly, the implication is true for $k = 1$. We argue by induction and verify that, if it is true for k, then so it is for $k + 1$. Suppose that $ND((k+1)/n;\mathbf{x}) \geq ND((k+1)/n;\mathbf{y})$, or equivalently that:

$$GL\left(\frac{k+1}{n};\mathbf{x}\right) + \frac{n-k-1}{n}\, x_{k+1} \geq GL\left(\frac{k+1}{n};\mathbf{y}\right) + \frac{n-k-1}{n}\, y_{k+1}, \quad (4.5)$$

If $x_{k+1} > y_{k+1}$, then the result follows from the fact that, by assumption, $GL(k/n;\mathbf{x}) \geq GL(k/n;\mathbf{y})$. If $x_{k+1} \leq y_{k+1}$, then we deduce immediately from (4.5) that rather $\mathbf{x} = (1, 4, 8, 9)$, $\mathbf{y} = (1, 5, 6, 10)$ and $\mathbf{z} = (1, 4, 7, 10)$.

$$GL\left(\frac{k+1}{n};\mathbf{x}\right) - GL\left(\frac{k+1}{n};\mathbf{y}\right) \geq -\frac{n-k-1}{n}\left[x_{k+1} - y_{k+1}\right] \geq 0. \quad (4.6)$$

Finally, consider distributions $\mathbf{x}\ :\ = (1, 4, 8, 9)$, $\mathbf{y}\ :\ = (1, 5, 6, 10)$ and $\mathbf{z} := (1, 4, 7, 10)$. One can easily verify that $\mathbf{x} \geq_{ND} \mathbf{z}$ but $\neg\,[\mathbf{x} \geq_{RO} \mathbf{z}]$, and that $\mathbf{y} \geq_{GL} \mathbf{z}$ but $\neg\left[\mathbf{y} \geq_{ND} \mathbf{z}\right]$, which confirms that the inclusions in statement (b) are strict. □

The non-deprivation quasi-ordering may therefore be considered an intermediate criterion halfway between the rank order and the generalised Lorenz quasi-orderings. As a result the non-deprivation criterion will provide a more partial—or to the best no more complete— ranking of the distributions under comparison than the one implied by the generalised Lorenz quasi-ordering. From a practical point of view, this may be considered a weakness of the non-deprivation quasi-ordering as compared with the generalised Lorenz one. As we will show below the non-deprivation quasi-ordering relies on value judgements that are less demanding—and hopefully more likely to be accepted—than the generalised Lorenz quasi-ordering. The loss in the discriminatory power of the non-deprivation quasi-ordering is the price to pay for its ability to reach a larger consensus. On the other hand, the extent of the loss in the conclusive rankings one experiences when the non-deprivation criterion is substituted for the generalised Lorenz quasi-ordering is ultimately an empirical matter.

There is evidence that the principle of transfers, that supports the generalised Lorenz criterion, is far from being unanimously accepted, as a number of experimental studies have demonstrated (see Amiel and Cowell 1992, 1999; Ballano and Ruiz-Castillo 1993; Harrison and Seidl 1994; Gaertner and Namezie 2003, among others). However, none of these studies provides information about the subjects' ethical preferences—with the exception that these preferences are at variance with the views captured by the principle of transfers—that might explain such a rejection. Nor do they propose alternatives to the principle of transfers that might be more in line with the subjects' attitudes towards inequality. We propose to weaken the principle of transfers by imposing a degree of solidarity among the individuals who take part in the redistribution process. However, solidarity is restricted to the individuals who give away a fraction of their incomes: if some income is taken from a rich individual, then in order for there to be solidarity, the same amount is taken from every individual who is as rich or richer than this individual. However, it is not necessary that the individuals who are poorer than the transfer recipient benefit also from an equal additional income.[5] By contrast, a progressive transfer imposes no solidarity at all among the individuals involved in the equalising transformation. It follows that the relative positions of the donors and beneficiaries of the transfers on the income scale will play a crucial role in the definition of our inequality reducing transformation. We emphasise that the idea of introducing the positions of the individuals for assessing the impact of a transfer is not new in the literature. Such an idea is at the heart of the notion of a *positional composite transfer* proposed by Chateauneuf and Wilthien (1999) and Zoli (2002).[6] Given two income distributions $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$, we will say that $\mathbf{x}$ is obtained from $\mathbf{y}$ by means of a *uniform on the right progressive transfer* if there exists an income amount $\Delta > 0$ and two individuals h, k $(1 \leq h < k \leq n)$ such that:

$$x_i = y_i, \ \forall\, i \in \{1, \ldots, h-1\} \cup \{h+1, \ldots, k-1\}; \tag{4.7a}$$

$$x_h = y_h + \Delta; \tag{4.7b}$$

$$x_i = y_i - \frac{\Delta}{n-k+1}, \ \forall\, i \in \{k, \ldots, n\}. \tag{4.7c}$$

If $k = n$, then a uniform on the right progressive transfer reduces to a usual progressive transfer, and this is actually the only case where both types of transformations coincide. Although in general uniform on the right progressive transfers and progressive transfers are different operations, it can be easily checked that the former can always be decomposed into a finite sequence of the latter, hence:

Proposition 4.2 *Let* $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$ *and suppose that* $\mathbf{x}$ *is obtained from* $\mathbf{y}$ *by means of a single uniform on the right progressive transfer. Then* $\mathbf{x}$ *can be obtained from* $\mathbf{y}$ *by means of a finite sequence of progressive transfers.*

It follows from Proposition 4.2 that, if a distribution is obtained from another by means of a uniform on the right progressive transfer, then the former will dominate the latter according to the generalised Lorenz criterion. But the fact that one distribution is ranked above another by the generalised Lorenz criterion does not imply that the former can be obtained from the latter by means of a sequence of uniform on the right progressive transfers. Choose $\mathbf{x} = (1, 4, 4, 7)$ and $\mathbf{y} = (1, 3, 5, 7)$: clearly $\mathbf{x} \geq_{GL} \mathbf{y}$ but it is impossible to transform $\mathbf{y}$ into $\mathbf{x}$ by means of uniform on the right progressive transfers. We find convenient at this stage to introduce an intermediate result that will prove useful later on.

Proposition 4.3 *Let* $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$. *Then* $\mathbf{x} \geq_{ND} \mathbf{y}$ *whenever* $\mathbf{x}$ *is obtained from* $\mathbf{y}$ *by means of a uniform on the right progressive transfer.*

Proof Suppose that $\mathbf{x}$ is obtained from $\mathbf{y}$ by means of a uniform on the right progressive transfer so that there exists $\Delta > 0$ and two individuals h, k $(1 \leq h < k \leq n)$ such

[5] One may conceive of other ways of introducing solidarity among the givers and receivers, which result in different possible equalising transformations (see Chateauneuf and Moyes 2006).

[6] The notion of a positional composite transfer generalises a suggestion of Kakwani (1980) that constitutes an alternative to the principle of diminishing transfers due to Kolm (1976).

that conditions (4.7a), (4.7b) and (4.7c), hold. Straightforward computation yields

$$ND\left(\frac{i}{n};\mathbf{x}\right) - ND\left(\frac{i}{n};\mathbf{y}\right) = \begin{cases} 0, & 1 \leq i < h, \\ \dfrac{(n-h+1)\,\Delta}{n} > 0, & i = h, \\ \dfrac{\Delta}{n} > 0, & h < i < k, \\ 0, & k \leq i \leq n, \end{cases} \tag{4.8}$$

hence $\mathbf{x} \geq_{ND} \mathbf{y}$. □

The following condition, which requires that social welfare does not decrease as the result of a uniform on the right progressive transfer, constitutes a weakening of the usual principle of transfers.

UNIFORM OF THE RIGHT PRINCIPLE OF TRANSFERS [URPT]. For all $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$, we have $W(\mathbf{x}) \geq W(\mathbf{y})$ and $I(\mathbf{x}) \leq I(\mathbf{y})$ whenever $\mathbf{x}$ is obtained from $\mathbf{y}$ by means of a uniform on the right progressive transfer.

We indicate by $\mathscr{W}^*$ the set of social welfare functions that satisfy MON and URPT and note that $\mathscr{W}^2 \subset \mathscr{W}^*$, which actually ensures that the class $\mathscr{W}^*$ is non-empty. For instance the social welfare function $W(\mathbf{x}) := H(ND(\mathbf{x}))$, where $ND(\mathbf{x}) := (ND(1/n;\mathbf{x}), \ldots, ND(n/n;\mathbf{x}))$ is the non-deprivation profile generated by distribution $\mathbf{x}$, verifies URPT provided that H be symmetric and monotone.[7] This social welfare function imposes a particular factorisation process and one might be willing to consider more conventional social welfare functions such as the utilitarian and extended Gini.

It is argued in Chateauneuf and Moyes (2006) that the utilitarian principle does not permit one to distinguish between the ethical observers who subscribe to the uniform on the right principle of transfers from those who agree with the principle of transfers.[8] By contrast the extended Gini approach makes it possible to identify the ethical observers who share the views reflected by either principle. We will say that $f \in \mathscr{F}$ is *star-shaped* if $f(q)/q$ is non-decreasing in q, for all $q \in (0, 1]$. It is well-known that the convexity of $f \in \mathscr{F}$ is a necessary and sufficient condition for welfare to increase as the result of a progressive transfer when the former is evaluated by means of the extended Gini social welfare function (see, e.g. Chateauneuf and Moyes 2004, Proposition 4.4). We show below that it is possible to find a similar justification for the star-shapedness property provided one is willing to consider the impact on social welfare of particular combinations of increments. Given two income distributions $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$, we will say that $\mathbf{x}$ is obtained from $\mathbf{y}$ by means of a

7 There is no need to impose that the aggregation function H be Schur-concave as it is done in Chakravarty and Mukherjee (1999) for social welfare as measured by $H(ND(\mathbf{x}))$ to increase as a result of an arbitrary uniform on the right progressive transfer.

8 More precisely, concavity of the utility function is shown to be a necessary and sufficient condition for the utilitarian social welfare function to satisfy the uniform on the right principle of transfers.

k-uniform on the right increment if there exists $\Delta > 0$ such that

$$x_i = y_i, \ \forall i = 1, 2, \ldots, k-1, \ \text{and} \tag{4.9a}$$

$$x_i = y_i + \frac{\Delta}{n-k+1}, \quad \forall i = k, k+1, \ldots, n. \tag{4.9b}$$

Clearly a k-uniform on the right increment results in a social welfare improvement in the extended Gini model provided that the weighting function f is increasing. Consider now an arbitrary distribution $\mathbf{y}$ and let $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ be two distributions such that $\mathbf{x}^{(1)}$ is obtained from $\mathbf{y}$ by means of a k-uniform on the right increment equal to $\Delta > 0$ while a $(k+1)$-uniform on the right increment of the same amount transforms $\mathbf{y}$ into $\mathbf{x}^{(2)}$. Computing the corresponding welfare changes, we obtain:

$$W_f\left(\mathbf{x}^{(1)}\right) - W_f(\mathbf{y}) = f\left(\frac{n-k+1}{n}\right)\frac{\Delta}{n-k+1}, \ \text{and} \tag{4.10a}$$

$$W_f\left(\mathbf{x}^{(2)}\right) - W_f(\mathbf{y}) = f\left(\frac{n-k}{n}\right)\frac{\Delta}{n-k}. \tag{4.10b}$$

Then the star-shapedness of f guarantees that $W_f\left(\mathbf{x}^{(1)}\right) - W_f(\mathbf{y}) \geq W_f\left(\mathbf{x}^{(2)}\right) - W_f(\mathbf{y})$: the lower is k, the greater the welfare improvement caused by a k uniform on the right increment of a given amount $\Delta > 0$. Observe further that, by construction, distributions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are such that $\mathbf{x}^{(1)}$ is derived from $\mathbf{x}^{(2)}$ by means of a uniform on the right progressive transfer of the amount $\Delta/(n-k+1) > 0$ from the individuals no poorer than $k+1$ to the individual k. The welfare change implied by this uniform on the right progressive transfer is equal to

$$W_f\left(\mathbf{x}^{(1)}\right) - W_f\left(\mathbf{x}^{(2)}\right) = \left[W_f\left(\mathbf{x}^{(1)}\right) - W_f(\mathbf{y})\right] - \left[W_f\left(\mathbf{x}^{(2)}\right) - W_f(\mathbf{y})\right] \geq 0. \tag{4.11}$$

The above argument generalises to an arbitrary uniform on the right progressive transfer by choosing appropriately the h-uniform on the right and k-uniform on the right increments so that $h < k$. Therefore a sufficient condition for a uniform on the right progressive transfer to improve social welfare as measured by the extended Gini welfare function is that the weighting function f is star-shaped.[9] For later reference we denote as $\mathscr{F}^*$ the set of weighting functions $f \in \mathscr{F}$ that are non-decreasing and star-shaped.

We are now in a position to state our main result, which establishes the connections between increments and uniform on the right progressive transfers, unanimity of value judgements among the ethical observers who subscribe to the uniform on the right principle of transfers, and non-deprivation domination.

[9] Imposing the further restriction that the weighting function is continuous over $[0, 1)$ it can be proven that star-shapedness is also a necessary condition by making use of a replication argument. However, in this case we can no longer restrict our attention to distributions of fixed dimension and we have to allow the population to vary.

Theorem 4.1 *Let* $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$. *The following four statements are equivalent:*

(a) $\mathbf{x}$ *is obtained from* $\mathbf{y}$ *by means of a finite sequence of increments and/or uniform on the right progressive transfers.*
(b) $W(\mathbf{x}) \geq W(\mathbf{y})$, *for all* $W \in \mathscr{W}^*$.
(b2) $W_f(\mathbf{x}) \geq W_f(\mathbf{y})$, *for all* $f \in \mathscr{F}^*$.
(c) $\mathbf{x} \geq_{ND} \mathbf{y}$.

Proof Letting $q_i := (n-i+1)/n$, for all $i = 1, 2, \ldots, n$, and $q_{n+1} := 0$, the extended Gini social welfare function can be rewritten as

$$W_f(\mathbf{x}) = \frac{f(q_1)}{q_1} q_1 x_1 + \sum_{i=2}^{n} \frac{f(q_i)}{q_i} q_i \left[x_i - x_{i-1}\right], \tag{4.12}$$

where we have made use of the fact that $f(q_1) = f(1) = 1$. Applying Abel's decomposition rule to (4.12) and using definition (4.1), we obtain

$$W_f(\mathbf{x}) = \sum_{k=1}^{n-1} \left[\frac{f(q_k)}{q_k} - \frac{f(q_{k+1})}{q_{k+1}}\right] ND\left(\frac{k}{n}; \mathbf{x}\right) + \frac{f(q_n)}{q_n} ND\left(\frac{n}{n}; \mathbf{x}\right). \tag{4.13}$$

(a) $\Longrightarrow$ (b). This follows directly from the definition of the class $\mathscr{W}^*$ of social welfare functions.
(b) $\Longrightarrow$ (b2). It is a consequence of the fact that $W_f \in \mathscr{W}^*$ whenever $f \in \mathscr{F}^*$.
(b2) $\Longrightarrow$ (c). Suppose that statement (b2) holds, which upon using (4.13) is equivalent to:

$$W_f(\mathbf{x}) - W_f(\mathbf{y}) = \sum_{k=1}^{n-1} \left[\frac{f(q_k)}{q_k} - \frac{f(q_{k+1})}{q_{k+1}}\right] \left[ND\left(\frac{k}{n}; \mathbf{x}\right) - ND\left(\frac{k}{n}; \mathbf{y}\right)\right] + \frac{f(q_n)}{q_n} \left[ND\left(\frac{n}{n}; \mathbf{x}\right) - ND\left(\frac{n}{n}; \mathbf{y}\right)\right] \geq 0. \tag{4.14}$$

Let $f^{(n)}(q) = q$, for all $q \in [0, 1]$, and consider the weighting function $f^{(1)}$ defined by $f^{(1)}(q) = 0$, for all $0 \leq q \leq (n-1)/n$, and $f^{(1)}(q) = n(q - (n-1)/n)$, for all $(n-1)/n < q \leq 1$. Similarly, for $k = 2, 3, \ldots, (n-1)$, let $f^{(k)}(q) = 0$, for all $0 \leq q \leq (n-k)/n$, $f^{(k)}(q) = (n-k+1)[q - (n-k)/n]$, for all $(n-k)/n < q \leq (n-k+1)/n$, and $f^{(k)}(q) = q$, for all $(n-k+1)/n < q \leq 1$. One can easily check that $f^{(k)} \in \mathscr{F}^*$, for $k = 1, 2, \ldots, n$. Upon substitution into (4.14), we obtain

$$W_{f^{(k)}}(\mathbf{x}) - W_{f^{(k)}}(\mathbf{y}) = ND\left(\frac{k}{n}; \mathbf{x}\right) - ND\left(\frac{k}{n}; \mathbf{y}\right) \geq 0, \tag{4.15}$$

for $k = 1, 2, \ldots, n$, from which we conclude that $\mathbf{x} \geq_{ND} \mathbf{y}$.
(c) $\Longrightarrow$ (a). Suppose that $\mathbf{x} \geq_{ND} \mathbf{y}$ in which case there are two possibilities. If $\mu(\mathbf{x}) = \mu(\mathbf{y})$, then $\mathbf{x} \geq_{ND} \mathbf{y}$ reduces to $ADP(k/n; \mathbf{x}) \leq ADP(k/n; \mathbf{y})$, for all $k = 1, 2, \ldots, n-1$, and it follows from Chateauneuf and Moyes (2006, Theorem 2)

that distribution $\mathbf{x}$ can be obtained from distribution $\mathbf{y}$ by means of a finite sequence of uniform on the right progressive transfers. If $\mu(\mathbf{x}) > \mu(\mathbf{y})$, then consider the distribution $\mathbf{z} := (z_1, \ldots, z_n)$ such that $z_i = y_i$, for all $i = 1, 2, \ldots, n-1$, and $z_n = y_n + n[\mu(\mathbf{x}) - \mu(\mathbf{y})]$. Distribution $\mathbf{z}$ is obtained from distribution $\mathbf{y}$ by means of a single increment, hence $\mathbf{z} \geq_{RO} \mathbf{y}$ and thus $\mathbf{z} \geq_{ND} \mathbf{y}$ by application of Proposition 4.1. Furthermore we have $ND(k/n; \mathbf{z}) = ND(k/n; \mathbf{y}) \leq ND(k/n; \mathbf{x})$, for all $k = 1, 2, \ldots, (n-1)$, and $ND(1; \mathbf{z}) = \mu(\mathbf{z}) = \mu(\mathbf{x}) = ND(1; \mathbf{x})$, hence $\mathbf{x} \geq_{ND} \mathbf{z}$ and $\mu(\mathbf{x}) = \mu(\mathbf{z})$. We are back to the previous case and we conclude that distribution $\mathbf{x}$ can be derived from distribution $\mathbf{z}$ by means of a finite sequence of uniform on the right progressive transfers. □

According to Theorem 4.1, if the non-deprivation curves of the distributions under comparison do not intersect, then there is no room for disagreement among ethical observers provided they subscribe to MON and URPT. More precisely, the distribution whose non-deprivation curve is located above that of the other distribution will be socially preferred from a welfare point of view. Requiring unanimity of value judgements over the subset of those ethical observers who make comparisons on the basis of the extended Gini social welfare function does not make a difference. The only restriction then will be that the weighting function, which captures the ethical observer's attitude towards inequality, must be star-shaped. A contrario, if the non-deprivation curves associated with two distributions intersect, then it is always possible to find two social welfare functions in the class $\mathscr{W}^*$—and therefore two star-shaped weighting functions—that will rank these distributions in the opposite way. More importantly, Theorem 4.1 identifies the equalising process that leads to dominance according to the non-deprivation quasi-ordering. If one distribution is ranked higher than another by the non-deprivation quasi-ordering, then it is always possible to obtain the dominating distribution from the dominated one by successive applications of uniform on the right progressive transfers and/or increments.

The non-deprivation quasi-ordering does not exhaust all the possibilities for checking whether one distribution will be judged as better than another distribution by all social welfare functions in the class $\mathscr{W}^*$. Chakravarty (1997) has already noted that the comparisons of the means of the censored distributions provide an alternative and equivalent test. Indeed, making use of (4.2), condition (c) in Theorem 4.1 reduces to $\mu\left(\mathbf{x}^k\right) \geq \mu\left(\mathbf{y}^k\right)$, for all $k = 1, 2, \ldots, n$. We know from Proposition 4.1 that the non-deprivation quasi-ordering is more incomplete than the generalised Lorenz quasi-ordering, which actually means that the former criterion is more demanding than the latter. Substituting (4.2) into condition (c) in Theorem 4.1, we obtain

$$GL\left(\frac{k}{n}; \mathbf{x}\right) - GL\left(\frac{k}{n}; \mathbf{y}\right) + \frac{n-k}{n}\left[x_k - y_k\right] \geq 0, \ \forall\, k = 1, 2, \ldots, n, \tag{4.16}$$

which indicates what is needed in addition to generalised Lorenz dominance for a distribution to be ranked above another by the non-deprivation quasi-ordering. Non-deprivation dominance necessitates that the vertical distance at any $p = k/n$ between the generalised Lorenz curves of the distributions under comparison be greater than a minimal value as the following result indicates:

Proposition 4.4 *Let* $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$. *Then* $\mathbf{x} \geq_{ND} \mathbf{y}$ *if and only if:*

$$GL\left(\frac{k}{n}; \mathbf{x}\right) - GL\left(\frac{k}{n}; \mathbf{y}\right) \geq \max\left\{0, -\frac{n-k}{n}\left[x_k - y_k\right]\right\}, \tag{4.17}$$

for all $k = 1, 2, \ldots, n$.

Proof In the light of (4.16), one can immediately check that condition (4.17) implies $\mathbf{x} \geq_{ND} \mathbf{y}$. To show the converse implication suppose that there exists $k \in \{1, 2, \ldots, n\}$ such that

$$GL\left(\frac{k}{n}; \mathbf{x}\right) - GL\left(\frac{k}{n}; \mathbf{y}\right) < \max\left\{0, -\frac{n-k}{n}\left[x_k - y_k\right]\right\}. \tag{4.18}$$

If $\max\{0, -((n-k)/n)\,[x_k - y_k]\} = 0$, then the above inequality simplifies to $GL(k/n; \mathbf{x}) - GL(k/n; \mathbf{y}) < 0$, which upon appealing to Proposition 4.1 implies that $\neg\left[\mathbf{x} \geq_{ND} \mathbf{y}\right]$. If $\max\{0, -((n-k)/n)\,[x_k - y_k]\} = -((n-k)/n)\,[x_k - y_k]$, then (4.18) reduces to $GL(k/n; \mathbf{x}) - GL(k/n; \mathbf{y}) < -((n-k)/n)\,[x_k - y_k]$, which appealing to (4.16) implies that $\neg\left[\mathbf{x} \geq_{ND} \mathbf{y}\right]$. □

4.2 The measurement of inequality

We indicate by $\mathscr{I}^*$ the set of ethical inequality indices that satisfy URPT and note that, in the extended Gini model, star-shapedness of f is a necessary and sufficient condition for I_f^R and I_f^A to fulfil URPT. The *relative non-deprivation curve* of distribution $\mathbf{x} \in \mathscr{Y}_n(D)$ with $D \subseteq \mathbb{R}_{++}$ is defined by $RND(p; \mathbf{x}) := ND(p; \hat{\mathbf{x}})$, for all $p \in [0, 1]$. Then, given two income distributions $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$ with $D \subseteq \mathbb{R}_{++}$, we will say that $\mathbf{x}$ *relative non-deprivation dominates* $\mathbf{y}$, which we write $\mathbf{x} \geq_{RND} \mathbf{y}$, if and only if $RND(k/n; \mathbf{x}) \geq RND(k/n; \mathbf{y})$, for all $k = 1, 2, \ldots, (n-1)$. Building upon Theorem 4.1 we obtain immediately:

Theorem 4.2 *Let* $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$ *with* $D \subseteq \mathbb{R}_{++}$. *The following four statements are equivalent:*

(a) $\mathbf{x}$ *is obtained from* $\mathbf{y}$ *by means of a scale transformation and/or a finite sequence of uniform on the right progressive transfers.*
(b) $I^R(\mathbf{x}) \leq I^R(\mathbf{y})$, *for all* $I^R \in \mathscr{I}^* \cap \mathscr{I}^R$.
(b2) $I_f^R(\mathbf{x}) \leq I_f^R(\mathbf{y})$, *for all* $f \in \mathscr{F}^*$.
(c) $\mathbf{x} \geq_{RND} \mathbf{y}$.

Turning next to absolute inequality we define the *absolute non-deprivation curve* of distribution $\mathbf{x} \in \mathscr{Y}_n(D)$ by $AND(p; \mathbf{x}) := ND(p; \tilde{\mathbf{x}})$, for all $p \in [0, 1]$. Then, given two income distributions $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$, we will say that $\mathbf{x}$ *absolute non-deprivation dominates* $\mathbf{y}$, which we write $\mathbf{x} \geq_{AND} \mathbf{y}$, if and only if $AND(k/n; \mathbf{x}) \geq AND(k/n; \mathbf{y})$, for all $k = 1, 2, \ldots, (n-1)$. The following result constitutes the counterpart of Theorem 4.2 when one is interested in absolute inequality.

Theorem 4.3 *Let* $\mathbf{x}, \mathbf{y} \in \mathscr{Y}_n(D)$. *The following four statements are equivalent:*

(a) $\mathbf{x}$ *is obtained from* $\mathbf{y}$ *by means of a translation transformation and/or a finite sequence of uniform on the right progressive transfers.*
(b) $I^A(\mathbf{x}) \leq I^A(\mathbf{y})$, *for all* $I^A \in \mathscr{I}^* \cap \mathscr{I}^A$.
(b2) $I_f^A(\mathbf{x}) \leq I_f^A(\mathbf{y})$, *for all* $f \in \mathscr{F}^*$.
(c) $\mathbf{x} \geq_{AND} \mathbf{y}$.

We note that no further restriction in addition to star-shapedness is needed for the relative (resp. absolute) inequality indices inherited from the extended Gini social welfare function to be consistent with the relative (resp. absolute) non-deprivation quasi-ordering.[10] Because URPT is a weaker condition than the usual PT, there is a high presumption that the relative and absolute non-deprivation quasi-orderings will provide far more incomplete rankings than the ones generated by the relative and absolute Lorenz quasi-orderings.[11]

5 Discussion of related results in the literature

It is fair to note that what we call the non-deprivation curve in this paper has already been known for some time in the literature as the *generalised satisfaction* curve introduced by Chakravarty (1997). However, it is not completely clear what the generalised satisfaction curve looks like. It is suggested in Chakravarty et al. (1995) that the satisfaction curve is continuously increasing over the interval [0, 1], while our definition of the non-deprivation curve insists that it is a stepwise function with a finite number of discontinuities. Actually, the non-deprivation curve is a modified version of the quantile curve, where the first differences in incomes are given different weights, which are decreasing at a constant rate with the individuals' ranks. From a practical point of view, the interpolation introduced implicitly by Chakravarty et al. (1995) in their definition of the generalised satisfaction curve does not pose a problem as long as one is interested in the comparisons of distributions for populations of *fixed* size. Things happen to be very different when the populations involved have different sizes, in which case Chakravarty (1997)'s generalised satisfaction criterion and our non-deprivation criterion may lead to different rankings of the distributions under comparison.

Up to the restriction that the distributions have the same means, our Theorem 4.1 looks at first sight quite similar to Theorem 3 in Chakravarty (1997). However, there is a major difference that concerns the procedure used in order to convert the dominated distribution into the dominating one: a *fair transformation* in Chakravarty (1997) and a sequence of *uniform on the right progressive transfer*s in our case. Actually, it can be easily checked that requiring that distribution $\mathbf{x}$ be obtained from distribution $\mathbf{y}$ by means of a fair transformation is but another way of imposing that $\mathbf{x}$ dominates

[10] This may be contrasted with the utilitarian framework where the consistency of the relative and absolute inequality indices with relative and absolute Lorenz dominance implies restrictions on the utility function that go far beyond concavity (see, e.g. Ebert 1988).

[11] The application of our criteria to the comparison of a range of OECD countries actually confirms this conjecture (see Magdalou and Moyes 2008).

y according to the non-deprivation—or equivalently, the generalised satisfaction—criterion. Therefore, the notion of a fair transformation adds little to our understanding of the implicit equalising process that leads to non-deprivation dominance. Although a uniform on the right progressive transfer is admittedly a more complex operation than a progressive transfer, it is far more elementary than a fair transformation. For this reason it is believed to be more informative than a fair transformation, making it easier to comprehend what is precisely meant by inequality reduction.

While Chateauneuf and Moyes (2006) are only concerned with the comparison of distributions of equal means, their results hint at a simple procedure for passing welfare judgements in the general case. Consider again our definition of the extended Gini social welfare function and notice that it can be equivalently rewritten as

$$W_f(\mathbf{x}) = \mu(\mathbf{x}) - \sum_{i=1}^{n-1} \left[\frac{f(q_i)}{q_i} - \frac{f(q_{i+1})}{q_{i+1}} \right] ADP\left(\frac{i}{n}; \mathbf{x}\right). \tag{5.1}$$

If the weighting function f is star-shaped, then it is sufficient for welfare to improve that mean income increases and that the absolute deprivation curve moves downwards. This suggests a two-stage procedure for evaluating income distributions: first the absolute deprivation curves of the distributions are compared and, if they do not cross, then the means are computed in a second stage. Formally this criterion can be written as:

$$\mu(\mathbf{x}) \geq \mu(\mathbf{y}) \text{ and } ADP\left(\frac{k}{n}; \mathbf{x}\right) \leq ADP\left(\frac{k}{n}; \mathbf{y}\right), \ \forall\, k = 1, 2, \ldots, (n-1). \tag{5.2}$$

This is reminiscent of the procedure based on the comparisons of the relative Lorenz curves and mean incomes that constituted the starting point of Shorrocks (1983). The problem with this way of proceeding—as was emphasised by Shorrocks (1983) in the Lorenz context—is that the distribution with the lower absolute deprivation curve may have also the lower mean, which makes the conclusion of this two-stage evaluation process ambiguous. However, the fact that two distributions do not pass the test described by (5.2) does not preclude the possibility that one distribution be ranked above the other one by all social welfare functions in the class $\mathscr{W}^*$. Consider for instance the distributions $\mathbf{x} = (1, 4, 5, 8)$ and $\mathbf{y} = (1, 2, 6, 7)$ and note that $\mathbf{x}$ and $\mathbf{y}$ cannot be ordered by the rank order criterion. One can easily verify that the absolute deprivation curves of distributions $\mathbf{x}$ and $\mathbf{y}$ intersect while $\mu(\mathbf{x}) > \mu(\mathbf{y})$, making it impossible to conclude whether $\mathbf{x}$ will be ranked above $\mathbf{y}$ or not by all ethical observers who subscribe to URPT and MON. However, the non-deprivation curve of $\mathbf{x}$ is everywhere located above that of $\mathbf{y}$, and Theorem 4.1 confirms that $W(\mathbf{x}) \geq W(\mathbf{y})$, for all $W \in \mathscr{W}^*$.

Finally we would like to contrast our approach with the alternative method proposed by Yitzhaki (1979) and further developed by Hey and Lambert (1980) for measuring overall deprivation in the society. The concepts of *individual* deprivation referred to in the two above papers and in ours are identical: an individual's deprivation is associated with the aggregate gap between her income and those of the individuals richer than her, up to a scale factor equal to the population size. It follows that the measures of individual non-deprivation give the same value for a given individual even though, in Yitzhaki (1979) an individual is associated with her income, while in our case she is identified by her relative position on the income scale. However, these two approaches differ in the way the individuals' deprivation levels are aggregated over the population in order to derive the overall deprivation in the society. While Hey and Lambert (1980) compare the individual non-deprivation profiles at each income level, we make this comparison for individuals located at the same position on the income scale. Under the condition that the distributions under comparison have equal means, Hey and Lambert (1980)'s dominance criterion reduces to

$$\frac{1}{n}\sum_{i=1}^{m(z;\mathbf{x})}(z-x_i) \leq \frac{1}{n}\sum_{i=1}^{m(z;\mathbf{y})}(z-y_i)\,,\ \forall\, z \in \mathbb{R}, \tag{5.3}$$

where $m(z;\mathbf{x})$ and $m(z;\mathbf{y})$ represent the number of individuals with an income no greater than z in situations $\mathbf{x}$ and $\mathbf{y}$, respectively. In other words, poverty in situation $\mathbf{x}$ must not exceed poverty in situation $\mathbf{y}$ for all possible poverty lines z, which appealing to Foster and Shorrocks (1988, Theorem 2) is equivalent to $\mathbf{x} \geq_{GL} \mathbf{y}$. Proposition 4.1 makes clear that our approach is less decisive than that of Hey and Lambert (1980) because non-deprivation dominance implies generalised Lorenz dominance but not the converse. On the other hand it is based on value judgements that might be considered less controversial than the ones implicitly adopted by Hey and Lambert (1980).

6 Concluding comments

Building on Chakravarty (1997) and Chateauneuf and Moyes (2006), we have proposed in this paper a method for making welfare and inequality comparisons based on the absence of deprivation. This method constitutes a natural alternative to the standard approach in normative economics, which consists in comparing the—generalised, relative and absolute—Lorenz curves of the distributions. The Lorenz criteria are all consistent with the principle of transfers, which requires that inequality decreases and welfare increases as the result of an *arbitrary* progressive transfer. The criteria we propose actually obey a weaker version of the principle of transfers: inequality decreases only for some *specific* combinations of the progressive transfers, where the positions of the donor(s) and the beneficiary of the transfer play a crucial role. Furthermore these criteria are related to the notion of deprivation that arises from the comparison for each individual of her situation with those of the individuals she considers as better-off.

Contrary to what might have been expected, preliminary empirical investigations suggest that the non-deprivation criterion performs rather well as compared with the generalised Lorenz criterion (see Magdalou and Moyes 2008). Although it is premature to generalise at this stage, this result is nevertheless worth noting given the general skepticism one may have to face when proposing criteria ethically more demanding than the generalised Lorenz quasi-ordering. Things are more contrasted as far as inequality comparisons are concerned but still there the non-deprivation based inequality quasi-orderings do not perform too badly by comparison with the standard relative and absolute Lorenz criteria. The application of the non-deprivation quasi-ordering may provide additional information about the nature of the equalising process that leads to the domination of a distribution by another. This is indisputable when the substitution of the non-deprivation quasi-ordering for the generalised Lorenz quasi-ordering does not affect the ranking of the distributions. For in this case, we get additional information concerning the structure of the modifications of the distributions that give rise to generalised Lorenz domination: these must be uniform on the right progressive transfers.

The criteria we have proposed do only generate partial rankings of the situations under comparison. While this might be considered satisfactory in some cases—for instance for the design of tax reforms—it generally provides insufficient grounds for making decisions. These criteria must therefore be considered a first round approach, which should be supplemented by the use of particular indices in the general classes we have identified. An avenue for future research would be to characterise by means of additional conditions particular social welfare functions and inequality indices among those that are consistent with the non-deprivation quasi-ordering.

The approach developed in this paper was partly motivated by the observation that the principle of transfers, which supports the generalised Lorenz criterion, does not achieve a consensus among those individuals having taken part in the questionnaire studies. It would therefore be interesting to check if the non-deprivation quasi-ordering rooted in the uniform on the right principle of transfers is more favourably accepted by individuals participating in such studies (see Magdalou 2006, for a preliminary investigation).

Acknowledgments We are indebted to Stephen Bazen for very useful conversations and suggestions. We also would like to thank an associate editor and two anonymous referees whose comments have helped us to improve the paper. The constructive comments of one referee in particular have permitted to simplify greatly the exposition and clarify a couple of issues. Needless to say, the authors bear the entire responsibility for remaining errors and deficiencies.

References

Amiel Y, Cowell F (1992) Measurement of income inequality: experimental test by questionnaire. J Publ Econ 47:3–26

Amiel Y, Cowell F (1999) Thinking about inequality. Cambridge University Press, Cambridge

Ballano C, Ruiz-Castillo J (1993) Searching by questionnaire for the meaning of income inequality. Revista Espanola Econ 10:233–259

Berrebi Z, Silber J (1985) Income inequality indices and deprivation: a generalization [relative deprivation and the Gini coefficient]. Quart J Econ 100:807–810

Blackorby C, Bossert W, Donaldson D (1999) Income inequality measurement: the normative approach. In: Silber J (ed) Handbook of income inequality measurement. Kluwer, Dordrecht pp 133–157

Chakravarty S (1997) Relative deprivation and satisfaction orderings. Keio Econ Stud XXXIV:17–32

Chakravarty S, Mukherjee D (1999) Measures of deprivation and their meaning in terms of social satisfaction. Theory Decis 47:89–100

Chakravarty S, Chattopadhyay N, Majumder A (1995) Income inequality and relative deprivation. Keio Econ Stud XXXII:1–15

Chateauneuf A, Moyes P (2004) Lorenz non-consistent welfare and inequality measurement. J Econ Inequality 2:61–87

Chateauneuf A, Moyes P (2006) Measuring inequality without the Pigou-Dalton condition. In: McGillivray M (ed) Inequality, poverty and well-being. Palgrave Macmillan, Basingtoke, Hampshire/New-York pp 22–65

Chateauneuf A, Wilthien P-H (1999) Third inverse stochastic dominance, Lorenz curves and favourable double-transfers. CERMSEM Discussion Paper, Université Paris I Panthéon-Sorbonne
Ebert U (1988) Measurement of inequality: an attempt at unification and generalization. Soc Choice Welf 5:59–81
Ebert U, Moyes P (2000) An axiomatic characterization of the Yitzhaki's index of individual deprivation. Econ Lett 68:263–270
Foster J (1985) Inequality measurement. In: Young H (ed) Fair allocation. American Mathematical Society, Providence pp 38–61
Foster J, Shorrocks A (1988) Poverty orderings and welfare dominance. Soc Choice Welf 5:91–110
Gaertner W, Namezie C (2003) Income inequality, risk, and the transfer principle: a questionnaire-experimental investigation. Math Soc Sci 45:229–245
Gastwirth J (1971) A general definition of the Lorenz curve. Econometrica 39:1037–1039
Harrison E, Seidl C (1994) Perceptional inequality and preference judgements: an empirical examination of distributional axioms. Public Choice 79:61–68
Hey J, Lambert P (1980) Relative deprivation and the Gini coefficient: comment. Quart J Econ 95:567–573
Kakwani N (1980) On a class of poverty measures. Econometrica 48:437–446
Kakwani N (1984) The relative deprivation curve and its applications. J Bus Econ Stat 2:384–405
Kolm S-C (1969) The optimal production of social justice. In: Margolis J, Guitton H (eds) Public economics. Macmillan, London pp 145–200
Kolm S-C (1976) Unequal inequalities I. J Econ Theory 12:416–442
Magdalou B (2006) Une Contribution Théorique et Expérimentale à la La Mesure des Inégalités: Gini et Lorenz au Regard de la Théorie de l'Absence de Privation. PhD Thesis, LAMETA, CNRS-Université Montpellier I, Montpellier, France
Magdalou B, Moyes P (2008) Deprivation and the measurement of welfare and inequality. GREThA Working Paper, CNRS-Université Montesquieu Bordeaux IV
Marshall A, Olkin I (1979) Inequalities: theory of majorization and its applications. Academic Press, New-York
Moyes P (1987) A new concept of Lorenz domination. Econ Lett 23:203–207
Moyes P (1999) Stochastic dominance and the Lorenz curve. In: Silber J (ed) Handbook of income inequality measurement. Kluwer, Dordrecht pp 199–222
Quiggin J (1993) Generalized expected utility theory. The rank-dependent model. Kluwer, Dordrecht
Runciman W (1966) Relative deprivation and social justice. Routledge and Kegan Paul, London
Sen A (1973) On economic inequality. Clarendon Press, Oxford
Shorrocks A (1983) Ranking income distributions. Economica 50:3–17
Weiss Y, Fershtman C (1998) Social status and economic performance: a survey. Euro Econ Rev 42:801–820
Weymark J (1981) Generalized Gini inequality indices. Math Soc Sci 1:409–430
Yaari M (1987) The dual theory of choice under risk. Econometrica 55:99–115
Yaari M (1988) A controversial proposal concerning inequality measurement. J Econ Theory 44:381–397
Yitzhaki S (1979) Relative deprivation and the Gini coefficient. Quart J Econ 93:321–324
Zoli C (2002) Inverse stochastic dominance, inequality measurement and Gini indices. In: Moyes P, Seidl C, Shorrocks A (eds) Inequalities: theory, experiments and applications. J Econ Suppl 9:119–161