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DISTRIBUTION OF THE VALUES OF ω IN SHORT INTERVALS

By

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Introduction. Let $\omega(m)$ denote the number of prime factors of m, $1 \leqq b(n) \leqq n$ be a sequence of integers and let

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy.$$

In [1], it is shown that

$$(1) \quad \frac{1}{b(n)} \text{ card } \{(n, n+b(n))] \cap (m: \omega(m) - \log\log m < x(\log\log m)^{1/2})\} \to \Phi(x)$$

as $n \to \infty$, provided $b(n) \geqq Tn^{\alpha}$ for some $T > 1$ and $0 < \alpha \leqq 1$. Similar results for general additive arithmetic functions are also proved in [1]. In this connection P. Erdős raised the following question. How small can one let $b(n)$ to be, so that (1) still holds? We have the following result in this direction.

THEOREM. *Let* $1 \leqq a(n) \leqq (\log\log n)^{1/2}$ *be a sequence of real numbers tending to infinity. Then* (1) *holds if* $b(n) \geqq n^{a(n)(\log\log n)^{-1/2}}$

NOTATIONS. Let Q denote the set of all primes and for any set E of integers let

$$\nu_n(E) = \frac{1}{b(n)} \text{ card } \{E \cap (n, n+b(n)]\}.$$

We require the following lemma (for a proof see [1]).

LEMMA (Lemma 1 of [1]). *Let* $\delta_p(m) = 1 - \frac{1}{p}$ *or* $-\frac{1}{p}$ *according as* $p|m$ *or not. Let* $\{a_p\}$ *be a sequence of real numbers and let* $r, k \leqq s$ *be integers. Then*

$$\Sigma'(\Sigma'' a_p \delta_p(m))^2 \leqq (2s + 8k^2)\Sigma'' \frac{1}{p} a_p^2,$$

where Σ' *denotes the sum over all integers* $m \in (r, r+s]$ *and* Σ'' *denotes the sum over all primes* $p < k$.

Proof of the theorem. To avoid repetitions of arguments of [1] and [2], we give only the main steps of the proof. Let $k = k_n = 1 + [(b(n))^{1/4}]$ and for any $t \geqq 1$,

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let $\omega_t(m)=\sum_{p|m,\, p<t} 1$. For $m\in(n, n+b(n)]$,

(2) $$|\omega(m)-\omega_k(m)| \leqq \sum_{p|m,\, k<p\leqq 2n} 1 \leqq 1+((\log 2n)/\log k) \ll \frac{(\log\log n)^{1/2}}{a(n)}$$

and

(3) $$0 \leqq \log\log 2n - \log\log m \leqq \log\log 2n - \log\log n \to 0$$

as $n\to\infty$. Since

$$\sum_{k<p\leqq n} \frac{1}{p} = \log\log n - \log\log k + o(1) \ll \log\log\log n,$$

by (2) and (3) it is enough to prove

(4) $$\nu_n\left\{m : \omega_k(m) - \sum_{p\leqq k}\frac{1}{p} < x\left(\sum_{p<k}\frac{1}{p}\right)^{1/2}\right\} \to \Phi(x).$$

Now let $r=r_n=\exp\left((\log n)(\log\log n)^{-1/2}\right)$. Clearly

(5) $$\left(\sum_{p\leqq r}\frac{1}{p}\right)\left(\sum_{p\leqq k}\frac{1}{p}\right)^{-1} \to 1$$

as $n\to\infty$. So by the lemma, it follows that for every $\varepsilon>0$,

(6) $$\nu_n\left\{m : \left|\omega_r(m)-\omega_k(m)+\sum_{r<p\leqq k}\frac{1}{p}\right| > \varepsilon\left(\sum_{p\leqq k}\frac{1}{p}\right)^{1/2}\right\} \to 0$$

as $n\to\infty$. In view of (5) and (6) it is enough to show that, as $n\to\infty$,

(7) $$F_n(x) = \nu_n\left\{m : \omega_r(m) - \sum_{p<r}\frac{1}{p} < x\left(\sum_{p<r}\frac{1}{p}\right)^{1/2}\right\} \to \Phi(x).$$

To prove (7), we introduce a sequence $\{\xi_p : p\in Q\}$ of independent random variables with

$$P\left(\xi_p = 1-\frac{1}{p}\right) = \frac{1}{p} \quad\text{and}\quad P\left(\xi_p = -\frac{1}{p}\right) = 1-\frac{1}{p}.$$

Put $\zeta_n=\left(\sum_{p\leqq r}\frac{1}{p}\right)^{-1/2}\sum_{p\leqq r}\xi_p$. It follows that

(8) $$\zeta_n \text{ converges weakly to } \Phi.$$

We shall now show that the distribution of ζ_n does not differ much from F_n. By a proof similar to the proof of lemma 3.1 of [2], we have for any integer $t\geqq 1$

(9) $$\left|E(\zeta_n^t) - \frac{1}{b(n)}\sum_{n<m\leqq n+b(n)}\left(\omega_r(m)-\sum_{p\leqq r}\frac{1}{p}\right)^k\right| \leqq \frac{2^t r^t}{b(n)}$$

and by lemma 3.2 of [2] we have

(10) $$|E(\zeta_n^t)| \leqq t!\, e^t.$$

Since $r_n^t/b(n)\to 0$ as $n\to\infty$ for any $t\geqq 1$, (7) follows from (8), (9), (10) above and Theorem 11.2 of [2]. This completes the proof.

REMARKS. If $c(m)\to\infty$, then, except possibly for $o(b(n))$ integers $m\in(n, n+b(n)]$, we have $|\omega(m)-\log\log m|<c(m)(\log\log m)^{1/2}$. In particular, by taking $b(n)=\exp\{\log n)(\log\log\log n/\log\log n)^{1/2}\}$ we have

$$|\omega(m)-\log\log m| < (\log\log\log m)(\log\log m)^{1/2}$$

for all but $o(b(n))$ many $m\in(n, n+b(n)]$.

As in [1] and [2], by going over to the Brownian motion we obtain, as $n\to\infty$, that

$$\frac{1}{b(n)}\operatorname{card}\left\{n < m \leqq n+b(n): \max_{t\leqq n}(\omega_t(m)-\log\log t) < x(\log\log m)^{1/2}\right\}\to$$

$$\to\sqrt{\frac{2}{\pi}}\int_0^x e^{-\frac{1}{2}y^2}\,dy.$$

As a consequence, in particular, it follows that

$$\left|\max_{t\leqq n}(\omega_t(m)-\log\log t)\right| < (\log\log\log m)(\log\log m)^{1/2}$$

for all but $o(b(n))$ many integers $m\in(n, n+b(n)]$.

One can show that many results of [1] still hold for general arithmetic functions, when the restriction $b(n)\geqq Tn^{\alpha}$ is weakened to $b(n)\geqq Tn^{\alpha(n)}$, where $\alpha(n)\to 0$ at an appropriate rate.

Some open problems. In this connection P. Erdős and I. Z. Ruzsa raised the following questions.

(a) What is the largest value of $a(n)$ such that, if $b(n)<a(n)$ for all n, then (1) fails to hold?

(b) Does (1) hold if $b(n)=n^{(\log\log n)^{-1/2}}$?

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References

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