INDIAN STATISTICAL INSTITUTE

First Semester Examination: 2019–2020

B. Stat. (Hons.) 1st Year. 1st Semester

Vectors and Matrices I

Date: November 25, 2019 | Maximum Marks: 60 | Duration: 3 hours

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• This question paper carries 70 points. Answer as much as you can. However, the maximum you can score is 60.

• **You should present all your arguments while answering a question.**

• You must state clearly any result stated (and proved) in class you may need in order to answer a particular question.

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1. Suppose $\mathbf{ABC} = \mathbf{0}$ and $\rho(\mathbf{AB}) = \rho(\mathbf{B})$. Show that $\mathcal{C}(\mathbf{C}) \subseteq \mathcal{N}(\mathbf{B})$. [8]

2. Let $\mathbf{A}$ and $\mathbf{B}$ be projectors of the same order. Then show that $\mathbf{A} + \mathbf{B}$ is a projector iff $\mathcal{C}(\mathbf{A}) \subseteq \mathcal{N}(\mathbf{B})$ and $\mathcal{C}(\mathbf{B}) \subseteq \mathcal{N}(\mathbf{A})$. [12]

3. Let V_1, V_2 and V_3 be finite-dimensional vector spaces over the same field F. Let B_i be a basis of V_i, $i = 1, 2, 3$. Let $T_1 : V_1 \to V_2$ and $T_2 : V_2 \to V_3$ be any two linear transformations. Let T_1 be represented by matrix M_1 with respect to B_1 and B_2, and let T_2 be represented by matrix M_2 with respect to B_2 and B_3. Show that the linear transformation represented by M_2M_1 with respect to B_1 and B_3 does not depend on B_2. [12]

4. Prove that $\rho(\mathbf{PAQ}) = \rho(\mathbf{A})$ iff $\rho(\mathbf{A}) = \rho(\mathbf{PA}) = \rho(\mathbf{AQ})$. [10]

5. Let $x_1, \ldots, x_n, y_1, \ldots, y_n (n > 2)$ be real numbers, none of which is zero. Let $\mathbf{A} = ((a_{ij}))_{n\times n}$ be defined by $a_{ij} = x_ix_j + y_iy_j$. Find the possible ranks of $\mathbf{A}$. [10]

6. Let $x_1, \ldots, x_n (n \geq 2)$ be real numbers, not all zero. Let $\mathbf{A} = ((a_{ij}))_{n\times n}$ be defined by $a_{ii} = i + x_i^2$ and $a_{ij} = x_ix_j$ if $i \neq j$. Find the inverse of $\mathbf{A}$ or show that $\mathbf{A}$ is singular. [10]

7. Let $n > 1$. Find two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ none of which is a zero-matrix but $\mathbf{AB} = \mathbf{0}$. [8]

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Backpaper Examination: 2019–2020

B. Stat. (Hons.) 1st Year. 1st Semester

Vectors and Matrices I

Date: 16|1|20 Maximum Marks: 100 Duration: 3 hours

- Answer all the questions.
- **You should present all your arguments while answering a question.**
- You must state clearly any result stated (and proved) in class you may need in order to answer a particular question.

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1. Let S and T be two subspaces of a vector space V. Prove the following:

$$d(S+T) + d(S \cap T) = d(S) + d(T). \qquad [12]$$

2. Show that for any matrix $\mathbf{A}$, the row rank of $\mathbf{A}$ equals the column rank of $\mathbf{A}$. [12]

3. Let $\mathbf{A}$ and $\mathbf{B}$ be matrices of orders $m \times n$ and $n \times p$ respectively. Show that $\rho(\mathbf{AB}) \geq \rho(\mathbf{A}) + \rho(\mathbf{B}) - n$, with equality iff $\mathcal{N}(\mathbf{A}) \subseteq \mathcal{C}(\mathbf{B})$. [10+6=16]

4. Consider a partitioned matrix

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$$

where $\mathbf{A}$ and $\mathbf{D}$ are non-singular. Show that $\mathbf{D} - \mathbf{CA}^{-1}\mathbf{B}$ is non-singular iff $\mathbf{A} - \mathbf{BD}^{-1}\mathbf{C}$ is non-singular. [14]

5. Determine all the values of α and β for which the vectors $(\alpha, \beta, \beta, \beta)$, $(\beta, \alpha, \beta, \beta)$, $(\beta, \beta, \alpha, \beta)$, and $(\beta, \beta, \beta, \alpha)$ of $\mathbb{R}^4$ are linearly dependent. [12]

6. Let $(\mathbf{P}, \mathbf{Q})$ be a rank-factorization of a non-null square matrix $\mathbf{A}$. Show that $\rho(\mathbf{A}) = \rho(\mathbf{A}^2)$ iff $\mathbf{QP}$ is non-singular. [14]

[P.T.O.]

7. Let $\mathcal{P}_n$ denote the vector space (over $\mathbb{R}$) of real-valued polynomials defined on $\mathbb{R}$ and of degree at most n. Let $x_1, x_2, \ldots, x_{n+1}$ be fixed distinct real numbers. Show that $\ell_1(t), \ell_2(t) \ldots, \ell_{n+1}(t)$ form a basis of $\mathcal{P}_n$, where $\ell_i(t) = \prod_{j \neq i}(t - x_j)$.
[10]

8. Consider the following linear operators on $\mathbb{R}^2$:

$$f(x_1, x_2) = (2x_1 + 3x_2, x_1 - x_2), \quad g(x_1, x_2) = (x_1, 2x_1 - 5x_2).$$

Find the matrix (representation) of $f \circ g$ with respect to the canonical basis.
[10]

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FIRST SEMESTER SEMESTRAL EXAMINATION (2019–20)

B. STAT. I YEAR

ANALYSIS I

Date : 18.11.2019 Maximum Marks : 100 Time : $3\frac{1}{2}$ hours

The question carries 120 marks. Maximum you can score is 100. Precisely justify all your steps. Carefully state all the results you are using.

1. Let $\{a_n\}$ be a bounded sequence of real numbers, and let $\alpha = \limsup_{n\to\infty} a_n$. Show that if $\{a_{n_k}\}$ is a convergent subsequence of $\{a_n\}$ with $\lim_{k\to\infty} a_{n_k} = a$, then $a \leq \alpha$. [10]

2. If $\sum_{n=1}^{\infty} a_n^2$ is convergent, then prove that $\sum_{n=1}^{\infty} \frac{a_n}{n}$ is absolutely convergent. [10]

3. If x is not an integer multiple of 2π, show that

$$\sum_{k=1}^{n} \cos kx = \frac{\sin(n+1/2)x - \sin(x/2)}{2\sin(x/2)}.$$

Using this show that $\sum_{n=1}^{\infty} \frac{\cos nx}{n^s}$ converges for all $s > 0$. [5 + 5 = 10]

4. Let $I \subseteq \mathbb{R}$ be an interval and $f, g : I \to \mathbb{R}$. Define

$$h(x) = \max\{f(x), g(x)\}, \quad x \in I.$$

(a) If f and g are continuous at $x_0 \in I$, show that h is continuous at x_0.

(b) If f and g are differentiable at x_0, is h differentiable at x_0? [10 + 5 = 15]

5. Let I be any interval and $f : I \to \mathbb{R}$ be a function. A point $a \in I$ is said to be a fixed point for f if $f(a) = a$.

(a) If $f : [0,1] \to [0,1]$ is continuous, then f has a fixed point.

(b) If, moreover, f is differentiable and $f'(x) \neq 1$ for any $x \in (0,1)$, then the fixed point is unique. [5 + 5 = 10]

[P.T.O.]

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. [10]

7. Let $I \subseteq \mathbb{R}$ be an interval and $x_0 \in I$. Let $f : I \to \mathbb{R}$ be continuous such that for all $x \neq x_0$, $f'(x)$ exists and $\lim_{x \to x_0} f'(x)$ exists. Is f differentiable at x_0? [10]

8. Let $f : (0, 1) \to [0, \infty)$ be a thrice differentiable function. If $f(x) = 0$ for *at least two* values of $x \in (0, 1)$, prove that $f'''(c) = 0$ for some $c \in (0, 1)$. [10]

9. Prove or disprove : [$3 \times 5 = 15$]

   (a) Any compact subset of $\mathbb{R}$ is either finite or uncountable.

   (b) Continuous image of an open set in $\mathbb{R}$ is open, i.e., if $A \subseteq \mathbb{R}$ is open, $f : A \to \mathbb{R}$ is continuous, then $f(A) \subseteq \mathbb{R}$ is open.

   (c) Continuous image of a closed set in $\mathbb{R}$ is closed, i.e., if $A \subseteq \mathbb{R}$ is closed, $f : A \to \mathbb{R}$ is continuous, then $f(A) \subseteq \mathbb{R}$ is closed.

   (d) Continuous image of a bounded set in $\mathbb{R}$ is bounded, i.e., if $A \subseteq \mathbb{R}$ is bounded, $f : A \to \mathbb{R}$ is continuous, then $f(A) \subseteq \mathbb{R}$ is bounded.

   (e) Continuous image of a closed and bounded set in $\mathbb{R}$ is bounded, i.e., if $A \subseteq \mathbb{R}$ is closed and bounded, $f : A \to \mathbb{R}$ is continuous, then $f(A) \subseteq \mathbb{R}$ is bounded.

10. Let $f : I \to \mathbb{R}$. We say that f is *locally bounded* if for every $x \in I$, there is an interval $(x - \delta, x + \delta)$ such that f is bounded on $(x - \delta, x + \delta) \cap I$. [$5 + 5 + 10 = 20$]

    (a) Show that any continuous $f : I \to \mathbb{R}$ is locally bounded.

    (b) Give an example to show that a locally bounded function need not be bounded.

    (c) Show that any locally bounded function on $I = [a, b]$ is bounded.

    [Hint : $[a, b]$ is compact.]

INDIAN STATISTICAL INSTITUTE

FIRST SEMESTER BACKPAPER EXAMINATION (2019–20)

B. STAT. I YEAR

ANALYSIS I

15.01.2020

Date : xx.xx.20xx Maximum Marks : 100 Time : 3 hours

Precisely justify all your steps. Carefully state all the results you are using.

1. Given nonempty subsets S and T of $\mathbb{R}$ such that

$$s < t, \qquad \text{for every } s \in S \text{ and } t \in T,$$

show that S has a supremum, T has an infimum and

$$\sup S \leq \inf T.$$

Can you actually conclude $\sup S < \inf T$? [$10 + 5 = 15$]

2. Test the convergence of the following series : [$5 + 5 = 10$]

$$(a) \ \sum_{n=2}^{\infty} \frac{n^3[\sqrt{2} + (-1)^n]^n}{3^n} \qquad (b) \ \sum_{n=2}^{\infty} \frac{2^n + n^2 + n}{2^{n+1} \cdot n(n+1)}$$

3. (a) Find all $x \in \mathbb{R}$ for which the power series $\sum_{n=1}^{\infty} nx^n$ converges.

   (b) Assuming that you are allowed to differentiate a geometric series "term by term", find a closed form expression for the sum of the above series.

   (c) Show that wherever the series converges, it converges to the above expression.

   [Hint : Show that the difference between the n-th partial sum and the "sum" converges to 0.] [$5 + 5 + 10 = 20$]

4. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function and define $g : [a, b] \longrightarrow \mathbb{R}$ by putting $g(a) = f(a)$ and

$$g(x) = \max\{f(t) : t \in [a, x]\}, \text{ for } x \in (a, b].$$

Show that g is continuous on $[a, b]$. [15]

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = \infty$. Prove that f attains its minimum on $\mathbb{R}$. [10]

6. Let I be an interval. Suppose $f : I \to \mathbb{R}$ is differentiable and the derivative is bounded on I. Show that f is uniformly continuous on I. [5]

7. Let f be a continuous real function on $\mathbb{R}$ such that for all $x \neq 0$, $f'(x)$ exists and $\lim_{x \to 0} f'(x) = 3$. Is f differentiable at 0? [5]

8. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. [10]

9. Let f be a thrice differentiable function on $(0, 1)$ such that $f(x) \geq 0$ for all $x \in (0, 1)$. If $f(x) = 0$ for *at least two* values of $x \in (0, 1)$, prove that $f'''(c) = 0$ for some $c \in (0, 1)$. [10]

Indian Statistical Institute
Semestral Examination
B-Stat (Hons.), 1st Year, 1st Semester (2019-20)
Subject : Statistical Methods I
Date and Time : November 29, 2019, 10.30am to 1.30pm

Answer all questions.

(1). Consider the bivariate data set $(3, 5)$, $(3, 7)$, $(8, 12)$, $(8, 4)$. Compute with relevant justification the simplicial median for this data.

[20 points]

(2). It is observed in a bivariate data that the conditional mean of Y given $X = x$ is $2 + 3x$, and the conditional ~~mean~~ variance of Y given $X = x$ is $3x^2$. Suppose that X takes values $2, 5, 7$ and 10, and the corresponding frequencies are $11, 21, 7$ and 9 respectively. Compute the coefficient of variation of Y.

[15 points]

(3). Prove that in any data set containing 500 data points, at most 55 data points will have Z-score above 3 or below -3.

[15 points]

(4). Suppose that the health authorities have claimed that only in 15% of the households in a big city with a few thousand households, one or more individuals are infected with dengue. Some NGO suspects the claim. They took a simple random sample of 400 households and found that in 75 households in their sample at least one individual is infected. If the claim of the health authorities is true, what will be the Z-score for the data obtained by the NGO? Do you think that there is any merit in the suspicion of the NGO?

[20 points]

(5). If two variables X and Y each takes only two distinct values, prove that they are independently distributed if and only if their correlation coefficient is zero.

[15 points]

(6). Assignments [15 points]

INDIAN STATISTICAL INSTITUTE

First-Semestral Examination: (2019 - 2020)

B. Stat 1st Year

<u>Introduction to Programming and Data Structures</u>

Date: 20.11.2019 **Maximum Marks:** 100 **Duration:** 3.5 hours

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The question paper is divided into following two groups: **<u>Group A:</u>** 70 marks, 2 hours duration; **<u>Group B:</u>** 30 marks, 1.5 hours duration.

<u>Group B</u>

Marks: 30 **Duration:** 1.5 hours

Write a C program to perform the following tasks: [6 + 8 + 12 + 4 = 30]

(1) *Create* two linked lists, namely, **L1** and **L2**, with ***m*** and ***n*** nodes, respectively. Each node stores a string, which is to be taken as input from the user.

[**L1** → **BFG** → **ABC** → **XZyw** → **abd** → **AA** → **de** → **MNy** → NULL and **L2** → **JFQ** → **cd** → **ABC** → **BC** → NULL]

(2) *Sort* **L1** and **L2** in alphabetical order.

[**L1** → **AA** → **ABC** → **abd** → **BFG** → **de** → **MNy** → **XZyw** → NULL and **L2** → **ABC** → **BC** → **cd** → **JFQ** → NULL]

(3) *Merge* two sorted lists, obtained in (2), in such a way that the merged linked list becomes a sorted list, in alphabetical order. The nodes with distinct strings are only to be included in merged linked list.

[**L1** → **AA** → **ABC** → **abd** → **BC** → **BFG** → **cd** → **de** → **JFQ** → **MNy** → **XZyw** → NULL]

(4) *Display* the linked list(s) for each of the above tasks.

You may use constant amount of memory for tasks (2) and (3). For task (2), you can only modify the link field of each node. For task (3), you cannot apply sorting algorithm on merged linked list. While writing the program, you need to take care of the following:

1. The names of the variables and functions should reflect their functionality.
2. Dynamic memory allocation/deallocation should be done.
3. Proper documentation of the program should be provided.

BSTAT I - Probability Theory I

Final Exam. / Semester I 2019-20

Date - November 22, 2019 / Time - 3 hours

Maximum Score - 50

NOTE : TOTAL MARKS ALLOTTED IS 57. ANSWER AS MUCH AS YOU CAN. SHOW ALL YOUR WORK TO GET THE FULL CREDIT. RESULTS USED MUST BE CLEARLY STATED.

1. (2+5+6=13) Each throw of an unfair die lands on each of the odd numbers 1, 3, 5 with probability C and on each of the even numbers with probability $2C$.

   (a) Find C.

   (b) Suppose the die is thrown once. Let $X = 1$, if the result is an odd number and $X = 0$, if the result is an even number. Further, let $Y = 1$, if the result is less than 4 and $Y = 0$, otherwise. Find the joint probability distribution of X and Y.

   (c) Suppose the die is thrown 24 times independently.

   (i) Find the probability that each of the six numbers occurs exactly 4 times.

   (ii) Find the probability that 8 of the outcomes are either one or two, 8 of the outcomes are either three or four and 8 of the outcomes are either five or six.

2. (7+7=14) $4n$ individuals, consisting of $2n$ married couples are to be seated at n different tables (numbered 1 to n) with four people at each table.

   (a) Let the seating is done at random. Find the probability that exactly 1 married couple is seating at table 1. Considering all the tables, find the expected number of married couples that are seated at the same table.

   (b) Let two women and two men are chosen randomly to be seated at each table. Find the probability that exactly 2 married couple is seating at table 1. Considering all the tables, find the expected number of married couples that are seated at the same table.

3. (5+5+3=13) An urn has n red balls and m white balls. A ball is chosen at random and its color is observed and put back in the urn along with another balls of same color. Let X_i be random variable that takes the value 1 if the ith draw yields the red color and 0 if it is white. Let T_k be another random variable that counts total number red ball drawn till the end of the kth draw.

   (a) Find the probability distribution of T_k.

(b) Show that $P(X_{j-1} = 1, X_j = 1) = P(X_1 = 1, X_2 = 1) = \frac{n(n+1)}{(n+m)(n+m+1)}$ for any $j \geq 2$.

(c) Find the expected number of instances in which a red ball immediately followed by a white one, at the end of the Mth draw.

4. ((4+5)+(4+4)=17) Two players, A and B are playing a game repeatedly and independently. Each game can result in a win, loss or a draw for the player A and each time the bet is one rupee. Let X_i be the profit made by A in the ith game, and it takes the values 1, 0, and -1 with (positive) probabilities p, r and q, respectively.

   (a) Suppose, initially A has 30 rupees and B has 70 rupees.

   (i) Find the probability that A is going to be ruined before B.

   (ii) Find the expected number of games that A can play before any of them is ruined.

   (b) Suppose, initially A has 30 rupees and B is a casino (i.e., one may consider it has infinite amount of money).

   (i) Find the probability that A is going to be ruined.

   (ii) Find the expected number of games that A can play before any of them is ruined, for $p < q$.

All the best.

INDIAN STATISTICAL INSTITUTE
REMEDIAL ENGLISH
B.STAT 1st Year

Date: 29.11.2019 **Marks: 100** **Time: 3 hours**

(1)Write a Paragraph (100 words) on any four topics: (4x5)

(a) Are security cameras an invasion of our privacy?

(b) How far is competition necessary in regards to the learning process?

(c)Is the boarding school system beneficial to children?

(d)Is it effective to censor parts of the media?

[e]Is global warming an issue?

(f)Is euthanasia justified?

(2) Complete these sentences with at, on, or in and the most likely word or phrase. (12X1)

(1) I bumped into Tim ________ I went to the other evening.

(2) The film was shot mainly ________ in North Africa.

(3) He was undoubtedly the best player _________ in the first half.

(4) Although he has been singing for ages, it will be the first time he has appeared _______.

(5) They live _______, so there's a lot of traffic going past.

(6) It will be the biggest event of its kind ever held ________.

(7) I know that people like to dress up ________, but that is ridiculous.

(8) Bill lived ________ of my street.

(9) The information ________ is out of date.

(10) Do you know that there's a rabbit _____, and it's eating your flowers?

(11) He put his hand _______ and took out some coins.

(12) Who has moved my briefcase? I left it ________-

The pitch, parties, this booklet, the table , the main road, a dinner, this country, his pocket, the top end, your lawn, the Opera House, Tunisia

Interviewer: I understand that your early life was not easy. Can you tell us a little about it? Where were you born?

Ruth: In Barnsley, in the north of England and that's where I __grew up__.

Interviewer: Were you lonely as a child?

Ruth: I had three sisters and two brothers so it was never quiet. There was always something (1) ______. The house was never empty because neighbours (2) _________ all the time.

Interviewer: Do you remember any particularly happy moments?

Ruth: Yes, when we went to bed my mother always told us stories. She didn't have a book- she just (3) ______ them ______ herself.

Interviewer: And then things went wrong. How did that (4) ______?

Ruth: Well, in the first place my father smoked a lot. He always said that he was going to (5) _______, but he never did. He got very ill and he was in hospital for several weeks. Even when the hospital (6) ______ him ________, he wasn't well. He had to (7) _____ and keep warm so that his bronchitis wouldn't start again. But at least he had the sense to finally (8) ______ smoking.

Interviewer: But things got worse.

Ruth: Yes, while he was recovering we heard that the factory where he worked had (9) _____ a lot of workers. At first, he wasn't affected but then we heard they were going to (10) ______ the factory ________.

Interviewer: And then things got better.

Ruth: Yes, my parents had to (11) ________ their new situation. They said that businesses could (12) ____________ factory workers but they would always need office staff. Luckily they had a savings account and every week they had (13) _____ something _______. Now they decided to (14) ______ their savings and (15) _______ a little business selling office equipment. It did quite well and when they retired I decided to (16) ____ it ____.

Interviewer: Well, that is a story with a happy end. Thank you for speaking to me.

Carry on, come about, cut down, do away with, draw out, drop in, face up to, give up, go on, grow up, lay off, let out, make up, pay in, set up, shut down, stay in.

(4) Write a letter to your younger brother who was caught copying in his exam (15)

(5) Write a conversation between two friends discussing about a subject as their exams are fast approaching: (15)

(6) In the following questions choose the word similar in meaning to the given word: (5x2)

(1) Bequest

(a) Parsimony
(b) Matrimony
(c) Heritage
(d) Patrimony

(2) Recuperate

(a)Recapture
(b) Reclaim
(c) Recover
(d) Regain

(3) Alms

(a) Blessings
(b) Charity
(c) Prayers
(d) Worship

(4) Vindictive

(a) Revengeful

(b) Triumphant

(c) Strategic

(d) Demonstrative

(5) Wrath

(a) Violence

(b) Anger

(c) Hatred

(d) Displeasing

(7)Here is a story about a day out for the Long family. Complete the story by choosing the correct option in each case. (12x1)

Mr Long is a careless driver. In fact he has a reputation as a dangerous driver because/~~owing to~~ the police have fined him three times for speeding. (1) Due to/Since he drives carelessly, his wife usually drives the family car, especially when the children are with them. The children often feel sick in the car (2) due to the fact that/owing to they are not good travellers, and when this happens Mrs Long has to stop the car (3)for/as them to have a break. Some people take pills for travel sickness, off course, but Mrs Long doesn't like the idea (4)because/due to she thinks the children will get addicted. One hot summer's day the family were on their way to visit Mrs. Long's mother (5)owing to the fact that/owing to it was her birthday. (6) Since/For it was a special day the children were wearing their best clothes, so it was obviously a bad day (7) for/because them to get dirty. Very soon the children were feeling sick, probably due (8) to/for the heat, so Mrs Long stopped the car several times (9) for/since them to get out and have a drink. When they finally arrived, grandmother said, "You're a bit late but I suppose that's (10) due to/ owing to the traffic." "Not really," said Mrs Long. "The journey took longer than usual (11) because/owing to the heat and we had to stop several times (12) to/for a break." At their grandmother's the children soon felt better and they had a great afternoon. After lunch they went for long walk with Grandma's dog, Queenie. On the way home they were tired and fell asleep straightaway in the car.