

## CONVERGENCE OF JOINT MOMENTS FOR INDEPENDENT RANDOM PATTERNED MATRICES

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It is known that the joint limit distribution of independent Wigner matrices satisfies a very special asymptotic independence, called freeness. We study the joint convergence of a few other patterned matrices, providing a framework to accommodate other joint laws. In particular, the matricial limits of symmetric circulants and reverse circulants satisfy, respectively, the classical independence and the half independence. The matricial limits of Toeplitz and Hankel matrices do not seem to submit to any easy or explicit independence/dependence notions. Their limits are not independent, free or half independent.

**1. Introduction.** Wigner [11] showed how the semicircular law arises as the limit of the empirical spectral distribution of a sequence of Wigner matrices. See, for example, [2] and [1] for such results and their variations. Then researchers studied the joint convergence of independent Wigner matrices and the limit is tied to the idea of free independence developed by Voiculescu [10].

It appears that the study of joint distribution of random matrices has been mostly concentrated on Wigner type matrices. In [7], joint limits of random matrices were studied and it was shown that there are some circumstances where the limit may exist but may not be free. As one of the Referees pointed out, [8] considers the joint distribution of Vandermonde matrices and diagonal matrices and emphasizes the importance of studying joint distribution of other patterned matrices.

We study the joint convergence of  $p$  independent symmetric matrices with identical pattern. In particular, we show that the tracial limit exists for any monomial when the patterned matrix is any one of, Toeplitz, Hankel, symmetric circulant or reverse circulant. The Wigner, symmetric circulant and reverse circulant limits are, respectively, free semicircular, classical independent normal and half independent Rayleigh. The Toeplitz and Hankel limits are not free, independent or half independent.

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In Section 2, we discuss some preliminaries on noncommutative probability spaces and recall the notions of independence, freeness and half independence. In Section 3, we introduce the notion of words and colored words and state our main result. In Proposition 1, we show that under a mild condition, if the marginal limit exists then the joint limit exists and can be expressed in terms of the marginals with the help of words and colored words. In Section 3.2, we discuss some examples. Finally, in Section 3.3, we give a proof of Proposition 1.

**2. Noncommutative probability spaces and independence.** A *noncommutative probability space* is a pair  $(\mathcal{A}, \phi)$  where  $\mathcal{A}$  is a unital complex algebra (with unity 1) and  $\phi: \mathcal{A} \rightarrow \mathbb{C}$  is a linear functional satisfying  $\phi(1) = 1$ .

Two important examples of such spaces are the following:

(1) Let  $(X, \mathcal{B}, \mu)$  be a probability space. Let  $L(\mu) = \bigcap_{1 \leq p < \infty} L^p(\mu)$  be the algebra of random variables with finite moments of all orders. Then  $(L(\mu), \phi)$  becomes a (commutative) probability space where  $\phi$  is the expectation operator, that is, integration with respect to  $\mu$ .

(2) Let  $(X, \mathcal{B}, \mu)$  be a probability space and let  $\mathcal{A} = \text{Mat}_n(L(\mu))$  be the space of  $n \times n$  complex random matrices with elements from  $L(\mu)$ . Then  $\phi$  equal to  $\frac{1}{n} E_\mu[\text{Tr}(\cdot)]$  or  $\frac{1}{n} [\text{Tr}(\cdot)]$  both yield noncommutative probabilities.

For any noncommuting variables  $x_1, \dots, x_n$ , let  $\mathbb{C}\langle x_1, x_2, \dots, x_n \rangle$  be the unital algebra of all complex polynomials in these variables. If  $a_1, a_2, \dots, a_n \in \mathcal{A}$ , then their joint distribution  $\mu_{\{a_i\}}$  is defined canonically by their mixed moments  $\mu_{\{a_i\}}(x_{i_1} \cdots x_{i_m}) = \phi(a_{i_1} \cdots a_{i_m})$ . That is,

$$\mu_{\{a_i\}}(P) = \phi(P(\{a_i\})) \quad \text{for } P \in \mathbb{C}\langle x_1, x_2, \dots, x_n \rangle.$$

*Convergence in law.* Let  $(\mathcal{A}_n, \phi_n)$ ,  $n \geq 1$ , and  $(\mathcal{A}, \phi)$  be noncommutative probability spaces and let  $\{a_i^n\}_{i \in J}$  be a sequence of subsets of  $\mathcal{A}_n$  where  $J$  is any finite subset of  $\mathbb{N}$ . Then we say that  $\{a_i^n\}_{i \in J}$  *converges in law* to  $\{a_i\}_{i \in J} \subset \mathcal{A}$  if for all complex polynomials  $P$ ,

$$\lim_{n \rightarrow \infty} \mu_{\{a_i^n\}_{i \in J}}(P) = \mu_{\{a_i\}_{i \in J}}(P).$$

To verify convergence in the above definition, it is enough to verify the convergence for all monomials  $q = x_{i_1} \cdots x_{i_k}$ ,  $k \geq 1$ .

*Independence and free independence of algebras.* Suppose  $\{\mathcal{A}_i\}_{i \in J} \subset \mathcal{A}$  are unital subalgebras. They are said to be *independent* if they commute and  $\phi(a_1 \cdots a_n) = \phi(a_1) \cdots \phi(a_n)$  for all  $a_i \in \mathcal{A}_{k(i)}$  where  $i \neq j \implies k(i) \neq k(j)$ .

These subalgebras are called *freely independent* or simply *free* if  $\phi(a_j) = 0$ ,  $a_j \in \mathcal{A}_{i_j}$  and  $i_j \neq i_{j+1}$  for all  $j$  implies  $\phi(a_1 \cdots a_n) = 0$ . The random variables (or elements of an algebra)  $(a_1, a_2, \dots)$  will be called independent (resp., free) if the subalgebras generated by them are independent (resp., free).

*Half independence of elements of an algebra.* Half independence arises in classification results for easy quantum groups and some quantum analogue of de Finetti’s theorem. We closely follow the developments in [3]. We shall see later how this half independence arises in the context of convergence of random matrices. To describe this, we need the concepts of half commuting elements and symmetric monomials.

*Half commuting elements.* Let  $\{a_i\}_{i \in J} \subset \mathcal{A}$ . We say that they *half commute* if  $a_i a_j a_k = a_k a_j a_i$ , for all  $i, j, k \in J$ . Observe that if  $\{a_i\}_{i \in J}$  half commute then  $a_i^2$  commutes with  $a_j$  and  $a_j^2$  for all  $i, j \in J$ .

*Symmetric monomials.* Suppose  $\{a_i\}_{i \in J} \subset \mathcal{A}$ . For any  $k \geq 1$ , and any  $\{i_j\} \subset J$ , let  $a = a_{i_1} a_{i_2} \cdots a_{i_k}$  be an element of  $\mathcal{A}$ . For any  $i \in J$ , let  $E_i(a)$  and  $O_i(a)$  be, respectively, the number of times  $a_i$  has occurred in the even positions and in the odd positions in  $a$ . The monomial  $a$  is said to be *symmetric* (with respect to  $\{a_i\}_{i \in J}$ ) if  $E_i(a) = O_i(a)$  for all  $i \in J$ . Else it is said to be nonsymmetric.

*Half independent elements.* Let  $\{a_i\}_{i \in J}$  in  $(\mathcal{A}, \phi)$  be half commuting. They are said to be half independent if (i)  $\{a_i^2\}_{i \in J}$  are independent and (ii) whenever  $a$  is nonsymmetric with respect to  $\{a_i\}_{i \in J}$ , we have  $\phi(a) = 0$ .

REMARK 1. The above definition is equivalent to that given in [3], although there is no notion of symmetric monomials there. As pointed out in [9], the concept of half independence does not extend to subalgebras.

EXAMPLE 1. This is Example 2.4 of [3]. Let  $(\Omega, \mathcal{B}, \mu)$  be a probability space and let  $\{\eta_i\}$  be a family of independent complex Gaussian random variables. Define  $a_i \in (M_2(L(\mu)), E[\text{Tr}(\cdot)])$  by

$$a_i = \begin{bmatrix} 0 & \eta_i \\ \bar{\eta}_i & 0 \end{bmatrix}.$$

These  $\{a_i\}$  are half independent.

REMARK 2. Let  $X, Y$  and  $Z$  be three self-adjoint elements of  $(\mathcal{A}, \phi)$  such that  $\phi(X) = \phi(Y) = \phi(Z) = 0$  but  $\phi(X^2), \phi(Y^2), \phi(Z^2) \neq 0$ . Following Remark 5.3.2 of [1]:

(i) If  $X, Y$  commute and are independent, then  $\phi(XY) = 0$  and  $\phi(XYXY) = \phi(X^2)\phi(Y^2) \neq 0$ .

(ii) If  $X, Y$  and  $Z$  are half independent, then  $\phi(XYZZYX) = 0$ . This happens since  $X$  appears two times in odd positions but zero times in even positions.

(iii) If  $X, Y, Z$  are free, then  $\phi(XYZXYZ) = 0$ . If they are half independent then  $\phi(XYZXYZ) = \phi(X^2)\phi(Y^2)\phi(Z^2) \neq 0$ .

### 3. Joint convergence of patterned matrices.

3.1. *Some preliminaries and the main result.* It is well known that the joint limit of independent Wigner matrices yields the freely independent semicircular law. At the same time, there are a host of results on the limiting spectrum of other matrices. Important examples include the sample variance–covariance, the Toeplitz and the Hankel matrices. The joint convergence of several sample variance–covariance matrices has also been investigated in the literature. However, joint convergence does not seem to have been addressed in any generality. In particular, it is not clear what other notions of independence are possible when we consider the joint limit of (independent) matrices.

Patterned matrices offer a general framework for which this question may be worth investigating. A general pattern matrix may be defined through a *link function*. Let  $d$  be a positive integer. Let  $\mathbb{Z}$  be the set of all integers and let  $\mathbb{N}$  be the set of all natural numbers. Let  $L_n : \{1, 2, \dots, n\}^2 \rightarrow \mathbb{Z}^d, n \geq 1$ , be a sequence of functions such that  $L_{n+1}(i, j) = L_n(i, j)$  whenever  $1 \leq i, j \leq n$ . We shall write  $L_n = L$  and call it the *link function* and by abuse of notation we write  $\mathbb{N}^2$  as the common domain of  $\{L_n\}$ . For our examples later, the value of  $d$  is either 1 or 2.

A typical patterned matrix is then of the form  $A_n = (x(L(i, j)))$  where  $\{x(i); i \geq 0\}$  or  $\{x(i, j); i, j \geq 1\}$  is a sequence of variables. In what follows, we only consider real symmetric matrices. Here are some well-known matrices and their link functions:

- (i) Wigner matrix  $W_n$ .  $L : \mathbb{N}^2 \rightarrow \mathbb{Z}^2$  where  $L(i, j) = (\min(i, j), \max(i, j))$ .
- (ii) Symmetric Toeplitz matrix  $T_n$ .  $L : \mathbb{N}^2 \rightarrow \mathbb{Z}$  where  $L(i, j) = |i - j|$ .
- (iii) Symmetric Hankel matrix  $H_n$ .  $L : \mathbb{N}^2 \rightarrow \mathbb{Z}$  where  $L(i, j) = i + j$ .
- (iv) Reverse circulant matrix  $RC_n$ .  $L : \mathbb{N}^2 \rightarrow \mathbb{Z}$  where  $L(i, j) = (i + j) \bmod n$ .
- (v) Symmetric circulant matrix  $SC_n$ .  $L : \mathbb{N}^2 \rightarrow \mathbb{Z}$  where  $L(i, j) = n/2 - |n/2 - |i - j||$ .

In general, we assume that the link function  $L$  satisfies Property B.

PROPERTY B.  $\Delta(L) = \sup_n \sup_{t \in \mathbb{Z}^d} \sup_{1 \leq k \leq n} \#\{l : 1 \leq l \leq n, L(k, l) = t\} < \infty$ .

In particular,  $\Delta(L) = 2$  for  $T_n, SC_n$ , and  $\Delta(L) = 1$  for  $W_n, H_n$  and  $RC_n$ .

Now let  $(\Omega, \mathcal{B}, \mu)$  be a probability space and let  $X_{i,n} : \Omega \rightarrow M_n$  for  $1 \leq i \leq p$  be symmetric patterned random matrices of order  $n$ . We shall refer to the  $p$  indices as  $p$  distinct *colors*. The  $(j, k)$ th entry of the matrix  $X_{i,n}$  will be denoted by  $X_{i,n}(L(j, k))$ .

ASSUMPTION I. Let the input sequence of each matrix in the collection  $\{X_{i,n}\}_{1 \leq i \leq p}$  be independent with mean zero and variance 1 and assume they are

also independent across  $i$ . Suppose the matrices have a common link function  $L$  which satisfies Property B and

$$\sup_{n \in \mathbb{N}} \sup_{1 \leq i \leq p} \sup_{1 \leq m \leq l \leq n} \mathbb{E}[|X_{i,n}(L(m, l))|^k] \leq c_k < \infty.$$

Suppressing the dependence on  $n$ , we shall simply write  $X_i$  for  $X_{i,n}$ . We view  $\{\frac{1}{\sqrt{n}} X_i\}_{1 \leq i \leq p}$  as elements of  $(\mathcal{A}_n = \text{Mat}_n(L(\mu)), \phi_n)$  where  $\phi_n = n^{-1} \mathbb{E}[\text{Tr}]$ . Denote the joint distribution of  $\{\frac{1}{\sqrt{n}} X_i\}_{1 \leq i \leq p}$  by  $\hat{\mu}_n$ . Then  $\{\frac{1}{\sqrt{n}} X_i\}_{1 \leq i \leq p}$  converges in law if

$$\begin{aligned} (1) \quad & \hat{\mu}_n(q) = \phi_n(q) \\ & = \frac{1}{n^{1+k/2}} \mathbb{E}[\text{Tr}(X_{i_1} \cdots X_{i_k})] \\ (2) \quad & = \frac{1}{n^{1+k/2}} \sum_{j_1, \dots, j_k} \mathbb{E}[X_{i_1}(L(j_1, j_2)) X_{i_2}(L(j_2, j_3)) \cdots X_{i_k}(L(j_k, j_1))] \end{aligned}$$

converges for all monomials  $q$  of the form  $q(\{X_i\}_{1 \leq i \leq p}) = X_{i_1} \cdots X_{i_k}$ .

To upgrade to almost sure convergence, we also define

$$\begin{aligned} \tilde{\mu}_n(q) &= \frac{1}{n^{1+k/2}} \text{Tr}[X_{i_1} \cdots X_{i_k}] \\ &= \frac{1}{n^{1+k/2}} \sum_{j_1, \dots, j_k} X_{i_1}(L(j_1, j_2)) X_{i_2}(L(j_2, j_3)) \cdots X_{i_k}(L(j_k, j_1)). \end{aligned}$$

All developments below are with respect to one fixed monomial  $q$  at a time.

*Circuit.* Any  $\pi : \{0, 1, 2, \dots, h\} \rightarrow \{1, 2, \dots, n\}$  with  $\pi(0) = \pi(h)$  is a *circuit* of length  $l(\pi) := h$ . The dependence of a circuit on  $h$  and  $n$  will be suppressed. A typical element in (2) can be now written as

$$(3) \quad \mathbb{E} \left[ \prod_{j=1}^k X_{i_j}(L(\pi(j-1), \pi(j))) \right].$$

If all  $L$ -values  $L(\pi(j-1), \pi(j))$  are repeated more than once in (3), then the circuit is *matched*. If  $L$  values are repeated exactly twice, then it is called *pair matched*. If the  $L$  values are repeated within the same color, then it is *color matched*.

For  $q = X_{i_1} X_{i_2} \cdots X_{i_k}$ , let for convenience, the corresponding sequence of colors be denoted by  $\{c_1, c_2, \dots, c_k\}$ . Also let  $H = \{\pi : \pi \text{ is a color matched circuit}\}$ . Define an equivalence relation on  $H$  by defining  $\pi_1 \sim_C \pi_2$  if and only if,  $c_i = c_j$  and

$$\begin{aligned} X_{c_i}(L(\pi_1(i-1), \pi_1(i))) &= X_{c_j}(L(\pi_1(j-1), \pi_1(j))) \\ \iff X_{c_i}(L(\pi_2(i-1), \pi_2(i))) &= X_{c_j}(L(\pi_2(j-1), \pi_2(j))). \end{aligned}$$

*Colored words.* An equivalence class induces a partition of  $\{1, 2, \dots, k\}$  and each block of the partition is associated with a color. Any such class can be expressed as a (colored) word  $w$  where letters appear in alphabetic order of their first occurrence and with a subscript to distinguish the color. For example, the partition  $(\{\{1, 3\}, 1\}, \{\{2, 4\}, 2\}, \{\{5, 7\}, 1\}, \{\{6, 8\}, 3\})$  is identified with the word  $a_1b_2a_1b_2c_1d_3c_1d_3$ . A typical position in a colored word would be referred to as  $w_{c_i}[i]$ .

Let the class of all (colored) circuits corresponding to a color matched word  $w$  and the class of all pair matched colored words be denoted, respectively, by

$$(4) \quad \begin{aligned} \Pi_{C_q}(w) &= \{\pi : w_{c_i}[i] = w_{c_j}[j] \iff X_{c_i}(L(\pi(i-1), \pi(i))) \\ &= X_{c_j}(L(\pi(j-1), \pi(j)))\}, \end{aligned}$$

$$(5) \quad \begin{aligned} CW_k(2) &= \{\text{all paired matched (within same color) words } w \text{ of length } k\} \\ &\hspace{20em} (k \text{ is even}). \end{aligned}$$

All the above notions have the corresponding noncolored versions. For instance, if we drop the colors from a colored word, then we obtain a *noncolored* word. For any monomial  $q$ , dropping the color amounts to dealing with only one matrix or in other words with the marginal distribution.

Let  $w[i]$  denote the  $i$ th entry of a noncolored word  $w$ . The equivalence class corresponding to  $w$  and the set of pair matched noncolored words will be denoted, respectively, by

$$(6) \quad \Pi(w) = \{\pi : w[i] = w[j] \iff L(\pi(i-1), \pi(i)) = L(\pi(j-1), \pi(j))\},$$

$$(7) \quad W_k(2) = \{\text{all paired matched words } w \text{ of length } k\} \quad (k \text{ is even}).$$

For any word  $w \in CW_k(2)$ , consider the noncolored word  $w'$  obtained by dropping the color. Then  $w' \in W_k(2)$ . Since we are dealing with one fixed monomial at a time, this yields a bijective mapping say

$$(8) \quad \psi_q : CW_k(2) \rightarrow W_k(2).$$

For any  $w \in CW_k(2)$ , define

$$p_{C_q}(w) = \lim_{n \rightarrow \infty} \frac{1}{n^{k/2+1}} |\Pi_{C_q}(w)| \quad \text{if the limit exists.}$$

PROPOSITION 1. Let  $\{X_i\}_{1 \leq i \leq p}$  be patterned matrices satisfying Assumption I. Fix any monomial  $q = X_{i_1} X_{i_2} \cdots X_{i_k}$ . Assume that, whenever  $k$  is even,

$$(9) \quad p(w) = \lim_{n \rightarrow \infty} \frac{1}{n^{k/2+1}} |\Pi(w)| \quad \text{exists for all } w \in W_k(2).$$

(a) Then  $p_{C_q}(w) = p(\psi_q(w))$  and for any  $k$ ,

$$\lim_{n \rightarrow \infty} \widehat{\mu}_n(q) = \sum_{w \in CW_k(2)} p_{C_q}(w) = \alpha(x_{i_1} \cdots x_{i_k}) \quad (\text{say})$$

with

$$|\alpha(x_{i_1} \cdots x_{i_k})| \leq \frac{k! \Delta(L)^{k/2}}{(k/2)! 2^{k/2}} \quad \text{if } k \text{ is even and each color} \\ \text{appear even number of times}$$

$$= 0 \quad \text{if } k \text{ is odd or a color appears odd number of times.}$$

(b)  $E[|\tilde{\mu}_n(q) - \hat{\mu}_n(q)|^4] = O(n^{-2})$  and hence  $\lim_{n \rightarrow \infty} \tilde{\mu}_n(q) = \alpha(x_{i_1} \cdots x_{i_k})$  almost surely.

REMARK 3. It is known from [4] that (9) holds true for Wigner, Toeplitz, Hankel, reverse circulant and symmetric circulant matrices. The quantity  $\frac{k!}{(k/2)! 2^{k/2}}$  above is the total number of noncolored pair matched words of length  $k$  ( $k$  even). Often not all pair matched words contribute to the limit and in such cases, this bound can be improved.

Consider the polynomial algebra  $\mathbb{C}\langle a_1, a_2, \dots, a_p \rangle$  in noncommutative indeterminates  $\{a_i\}_{1 \leq i \leq p}$  and define a linear functional  $\phi$  on it by

$$\phi(a_{i_1} \cdots a_{i_k}) = \lim_{n \rightarrow \infty} \hat{\mu}_n(X_{i_1} \cdots X_{i_k}).$$

Then Proposition 1 implies we have convergence in law of  $(\{\frac{1}{\sqrt{n}} X_i\}_{1 \leq i \leq p}, \phi_n)$  to  $(\{a_i\}_{1 \leq i \leq p}, \phi)$  where  $\phi_n$  equals  $\frac{1}{n} E[\text{Tr}(\cdot)]$  or  $\frac{1}{n} [\text{Tr}(\cdot)]$ . In the latter case, the convergence is almost sure.

REMARK 4. Proposition 1 shows that the joint moments of pattern matrices can be expressed as functions of pair matched words or, in other words, pair partitions. [5] consider bosonic, fermionic and  $q$ -Brownian motions and show that the joint distribution of certain operators (in some appropriate sense) can be expressed as functions on the set of pair partitions. It would be interesting to investigate if there are connections between the two types of models.

3.2. *Some examples.* From the above result,  $\lim \hat{\mu}_n(q) = 0$  when  $k$  is odd or when there is a color which appears an odd number of times in the monomial  $q$ . Henceforth, we thus assume that the order of the monomial is even and each color appears an even number of times.

EXAMPLE 2 (Wigner matrices). Joint convergence of the Wigner matrices was first studied in [10] and later many authors extended it. For details of the classical proof and further extensions, we refer the readers to [1]. Here we give a quick partial proof essentially translating the concept of noncrossing partitions that is used in the standard proof into words.

*Colored Catalan words.* Fix  $k \geq 2$ . If for a  $w \in CW_k(2)$ , sequentially deleting all double letters of the same color leads to the empty word then we call  $w$  a *colored Catalan word*. For example, the monomial  $X_1X_2X_2X_1X_1X_1$  has exactly two colored Catalan words  $a_1b_2b_2a_1c_1c_1$  and  $a_1b_2b_2c_1c_1a_1$ . A colored Catalan word associated with  $X_1X_2X_2X_1X_1X_1X_2X_2$  is  $a_1b_2b_2a_1c_1c_1d_2d_2$  which is not even a valid colored word for the monomial  $X_1X_1X_1X_1X_2X_2X_2X_2$ .

Let  $\{W_i\}_{1 \leq i \leq p}$  be an independent sequence of  $n \times n$  Wigner matrices satisfying Assumption I. Then from Proposition 1 and Remark 3,  $\{n^{-1/2}W_i\}_{1 \leq i \leq p}$  converges in law to  $\{a_i\}_{1 \leq i \leq p}$ . We show that  $\{a_i\}_{1 \leq i \leq p}$  are free and the marginals are distributed according to the semicircular law.

From Table 1 of [4], for noncolored words,  $p(w)$  equals 1 if  $w$  is a Catalan word and otherwise  $p(w) = 0$ . As a consequence, the marginals are semicircular. Now fix any monomial  $q = x_{i_1}x_{i_2} \cdots x_{i_{2k}}$  where each color appears an even number of times in the monomial.

Let  $w$  be a colored Catalan word. It remains Catalan when we ignore the colors. Hence from above,  $p_{C_q}(w) = p(\psi_q(w)) = 1$ . Likewise, if  $w$  is not colored Catalan then the word  $\psi_q(w)$  cannot be Catalan and hence  $p_{C_q}(w) = p(\psi_q(w)) = 0$ . Hence, if  $CAT_q$  denotes the set of colored Catalan words corresponding to a monomial  $q$  then from the above discussion,

$$\lim_{n \rightarrow \infty} \widehat{\mu}_n(q) = |CAT_q|.$$

Any double letter corresponds to a pair partition (within the same color) by the equivalence relation  $\sim_C$ . It is known that the number of Catalan words of length  $2k$  is same as the number of noncrossing pair partitions [denoted by  $NC_{2k}(2)$ ] of length  $2k$ . See [4] and Chapter 1 of [1] for proofs.

Since the elements of the same pair partition must belong to the same color, we have

$$|CAT_q| = \sum_{\pi \in NC_{2k}(2)} \prod_{(j, j') \in \pi} \mathbb{I}_{c_j = c_{j'}}.$$

This is precisely the free joint semicircular law corresponding to  $q$  (see Theorem 5.4.2 of [1]).

Incidentally, since the number of noncolored Catalan words of length  $2k$  is  $\frac{2k!}{k!(k+1)!}$ , we have  $|\alpha(x_{i_1} \cdots x_{i_{2k}})| \leq \frac{2k!}{k!(k+1)!}$ . Corollary 5.2.16 of [1] can hence be applied to claim the existence of a  $C^*$ -probability space with a state  $\phi$  and free semicircular random variables  $\{a_i\}$  in it.

**EXAMPLE 3 (Symmetric circulants).** The case of symmetric circulant is rather easy. These matrices are commutative and so the limit is also commutative.

Let  $\{SC_i\}_{1 \leq i \leq p}$  be an independent sequence of  $n \times n$  symmetric circulant matrices satisfying Assumption I. Then  $\{n^{-1/2}SC_i\}_{1 \leq i \leq p}$  converges in law to  $\{a_i\}_{1 \leq i \leq p}$  which are independent and the marginals are distributed according to the standard



Gaussian law. To see this, first recall that the total number of (noncolored) pair matched words of length  $2k$  equals  $\frac{2k!}{k!2^k} = C_k$  (say). Further (see [4]), for *any* pair matched word  $w \in W_{2k}(2)$ ,

$$p(w) = \lim_n \frac{1}{n^{1+k}} |\Pi(w)| = 1.$$

Now consider an order  $2k$  monomial where each color appears an even number of times. Hence, from Proposition 1, for any fixed monomial  $q$ ,

$$\lim_{n \rightarrow \infty} \widehat{\mu}_n(q) = \sum_{w \in CW_{2k}(2)} p_{C_q}(w) = |CW_{2k}(2)|.$$

Let  $l$  be the total number of distinct colors (distinct matrices) in the monomial  $q = x_{i_1} x_{i_2} \cdots x_{i_{2k}}$ . Let  $2 \times n_i$  be the number of matrices of the  $i$ th color. Then the set of all pair matched colored words of length  $2k$  is obtained by forming pair matched subwords of color  $i$  of lengths  $2n_i$ ,  $1 \leq i \leq l$ . Hence,

$$(10) \quad \phi(a_{i_1} \cdots a_{i_{2k}}) = \prod_{i=1}^l C_{n_i}.$$

Thus if  $\{a_1, \dots, a_p\}$  denotes i.i.d. standard normal random variables, then the above is the mixed moment  $E[\prod_{i=1}^l a_i^{2n_i}]$ .

EXAMPLE 4 (Reverse circulant). It can be easily observed using the link function that the reverse circulant matrices are half commuting. This motivates the next theorem.

THEOREM 1. Let  $\{RC_i\}_{1 \leq i \leq p}$  be an independent sequence of  $n \times n$  reverse circulant matrices satisfying Assumption I. Then  $\{n^{-1/2}RC_i\}_{1 \leq i \leq p}$  converges in law to half independent  $\{a_i\}_{1 \leq i \leq p} \in (M_2(L(\mu)), E[\text{Tr}(\cdot)])$  where  $a_i = \begin{bmatrix} 0 & \eta_i \\ \bar{\eta}_i & 0 \end{bmatrix}$  and  $\eta_i$  are i.i.d. complex Gaussian.

To prove the result, we need the following notion.

Colored symmetric words. Fix  $k \geq 2$ . A word  $w \in CW_k(2)$  is called *colored symmetric* if each letter occurs once each in an odd and an even position *within the same color*. Clearly, every colored Catalan word is a colored symmetric word.

PROOF OF THEOREM 1. Consider a monomial  $q$  of length  $2k$  where each color appears an even number of times. From the single matrix case, it follows that  $p(w) = 0$  if  $w$  is not a symmetric word (see Table 1 of [4]). If  $w$  is not a colored symmetric word, then  $\psi_q(w)$  is not a symmetric word and hence for such  $w$ ,  $p_{C_q}(w) = p(\psi_q(w)) = 0$ . Hence, we may restrict to colored symmetric words and then we have by Proposition 1(a) that

$$\lim_{n \rightarrow \infty} \widehat{\mu}_n(q) = |CS_q(w)|,$$

where  $CS_q(w)$  is the collection of all colored symmetric words of length  $2k$ .

The number of symmetric words of length  $2k$  is  $k!$ . Let, as before,  $l$  be the number of distinct colors in the monomial and  $2n_i$  be the number of matrices of the  $i$ th color. All symmetric words are obtained by arranging the  $2n_i$  letters of the  $i$ th color in a symmetric way for  $i = 1, 2, \dots, l$ .

It is then easy to see that these arguments imply that

$$(11) \quad |CS_q(w)| = n_1! \times n_2! \times \dots \times n_l!$$

First, observe that if the monomial  $a_{i_1}a_{i_2} \dots a_{i_k} \in (M_2(L(\mu)), E[\text{Tr}(\cdot)])$  is non-symmetric, then

$$E(\text{Tr}[a_{i_1}a_{i_2} \dots a_{i_k}]) = 0.$$

If instead  $q(\{a_i\}) = a_{i_1}a_{i_2} \dots a_{i_{2k}}$  is symmetric, then we have by half independence (Example 1),

$$E(\text{Tr}[a_{i_1} \dots a_{i_{2k}}]) = n_1! \times n_2! \times \dots \times n_l! = \lim_{n \rightarrow \infty} \widehat{\mu}_n(q).$$

So it follows from (11) that the joint limit is asymptotically half independent. Incidentally the moments  $\{k!, k \geq 1\}$  are the  $(2k)$ th moments of the symmetrized Rayleigh distribution.  $\square$

EXAMPLE 5 (Toeplitz and Hankel). Consider first the Toeplitz matrix. Since  $p(w)$  exists, from Proposition 1, we have the joint convergence for Toeplitz matrices. For any fixed monomial  $q$ , let  $SNC_q$  be the colored symmetric words which are not Catalan. Then we obtain the following:

$$\begin{aligned} \phi(a_{i_1} \dots a_{i_k}) &= \sum_{w \in CW_k(2)} p_{C_q}(w) \\ &= \sum_{w \in \text{CAT}_q} p_{C_q}(w) + \sum_{w \in SNC_q} p_{C_q}(w) \\ &\quad + \sum_{\text{other pair matched colored words}} p_{C_q}(w) \\ &= |\text{CAT}_q| + \sum_{w \in SCN_q} p_{C_q}(w) \\ &\quad + \sum_{\text{other pair matched colored words}} p_{C_q}(w). \end{aligned}$$

Consider  $q(X_1, X_2, X_3) = X_1X_2X_3X_1X_2X_3$  where  $X_1, X_2$  and  $X_3$  are scaled independent Toeplitz matrices. From Table 4 of [4],

$$p_{C_q}(a_1b_2c_3a_1b_2c_3) = p(\psi_q(a_1b_2c_3a_1b_2c_3)) = p(abcabc) = \frac{1}{2}.$$

For this monomial, the only pair matched colored word possible is  $a_1b_2c_3a_1b_2c_3$  and hence  $\phi(a_1a_2a_3a_1a_2a_3) = \frac{1}{2} \neq 0$ . Thus, the limit is not free.

Now let  $q(X_1, X_2, X_3) = X_1X_2X_3X_2X_3X_1$ . Then the only pair of matched colored word is  $a_1b_2c_3b_2c_3a_1$  and  $\phi(a_1a_2a_3a_2a_3a_1) = p_{C_q}(a_1b_2c_3b_2c_3a_1) = p(abc bca) = \frac{2}{3}$ . On the other hand we have already seen that  $\phi(a_1a_2a_3a_1a_2a_3) = \frac{1}{2}$ . Since the two contributions are not equal, the Toeplitz limit is not independent.

If they had been half independent, then  $\phi(a_1a_2a_3a_1a_2a_3) = \phi(a_1^2)\phi(a_2^2) \times \phi(a_3^2) = 1$ , but that is not the case. Thus, the Toeplitz limit is not free, independent or half independent.

For Hankel matrices, the colored nonsymmetric words *do not* contribute to the limit. So for any fixed monomial  $q$  we have

$$\phi(a_{i_1} \cdots a_{i_k}) = |\text{CAT}_q| + \sum_{w \in \text{SNC}_q} p_{C_q}(w).$$

That the Hankel limit is also not free, half independent or independent can be checked along the above lines by considering appropriate monomials and their contributions. It is interesting to note that Hankel matrices do not half commute and that is why even though the limits vanish on nonsymmetric words they are not half independent.

3.3. *Proof of Proposition 1.* (a) Fix a monomial  $q = q(\{X_i\}_{1 \leq i \leq p}) = X_{i_1} \cdots X_{i_k}$ . Since  $\psi_q$  is a bijection,

$$\Pi_{C_q}(w) = \Pi(\psi_q(w)) \quad \text{for } w \in CW_k(2).$$

Hence using (9),

$$\lim_{n \rightarrow \infty} \frac{1}{n^{k/2+1}} |\Pi_{C_q}(w)| = \lim_{n \rightarrow \infty} \frac{1}{n^{k/2+1}} |\Pi(\psi_q(w))| = p(\psi_q(w)) = p_{C_q}(w).$$

For simplicity, denote

$$\mathbb{T}_{\mathbf{j}} = \mathbb{E}[X_{i_1}(L(j_1, j_2))X_{i_2}(L(j_2, j_3)) \cdots X_{i_k}(L(j_k, j_1))] \quad \text{for } \mathbf{j} = (j_1, \dots, j_k).$$

Then

$$(12) \quad \widehat{\mu}_n(q) = \frac{1}{n^{k/2+1}} \sum_{j_1, \dots, j_k} \mathbb{T}_{\mathbf{j}}.$$

In the monomial, if any color appears once, then by independence and mean zero condition,  $\mathbb{T}_{\mathbf{j}} = 0$  for every  $\mathbf{j}$ . Hence,  $\widehat{\mu}_n(q) = 0$ .

So henceforth, assume that each color appearing in the monomial, appears at least twice. Now again, if  $\mathbf{j}$  belongs to a circuit which is not color matched, then  $\mathbb{T}_{\mathbf{j}} = 0$ .

Now form the following matrix  $M$ :

$$M(L(i, j)) = |X_{i_1}(L(i, j))| + |X_{i_2}(L(i, j))| + \cdots + |X_{i_k}(L(i, j))|.$$

Observe that

$$|\mathbb{T}_{\mathbf{j}}| \leq \mathbb{E}[M(L(j_1, j_2) \cdots M(L(j_k, j_1))].$$

From Lemma 1 of [4], it is known that the total contribution of all circuits which have at least one three match, is zero in the limit.

As a consequence of the above discussion, if  $k$  is odd, then  $\widehat{\mu}_n(q) \rightarrow 0$ . So assume  $k$  is even. In that case, we need to consider only circuits which are pair matched. Further this pair matching must occur within the same color. If  $\mathbf{j}$  belongs to any such circuit, then by independence, mean zero and variance one condition,  $\mathbb{T}_{\mathbf{j}} = 1$ .

Then using all the facts established so far,

$$\begin{aligned} \lim_{n \rightarrow \infty} \widehat{\mu}_n(q) &= \lim_{n \rightarrow \infty} \frac{1}{n^{k/2+1}} \sum_{\substack{\pi : \pi \text{ pair matched} \\ \text{within colors}}} \mathbb{E}[X_{i_1}(L(\pi(0), \pi(1))) \cdots \\ &\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \times X_{i_k}(L(\pi(k-1), \pi(k)))] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^{k/2+1}} \sum_{w \in CW_k(2)} \sum_{\pi \in \Pi_{C_q}(w)} \mathbb{E}[X_{i_1}(L(\pi(0), \pi(1))) \cdots \\ &\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \times X_{i_k}(L(\pi(k-1), \pi(k)))] \\ &= \sum_{w \in CW_k(2)} p_{C_q}(w). \end{aligned}$$

The last claim in part (a) follows since

$$\sum_{w \in CW_{2k}(2)} p_{C_q}(w) = \sum_{w \in CW_{2k}(2)} p(\psi_q(w)) \leq \sum_{w \in W_{2k}(2)} p(w) \leq \frac{(2k)! \Delta(L)^k}{k! 2^k}.$$

The last inequality above is shown in [4].

(b) For part (b), the following notions will be useful:  $l$  circuits  $\pi_1, \pi_2, \dots, \pi_l$  are said to be *jointly matched* if each  $L$ -value occurs at least twice across all circuits. They are said to be *cross matched* if each circuit has at least one  $L$ -value which occurs in at least one of the other circuits. We can write

$$(13) \quad \mathbb{E}[|\widetilde{\mu}_n(q) - \widehat{\mu}_n(q)|^4] = \frac{1}{n^{2k+4}} \sum_{\pi_1, \pi_2, \pi_3, \pi_4} \mathbb{E}\left[\prod_{l=1}^4 (\mathbb{X}_{\pi_l} - \mathbb{E}\mathbb{X}_{\pi_l})\right],$$

where

$$\mathbb{X}_{\pi} = X_{i_1}(L(\pi(0), \pi(1))) \cdots X_{i_k}(L(\pi(k-1), \pi(k))).$$

If  $(\pi_1, \pi_2, \pi_3, \pi_4)$  are not jointly matched, then one of the circuits, say  $\pi_j$ , has an  $L$  value which does not occur anywhere else. Also note that  $\mathbb{E}\mathbb{X}_{\pi_j} = 0$ . Hence,

using independence

$$(14) \quad \mathbb{E} \left[ \prod_{l=1}^4 (\mathbb{X}_{\pi_l} - \mathbb{E}\mathbb{X}_{\pi_l}) \right] = \mathbb{E} \left[ \mathbb{X}_{\pi_j} \prod_{l=1, l \neq j}^4 (\mathbb{X}_{\pi_l} - \mathbb{E}\mathbb{X}_{\pi_l}) \right] = 0.$$

If  $(\pi_1, \pi_2, \pi_3, \pi_4)$  is jointly matched but is not cross matched then one of the circuits, say  $\pi_j$  is only self-matched, that is, none of the  $L$ -values is shared by the other circuits. Then by independence,

$$(15) \quad \mathbb{E} \left[ \prod_{l=1}^4 (\mathbb{X}_{\pi_l} - \mathbb{E}\mathbb{X}_{\pi_l}) \right] = \mathbb{E} \left[ (\mathbb{X}_{\pi_j} - \mathbb{E}\mathbb{X}_{\pi_j}) \prod_{l=1, l \neq j}^4 (\mathbb{X}_{\pi_l} - \mathbb{E}\mathbb{X}_{\pi_l}) \right] = 0.$$

Since  $\{X_{i,n}\}_{1 \leq i \leq n}$  satisfy Assumption I,  $\mathbb{E}[\prod_{l=1}^4 (\mathbb{X}_{\pi_l} - \mathbb{E}\mathbb{X}_{\pi_l})]$  is uniformly bounded over all  $(\pi_1, \pi_2, \pi_3, \pi_4)$ .

The arguments given in [6] for Toeplitz and Hankel matrices can be extended to our set up easily to yield the following: let  $Q_{k,4}$  be the number of quadruples of circuits  $(\pi_1, \pi_2, \pi_3, \pi_4)$  of length  $k$  such that they are jointly matched and cross matched with respect to  $L$ . If  $L$  satisfy Property B, then there exists a constant  $K$  such that  $Q_{k,4} \leq Kn^{2k+2}$ . Using this, and (13)–(15),

$$\mathbb{E}[|\tilde{\mu}_n(q) - \hat{\mu}_n(q)|^4] \leq K \frac{n^{2k+2}}{n^{2k+4}} = O(n^{-2}).$$

Now by an easy application of Borel–Cantelli lemma  $\tilde{\mu}_n(q)$  converges almost surely.

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