Indian Statistical Institute
Vectors and Matrices I
B-I, Midsem

Date: Sep 04, 2017 Duration: 3hrs.

Attempt all questions. The maximum you can score is 50. Justify all your steps. This is a closed book, closed notes examination. Use of determinants (not covered in class) is discouraged.

If copying is detected in the solution for any problem, all the students involved in the copying will get 0 for that problem. Also an additional penalty of 5 will be subtracted from the overall aggregate of each of these students.

1. Consider the subspace $S = span\{(1,1,0,0),(0,1,1,0),(-2,1,3,0)\} \le \mathbb{R}^4$. Use GSO to find an ONB of S. Extend it to an ONB of $\mathbb{R}^4$. [10+6 marks]

2. State true (providing proof) or false (providing counterexample):

   (a) If $span\{\vec{x}_1, ..., \vec{x}_n\} = span\{\vec{x}_1, ..., \vec{x}_{n-1}\}$, then $\{\vec{x}_1, ..., \vec{x}_n\}$ must be linearly dependent. [3]

   (b) If Let A be an 5×6 matrix. Then nullity(A) =nullity(A'). [3]

   (c) If A, B are square matrices of the same size, and $r(A) = r(B)$, then we must have $r(A^2) = r(B^2)$. [3]

   (d) $S = \{(x,y) \ : \ xy = 0\}$ is a subspace of $\mathbb{R}^2$. [3]

3. Consider the following matrix

$$\begin{bmatrix} 0 & 1 & 0 & 4 & 4 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   Is it in the reduced row echelon form? Justify your answer. Find a basis for its null space. [3+7]

4. Show that for any real matrix $A_{m \times n}$ we must have $\mathcal{N}(A) = \mathcal{N}(A'A)$. Hence, or otherwise, prove that $r(A'A) = r(A)$. [4+3]

5. Let A be an 8×5 matrix with rank 3. Let $\vec{b}$ be a nonzero vector in $\mathcal{N}(A')$.

   (a) Show that the system $A\vec{x} = \vec{b}$ must be inconsistent.

   (b) How many least squares solution will the system $A\vec{x} = \vec{b}$ have?

   [5+5]

INDIAN STATISTICAL INSTITUTE

MID-TERM EXAMINATION (2017-18)

B. STAT. I YEAR

ANALYSIS I

Date : 05.09.2017 Maximum Marks : 60 Time : 2 hours

The question carries 65 marks. Maximum you can score is 60. Precisely justify all your steps. Carefully state all the results you are using.

1. In each of the following cases, decide whether the given set $A \subseteq \mathbb{R}$ is bounded above (and/or below) and if it is, find its supremum (and/or infimum).

   (a) $A = \{x \in \mathbb{R} : 3x^2 - 10x + 3 < 0\}$.

   (b) $A = \left\{ \left| x + \frac{1}{x} \right| : x \neq 0 \right\}$. [$2 + 3 = 5$]

2. Let $A \subseteq \mathbb{R}$ be bounded above. Show that there is a sequence $\{a_n\}$ taking values in A such that $\lim_{n \to \infty} a_n = \sup A$. [5]

3. In each of the following cases, decide whether the given sequence $\{a_n\}$ is convergent or not and if it is, find its limit.

   (a) $a_n = \dfrac{3^n + (-2)^n}{3^{n+1} + (-2)^{n+1}}$

   (b) $a_n = \dfrac{n^{2/3} \sin(n!)}{n+1}$

   (c) $a_n = \dfrac{(-1)^n}{n} + \dfrac{1 + (-1)^n}{2}$ [$5 + 5 + 5 = 15$]

4. Find the lim sup and lim inf of the following sequences :

   $(a)\ a_n = (5^n + 7^n)^{1/n}$ $(b)\ a_n = \frac{n}{7} - \left[\frac{n}{7}\right]$,

   where $[x]$ is the largest integer $\leq x$. [$5 + 5 = 10$]

5. Let $0 < \alpha < 1$. Given $x_0, x_1 \in \mathbb{R}$, define $x_{n+1} = \alpha x_n + (1 - \alpha) x_{n-1}$ for all $n \geq 1$.

   Prove that $\{x_n\}$ is Cauchy and hence converges.

   Find $\lim_{n \to \infty} x_n$ in terms of x_0, x_1 and α. [$6 + 4 = 10$]

6. Test the following series for convergence. [$5 + 5 = 10$]

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3[\sqrt{2} + (-1)^n]^n}{3^n}$.

   (b) $\displaystyle\sum_{n=1}^{\infty} n^p(\sqrt{n+1} - \sqrt{n}), p \in \mathbb{R}$.

7. Find all real x and s such that the power series $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^s}$ converges. For what values of x and s does the series converge absolutely? [10]

Probability theory 1

Midterm exam
B. Stat 1st year
Maximum marks: 40
Time: 2 hours

September 6, 2017

1. **(8 points)** Consider a coin with probability of Heads being p. The coin is tossed repeatedly, until two consecutive tosses give the same outcome. Write down the sample space, and the probability of each outcome in the sample space.

2. **(10 points)** Suppose that for each n, A_n, B_n, C_n are pairwise independent events such that

$$P(A_n) = P(B_n) = P(C_n) = p_n > 0\,.$$

If it holds that

$$\lim_{n\to\infty} p_n = 0\,,$$

then calculate

$$\lim_{n\to\infty} \frac{P(A_n \cup B_n \cup C_n)}{p_n}$$

3. **(12 points)** A fair die is rolled 20 times. What is the probability that the sum of the numbers obtained is strictly larger than 25 ?

4. **(10 points)** An urn contains W white balls and B black balls, with both W and B being positive integers. Balls are drawn from the urn, one by one, without replacement, until at least one ball of both colours is drawn. Denote by Y the number of draws needed. Calculate $\mathrm{E}(Y)$.

INDIAN STATISTICAL INSTITUTE
B. Stat. I: 2017–2018
Introduction to Programming & Data Structure
Semestral Examination

Date: 07. 09. 2017 | Marks: 50 | Time: 3 Hours

Answer any part of any question. The question is of 55 marks. The maximum marks you can get is 50. Please write all the part answers of a question at the same place.

1. (a) Write the recursive as well as non-recursive version of a function in C that can evaluate $f(n) = 2f(n-1) + 3f(n-2)$ with initial condition $f(0) = 0, f(1) = 1$.
   (b) Write a function in C to check whether a string contains at least one vowel.
   (c) Write a function in C to calculate $a^b \bmod n$, where a, b, n are positive integers. Do not use the 'pow' function available in C.

   3+3+4 = 10

2. (a) Write down the algorithm for heap sort.
   (b) Derive the time complexity to construct a heap with n many points.
   (c) Construct a heap with the data set 1201, 137, 166, x, 1210, 191, 981, 367, 485, where x is two least significant digits of your roll number.

   10+5+5 = 20

3. (a) Explain an efficient data structure for implementing polynomials.
   (b) Write a function in C programming language to add two polynomials.
   (c) Show how your function works (step by step) when the polynomials $x^{101} + 3x^5 + 7$ and $x^{98} + x^5 + 18$ are added.

   2+8+5 = 15

4. (a) How many three input three output Boolean functions are there? Out of those how many are reversible?
   (b) Prove that any n-input m-output Boolean function can be implemented using two-input NAND gates only.
   (c) Efficiently implement the four-input one-output Boolean function

   $$\sum\{1, 2, 4, 7, 8, 11, 13, 14\} \text{ (SOP)}$$

   with any logic gate of your choice.

   2+4+4 = 10

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination: 2017-18

B. STAT 1st Year

STATISTICAL METHODS I

08.09.2017 Full Marks: 30 Duration: 1 hr 30 minutes

[Calculators are allowed; answer all questions.]

1. The life of electric bulbs manufactured by Company A for 50 bulbs (in 100 hour unit) is given below.

9.5	15.2	25.1	0.1	4.5	15.1	10.3	1.5	5.5	3.3	8.5
10.6	7.0	8.4	4.0	7.7	23.4	10.3	5.7	3.7	2.7	1.2
4.3	0.1	15.2	21.6	0.6	6.6	21.2	19.2	4.5	5.9	3.9
21.4	5.9	15.0	10.2	23.3	5.2	12.6	5.9	3.5	14.2	22.7
16.5	1.8	26.2	3.7	17.3	15.4					

A frequency distribution of the same for a number of bulbs manufactured by Company B is also available and given in Table 1.

Table 1: Table showing frequency distribution of life of electric bulbs

Class intervals for life (in 100 hour unit)	Frequency
0.1 — 3.0	12
3.1 — 6.0	21
6.1 — 9.0	10
9.1 — 12.0	8
12.1 — 15.0	5
15.1 — 18.0	3
18.1 — 21.0	2
21.1 — 24.0	4
24.1 — 27.0	3
27.1 — 30.0	4

(a) Compare the two frequency distributions and write a short report. Each comment in your report must be justified properly with supporting measures and / or diagrams.

(b) Later it was discovered that during the collection of data, information of 21 bulbs manufactured by Company B was not included in the dataset, because 16 bulbs were found to be defective and the life hours of 5 bulbs were found to be 42.5, 58.7, 37.6, 43.2, and 65.1. But a statistician wants to include this information in the second dataset. Assume that the life of each of 16 defective bulbs can be taken as 0. Do you want to modify your analysis and hence the report? If yes, do it; if not, justify your reason for not modifying this analysis and report. Supporting measures and / or diagrams must be provided.

(c) Calculate a suitable measure of central tendency for of the final dataset for Company A and Company B and comment.

((4+4+4) + (10) + 4 = 26)

2. Develop a formula to calculate mode for a frequency distribution of a continuous variable where no two class-widths are equal. (4)

Indian Statistical Institute

Vectors and Matrices I

B-I, Midsem

Date: Oct 23. 2017 Duration: 3hrs.

Attempt all questions. The maximum you can score is 30. Justify all your steps. This is a closed book, closed notes examination. Use of determinants (not covered in class) is discouraged.

1. Let $A_{m\times n}$ and $\vec{b}_{m\times 1}$ be such that the system $A\vec{x} = \vec{b}$ has infinitely many solutions. Then prove or disprove this statement: for every $\vec{c}_{m\times 1}$ the system $A\vec{x} = \vec{c}$ must also have infinitely many solutions. [8]

2. Let $A\vec{x} = \vec{b}$ be a (possibly inconsistent) system of equations. Here $\vec{b} = (1, 2, 3)'$, and A has QR-decomposition given by

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix} \text{ and } R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

   Find a least square solution of the system. Is this solution unique? Explain. [10+5]

3. Let A be a 3×4 matrix with columns $\vec{a}_1, ..., \vec{a}_4$. If

$$\vec{b} = \vec{a}_1 + \vec{a}_2 + \vec{a}_3 + \vec{a}_4,$$

   then what can you conclude about the number of solutions to the linear system $A\vec{x} = \vec{b}$? Explain. [10]

4. $A_{m\times n}$, $\vec{w}_{n\times 1}$ and $\vec{b}_{m\times 1}$ are such that $\vec{b} - A\vec{w}$ is orthogonal to $A\vec{w}$. Prove or disprove: $\vec{w}$ must be a least square solution of the system $A\vec{x} = \vec{b}$. [10]

5. Find the reduced row echelon form R of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}.$$

   Do A and R have the same column space? Justify your answer. [5+2]

INDIAN STATISTICAL INSTITUTE

SUPPLEMENTARY MID-TERM EXAMINATION (2017–18)

B. STAT. I YEAR

ANALYSIS I

Date : 25.10.2017 Maximum Marks : 60 Time : 2 hours

The question carries 65 marks. Maximum you can score is 60. Precisely justify all your steps. Carefully state all the results you are using.

1. Given two sequences $\{a_n\}$ and $\{b_n\}$ of real numbers, define a sequence $\{c_n\}$ as follows : $c_{2n} = a_n$ and $c_{2n-1} = b_n$.

   (a) If $\{a_n\}$ converges and $\lim_{n\to\infty}(a_n - b_n) = 0$, prove that $\{c_n\}$ converges. [5]

   (b) Can you drop the condition "$\{a_n\}$ converges" in (a)? [5]

2. Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of positive numbers such that $\{a_n\}$ is a bounded sequence and $\lim_{n\to\infty} a_n/b_n = \infty$. What can you conclude about $\{b_n\}$? [10]

3. (a) For every $m \in \mathbb{N}$, show that $\dfrac{1}{\sqrt{m+1}} < 2(\sqrt{m+1} - \sqrt{m}) < \dfrac{1}{\sqrt{m}}$.

   (b) Deduce that for every $n \in \mathbb{N}$, $2(\sqrt{n} - 1) < \sum_{k=1}^{n} \dfrac{1}{\sqrt{k}} < 2\sqrt{n} - 1$.

   (c) If $a_n = 2\sqrt{n} - \sum_{k=1}^{n} \dfrac{1}{\sqrt{k}}$, show that $\{a_n\}$ converges. [5 + 5 + 5 = 15]

4. Find $\lim_{n\to\infty} \left(\dfrac{n^n}{n!}\right)^{1/n}$. [5]

5. Given that $\sum_{n=1}^{\infty} a_n$ converges absolutely, show that each of the following series also converges absolutely [6 + 7 + 2 = 15]

   $(a)\ \sum_{n=1}^{\infty} a_n^2 \qquad (b)\ \sum_{n:a_n\neq -1} \dfrac{a_n}{a_n+1} \qquad (c)\ \sum_{n=1}^{\infty} \dfrac{a_n^2}{a_n^2+1}$

6. Test the following series for convergence. [5 + 5 = 10]

   (a) $\sum_{n=1}^{\infty} n^{-1-1/n}$.

   (b) $\sum_{n=1}^{\infty} \sin\left(\dfrac{n^2}{2^n}\right)$.

Mid Sem

B Stat I
26.10.2017

Time: $1\frac{1}{2}$ hours
Total Marks: 50

1. Write a C Program to check whether an integer is palindrome. 10

2. Write an algorithm to describe how one can delete an element from a heap 15

3. (a) Write an algorithm to explain how a linked list can be used as a stack.
   (b) Implement your algorithm with proper data structure in C programming language
   5+10=15

4. (a) Write a C ~~function~~ program two Calculate the ~~g~~ GCD and LCM of two integers.
   (b) Explain whether your function always provides correct result.
   5+5 =10

Sutharny Maitra
26/10/2017

INDIAN STATISTICAL INSTITUTE

Mid-Semester (Sypplementary) Examination: 2017-18

B. STAT 1st Year

STATISTICAL METHODS I

27.10.2017 Full Marks: 30 Duration: 1 hr 30 minutes

[Calculators are allowed; answer all questions.]

1. Amount of iron per unit mass in a chemical compund needs to be evaluated for randomly selected samples of that compound. Two technicians A and B are employed for this purpose. A and B made respectively 50 and 32 determinations of the amount of iron per suitable unit. Note that it can take any value within the feasible range; the measuring instruments used by the two technicians can record only up to one place of decimal. Their results are given in the following table:

Determination by A	Determination by B
8.6, 7.9, 8.7, 9.1, 7.7, 8.2, 8.4, 9.2, 8.4, 8.5, 9.1, 8.7, 8.4, 7.9, 6.9, 7.3, 8.1, 9.3, 9.4, 7.3, 7.9, 9.0, 9.1, 8.4, 7.7, 8.3, 7.7, 8.1, 8.3, 8.2, 7.8, 8.1, 8.5, 7.8, 8.2, 7.0, 9.1, 9.1, 7.4, 7.4, 9.5, 9.0, 7.1, 9.0, 6.7, 6.7, 8.8, 7.7, 7.5, 7.6	9.1, 8.2, 7.6, 8.4, 8.7, 6.8, 7.4, 8.3, 9.1, 8.6, 7.1, 8.2, 8.5, 8.6, 8.8, 8.7, 7.9, 9.2, 8.4, 8.4, 7.8, 8.7, 8.4, 9.0, 7.7, 7.5, 7.6, 7.5, 7.5, 8.4, 8.7, 9.1

(a) Compare the frequency distributions and write a short report. Each comment in your report must be justified properly with supporting measures and / or diagrams.

(b) Later it was found that the instrument used by technician B rounded off the values to 10 as soon as it encounters a a piece of compound containing more than 9.8. As such he observed 12 such values (equal to 10) and ignored by thinking that the values are erroneous. Later a statistician wants to retain these observations and merge with the second dataset. Do you agree with the statistician and redo your comparative study with first dataset? Whether your answer is 'yes' or 'no', justify your argument and present a final anlaysis along with a short report. Supporting measures and /or diagrams must be provided in favour your analysis/explanation. Also write a final report on this comparative study. [15 + 15 = 30]

First Semester Examinations (2017-2018)
B. Stat. – 1st yr
Remedial English
100 Marks
1½ hours

Date: 17th November 2017

1) Write an essay on any one of the following topics. Five paragraphs are expected: 60 marks
a) Libraries
b) Cricket versus other games in India
c) The decline of literature is the decline of a nation.

2) Fill in the blanks with appropriate prepositions (Write the full sentence): 20 marks
a) The dog ran _______ the cat.
b) The bus stopped _____ the crossing.
c) Tears flowed _______ his cheek.
d) I was flying _______ Bagdogra ________ Kolkata.
e) The kingfisher dived ________ the stream.
f) The drain _________ the road had overflowed.
g) Please try _________ think _________ the agenda _________the meeting.
h) My elder brother was born a year __________ me.
i) Keep the jug ______ the table.
j)I am obliged______you______ your kindness.
k) I am not aware _______ their plans.
l) He had last attended office_________Monday.
m) The suggestion was not acceptable _______ us.
n) You must abstain _________ smoking and drinking.
o)He was admitted _________ the hospital.
p)I was amazed _________ her performance.

3) Fill in the blanks with appropriate words (Write the full sentence): 20 marks

There ____ a thin, crisp, continuous patter ____ somewhere in the heart of the crawling bank. The cloud was within fifty yards of where we lay, and we glared at it, uncertain of what ______ was about to break from the heart of it. I was at Holmes's elbow, and I glanced for an _______at his face. It was pale and exultant, his ____ shining brightly in the moonlight. But ________ they started forward in a rigid, fixed ______ and his ____ parted in amazement. At the _____ instant Lestrade gave a yell of ______ and threw himself ____ downward upon the _____. I sprang to my ____, my inert hand grasping my pistol, my _____paralyzed by the dreadful shape which had_______ out upon us from the shadows of the fog. A hound it was, an enormous coal-black hound, but not such a hound as mortal ____ have ever seen. Fire burst from its open ______, its ____ glowed with a smouldering glare, its muzzle and hackles and dewlap were outlined in flickering flame. Never in the delirious _____ of a disordered brain could anything more savage, more appalling, more hellish be conceived than that dark form and savage face which broke _______ us out of the wall of fog.

INDIAN STATISTICAL INSTITUTE

Semester Examination: 2017-18

B. STAT 1st Year

STATISTICAL METHODS I

20.11.2017 Full Marks: 85 Duration: 3 hours

[Calculators are allowed; answer all questions; You can score mximum 70 marks.]

1. Amount of iron per unit mass in a chemical compund needs to be evaluated for randomly selected samples of that compound. Two technicians A and B are employed for this purpose. A and B made respectively 50 and 32 determinations of the amount of iron per suitable unit. Note that it can take any value within the feasible range; the measuring instruments used by the two technicians can record only up to one place of decimal. Their results are given in the following table:

Determination by A	Determination by B
8.6, 7.9, 8.7, 9.1, 7.7, 8.2, 8.4, 9.2, 8.4, 8.5, 9.1, 8.7, 8.4, 7.9, 6.9, 7.3, 8.1, 9.3, 9.4, 7.3, 7.9, 9.0, 9.1, 8.4, 7.7, 8.3, 7.7, 8.1, 8.3, 8.2, 7.8, 8.1, 8.5, 7.8, 8.2, 7.0, 9.1, 9.1, 7.4, 7.4, 9.5, 9.0, 7.1, 9.0, 6.7, 6.7, 8.8, 7.7, 7.5, 7.6	9.1, 8.2, 7.6, 8.4, 8.7, 6.8, 7.4, 8.3, 9.1, 8.6, 7.1, 8.2, 8.5, 8.6, 8.8, 8.7, 7.9, 9.2, 8.4, 8.4, 7.8, 8.7, 8.4, 9.0, 7.7, 7.5, 7.6, 7.5, 7.5, 8.4, 8.7, 9.1

(a) Compare the frequency distributions and write a short report. Each comment in your report must be justified properly with supporting measures and / or diagrams.

(b) Later it was found that the instrument used by technician B rounded off the values to 10 as soon as it encounters a a piece of compound with amount of iron per mass greater than 9.8. As such he observed 12 such values (equal to 10) and ignored by thinking that the values are erroneous. Later a statistician wants to retain these observations and merge with the second dataset. Do you agree with the statistician and redo your comparative study with first dataset? Whether your answer is 'yes' or 'no', justify your argument and present a final anlaysis along with a short report. Supporting measures and /or diagrams must be provided in favour your analysis/explanation. (18 + 17 = 35)

2. Iris flowers are collected for two species 'virginica' and 'setosa' and measured for petal length and petal width. For each species, 'n' pairs of observations are recorded. In order to see the dependence of petal width on petal length for each species, one can fit linear regression equations based on the information given below.

(a) First fit two linear regression equations separately for two species of flowers.

(b) Looking at the results of (a), you guess that the true regression equations might be parallel. Modify your analysis by fitting an improved linear regression equation to the dataset.
(c) Finally suggest and calculate a measure that can identify which of the solutions (i.e. (a) or (b)) would be better.

Let X_1 and X_2 denote the petal length for setosa and virginica flowers respectively and Y_1 and Y_2 denote the petal width for setosa and virginica flowers respectively. Few summary measures are given based 50 pairs of observations for each species as:

$$\sum_{i=1}^{n} X_{1i} = 73.1,\ \sum_{i=1}^{n} X_{1i}^2 = 108.35,\ \sum_{i=1}^{n} Y_{1i} = 12.3,\ \sum_{i=1}^{n} Y_{1i}^2 = 3.57,$$

$$\sum_{i=1}^{n} X_{2i} = 277.6,\ \sum_{i=1}^{n} X_{2i}^2 = 1556.16,\ \sum_{i=1}^{n} Y_{2i} = 101.3,\ \sum_{i=1}^{n} Y_{2i}^2 = 208.93,$$

$$\sum_{i=1}^{n} X_{1i}X_{2i} = 406.33,\ \sum_{i=1}^{n} X_{1i}Y_{1i} = 18.28,\ \sum_{i=1}^{n} X_{2i}Y_{2i} = 564.81,\ \sum_{i=1}^{n} Y_{1i}Y_{2i} = 25.03$$

(15+15 = 30)

3. In presence of extreme values, define two new measures of central tendency as follows. Assume that the frequency distribution is symmetric and unimodal and the total frequency is an odd number.

(i) Trimmed mean: Arithmetic mean of the remaining observations ignoring all values lower than the first quartile and higher than the third quartile.
(ii) Winsorised mean: Arithmetic mean after replacing all values lower than the first quartile by the value of first quartile and all values higher than the third quartile by the value of third quartile.

Then,

(a) Calculate these new means and compare to the usual arithmetic mean based on all observations.
(b) What happens when the distribution is skewed instead of symmetric?
(c) Also compare the standard deviations based on these two concepts (trimming and winsorising) with the usual standard deviation based on the actual observations. (6+6+8=20)

Indian Statistical Institute
Vectors and Matrices I
B-I, First semestral examination, 2017-2018

Date: Nov 23, 2017 Duration: 3hrs.

This paper carries 55 marks. Attempt all questions. The maximum you can score is 50. Justify all your steps. This is an open book, open notes examination.

If copying is detected in the solution for any problem, all the students involved in the copying will get 0 for that problem. Also an additional penalty of 10 will be subtracted from the overall aggregate for this subject of each of these students.

1. Let $\{\vec{v}_1, ..., \vec{v}_n\}$ be any orthonormal basis of $\mathbb{R}^n$. Then we know that for all $\vec{x} \in \mathbb{R}^n$ we have unique $\alpha_1, ..., \alpha_n \in \mathbb{R}$ such that

$$\vec{x} = \alpha_1\vec{v}_1 + \cdots \alpha_n\vec{v}_n.$$

Consider the map $\vec{x} \mapsto (\alpha_1, ..., \alpha_n)'$. Is this map a linear transformation? If so, obtain its matrix in terms of $\vec{v}_1, ..., \vec{v}_n$. If not, then prove why not.

[10]

2. Find all eigenvalues and corresponding eigenspaces of the matrix $A_{5\times5} = \vec{b}\,\vec{b}'$, where $\vec{b} = (1, 2, 3, 4, 5)'$.

[10]

3. $A_{m\times5}$ is a matrix with all rows non-null and $m \geq 5$. Let B be the reduced row echelon form of A > If A and B have the same column space, then find m. Justify your answer.

[10]

4. Let S be a subspace of $\mathbb{R}^n$. Prove or disprove: If B is any basis of $\mathbb{R}^n$ then there must exist a subset $C \subseteq B$ such that C is a subspace of S.

[5]

5. Let $\vec{v} \in \mathbb{R}^n$ be any nonzero vector. Let S be the intersection of all 2-dimensional subspaces of $\mathbb{R}^n$ containing $\vec{v}$. Prove that S is a subspace of $\mathbb{R}^n$. Also find its dimension. Justify your answer.

[5+7]

6. Express $x^2 + 2xy$ as a quadratic form in $(x, y)'$. Obtain numbers , b, c and d such that the same expression can be written as a diagonal quadratic form w.r.t. (u, v) where

$$u = ax + by \text{ and } v = cx + dy.$$

[3+5]

INDIAN STATISTICAL INSTITUTE

First Semester Examination: 2017 – 18

B. Stat 1st Year
Probability theory 1

Date: 27/11/17 **Maximum Marks: 60** **Duration: 3 hours**

Answer **any two questions in Group A** and **any three question in Group B**. State clearly any result that you use. Usage of calculator is strictly disallowed.

Group A

1. **(15 points)** Let $(S_n : n \geq 0)$ be a simple symmetric random walk starting from 0. Define, for a fixed n,

$$M := \max\{S_0, \dots, S_{2n}\}.$$

Show that

$$\mathrm{E}(M|S_{2n} = 0) = \frac{1}{2}\left(\binom{2n}{n}^{-1} 4^n - 1\right).$$

2. **(15 points)** A particular kind of bacterium either splits into 3 bacteria of the same kind or dies, with probability p and $1-p$ respectively. At any point of time, all such bacteria that exist behave in the described way, independently. If a system contains 1 bacterium of this kind, find the probability that eventually the system will be free of such bacteria, assuming that bacteria from outside the system cannot enter it.

3. **(15 points)** Consider a simple symmetric random walk $(S_n : n \geq 0)$ starting from 0. Define

$$T := \inf\{n \geq 1 : S_n = 0\}.$$

Calculate

$$\lim_{n\to\infty} n^{3/2} P(T = 2n).$$

Group B

4. **(10 points)** Let X follow Geometric(p) for some $0 < p < 1$. Suppose that Y follows Bernoulli(p) independently of X. Find the probability generating function of XY.

5. **(10 points)** The number of accidents on a particular day in Kolkata follows Poisson(λ) for some $\lambda > 0$. Each accident is reported to the police with probability p, independently of the others. If no accident is reported on a day, find the conditional distribution of the number of accidents that occurred.

6. **(10 points)** Let $Z_1, Z_2, \ldots$ be i.i.d. random variables with

$$P(Z_1 = n) = \frac{1}{n(n+1)},\ n \in \mathbb{N}.$$

For $n = 1, 2, \ldots$, define

$$X_n := \sum_{j=1}^{n} \mathbf{1}(Z_j \geq n).$$

For every fixed $r \in \mathbb{N} \cup \{0\}$, calculate

$$\lim_{n \to \infty} P(X_n = r).$$

7. **(10 points)** A random sample of size n is drawn from $\{1, \ldots, N\}$ with replacement. Calculate the expected number of distinct units in the sample.

INDIAN STATISTICAL INSTITUTE

FIRST SEMESTER SEMESTRAL EXAMINATION (2017–18)

B. STAT. I YEAR

ANALYSIS I

Date : 29.11.2017 Maximum Marks : 100 Time : $3\frac{1}{2}$ hours

The question carries 110 marks. Maximum you can score is 100. Precisely justify all your steps. Carefully state all the results you are using.

1. (a) Given nonempty subsets S and T of $\mathbb{R}$ such that

$$s \le t, \qquad \text{for every } s \in S \text{ and } t \in T, \tag{1}$$

show that S has a supremum, T has an infimum and

$$\sup S \le \inf T. \tag{2}$$

(b) "If the inequality in (1) is strict for every $s \in S$ and $t \in T$, then the inequality in (2) is also strict." — True or false? Justify. [$10 + 5 = 15$]

2. If x is not an integer multiple of 2π, show that

$$\sum_{k=1}^{n} \cos kx = \frac{\sin(n+1/2)x - \sin(x/2)}{2\sin(x/2)}.$$

Using this show that $\sum_{n=1}^{\infty} \frac{\cos nx}{n^s}$ converges for all $s > 0$. [$5 + 5 = 10$]

3. Let $f, g : I \to \mathbb{R}$. Define

$$h(x) = \max\{f(x), g(x)\}, \quad x \in I.$$

(a) If f and g are continuous at $x_0 \in I$, show that h is continuous at x_0.

(b) If f and g are differentiable at x_0, is h differentiable at x_0? [$10 + 5 = 15$]

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty$. Prove that f attains its minimum on $\mathbb{R}$. [10]

5. Let f be a continuous real function on $\mathbb{R}$ such that for all $x \neq 0$, $f'(x)$ exists and $\lim_{x \to 0} f'(x) = 3$. Is f differentiable at 0? [5]

6. Let $f(x) = 1 - x^{2/3}$. Show that $f(1) = f(-1) = 0$. Can the derivative of f ever be zero on $[-1, 1]$? This seems to contradict a theorem you may know. Which theorem? Does this really contradict the theorem? Justify! [10]

7. Let $f : (0, 1] \to \mathbb{R}$ be a differentiable function with f' bounded on $(0, 1]$.

   (a) Show that f is uniformly continuous.

   (b) Define
   $$a_n = f\left(\frac{1}{n}\right), \quad n \geq 1.$$
   Show that $\{a_n\}$ is a convergent sequence. $[5 + 10 = 15]$

8. Let $f : (0, 1) \to [0, \infty)$ be a thrice differentiable function. If $f(x) = 0$ for *at least two* values of $x \in (0, 1)$, prove that $f'''(c) = 0$ for some $c \in (0, 1)$. [10]

9. Let $f : I \to \mathbb{R}$. We say that f is *locally bounded* if for every $x \in I$, there is an interval $(x - \delta, x + \delta)$ such that f is bounded on $(x - \delta, x + \delta) \cap I$. $[5 + 5 + 10 = 20]$

   (a) Show that any continuous $f : I \to \mathbb{R}$ is locally bounded.

   (b) Give an example to show that a locally bounded function need not be bounded.

   (c) Show that any locally bounded function on $I = [a, b]$ is bounded.

INDIAN STATISTICAL INSTITUTE
B. Stat. I: 2017–2018
Introduction to Programming & Data Structures
Semestral Examination

Date: 01. 12. 2017 Marks: 100 Time: 3 Hours

Answer any 5 (out of 7) questions. Each question is of 20 marks. Please write all the part answers of a question at the same place.

1. (a) Write a recursive as well as non-recursive version of a function in C that can evaluate $f(n) = f(n-1) + f(n-2)$ with initial conditions $f(0) = 0, f(1) = 1$.
   (b) Write a function in C to check whether a given integer is a palindrome.
   (c) Write a program in C to calculate the GCD and LCM of two unsigned integers. Can your program provide an incorrect answer for some valid inputs? Explain.

   (3+3)+5+(5+4) = 20

2. (a) Write down the algorithm for insertion of an element into a heap.
   (b) Derive the time complexity to construct a heap with n many elements.
   (c) Explain each step to construct a heap with the data set 1, 35, 66, x, 36, 91, 98, 37, 48, where x is two least significant digits of your roll number. Insert the elements one by one to provide detailed explanation.

   10+5+5 = 20

3. (a) Explain an efficient data structure for implementing a linked list.
   (b) Write a complete C program to implement a queue using a linked list. You should implement all the necessary functions for this.
   (c) Explain the insert and delete operation in details with proper examples.

   2+14+4 = 20

4. (a) Explain an efficient data structure for implementing a binary search tree.
   (b) Write a C program to implement the deletion of a node in such a binary search tree.
   (c) Explain your program by deleting the root in a binary search tree with 8 elements and the root having both the children.

   2+14+4 = 20

5. The pointer to the root of a binary search tree is given. You need to return the root of another copy of the tree (without disturbing the original tree) where the integer (information) in each node of the new tree will be one more than the corresponding node in the original tree. Write the C function of prototype "struct node * newtree(struct node *)", where the parameter passed through the function is the root of the original tree and the parameter returned provides the root of the new tree. Provide proper explanation of your code.

   20

P. T. O

6. (a) How many four input four output Boolean functions are there? Out of those how many are reversible?
   (b) Prove that any n-input m-output Boolean function can be implemented using two-input NOR gates only.
   (c) Efficiently implement the four-input one-output Boolean function

$$\sum\{1, 2, 4, 7, 8, 11, 13, 14, 15\} \text{ (SOP)}$$

   with any kind of logic gates of your choice.
   (d) Explain with examples how any n-variable Boolean function can be implemented with a suitable multiplexer.

(2+3)+5+5+5 = 20

7. (a) Explain how to design a 3-bit counter with T flip-flops only (and necessary combinational gates).
   (b) Explain how to design the same counter with D flip-flops only (and necessary combinational gates).
   (c) Which design is more efficient and why?

8+8+4 = 20

First Semester Examination (2017-2018)
B. Stat. - 1st year
Remedial English
100 marks
1.5 hours
(Back Paper)

Date: 29/12/2017

1. Write an essay on any of the following topics. Five paragraphs are expected: (60 marks)
   a) Cricket
   b) My favourite book
   c) My school

2. Fill in the blanks with appropriate prepositions (Write the full sentence) (20 marks)
   a) Look ________ you leap.
   b) She is _______ a cab.
   c) The geese flew ______ the houses.
   d) Hang the picture _______ the wall.
   e) The teacher pointed _______ the blackboard.
   f) I was born ________ March.
   g) Lets meet ______ 3:30pm.
   h) The frog jumped ______ the wall.
   i) I journeyed ______ Darjeeling.
   j) The drain ______ the road had overflowed.
   k) Please try ____ think _____ the agenda ____ the meeting.
   l) I am obliged _____ you _____ your kindness.
   m) He had last attended office _____ Monday.
   n) The suggestion was not acceptable _____ us.
   o) You must abstain ____ smoking and drinking.
   p) I was amazed _____ her performance.

3. Fill in the blanks with appropriate words (Write the full sentence) (20 marks)

A farmer ________ a goose ________ the village fair. His wife _____ _______ ______ to have the goose. They _____ the _____ in their barn. The goose _____ a golden _____ ______ day. The farmer and his wife ____ _____ _____. Each day ____ _____ sold ____ _____ in the village _______. The farmer and his wife _____ became _____. The farmer's wife became _______.

INDIAN STATISTICAL INSTITUTE

Semester Examination - Backpaper: 2017-18

B. STAT 1st Year

STATISTICAL METHODS I

Date: 01/01/2018 Full Marks: 80 Duration: 3 hrs

1. Express diagrammatically the following data on employment status of people of India by their broad categories during 1989-90, justifying your choice of diagram(s): (12)

Employment status	Rural		Urban	
	Male	Female	Male	Female
Unemployment	961	708	977	826
Employment	11	9	7	8
Not in labour force	28	283	16	166

2. What is a frequency curve? Draw a rough sketch of three different types of frequency curves with a real-life example for each. Justify your examples and the corresponding frequency curves. (5+3×3=14)

3. The life of electric bulbs manufactured by Company A for 50 bulbs (in 100 hour unit) is given below.

9.5	15.2	25.1	0.1	4.5	15.1	10.3	1.5	5.5	3.3	8.5
10.6	7.0	8.4	4.0	7.7	23.4	10.3	5.7	3.7	2.7	1.2
4.3	0.1	15.2	21.6	0.6	6.6	21.2	19.2	4.5	5.9	3.9
21.4	5.9	15.0	10.2	23.3	5.2	12.6	5.9	3.5	14.2	22.7
16.5	1.8	26.2	3.7	17.3	15.4					

A frequency distribution of the same for a number of bulbs manufactured by Company B is also available and given in Table 1.

Table 1
Table showing frequency distribution of life of electric bulbs

Class intervals for life (in 100 hour unit)	Frequency
0.1 — 3.0	12
3.1 — 6.0	21
6.1 — 9.0	10
9.1 — 12.0	8
12.1 — 15.0	5
15.1 — 18.0	3
18.1 — 21.0	2
21.1 — 24.0	4
24.1 — 27.0	3
27.1 — 30.0	4

P.T.O

(a) Compare the two frequency distributions and write a short report. Each comment in your report must be justified properly with supporting measures and / or diagrams.

(b) Later it was discovered that during the collection of data, the information on 21 bulbs manufactured by Company B was not included in the dataset, because 16 bulbs were found to be defective and the life hours of 5 bulbs were found to be 42.5, 58.7, 37.6, 43.2, and 65.1. But a statistician wants to include this information in the second dataset. Assume that the life of each of 16 defective bulbs can be taken as 0. Do you want to modify your analysis and hence the report? If yes, do it; if not, justify your reason for not modifying this analysis and report. Supporting measures and / or diagrams must be provided. (16+19=35)

4. Develop a formula to calculate mode for a frequency distribution of a continuous variable where no two class-widths are equal. (7)

5. Explain correlation coefficient, correlation ratio, and correlation index. Write down the formula and range of each of the measures. (12)

6. Let r be the correlation coefficient based on n pairs of observations $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$. Later it is found that (x_1, y_1) is recorded wrongly and hence r was calculated wrongly. The correct observation for this pair would be (x_0, y_0). Derive the expression for the correct correlation coefficient in terms of r and other suitable measures. (12)

7. Suppose the mean and the standard deviation of a set of n observations are $\bar{x}$ and s respectively. When a new observation is introduced the mean decreases but the variance remains the same. Express the new observation in terms of $\bar{x}$ and s. (8)

INDIAN STATISTICAL INSTITUTE
B. Stat. I: 2017-2018
Introduction to Programming & Data Structures
Semestral Examination

Date: 01. 12. 2017 Marks: 100 Time: 3 Hours

Answer any 5 (out of 7) questions. Each question is of 20 marks. Please write all the part answers of a question at the same place.

1. (a) Write a recursive as well as non-recursive version of a function in C that can evaluate $f(n) = f(n-1) + f(n-2)$ with initial conditions $f(0) = 0, f(1) = 1$.
   (b) Write a function in C to check whether a given integer is a palindrome.
   (c) Write a program in C to calculate the GCD and LCM of two unsigned integers. Can your program provide an incorrect answer for some valid inputs? Explain.

   (3+3)+5+(5+4) = 20

2. (a) Write down the algorithm for insertion of an element into a heap.
   (b) Derive the time complexity to construct a heap with n many elements.
   (c) Explain each step to construct a heap with the data set 1, 35, 66, x, 36, 91, 98, 37, 48, where x is two least significant digits of your roll number. Insert the elements one by one to provide detailed explanation.

   10+5+5 = 20

3. (a) Explain an efficient data structure for implementing a linked list.
   (b) Write a complete C program to implement a queue using a linked list. You should implement all the necessary functions for this.
   (c) Explain the insert and delete operation in details with proper examples.

   2+14+4 = 20

4. (a) Explain an efficient data structure for implementing a binary search tree.
   (b) Write a C program to implement the deletion of a node in such a binary search tree.
   (c) Explain your program by deleting the root in a binary search tree with 8 elements and the root having both the children.

   2+14+4 = 20

5. The pointer to the root of a binary search tree is given. You need to return the root of another copy of the tree (without disturbing the original tree) where the integer (information) in each node of the new tree will be one more than the corresponding node in the original tree. Write the C function of prototype "struct node * newtree(struct node *)", where the parameter passed through the function is the root of the original tree and the parameter returned provides the root of the new tree. Provide proper explanation of your code.

   20

P.T.O

is the root of the original tree and the parameter returned provides the root of the new tree. Provide proper explanation of your code.

20

6. (a) How many five input four output Boolean functions are there? Out of those how many are reversible?

   (b) Prove that any n-input m-output Boolean function can be implemented using two-input NOR gates only.

   (c) Efficiently implement the four-input one-output Boolean function

$$\sum\{2, 4, 7, 8, 11, 13, 14, 15\} \text{ (SOP)}$$

   with any kind of logic gates of your choice.

   (d) Explain with example how any 4-variable Boolean function can be implemented with a suitable multiplexer.

(2+3)+5+5+5 = 20

7. (a) Explain how to design a 4-bit counter with T flip-flops only (and necessary combinational gates).

   (b) Explain how to design the same counter with D flip-flops only (and necessary combinational gates).

   (c) Which design is more efficient and why?

8+8+4 = 20

INDIAN STATISTICAL INSTITUTE

FIRST SEMESTER BACKPAPER EXAMINATION (20017–18)

B. STAT. I YEAR

ANALYSIS I

Date : 03.01.2018 Maximum Marks : 100 Time : 3 hours

Precisely justify all your steps. Carefully state all the results you are using.

1. Let $\{a_n\}$ be a bounded sequence of real numbers, and let $\alpha = \limsup_{n\to\infty} a_n$. Show that if $\{a_{n_k}\}$ is a convergent subsequence of $\{a_n\}$ with $\lim_{k\to\infty} a_{n_k} = a$, then $a \leq \alpha$. [10]

2. Test the convergence of the following series : [5 + 5 = 10]

   $$(a)\ \sum_{n=2}^{\infty} \frac{n^3[\sqrt{2}+(-1)^n]^n}{3^n} \qquad (b)\ \sum_{n=2}^{\infty} \frac{2^n+n^2+n}{2^{n+1}\cdot n(n+1)}$$

3. (a) $\{a_n\}$, $\{b_n\}$ are such that $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ are both convergent, then prove that $\sum_{n=1}^{\infty} a_n b_n$ is absolutely convergent.

   (b) Given a convergent series $\sum_{n=1}^{\infty} a_n$ of non-negative terms, prove that $\sum_{n=1}^{\infty} \sqrt{a_n}\cdot n^{-p}$ converges, if $p > \frac{1}{2}$. Give a counterexample for $p = \frac{1}{2}$. [10 + (4 + 6)] = [20]

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty$. Prove that f attains its minimum on $\mathbb{R}$. [10]

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| \leq (x-y)^2$ for all $x, y \in \mathbb{R}$. [10]

6. Let I be any interval and $f : I \to \mathbb{R}$ be a function. A point $a \in \mathbb{R}$ is said to be a fixed point for f if $f(a) = a$.

   (a) If $f : [0,1] \to [0,1]$ is continuous, then f has a fixed point.

   (b) If, moreover, f is differentiable and $f'(x) \neq 1$ for any $x \in (0,1)$, then the fixed point is unique. [8 + 7 = 15]

7. (a) Find all real x such that $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges.

   (b) Prove that the n-th partial sum of the above series coincides with the Taylor polynomial of degree n around 0 of the function $f(x) = \log \frac{1}{1-x}$. [5 + 10 = 15]

8. Let f be a thrice differentiable function on $(0, 1)$ such that $f(x) \geq 0$ for all $x \in (0, 1)$. If $f(x) = 0$ for *at least two* values of $x \in (0, 1)$, prove that $f'''(c) = 0$ for some $c \in (0, 1)$. [10]

Indian Statistical Institute
Vectors and Matrices I
B-I, First back paper examination, 2017-2018

Date: 04 · 01 · 18 Duration: 3hrs.

This paper carries 100 marks. Attempt all questions. The maximum you can score is 45. Justify all your steps. This is an open book, open notes examination.

1. Let A, B be two matrices of the same size, and having the same null space. Then prove or disprove (with a counterexample) each of the following statements:

   (a) $\mathcal{C}(A) = \mathcal{C}(B)$.

   (b) $\mathcal{R}(A) = \mathcal{R}(B)$.

   [8+7]

2. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. If for every linearly independent set $\{\vec{v}_1, ..., \vec{v}_k\} \subseteq \mathbb{R}^n$ the set $\{f(\vec{v}_1), ..., f(\vec{v}_k)\} \subseteq \mathbb{R}^n$ is also linearly independent, then show that f must be a bijection.

   [10]

3. Let A, B and C be 3×3 matrices denoting, respectively, counterclockwise rotation by $90°$ around the x-, y- and z-axis. Find the axis for the composite rotation given by the matrix ABC. [15]

4. How many orthogonal projections $f : \mathbb{R}^3 \to \mathbb{R}^3$ are there such that $f(\vec{e}_1) = f(\vec{e}_2) = f(\vec{e}_3)$? Justify your answer. Find the matrix associated with any one of them.

   [5+10]

5. Obtain *all* least squares solutions of the system $A\vec{x} = \vec{b}$, where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$$

   Justify that you have obtained *all* the least squares solutions. [10+5]

6. If $A_{3\times3}$ is a square real matrix and B is its reduced row echelon form, then in is true that A and B must have the same eigenvalues? Prove or provide a counteraxample.

   [5]

7. Let $\vec{u}, \vec{v} \in \mathbb{R}^{10}$ be two vectors. Consider the matrix $A_{10\times 5}$ given by

$$A = \begin{bmatrix} \vec{u} & \vec{u}+\vec{v} & 2\vec{v} & 3\vec{v}-5\vec{u} & \vec{u}-\vec{v} \end{bmatrix}.$$

If A has rank 2, then find a rank factorisation of A. What if A had rank 1? [10+5]

8. If $\vec{x}_1$ and $\vec{x}_2$ are two solutions of $A\vec{x} = \vec{b}$, then show that $\vec{x}_3 = \frac{1}{2}(\vec{x}_1 + \vec{x}_2)$ is also a solution. Hence, or otherwise, deduce that for any consistent system $A\vec{x} = \vec{b}$, we have exactly one solution with minimum norm.

[2+8]

INDIAN STATISTICAL INSTITUTE

First Semester Examination (Back paper): 2017 – 18

B. Stat 1st Year
Probability theory 1

Date: 05.01.18 **Maximum Marks: 100** **Duration: 3 hours**

Answer **any five** of the following questions. Each question caries 20 marks. State clearly any result that you use. Usage of calculator is strictly disallowed.

1. A fair die is thrown till the first six is obtained. Find the expectation of the sum of the numbers obtained on all the throws, excluding the last one which yields the six.

2. A sample of size n is drawn from $\{1,\ldots,10\}$ with replacement. Let A_n be the event that the value 5 has been drawn at least twice. Show that
$$\lim_{n\to\infty} P(A_n) = 1\,.$$

3. Calculate the extinction probability of a branching process with progeny distribution Binomial $(3, 1/2)$.

4. The number of misprints in a paper follows Negative Binomial (k, p), for some $k \in \mathbb{N}$ and $0 < p < 1$. While proofreading the paper, its author misses each misprint with probability p, independently of the other misprints. What is the probability that the author won't catch a single misprint?

5. Let $(S_n : n \geq 0)$ be a simple symmetric random walk. Show that
$$P(S_1 \neq 1, \ldots, S_n \neq 1) = P(0 \leq S_n \leq 1)\,.$$

6. The events A, B and C are pairwise independent, each occurring with probability p. Assume that at least one of the above three events necessarily occurs. Calculate $P(A \cap B \cap C)$.

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination: 2017-18

B. STAT 1st Year

STATISTICAL METHODS II

26.02.2018 Full Marks: 30 Duration: 1 hr 30 mins

[Answer all questions. Symbols have their usual meanings.]

1. Multiple correlation coefficient is defined as the product moment correlation between x_1 and $X_{1.23...p}$; but in bivariate case, it is defined as simple product moment correlation between x_1 and x_2. Explain this apparent confusion, both mathematically and intuitively. (8)

2. Given three variable x_1, x_2, x_3, you are to find a linear regression equation for x_1. You need to decide whether to use (i) a simple regression with either x_2 or x_3 as regressor, or (ii) a multiple regression. Which would you choose in the following situations? Justify your answer.

   (i) $r_{23} = 0$ (ii) $r_{23} = 0, r_{12} = r_{13}$ (5+5=10)

3. Prove the relation:

$$1 - r_{1.23...p}^2 = (1 - r_{12}^2)(1 - r_{13.2}^2)(1 - r_{14.23}^2) \quad (12)$$

   [You need not derive all the results that are required to prove this; but you must state them clearly so that it can be understood.]

Indian Statistical Institute
Vectors and Matrices II
B-I, Midsem

Date: 27·02·18 Duration: 2hrs.

Attempt all questions. The maximum you can score is 50. Justify all your steps. This is a closed book, closed notes examination.

If copying is detected in the solution for any problem, all the students involved in the copying will get 0 for that problem. Also an additional penalty of 10 will be subtracted from the overall aggregate of each of these students.

1. A linear system $A_{10\times 7}\vec{x} = \vec{b}$ over a field F has exactly 64 solutions. If k denotes the size of F, and r denotes the rank of A, then find (with justification) all possible pairs (k, r). Also find, in each of these cases, the number of vectors $\vec{c}$ such that the system $A\vec{x} = \vec{c}$ is inconsistent.

   [5+5 marks]

2. Consider the inner product space $C[0,1]$ consisting of all continuous real-valued functions under pointwise operations and the inner product

   $$\langle f, g\rangle = \int_0^1 f(x)g(x)dx.$$

   Obtain the orthogonal projection of e^x on $span\{1, x, x^2\}$.

   [10 marks]

3. If V is a vector space of dimension n over $\mathbb{C}$, is it true that V must be a vector space of dimension $2n$ over $\mathbb{R}$? If V is the vector space (over $\mathbb{R}$) of all *complex* polynomials of degree ≤ 2 then find a basis of V.

   [5+5 marks]

4. Let V be the vector space of all 5×5 matrices over a field F. If $f : V \to F$ is a function such that $f(A)$ is alternating and multilinear in the rows of A, then prove that $f(A) = \alpha \cdot det(A)$, for some some $\alpha \in F$.

   [15 marks]

5. Submit the Lights Out assignment.

   [10 marks]

INDIAN STATISTICAL INSTITUTE

MID-TERM EXAMINATION (2017–18)

B. STAT. I YEAR

ANALYSIS II

Date : 28.02.2018 Maximum Marks : 60 Time : 2 hours

The question carries 70 marks. Maximum you can score is 60. Precisely justify all your steps. Carefully state all the results you are using.

1. Let $f : [0,1] \to [0,1]$ be defined by

$$f(x) = \begin{cases} x & \text{if} \quad x \text{ is rational} \\ 1-x & \text{if} \quad x \text{ is irrational} \end{cases}$$

Compute $\underline{\int_0^1} f(x)\,dx$, $\overline{\int_0^1} f(x)\,dx$ and decide whether $f \in \mathcal{R}[0,1]$. [15]

2. Evaluate the limit $\lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]^{1/n}$. [7]

3. Let $f : [a,b] \to \mathbb{R}$ be continuous. Show that for any partition P of $[a,b]$, there is a marking of P such that the Riemann sum $S(P,f) = \int_a^b f(x)\,dx$. [7]

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous periodic function of period $p > 0$, *i.e.*,

$$f(x+p) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

Show that

$$\int_a^{a+p} f(x)dx = \int_b^{b+p} f(x)dx \quad \text{for all } a, b \in \mathbb{R}.$$ [8]

5. Let $f : [a,b] \to (0,\infty)$ be a continuous function. Let $M = \sup\{f(x) : x \in [a,b]\}$ and $m = \inf\{f(x) : x \in [a,b]\}$. Find the limit

$$\lim_{n\to\infty} \left(\int_a^b [f(x)]^n dx\right)^{1/n}.$$ [10]

6. Test the convergence of the integral $\int_0^1 |\log x|\,dx$. [8]

7. Show that the improper integral

$$\int_0^\infty \frac{1}{x^p(1+x)^q}\,dx$$

converges if and only if $0 < 1-p < q$. [15]

INDIAN STATISTICAL INSTITUTE

Second Semester Examination: 2017 – 18

B. Stat 1st Year
Probability theory 2

Date: 02·03·18 **Maximum Marks: 40** **Duration: 2 hours**

Answer **any four** questions. Each question carries 10 marks.

1. Let X and Y be i.i.d. from Pareto (α), that is, each of them has the density
$$f(x) = \alpha x^{-\alpha-1}\mathbf{1}(x \geq 1),\, x \in \mathbb{R}\,,$$
for some $\alpha > 0$. Calculate $P(XY > z)$ for every $z \geq 1$.

2. Suppose that (U, V) has joint density
$$f(u, v) = c(\lambda)e^{-\lambda v}\mathbf{1}(0 < u < v),\, u, v \in \mathbb{R}\,,$$
for some $\lambda > 0$, where $c(\lambda)$ is a constant depending only on λ.
   (a) (**3 points**) Calculate $c(\lambda)$.
   (b) (**5 points**) Find the joint density of U and W, where
$$W = V - U\,.$$
   (c) (**1 point**) Are U and W independent?
   (d) (**1 point**) Are U and V independent?

3. A stick of length 1 metre is broken at three points, chosen uniformly at random, independently of each other. Calculate the density of the sum of the lengths of the two leftmost parts.

4. Let (X_1, X_2) follow bivariate normal, with
$$\mathrm{E}(X_1) = \mathrm{E}(X_2) = 0\,,$$
$$\mathrm{Var}(X_1) = 1,\, \mathrm{Var}(X_2) = 2\,,$$
and
$$\mathrm{Corr}(X_1, X_2) = \frac{1}{2}\,.$$

P.T.O

Let

$$Y_i = e^{X_i},\ i = 1, 2\,.$$

Calculate $P(Y_1 < Y_2^2)$.

5. Suppose that X and Y are i.i.d. from Beta(2, 1). Calculate

$$\mathrm{E}\left(\frac{X}{Y}\right)\,.$$

P.S.: Recall that the density of Beta(a, b) is

$$f(x) = \frac{1}{\beta(a,b)} x^{a-1}(1-x)^{b-1} 1(0 < x < 1),\ x \in \mathbb{R}\,.$$

INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination: 2017-18

Course Name: B. Stat. 1st year

Subject: Numerical Analysis

Date: 05.03.2018

Full Marks: 60 Duration: 2.5 hrs.

Answer any four questions.

1. a) For a fixed point iteration of the form $x_{n+1} = g(x_n)$, show that $g(x) = \ln(1+2x)$ has a unique fixed point on [1, 2]. Estimate a bound on the number of iterations required to achieve an accuracy up to 11 decimal places for this fixed point iteration, starting with the initial point $x_0 = 1$. Show all your work.

b) Consider the following table:

x_i	10	15	17
$f(x_i)$	35	10	14

Using the second Lagrange interpolating polynomial, find an approximate solution of the equation $f(x) = 11$. Comment on the bound of the error being caused due to the approximation.

(5+4)+(4+2) = 15

2. a) Find coefficients a_0, a_1 and a_2 such that the quadrature rule:

$$\int_{-1}^{1} |x| f(x) dx = a_0 f(-1) + a_1 f(0) + a_2 f(1)$$

is exact for polynomials of degree less than or equal to 2. Show that, in fact, the rule thus obtained remains valid for any polynomial of degree less than or equal to 3.

b) Determine the values of n and h required to approximate the integral $\int_0^2 e^{2x} \sin 3x \, dx$ to within 10^{-4} by using the composite Simpson's rule.

(4+3)+8 = 15

3. Consider the linear system of equations $Ax = b$ where the matrix A is given by:

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

P.T.O

The iterative procedures for solving the system can be expressed as: $x_{n+1} = Tx_n + d$, where the matrix T is called an iteration matrix.

a) Derive the iteration matrices for the Jacobi and Gauss-Seidel iterative methods.

b) Hence determine which of these iterations converges and which one diverges. Show all your work.

(3+4)+(4+4) = 15

4. a) Suppose that $f:[0,1] \to \Re$ is twice continuously differentiable, and that p is a polynomial of degree 1 which interpolates f at the points 0 and 1. Show that

$$|f(x) - p(x)| \leqslant \frac{1}{8} \max_{0 \leqslant x \leqslant 1} |f''(x)| .$$

b) Suppose that $x_0 = 0, x_1 = 1, x_2 = 2,$ and $x_3 = 3$. Find the coefficients $a_0, a_1, a_2,$ and a_3 such that graph of the following polynomial of degree 3:

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$

may pass through the points $p(x_0) = 3, p(x_1) = 1, p(x_2) = 3,$ and $p(x_3) = 15$.

c) Neville's method is used to approximate a function $f(x)$ at $x = 0.5$. Complete the following table by finding out the values of a and b.

i	x_i	$Q_{i,0}$	$Q_{i,1}$	$Q_{i,2}$
0	0	$0 = Q_{0,0}$		
1	0.4	$2.8 = Q_{1,0}$	$3.5 = Q_{1,1}$	
2	0.7	$a = Q_{2,0}$	$b = Q_{2,1}$	$27/7 = Q_{2,2}$

6+5+4 = 15

5. a) Consider the following matrix:

$$A = \begin{pmatrix} 3 & 0 & 3 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{pmatrix}$$

Compute an LU decomposition of the matrix and show your steps. Use the LU decomposition to solve the following system of linear equations. Show all your work.

$$\begin{pmatrix} 3 & 0 & 3 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

b) Obtain an expression for the n-th order divided difference for the function $f(x) = 1/x$ using the $n + 1$ distinct nodes $x_0, x_1, \ldots, x_n$.

(7+5)+3 = 15

P.T.O

6. a) Consider the following set of non-linear simultaneous equations:

$$x_1 + 3\log_{10} x_1 - x_2^2 = 0, \qquad 2x_1^2 - x_1 x_2 - 5x_1 + 1 = 0.$$

Hand trace the first 3 iterations of the Newton's method for solving the equations, starting with $x^{(0)} = (3.4 \quad 2.2)^T$. Clearly show all your intermediate computations.

b) Show that the equation $x = 3 + \dfrac{\sin x}{2}$ has a root in the interval [3, 4]. Hence, determine a bound on the number of iterations required to solve the equation by bisection method with an accuracy of two decimal places.

9+(3+3) = 15

INDIAN STATISTICAL INSTITUTE

Second Semester Final Examination: 2017 – 18

B. Stat 1st Year
Probability theory 2

Date: May 2, 2018 Maximum Marks: 60 Duration: 3 hours

Answer **any five** questions. Each question carries 12 marks.

1. Suppose that (X, Y) has joint density

$$f(x,y) = \frac{1}{2\sqrt{\pi}} \exp\left(-x^2 - |x-y|\right), \; x, y \in \mathbb{R}.$$

   Find the distribution of $Y - X$.

2. Suppose that X_1 and X_2 are independent standard normal random variables. Let

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = A \begin{bmatrix} X_1 \\ X_2 \end{bmatrix},$$

   where A is a 2×2 symmetric matrix with

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

   Find the joint distribution of (Y_1, Y_2).

3. Suppose that X and Y are i.i.d. from the standard exponential distribution. For $z > 0$, find the conditional distribution of X given $X + Y = z$.

4. Suppose that Y follows chi-squared distribution with $2n$ degrees of freedom for some $n \in \mathbb{N}$. Let

$$X = \frac{1}{2} Y.$$

   (a) **(6 marks)** Calculate the moment generating function of X.

   (b) **(6 marks)** Hence or otherwise, show that for all $k \geq 1$,

$$\mathrm{E}(X^k) = \frac{(n+k-1)!}{(n-1)!}.$$

5. Suppose that X and Y are i.i.d. random variables. The characteristic function of

$$Z = X + Y\,,$$

is

$$\phi_Z(t) = (1 - |t|)^2 \mathbf{1}(|t| \le 1),\ t \in \mathbb{R}\,.$$

Find the density of X.

6. Let $X_1, X_2, \ldots$ be non-negative random variables with

$$\begin{aligned} \mathrm{E}(X_n) &= \mu,\ n \ge 1\,, \\ \mathrm{Var}(X_n) &= \sigma^2 < \infty,\ n \ge 1\,, \\ \mathrm{Cov}(X_m, X_n) &\le 0,\ 1 \le m < n\,. \end{aligned}$$

Define

$$S_n = X_1 + \ldots + X_n,\ n \ge 1\,.$$

(a) **(6 marks)** Show that as $n \to \infty$,

$$\frac{1}{n^2} S_{n^2} \to \mu \text{ a.s.}$$

(b) **(6 marks)** Hence or otherwise, prove that as $n \to \infty$,

$$\frac{1}{n} S_n \to \mu \text{ a.s.}$$

INDIAN STATISTICAL INSTITUTE

Second Semester Examination: 2017-18

B STAT, FIRST YEAR

Statistical Methods - II

Date: 04/05/2018 Maximum Marks: 70 Duration: 3 hours

1. In a well shuffled cards containing 52 cards, the top-most card is placed randomly in the pack. After one placement, the top-most card is changed and again it is placed randomly in the pack. And this process continues. Let X be a random variable denoting the number of such placements so that the last card will move up to the first spot.

   (a) Write an algorithm in detail to calculate the expected number of such placements by simulation.

   (b) Actually calculate this expected value of X by simulation using calculator assuming that there is a total of 4 cards (instead of 52 cards). You have to write all the random numbers that you generate in the form of a table and relate them to your decision at several steps. (7+7=14)

2. n pairs of brothers and sisters are selected from n unrelated families. Each one is asked to draw a circle by choosing a diameter of his/her choice from a ruler. Let X_i and Y_i $(i = 1, \ldots, n)$ be the diameters chosen by the brother and sister from i-th family. A brother-sister pair will get a coloured pencil box if the difference between the areas of the circles they drew would be less than 15 sq inches. Assume that $(X_i, Y_i) \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ for $i = 1 \ldots, n$.

   (a) Describe an algorithm to simulate this experiment.

   (b) How do you estimate the parameters of the above distribution? Describe in detail.

(c) Extending your simulation scheme as in (a), give an estimate of expected number of coloured pencil boxes that the organiser needs to buy. (5+8+6=19)

3. Suppose $(X_i, Y_i) \sim N_2(0, 0, 1, 1, \rho)$ for $i = 1\ldots, n$. Now in order to estimate ρ^2, Statistician A proposes to use r^2 where r is the sample correlation coefficient based on $\{(X_i, Y_i),\ i = 1, \ldots, n\}$. Statistician B proposes to use sample correlation between X_i^2 and Y_i^2, $i = 1, \ldots, n$. How do you justify the choices of the two statisticians through simulation? Based on your simulation scheme, which one you would prefer and why? (7)

4. Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$.

   (a) Find MLE of μ/σ.

   (b) Check whether the above MLE is unbiased. If not, suggest an unbiased estimator with justification.

   (c) Derive the variance of the estimator you proposed. Check its behaviour when the sample size n becomes very large.

   (d) Suggest an estimator of $\mu^2 + \sigma^2$ using sample moments. Check whether it is unbiased. If not, find out its unbiased estimator, with justification. (5+6+5+7=23)

5. In an agricultural experiment, three types of manures are given to each plot. Yield of paddy (x_1) are collected from 25 such plots. The person who initiated the experiment wants to have some information about this entire experiment with three types of manures. Based on his preliminary knowledge in statistics, he used a multiple linear regression model $x_1 = \beta_0 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_4$, where x_2, x_3, and x_4 denotes the amounts of three types of manures. Based on the data he got the fitted model as $X_{1.234} = -3.281 + 0.031\, x_2 + 2.485\, x_3 + 3.025\, x_4$, and the correlation matrix is given by,

$$\begin{pmatrix} 1 & 0.059 & 0.554 & 0.688 \\ 0.059 & 1 & 0.028 & 0.04 \\ 0.554 & 0.028 & 1 & -0.046 \\ 0.688 & 0.04 & -0.046 & 1 \end{pmatrix}$$

He shared this information to you and asked to do the following.

(a) Calculate $r_{1.234}$ and interpret the result.

(b) Calculate $r_{12.34}$, $r_{13.24}$, and $r_{14.23}$, and interpret the results.

(c) Do you want to remove any variable from the equation? Justify your answer based on the above calculations in (a), (b), and (c).

(d) Define sum of squares due to errors (SSE) as $SSE = \sum_{i=1}^{n}(x_{1i} - \hat{\beta}_0 - \hat{\beta}_2\, x_{2i} - \hat{\beta}_3\, x_{3i} - \hat{\beta}_4\, x_{4i})^2$. How many degrees of freedom are associated with this SSE?

You can do some more calculation, justifying each one of them. Finally write a report on the findings of this entire analysis.

(5+9+4+1+3=22)

INDIAN STATISTICAL INSTITUTE

End-Semestral Examination: 2017-18

Course Name: B. Stat. 1st year

Subject: Numerical Analysis

Date: 07.05.2018

Full Marks: 100 Duration: 3 hrs.

Answer as much as you can.

1. a) The "error function"

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds$$

arises often in mathematics and physics. A few approximate values of $erf(x)$ are tabulated below.

x	0	0.1	0.2	0.3
$\mathrm{erf}(x)$	0	0.1124	0.2227	0.3286

i) Use linear interpolation to estimate $erf(0.17)$.

ii) Derive an upper bound of the error in your answer of Part (i).

b) Consider the following family of methods for solving Initial Value Problems of the form $y' = f(t, y), y(t_0) = y_0$:

$$y_{k+1} = y_k + \Delta t[(1-w)f(t_k, y_k) + wf(t_{k+1}, y_{k+1})] \quad (1)$$

In equation (1), the constant w can take any value in the interval [0, 1]; different values produce different methods. We can see, for example, that $w = 0$ corresponds to Forward Euler, $w = 0.5$ is Trapezoidal Rule and $w = 1$ produces the Backward Euler Rule. Show that for $0.5 \le w \le 1$, the method of equation (1) is unconditionally stable on the equation $\frac{dy}{dt} = \lambda y$, $\lambda < 0$.

[Hint: Note that stability of a method on this model equation is equivalent to showing that $y_k \to 0$ as $k \to \infty$]

c) Find the value of $y(1.1)$ by using the fourth-order Runge-Kutta method, from the differential equation:

$$\frac{dy}{dx} = x - y$$

with the initial condition $y(1) = 1$. Assume step-size $h = 0.1$.

(3+6)+6+5 = 20

P. T. O

2. a) Carry out two steps of the inverse iteration for computing an eigenvalue and the corresponding eigenvector of the matrix: $A = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$ using an estimate of the eigenvalue $\lambda = 5$ and the initial vector $v^{(0)} = [1 \quad 1]^T$. Show all the steps of your computation.

b) Evaluate the Rayleigh quotient using the vector $v^{(2)}$ obtained at the end of second inverse iteration in Part (a) and compute the percentage error with respect to the true eigenvalue.

c) Let a matrix $A \in R^{n \times n}$ be given, and is nonsingular. Then, for any singular matrix $B \in R^{n \times n}$, prove that

$$\frac{1}{\kappa(A)} \le \frac{\|A - B\|}{\|A\|},$$

where $\kappa(A)$ denotes the condition number of the matrix A.

10+4+6 = 20

3. a) Perform the Singular Value Decomposition (SVD) on the matrix and show the necessary steps:

$$A = \begin{pmatrix} 2 & 1 \\ 2 & -1 \\ 1 & 0 \end{pmatrix}$$

b) Compute the Moore-Penrose pseudo-inverse of A by using the SVD obtained in Part (a).

c) Find the minimal-length least-squares solution of the following set of linear equations by using the SVD:

$$Ax = b = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

10+4+6 = 20

4. a) Use the QR factorization to approximate all the eigenvalues of the matrix and show the steps of your work. You may hand trace the process up to a suitable number of iterations.

$$A = \begin{pmatrix} 3 & 3 \\ 3 & 5 \end{pmatrix}$$

b) Use the Gerschgorin Theorem to determine the approximate location of the eigenvalues of the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 5 & 1 \\ -2 & -1 & 9 \end{bmatrix}$$

Can you use the results from the same theorem to infer whether the eigenvalues are real or complex? Explain.

12+8 = 20

5. a) A binary machine that carries 30 bits in the mantissa of each floating-point number is designed to round a given real number up or down correctly to get the nearest representable floating-point number. Derive a simple upper bound for the relative error in the rounding process.

b) A modification of the Newton's method (called the *Steffenson's method*) can be suggested by using the following approximation to compute the derivative $f'(x_n)$:

$$x_{n+1} = x_n - \frac{f(x_n)}{D(x_n)}, \qquad D(x_n) = \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)}, n \geq 0.$$

Assuming $f'(\alpha) \neq 0$, where α is a root of $f(x) = 0$, find the convergence order of the above mentioned method.

c) Find x_0, x_1, c_1 such that the following approximation of the integral: $\int_0^1 f(x)dx = \frac{1}{2}f(x_0) + c_1 f(x_1)$ is precise to as high a degree as possible w.r.t a polynomial function.

3+10+7 = 20

6. a) Let λ and x be an eigenvalue and the corresponding eigenvector for the matrix $A \in R^{n\times n}$, and let Q be an orthogonal matrix satisfying $Qx = e_1$, where $e_1 = [1\,0\,0....0]^T$. Assume that $\|x\|_2 = 1$. Letting $a \in R^n$ and the matrix $A_2 \in R^{(n-1)\times(n-1)}$, show that

$$B = QAQ^T = \begin{pmatrix} \lambda & a^T \\ 0 & A_2 \end{pmatrix}$$

b) Consider the $n \times n$ linear system of equations $Ax = b$.

i) Show that if A is diagonal, the Jacobi method converges just after one iteration.

ii) Show that if A is lower triangular, then the Gauss-Seidel method converges after just one iteration.

c) Consider minimization of the function $f(x_1, x_2) = (x_1 - 2)^4 + 3(x_2 + 3)^2$ by the method of steepest descent with an adaptive learning rate parameter η_k. Suppose the update formula for η_k at the k-th iteration is given by:

$$\mu_k = \frac{1}{\left|\nabla^T(F^{(k)}).\nabla(F^{(k)})\right|},$$

where $F^{(k)} = f(x_1^{(k)}, x_2^{(k)})$ is the value of the function at the k-th iteration and $\nabla(F^{(k)})$ is the corresponding gradient vector. Starting with the initial solution $X^{(0)} = (1, -2)^T$, trace the first 3 iterations of the algorithm by hand and tabulate the results.

7+(3+3)+7 = 20

INDIAN STATISTICAL INSTITUTE

SECOND SEMESTER SEMESTRAL EXAMINATION (2017–18)

B. STAT. I YEAR

ANALYSIS II

Date : 09.05.2018 Maximum Marks : 100 Time : $3\frac{1}{2}$ hours

The question carries 115 marks. Maximum you can score is 100. Precisely justify all your steps. Carefully state all the results you are using.

1. (a) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Define

$$f_+(x) = \begin{cases} f(x) & \text{if } \quad f(x) > 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_-(x) = \begin{cases} -f(x) & \text{if } \quad f(x) < 0 \\ 0 & \text{otherwise} \end{cases}$$

Show that $f \in \mathcal{R}[a,b]$ if and only if both $f_+ \in \mathcal{R}[a,b]$ and $f_- \in \mathcal{R}[a,b]$. Moreover,

$$\int_a^b f(x)dx = \int_a^b f_+(x)dx - \int_a^b f_-(x)dx. \qquad [10]$$

(b) Let $f : [a,b] \to \mathbb{R}$ be such that f' is continuous. Show that f is the sum of a continuous increasing function and a continuous decreasing function. [7]

2. For various values of s, test the following improper integral for convergence

$$\int_2^\infty \frac{dx}{x(\log x)^s}. \qquad [10]$$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Define

$$f_n(x) = f\left(x + \frac{1}{n}\right), \quad x \in \mathbb{R}, n \geq 1.$$

Does $\{f_n\}$ converge uniformly to f on $\mathbb{R}$? If yes, prove it.

If not, will it work under some stronger assumption? Justify! [10]

4. Let $g : [0,1] \to \mathbb{R}$ be continuous. Let $f_n(x) = x^n g(x)$ for $x \in [0,1]$. Show that $\{f_n\}$ converges uniformly on $[0,1]$ if and only if $g(1) = 0$. [10]

5. Decide whether the power series $\sum_{n=1}^{\infty} \frac{n^3[\sqrt{2}+(-1)^n]^n}{3^n} x^n$ converges or diverges at the points $x = 1$ and $x = 2$. [8]

6. Starting from a geometric series and precisely justifying all your steps prove that $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$. [10]

7. Suppose $f : (-\pi, \pi) \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0 & \text{if} \quad x \in (-\pi, 0) \\ 2 & \text{if} \quad x \in (0, \pi) \end{cases}$$

(a) Determine the value of f at the points $-\pi$, 0 and π, so that the Fourier series for f converges to f at every $x \in [-\pi, \pi]$? Briefly justify your answer. [10]

(b) Compute the Fourier series of f and verify your result in (a). [15]

(c) With these values, is the convergence uniform on $[-\pi, \pi]$? [5]

(d) Using f and its Fourier series, show that

(i) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4}$. [5]

(ii) $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$ [10]

(iii) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. [5]

Indian Statistical Institute

Vectors and Matrices II

B.Stat (hons) 1st year

Second Semestral Examination

Date: May 11, 2018 Duration: 3hrs.

This paper carries 55 marks. Attempt all questions. The maximum you can score is 50. Justify all your steps. This is a closed book, closed notes examination. You are not allowed to use the exercises from the class webpage without proof.

If copying is detected in the solution of any problem, all the students involved in the copying will get 0 for that problem.

1. For any real square matrix A, show that the matrices $A'A$ and AA' are similar.

   [10 marks]

2. Let A be a square matrix with nonnegative real entries, with all the row sums equal to 5. Show that 5 must be an eigenvalue of A. Also show that if λ is any eigenvalue of A then $|\lambda| \leq 5$. [3+7 marks]

3. Let A and B be positive definite matrices. Show that

   $$\max\left\{\frac{\vec{x}'A\vec{x}}{\vec{x}'B\vec{x}} \ : \ \vec{x} \neq \vec{0}\right\}$$

   is the largest eigenvalue of AB^{-1}. [10 marks]

4. If P is an orthogonal projector then show that $P^+ = P$. Here P^+ denotes the Moore-Penrose pseudoinverse of P. [5 marks]

5. If the Jordan Canonical Form of a complex square matrix A contains a Jordan block of size 7 with diagonal entries 3, then show that the minimal polynomial $p(\lambda)$ of A must be divisible by $(\lambda - 3)^7$. [10 marks]

6. Submit the "QR decomposition by Householder method" project.

   [10 marks]