Simultaneous Tests for Average and Dispersion by Combined Control Charts

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1. INTRODUCTION

A usual process control plan involves the computation of two statistics, one describing central tendency and a second measuring dispersion of the measured quality characteristic (x) and observing whether values of both obtained in rational sub-groups inspected from the process lie within their respective control limits. Assuming x to have a Normal distribution, the standard values of m and σ required for determining the control limits are say m_0 and σ_0. An assignable cause of variation will result in a population $N(m, \sigma)$. When a small sample has been obtained from this population, the problem is to test whether this population differs significantly from the standard population i.e. to test the 2-parameter simple hypothesis, $H_0 : m = m_0$, $\sigma = \sigma_0$.

The two control charts (say for $\bar{x}$ and R or for $\bar{x}$ and s) must be interpreted simultaneously, since the probability of $\bar{x}$ lying outside its limits depends not only on how m compares with m_0 but also on the values of σ and σ_0. And for this it is reasonable to set the limits on the two charts in such a way that the probability of a sample point going beyond the U.C.L. or below the L.C.L. on either the $\bar{x}$ or the s (or R) chart is the same, say, γ when H_0 is true. A lack of control will be indicated or H_0 will be rejected whenever a sample measure exceeds any of the limits on either chart. Either statistic will lie within its control limits with probability $1-2\gamma$; assuming the measures of central tendency and dispersion used to be independent, the probability that both statistics will lie within their respective control limits is $(1-2\gamma)^2$ and thus, when H_0 is true, the probability of a sample measure going outside of control limits on either chart is $1-(1-2\gamma)^2$. Thus we may set limits by considering $\alpha = 1-(1-2\gamma)^2$.

$$\text{or} \qquad 4\gamma^2-4\gamma+\alpha = 0$$

$$\text{or} \qquad \gamma = \frac{4 \pm \sqrt{16-16\alpha}}{8} = \frac{1 \pm \sqrt{1-\alpha}}{2},$$

so as to give a preassigned size α to the joint test. (Obviously, the negative sign alone can be considered.) In the case of statistics not distributed independently, the joint distribution must be used to obtain α, given γ.

Walsh (1952) considered three joint tests for $H_0 : m = m_0$, $\sigma = \sigma_0$, viz., $\bar{x}$ and s, t and s, and t and $s(m)$ where

$$s = \sqrt{\frac{1}{n-1}\,\Sigma(x_i-\bar{x})^2} \text{ and } s(m) = \sqrt{\frac{1}{n}\,\Sigma\,(x_i-m_0)^2}$$

By computing powers, he concluded that each of the three tests is found to have regions where its O.C.'s are poor. No one test has uniformly better O.C. than the others. On the whole, tests based on the t-statistic and s appeared to be inferior to the other two, which are roughly equivalent.

The importance of range as a measure of dispersion has always been recognized in Quality Control practice for the facility in its computation. However it remains to investigate power properties of the joint test involving $\bar{x}$ and R. The present paper attempts to study the O.C. surface of this test and to compare the OC of this test with those obtained earlier.

2. OPERATING CHARACTERISTIC OF THE $\bar{x}$ AND R CHARTS USED SIMULTANEOUSLY

Control limits for $\bar{x}$ and R charts used jointly to give a probability α of falsely rejecting the state of control $m = m_0$, $\sigma = \sigma_0$ are to be set at $\bar{\bar{x}} \pm A_1 \bar{R}$ and ($D_3 \bar{R}$ and $D_4 \bar{R}$) when process standards are not given and at $m_0 \pm A\sigma_0$ and ($D_1\sigma_0$ and $D_2\sigma_0$) if standards are provided. Constants A, D_1 and D_2 are defined from the relations :

$$\gamma = \Pr\{\bar{x} \leqslant m_0 - A\sigma_0/m_0, \sigma_0\}$$

$$= \Pr\{\bar{x} \geqslant m_0 + A\sigma_0/m_0, \sigma_0\}$$

$$= \Pr\{R \leqslant D_1\sigma_0/\sigma_0\} = \Pr\{R \geqslant D_2\sigma_0/\sigma_0\},$$

$$A_1 = A/d_2, \quad D_3 = D_1/d_2 \quad \text{and} \quad D_4 = D_2/d_2$$

where $E(R) = d_2\sigma$.

Tables I(a) and I(b) give values of the factors A, D_1, D_2, A_1, D_3 and D_4 for samples of sizes 3, 4 and 5 and corresponding to probabilities ·005, ·01, ·02, ·05 and ·10 of type I errors using the joint test. These were obtained by linear interpolation in tables of the incomplete integrals for the distributions of the standardised normal deviate and of the standardised range.

For a sample drawn from a population $N(m, \sigma)$ the operating characteristic of the $\bar{x}$ and R charts is

$$\beta = \Pr\{m_0 - A\sigma_0 \leqslant \bar{x} \leqslant m_0 + A\sigma_0/m, \sigma\} \times$$

$$\Pr\{D_1\sigma_0 \leqslant R \leqslant D_2\sigma_0/\sigma\}$$

since $\bar{x}$ and R in random samples from a Normal population are distributed independently (Daly, 1946). It is convenient to define the class of alternatives (m, σ) in terms of the parameters k and l where

$$k = \frac{m-m_0}{\sigma_0/\sqrt{n}} \quad \text{and} \quad l = \sigma_0/\sigma.$$

In terms of these values the O.C. of the joint test using ($\bar{x}$ and R) charts is

$$\beta = \int_{-(\sqrt{n}A+k)l}^{(\sqrt{n}A-k)l} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt \times \int_{D_1 l}^{D_2 l} f(w)dw,$$

w being the standardized range R/σ. The O.C. function is easily found to be the same for k and $-k$. To visualize the probabilities, a three-dimensional O.C. surface is necessary. Values of O.C. for $n = 3, 4$ and 5 for certain values of k and l and corresponding to $\alpha = \cdot 005$, $\cdot 01$, $\cdot 02$ and $\cdot 05$ are presented in Table 2. The choice of k and l values was made to facilitate comparison with results obtained by Walsh.

On the plane for each value of σ_0/σ, the curve is symmetrical and roughly bell-shaped. Thus for a given value of σ, the probability of accepting a state of control diminishes as k increases or as m departs from m_0. The curves on planes for large values of l are platykurtic not curving down much below $k = 4{\cdot}0$. Thus for small values of σ (compared to σ_0) the probability of accepting the process continues to remain appreciable even for moderately large shift in process mean. For $\sigma \geqslant 4\,\sigma_0$, however, the test has a very large power to detect lack of control.

On the plane for each value of $k \leqslant 3{\cdot}0$, the curves are positively skew. Thus when m remains within 3σ-limits on the chart for $\bar{x}$, the probability of accepting the process gets smaller as σ becomes larger and then it declines very slowly. For any positive shift in the process variability ($\sigma > \sigma_0$) the probability of accepting the process is much less compared to a negative shift ($\sigma < \sigma_0$) to an equal extent.

It is found that O.C. values for the same deviations (k, l) vary remarkably for different sizes of the joint test. A decrease in the sample size naturally leads to a rise in O.C. values and appreciably so for larger values of l $(\sigma < \sigma_0)$ and moderate deviations in process mean $(k \leqslant 2{\cdot}0)$. The test is almost ineffective for σ near about $2\sigma_0$.

When σ remains at σ_0 the joint test using $\bar{x}$ and R charts is slightly worse than the single test using $\bar{x}$ for detecting deviations in m, the largest inefficiency being observed near $k = 2$. It is remarkable, however, that when m is m_0, the joint test is uniformly more powerful than the single test using R for the class of situations $l < 1$ i.e., $\sigma > \sigma_0$. Thus when there is no deviation in process mean, the joint control chart is superior to the single R-chart for detecting positive shifts in process standard deviation.

3. COMPARISON OF O. C. FUNCTIONS

When σ remains at σ_0 the $(\bar{x}-R)$ charts have exactly the same O.C. as the $(\bar{x}-s)$ charts for all sizes of the joint test. With no change in the process mean, the $(\bar{x}-R)$ charts are almost as effective as $(\bar{x}-s)$ charts, the fall in power diminishing with increase in the size of the test and the largest fall being observed at $\sigma = 2\sigma_0$. Reduction in efficiency for $n = 5$ is 3% at the most. However with the class of alternatives $\sigma < \sigma_0$, the $(\bar{x}-R)$ charts are far more powerful than the $(\bar{x}-s)$ charts, at least for moderate deviations in m and smaller test sizes, since for larger deviations in mean and more stringent tests (with smaller α) even the $(\bar{x}-s)$ charts have high probabilities for rejecting the hypothesis of control. The $(\bar{x}-s)$ test is almost insensitive to moderate deviations in process mean under such situations.

Sometimes, however, we have to consider an increase in the variability of a manufacturing process and pay attention to situations where $\sigma > \sigma_0$. It is found that the $(\bar{x}-R)$ charts are decidedly less effective than the $(\bar{x}-s)$ charts for $\sigma = 2\sigma_0$.

As σ increases further both the joint tests have very small probability of accepting the process and O.C. values for the two tests compare very favourably. It is remarkable to note that under the class of situations $\sigma > \sigma_0$, the $\bar{x}-R$ charts have nowhere a better performance than the $(\bar{x}-S)$ charts.

In Walsh's set, t and s and $\bar{x}$ and $s(m)$ are not statistically independent and much effort is to be put up to obtain their O.C. values. Anyway, tests based on $\bar{x}$ and $s(m)$ are almost equivalent to $(\bar{x}-s)$ and $(\bar{x}-R)$ charts. But (t and s) give mostly unfavourable values. Comparing the O.C. values for $n = 5$ and $\alpha = \cdot 01$ it is found that the (t, s) joint test has a very poor performance so long as $\sigma = \sigma_0$. It is better than the present joint test using ($\bar{x}$ and R) for large k and small l and for some moderate k with large l. For all other situations the $(\bar{x}-R)$ charts are superior to the $(t-s)$ tests.

4. COMPARISON WITH THE SINGLE CHART FOR EXTREME VALUES

It was as early as 1949 that Howell proposed a single control chart for the smallest (s) and the largest (L) observations to describe both central tendency and dispersion. He gave 3σ-control limits as $m \pm A_4\sigma$ (standards given) and $\bar{\bar{x}} \pm A_3\bar{R}$ (no standards given) for sample sizes $n = 2$ to 10.

On this chart probability limits $m \pm A_4\sigma$ or $\bar{\bar{x}} \pm A_3\bar{R}$ may be given from the relations

$$\Pr\{m_0 - A_4\sigma_0 < S,\ L < m_0 + A_4\sigma_0 / m_0, \sigma_0\} = 1-\alpha$$

or

$$1-\alpha, = \left[\int_{-A_4}^{A_4'} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt\right]^n$$

and

$$A_3 = A_4/d_2.$$

The O.C. of this chart for $k = \dfrac{m-m_0}{\sigma_0}$ and $l = \sigma_0/\sigma$ is

$$\Pr \{m_0 - A_4\sigma_0 < S,\ L < m_0 + A_4\sigma_0 / m_0 + k\sigma_0,\ \sigma_0/l\}.$$

Thus

$$\text{O.C. } (\beta) = \left[\int_{-l(A_4+k)}^{l(A_4-k)} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \right]^n$$

Values of probability limits A_4 and A_3 for $n = 3$, 4 and 5 and $\alpha = \cdot 01$, $\cdot 05$ and $\cdot 10$ are given in Table 3. Table 4 gives O.C. values of this chart compared with those for $(\bar{x} - R)$ charts for $n = 5$ and $\alpha = \cdot 05$ for several values for k and l.

It is found that when $l = 1$, i.e., when σ does not change from its initial setting, this test is uniformly better than the $(\bar{x} - R)$ or $(\bar{x} - s)$ charts. For $l < 1$, this test though worse than the other tests for lower values of k, is much better than the $(\bar{x} - s)$ test for larger deviations k, since for such situations, the $(\bar{x} - s)$ test is almost inefficient for $\alpha = \cdot 02$ or $\cdot 05$. For $k = 0$, i.e., for m remaining at its initial setting m_0 the test is very poor for situations $\sigma < \sigma_0$. This test, however, is almost insensitive to changes in process mean $k \leqslant 1{\cdot}25$ below $\sigma = \sigma_0/3$.

From the point of computational facility, the joint test based on range and midrange might have been considered. But there does not exist any distribution with a limited first and a continuous second derivative for which range and midrange are statistically independent (Frechet, 1954). Although the joint distribution of these two statistics in random samples from a normal population has been derived (Pillai, 1950), the moments and incomplete probability integrals have yet to be calculated from the infinite series given.

5. CONCLUDING REMARKS

Detection of lack of control in a manufacturing process involves a test for the two parameter simple statistical hypothesis $H_0 : m = m_0$, $\sigma = \sigma_0$, where m_0 and σ_0 are standards used for setting up control limits for relevant charts. It has been shown (Lehmann, 1952) under certain regularity conditions that unbiased tests of H_0 do not exist. Tests of minimum bias and other types of minimax tests have been given under suitable conditions of monotonicity. But they cannot conveniently be applied for routine quality control. Tests using type C critical regions of Neyman and Pearson, and later developed by Isaacson, are only locally most powerful. The simultaneous use of control charts for $\bar{x}$ and R or for $\bar{x}$ and s may provide useful tests for ready application. Both these joint tests have some "weak points" with small power. On the whole $\bar{x}$ and R charts may be used for situations $\sigma < \sigma_0$ while $\bar{x}$ and s charts are preferable for situations $\sigma > \sigma_0$. The single chart for extreme values can be definitely recommended for detecting deviations in process mean alone, σ being equal to σ_0.

Acknowledgment

The authors would like to thank Sri S. Ganguly for his assistance in computing the O.C. values.

TABLE 1

Factors for obtaining control limits on the control chart for $\bar{x}$ and R to be used simultaneously

(a) Standards given

α	γ	$n = 3$			$n = 4$			$n = 5$		
		A	D_1	D_2	A	D_1	D_2	A	D_1	D_2
.005	.0012, 521	1.7453	.0631	4.9740	1.51145	.2126	5.2370	1.3519	.3877	5.4120
.01	.0025, 063	1.6202	.0930	4.7016	1.4031	.2681	4.9656	1.2550	.4639	5.1492
.02	.0050, 253	1.4862	.1327	4.4229	1.28705	.3428	4.6910	1.1512	.5556	4.8839
.05	.0126, 603	1.2912	.2136	4.0131	1.11825	.4690	4.2980	1.0002	.7078	4.5014
.10	.0256, 584	1.1251	.3067	3.6688	.9744	.5996	3.9719	.8715	.8056	4.1847

(b) Standards not given

α	γ	A_1	D_3	D_4	A_1	D_3	D_4	A_1	D_3	D_4
.005	.0012, 521	1.0311	.0377	2.9386	.7341	.1033	2.5436	.5812	.1667	2.3266
.01	.0025, 063	.9572	.0549	2.7777	.6815	.1302	2.4118	.5395	.1994	2.2136
.02	.0050, 253	.8780	.0784	2.6130	.6251	.1665	2.2784	.4949	.2389	2.0996
.05	.0126, 603	.7628	.1262	2.3709	.5431	.2278	2.0875	.4300	.3043	1.9352
.10	.0256, 584	.6647	.1812	2.1675	.4733	.2912	1.9292	.3747	.3463	1.7990

TABLE 2

Operating characteristic for the joint test based on control charts for $\bar{x}$ and R

$K = \frac{m - m_0}{\sigma_0/\sqrt{n}}$	α	$n = 5$						
		$l = .125$	$l = .250$	$l = .500$	$l = 1.00$	$l = 2.00$	$l = 4.00$	$l = 8.00$
0	.005	.00315	.06911	.59986	.99500	.98210	.80870	.18260
	.01	.00244	.05532	.53571	.99000	.96543	.68370	.06713
	.02	.00184	.04315	.46699	.98001	.93477	.51580	.01448
	.05	.00117	.02806	.36565	.95004	.85489	.26500	.00060
1.0	.005	.00313	.06735	.56718	.97600	.98207	.80870	.18260
	.01	.00242	.05388	.50306	.95967	.96528	.68370	.06713
	.02	.00186	.04199	.43525	.93266	.93400	.51580	.01448
	.05	.00116	.02727	.33708	.86876	.84918	.26500	.00060
2.0	.005	.00306	.06234	.47577	.84477	.96210	.80868	.18260
	.01	.00237	.04977	.41384	.78603	.91388	.68327	.06713
	.02	.00179	.03870	.35050	.70994	.81762	.51023	.01448
	.05	.00113	.02505	.26312	.57856	.58313	.21948	.00058
3.0	.005	.00298	.05479	.34730	.50798	.50922	.43425	.10474
	.01	.00228	.04359	.29330	.42112	.33723	.14997	.00408
	.02	.00172	.03378	.24054	.33183	.18446	.02287	0
	.05	.00109	.02245	.17223	.22586	.05425	.00030	0
4.0	.005	.00280	.04572	.21556	.16393	.02492	.00004	0
	.01	.00216	.03619	.17549	.11574	.00820	0	0
	.02	.00163	.02792	.13829	.07622	.00204	0	0
	.05	.00103	.01843	.09337	.03795	.00018	0	0

TABLE 2—*contd.*

$K \backslash l$	α	$n = 4$						
		.125	.25	.50	1.00	2.00	4.00	8.00
0	.005	.00975	.11538	.65350	.99491	.99050	.93180	.62550
	.01	.00780	.09497	.59144	.99000	.98132	.87300	.42743
	.02	.00610	.07650	.52400	.98001	.96236	.76672	.21172
	.05	.00414	.05243	.42184	.95002	.91079	.54595	.03985
1	.005	.00968	.11245	.61790	.97590	.99047	.93180	.62550
	.01	.00774	.09249	.55540	.95966	.98118	.87300	.42743
	.02	.00605	.07445	.48838	.93265	.96157	.76672	.21172
	.05	.00410	.05096	.38888	.86875	.90471	.54595	.03985
2	.005	.00946	.10408	.51831	.84469	.97033	.93178	.62550
	.01	.00757	.08543	.45689	.78603	.92893	.87245	.42743
	.02	.00591	.06862	.39329	.70994	.84176	.75844	.21172
	.05	.00401	.04681	.30355	.57855	.62126	.45217	.03869
3	.005	.00912	.09148	.37835	.50793	.51358	.50035	.35880
	.01	.00729	.07483	.32382	.42112	.34278	.19149	.02595
	.02	.00570	.05989	.26990	.33183	.18990	.03400	.00007
	.05	.00386	.04194	.19870	.22585	.05780	.00062	0
4	.005	.00865	.07633	.23484	.16391	.02513	.00004	0
	.01	.00692	.06213	.19375	.11574	.00833	0	0
	.02	.00540	.04949	.15517	.07622	.00210	0	0
	.05	.00366	.03444	.10772	.03795	.00019	0	0

TABLE 2—*contd.*

K \ l	α	.125	.25	.50	1.00	2.00	4.00	8.00
					n = 3			
0	.005	.02977	.19065	.70714	.99500	.99540	.98250	.93210
	.01	.02492	.16339	.65432	.99076	.99034	.96244	.85848
	.02	.02034	.13100	.59113	.98001	.98062	.92515	.73321
	.05	.01234	.08478	.44189	.93735	.95080	.81778	.44828
1	.005	.29552	.18581	.66862	.97600	.99537	.98250	.93210
	.01	.02473	.15913	.61444	.96040	.99019	.96244	.85848
	.02	.02019	.13313	.55095	.93266	.97982	.92515	.73321
	.05	.01225	.08241	.40736	.85716	.94445	.81778	.44828
2	.005	.02890	.17198	.56086	.84477	.97513	.98248	.93210
	.01	.02418	.14694	.50546	.78663	.93746	.96184	.85848
	.02	.01974	.12289	.44368	.70994	.85773	.91517	.73321
	.05	.01197	.07569	.31798	.57084	.64855	.67731	.43518
3	.005	.02784	.15116	.40941	.50798	.51612	.52757	.53467
	.01	.02329	.12874	.35824	.42144	.34593	.21111	.05212
	.02	.01901	.10725	.30448	.33183	.19351	.04102	.00024
	.05	.01152	.06782	.20814	.22284	.06034	.00093	0
4	.005	.02643	.12612	.25412	.16393	.02526	.00005	0
	.01	.02210	.10690	.21435	.11582	.00841	0	0
	.02	.01803	.08863	.17505	.07622	.00214	0	0
	.05	.01093	.05568	.11283	.03744	.00020	0	0

TABLE 3

Values of A_3 and A_4

A_4 (Standards given)

α \ n	3	4	5
.10	2.1141	2.2263	2.3110
.05	2.3878	2.4909	2.5688
.01	2.9342	3.0222	3.0891

A_3 (Standards not given)

α \ n	3	4	5
.10	1.2490	1.0813	0.9935
.05	1.4107	1.2098	1.1043
.01	1.7335	1.4679	1.3280

TABLE 4

O.C. of control charts for extreme values and of (x, R) charts

$n = 5 \quad \alpha = \cdot 05$

Chart	K \ l	.25	.50	1.00	1.50	2.00	4.00
Extreme Values	0	.025242	.329429	.950030	.999500	1.000000	1.000000
$\bar{x}$, R		.028910	.365066	.950014	.943649	.854893	.265000
Extreme Values	.5	.024411	.301716	.902537	.995010	.999915	1.000000
$\bar{x}$, R		.027911	.330222	.845944	.900283	.010814	.264999
Extreme Values	1.0	.022046	.231509	.739779	.954357	.995753	1.000000
$\bar{x}$, R		.025116	.241950	.487502	.472426	.427723	.132669
Extreme Values	1.5	.018597	.147869	.463359	.755627	.921305	1.000000
$\bar{x}$, R		.021064	.141750	.128535	.044228	.010860	.000001
Extreme Values	2.0	.014650	.078150	.187258	.334286	.505443	.944285
$\bar{x}$, R		.016460	.065242	.012369	.000377	.000003	0
Extreme Values	2.5	.010789	.033798	.040842	.046343	.052601	.083563
$\bar{x}$, R		.011983	.023210	.000389	0	0	0
Extreme Values	3.0	.007416	.011842	.004107	.001161	.000277	.000003
$\bar{x}$, R		.008125	.006294	.000004	0	0	0

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(Received Feb. 2, 1966. Revised May 18, 1967.)