Some topics in Leavitt path algebras and their generalizations

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September 2020

Indian Statistical Institute

Doctoral Thesis

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A thesis submitted to the Indian Statistical Institute
in partial fulfilment of the requirements for
the degree of
Doctor of Philosophy (in Mathematics)

Theoretical Statistics & Mathematics Unit

Indian Statistical Institute, Bangalore Centre

September 2020

Dedicated to my parents

Acknowledgements

காலத்தி னாற்செய்த நன்றி சிறிதெனினும்
ஞாலத்தின் மாணப் பெரிது

குறள் 102

[A timely favour, however trivial its material value is, is invaluable.]

I sincerely thank my supervisor Ramesh Sreekantan foremost for his invaluable support and guidance. I wish to thank B.N. Suhas who agreed to collaborate with me and who helped me to finalise the doctoral thesis. I thank B. Sury for his engaging mathematical discussions throughout the course of my doctoral degree.

I sincerely thank National Board for Higher Mathematics (Department of Atomic Energy) for providing me with PhD scholarship 2014-2019. I also thank Research Fellows Advisory Committee, and office staff of Indian Statistical Institute Bangalore Centre for their constant support.

I wish to thank Roozbeh Hazrat, Cristóbal Gil Canto, Raimund Preusser, Kulumani M. Rangaswamy, Gene Abrams, Pere Ara, Rajesh Kannan, Apoorva Khare, and A.A. Ambily for their comments and suggestions towards improving my research work. I thank R. Venkatesh, Mamta Balodi for their support in running a series of seminars on Leavitt path algebras which helped me to learn the subject.

I specially thank Aditya Challa for helping me with Python programming language which was crucial in my first research manuscript. I wish to thank Tiju Cherian John for hosting me during a crucial juncture of my life which helped me to get admission into the doctoral program.

My sincere salutations to my mathematics teachers A. Joseph, D.S. Narayana, B.S. Upadhyaya, B.S.P. Raju, S. Kumaresan, S.H. Kulkarni, and T.E. Venkata Balaji.

I sincerely thank Vijaya Kumar, Sankar. T. R, Manish Kumar, Muthukumar, Shankar, Deepak Johnson, Subhadarshee Nayak, and all other friends in Indian Statistical Institute Bangalore. My thanks are also due to my friends Susobhan, Sairam, and Bharaneedhran from Indian Institute of Techonology Madras.

Special thanks to Jishnu Biswas and Rema Krishnaswamy; Vaibhav Vaish, Rajarama Bhat, C.S. Aravinda and Bhāvanā magazine team for their constant encouragement.

I thank my parents and in-laws for their constant support throughout my PhD course. I cannot thank enough my friend Aswath Balakrishnan for taking care of me during most depressing period of my life and made sure I remained sane. To my partner Vijitha Rajan, I love you.

Mohan. R
14 February 2020

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Chapter 1

Preliminaries

1.1 History and overview

The purpose of this section is to motivate the historical development of Leavitt algebras, Leavitt path algebras and their various generalizations and thus provide a context for this thesis. There are two historical threads which resulted in the definition of Leavitt path algebras. The first one is about *the realization problem for von Neumann regular rings* and the second one is about studying algebraic analogs of *graph C^*-algebras*. In what follows we briefly survey these threads and also introduce important concepts and terminology which will recur throughout.

1.1.1 The first historical thread: Leavitt algebras and graph monoids

1.1.1.1 Invariant basis number and Leavitt algebras

Let R be a unital ring and M a left R-module with minimal generating set X. If X is infinite, then any generating set of M has at least $|X|$ elements. In particular any two minimal generating sets of M have the same cardinality. However if X is finite, this need not be the case. Therefore a free module on a finite generating set may have minimal generating sets of different sizes. For a positive integer n, we say the left R-module R^n has **unique rank** n if it is not isomorphic to R^m for any positive integer $m \neq n$. Most of the examples that are encountered in a first course on ring theory such as $\mathbb{Z}$, K, $K[X]$, $K[X, X^{-1}]$, $M_n(K)$ all have *invariant basis number*:

Definition 1.1.1. A unital ring R is said to have the **Invariant Basis Number** (IBN) if every free left (or right) R-module has a unique rank.

A wide class of unital rings have IBN; examples are commutative rings and Noetherian rings. But there are also examples of rings which do not have IBN.

Example 1.1.2. Let V be a countably infinite dimensional vector space over a field K, and let R denote $\mathrm{End}_K(V)$, the K-algebra of all linear transformations from V to itself. Then the left modules R and R^2 are isomorphic (and hence $R^i \cong R^j$ for any pair $i, j \in \mathbb{N}$).

Let R be any unital ring and suppose that R^n has a generating set consisting of m elements, for some $m, n \in \mathbb{N}$. Then we have a surjection $R^m \to R^n$, giving rise to an exact sequence

$$0 \to M \to R^m \to R^n \to 0.$$

Since R^n is free, the sequence splits and so $R^m \cong R^n \oplus M$. Hence IBN property can be stated as follows:

$$\text{For any } m, n \in \mathbb{N},\ R^m \cong R^n \text{ implies } m = n.$$

By describing the change of basis, we can also express IBN property in matrix form:

$$\text{For any } A \in M_{m\times n}(R), B \in M_{n\times m}(R), \text{ if } AB = I_m \text{ and } BA = I_n, \text{ then } m = n.$$

It is direct that for a non-zero unital ring R *without* IBN there exist $h, k \in \mathbb{N}$ such that $R^h \cong R^{h+k}$. The first pair of positive integers (m, n) with $m < n$ in the lexicographic ordering such that $R^m \cong R^n$ is called the **module type** of the ring R.

Lemma 1.1.3. *Let R, S be two non-zero unital rings, $m, n \in \mathbb{N}$ and $m < n$.*

1. *If R has module type (m, n), then $R^h \cong R^k$ holds if and only if $h = k$ or $h, k \geq m$ and $h \equiv k$ mod n.*

2. *Let $f : R \to S$ be a ring homomorphism (which preserves unit). If S has IBN then R also has IBN.*

Proof. 1. Follows from the definition of module type.

2. We show that if R does not have IBN, then S does not have IBN. Let R be of module type (m, n). Then there exists $A \in M_{m\times n}(R), B \in M_{n\times m}(R)$ satisfying $AB = I_m, BA = I_n$. Applying f entry wise we get such matrices over S, whence it follows that $S^m \cong S^n$, so S cannot have IBN. In particular this means that if the module type of S is (m', n') then $m' \leq m$ and $n' - m' \mid n - m$.

□

In a series of papers [47, 48, 49, 50], Leavitt constructed and studied 'canonical' examples of non-zero unital rings that do not have Invariant Basis Number. For a field K and $m, n \in \mathbb{N}$ with $m < n$, he constructed K-algebras (which we now call) **Leavitt algebras** $L_K(m, n)$ of module type (m, n). Also $L_K(m, n)$ is universal K-algebra with this property in the sense that if T is any K-algebra having module type (m, n), then there exists a non-zero K-algebra homomorphism $\varphi : L_K(m, n) \to T$ such that the isomorphism $f : T^m \to T^n$ is equal to $g \otimes_\varphi \mathrm{id}_T$ where $g : (L_K(m, n))^m \xrightarrow{\sim} (L_K(m, n))^n$.

$L_K(m, n)$ is explicitly presented as K-algebra with $2mn$ generators x_{ij}, x^*_{ij} where $1 \leq i \leq m$, $1 \leq j \leq n$, and relations

$$\sum_{j=1}^{n} x_{lj}x^*_{jk} = \delta_{lk} \quad \text{and} \quad \sum_{j=1}^{m} x^*_{lj}x_{jk} = \delta_{lk}.$$

Leavitt also proved that $L_K(m, n)$ is simple if $m = 1$ and that $L_K(m, n)$ is domain for all $m > 1$.

$L_K(m, n)$ can be equivalently viewed as the the quotient of associative free K-algebra generated by x_{ij}, x^*_{ij} where $1 \leq i \leq m$, $1 \leq j \leq n$ modulo the matrix relations given by

$$AA^* = I_m \quad \text{and} \quad A^*A = I_n,$$

where $A \in M_{m\times n}(R)$, $(A)_{ij} = x_{ij}$ and $A^* \in M_{n\times m}(R)$, $(A^*)_{ij} = x^*_{ij}$.

1.1.1.2 $\mathcal{V}$-monoid and Bergman's universal ring constructions

Let R be a unital ring; a left (or right) R-module P is called **projective module** over R if it is isomorphic to a direct summand of a free R-module. If P is a finitely generated projective module, generated by n elements, then we have $P \oplus P' \cong R^n$ for some (projective) module P'. The projection of R^n on P is given by an idempotent $n \times n$ matrix E and we may write $P = R^n E$. We note the conditions for two idempotent matrices to define isomorphic projective modules.

Proposition 1.1.4. *[29, Proposition 0.3.1] Let R be a unital ring and let $E \in M_{n\times n}(R)$, $F \in M_{m\times m}(R)$ be idempotent matrices. Then $E = XY$, $F = YX$ for some $X \in M_{n\times m}(R)$, $Y \in M_{m\times n}(R)$ if and only if the projective left (or right) R-modules defined by E and F are isomorphic: $R^n E \cong R^m F$.*

Definition 1.1.5. For a unital ring R, let $\mathcal{V}(R)$ denote the set of isomorphism classes of finitely generated projective left R-modules, and define a binary operation $\oplus$ on $\mathcal{V}(R)$ by setting $[P] \oplus [Q] = [P \oplus Q]$. Then $(\mathcal{V}(R), \oplus)$ is an abelian monoid with zero element $[0]$.

The monoid $\mathcal{V}(R)$ can equivalently defined in terms of idempotent matrices. For any unital ring R, a finitely generated projective left R-module P is generated by the rows of an $n \times n$ idempotent matrix E. Let the idempotent matrices $E \in M_{n\times n}(R)$ and $F \in M_{m\times m}(R)$ correspond to P and Q respectively. In view of Proposition 1.1.4, define E and F are isomorphic if there exists matrices $X \in M_{n\times m}(R), Y \in M_{m\times n}(R)$ such that $XY = E$ and $YX = F$. Define the diagonal sum $E \oplus F$ to be the matrix $\begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix}$. Then $E \oplus F$ corresponds to $P \oplus Q$. Hence $\mathcal{V}(R)$ may be defined as the set of isomorphism classes of idempotent matrices with the operation $\oplus$. This also helps us to extend the definition of $\mathcal{V}$-monoid to a non-unital ring as follows.

Definition 1.1.6. Let R be a ring, and let $M_\infty(R)$ denote the set of all $\omega \times \omega$ matrices over R with finitely many nonzero entries, where ω varies over $\mathbb{N}$. For idempotents $e, f \in M_\infty(R)$, the **Murray-von Neumann equivalence** $\sim$ is defined as follows: $e \sim f$ if and only if there exists $x, y \in M_\infty(R)$ such that $xy = e$ and $yx = f$.

Let $\mathcal{V}(R)$ denote the set of all equivalence classes $[e]$, for idempotents $e \in M_\infty(R)$. Define $[e] + [f] = [e \oplus f]$, where $e \oplus f$ denotes the block diagonal matrix $\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$.

Under this operation, $\mathcal{V}(R)$ is an abelian monoid, and it is conical, that is, $a + b = 0$ in $\mathcal{V}(R)$ implies $a = b = 0$. An abelian monoid M is said to have a **distinguished**

element d if for any $x \in M$, there exists $y \in M$ such that $x + y = nd$ for some $n \in \mathbb{N}$. Moreover, if R is unital then $[R] \in \mathcal{V}(R)$, which is also a distinguished element. Also $\mathcal{V}(\_)$:**Rings**$\to$**AbMon** is a continuous functor. That is, it maps direct limits to direct limits. Thus if unital rings R and S are isomorphic then there exists a monoid isomorphism $\mathcal{V}(R) \to \mathcal{V}(S)$ for which $[R] \mapsto [S]$. We denote such an isomorphism by $(\mathcal{V}(R), [R]) \cong (\mathcal{V}(S), [S])$.

Theorem 1.1.7. *[67, Theorem 1.1.3] Let S be a commutative semigroup (not necessarily having zero). There is a unique abelian group $G(S)$, called the **Grothendieck group of** S, together with a semigroup homomorphism $\varphi : S \to G(S)$, such that for any group G and homomorphism $\psi : S \to G$, there is a unique group homomorphism $\theta : G(S) \to G$ with $\psi = \theta \circ \varphi$.*

Definition 1.1.8. For a unital ring R, the Grothendieck group of $\mathcal{V}(R)$ is called the **Grothendieck group of** R and denoted by $K_0(R)$. In other words $K_0(R) := G(\mathcal{V}(R))$.

We note that $K_0(\_)$:**Rings**$\to$**AbGroups** is a continuous functor. For, if $\varphi : R \to S$ is homomorphism, then it induces a monoid homomorphism $\mathcal{V}(R) \to \mathcal{V}(S)$ defined by $[P] \mapsto [S \otimes_\varphi P]$. To see that this map is well defined let $P \oplus Q \cong R^n$, then

$$(S \otimes_\varphi P) \oplus (S \otimes_\varphi Q) \cong S \otimes_\varphi (P \oplus Q) \cong S \otimes_\varphi R^n = S^n.$$

Since tensor product commutes with direct sums, the map is well defined. Now from the definition of K_0, it can be verified that the map $K_0(\varphi) : K_0(R) \to K_0(S)$ is a group homomorphism which satisfies the usual functorial conditions.

Definition 1.1.9. Let R, S be a unital rings. If the module categories R-***Mod*** and S-***Mod*** are equivalent, then R and S are said to be **Morita equivalent**.

For example, any unital ring R is Morita equivalent to the matrix ring $M_{n\times n}(R)$, for any $n \in \mathbb{N}$. If R, S are Morita equivalent, then there exists an isomorphism $\varphi : \mathcal{V}(R) \to \mathcal{V}(S)$. However, $\varphi([R])$ need not be equal to $[S]$.

In [28], Cohn studied properties that are successively stronger than IBN which are given here. Let R be a unital ring. In the following discussion by an R-module we mean an one sided R-module.

I. Every free R-module has a unique rank.

II. A free R-module of rank n cannot be generated by less than n elements.

III. In a free R-module of any rank n, any generating set of n elements is free.

Reformulating we get the following equivalent properties.

I′. For every $m, n \in \mathbb{N}$, $R^m \cong R^n$ implies $m = n$.

II′. For every $m, n \in \mathbb{N}$, $R^m \cong R^n \oplus P$ implies $m \geq n$.

III′. For every $n \in \mathbb{N}$, $R^n \cong R^n \oplus P$ implies $P = 0$.

Rewriting again in terms of matrices we get

IBN: For any $A \in M_{m\times n}(R), B \in M_{n\times m}(R)$, if $AB = I_m$ and $BA = I_n$ then $m = n$.

UGN: For any $A \in M_{m\times n}(R), B \in M_{n\times m}(R)$, if $AB = I_m$ then $n \geq m$.

WF: For any $A, B \in M_{n\times n}(R)$, if $AB = I_n$, then $BA = I_n$.

Clearly

$$\text{WF} \Rightarrow \text{UGN} \Rightarrow \text{IBN}.$$

Definition 1.1.10. A unital ring R is said to have **Unbounded Generating Number** (*UGN*) if the above condition *UGN* is satisfied. R is called **weakly finite** if the condition *WF* is satisfied. We say R is **weakly n-finite** if condition *WF* holds only for n. Thus R is weakly finite if and only if it is weakly n-finite for every $n \in \mathbb{N}$. A weakly 1-finite ring is called **directly finite**.

Cohn constructed and studied 'canonical' examples of non-zero unital rings that do not have IGN, UGN and WF properties. For a field K and $m, n \in \mathbb{N}$ with $m \leq n$ define $V_K(m, n)$ as the K-algebra generated by the $2mn$ symbols a_{ij}, a^*_{ij} with defining relations

$$\sum_{k=1}^{n} a_{ik}a^*_{kj} = \delta_{ij} \quad (1 \leq i, j \leq m), \tag{1.1.1}$$

$$\sum_{k=1}^{n} a^*_{ik}a_{kj} = \delta_{ij} \quad (1 \leq i, j \leq n). \tag{1.1.2}$$

In matrix notation the defining relations of $V_K(m,n)$ are

$$AA^* = I_m \quad \text{and} \quad A^*A = I_n$$

where $A \in M_{m\times n}(R)$ with $A_{ij} = a_{ij}$ and $A^* \in M_{n\times n}(R)$ with $A^*_{ij} = a^*_{ij}$.

Define the **Cohn K-algebra** of type (m,n), $C_K(m,n)$, to be the K-algebra on the same generators but with only defining relations 1.1.1. Clearly, $V_K(m,n)$ is a quotient of $C_K(m,n)$. For $m < n$, $V_K(m,n) = L_K(m,n)$, Thus Cohn's constructions extend Leavitt algebras.

It will be useful to consider the negations of IBN, UGN and WF:

$\alpha_{m,n}$: There exists $A \in M_{m\times n}(R)$ and $B \in M_{n\times m}(R)$ such that $AB = I_m$ and $BA = I_n$.

$\beta_{m,n}$: There exists $A \in M_{m\times n}(R)$ and $B \in M_{n\times m}(R)$ such that $AB = I_m$.

γ_n: There exists $A, B \in M_{n\times n}(R)$ such that $AB = I_n$ and $BA \neq I_n$.

$V_K(m,n)$ is universal for the K-algebra satisfying $\alpha_{m,n}$ and $C_K(m,n)$ is universal for K-algebras satisfying $\beta_{m,n}$. Cohn described the normal forms (see Section 3.4) for $V_K(m,n)$ and $C_K(m,n)$ and using normal forms he showed that $V_K(m,n)$ and $C_K(m,n)$ are domains for $m > 1$.

Cohn proved that for $m < n$, $C_K(m,n)$ satisfies $\beta_{n,m}$ but not $\beta_{k,h}$ with $h < m, k < n$ and therefore $C_K(m,n)$ does not have UGN but has IBN. He also proved that for $C_K(n,n)$ satisfies γ_n but not γ_m for $m < n$. Thus $C_K(n,n)$ is not WF, but satisfies UGN.

Later in [68], Skornyakov studied K-algebras $W_K(n)$ with a universal idempotent $n \times n$ matrix. In other words $W_K(n)$ is presented by n^2 generators x_{ij} and relations obtained from the matrix relation $A^2 = A$, where $(A_{ij}) = x_{ij}$.

In [24], Bergman generalized Cohn's constructions by considering the rings obtained from a given K-algebra R by 'adjoining to R' universal homomorphisms, isomorphisms, left-invertible maps, and idempotent endomorphisms between finitely generated projective R-modules.

Given any K-algebra R, a K-algebra S is called an R-ring$_K$ if there is a K-algebra homomorphism $R \to S$. If S is an R-ring$_K$ and M an R-module, then the S-module $M \otimes_R S$ is denoted by $\overline{M}$. If $f : M \to N$ is a R-module homomorphism then $f \otimes S$ is denoted by $\overline{f} : \overline{M} \to \overline{N}$. Bergman described the following universal module map constructions:

Theorem 1.1.11. *[24, Theorem 3.1 - 3.2]*

1. ***Adjoining maps**: Let R be a K-algebra, M be any R-module, and P be a projective R-module. Then there exists an R-ring$_K$ S, having a universal module homomorphism $f : M \otimes S \to P \otimes S$. We denote S by $R\langle f : \overline{M} \to \overline{P}\rangle$.*

2. ***Imposing relations**: Let R be a K-algebra, M be any R-module, and P be a projective R-module. If $f : M \to P$ is any module homomorphism then there exists an R-ring$_K$ S such that $f \otimes S = 0$, and universal with that property. We denote S by $R\langle \overline{f} = 0\rangle$.*

In the theorem, universal means the following: Given any R-ring$_K$ T with the property given in the theorem there exists a unique homomorphism of R-rings, $S \to T$. Moreover, in the construction of adjoining maps, S can be obtained by adjoining to R a family of generators subject to certain relations, and in the construction of imposing relations, S is obtained as a quotient of R. More generally, given a family of such pairs M_i, P_i ($i \in I$ an indexing set), there exists an R-ring$_K$ S having a universal family of homomorphisms $f_i : M_i \otimes S \to P_i \otimes S$ with the same universal property and given a family of maps $f_i : M_i \to P_i$, there exists an R-ring$_K$ S, universal for the property that $f_i \otimes S = 0$ for all $i \in I$.

Using these constructions Bergman then described more complicated constructions. Let R be a K-algebra and P, Q be a two non-zero finitely generated projective R-modules. We can also adjoin a universal isomorphism between $\overline{P}$ and $\overline{Q}$ by first adjoining maps $i : \overline{P} \to \overline{Q}$ and $i^{-1} : \overline{Q} \to \overline{P}$ and then imposing the relations $ii^{-1} = \mathrm{id}_{\overline{Q}}$ and $i^{-1}i = \mathrm{id}_{\overline{P}}$. We denote such a ring by $R\langle i, i^{-1} : \overline{P} \cong \overline{Q}\rangle$. We can adjoin a map $\overline{P} \to \overline{Q}$ and a one-sided inverse only getting a ring $R\langle i : \overline{P} \to \overline{Q}, j : \overline{Q} \to \overline{P}; ji = \mathrm{id}_{\overline{P}}\rangle$. We can adjoin a universal idempotent $e : \overline{P} \to \overline{P}$ by first adjoining a map $e : \overline{P} \to \overline{P}$ and then imposing the relation $e^2 = e$ to obtain $R\langle e : \overline{P} \to \overline{P};\ e^2 = e\rangle$.

Homological algebra classifies rings (resp. algebras) according to their *global dimension*, i.e. the length of projective resolutions of modules. The case of zero dimension (semisimple rings) is fairly well known. The ring has global dimension 1 precisely when all submodules of projective modules are projective but the ring is not semisimple. It is well known that this holds for left modules if all left ideals are projective. A ring R is called **left hereditary** (resp. **left semi-hereditary** if every left ideal (resp. finitely generated left ideal) of R is projective. Corresponding definitions apply for the right ideals as well.

Now it is direct that

$$\begin{aligned} V_K(m,n) &= K\langle i, i^{-1} : \overline{K^n} \cong \overline{K^m}\rangle, \\ C_K(m,n) &= K\langle i : \overline{K^n} \to \overline{K^m}, j : \overline{K^m} \to \overline{K^n}; ji = \mathrm{id}_{\overline{K^n}}\rangle. \\ W_K(n) &= K\langle e : \overline{K^n} \to \overline{K^n}; e^2 = e\rangle. \end{aligned}$$

Theorem 1.1.12. *[24, Theorem 5.1 - 5.4] Let R be a K-algebra, P and Q be a non-zero finitely generated projective R-modules.*

1. *Let $S = R\langle e : \overline{P} \to \overline{P};\ e^2 = e\rangle$. Then $\mathcal{V}(S)$ is obtained from $\mathcal{V}(R)$ by adjoining two new generators $[P_1]$ and $[P_2]$ and one relation $[P_1] + [P_2] = [P]$.*
2. *Let $S = R\langle i, i^{-1} : \overline{P} \cong \overline{Q}\rangle$. Then $\mathcal{V}(S)$ is obtained from $\mathcal{V}(R)$ by imposing one relation $[P] = [Q]$.*
3. *Let $S = R\langle f : \overline{P} \to \overline{Q}\rangle$. Then $\mathcal{V}(S) \cong \mathcal{V}(R)$ and under the map $[M] \mapsto [\overline{M}]$.*
4. *Let $S = R\langle i : \overline{P} \to \overline{Q}, j : \overline{Q} \to \overline{P}; ji = 1_{\overline{P}}\rangle$. Then $\mathcal{V}(S)$ is obtained from $\mathcal{V}(R)$ by adjoining a generator $[Q']$ and one relation, $[P] + [Q'] = [Q]$.*

Corollary 1.1.13. *[24, Theorem 6.1] Let K be a field and $m, n \in \mathbb{N}$. Then we have the following presentations*

$$\mathcal{V}(C_K(m,n)) = \langle I, J \mid mI = nI + J\rangle$$

$$\mathcal{V}(V_K(m,n)) = \langle I \mid mI = nI\rangle$$

$$\mathcal{V}(W_K(n)) = \langle I, P, Q \mid nI = P + Q\rangle$$

where $I \in \mathcal{V}(R)$ denotes the isomorphic class $[R]$.

Bergman established the following remarkable theorem.

Theorem 1.1.14. *[24, Theorem 6.2] Let M be a finitely generated abelian conical monoid with distinguished element $d \neq 0$, and let K be any field. Then there exists a hereditary K-algebra $B = B_K(M, d)$ such that $(\mathcal{V}(B), [B]) \cong (M, d)$. Moreover, B has the weak universal property that for any K-algebra S and any homomorphism $\varphi : M \to \mathcal{V}(S)$ such that $d \mapsto [S]$, there is a (generally nonunique) K-algebra homomorphism $\Phi : B \to S$ such that φ is equal to the induced map $\overline{\Phi} : S \otimes_B \_ : \mathcal{V}(B) \cong M \to \mathcal{V}(S)$.*

The construction of $B_K(M, d)$ depends on the specific presentation of M as $\mathcal{F}/\langle\mathcal{R}\rangle$, where $\mathcal{F}$ is a finitely generated free abelian monoid, and $\mathcal{R}$ is a finite set of relations in $\mathcal{F}$. Given $\mathcal{F}$ and $\mathcal{R}$, $B(\mathcal{F}/\langle\mathcal{R}\rangle, d)$ is constructed explicitly in a finite sequence of steps consisting of adjoining maps and relations. We refer to $B = B_K(M, d) = B_K(\mathcal{F}/\langle\mathcal{R}\rangle, d)$ as the **Bergman algebra** of $(\mathcal{F}/\langle\mathcal{R}\rangle, d)$. We note that $\mathcal{R}$ could be a multi-set (see Example 1.1.15). We often refer to the process of obtaining the Bergman algebra from conical monoids (and vice versa) as **Bergman machinery**.

Example 1.1.15.

1. If $(M, d) = (\mathbb{Z}^+/\langle\emptyset\rangle, 1)$, then $B_K(M, d) = K$.
2. If $(M, d) = (\mathbb{Z}^+/\langle 1 = 1\rangle, 1)$, then $B_K(M, d) = K[X, X^{-1}]$.
3. If $(M, d) = (\mathbb{Z}^+/\langle 1 = 1,\ 1 = 1\rangle, 1)$, then $B_K(M, d)$ is the free product $K[X, X^{-1}] * K[Y, Y^{-1}]$.
4. Let $m, n \in \mathbb{N}$. If $(M, d) = (\mathbb{Z}^+/\langle m = n\rangle, 1)$, then $B_K(M, d) = V_K(m, n)$.

Bergman's theorem thus solves the *realization problem for hereditary algebras*(see below) in positive when the $\mathcal{V}$-monoid is finitely generated.

Problem 1.1.16 (Realization problem for hereditary rings). Is every conical abelian monoid with a order-unit realizable by a unital hereditary ring?

Later in [26], Bergman and Dicks completely solved the problem in positive.

1.1.1.3 The realization problem for von Neumann regular rings and graph monoids

Now, we focus our attention on von Neumann regular rings. A good reference for von Neumann regular rings is [35].

Definition 1.1.17. A unital ring R is said to be **von Neumann regular** if for any $x \in R$ there exists $y \in R$ such that $xyx = x$.

Theorem 1.1.18. *The following are equivalent for a von Neumann regular ring.*

1. *R is von Neumann regular.*

2. *Every principal left ideal is generated by an idempotent element.*

3. *Every principal left ideal is a direct summand of the left R-module R.*

4. *Every finitely generated submodule of a projective left R-module P is a direct summand of P.*

5. *R being absolutely flat. That is, every left R-module is flat.*

A monoid M is said to be a **refinement monoid** in case any equality $x_1+x_2 = y_1+y_2$ admits a refinement, that is, there are $z_{ij}, 1 \leq i,j \leq 2$ such that $x_i = z_{i1} + z_{i2}$ and $y_j = z_{1j} + z_{2j}$ for all i, j. If R is a von Neumann regular ring, then the monoid $\mathcal{V}(R)$ is a refinement monoid by [35, Theorem 2.8]. In [36], Goodearl posed the following problem:

Fundemental open problem: Which monoids arise as $\mathcal{V}(R)$'s for a von Neumann regular ring R?

It was shown by Wehrung in [71] that there exist conical refinement monoids of size $\aleph_2$ which cannot be realized as von Neumann regular rings. Thus we have the following problem.

Problem 1.1.19 (Realization problem for von Neumann regular rings). Is every countable conical abelian refinement monoid realizable by a von Neumann regular ring?

Now, we discuss about a natural monoid associated to a graph. For terminology used here, the reader is refered to section 1.2.

Let $E = (E^0, E^1, s, r)$ be a row-finite graph (i.e. each vertex emits only finitely many edges) and A_E be the adjacency matrix of E. Then the **graph monoid** M_E of E is the abelian monoid presented by generating set E^0 and the following relations.

$$\text{for every non-sink } v \in E^0,\ v = \sum_{w \in E^0} A_E(v,w)w$$

where $A_E(v,w) = |\{e \in E^1 \mid s(e) = v,\ r(e) = w\}|$.

It was shown in [18] that for every finite graph E, M_E is conical refinement monoid with the order-unit $\sum\limits_{v \in E^0} v$. Hence the authors considered addressing the realization problem 1.1.19 for graph monoids. Using Bergman's machinery they obtained a K-algebra $L_K(E)$, called Leavitt path algebra of the graph E, for any finite graph E such that $\mathcal{V}(L_K(E)) \cong M_E$. It was understood immediately that Leavitt algebras of type $(1,m)$ for any $m \in \mathbb{N}$ is an example of Leavitt path algebra for a class of graphs R_m called *rose with m petals*.

In [14], it was shown that $L_K(E)$ is von Neuman regular if and only if the underlying graph is acylic. However, in [18] for any row-finite graph E, a von Neuman regular K-algebra $Q_K(E)$ is constructed by *universal localization* of $L_K(E)$ such that $L_K(E)$ can be embedded in $Q_K(E)$ and $\mathcal{V}(L_K(E)) \cong \mathcal{V}(Q_K(E))$. Thus an attempt to solve the realization problem in the case of finitely generated refinement monoid has been made in the positive direction.

This raised the question whether every finitely generated conical refinement monoids can be represented as graph monoids. The answer to this question is negative as it was shown in [21] that even the basic example such as $M = \langle p, a, b \mid p = a + p = b + p\rangle$ does not occur as a graph monoid. To rectify this issue, in [20], Ara and Goodearl defined graphs with additional structure, called *separated graphs*, so that any conical monoid can be represented as (separated) graph moniods. Thus Leavitt path algebras of separated graphs were the promising candidates towards the positive solution of the realization problem. Recently, in [19], it was announced that the realization problem is solved in the case of finitely generated refinement monoids in positive by considering Leavitt path algebras of a class of separated graph called *adaptable separated graphs*.

1.1.2 The second historical thread: Graph C^*-algebras

For undefined terminology used here the reader may consult any graduate level books on C^*-algebras such as [33], [54] or [70].

In [30], for any natural number $n > 1$, Cuntz constructed the C^*-algebras $\mathcal{O}_n$. His original motivation was to study 'simple separable C^*-algebras with unit infinites'. Later in [31], he also computed K-theory of these algebras. Now these algebras are called Cuntz algebras. From the definition of Cuntz algebras one can easily realize that $\mathcal{O}_n$ are, in fact, C^*-algebra analogs of Leavitt algebras $L_{\mathbb{C}}(1,n)$ of type $(1,n)$.

In [27], Brown introduced two classes of C^*-algebras with an intention to generalize Atiyah's proof of the Künneth theorem of K-theory to non-commutative C^*-algebras. First of these is the *non-commutative Grassmanian* G_n^{nc} where n is any natural number. G_n^{nc} is presented by an identity and elements P_{ij}, $i,j=1,\dots,n$, subject to the relations that makes the $n\times n$ matrix $[P_{ij}]$ a projection. In other words $P_{ij}^*=Pji$ and $P_{ij}=\sum_k P_{ik}P_{kj}$. It can be seen that G_n^{nc} are actually C^*-algebra analogs of the Bergman algebra $W_{\mathbb{C}}(n)$. The second class of examples are as follows: For any $n\in\mathbb{N}$, let U_n^{nc} be the C^*-algebra generated by an identity and elements U_{ij}, $i,j=1,\dots,n$ subject to the relations that make the matrix $[U_{ij}]$ unitary. In other words, U_n^{nc} are C^*-algebra analogs of Bergman algebras $V_{\mathbb{C}}(n,n)$. Later McClanahan studied K-theory of G_n^{nc} and U_n^{nc} in [51]. In [52], McClanahan introduced the notion of *rectangular unitary C^*-algebras $U_{m,n}^{nc}$* and studied their Ext and K-theory. For any natural numbers $m<n$ it can be seen that $U_{m,n}^{nc}$ are C^*-algebra analogs of Leavitt algebras $L_{\mathbb{C}}(m,n)$ of type (m,n).

In the early 1980s Cuntz and Krieger considered a class of C^*-algebras that arose in the study of topological Markov chains ([32]). These Cuntz-Krieger algebras $\mathcal{O}_A$ are generated by partial isometries whose relations are determined by a finite matrix A with entries $\{0,1\}$. In order for these C^*-algebras to be unique, the author further assumed a non-degeneracy condition called Condition (I) on A. Since their introduction, Cuntz-Krieger algebras are generalized in various ways and considered in the study of classification of C^*-algebras.

In [69], Watatani noted that $\mathcal{O}_A$ can be viewed as the C^*-algebra associated to a finite directed graph with adjacency matrix A and the condition (I) corresponds to the property that the graph has no sinks or sources. In the late 1990s, the generalization of these C^*-algebras were considered for possibly infinite graphs that were allowed to contain sources and sinks. Originally, a definition was given only for graphs that are row-finite ([45, 44, 23]), and later generalized to arbitrary graphs ([34]).

Definition 1.1.20 (Graph C^*-algebras)**.** For a graph E, the graph C^*-algebra $C^*(E)$ is the universal C^*-algebra generated by mutually orthogonal projections $\{p_v \mid v\in E^0\}$ and partial isometries with mutually orthogonal ranges $\{s_e \mid e\in E^1\}$ satisfying

1. $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$,
2. $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$,
3. $p_v = \sum_{e \in s^{-1}(v)} s_e s_e^*$ for all $v \in E^0$ and $0 < |s^{-1}(v)| < \infty$.

The interested reader is refered to [64] or [58] to learn more about graph C^*-algebras.

Though it is direct to see that the Cuntz algebras $\mathcal{O}_n$ are C^*-algebra analogs of Leavitt algebras $L_{\mathbb{C}}(1, n)$, this realization came only in early 2000s during an NSF-CBMS conference on "Graph algebras: Operator Algebras We Can See" when a group of ring theorists attended the conference (cf. [5, p. 68-69]). This led to the consideration of study of algebraic analogs of graph C^*-algebras, which are now termed as 'Leavitt path algebras'. In [7], Abrams and Pino defined Leavitt path algebras of row-finite graphs and characterized simplicity of these algebras in terms of underlying graphs. Later in [2], the definition of Leavitt path algebras was extended to arbitrary graphs. Initially the focus was to study if the dictionary between graph properties which translate to C^*-algebra properties also translate to purely algebraic properties. Though many of such properties exist, not all properties directly translate. However, then the focus turned into algebraic study of Leavitt path algebras such as multiplicative ideal theory(cf. [66]), module theory (cf. [65]), chain conditions ([10]), finiteness conditions ([3, 4]), representation theory in terms of underlying quiver represenatations ([41, 40]), etc.

1.2 Graph theory preliminaries

A **graph** E is a 4-tuple (E^0, E^1, s, r) where E^0, E^1 are sets and $r, s : E^1 \to E^0$ are functions. The elements of E^0 are called **vertices** of E and the elements of E^1 are called **edges** of E. We place no restriction on the cardinalities of E^0 and E^1. For each edge e, $s(e)$ is called the **source** of e and $r(e)$ is called the **range** of e; if $s(e) = v$ and $r(e) = w$ we also say that v **emits** e and that w **receives** e or that e is an edge from v to w. We represent this visually as follows:

$$s(e) \bullet \xrightarrow{\quad e \quad} \bullet \; r(e)$$

A graph is also called 'oriented multi-graph' in graph theory, a 'diagram' in category theory, and a 'quiver' in representation theory. If there is more than one graph, we write s_E and r_E to emphasize that they are the range and source maps of E.

We say a graph E is called **finite** (or **countable**) if both E^0 and E^1 are finite (or countable). E is **simple** if both s and r are injective. E is called **row-finite** (or **column-finite**) if the set $s^{-1}(v)$ is finite for every $v \in E^0$ (respectively if $r^{-1}(v)$ is finite for every $v \in E^0$). An edge e for which $s(e) = r(e) = v$ is called a **loop based at** v. A vertex which does not receive any edges is called a **source** (not to be confused with source map). A vertex which emits no edges is called a **sink**. A graph E is called **sink-free** (resp. **source-free**) if it has no sinks (resp. no sources).

We set

$$E^1(v,w) := \{e \in E^1 \mid s(e) = v, r(e) = w\}.$$

Hence for $v, w \in E^0$, we have $s^{-1}(v) = \bigsqcup_{w \in E^0} E^1(v,w)$ and $r^{-1}(v) = \bigsqcup_{w \in E^0} E^1(v,w)$. A vertex $v \in E^0$ is called **row-regular** (resp. **column regular**) if $0 < |s^{-1}(v)| < \infty$ (resp. if $0 < |r^{-1}(v)| < \infty$). The set of all row-regular (resp. column-regular) vertices of E is denoted by $\mathrm{RReg}(E)$ (resp. $\mathrm{CReg}(E)$).

The **adjacency matrix** A_E of the graph E is the $|E^0| \times |E^0|$ matrix defined by

$$A_E(v,w) = |E^1(v,w)|.$$

Thus the graph is row-finite if and only if each row sum of A_E is finite.

A **subgraph** $F = (F^0, F^1, r_F, s_F)$ of $E = (E^0, E^1, r_E, s_E)$ is a graph such that $F^0 \subseteq E^0$, $F^1 \subseteq E^1$, r_F is the restriction of r_E on F^1 and s_F is the restriction of s_E on F^1. Let V be a subset of E^0. The **induced subgraph** on V is the subgraph $E_V = (V, E^1_V, r_V, s_V)$ such that $E^1_V := s^{-1}(V) \cap r^{-1}(V)$, r_V and s_V are restrictions of r_E and s_E on E^1_V respectively. A subgraph is **full** if it is induced on its set of vertices.

A **graph morphism** $\phi : F = (F^0, F^1, r_F, s_F) \to E = (E^0, E^1, r_E, s_E)$ is a pair of maps $\phi^0 : F^0 \to E^0$ and $\phi^1 : F^1 \to E^1$ such that $r_E(\phi^1(e)) = \phi^0(r_F(e))$ and $s_E(\phi^1(e)) = \phi^0(s_F(e))$, for every $e \in F^1$. That is the following diagrams commute.

$$\begin{tikzcd} F^1 \arrow[r, "\phi^1"] \arrow[d, "r_F"'] & E^1 \arrow[d, "r_E"] \\ F^0 \arrow[r, "\phi^0"'] & E^0 \end{tikzcd} \qquad \begin{tikzcd} F^1 \arrow[r, "\phi^1"] \arrow[d, "s_F"'] & E^1 \arrow[d, "s_E"] \\ F^0 \arrow[r, "\phi^0"'] & E^0 \end{tikzcd}$$

We denote the category of graphs along with graph morphisms by **Gra**. Given a family of graphs $\{E_i\}_{i\in I}$ in **Gra**, we define their disjoint union $\bigsqcup_{i\in I} E_i$ to be the graph whose vertex set is $\bigsqcup_{i\in I} E_i^0$, edge set is $\bigsqcup_{i\in I} E_i^1$, and the source and range maps are trivial extensions of s_i and r_i respectively for all $i \in I$.

A **path** μ in a graph E is either a vertex $v \in E^0$ or a finite sequence of edges $\mu = e_1e_2\dots e_n$ such that $r(e_i) = s(e_{i+1})$, for $i = 1,\dots,n-1$. The set of all paths in E is denoted by $E^\star$. We define the length function $l(\_) : E^\star \to \mathbb{Z}^+$ by

$$l(\mu) = \begin{cases} 0, & \text{if} \quad \mu = v \in E^0, \\ n, & \text{if} \quad \mu = e_1e_2\dots e_n. \end{cases}$$

We denote the set of all paths in E of length n by E^n, and hence $E^\star = \bigcup_{n\geq 0} E^n$. The source and range functions s, r can be extended to $E^\star$ as follows:

$$\text{for } v \in E^0,\ s(v) := v \text{ and } r(v) := v,$$

$$\text{if } \mu = e_1e_2\dots e_n,\ s(\mu) := s(e_1) \text{ and } r(\mu) := r(e_n).$$

For a path $\mu \in E^\star$ the set $\{s(e_1), r(e_1), r(e_2), \dots, r(e_n)\}$ is called the **support** of μ. Let $\mu = e_1e_2\dots e_n \in E^\star$ (that is $|\mu| \geq 1$). If $v = s(\mu) = r(\mu)$, then μ is called a **closed path based at** v. A closed path $\mu = e_1\dots e_n$ based at v such that $s(e_i) \neq s(e_j)$ for every $i \neq j$ is called a **cycle based at** v. E is called **acyclic** if it does not have any cycles based at any vertex of E. An edge $e \in E^1$ is called an **exit** of μ (resp. **entry** of μ) if there exists an i $(1 \leq i \leq n)$ such that $s(e) = s(e_i)$ and $e \neq e_i$ (resp. $r(e) = r(e_i)$

and $e \neq e_i$). For any $v, w \in E^0$ and $n \in \mathbb{N}$ we set

$$E^n(v,w) := \{\mu \in E^n \mid s(\mu) = v, r(\mu) = w\}$$

Note that if E is finite then $|E^n(v,w)| = A_E^n(v,w)$, where A_E is the adjacency matrix of E. Hence a finite graph E is acyclic if and only if A_E is nilpotent.

A graph E is **strongly connected** if for any $v, w \in E^0$, there exists $\mu \in E^\star$ such that $s(\mu) = v$ and $r(\mu) = w$.

The **(free) path category** $\mathcal{C}_E$ generated by a graph E is the small category with $\mathrm{Ob}(\mathcal{C}_E) = E^0$ and for $v, w \in E^0$, $\mathrm{Mor}(v,w) = E^\star(v,w) := \{\mu \in E^\star \mid s(\mu) = v, r(\mu) = w\}$. In other words, the elements of $\mathcal{C}_E$ are paths in E and the partial multiplication is defined by path concatenation.

We remark that the path category of a graph is a natural generalization of the free monoid of words of a set as follows: If X is a set, we denote the set of all words with letters in X along with an 'empty word' is denoted by $X^\star$. It is direct that $X^\star$ becomes a monoid with respect to multiplication defined by word concatenation (say on right). Then $(\_)^\star$: **Sets** $\to$ **Mon** is a functor such that $X \overset{i}{\hookrightarrow} X^\star$ and satisfies the following universal property: for any monoid M and any set map $f : X \to M$, there exists a unique monoid morphism $\overline{f} : X^\star \to M$ such that $\overline{f} \circ i = f$. Let **Cat** denote the category of small categories. Then $\mathcal{C}_{(\_)}$:**Gra**$\to$**Cat** is a functor such that E embeds into $\mathcal{C}_E$ as a subgraph and satisfies the following universal property: for any small category $\mathcal{D}$, the graph morphism $\phi : E \to \mathcal{D}$ factors through the embedding.

Let X be a set and $F(X)$ denote the free group generated by X. Let $\overline{X} := \{x^* \mid x \in X\}$ and $\overline{X^\star} := (\overline{X})^\star$. Then the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d, hook] & X^\star \arrow[d, hook] \\
\overline{X^\star} \arrow[r, hook] & F(X)
\end{tikzcd}$$

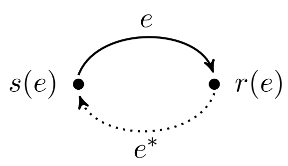
In fact, $F(X)$ has the following monoid presentation: the generating set is $X^\star \sqcup \overline{X^\star}$ and the relations are $xx^* = 1$ and $x^*x = 1$, where $1 = 1^*$ denotes the empty word over X.

For a graph $E = (E^0, E^1, r, s)$, the **double graph** $\widehat{E}$ of E is a new graph $(E^0, E^1 \sqcup \overline{E^1}, \widehat{s}, \widehat{s})$, where $\overline{E^1} := \{e^* \mid e \in E^1\}$ and

$$\widehat{s}(e) = s(e) \text{ if } e \in E^1 \text{ and } \widehat{s}(e^*) = r(e) \text{ if } e^* \in \overline{E^1}$$

$$\widehat{r}(e) = r(e) \text{ if } e \in E^1 \text{ and } \widehat{r}(e^*) = s(e) \text{ if } e^* \in \overline{E^1}$$

For visual representation of the double of a graph E, for each edge e in E we add a new dotted edge e^* in the reverse orientation.



A path μ in $\widehat{E}$ is called a *generalized path.* A graph E is **connected** if $\widehat{E}$ is strongly connected. That is, for any $v, w \in E^0$, there is a generalized path $\mu \in \widehat{E}^\star$ such that $s(\mu) = v$ and $r(\mu) = w$. The *connected components* of E are the graphs $\{E_j\}_{j\in J}$ such that $E = \bigsqcup\limits_{j\in J} E_j$, where every E_j is connected.

A natural generalization of free group on a set is the notion of **free groupoid on a graph**, which we define here: Given a graph E the free groupoid on E, denoted by $\mathcal{F}(E)$ is obtained by imposing the following relations on $C_{\widehat{E}}$:

1. $v^* = v$ for every $v \in E^0$.
2. $s(e)e = e = er(e)$ for every $e \in E^1$.
3. $r(e)e^* = e^* = e^*s(e)$ for every $e \in E^1$.
4. $e^*e = r(e)$ and $ee^* = s(e)$ for every $e \in E^1$.

Given a set X the free 0-monoid $X_0^\star$ is the set $X^\star \sqcup \{0\}$ such that multiplication is extended from that of word concatenation by defining $0 \cdot x = x \cdot 0 = 0$ for every $x \in X$. Similarly one can define the free 0-group on X. An important generalization of free 0-groups is the concept of the **polycyclic inverse monoids** introduced by Nivat and Perrot in [57].. The following definition is taken from [46]: For any $n \geq 2$ the polycyclic monoid P_n is a 0-monoid presented by

$$P_n = \langle x_1 \ldots, x_n, x_1^*, \ldots, x_n^* \mid x_i^* x_j = \delta_{ij} \rangle$$

This construction can be easily generalized to any set X as follows: For a set X, first consider the free 0-monoid on $X^\star \sqcup \overline{X^\star}$ and then impose the relations $x^* y = \delta_{xy}$. We denote the polycyclic monoid on X by P_X.

Recall that a semigroup $\mathcal{S}$ is said to be **von Neumann regular** if for every $x \in \mathcal{S}$ there exists $y \in \mathcal{S}$, called an inverse of x such that $xyx = x$. A regular semigroup $\mathcal{S}$ is called an **inverse semigroup** if every element has unique inverse. Note that polycyclic monoids are inverse semigroups.

A generalization of polycyclic monoids to the case of graph was considered by Ash and Hall in [22]. They defined a notion of *inverse semigroups associated to graphs.* For any small category $\mathcal{C}$ we associate a 0-semigroup $\mathcal{S}^0(\mathcal{C})$ as follows: As a set $\mathcal{S}^0(\mathcal{C})$ is $\mathcal{C}^\star \sqcup \{0\}$, where $\mathcal{C}^\star$ is the set of all morphisms in $\mathcal{C}$ and multiplication is defined by extension of partial multiplication in $\mathcal{C}$ as follows: $\mu\nu = 0$ if $\mathrm{cod}(\mu) \neq \mathrm{dom}(\nu)$. Given a graph E the **graph inverse semigroup** on E is presented by $\mathcal{S}^0(\mathcal{C}_{\widehat{E}})$ modulo the following relations

1. $s(\mu)\mu = \mu = \mu r(\mu)$ for any $\mu \in E^\star$.
2. $r(\mu)\mu^* = \mu^* = \mu^* r(\mu)$ for any $\mu \in E^\star$.
3. $\mu^*\nu = \delta_{\mu\nu} r(\mu)$.

1.3 Algebra preliminaries

For a field K a K**-algebra** is an associative (not necessarily unital) ring R given with a homomorphism of K into its center $Z(R)$. A K-category is a category in which every morphism set $\mathrm{Mor}(v, w)$ is given a structure of K-module, such that the composition maps $\mathrm{Mor}(v, w) \times \mathrm{Mor}(u, v) \to \mathrm{Mor}(u, w)$ are K-bilinear. A K-linear functor will mean a functor between K-categories that maps morphism sets by K-module homomorphisms.

An associative ring R is said to have a **set of local units** U if U is a set of idempotents in R having the property that, for each finite subset $r_1, \ldots, r_n$ of R, there exists a $u \in U$

for which $ur_iu = r_i$, for $1 \leq i \leq n$. Also, an associative ring R is said to have **enough idempotents** if there exists a set of nonzero orthogonal idempotents $\mathcal{I}$ in R for which the set F of finite sums of distinct elements of $\mathcal{I}$ is a set of local units for R. By **K-Alg**, we mean the category whose objects are K-algebras with enough idempotents and whose morphisms are K-algebra morphisms which map local units to local units. If R is a ring with enough idempotents, then we have

$$R = \bigoplus_{e \in I} eR = \bigoplus_{f \in I} Rf = \bigoplus_{e,f \in I} eRf$$

as additive groups.

Let E be a graph. The **Path K-algebra of** E, denoted by $K(E)$, is defined to be the quotient of the free associative K-algebra generated by $E^\star$ modulo the following relations:

$$vw = \delta_{vw}v, \text{ for all } v, w \in E^0, \tag{V}$$

$$s(\mu)\mu = \mu r(\mu) = \mu, \text{ for all } \mu \in E^\star. \tag{U}$$

In other words, $K(E)$ is obtained as the contracted K-algebra of the graph 0-semigroup $S^0(E)$. (That is, zero element of $K(E)$ is identified with 0 of $S^0(E)$.

Proposition 1.3.1. *For a graph E, the path K-algebra $K(E)$ has enough idempotents, where the set of nonzero orthogonal idempotents is E^0. Moreover, $K(E)$ is unital if and only if E^0 is finite, in which case $\sum_{v \in E^0} v$ is the unit.*

Proof. By V, $\sum\limits_{v \in V} v$, where $V \subseteq E^0$ is a finite subset, is an idempotent. Let A be a finite subset of $K(E)$. Then each element $a \in A$ is of the form $\sum\limits_{i=1}^{m} k_i^a \mu_i^a$, where $k_i^a \in K$ and $\mu_i^a \in E^\star$. Let the support of A, $\mathrm{supp}(A)$, be the union of supports of μ_i^a over A. Then $\mathrm{supp}(A)$ is finite and by U, $u_A = \sum\limits_{v \in \mathrm{supp}(A)} v$ satisfies $u_A a u_A = a$ for any $a \in A$. Thus E^0 is the set of enough idempotents of $K(E)$.

For E^0 finite, the sum of all vertices is finite and it is direct to check that $\sum\limits_{v \in E^0} v$ is the unit of $K(E)$. If E^0 is infinite, then since the vertices form the set of orthogonal idempotents, there is no element of $K(E)$ which acts as an identity on each vertex. $\square$

By **K-Alg** we mean the category whose objects are K-algebras with enough idempotents and whose morphisms are K-algebra morphisms which map local units to local units. We note that $K(\_)$ is *not* a functor from **Gra** to **K-Alg**. This is because a graph morphism $\phi : F \to E$ can map two distinct vertices $v, w \in F^0$ to a same vertex in E, in which case $K(\phi)(vw) \neq 0$ in $K(E)$, but $vw = 0$ in $K(F)$. However, let **Gr** denote the category whose objects are graphs and morphisms are graph morphisms $\phi = (\phi^0, \phi^1)$ such that ϕ^0 is injective. Then it is easy to verify that $K(\_)$ is a continuous functor from **Gr** to **K-Alg**. For any graph E, it is easy to see that $K(E) = \bigoplus_{j\in J} K(E_j)$, where $\{E_j\}_{j\in J}$ are connected components of E.

We introduce an important tool called 'Bergman's diamond lemma for rings' to compute K-linear basis of a K-algebra which is presented by generators and relations. Given a set W, let $\langle W\rangle$ denote the semigroup of all nonempty words over W (with juxtaposition) and $\overline{\langle W\rangle}$ denote $\langle W\rangle \cup \{\text{empty word}\}$. Further, let $K\langle W\rangle$ denote the **free K-algebra generated by W**.

Let Σ be a set of pairs of the form $\sigma = (w_\sigma, f_\sigma)$, where $w_\sigma \in \langle W\rangle$ and $f_\sigma \in K\langle W\rangle$. Then Σ is called a **reduction system** for $K\langle W\rangle$. For any $\sigma \in \Sigma$ and $A, B \in \overline{\langle W\rangle}$, let $r_{A\sigma B}$ denote the endomorphism of $K\langle W\rangle$ that maps $Aw_\sigma B$ to $Af_\sigma B$ and fixes all other elements of $\langle W\rangle$. The maps $r_{A\sigma B} : K\langle W\rangle \to K\langle W\rangle$ are called **reductions**.

We shall say a reduction $r_{A\sigma B}$ acts **trivially** on an element $a \in K\langle W\rangle$ if the coefficient of $Aw_\sigma B$ in a is zero, and we shall call a **irreducible (under Σ)** if every reduction is trivial on a. The K linear subspace of all irreducible elements of $K\langle W\rangle$ will be denoted by $K\langle W\rangle_{\text{irr}}$. A finite sequence of reductions $r_1, \dots, r_n$ will be said to be **final** on $a \in K\langle W\rangle$ if $r_n \dots r_1(a) \in K\langle W\rangle_{\text{irr}}$.

An element $a \in K\langle W\rangle$ will be called **reduction-finite** if for every infinite sequence $r_1, r_2, \dots$ of reductions, r_i acts trivially on $r_{i-1} \dots r_1(a)$, for all sufficiently large i. If a is reduction-finite, then any maximal sequence of reductions r_i, such that each r_i acts nontrivially on $r_{i-1} \dots r_1(a)$, will be finite, and hence a final sequence. It follows from their definition that the reduction-finite elements form a K linear subspace of $K\langle W\rangle$. We shall call an element $a \in K\langle W\rangle$ **reduction-unique** if it is reduction-finite, and if its images under all finite sequences of reductions are the same This common value will be denoted $r_\Sigma(a)$. The set of reduction-unique elements of $K\langle W\rangle$ forms a K linear subspace, and r_Σ is a bilinear map of this subspace into $K\langle W\rangle_{\text{irr}}$.

A 5-tuple $(\sigma, \Theta, A, B, C)$ with $\sigma, \Theta \in \Sigma$ and $A, B, C \in \langle W \rangle$, such that $w_\sigma = AB$ and $w_\Theta = BC$ is called an **overlap ambiguity** of Σ. We shall say the overlap ambiguity $(\sigma, \Theta, A, B, C)$ is **resolvable** if there exist compositions of reductions r and r', such that $r(f_\sigma C) = r'(A f_\Theta)$. Similarly, a 5-tuple $(\sigma, \Theta, A, B, C)$ with $\sigma \neq \Theta$ and $A, B, C \in \overline{\langle W \rangle}$ will be called an **inclusion ambiguity** if $w_\sigma = B, w_\Theta = ABC$ and such an ambiguity will be called **resolvable** if $A f_\sigma C$ and f_Θ can be reduced to a common expression.

By a **semigroup partial ordering** on $\langle W \rangle$, we shall mean a partial order $\leq$ such that

$$B < B' \Rightarrow ABC < AB'C,$$

for any $B, B' \in \langle W \rangle, A, C \in \overline{\langle W \rangle}$. We call $\leq$ **compatible** with Σ if for all $\sigma \in \Sigma$, f_σ is a linear combination of monomials $< w_\sigma$.

We now state the Bergman's diamond lemma which will be used to find a basis for Cohn-Leavitt path algebras of $\mathcal{A}$-graphs.

Theorem 1.3.2 (Bergman's diamond lemma). *[25, Theorem 1.2] Let $\leq$ be a semigroup partial ordering on $\langle W \rangle$ compatible with Σ and having descending chain condition. Then the following conditions are equivalent:*

1. *All ambiguities of Σ are resolvable.*
2. *All ambiguities of $K\langle W \rangle$ are reduction-unique under Σ.*
3. *$K\langle W \rangle_{\mathrm{irr}}$ is a set of representatives for the elements of the K-algebra $K\langle W \rangle / I$, where I is the ideal of $K\langle W \rangle$ generated by the elements $w_\sigma - f_\sigma (\sigma \in \Sigma)$.*

When these conditions hold, $K\langle W \rangle / I$ may be identified with the K-linear space $K\langle W \rangle_{\mathrm{irr}}$, made a K-algebra by the multiplication $a \cdot b = r_\Sigma(ab)$.

Proposition 1.3.3. *For a graph E, $E^\star$ forms a K-linear basis for $K(E)$.*

Proof. In order to apply Theorem 1.3.2, we replace the defining relations by the following:

$1'$: For any $v, w \in E^0$,

$$vw = \delta_{v,w} v.$$

$2'$: For any $v \in E^0, e \in E^1$,

$$ve = \delta_{v,s(e)} e,$$

$$ev = \delta_{v,r(e)} e,$$

$3'$: For any $e, f \in E^1$,

$$ef = 0 \text{ if } r(e) \neq s(f)$$

Denote by Σ the reduction system consisting of all pairs $\sigma = (w_\sigma, f_\sigma)$, where w_σ equals the LHS of an equation above and f_σ the corresponding RHS. Let $\langle \bar{P} \rangle$ be the monoid consisting of all words formed by letters in $E^0 \cup E^1$. Define a partial order $\leq$ on $\langle P \rangle$ by $A \leq B$ if $A = B$ or $l(A) < l(B)$. Then clearly $\leq$ is a semigroup partial order on $\langle P \rangle$ compatible with Σ and also the descending chain condition is satisfied. It remains to show that all ambiguities of Σ are resolvable.

In the following table we list all types of ambiguities which may occur:

Ambiguities			
	$1'$	$2'$	$3'$
$1'$	uvw	vwe	$-$
$2'$	evw	vew	vef
$3'$	$-$	efv	efg

We note that there are no inclusion ambiguities. We only show how to resolve ambiguity of type $2' - 3'$ leaving other similar cases to the reader.

$$\begin{tikzcd}
 & \left[\delta_{v,s(e)} e\right] f \arrow[dr, bend left, "3'"] & \\
vef \arrow[ur, bend left, "2'"] \arrow[dr, bend right, "3'"'] & & \delta_{v,s(e)}\delta_{s(e),r(f)} ef \\
 & \left[\delta_{r(e),s(f)} ef\right] \arrow[ur, bend right, "2'"'] &
\end{tikzcd}$$

This proves the confluence condition and hence the reduction finiteness as well. $\square$

A right R-module M over a K-algebra R is called **unital** if $MR = R$. That is for any $m \in M$ we can find $r_1, r_2, \ldots, r_n \in R$ and $m_1, m_2, \ldots, m_n \in M$ so that $m = m_1 r_1 + m_2 r_2 + \cdots + m_n r_n$. Note that this condition is equivalent to the standard definition of unital module (when R has a 1) since $m1 = (m_1r_1 + m_2r_2 + \cdots + m_nr_n)1 = m_1r_11 + m_2r_21 + \cdots + m_nr_n1 = m_1r_1 + m_2r_2 + \cdots + m_nr_n = m$. Let $\mathfrak{M}_R$ denote the category of unital right R-modules with R-module homomorphisms. Note that the category $\mathfrak{M}_R$ has a natural K-linear structure, hence so do the full subcategories $\mathfrak{M}_R^{fg}$ of finitely generated right unital R-modules, and $\mathfrak{M}_R^{proj}$ of finitely generated projective right unital R-modules.

Note that for a graph E the right unital module $vK(E)$ generated by $v \in E^0$ is projective and also that $\mathcal{V}(K(E))$ is free abelian monoid generated by E^0.

Let E be a graph. The category of unital right $K(E)$-modules is denoted by $\mathfrak{M}_E$ (since K is fixed). Note that $\mathfrak{M}_E$ is equivalent to the category of **quiver representations** of E, whose objects are the functors from the path category $\mathcal{C}_E$ of E to the category of K-vector spaces, and morphisms are natural transformations between two such functors. That is, a quiver representation ρ assigns a (possibly infinite dimensional) vector space $\rho(v)$ to each vertex and a linear transformation $\rho(\mu) : \rho(s(\mu)) \to \rho(r(\mu))$ to each path $\mu \in E^\star$. A morphism of quiver representations $\varphi : \rho \to \rho'$ is a family of linear transformations $\{\varphi_v : \rho(v) \to \rho'(v)\}_{v \in E^0}$ such that for each $\mu \in E^\star$ the following diagram commutes.

$$\begin{array}{ccc} \rho(s(\mu)) & \xrightarrow{\rho(\mu)} & \rho(r(\mu)) \\ {\scriptstyle \phi_{s(\mu)}}\big\downarrow & & \big\downarrow{\scriptstyle \phi_{r(\mu)}} \\ \rho'(s(\mu) & \xrightarrow{\rho'(\mu)} & \rho'(r(\mu)) \end{array}$$

For a unital right $K(E)$-module M, observe that $M = \bigoplus\limits_{v \in E^0} Mv$. The **support subgraph** of M is the full subgraph of E induced on $V_M := \{v \in E^0 \mid Mv \neq 0\}$.

A $*$**-ring** is an associative unital ring R with an anti-automorphism $* : R \to R$ that is also an involution. That is, $*$ satisfies the following properties: for every $x, y \in R$, $(x + y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, $(x^*)^* = x$ and $1^* = 1$. Let K be a $*$-field with involution $\overline{} : K \to K$. A $*$**-algebra** is a K-algebra R that is also a $*$-algebra such that $(kr)^* = \overline{k}r^*$ for every $k \in K$ and $r \in R$.

A K-algebra R is called a G**-graded algebra** if $R = \bigoplus\limits_{g \in G} R_g$, where G is a group and each R_g is a K-subspace of R and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. The set $R^h = \bigcup_{g \in G} R_g$ is called the set of **homogeneous elements** of R. R_g is called the g-component of R and the nonzero elements of R_g are called **homogeneous of degree** g. We write

$\deg(r) = g$ if $r \in A_g - \{0\}$. We call the set $G_R = \{g \in G \mid R_g \neq 0\}$, the support of R. We say the G-graded ring R has a trivial grading if $G_R = \{e\}$, where e is the identity of G. i.e, $R_e = A$. For G-graded algebras R and S, a G-graded algebra homomophism $f : R \to S$ is a K-algebra homomorphism such that $f(R_g) \subseteq S_g$ for all $g \in G$. A graded homomorphism f is called a graded isomorphism if f is bijective, in which case we write $R \cong_{gr} S$. It is easy to see that if f is a graded isomorphism, f^{-1} is also a graded homomorphism.

Let R be a G-graded ring. A **graded unital right R-module** M is defined to be a right R-module M with a direct sum decomposition $M = \bigoplus_{g \in G} M_g$, where each M_g, is an additive subgroup of M such that $M_g R_h \subseteq M_{gh}$ for all $g, h \in G$. For G-graded unital right R-modules M and N, a G-graded module homomorphism $f : M \to N$ is a module homomorphism such that $f(M_g) \subseteq N_g$ for all $g \in G$. A graded homomorphism f is called a **graded module isomorphism** if f is bijective and, when such a graded isomorphism exists, we write $M \cong_{\mathrm{gr}} N$.

Recall that for a graph E, $\widehat{E}$ denotes the double of E. Let K be a field with an involution. Then there is a $\mathbb{Z}$-grading on $K(\widehat{E})$ given by

$$\deg(\mu) := \begin{cases} l(\mu), & \text{if} \quad \mu \in E^\star, \\ -|l(\mu), & \text{if} \quad \mu \in \overline{E^\star}. \end{cases}$$

Note that the linear extension of $*$ induces a grade-reversing involutive anti-automorphism. That is, $\deg(\mu)^* = -\deg(\mu)$ and $(\mu\nu)^* = \nu^*\mu^*$. Hence $K(\widehat{E})$ is $\mathbb{Z}$-graded $*$-algebra. Also the (graded) categories of left unital $K(\widehat{E})$-modules and right unital $K(\widehat{E})$-modules are equivalent.

1.4 Leavitt path algebras

This section can be considered as a brief survey on Leavitt path algebras. Throughout this section we fix K to be a field.

Definition 1.4.1. Let E be a graph, $S \subseteq \mathrm{RReg}(E)$ and K be a field. The **Cohn-Leavitt path algebra** of E relative to S with coefficients from K, denoted by $C_K^S(E)$, is defined as the quotient of $K(\widehat{E})$ modulo the the following relations:

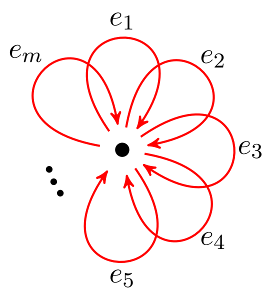
$$\forall\, e, f \in E^1,\ e^* f = \delta_{ef} r(e), \tag{CK 1}$$

$$\forall\, v \in S,\ v = \sum_{e \in s^{-1}(v)} ee^*. \tag{CK 2}$$

The algebra $C_K(E) := C_K^{\emptyset}(E)$ is called the **Cohn path algebra** of E and the algebra $L_K(E) := C_K^{\mathrm{RReg}(E)}(E)$ is called the **Leavitt path algebra** of E.

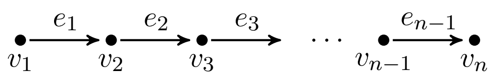
From definition it is clear that for a graph E, the Cohn path algebra $C_K(E)$ is the contracted K-algebra of the graph inverse semigroup of E. Also note that for an $S \subseteq \mathrm{RReg}(E)$, $C_K^S(E)$ is a quotient of $C_K(E)$. In particular, $L_K(E)$ is also a quotient of $C_K(E)$. However, it was shown in [6, Theorem 1.5.18] that for any graph E and $S \subseteq \mathrm{RReg}(E)$ there exists a graph $E(S)$ such that $C_K^S(E) \cong L_K(E(S))$ as K-algebras.

Example 1.4.2 (Leavitt algebras of type $(1, m)$)**.** For $m \in \mathbb{Z}^+$ let R_m denote the *rose with m petals* - a graph having one vertex and m loops:

$R_m \quad =$


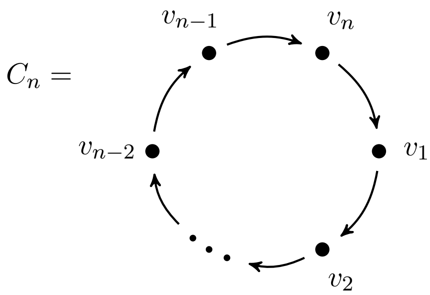
Then from the defining relations it is direct that $L_K(R_0) \cong K$, $L_K(R_1) \cong K[X, X^{-1}]$, and $L_K(R_m) \cong L_K(1, m)$ for $m \geq 2$. Also $C_K(R_m) \cong C_K(1, m)$ for any $m \in \mathbb{N}$.

Example 1.4.3 (Matrix algebras)**.** For $n \in \mathbb{N}$ let A_n denote the *oriented n-line graph* having n vertices and $n-1$ edges:

$A_n =$


Then it can be shown that $L_K(A_n) \cong M_n(K)$ (cf. [6, Proposition 1.3.4]).

Example 1.4.4 (Leavitt path algebra of a circle)**.** For $n \in \mathbb{N}$ let C_n denote the *n-circle graph* having n vertices and n edges:



Then from [4, Theorem 3.3], it follows that $L_K(C_n) \cong M_n(K[X, X^{-1}])$.

As we mentioned earlier, an important motivation to study Leavitt path algebras was the realization problem for von Neumann regular rings. For a row-regular graph E, let M_E denote the graph monoid of E. Then it was shown in [18] that $\mathcal{V}(L_K(E)) \cong M_E$ as monoids. More generally we have

Theorem 1.4.5. *[20, Theorem 4.3] Let E be any graph and $S \subseteq RReg(E)$. Then the $\mathcal{V}$-monoid of Cohn-Leavitt path algebra is generated by the set $S \sqcup \{q_v \mid v \in RReg(E) - S\} \sqcup \{q_v^X \mid v$ is an infinite emitter and X is a finite subset of $s^{-1}(v)\}$ modulo the following relations*

1. *for every $v \in S$,*
$$v = \sum_{e \in s^{-1}(v)} r(e),$$
2. *for every $v \in RReg(E) - S$,*
$$v = \sum_{e \in s^{-1}(v)} r(e) + q_v,$$
3. *for every infinite emitter $v \in E^0$ and finite subset X of $s^{-1}(v)$,*
$$v = \sum_{w \in X} w + q_v^X.$$

Note that the defining relations of Cohn-Leavitt path algebra of a graph E are of homogeneous degree 0 and hence $K(\widehat{E}) \to C_K^S(E)$ is a $\mathbb{Z}$-graded algebra homomorphism.

To find a K-basis for $C_K^S(E)$, we can use Bergman's diamond lemma and can check that there are no inclusion ambiguities and all ambiguities are resolvable. We have

Proposition 1.4.6 (cf. [6]). *Let E be a graph and $S \subseteq RReg(E)$. Then*

1. *$C_K^S(E)$ is a $\mathbb{Z}$-graded $*$-algebra with enough idempotents E^0. Moreover, $L_K(E)$ is unital if and only if E^0 is finite in which case $\sum_{v\in E^0} v$ is the unit.*

2. *For each $v \in S$ choose an edge $e_v \in s^{-1}(v)$. Then a K-basis of $C_K^S(E)$ is given by the set*

$$\{\mu\nu^* \in \widehat{E}^\star \mid \mu,\nu \in E^\star, r(\mu)=r(\nu)\} - \{\lambda e_v e_v^* \kappa^* \in \widehat{E}^\star \mid \lambda,\kappa \in E^\star, r(\lambda)=r(\kappa)=v, v\in S\}.$$

From part 2 of Proposotion 1.4.6, it is clear that $K(E)$ embeds into $L_K(E)$. Hence any right unital $L_K(E)$-module can also be viewed as a module over $K(E)$ and it is interesting to study representations of Leavitt path algebras in terms of quiver representations of E. This question was taken up in [41] and it was shown that when E is row-finite, the category of $L_K(E)$-modules is equivalent to a full subcategory and also a retract of quiver representations of E.

A major theme in the area focus on passing structural information from the directed graph E to the Leavitt path algebra $L_K(E)$, and vice-versa:

$$E \text{ has graph property } \mathcal{P} \Leftrightarrow L_K(E) \text{ has algebraic property } \mathcal{Q}.$$

We list some theorems which illustrate this point.

Theorem 1.4.7. *For a graph E the following are equivalent.*

1. *E is acyclic.*

2. *$L_K(E)$ is right (or left) Artinian [10, Theorem 2.6].*

3. *$L_K(E)$ is von Neumann regular [14, Theorem 1].*

4. *$L_K(E)$ is finite dimensional [3, Corollaries 3.6, 3.7].*

5. *$L_K(E)$ is isomorphic to direct sum of matrix algebras over K.*

Moreover, we have

Theorem 1.4.8 (Structure theorem for finite acyclic graphs)**.** *[5, Theorem 9] Let E be a finite acyclic graph. Let $w_1, \ldots, w_t$ denote the sinks of E. For each w_i, let N_i denote the number of elements of $E^\star$ having range vertex w_i. Then*

$$L_K(E) \cong \bigoplus_{i=1}^{t} M_{N_i}(K).$$

Theorem 1.4.9. *For a graph E the following are equivalent*

1. *The cycles of E have no exits.*
2. *$L_K(E)$ is right (or left) Noetherian [10, Theorem 3.8].*
3. *$L_K(E)$ is locally finite dimensional (each homogeneous summand is finite dimensional)[4, Theorem 1.8].*
4. *$L_K(E)$ is principal ideal ring [13, Proposition 23].*
5. *$L_K(E)$ is isomorphic to direct sum of matrix algebras over K and matrix algebras over $K[X, X^{-1}]$. [4, Theorems 3.8, 3.10].*

Definition 1.4.10. Let E be a graph and $H \subseteq E^0$.

1. H is **hereditary** if whenever $v \in H$ and $w \in E^0$ for which there exists a path μ such that $s(\mu) = v$ and $r(\mu) = w$, then $w \in H$.
2. H is **saturated** if whenever $v \in E^0$ is regular such that $\{r(e) \mid e \in E^1, s(e) = v\} \subseteq H$, then $v \in H$.
3. E satisfies **condition (L)** if every cycle in E has an exit.

Clearly the sets $\emptyset$ and E^0 are hereditary and saturated subsets of E^0 and intersection of any family of hereditary and saturated subsets of E^0 is also hereditary and saturated. For $H \subseteq E^0$ let $\overline{H}$ denote the hereditary saturated closure of H. In fact, the set of all hereditary and saturated subsets of E^0 forms a complete lattice with respect to set inclusion, supremum given by $\bigvee\limits_i H_i = \overline{\bigcup\limits_i H_i}$ and $\bigwedge\limits_i H_i = \bigcap\limits_i H_i$. By the defining relations of Leavitt path algebra, we can verify easily that for any ideal I of $L_K(E)$, $I \cap E^0$ is hereditary and saturated subset of E^0.

Theorem 1.4.11. *[6, Theorem 2.5.9] Let E be a row-finite graph. Then the map $I \mapsto I \cap E^0$ is a lattice isomorphism from the lattice of graded ideals of $L_K(E)$ to the lattice of hereditary and saturated subsets of E^0.*

One of the first theorems proved in the area of Leavitt path algebras is the simplicity theorem.

Theorem 1.4.12. *[6, Theorem 2.9.1] For any graph E, the Leavitt path algebra $L_K(E)$ is simple if and only if E satisfies Condition (L) and only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0*

Definition 1.4.13. A unital K-algebra A is called *purely infinite simple* if A is not a division ring, and A has the property that for every nonzero element x of A there exists $b, c \in A$ for which $bxc = 1_A$.

The finite graphs E for which the Leavitt path algebra $L(E)$ is purely infinite simple have been explicitly described in [8].

Theorem 1.4.14. *$L(E)$ is purely infinite simple if and only if E is sink-free, satisfies Condition (L), and only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.*

In other words, the graph E satisfies the following properties: every vertex in E connects to every cycle of E; every cycle in E has an exit; and E contains at least one cycle.

It is shown in [17, Corollary 2.2], that if A is a unital purely infinite simple K-algebra, then the semigroup $(\mathcal{V}(A)^*, \oplus)$ is in fact a group, and moreover, that $\mathcal{V}(A)^* \cong K_0(A)$, the Grothendieck group of A. For unital Leavitt path algebras, the converse is true as well: if $\mathcal{V}(L(E))^*$ is a group, then $L(E)$ is purely infinite simple. (This converse is not true for general K-algebras.)

Theorem 1.4.15. *If $L(E)$ is unital purely infinite simple, then*

$$K_0(L(E)) \cong \mathcal{V}(L(E))^* \cong M_E^*.$$

1.5 Chapter-wise summary

In chapter 2, we introduce the notion of weighted Cayley graph $\mathsf{Cay}(G, S, w)$ of a group G with respect to a generating set S and a weight function $w : S \to \mathbb{N}$. Then we study Leavitt path algebras $L_K(C_n(S, w))$, where $C_n(S, w)$ denotes the weighted Cayley graph of $\mathbb{Z}_n$ with respect to a generating set S of $\mathbb{Z}_n$ and weight function $w : S \to \mathbb{N}$. The algebras $L_K(C_n(S, w))$ satisfy a useful algebraic property known as purely infinite simpleness and due to an important theorem called *algebraic Kirchberg-Philips theorem* such algebras can be classified by computing their Grothendieck groups under mild hypothesis. We present a method to compute the Grothendieck group. Specifically, we find conditions under which the hypothesis of algebraic KP theorem is satisfied and also provide a method to reduce the computation of the Grothendieck group. Finally, we illustrate the method by considering some simple cases when $|S| = 1, 2$ or n.

In chapter 3, we define bi-separated graphs $\dot{E}$ and their Cohn-Leavitt path algebras $\mathcal{A}_K(\dot{E})$ and state some very basic results that follow from the definitions. We also show how the generalizations introduced in previous section are special cases of $\mathcal{A}_K(\dot{E})$. In section 3.3, we define the category **BSG** of bi-separated graphs and approrpiate morphisms such that $\mathcal{A}_K(\_)$ is a continuous functor from **BSG** to **K-Alg**. Also we show that every object in this category is a direct limit of countable complete sub-objects (see Proposition 3.3.7). However, this statement does not hold true if we replace countable by finite. We then define a new sub-category of **BSG**, which we call "tame category **tBSG**" and show that this category characterizes all objects of **BSG** which are direct limits of finite 'complete' sub-objects. Thus if $\dot{E}$ is a tame bi-separated graph then $\mathcal{A}_K(\dot{E})$ is a direct limits of unital sub Cohn-Leavitt path algebras (of corresponding finite complete sub bi-separated graphs). Section 3.4 deals with computation of normal forms of $\mathcal{A}_K(\dot{E})$ using Bergman's diamond lemma and some of it's applications. In particular, we find bi-separated graph theoretic conditions to study algebraic properties of Cohn-Leavitt path algebras such as simplicity, semiprimitivity, von Neumann regularity, growth and finiteness and also characterize the algebras which are domains.

In chapter 4, we focus our attention to the study of B-hypergraphs. In section **??**, we define B-hypergraphs $(\dot{E}, \Lambda)$ and their H-monoids and show that H-monoids are isomorphic to the $\mathcal{V}$-monoids of the corresponding Cohn-Leavitt path algebras. In section 4.2, we introduce the partially ordered set of admissible triples $\mathrm{AT}(\dot{E}, \Lambda)$ for each B-hypergraph $(\dot{E}, \Lambda)$ and show that this poset is a lattice. We further show that the lattice of order-ideals of H-monoid of $(\dot{E}, \Lambda)$ is isomorphic to the lattice $\mathrm{AT}(\dot{E}, \Lambda)$, which establishes that $\mathrm{AT}(\dot{E}, \Lambda)$ is isomorphic to the complete lattice of trace ideals of Cohn-Leavitt path algebra of $(\dot{E}, \Lambda)$. In section 4.3, we study the representations of Leavitt path algebras of regular hypergraphs and show that the category of unital right modules of these algebras is a full subcategory and a retract of quiver representations of underlying graphs of the hypergraphs. Also, we give a characterization of Leavitt path algebras of regular hypergraphs to have a finite dimensional representation in terms of their H-monoids. Finally, in section 4.4, we provide a matrix criteria for a Leavitt path algebra of a finite hypergraph to have invariant basis number.

Chapter 2

Leavitt path algebras of weighted Cayley graphs $C_n(S, w)$

2.1 Introduction

For a finite group G and a subset $S \subseteq G$, let the associated Cayley graph be denoted by $\mathsf{Cay}(G, S)$. When the given group is $\mathbb{Z}_n$ we write $C_n(S) = \mathsf{Cay}(\mathbb{Z}_n, S)$. Leavitt path algebras of Cayley graphs of the finite cyclic group $\mathbb{Z}_n$ with respect to the subset $S = \{1, n-1\}$ were initially studied in [15]. It was shown that there are exactly four isomorphism classes represented by the collection $\{L(C_n(1, n-1)) \mid n \in \mathbb{N}\}$.

Subsequently, in [9], the authors computed the important integers $|K_0(L(C_n(1, j)))|$ and $\det(I_n - A^t_{C_n(1,j)})$, where $A_{(-)}$ denotes the adjacency matrix of a directed graph, and $K_0(-)$ denotes the Grothendieck group of a ring. Also in [9], the collections of K-algebras $\{L(C_n(1, j)) \mid n \in \mathbb{N}\}$ for $j = 0, 1, 2$ were described upto isomorphism. The descriptions of all these algebras follow from an application of the powerful tool known as the (Restricted) Algebraic Kirchberg-Philips Theorem.

In [11], the study was extended and a method to compute the Grothendieck group of the Leavitt path algebra $L(C_n(1, j))$ to the case where $0 \leq j \leq n-1$ and $n \geq 3$ was derived. Specifically, a method was given to reduce the computation of the Smith Normal Form of the $n \times n$ matrix $I_n - A^t_{C_n(1,j)}$ to that of calculating the Smith Normal Form of a $j \times j$ matrix $(M^n_j)^t - I_j$. Further a description of $K_0(L(C_n(1, j))$ was also given.

In this chapter we generalize the work done in [11] to study $L(C_n(S,w))$, where S is any nonempty generating subset of $\mathbb{Z}_n$, $w : S \to \mathbb{N}$ is a map and $C_n(S,w)$ is the weighted Cayley graph. In section 2.2, we recall the background information required. In Section 2.3, we present a method to compute the Grothendieck group. Specifically we find the conditions to determine the sign of $\det(I_n - A^t_{C_n(S,w)})$ and also the cardinality of $K_0(L(C_n(S,w)))$. Also we find a method to reduce the computation of the Smith Normal form of the $n \times n$ matrix $I_n - A^t_{C_n(S,w)}$ to that of calculating the Smith Normal form of a square matrix of smaller size if $0 \notin S$ (Theorem 2.3.9). In Section 2.4, we use the method developed in Section 2.3 to study the following simple cases when $\langle S \rangle = \mathbb{Z}_n$:

Case 1 : $|S| = 1$,

Case 2 : $|S| = 2$,

Case 3 : $|S| = n$.

Moreover, we recover the results studied in [15],[9], and [11] as special cases and get some new results. Among these new results, in particular, we show that $L(K_n) \cong L(1,n)$ where K_n is the unweighted complete n-graph (See 2.4.1 for definition) and $L(1,n)$ is the Leavitt algebra. We also show that the main result of [15] holds true if $C_n(1,n-1)$ is replaced by D_n for every $n \in \mathbb{N}$, where D_n denotes the Cayley graph of Dihedral group with respect to the usual generating set.

2.2 Background information

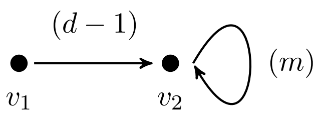
2.2.1 The Algebraic KP theorem

The following important theorem will be used to yield a number of key results in the subsequent sections:

Theorem 2.2.1 ((Restricted) Algebraic KP Theorem). *[12, Corollary 2.7] Suppose E and F are finite graphs for which the Leavitt path algebras $L(E)$ and $L(F)$ are purely infinite simple. Suppose that there is an isomorphism $\varphi : K_0(L(E)) \to K_0(L(F))$ for which $\varphi([L(E)]) = [L(F)]$, and suppose also that the two integers $\det(I_{|E^0|} - A^t_E)$ and $\det(I_{|F^0|} - A^t_F)$ have the same sign. Then $L(E) \cong L(F)$ as K-algebras.*

Example 2.2.2 (Leavitt algebras)**.** For any integer $m \geq 2$, $L(1, m)$ denote the Leavitt algebra of type $(1, m)$. It is easy to see that for $m > 2$, if R_m is the graph having one vertex and m loops, then $L(R_m) \cong L(1, m)$. From Theorem 1.4.14 it follows that $L(R_m)$ is unital purely infinite simple and hence $K_0(L(R_m)) \cong M^*_{R_m}$ is the cyclic group $\mathbb{Z}_{m-1}$, where the regular module $[L(R_m)]$ in $K_0(L(R_m))$ corresponds to 1 in $\mathbb{Z}_{m-1}$.

Unital purely infinite simple Leavitt path algebras $L(E)$ whose corresponding K_0 groups are cyclic and for which $\det(I_{|E^0|} - A^t_E) \leq 0$ are relatively well-understood, and arise as matrix rings over the Leavitt algebras $L(1, m)$, as follows: Assume $d \geq 2$, and consider the graph R^d_m having two vertices v_1, v_2; $d - 1$ edges from v_1 to v_2; and m loops at v_2.

$R^d_m =$ 

It is shown in [1] that $L(R^d_m)$ is isomorphic to the matrix algebra $M_d(L(1, m))$. By standard Morita equivalence theory, we have that $K_0(M_d(L(1, m))) \cong K_0(L(1, m))$. Moreover, the element $[M_d(L(1, m))]$ of $K_0(M_d(L(1, m)))$ corresponds to the element d in $\mathbb{Z}_{m-1}$. In particular, the element $[M_{m-1}(L(1, m))]$ of $K_0(M_{m-1}(L(1, m)))$ corresponds to $m - 1 \equiv 0$ in $\mathbb{Z}_{m-1}$. Finally, an easy computation yields that $\det(I_2 - A^t_{R^d_m}) = -(m - 1) \leq 0$ for all m, d. Therefore, by invoking the Algebraic KP Theorem, the previous discussion immediately yields the following.

Proposition 2.2.3. *Suppose that E is a graph for which $L(E)$ is unital purely infinite simple. Let M^*_E be isomorphic to the cyclic group $\mathbb{Z}_{m-1}$, via an isomorphism which takes the element $\sum_{v \in E^0}[v]$ of M^*_E to the element d of $\mathbb{Z}_{m-1}$. Finally, suppose that $det(I_{|E^0|} - A^t_E) \leq 0$. Then $L(E) \cong M_d(L(1, m))$.*

2.2.1.1 Computation of Grothendieck group

Let E be a finite directed graph for which $|E^0| = n$. We view $I_n - A^t_E$ both as a matrix, and as a linear transformation $I_n - A^t_E : \mathbb{Z}^n \to \mathbb{Z}^n$, via left multiplication (viewing elements of $\mathbb{Z}^n$ as column vectors). As discussed in [1, Section 3], we have

Proposition 2.2.4. *If $L(E)$ is purely infinite simple, then*

$$M_E^* \cong K_0(L(E)) \cong \mathbb{Z}^n/\mathrm{Im}(I_n - A_E^t) = \mathrm{Coker}(I_n - A_E^t).$$

Under this isomorphism $[v_i] \mapsto \vec{b_i} + \mathrm{Im}(I_n - A_E^t)$, where $\vec{b_i}$ is the element of $\mathbb{Z}^n$ which is 1 in the i^{th} coordinate and 0 elsewhere.

Let $M \in M_n(\mathbb{Z})$ and view M as a linear transformation $M : \mathbb{Z}^n \to \mathbb{Z}^n$ via left multiplication on columns. The cokernel of M is a finitely generated abelian group, having at most n summands; as such, by the invariant factors version of the Fundamental Theorem of Finitely Generated Abelian Groups, we have

$$\mathrm{Coker}(M) \cong \mathbb{Z}_{s_l} \oplus \mathbb{Z}_{s_{l+1}} \oplus \cdots \oplus \mathbb{Z}_{s_n},$$

for some $1 \le l \le n$, where either $n = l$ and $s_n = 1$ (i.e., $\mathrm{Coker}(M)$ is trivial group), or there are (necessarily unique) nonnegative integers $s_l, s_{l+1}, \ldots, s_n$, for which the nonzero values $s_l, s_{l+1}, \ldots, s_r$ satisfy $s_j \ge 2$ for $1 \le j \le r$ and $s_i | s_{i+1}$ for $l \le i \le r-1$, and $s_{r+1} = \cdots = s_n = 0$. $\mathrm{Coker}(M)$ is a finite group if and only if $r = n$. In case $l > 1$, we define $s_1 = s_2 = \cdots = s_{l-1} = 1$. Clearly then we have

$$\mathrm{Coker}(M) \cong \mathbb{Z}_{s_1} \oplus \mathbb{Z}_{s_2} \oplus \cdots \oplus \mathbb{Z}_{s_l} \oplus \cdots \oplus \mathbb{Z}_{s_n},$$

since any additional direct summands are isomorphic to the trivial group $\mathbb{Z}_1$.

We note that if P, Q are invertible in $M_n(\mathbb{Z})$ (hence their determinant is ± 1), then $\mathrm{Coker}(M) \cong \mathrm{Coker}(PMQ)$. In other words, if $N \in M_n(\mathbb{Z})$ is a matrix which is constructed by performing any sequence of $\mathbb{Z}$-elementary row (or column) operations starting with M, then $\mathrm{Coker}(M) \cong \mathrm{Coker}(N)$ as abelian groups.

Definition 2.2.5. Let $M \in M_n(\mathbb{Z})$, and suppose $\mathrm{Coker}(M) \cong \mathbb{Z}_{s_1} \oplus \mathbb{Z}_{s_2} \oplus \cdots \oplus \mathbb{Z}_{s_n}$ as described above. The **Smith Normal Form** of M ($\mathrm{SNF}(M)$ in short), is the $n \times n$ diagonal matrix $\mathrm{diag}(s_1, s_2, \ldots, s_r, 0, \ldots, 0)$.

For any matrix $M \in M_n(\mathbb{Z})$, the Smith Normal Form of M exists and is unique. If $D \in M_n(\mathbb{Z})$ is a diagonal matrix with entries $d_1, d_2, \ldots, d_n$, then clearly $\mathrm{Coker}(D) \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_n}$. We also note the following:

Proposition 2.2.6. *Let $M \in M_n(\mathbb{Z})$, and let S denote the Smith Normal Form of M. Suppose the diagonal entries of S are $s_1, s_2, \dots, s_n$. Then*

$$\mathrm{Coker}(M) \cong \mathbb{Z}_{s_1} \oplus \mathbb{Z}_{s_2} \oplus \cdots \oplus \mathbb{Z}_{s_n}.$$

In particular, if there are no zero entries in the Smith Normal Form of M, then $|\mathrm{Coker}(M)| = s_1 s_2 \dots s_n = |\det(S)| = |\det(M)|$.

Proposition 2.2.6 yields the following:

Proposition 2.2.7. *Let E be a finite graph with $|E^0| = n$ and adjacency matrix A_E. Suppose that $L(E)$ is purely infinite simple. Let S be the Smith Normal Form of the matrix $I_n - A_E^t$, with diagonal entries $s_1, s_2, \dots, s_n$. Then*

$$K_0(L(E)) \cong \mathbb{Z}_{s_1} \oplus \mathbb{Z}_{s_2} \oplus \cdots \oplus \mathbb{Z}_{s_n}.$$

Moreover, if $K_0(L(E))$ is finite, then an analysis of the Smith Normal Form of the matrix $I_n - A_E^t$ yields

$$|K_0(L(E))| = \left|\det(I_n - A_E^t)\right|,$$

Conversely, $K_0(L(E))$ is infinite if and only if $\det(I_n - A_E^t) = 0$ and in this case $\mathrm{rank}(K_0(L(E)) = \mathrm{nullity}(I_n - A_E^t)$.

We record the following theorem which will be used in computations of Smith Normal Forms in later sections:

Theorem 2.2.8 (**Determinant Divisors Theorem**). *[56, Theorem II.9]*
Let $M \in M_n(\mathbb{Z})$. Define $\alpha_0 = 1$, and for each $1 \le i \le n$, define the i^{th} determinant divisor of M to be the integer

$$\alpha_i = \textit{the greatest common divisor of the set of all } i \times i \textit{ minors of } M.$$

Let $s_1, s_2, \dots, s_n$ denote the diagonal entries of the Smith Normal Form of M, and assume that each s_i is nonzero. Then

$$s_i = \frac{\alpha_i}{\alpha_{i-1}},$$

for each $1 \le i \le n$.

2.2.2 Weighted Cayley graphs and circulant matrices

Recall that given a group G, and a subset $S \subseteq G$, the *associated Cayley graph* $\mathsf{Cay}(G, S)$ is the directed graph $E(G, S)$ with vertex set $\{v_g \mid g \in G\}$, and in which there is an edge $e(g, h)$ from v_g to v_h if there exists (a necessarily unique) $s \in S$ with $h = gs$ in G. Thus, in $\mathsf{Cay}(G, S)$, at every vertex v_g, the number of edges emitted is $|S|$. The identity of G is in S if and only $\mathsf{Cay}(G, S)$ contains a loop at every vertex.

Definition 2.2.9. Let G be a group, $S \subseteq G$ and $w : S \to \mathbb{N}$ be a map. Then w induces a map (also denoted by w) from the set of edges of $\mathsf{Cay}(G, S)$ to $\mathbb{N}$ by $e(g, h) \mapsto w(s)$ whenever $h = gs$. The weighted graph of $\mathsf{Cay}(G, S)$ associated to the map w is called the **weighted Cayley graph** (or **w-Cayley graph**) and is denoted by $\mathsf{Cay}(G, S, w)$.

In particular, $\mathsf{Cay}(G, S)$ is a special case of $\mathsf{Cay}(G, S, w)$ when w is the constant map $w(e) = 1$ for every edge e. In this case we say $\mathsf{Cay}(G, S)$ is unweighted.

Remark 2.2.10. $\mathsf{Cay}(G, S, w)$ is strongly connected if and only if $\langle S\rangle = G$. In particular, $\mathsf{Cay}(\langle S\rangle, S, w)$ is a connected component of $\mathsf{Cay}(G, S, w)$, where $\langle S\rangle$ is the subgroup generated by S.

Notation 2.2.11. For a positive integer n, let $G = \mathbb{Z}_n$, and S be any non-empty subset of G. We denote the w-Cayley graph $\mathsf{Cay}(G, S, w)$ simply by $C_n(S, w)$.

In other words, if $S = \{s_1, s_2, \dots s_k\}$ then the w-Cayley graph $C_n(S, w)$ is the directed graph with the vertex set $\{v_0, v_1, v_2, \dots, v_{n-1}\}$ and the edge set $\{e_l(i, s_j) \mid 0 \le i \le n-1, 1 \le j \le k, 1 \le l \le w(s_j)\}$ for which $s(e_l(i, s_j)) = v_i$, and $r(e_l(i, s_j)) = v_{i+s_j}$, where the indices are interpreted modulo n. Therefore $C_n(S, w)$ is a finite graph.

Definition 2.2.12. For a positive integer n, let $\mathbf{c} = (c_0, c_1, \dots, c_{n-1}) \in \mathbb{Q}^n$. Consider the shift operator $T : \mathbb{Q}^n \to \mathbb{Q}^n$, defined by $T(c_0, c_1, \dots, c_{n-1}) = (c_{n-1}, c_0, \dots, c_{n-2})$. The *circulant matrix* $circ(c)$, associated with $\mathbf{c}$ is the $n \times n$ matrix C whose kth row is $T^{k-1}(\mathbf{c})$, for $k = 1, 2, \dots .n$. Thus C is of the form

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \dots & c_{n-2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ c_1 & c_2 & c_3 & \dots & c_0 \end{pmatrix}$$

In other words, a circulant matrix is obtained by taking an arbitrary first row, and shifting it cyclically one position to the right in order to obtain successive rows. The (i, j) element of C is c_{j-i}, where subscripts are taken modulo n.

Note that $A_{C_n(S,w)}$ is the $n \times n$ matrix with the $(i, j)^{th}$ entry as $w(s)$ if $i + s = j$ modulo n, for some $s \in S$, and 0, otherwise. Hence $A_{C_n(S,w)}$ is a circulant matrix with non-negative integer entries. In the case of unweighted Cayley graph $C_n(S)$, the adjacency matrix is binary circulant matrix.

Definition 2.2.13. For $\mathbf{c} \in \mathbb{Q}^n$, let $C = circ(\mathbf{c})$. The **representer polynomial** of C is defined to be the polynomial $P_C(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} \in \mathbb{Q}[x]$.

Lemma 2.2.14. *Let $C = circ(\boldsymbol{c})$ be a circulant matrix. Then the eigenvalues of C equal $P_C(\zeta_n^k) = c_0 + c_1\zeta_n^k + \cdots + c_{n-1}\zeta_n^{k(n-1)}$ for $k = 0, 1, \ldots, n-1$, where $\zeta_n = e^{\frac{2\pi i}{n}}$, the primitive n^{th} root of unity. Further*

$$\det(C) = \prod_{l=0}^{n-1} \left(\sum_{k=0}^{n-1} c_j \zeta_n^{lk} \right).$$

For a proof of Lemma 2.2.14, we refer the reader to [42, Theorem 6].

Note that the n^{th} cyclotomic polynomial, denoted by

$$\Phi_n(x) = \prod_{\substack{1 \leq a < n \\ \gcd(a,n)=1}} (x - \zeta_n^a),$$

is an element of $\mathbb{Z}[x]$. Also, $x^n - 1 = \prod_{d|n} \Phi_d(x)$. Since $\Phi_n(x)$ is the minimal polynomial of ζ_n, $f(\zeta_n) = 0$ for some $f(x) \in \mathbb{Z}[x]$ implies $\Phi_n(x)$ divides $f(x)$. By applying Lemma 2.2.14 we get

Lemma 2.2.15. *Let $C = circ(\boldsymbol{c})$. Then the following are equivalent.*

(a) C is singular.

(b) $P_C(\zeta_n^k) = 0$ for some $k \in \mathbb{Z}$.

(c) The polynomials $P_C(x)$ and $x^n - 1$ are not relatively prime.

2.3 Leavitt path algebras of $C_n(S, w)$

Theorem 2.3.1. *Let G be a finite group, S its generating set and $w : S \to \mathbb{N}$ a weight function. Let $W = \sum_{s\in S} w(s)$. Then the following are equivalent:*

1. *$L(\mathsf{Cay}(G, S, w))$ is purely infinite simple.*
2. *$W \geq 2$.*
3. *$L(\mathsf{Cay}(G, S, w))$ does not have Invariant Basis Number.*

Proof. (1) $\Rightarrow$ (2) $\Leftarrow$ (3). Let $|G| = n$. If $W = 1$, then $|S| = 1$. Setting $S = \{g\}$, we have G is cyclic group generated by g. Hence $\mathsf{Cay}(G, S, w)$ is the graph C_n, which is cycle of length n and which does not satisfy condition L. This contradicts (1). By [4, Theorem 3.8 and 3.10] $L(C_n) \cong M_n(K[x, x^{-1}])$ which has Invariant Basis Number.

(2) $\Rightarrow$ (1). Let $W \geq 2$. In $\mathsf{Cay}(G, S, w)$, the number of edges emitted at each vertex v_g is W. So there are at least two edges emitted from each vertex. This also implies condition (L). Since $\langle S\rangle = G$, $\mathsf{Cay}(G, S, w)$ is strongly connected. Hence for any vertex v_g there is a non-trivial path connecting v_g to v_1 and vice versa. Therefore $\mathsf{Cay}(G, S, w)$ contains a cycle and there is no non-trivial hereditary subset of vertices.

(2) $\Rightarrow$ (3). For a finite graph E, $L(E)$ has Invariant Basis Number if and only if for each pair of positive integers m and n,

$$m \sum_{v\in E^0} [v] = n \sum_{v\in E^0} [v] \text{ in } M_E \Rightarrow m = n.$$

In $M_{\mathsf{Cay}(G,S,w)}$, for each v_g we have $[v_g] = \sum_{s\in S} w(s)[v_{gs}]$ and hence,

$$\sum_{g\in G}[v_g] = \sum_{g\in G}\sum_{s\in S} w(s)[v_{gs}] = \sum_{s\in S}\sum_{g\in G} w(s)[v_{gs}] = \sum_{s\in S} w(s) \sum_{g\in G}[v_{gs}] = W \sum_{g\in G}[v_{gs}].$$

Since G is a finite group we have $G = \{gs \mid g \in G\}$ and hence $\sum_{g\in G}[v_{gs}] = \sum_{g\in G}[v_g]$. Hence we have $\sum_{g\in G}[v_g] = W \sum_{g\in G}[v_g]$. If $W \geq 2$ then $L(\mathsf{Cay}(G, S, w))$ does not have Invariant Basis Number. $\square$

Corollary 2.3.2. *[55, Proposition 4.1, Theorem 4.2] Let G be a finite group, S its generating set. Then the following are equivalent:*

1. *$L(\mathsf{Cay}(G,S))$ is purely infinite simple.*

2. *$L(\mathsf{Cay}(G,S))$ does not have Invariant Basis Number.*

3. *$|S| \geq 2$.*

Proof. In this case $W = |S|$. $\square$

From now on, we work with the following assumption:

Assumption 2.3.3. Let $n \in \mathbb{N}$, $S = \{s_1, s_2, \dots s_k\} \subseteq \mathbb{Z}_n$. $s_1 < s_2 < \cdots < s_k$. Further set $W = \sum\limits_{s_j \in S} w(s_j)$.

Theorem 2.3.4. *Let $\langle S \rangle = \mathbb{Z}_n$ and $W \geq 2$. Then in the group $M^*_{C_n(S,w)}$, the order of $\sum_{i=0}^{n-1}[v_i]$ divides $W-1$. Further, if $\gcd(W-1,n) = 1$ then order of $\sum_{i=0}^{n-1}[v_i]$ is $W-1$.*

Proof. Let $S = \{s_1, s_2, \dots, s_k\}$. Then in $M^*_{C_n(S,w)}$, we have the following relations

$$[v_i] = \sum_{s_j \in S} w(s_j)[v_{i+s_j}].$$

Let $\sigma = \sum_{i=0}^{n-1}[v_i]$. Then using the defining relations in $M^*_{C_n(S),w}$, we have

$$\begin{aligned}
\sigma = \sum_{i=0}^{n-1}[v_i] &= \sum_{i=0}^{n-1}\left(\sum_{s_j \in S} w(s_j)[v_{i+s_j}]\right) = \sum_{s_j \in S} w(s_j)\left(\sum_{i=0}^{n-1}[v_{i+s_j}]\right) \\
&= \sum_{s_j \in S} w(s_j)\left(\sum_{i=0}^{n-1}[v_i]\right) = \left(\sum_{s_j \in S} w(s_j)\right)\sigma = W\sigma.
\end{aligned}$$

Thus, in the group $M^*_{C_n(S,w)}$, we have $(W-1)\,\sigma = 0$. This proves the first part of the theorem.

By Theorem 2.2.4, $M^*_{C_n(S,w)} \cong \mathrm{Coker}(I_n - A^t_{C_n(S,w)})$, and under the isomorphism $[v_i] \mapsto \vec{b_i} + \mathrm{Im}(I_n - A^t_{C_n(S,w)})$, where $\vec{b_i}$ is the element of $\mathbb{Z}^n$ which has 1 in the i^{th} coordinate and 0 elsewhere.

Hence for a natural number d, $d\sigma = 0$ in $M^*_{C_n(S,w)}$ if and only if $d\vec{v} \in \mathrm{Im}(I_n - A^t_{C_n(S,w)})$ where $\vec{v} = (1,1,\dots,1)^t$. This is equivalent to $\vec{u} - A^t\vec{u} = d\vec{v}$ for some $\vec{u} = (u_0, u_1, \dots, u_{n-1}) \in \mathbb{Z}^n$, which in turn is equivalent to

$$u_l - \sum_{s_j \in S} w(s_j) u_{n-s_j+l} = d \qquad 0 \leq l \leq n-1.$$

Adding all the above equations, we get

$$\sum_{l=0}^{n-1} u_l - \sum_{l=0}^{n-1}\sum_{s_j\in S} w(s_j)u_{n-s_j+l} = nd.$$

$$\begin{aligned}
\text{LHS} &= \sum_{l=0}^{n-1} u_l - \sum_{s_j\in S}\sum_{l=0}^{n-1} w(s_j)u_{n-s_j+l}\\
&= \sum_{l=0}^{n-1} u_l - \sum_{s_j\in S} w(s_j)\sum_{l=0}^{n-1} u_{n-s_j+l}\\
&= \left(1-\sum_{s_j\in S} w(s_j)\right)\sum_{l=0}^{n-1} u_l\\
&= (1-W)\sum_{l=0}^{n-1} u_l.
\end{aligned}$$

Thus $W-1$ divides nd. If $\gcd(W-1,n)=1$, then $W-1$ divies d. In particular, when $\gcd(W-1,n)=1$ order of $\sum\limits_{i=0}^{n-1}[v_i]$ is $W-1$.

□

Assumption 2.3.5. In what follows, we always assume that $\langle S\rangle = \mathbb{Z}_n$ and $W \geq 2$.

As we noted in 2.2.14, for a circulant matrix C,

$$\det(C) = \prod_{l=0}^{n-1}\left(\sum_{j=0}^{n-1} c_j\zeta_n^{lj}\right),$$

where $\zeta_n = e^{\frac{2\pi i}{n}}$, the primitive n^{th} root of unity. For $C_n(S,w)$, the adjacency matrix $A_{C_n(S,w)}$ is circulant. Also $I_n - A^t_{C_n(S,w)}$ is circulant (with integer entries). Let $S=\{s_1,s_2,\ldots,s_k\}$. Then,

$$\det(I_n - A^t_{C_n(S)}) = \det(I_n - A_{C_n(S)}) = \prod_{l=0}^{n-1}\left(1-\sum_{s_j\in S} w(s_j)\zeta_n^{ls_j}\right).$$

Proposition 2.3.6. *Let* $S_0 := \{j\in S \mid j\equiv 0 \pmod 2\}$, $S_1 := \{j\in S\mid j\equiv 1 \pmod 2\}$, $W_0 := \sum_{s_j\in S_0} w(s_j)$, *and* $W_1 := \sum_{s_j\in S_1} w(s_j)$. *Then* $\det(I_n - A^t_{C_n(S,w)}) > 0$ *if and only if n is even and* $1+W_1 < W_0$.

Proof. Let $P(x) = 1 - \sum_{s_j} w(s_j)x^{s_j}$ be the representer polynomial of $I_n - A^t_{C_n(S,w)}$. Let $z_l = P(\zeta_n^l) = 1 - \sum_{s_j} w(s_j)\zeta^{ls_j}$. It is easy to see that $z_0 = 1 - \sum_{s_j \in S} w(s_j) = 1 - W < 0$ and $z_{n-l} = \overline{z_l}$ for all l. Thus $\det(I_n - A^t_{C_n(S)}) > 0$ if and only if n is even and $z_{\frac{n}{2}} < 0$. Since

$$z_{\frac{n}{2}} = 1 - \sum_{j \quad even} w(s_j) + \sum_{j \quad odd} w(s_j),$$

Thus $z_{\frac{n}{2}} < 0$ iff $1 + W_1 < W_0$. $\square$

Proposition 2.3.7. *Let $P(x) \in \mathbb{Z}[x]$ be the representer polynomial associated with the circulant matrix $I_n - A^t_{C_n(S,w)}$. Then $K_0(L(C_n(S,w)))$ is infinite if and only if $P(x)$ and $x^n - 1$ are relatively prime.*

Proof. Follows from Lemma 2.2.15 and Proposition 2.2.7 $\square$

In order to compute the Grothendieck group of the Leavitt path algebra of $C_n(S,w)$, we look at the generating relations for $M^*_{C_n(S,w)}$

$$[v_i] = \sum_{s_j \in S} w(s_j)[v_{i+s_j}].$$

where $0 \leq i \leq n-1$, (subscripts are modulo n) and $S = \{s_1, s_2, \dots, s_k\}$, ($s_l < s_m$ for $l < m$). Any statement about $[v_0]$ in $M^*_{C_n(S,w)}$, has an analogous statement for $[v_k]$ for $0 \leq k \leq n-1$, by symmetry of relations.

Definition 2.3.8. The **companion matrix** of the monic polynomial $p(t) = c_0 + c_1 t + \cdots + c_{n-1}t^{n-1} + t^n$, is a $n \times n$ matrix defined as

$$T(p) = \begin{pmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{n-1} \end{pmatrix}$$

Let a linear recursive sequence be of the form

$$u_{n+k} - c_{n-1}u_{n+k-1} - \cdots - c_0 u_n = 0 \quad (n \geq 0),$$

where $c_0, c_1, \dots, c_{n-1}$ are constants. The **characteristic polynomial** of the above linear recursive sequence is defined as $p(t) = t^n - c_{n-1}t^{n-1} - \cdots - c_1 t - c_0$ whose companion matrix is

$$T(p) = \begin{pmatrix} 0 & 0 & \dots & 0 & c_0 \\ 1 & 0 & \dots & 0 & c_1 \\ 0 & 1 & \dots & 0 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & c_{n-1} \end{pmatrix}$$

This matrix generates the sequence in the sense that,

$$\begin{pmatrix} a_k & a_{k+1} & \dots & a_{k+n-1} \end{pmatrix} T(p) = \begin{pmatrix} a_{k+1} & a_{k+2} & \dots & a_{k+n} \end{pmatrix}$$

In particular, the $(n, n)^{th}$ entry of $T(p)^k$ is u_{n+k-2}.

When $0 \notin S$, from the linear recursive relation in $M^*_{C_n(S,w)}$, we have the characteristic polynomial $p(S, w, t) = t^{s_k} - \sum_{s_j \in S} w(s_j) t^{s_k - s_j}$. The companion matrix of $p(S, w, t)$ is denoted by $T_{C_n(S,w)}$, is then the $s_k \times s_k$ matrix

$$T_{C_n(S,w)} = \begin{pmatrix} 0 & 0 & \dots & 0 & \\ 1 & 0 & \dots & 0 & \\ 0 & 1 & \dots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \mathbf{c} \\ & & & & \\ 0 & 0 & \dots & 1 & \end{pmatrix}$$

where $\mathbf{c}$ is the last column of $T_{C_n(S,w)}$ which contains entry $w(s_j)$ at positions $s_k - s_j + 1$ and 0 elsewhere.

In $M^*_{C_n(S,w)}$, we observe that by writing the generating relations and then expanding the equation such that the subscripts are kept in *increasing order*, at i^{th} step we get the coefficients to be the last column of $T^i_{C_n(S,w)}$.

The computation of the Smith Normal Form of $I_n - A^t_{C_n(S,w)}$ is the key tool for determining the K_0 of the Leavitt path algebra of $C_n(S, w)$. We show that this computation reduces to calculating the Smith Normal Form of an $s_k \times s_k$ matrix.

Theorem 2.3.9. *Let $n \in \mathbb{N}, S = \{s_1, s_2, \dots, s_k\} \subsetneq \mathbb{Z}_n$ such that $\langle S \rangle = \mathbb{Z}_n, 0 \notin S$, and $W \geq 2$. Then* $\operatorname{Coker}(I_n - A^t_{C_n(S,w)}) \cong \operatorname{Coker}(T^n_{C_n(S,w)} - I_n)$.

Proof. Since the Smith normal form of $I_n - A^t_{C_n(S,w)}$ and $A_{C_n(S,w)} - I_n$ are the same, their cokernels are same and we only show that $\operatorname{Coker}(A_{C_n(S,w)} - I_n) \cong \operatorname{Coker}(T^n_{C_n(S,w)} - I_n)$. For simplicity, we write $B = A_{C_n(S,w)} - I_n$ and $T = T_{C_n(S,w)}$. First we observe that

$$B_{pq} = \begin{cases} -1, & \text{if} \quad q = p, \\ w(s_j), & \text{if} \quad q = p + s_j, \\ 0, & \text{otherwise.} \end{cases}$$

Let P be a $(s_k \times s_k)$ lower triangular matrix given by

$$P_{pq} = \begin{cases} 0, & \text{if} \quad q > p, \\ w(s_j), & \text{if} \quad p - q = s_k - s_j, \\ 0, & \text{otherwise.} \end{cases}$$

and let Q be a $(s_k \times s_k)$ upper triangular matrix given by

$$Q_{pq} = \begin{cases} 0, & \text{if} \quad q < p, \\ -1, & \text{if} \quad q = p, \\ w(s_j), & \text{if} \quad q - p = s_j, \\ 0, & \text{otherwise.} \end{cases}$$

It is direct that P and Q are invertible. Let $R = -Q^{-1}$. Then a direct computation yields $PR = T^{s_k}$, and also $QR = -I_{s_k}$. Let P' be the block matrix $\left[P \mid 0_{s_k \times (n - s_k)}\right]$ and Q' be the block matrix $\left[0_{s_k \times (n-s_k)} \mid Q\right]$. The $(s_k \times (n - s_k))$ submatrix of B consisting of bottom $s - k$ rows can be written as $P' + Q'$.

The first $(n-s_k)$ reduction steps of the Smith normal form will result in an $(n-s_k)\times(n-s_k)$ identity submatrix in the upper left corner. On the bottom s_k rows, the i^{th} reduction step adds the i^{th} column to the sum of $w(s_j)$ times $(i+s_k)^{th}$ columns, then zeros out the i^{th} column. The matrix that accomplishes this reduction step is

$$P^{-1}TP=\begin{pmatrix} & & \mathbf{r} & & & \\ 1 & 0 & 0 & \dots & 0 & 0\\ 0 & 1 & 0 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & 1 & 0\end{pmatrix}$$

where $\mathbf{r}$ is the first row contains entry $w(s_j)$ at positions s_k and 0 elsewhere. After i reduction steps, the first $(s_k\times s_k)$ submatrix with nonzero column vectors on the bottom s_k rows will be

$$P\cdot(P^{-1}TP)^i=T^iP.$$

Therefore the first $(n-s_k)$ reduction steps of the Smith Normal Form will result in the following form.

$$B\sim\begin{pmatrix} I_{(n-s_k)} & 0_{(n-s_k)\times s_k}\\ 0_{s_k\times(n-s_k)} & T^{n-s_k}P+Q\end{pmatrix}$$

Since $(T^{n-s_k}P+Q)R=T^n-I_{s_k}$,

$$B\sim\begin{pmatrix} I_{(n-s_k)} & 0_{(n-s_k)\times s_k}\\ 0_{k\times(n-s_k)} & T^n-I_{s_k}\end{pmatrix}$$

Hence $\mathrm{Coker}(B)\cong\mathrm{Coker}(T^n-I_{s_k})$.

□

2.4 Illustrations

As illustrations of the above discussion we consider some simple cases when $W\geq 2$ and $\sum\limits_{i=0}^{n-1}[v_i]$ is the identity in $M^*_{C_n(S,w)}$, which recovers the examples obtained in [15],[9],and [11].

2.4.1 $S = \mathbb{Z}_n$

In this subsection, we only look at the following two simple cases when $\sum_{i=0}^{n-1}[v_i]$ is the identity in $M^*_{C_n(\mathbb{Z}_n,w)}$

Definition 2.4.1. Let n, l be two positive integers. We define $K_n^{(l)}$ to be the graph with n vertices $v_0, v_1, \dots v_{n-1}$, in which there is exactly one edge from v_i to v_j for each $0 \le i \ne j \le n-1$ and l loops at each vertex. We call $K_n^{(l)}$ the **complete n-graph with l loops**.

Theorem 2.4.2. *Let $n \geq 2$ be a positive integer.*

1. *$L(K_n^{(1)}) \cong L(1, n)$.*
2. *Let E be a finite graph such that $L(E)$ is purely infinite simple. If $K_0(L(E)) \cong \mathbb{Z}^n$ and $[L(E)]$ is identity in $K_0(L(E))$, then $L(E) \cong L(K_{n+1}^{(2)})$.*

Proof. Let $w_l : S \to \mathbb{N}$ be the weight function defined by $w_l(0) = l$ and $w_l(i) = 1$ for $1 \le i \le n-1$. Then it is direct that $C_n(\mathbb{Z}_n, w_l) \cong K_n^{(l)}$.

1. We note that

$$\det(I_n - A^t_{K_n^{(1)}}) = \prod_{l=0}^{n-1}(-1)(\sum_{j=1}^{n-1}\zeta^{lj}) = -(n-1) < 0.$$

Also we have $W - 1 = |S| - 1 = n - 1$. So $\gcd(W - 1, n) = 1$ and hence $\sum_{i=0}^{n-1}[v_i]$ is the identity in $M^*_{K_n^{(1)}}$. Also determinant divisors theorem yields that

$$\mathrm{SNF}(I_n - A^t_{K_n^{(1)}}) = \mathrm{diag}(1, 1, \dots, 1, n-1).$$

Hence, $K_0(L(K_n^{(1)})) \cong \mathbb{Z}_{n-1}$. By Proposition 2.2.3, the result follows.

2. We note that $(I_n - A^t_{K_n^{(2)}})$ is the $n \times n$ matrix with every entry -1. Hence $\det(I_n - A^t_{K_n^{(2)}}) = 0$ and $\mathrm{rank}(I_n - A^t_{K_n^{(2)}}) = 1$. Therefore if $n \geq 2$, then

$$K_0(L(K_n^{(2)})) \cong \mathbb{Z}^{n-1}.$$

Also in $K_0(L(K_n^{(2)}))$,

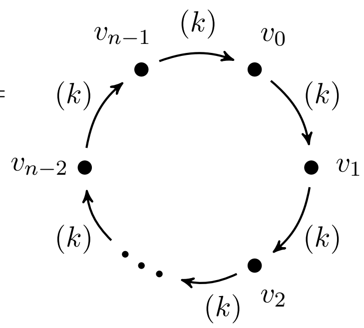
$$\sigma = \sum_{i=0}^{n-1}[v_i] = [v_0] + \sum_{i=1}^{n-1}[v_i] = \left(2[v_0] + \sum_{i=1}^{n-1}[v_i]\right) + \sum_{i=1}^{n-1}[v_i] = 2\sum_{i=0}^{n-1}[v_i] = 2\sigma.$$

Hence $\sum\limits_{i=0}^{n-1}[v_i]$ is the identity in $K_0(L(K_n^{(2)}))$. Applying Algebraic KP Theorem, we have the result. □

2.4.2 $|S| = 1$

Let $S = \{i\}$. Since $\langle S\rangle = \mathbb{Z}_n$, $\gcd(i,n) = 1$ and the weight function $w : S \to \mathbb{N}$ is given by $w(i) = W$. Let D_n^k be the graph with n vertices $v_0, v_1, \ldots, v_n$ and kn edges such that every vertex v_i emit k edges to v_{i+1}. We call D_n^k an k-cycle of length n.

$D_n^k =$



It is easy to see that $C_n(S,w) \cong D_n^W$.

The generating relations for $M^*_{D_n^W}$ are given by

$$[v_i] = W[v_{i+1}]$$

for $0 \leq i \leq n$, where the subscripts are interpreted mod n. So for each $0 \leq i \leq n$ we have that

$$[v_i] = W[v_{i+1}] = W^2[v_{i+2}] = \cdots = W^{n-i}[v_{n-1}] = W^{n+1-i}[v_0].$$

In particular, each $[v_i]$ is in the subgroup of $M^*_{D_n^W}$ generated by $[v_0]$. Since the set $\{[v_i] \mid 0 \leq i \leq n-1\}$ generates $M^*_{D_n^W}$, we conclude that $M^*_{D_n^W}$ is cyclic, and $[v_0]$ is a generator.

We also observe that

$$\det(I_n - A^t_{D_n^W}) = \prod_{l=0}^{n-1}(1 - W\zeta^l) = 1 - W^n < 0.$$

We conclude that $|K_0(L(D_n^W))| = W^n - 1$. Thus we have

$$K_0(L(C_n(S,w)) \cong M^*_{D_n^W} \cong \mathbb{Z}_{W^n-1}.$$

Proposition 2.4.3. *Let* $S = \{i\}, \gcd(i,n) = 1,$ *and* $\gcd(W-1,n) = 1$*. Then*

$$L(C_n(S,w)) \cong M_{W^n-1}(L(1,W^n)).$$

Proof. $\sum_{i=0}^{n-1}[v_i]$ is the identity in the group $M^*_{C_n(S,w)}$. Hence by Proposition 2.2.3 the result follows. □

Corollary 2.4.4. *([9], Proposition 3.4) Assume the hypothesis of Proposition 2.4.3 and* $W = 2$*. Then* $L(C_n(S,w)) \cong M_{2^n-1}(L(1,2^n))$

2.4.3 $|S| = 2$

Let $S = \{s_1, s_2\}$ with $s_1 < s_2$. Let $a, b \in \mathbb{N}$. We define $w(s_1) = a$ and $w(s_2) = b$. Thus, $W = a + b \geq 2$. Since $\langle S \rangle = \mathbb{Z}_n$, it is sufficient to consider only the following subcases:

1. $s_1 = 0$ and $s_2 = 1$.
2. $s_1 = 1$.
3. s_1 and s_2 divide n with $1 < s_1 < s_2$, and $\gcd(s_1, s_2) = 1$.

In what follows we consider these subcases separately.

Lemma 2.4.5. *In each of the above subcases if* $a = b = 1$*, then* $\sum\limits_{i=0}^{n-1}[v_i]$ *is the identity in* $M^*_{C_n(S,w)}$.

Proof. Since $W - 1 = |S| - 1 = 1$ in these subcases we have $\gcd(W-1,n) = 1$ and the result follows from Theorem 2.3.4. □

Proposition 2.4.6. *Let $n, a, b \in \mathbb{N}$ be fixed. Let $0 \le s_1 < s_2 \le n-1$. Consider the w-Cayley graph $C_n(S,w)$, where $S = \{s_1, s_2\}$ and $w(s_1) = a, w(s_2) = b$. Then $\det(I_n - A^t_{C_n(S,w)}) = 0$ if and only if exactly one of the following occurs:*

1. $a = b = 1, n \equiv 0 \pmod 6, s_2 \equiv 5s_1 \pmod 6$.
2. $a = b + 1$, *n is even, s_1 is even, s_2 is odd.*
3. $b = a + 1$, *n is even, s_1 is odd, s_2 is even.*

Proof. Let $\Delta = \det(I_n - A^t_{C_n(S,w)})$ and $z_l = a\zeta^{ls_1} + b\zeta^{ls_2}$. Since

$$\Delta = \prod_{l=0}^{n-1} \left(1 - a\zeta^{ls_1} - b\zeta^{ls_2}\right),$$

We see that $\Delta = 0$ if and only if $z_l = 1$ for some l. We observe that $z_0 = a + b > 1$ and $z_{n-l} = \overline{z_l}$. So we can write

$$\Delta = \begin{cases} (1 - z_0) \prod\limits_{l=1}^{\frac{n-1}{2}} (1 - z_l)(\overline{1 - z_l}), & \text{if } n \quad \text{is odd,} \\ (1 - z_0)(1 - a(-1)^{s_1} - b(-1)^{s_2}) \prod\limits_{l=1}^{\frac{n}{2}-1} (1 - z_l)(\overline{1 - z_l}), & \text{if } n \quad \text{is even.} \end{cases}$$

Hence we can assume $1 \le l \le [\frac{n}{2}]$, where $[\frac{n}{2}]$ is the integer part of $\frac{n}{2}$. Further, $z_l = 1$ implies $\left|a\zeta^{ls_1} + b\zeta^{ls_2}\right| = 1$. So $1 = \left|a\zeta^{ls_1} + b\zeta^{ls_2}\right| \ge ||a| - |b|| = |a - b| \ge 0$. Since $a, b \in \mathbb{N}$, only possiblities are $a = b, a = b + 1$, or $b = a + 1$.

Case 1: $a = b$

Let $\theta = \frac{2\pi l}{n}$. Then $z_l = 1$ if and only if

$$a\left(\cos s_1\theta + \cos s_2\theta\right) = 1 \quad \text{and} \quad a\left(\sin s_1\theta + \sin s_2\theta\right) = 0.$$

The second equation implies that $s_1\theta \equiv -s_2\theta \pmod{2}\pi$. Substituting back in first equation we get,

$$1 = a\left(\cos(-s_2\theta) + \cos s_2\theta\right) = 2a\cos s_2\theta$$

$$\Rightarrow \cos s_2\theta = \frac{1}{2a} \Rightarrow \frac{2\pi l}{n}s_2 = \arccos\left(\frac{1}{2a}\right).$$

Thus $n \in \mathbb{N}$ only if $a = 1$. Assuming $a = 1$, we have $\arccos\frac{1}{2} = \frac{\pi}{3}$ or $\frac{5\pi}{3}$. Substituting back, we see that

$$\frac{2\pi l s_2}{n} = \frac{\pi}{3} \Rightarrow n = 6s_2 l, \quad \text{or} \quad \frac{2\pi l s_2}{n} = \frac{5\pi}{3} \Rightarrow 5n = 6s_2 l.$$

In either case, $n \equiv 0 \pmod 6$. Also, $s_2\theta \equiv -s_1\theta \pmod 2\pi$ implies that for some integer m,

$$(s_2 + s_1)\frac{\pi}{3} = 2\pi m \Rightarrow s_2 + s_1 = 6m \quad \text{or} \quad (s_2 + s_1)\frac{5\pi}{3} = 2\pi m \Rightarrow 5(s_2 + s_1) = 6m.$$

In either case, $s_2 + s_1 \equiv 0 \pmod 6$, or $s_2 \equiv 5s_1 \pmod 6$.

Conversely, when $a = 1, n \equiv 0 \pmod 6$ and $s_2 \equiv 5s_1 \pmod 6$, then letting $l = 6$ implies that

$$z_l = \omega^{ls_1} + \omega^{ls_2} = \left(e^{\frac{2\pi i}{6}}\right)^{s_1} + \left(e^{\frac{2\pi i}{6}}\right)^{-s_1} = 1$$

Case 2: $a = b + 1$

As in case 1, let $\theta = \frac{2\pi l}{n}$. Then $z_l = 1$ if and only if

$$(b+1)\cos s_1\theta + b\cos s_2\theta = 1 \quad \text{and} \quad (b+1)\sin s_1\theta + b\sin s_2\theta = 0.$$

The second equation implies that $s_1\theta = \arcsin\left(\frac{-b}{b+1}\sin s_2\theta\right)$. Substituting back in the first equation we get,

$$(b+1)\cos\left(\arcsin\left(\frac{-b}{b+1}\sin s_2\theta\right)\right) + b\cos s_2\theta = 1.$$

Since $\cos(\arcsin x) = \sqrt{1-x^2}$, we have

$$(b+1)\sqrt{1 - \left(\frac{b}{b+1}\sin s_2\theta\right)^2} + b\cos s_2\theta = 1.$$

Hence,

$$\sqrt{b^2 + 2b + 1 - b^2\sin^2 s_2\theta} = 1 - b\cos s_2\theta.$$

Squaring both sides,

$$b^2\cos^2 s_2\theta + 2b + 1 = b^2\cos^2 s_2\theta - 2b\cos s_2\theta + 1 \Rightarrow \cos s_2\theta = -1.$$

Therefore, $s_2\theta \equiv \pi \pmod{2}\pi$. Substituting $\theta = \frac{2\pi l}{n}$, we see that n is even. Also, $s_2\theta \equiv \pi \pmod{2}\pi$ implies that $(s_2-1)\pi = 2\pi m$ for some integer m. So, $s_2 = 2m+1$ or s_2 is odd. Also since, $s_1\theta = \arcsin\left(\frac{-b}{b+1}\sin s_2\theta\right) = \arcsin(0)$, $s_1\pi = 0$ or π. $(b+1)\cos s_1\pi - b = 1 \Rightarrow s_1\pi \equiv 0 \pmod{2}\pi$. Hence s_1 is even.

Conversely, let n, s_1 be even and s_2 be odd then by taking $l = \frac{n}{2}$, we get

$$z_l = (b+)\omega_l^{s_1} + b\omega_l^{s_2} = (b+1)(-1)^{s_1} + b(-1)^{s_2} = b+1-b = 1.$$

Case 3: $b = a+1$

The proof is similar to that of case 2.

$\square$

Corollary 2.4.7. *Assume the hypothesis of Proposition 2.4.6. Further assume that $L(C_n(S,w))$ is unital purely infinite simple. Then $K_0(L(C_n(S,w)))$ is infinite abelian group if and only if one of the following holds:*

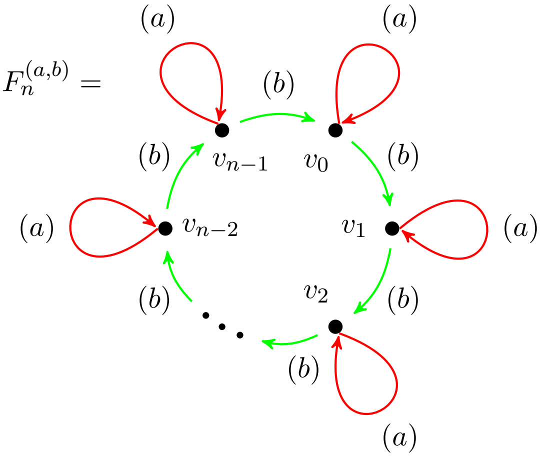
1. $a = b = 1, n \equiv 0 \pmod{6}, s_2 \equiv 5s_1 \pmod{6}$.
2. $a = b+1$, *n is even, s_1 is even, s_2 is odd.*
3. $b = a+1$, *n is even, s_1 is odd, s_2 is even.*

In which case $\operatorname{rank}(K_0(L(C_n(S,w)))) = n - \operatorname{rank}(I_n - A_{C_n(S,w)})$.

2.4.3.1 Subcase 2.1: $S = \{0,1\}$

Let $F_n^{(a,b)}$ be the graph with n vertices $v_0, v_1, \ldots, v_{n-1}$ and $ak+bk$ edges such that at every vertex v_l, there are a loops and b edges getting emitted into v_{l+1} (subscripts are mod n).

$F_n^{(a,b)} =$



Then $C_n(S, w) \cong F_n^{(a,b)}$ when $S = \{0, 1\}$. We note that

$$\det(I_n - A^t_{F_n^{(a,b)}}) = \prod_{l=0}^{n-1}(1 - a - b\zeta^l) = (1 - a)^n - b^n.$$

Lemma 2.4.8. *Let $n, a, b \in \mathbb{N}$. Then*

$$\det(I_n - A^t_{F_n^{(a,b)}}) \geq 0 \textit{ if and only if } n \textit{ is even and } a \geq b + 1.$$

Moreover, $\det(I_n - A^t_{F_n^{(a,b)}}) = 0$ *if and only if n is even and $a = b + 1$.*

Proof. We refer to the proof of Proposition 2.4.6. We need to substitute $s_1 = 0$, and $s_2 = 1$. Since,

$$\Delta = \begin{cases} (1 - z_0) \prod_{l=1}^{\frac{n-1}{2}} (1 - z_l)(\overline{1 - z_l}), & \text{if } n \quad \text{is odd,} \\ (1 - z_0)(1 - a(-1)^j - b(-1)^k) \prod_{l=1}^{\frac{n}{2}-1} (1 - z_l)(\overline{1 - z_l}), & \text{if } n \quad \text{is even,} \end{cases}$$

We see that $\Delta \geq 0$ if and only if n is even and $a \geq b+1$, in which case $1-z_{\frac{n}{2}} = 1-a+b \leq 0$. Also, it follows that, $\det(I_n - A^t_{F_n^{(a,b)}}) = 0$ if and only if n is even and $a = b + 1$.

$\square$

We describe the Smith Normal Form of $I_n - A^t_{F_n^{(a,b)}}$.

Lemma 2.4.9. *Suppose $n \in \mathbb{N}$. Let T be the $n \times n$ circulant matrix whose first row is $\vec{t} = ((1-a), -b, 0, \ldots, 0)$. Let $\gcd(1-a, b) = d$. Then the Smith Normal Form*

$$\mathrm{SNF}(T) = \mathrm{diag}\left(d, d, \ldots, d, \frac{|(1-a)^n - b^n|}{d^{n-1}}\right).$$

Proof. In order to compute Smith Normal Form of T, we use the determinant divisors theorem and look at $i \times i$ minors of T for each $1 \le i \le n$. Let α_i be the gcd of the set of all $i \times i$ minors of T and $\alpha_0 = 1$. Then

$$\mathrm{SNF}(T) = \mathrm{diag}\left(\frac{\alpha_1}{\alpha_0}, \frac{\alpha_2}{\alpha_1}, \ldots, \frac{|\det(T)|}{\alpha_{n-1}}\right).$$

By the definition of T, it is easy to observe that $\alpha_i = \gcd\left((a-1)^i, b^i\right) = \gcd(a,b)^i = d^i$ for $1 \le i \le n-1$ and $|\det(T)| = |(1-a)^n - b^n|$. Therefore

$$\begin{aligned}\mathrm{SNF}(T) &= \mathrm{diag}\left(\frac{d}{1}, \frac{d^2}{d}, \frac{d^3}{d}, \ldots, \frac{|(1-a)^n - b^n|}{d^{n-1}}\right)\\ &= \mathrm{diag}\left(d, d, d, \ldots, \frac{|(1-a)^n - b^n|}{d^{n-1}}\right)\end{aligned}$$

□

Theorem 2.4.10. *Let $n, a, b \in \mathbb{N}$ be fixed. Suppose $S = \{0, 1\} \subset \mathbb{Z}_n$, $w : S \to \mathbb{N}$ is defined by $w(0) = a$ and $w(1) = b$. Let $d = \gcd(a-1, b)$. Then*

$$K_0(L(C_n(S,w))) \cong \begin{cases} (\mathbb{Z}_d)^{n-1} \oplus \mathbb{Z}, & \textit{if } a = b+1 \quad \textit{and} \quad n \quad \textit{is even,} \\ (\mathbb{Z}_d)^{n-1} \oplus \mathbb{Z}_{\frac{|(1-a)^n - b^n|}{d^{n-1}}}, & \textit{otherwise.} \end{cases}$$

Proof. Follows from the above lemmas 2.4.8 and 2.4.9. □

Example 2.4.11. $L(C_n(0,1)) \cong L(1,2)$.

Proof. In Theorem 2.4.10 we take $a = b = 1$. Then $\gcd(a-1, b) = 1$. Hence $\det(I_n - A^t_{C_n(0,1)}) = -1 < 0$ and $K_0(L(C_n(0,1)))$ is trivial. By Proposition 2.2.3 we have $L(C_n(0,1)) \cong L(1,2)$. □

The above example was observed in [9], Proposition 3.3.

2.4.3.2 Subcase 2.2: $S = \{1, j\}$ with $j > 1$

We note that by Proposition 2.3.6 $\det(I_n - A^t_{C_n(S,w)}) > 0$ if and only if n, j are even and $b > a+1$. Also by Proposition 2.4.6 $\det(I_n - A^t_{C_n(S,w)}) = 0$ if and only if one of the following occurs:

1. $a = b = 1, n \equiv 0 \pmod 6, j \equiv 5 \pmod 6$
2. $b = a+1, n, j$ are even.

In order to compute $K_0(L(C_n(S,w)))$, we apply Theorem 2.3.9 and compute the Smith normal form of $T^n_{C_n(S,w)} - I_n$. This procedure is performed for unweighted Cayley graph in [11]. However, we record an interesting example here.

2.4.3.3 Leavitt Path algebras of Cayley graphs of Dihedral groups

Let $\tilde{D}_n$ be the dihedral group of order $2n$. i.e. $\tilde{D}_n = \langle r, s \mid r^n = s^2 = e, rsr = s \rangle$. Let D_n denote the Cayley graph of $\tilde{D}_n$ with respect to the generating subset $S = \{r, s\}$.

The following discussion is taken from [12]. A graph transformation is called standard if it is one of the following types: in-splitting, in-amalgamation, out-splitting, out-amalgamation, expansion, or contraction. For definitions the reader is referred to [6]. If E and F are graphs having no sources and no sinks, a flow equivalence from E to F is a sequence $E = E_0 \to E_1 \to \cdots \to E_n = F$ of graphs and standard graph transformations which starts at E and ends at F.

Proposition 2.4.12. *[6, Corollary 6.3.13] Suppose E and F are finite graphs with no sources whose corresponding Leavitt path algebras are purely infinite simple. Then E is flow equivalent to F if and only if* $\det(I_{|E|} - A_E) = \det(I_{|F|} - A_F)$ *and* $\mathrm{Coker}(I_{|E|} - A_E) \cong \mathrm{Coker}(I_{|F|} - A_F)$.

Definition 2.4.13 (In-splitting)**.** Let $E = (E^0, E^1, r, s)$ be a directed graph. For each $r^{-1}(v) \neq \phi$, partition the set $r^{-1}(v)$ into disjoint nonempty subsets $\mathcal{E}^v_1, \ldots, \mathcal{E}^v_{m(v)}$ where $m(v) \geq 1$. If v is a source then set $m(v) = 0$. Let $\mathcal{P}$ denote the resulting partition of E^1. We form the in-split graph $E_r(\mathcal{P})$ from E using the partition $\mathcal{P}$ as follows:

$$E_r(\mathcal{P})^0 = \{v_i \mid v \in E^0, 1 \leq i \leq m(v)\} \cup \{v \mid m(v) = 0\},$$
$$E_r(\mathcal{P})^1 = \{e_j \mid e \in E^1, 1 \leq j \leq m(s(e))\} \cup \{e \mid m(s(e)) = 0\},$$

and define $r_{E_r(\mathcal{P})}, s_{E_r(\mathcal{P})} : E_r(\mathcal{P})^1 \to E_r(\mathcal{P})^0$ by

$$s_{E_r(\mathcal{P})}(e_j) = s(e)_j \text{ and } s_{E_r(\mathcal{P})}(e) = s(e)$$
$$r_{E_r(\mathcal{P})}(e_j) = r(e)_i \text{ and } s_{E_r(\mathcal{P})}(e) = s(e)_i \text{ where } e \in \mathcal{E}_i^{r(e)}.$$

We observe that D_n can be obtained from C_n^{n-1} by the standard operation in-splitting with respect to the partition $\mathcal{P}$ of the edge set of C_n^{n-1} that places each edge in its own singleton partition class. In [15] the collection of Leavitt path algebras $\{L(C_n^{n-1}) \mid n \in \mathbb{N}\}$ is completely described and by Proposition 2.4.12 we have that the same description holds true if we replace C_n^{n-1} with D_n for every $n \in \mathbb{N}$. Hence we have

Theorem 2.4.14. *For each $n \in \mathbb{N}$, $\det(I_n - A^t_{D_n}) \leq 0$. And*

1. *If $n \equiv 1$ or $5 \pmod 6$ then $K_0(L(D_n)) \cong \{0\}$ and $L(D_n) \cong L(1, 2)$.*
2. *If $n \equiv 2$ or $4 \pmod 6$ then $K_0(L(D_n)) \cong \mathbb{Z}/3\mathbb{Z}$ and $L(D_n) \cong M_3(L(1, 4))$.*
3. *If $n \equiv 3 \pmod 6$ then $K_0(L(D_n)) \cong (\mathbb{Z}/2\mathbb{Z})^2$*
4. *If $n \equiv 0 \pmod 6$ then $K_0(L(D_n)) \cong \mathbb{Z}^2$ and $L(D_n) \cong L(K_3^{(2)})$*

2.4.3.4 Subcase 2.3: $S = \{s_1, s_2\}$ where s_1, s_2 divide n, $1 < s_1 < s_2$ and $\gcd(s_1, s_2) = 1$

By Proposition 2.4.6 and by Proposition 2.3.6, we have that $\det(I_n - A^t_{C_n(S,w)}) = 0$ if and only if one of the following occurs:

1. $a = b = 1, n \equiv 0 \pmod 6, d_2 \equiv 5d_1 \pmod 6$.
2. $a = b + 1$, n, d_1 are even, d_2 is odd.
3. $b = a + 1$, n, d_2 are even, d_1 is odd.

and $\det(I_n - A^t_{C_n(S,w)}) > 0$ if and only if one of the following occurs:

1. $a > b+1$, n, d_1 are even, d_2 is odd.

2. $b > a+1$, n, d_2 are even, d_1 is odd.

In order to compute $K_0(L(C_n(S,w)))$, we apply Theorem 2.3.9 and compute the Smith Normal form of $T^n_{C_n(S,w)} - I_n$.

We illustrate this when $S = \{d_1, d_2\}$, where d_1, d_2 divides n, $\gcd(d_1, d_2) = 1$ and $a = b = 1$. In this special case we have $\det(I_n - A^t_{C_n(d_1,d_2)}) = 0$ if and only if $n \equiv 0 \pmod 6$ and $d_2 \equiv 5d_1 \pmod 6$. In all other cases, we have $\det(I_n - A^t_{C_n^k}) < 0$. Define $H_{(d_1,d_2)}(n) := |\det(I_n - A^t_{C_n^k})|$. In order to compute $K_0(L(C_n(d_1,d_2)))$, we apply Theorem 2.3.9 and compute the Smith normal form of $T^n_{C_n(d_1,d_2)} - I_n$.

For $1 \leq j, k \in \mathbb{N}$ let us define a sequence $F_{(j,k)}$ recursively as follows:

$$F_{(j,k)}(n) = \begin{cases} 0, & \text{if} \quad 1 \leq n \leq k-2, \\ 1, & \text{if} \quad n = k-1, \\ 0, & \text{if} \quad n = k, \\ F_{(j,k)}(n-j) + F_{(j,k)}(n-k), & \text{if} \quad n \geq k+1. \end{cases}$$

In $M^*_{C_n(d_1,d_2)}$, we have

$$\begin{aligned} [v_0] &= [v_j] + [v_k] \\ &= [v_{2j}] + [v_k] + [v_{j+k}] \\ &= [v_{3j}] + [v_k] + [v_{j+k}] + [v_{2j+k}] \\ &= \dots \end{aligned}$$

The coefficients appearing in the above equations are terms in the sequence $F_{(d_1,d_2)}$ and corresponding $T_{C_n(d_1,d_2)}$ is given by the following:

Lemma 2.4.15. *For fixed d_1, d_2, let $d_2 - d_1 = k$. Let $T = T_{C_n(d_1,d_2)}$. Suppose $G(n) := F_{(d_1,d_2)}(n)$ is the sequence defined above. Then for each $n \in \mathbb{N}$,*

$$T^n = \begin{pmatrix} G(n-1) & G(n) & \dots & G(n+d_2-2) \\ G(n-2) & G(n-1) & \dots & G(n+d_2-3) \\ \vdots & \vdots & \dots & \vdots \\ G(n+d_1-1) & G(n+d_1) & \dots & G(n+d_2+d_1-2) \\ \vdots & \vdots & \dots & \vdots \\ G(n) & G(n+1) & \dots & G(n+d_2-1) \end{pmatrix}$$

where the highlighted row is $(k+1)^{th}$ row.

Proof. We prove the lemma by induction on n. We extend the definition of G to the negative integers as well. Then,

$$T = \begin{pmatrix} 0 & 0 & \dots & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 & 1 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \dots & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} G(0) & G(1) & \dots & G(k-1) & \dots & G(d_2-2) & G(d_2-1) \\ G(-1) & G(0) & \dots & G(k-2) & \dots & G(d_2-3) & G(d_2-2) \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ G(d_1) & G(d_1+1) & \dots & G(d_2-1) & \dots & G(d_2+d_1-2) & G(d_2+d_1-1) \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ G(2) & G(3) & \dots & G(k+1) & \dots & G(d_2) & G(d_2+1) \\ G(1) & G(2) & \dots & G(k) & \dots & G(d_2-1) & G(d_2) \end{pmatrix}$$

where highlighted column is k^{th} column.

Thus we have the statement true for $n = 1$. Now suppose

$$T^{n-1} = \begin{pmatrix} G(n-2) & G(n-1) & \dots & G(n+d_2-3) \\ G(n-3) & G(n-2) & \dots & G(n+d_2-4) \\ \vdots & \vdots & \dots & \vdots \\ G(n+d_1-2) & G(n+d_1-1) & \dots & G(n+d_2+d_1-3) \\ \vdots & \vdots & \dots & \vdots \\ G(n-1) & G(n) & \dots & G(n+d_2-2) \end{pmatrix}$$

Then,

$$T^n = T^{n-1}T$$

$$= \begin{pmatrix} G(n-2) & \dots & G(n+k-2) & \dots & G(n+d_2-3) \\ G(n-3) & \dots & G(n+k-3) & \dots & G(n+d_2-4) \\ \vdots & & \vdots & & \vdots \\ G(n+d_1-2) & \dots & G(n+d_2-2) & \dots & G(n+d_2+d_1-3) \\ \vdots & & \vdots & & \vdots \\ G(n-1) & \dots & G(n+k-2) & \dots & G(n+d_2-2) \end{pmatrix} \begin{pmatrix} 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} G(n-1) & G(n) & \dots & G(n-2)+G(n+k-2) \\ G(n-2) & G(n-1) & \dots & G(n-3)+G(n+k-3) \\ \vdots & \vdots & \dots & \vdots \\ G(n+d_1-1) & G(n+d_1) & \dots & G(n+d_1-2)+G(n+d_2-2) \\ \vdots & \vdots & \dots & \vdots \\ G(n) & G(n+1) & \dots & G(n-1)+G(n+k-2) \end{pmatrix}$$

$$= \begin{pmatrix} G(n-1) & G(n) & \dots & G(n+d_2-2) \\ G(n-2) & G(n-1) & \dots & G(n+d_2-3) \\ \vdots & \vdots & \dots & \vdots \\ G(n+d_1-1) & G(n+d_1) & \dots & G(n+d_2+d_1-2) \\ \vdots & \vdots & \dots & \vdots \\ G(n) & G(n+1) & \dots & G(n+d_2-1) \end{pmatrix}$$

$\square$

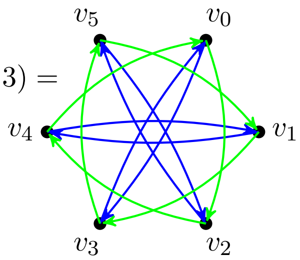
Using the determinant divisors theorem, the Smith normal form of $T^n_{C_n(d_1,d_2)} - I_k$ can be reduced to

$$SNF(T^n_{C_n(d_1,d_2)} - I_k) = \begin{pmatrix} \alpha_1(n) & & & \\ & \frac{\alpha_2(n)}{\alpha_1(n)} & & \\ & & \ddots & \\ & & & \frac{\alpha_{d_2}(n)}{\alpha_{d_2-1}(n)} \end{pmatrix}$$

where α_i is the greatest common divisor of the set of all $i \times i$ minors of $T^n_{C_n(d_1,d_2)}$.

Example 2.4.16. Let $n = 6, d_1 = 2, d_2 = 3$. The corresponding Cayley graph is

$C_6(2,3) =$



The corresponding companion matrix is given by

$$T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and,

$$T^6 - I_3 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 2 & 1 \end{pmatrix}$$

whose Smith normal form is given by

$$SNF(T^6 - I_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Hence, $K_0(L(C_6(2,3))) \cong \mathbb{Z}_7$ and $L(C_6(2,3)) \cong L(1,8)$.

Chapter 3

Cohn-Leavitt path algebras of bi-separated graphs

3.1 Various generalizations of Leavitt path algebras

In this section, we introduce various generalizations of Leavitt path algebras such as weighted Leavitt path algebras of weighted graphs, Cohn-Leavitt path algebras of separated graphs and Leavitt path algebras of hypergraphs.

3.1.1 Weighted Leavitt path algebras

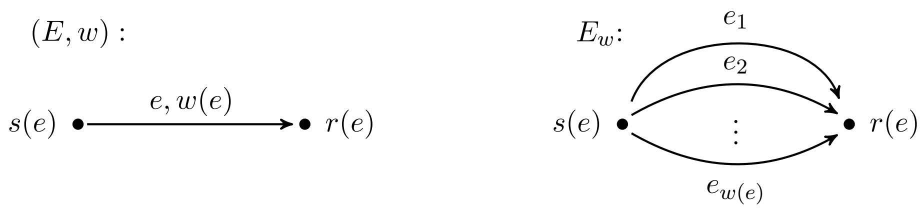
Leavitt algebras of module type (m,n) for any $m,n \in \mathbb{N}$ with $1 < m < n$ are not examples of Leavitt path algebras of any graphs. This is because the former is a domain [28, Theorem 6.1], whereas, the latter is a domain if and only if the graph is either a single vertex or a single loop. In [37], Hazrat introduced the concept of weighted graphs and the associated weighted Leavitt path algebras to be able to express any Leavitt algebras as examples of graph algebras.

Definition 3.1.1. Let E be a row-finite graph and $w : E^1 \to \mathbb{N}$ be a (weight) function. The graph $E_w = (E^0, E^1_w, r_{E_w}, s_{E_w})$ where $(E_w)^1 := \{e_1, \ldots, e_{w(e)} \mid e \in E^1\}, r_{E_w}(e_i) = r(e)$ and $s_{E_w}(e_i) = r(e)$ for each $e_i \in (E_w)^1$, is called the **weighted graph** of E with respect to w.

In other words, E^w is obtained from (E, w) by replacing an edge $e \in E^1$ which has weight $w(e)$ with $w(e)$ number of edges $e_1, \ldots, e_{w(e)}$ from $s(e)$ to $r(e)$.

For each $e \in E^1$, $w(e)$ is called the weight of e. Define $w(v) = \max\{w(e) \mid e \in s^{-1}(v)\}$. The graph (E, w) is **vertex weighted** if $w(e) = w(v)$ for every $e \in s^{-1}(v)$.

We represent the weighted graph E_w visually as follows:



Definition 3.1.2. Let E be a row-finite graph and $w : E^1 \to \mathbb{N}$ be a weight function. The **weighted Leavitt path algebra** $L_K^w(E, w)$ of the weighted graph E_w is the quotient of $K(\widehat{E_w})$ modulo the following relations:

$$\forall e, f \in E^1 \text{ if } s(e) = s(f) = v \in E^0 \text{ then } \sum_{1 \leq i \leq w(v)} e_i^* f_i = \delta_{ef} r(e) \tag{wCK1}$$

$$\forall v \in E^0 \text{ and } 1 \leq i, j \leq w(v) \quad \sum_{e \in s^{-1}(v)} e_i e_j^* = \delta_{ij} v \tag{wCK2}$$

where we set e_i and e_i^* to be zero whenever $i > w(e)$.

First of all, note that $L_K^w(E, w)$ is not the same as $L_K(E_w)$. Let R_n be a *rose with n petals* and let w_m be a weight function on R_n such that $w_m(e) = m \in \mathbb{N}$ for every edge e. Then it is direct that $L_K^w(R_n, w_m) \cong L_K(m, n)$. In [38], the normal forms of weighted Leavitt path algebras were computed. As an application, it was shown that weighted Leavitt path algebra is simple if and only if it is isomorphic to a simple Leavitt path algebra of a unweighted graph. The authors also defined the notion of local valuation and characterized weighted Leavitt path algebras which are domains in terms of underlying graphs and weight functions.

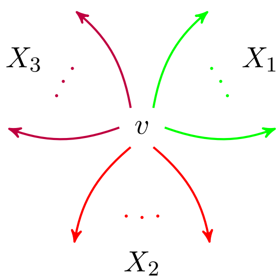
Preusser took up the study of weighted Leavitt path algebras in a series of articles. In [60], weighted Leavitt path algebras of finite Gelfand-Kirillov dimensions were studied and as an application it was shown that weighted Leavitt path algebras are finite dimensional if and only if they contains no *quasi-cycles*. In [59], locally finite weighted Leavitt path algebras were shown to be Noetherian and also that $L_K^w(E, w)$ being locally finite is equivalent to their GK dimension being either 0 or 1. In [62], the $\mathcal{V}$-monoid of weighted Leavitt path algebras were computed using Bergman's machinery. In [63], weighted Leavitt path algebras which are isomorphic to Leavitt path algebras were characterized. In particular, it was shown that if a weighted Leavitt path algebra satisfies finiteness properties (such as locally finite or Noetheiran or Artinian or finite GK dimension) or regularity then it is isomorphic to a Leavitt path algebra of an undirected graph.

3.1.2 Cohn-Leavitt path algebras of separated graphs

As mentioned previously, the notion of separated graphs arise in the study of realization problem of von Neumann regular rings (see 1.1.19).

Definition 3.1.3. A **separated graph** is a pair (E, C), where E is a graph and $C = \bigsqcup_{v\in E^0} C_v$, where C_v is a partition of $s^{-1}(v)$, for each vertex $v \in E^0$.

We note that in case $v \in E^0$ is a sink, C_v can be taken to be empty family of subsets of $s^{-1}(v)$. If all the sets in C are finite, we say that (E, C) is **finitely separated**. In case $C_v = \{s^{-1}(v)\}$ for each non-sink $v \in E^0$, we say (E, C) is **trivially separated**. We also define $C_{\text{fin}} = \{X \in C \mid |X| < \infty\}$. In the visual representation of (E, C) we use different colors for each element of C. For example in the following graph we have $C_v = \{X_1, X_2, X_3\}$ which are represented by colors green, red and purple.



Definition 3.1.4. Let K be a field, (E, C) be a separated graph and $S \subseteq C_{\text{fin}}$. Then the **Cohn-Leavitt path algebra** $CL_K(E, C, S)$ **of** (E, C) **relative to** S is defined as the quotient of $K(\widehat{E})$ obtained by imposing the following relations:

$$\forall e, f \in X,\ X \in C \quad e^*f = \delta_{ef}r(e), \tag{SCK1}$$

$$\forall X \in S,\ v \in E^0 \quad v = \sum_{e\in X} ee^*. \tag{SCK2}$$

The Cohn-Leavitt path algebra is called Leavitt path algebra if $S = C_{\text{fin}}$ and in this case, we simply write $L_K(E,C)$ in place of $CL_K(E,C,S)$. It is direct that if (E,C) is trivially separated and $S = C_{\text{fin}}$ then $L_K(E,C)$ is the usual Leavitt path algebra $L_K(E)$. The following remarkable theorem was also established:

Theorem 3.1.5. *[20, Proposition 4.4] For any conical abelian monoid M, there exists a finitely separated graph (E,C) such that $M \cong \mathcal{V}(L_K(E,C))$.*

We list a few important examples.

Example 3.1.6. For any $m, n \in \mathcal{N}$ consider the separated graph $(E(m,n), C(m,n))$, where

1. $E(m,n)^0 := \{v, w\}$,
2. $E(m,n)^1 := \{e_1, \dots, e_n, f_1, \dots, f_m\}$,
3. $s(e_i) = s(f_j) = v$ and $r(e_i) = r(f_j) = w$ for all i, j and
4. $C(m,n) := \{\{e_1, \dots, e_n\}, \{f_1, \dots, f_m\}\}$.

Let $A_{m,n} := L_K(E(m,n), C(m,n))$. Then we have by [20, Proposition 2.12] that $A_{m,n} \cong M_{n+1}(L_K(m,n)) \cong M_{m+1}(L_K(m,n))$, $vA_{m,n}v \cong M_m(L_K(m,n)) \cong M_n(L_K(m,n))$ and $wA_{m,n}w \cong L_K(m,n)$. As the authors mentioned there, $A_{m,n}$ are Leavitt path algebra versions of $L_K(m,n)$ which are generated by 'partial isometries'.

Example 3.1.7. [20, Proposition 2.10] Let (E,C) be a separated graph with $|E^0| = 1$. Then $L_K(E,C)$ is the free product of Leavitt path algebras of type $(1, |X|)$, for $X \in C$.

In particular, for any set A the K-algebra of the free group $F(A)$ on A is an example of Leavitt path algebra of a separated graph (Take $X \in C$ to be singletons $\{a\}$ for each $a \in A$).

Later in [16], the C^*-algebra analogs of Leavitt path algebras of separated graphs were considered and in particular, K-theory of these algebras were computed and proved a conjecture posed in [52].

3.1.3 Leavitt path algebras of hypergraphs

Recently in [61], Preusser initiated a unified approach to study Leavitt path algebras of separated graphs and weighted Leavitt path algebras of vertex weighted graphs by introducing the notion of hypergraphs.

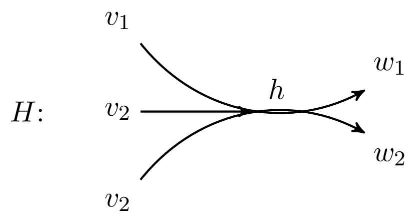
Definition 3.1.8 (**Leavitt path algebras of Hypergraphs**). Let I and X be sets. Recall that a function $x : I \to X$, given by $i \mapsto x_i = x(i)$ is called a family of elements in X indexed by I. We denote a family x of elements in X indexed by I by $(x_i)_{i\in I}$.

A **hypergraph** is a quadruple $\mathcal{H} = (\mathcal{H}^0, \mathcal{H}^1, s, r)$ where $\mathcal{H}^0$ and $\mathcal{H}^1$ are sets called the set of vertices and the set of hyperedges respectively. For each $h \in \mathcal{H}^1$ there exists a pair of non-empty indexing sets I_h, J_h such that $s(h) : I_h \to \mathcal{H}^0$, and $r(h) : J_h \to \mathcal{H}^0$ are families of vertices.

Let $\mathcal{H}$ be a hypergraph. A hyperedge $h \in \mathcal{H}^1$ is called **source regular** (resp. **range regular**) if I_h is finite (resp. J_h is finite). The set of all source regular hyperedges of $\mathcal{H}$ is denoted by $\mathcal{H}^1_{\text{sreg}}$ and the set of all range regular hyperedges of $\mathcal{H}$ is denoted by $\mathcal{H}^1_{\text{rreg}}$. The hypergraph $\mathcal{H}$ is said to be **regular** if $\mathcal{H}^1 = \mathcal{H}^1_{\text{sreg}} = \mathcal{H}^1_{\text{sreg}}$.

Visualization become a little tricky in the case of hypergraphs. We give a simple example below to illustrate the last statement.

Example 3.1.9. Consider the hypergraph $H = (H^0, H^1, r, s)$ where $H^0 = \{v_1, v_2, w_1, w_2\}$, $H^1 = \{h\}$, $s(h) = \{v_1, 2 \cdot v_2\}$ and $r(h) = \{w_1, w_2\}$. We might visualize H as follows:



The **Leavitt path algebra** $L_K(\mathcal{H})$ **of the hypergraph** $\mathcal{H}$ is the K-algebra presented by the generating set $\{v, h_{ij}, h^*_{ij} \mid v \in \mathcal{H}^0, h \in \mathcal{H}^1, i \in I_h, j \in J_h\}$ and the relations

1. $uv = \delta_{u,v}u$, for every $u, v \in \mathcal{H}^0$,

2. $s(h)_i h_{ij} = h_{ij} = h_{ij} r(h)_j$ and $r(h)_j h^*_{ij} = h^*_{ij} = h^*_{ij} s(h)_i$, for every $h \in \mathcal{H}^1$, $i \in I_h$, and $j \in J_h$,

3. $\sum\limits_{j \in J_h} h_{ij} h^*_{kj} = \delta_{ik} s(h)_i$, for every $h \in \mathcal{H}^1_{\text{rreg}}$ and $i, k \in I_h$,

4. $\sum\limits_{i \in I_h} h^*_{ij} h_{ik} = \delta_{jk} r(h)_j$, for every $h \in \mathcal{H}^1_{\text{sreg}}$ and $j, k \in J_h$.

Preusser investigated Leavitt path algebras of hypergraphs in terms of linear bases, Gelfand-Kirillov dimension, ring theoretic properties such as simplicity, von Neumann regularity and Noetherianess, and non-stable (graded) K-theory.

Remark 3.1.10. Let L denote any one of the K-algebras appearing in the subsections 3.1.1, 3.1.2 and 3.1.3. Then note that L satisties the following properties.

1. The algebra L is unital if and only if the set of vertices V in the underlying graph is finite. In this case, the unit is the sum of vertices. In general (L, V) is a K-algebra with enough idempotents.

2. If $\overline{} : K \to K$ is an involution on the field K, then L is a $*$-algebra (with respect to the involution $* : L \to L$).

3. L is a graded quotient algebra of $K(\widehat{E})$ with respect to standard $\mathbb{Z}$-grading given by length of paths.

In the following sections we provide a common framework for studying various generalizations of Leavitt path algebras. We first define Cohn-Leavitt path algebras of graphs with an additional structure called bi-separated graphs. We then define and study the category **BSG** of bi-separated graphs with appropriate morphisms so that the functor which associates bi-separated graphs to their Cohn-Leavitt path algebras is continuous. Next, we define two sub-categories of **BSG**, compute basis for the algebras corresponding to one of those subcategories and study some algebraic properties in terms of bi-separated graph-theoretic properties.

3.2 The algebras $\mathcal{A}_K(\dot{E})$

In this section, we introduce the notions of bi-separated graphs and their Cohn-Leavitt path algebras. The aim is to provide a unified framework for studying various generalizations introduced in the previous section.

Definition 3.2.1. A **bi-separated graph** is a triple $\dot{E} = (E, C, D)$ such that

1. $E = (E^0, E^1, r, s)$ is a graph,
2. $C = \bigsqcup_{v \in E^0} C_v$, where C_v is a partition of $s^{-1}(v)$ for every non-sink $v \in E^0$,
3. $D = \bigsqcup_{v \in E^0} D_v$, where D_v is a partition of $r^{-1}(v)$ for every non-source $v \in E^0$,
4. $|X \cap Y| \leq 1$, for every $X \in C$ and $Y \in D$.

In the above definition, C is called row-separation of E, D is called column-separation of E and (C, D) is called bi-separation of E. The elements of C are called rows and the elements of D are called columns. Let $C_{\text{fin}} := \{X \in C \mid |X| < \infty\}$ and $D_{\text{fin}} := \{Y \in D \mid |Y| < \infty\}$. A bi-separated graph $\dot{E}$ is called **finitely row-separated** (resp. **finitely column-separated**) if $C = C_{\text{fin}}$ (resp. if $D = D_{\text{fin}}$) and is called **finitely bi-separated** if both $C = C_{\text{fin}}$ and $D = D_{\text{fin}}$.

In the above definition we follow the convention that if S is a set, by a partition P of S we mean a family of pairwise disjoint nonempty subsets of A, whose union is S. For any non-empty set S there always exist two trivial partitions: the partition P_1 on S called the **discrete partition**, if each element of P_1 is singleton and the partition P_S called **full partition**, if S is the only element of P_S.

Example 3.2.2. (**Standard bi-separation of a simple graph**) Let E be a *simple* graph. That is, if $e, f \in E^1$ such that $s(e) = s(f)$ and $r(e) = r(f)$ then $e = f$. (In other words, there are no multiedges allowed between any two vertices). We can obtain a canonical bi-separation on E by considering both C_v and D_v to be full partitions. In other words, $C_v = \{s^{-1}(v)\}$ for every non-sink $v \in E^0$ and $D_v = \{r^{-1}(v)\}$ for every non-source $v \in E^0$. This bi-separation is called standard.

In the following examples E denotes an arbitrary graph.

Example 3.2.3. (**Trivial bi-separation of a graph**) By trivial bi-separation on a graph E, we mean both C_v and D_v are discrete partitions. i.e. $C_v = \{\{e\} \mid e \in s^{-1}(v)\}$ for every non-sink $v \in E^0$ and $D_v = \{\{e\} \mid e \in r^{-1}(v)\}$ for every non-source $v \in E^0$.

Example 3.2.4. (Cuntz-Krieger bi-separation of a graph) We can obtain another canonical bi-separation on E by combining full row-separation and discrete column-separation on E as follows: Consider $C_v = \{s^{-1}(v)\}$ for every non-sink $v \in E^0$ and $D_v = \{\{e\} \mid e \in r^{-1}(v)\}$ for every non-source $v \in E^0$.

Example 3.2.5. (Separated graphs) A bi-separated graph (E, C, D) in which the column-separation is discrete is called a **row-separated graph** or simply **separated graph** (cf. Definition 3.1.4). Separated graphs are denoted by (E, C).

Example 3.2.6. (Weighted graphs) Let E be a row-finite graph and $w : E^1 \to \mathbb{N}$ be a weight map on E. Consider the weighted graph $E_w = (E_w^0, E_w^1, r_w, s_w)$. We associate a bi-separation on E_w as follows: For every $v \in \mathrm{RReg}(E)$ and $1 \le i \le w(v)$ define $X_v^i := \{e_i \mid e \in s^{-1}(v), w(e) \ge i\}$. For every $e \in E^1$ define $Y^e := \{e_i \mid 1 \le i \le w(e)\}$. Now consider $C_v := \{X_v^i \mid 1 \le i \le w(v)\}$ and $D_v := \{Y^e \mid e \in r^{-1}(v)\}$. Here $C = C_{\mathrm{fin}}$, since E is row-finite and $D = D_{\mathrm{fin}}$, since w takes natural numbers as values.

Example 3.2.7. (**Hypergraphs**) We show that any hypergraph $\mathcal{H}$ can be associated to a bi-separated graph $\dot{E} = (E, C, D)$ as follows: Define $E = (\mathcal{H}^0, E^1, s', r')$, where $E^1 = \{h_{ij} \mid h \in \mathcal{H}^1, i \in I_h, j \in J_h\}$, $s'(h_{ij}) = s(h)_i$ and $r'(h_{ij}) = r(h)_j$. For an arbitrary $h \in \mathcal{H}^1$, if $i \in I_h$ then $X_h^i := \{h_{ij} \mid j \in J_h\}$ and if $j \in J_h$ then $Y_h^j := \{h_{ij} \mid i \in I_h\}$. For $v \in E^0$, define $C_v = \{X_h^i \mid h \in \mathcal{H}^1, i \in I_h, v = s(h)_i\}$ and $D_v = \{Y_h^j \mid h \in \mathcal{H}^1, j \in J_h, v = r(h)_j\}$. By construction, $C = C_{\mathrm{fin}}$ and $D = D_{\mathrm{fin}}$.

Notation 3.2.8. Given a bi-separated graph $\dot{E} = (E, C, D)$, the maps s and r can be extended to C and D respectively in well-defined manner as follows: For $X \in C$, define $s(X) := s(e)$ where $e \in X$ and for $Y \in D$, define $r(Y) := r(e)$ where $e \in Y$.

Also, for each $X \in C$ and $Y \in D$ we set

$$XY = YX = \begin{cases} e, & \text{if} \quad X \cap Y = \{e\}, \\ 0, & \text{otherwise.} \end{cases}$$

We interchangeably use XY and $X \cap Y$, wherever there is no cause for confusion.

Definition 3.2.9. Let $\dot{E} = (E, C, D)$ be a bi-separated graph. The **Leavitt path algebra of** $\dot{E}$ with coefficients over K, denoted by $L_K(\dot{E})$, is the quotient of $K(\widehat{E})$ obtained by imposing the following relations:

*L*1: for every $X, X' \in C_{\mathrm{fin}}$,

$$\sum_{Y \in D} (XY)(YX')^* = \delta_{XX'} s(X),$$

$L2$: for every $Y, Y' \in D_{\text{fin}}$,

$$\sum_{X \in C} (YX)^*(XY') = \delta_{YY'} r(Y),$$

Example 3.2.10. (**Leavitt path algebra of a standard bi-separated simple graph**)

Let E be a simple graph and consider the standard bi-separation (C, D) on E. Let the set of all non-sinks of E be denoted by E_α and the set of all non-sources be denoted by E_ω. Then $|C| = |E_\alpha|$ and $|D| = |E_\omega|$. Recall that a vertex $v \in E^0$ is called row-regular if $0 < |s^{-1}(v)| < \infty$ and $w \in E^0$ is called column-regular if $0 < |r^{-1}(w)| < \infty$. The set of all row-regular vertices is denoted by $\mathrm{RReg}(E)$ and the set of all column regular vertices is denoted by $\mathrm{CReg}(E)$. Note that $|C_{\text{fin}}| = |\mathrm{RReg}(E)|$ and $|D_{\text{fin}}| = |\mathrm{CReg}(E)|$.

Let A be a $|C_{\text{fin}}| \times |D|$ matrix over $K(\widehat{E})$ with entries

$$A(v, w) = \begin{cases} e, & \text{if} \quad e \in E^1 \text{ such that } s(e) = v, r(e) = w, \\ 0, & \text{otherwise}, \end{cases}$$

where $v \in \mathrm{RReg}(E)$ and $w \in E_\omega$. Let A^* denote the 'adjoint transpose' of A. Similiarly, let B be a $|C| \times |D_{\text{fin}}|$ matrix over $K(\widehat{E})$ with entries

$$B(v, w) = \begin{cases} e, & \text{if} \quad e \in E^1 \text{ such that } s(e) = v, r(e) = w, \\ 0, & \text{otherwise}, \end{cases}$$

where $v \in E_\alpha$ and $w \in \mathrm{CReg}(E)$. Let B^* denote the 'adjoint transpose' of B.

Then the defining relations L1 and L2 of Leavitt path algebras are obtained by imposing the following matrix relations:

$$\begin{aligned} (L1): \quad AA^* &= V, \\ (L2): \quad B^*B &= U. \end{aligned}$$

where V is the $|\mathrm{RReg}(E)| \times |\mathrm{RReg}(E)|$ diagonal matrix with diagonal entries $V(v, v) = s(v)$ and and U is the $|\mathrm{CReg}(E)| \times |\mathrm{CReg}(E)|$ diagonal matrix with diagonal entries $U(w, w) = r(w)$.

In particular, if E is finite simple graph, then the matrix A (resp. the matrix B) is obtained from the adjacency matrix of E by removing the zero rows (resp. zero columns) and replacing 1's with corresponding edges. We illustrate this with a few examples below.

(1) For $n \geq 1$, let Σ_n be the following line graph with n vertices and $n-1$ edges:

$$\Sigma_n = \quad \underset{v_1}{\bullet} \xrightarrow{e_1} \underset{v_2}{\bullet} \xrightarrow{e_2} \underset{v_3}{\bullet} \xrightarrow{e_3} \quad \cdots \quad \underset{v_{n-1}}{\bullet} \xrightarrow{e_{n-1}} \underset{v_n}{\bullet}$$

Then

$$A = \begin{pmatrix} e_1 & 0 & \dots & 0 \\ 0 & e_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & e_{n-1} \end{pmatrix},$$

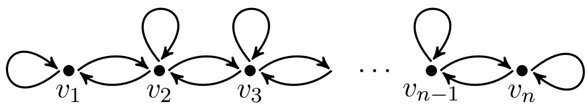
and the relations obtained are

$$e_i e_i^* = v_i \quad \text{and} \quad e_i^* e_i = v_{i+1},$$

where $1 \leq i \leq n-1$. In this case, it is easy to see that

$$L_K(\dot{\Sigma}_n) \cong L_K(\Sigma_n) \cong M_n(K).$$

(2) For $n \geq 3$, let Γ_n denote the following graph with n vertices:

$\Gamma_n =$ 

Then the adjacency matrix of Γ_n has at least two entries and at most three entries in both rows and columns, i.e.,

$$A = \begin{pmatrix} * & * & 0 & 0 & 0 & \dots & 0 & 0 \\ * & * & * & 0 & 0 & \dots & 0 & 0 \\ 0 & * & * & * & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & * & * \end{pmatrix},$$

where $*$'s are nonzero entries filled by the corresponding edges. In this case, though, the explicit description of the Leavitt path algebra $L_K(\dot{\Gamma_n})$ is not known.

Example 3.2.11. (**Groupoid algebra of a free groupoid**) Let E be a graph and consider the trivial bi-separation (C, D) on E. The defining relations of Leavitt path algebra of (E, C, D) turns the free path category of $\widehat{E}$ into a free groupoid and hence $L_K(E, C, D)$ is the groupoid algebra of this free groupoid. In particular if E has only one vertex then with respect to trivial bi-separation, the Leavitt path algebra is the group algebra of the free group with generators as elements of E^1 (Here we identified the vertex with the group identity).

Example 3.2.12. (**Leavitt path algebra of a graph**) Let E be any graph and $\dot{E}$ be the associated bi-separated graph with respect to Cuntz-Kreiger bi-separation on E. Then we have $L_K(\dot{E}) \cong L_K(E)$.

Example 3.2.13. (**Leavitt path algebra of a separated graph**) Let $\dot{E} = (E, C)$ be a (row) separated graph. Then it is direct that $L_K(\dot{E}) \cong L_K(E, C)$.

Example 3.2.14. (**Weighted Leavitt path algebra of a weighted graph**) Let E be a row-finite graph and $w : E^1 \to \mathbb{N}$ be a weight map. Consider $\dot{E} = (E, C, D)$ where (C, D) is the weighted bi-separation on E as in example 3.2.6. Then it is immediate that $L_K(\dot{E}) \cong WL_K(E_w)$.

It has been noted in [20, page 171] that neither weighted Leavitt path algebras nor Leavitt path algebras of separated graphs are particular cases of the each other. One can mix the above two examples and construct new algebras as follows:

Example 3.2.15 (**Weighted Cohn-Leavitt path algebras of finitely separated graphs**)**.** Let (E, C) be a finitely row-separated graph (i.e. a separated graph in which $C = C_{\text{fin}}$). Let $w : E^1 \to \mathbb{N}$ be a function and E_w be the associated weighted graph. For $X \in C$, set $w(X) = \max\{w(e) \mid e \in X\}$. The weighted Cohn-Leavitt path algebra $CL_K(E_w, C)$ of (E_w, C) can be defined as the quotient of $P_K(\widehat{E}_w)$ by factoring out the following relations:

wSCK1: $\sum\limits_{1 \le i \le w(X)} e_i^* f_i = \delta_{e,f} r(e)$, for every $e, f \in X$ and $X \in C$,

wSCK2: $\sum\limits_{e \in X} e_i e_j^* = \delta_{i,j} s(e)$, for each $X \in C, 1 \le i, j \le w(X)$, where we set e_i and e_i^* to be zero whenever $i > w(e)$.

Given a weighted finitely separated graph (E_w, C), we get a canonical bi-separated graph as follows: For $X \in C$ and $1 \leq i \leq w(X)$, define $\widetilde{X}^i = \{e_i \mid e \in X\}$ and set $\widetilde{C}_v = \{\widetilde{X}^i \mid X \in C_v \text{ and } 1 \leq i \leq w(X)\}$. Here $\widetilde{C} = \widetilde{C}_{\text{fin}}$, since E is finitely separated. Now, for $e \in X$, define $\widetilde{Y}_X^e = \{e_i \mid 1 \leq i \leq w(e)\}$ and $\widetilde{D}_{vw} = \{\widetilde{Y}_X^e \mid e \in X\}$. Observe that $\widetilde{D} = \widetilde{D}_{\text{fin}}$, since w is natural number valued. Now setting $\dot{E} = (E, \widetilde{C}, \widetilde{D})$, we immediately get

$$L_K(E, \widetilde{C}, \widetilde{D}) \cong L_K(E_w, C).$$

Example 3.2.16 (**Leavitt path algebra of a hypergraph**). Given any hypergraph $\mathcal{H}$, consider the associated bi-separated graph $\dot{E_H}$ as in example 3.2.7. Then we have $L_K(\dot{E}) \cong L_K(\mathcal{H})$.

Definition 3.2.17. Let $\dot{E} = (E, C, D)$ be bi-separated graph. Let $S \subseteq C_{\text{fin}}$ and $T \subseteq D_{\text{fin}}$ be two distinguished sets. The **Cohn-Leavitt path algebra of** $\dot{E}$ with coefficients over K relative to (S, T), denoted by $\mathcal{A}_K(E, (C, S), (D, T))$, is the quotient of $K(\widehat{E})$ obtained by imposing the following relations:

$\mathcal{A}1$: for every $X, X' \in S$,

$$\sum_{Y \in D} (XY)(YX')^* = \delta_{XX'} s(X),$$

$\mathcal{A}2$: for every $Y, Y' \in T$,

$$\sum_{X \in C} (YX)^*(XY') = \delta_{YY'} r(Y).$$

For notational convenience we denote the bi-separated graph with given distinguished subsets as in the above definition as a 5-tuple $\dot{E} = (E, (C, S), (D, T))$ and again call it bi-separated graph if there is no confusion and denote the Cohn-Leavitt path algebra also as $\mathcal{A}_K(\dot{E})$. Whenever we want to distinguish the case that $S = C_{\text{fin}}$ and $T = D_{\text{fin}}$ we simply call the Cohn-Leavitt path algebra as Leavitt path algebra.

Proposition 3.2.18 (**Universal property of $\mathcal{A}_K(\dot{E})$**). *Let $\dot{E} = (E, (C, S), (D, T))$ be a bi-separated graph. Suppose $\mathfrak{A}$ is a K-algebra which contains a set of pairwise orthogonal idempotents $\{A_v \mid v \in E^0\}$, two sets $\{A_e \mid e \in E^1\}$, $\{B_e \mid e \in E^1\}$ for which the following hold.*

1. *$A_{s(e)}A_e = A_eA_{r(e)} = A_e$, and $A_{r(e)}B_e = B_eA_{s(e)} = B_e$ for all $e \in E^1$.*

2. *for every* $X, X' \in S$, $\sum\limits_{Y \in D} A_{XY}B_{YX'} = \delta_{XX'}A_{s(X)}$,

3. *for every* $Y, Y' \in T$, $\sum\limits_{X \in C} B_{YX}A_{XY'} = \delta_{YY'}A_{r(X)}$.

Then there exists a unique map $\psi : \mathcal{A}_K(\dot{E}) \to \mathfrak{A}$ *such that* $\psi(v) = A_v$, $\psi(e) = A_e$, *and* $\psi(e^*) = B_e$ *for all* $v \in E^0$ *and* $e \in E^1$.

Example 3.2.19 (**Cohn-Leavitt path algebra of a graph**)**.** Let E be a graph and let $S \subseteq \mathrm{RReg}(E)$. Then the Cohn-Leavitt path algebra $CL_K^S(E)$ of E can be realized as Cohn-Leavitt path algebra $\mathcal{A}_K(\dot{E})$ of the bi-separated graph $\dot{E} = (E, (C, S), (D, T))$, where (C, D) is the Cuntz-Krieger bi-separation on E, $\mathrm{RReg}(E) = C_{\text{fin}}$ and $T = D$.

Example 3.2.20. (**Cohn-Leavitt path algebra of a separated graph**) Let (E, C) be a separated graph and $S \subseteq C_{\text{fin}}$. Set $\dot{E} = (E, (C, S), (D, T))$, where $T = D_{\text{fin}} = D$. Then from definition it is clear that $\mathcal{A}_K(\dot{E}) \cong CL_K(E, C, S)$.

We say a bi-separated graph $\dot{E} = (E, (C, S), (D, T))$ is *connected* if the underlying graph E is connected. Because of the following proposition we assume that every bi-separated graph is connected henceforth.

Proposition 3.2.21. *Let* $\dot{E}$ *be a bi-separated graph. Suppose* $\dot{E} = \bigsqcup\limits_{j \in J} E_j$ *is a decomposition of* $\dot{E}$ *into its connected components. Then* $\mathcal{A}_K(\dot{E}) \cong \bigoplus\limits_{j \in J} \mathcal{A}_K(\dot{E}_j)$, *where* $\dot{E}_j$ *is the bi-separated graph structure on* E_j *induced by the bi-separated graph structure on* E.

Proof. Follows from universal property of $\mathcal{A}_K(\dot{E})$. □

Lemma 3.2.22. *Let* $\dot{E}$ *be a bi-separated graph.*

1. *The algebra* $\mathcal{A}_K(\dot{E})$ *is unital if and only if* E^0 *is finite. In this case,*

$$1_{\mathcal{A}_K(\dot{E})} = \sum_{v \in E^0} v.$$

2. *For each* $\alpha \in \mathcal{A}_K(\dot{E})$, *there exists a finite set of distinct vertices* $V(\alpha)$ *for which* $\alpha = f\alpha f$, *where* $f = \sum\limits_{v \in V(\alpha)} v$. *Moreover, the algebra* $(\mathcal{A}_K(\dot{E}), E^0)$ *is a ring with enough idempotents.*

3. *Let* $\overline{} : K \to K$ *be an involution on the field* K. *Then with respect to the involution* $* : \mathcal{A}_K(\dot{E}) \to \mathcal{A}_K(\dot{E})$, $\mathcal{A}_K(\dot{E})$ *is a* $*$*-algebra.*

4. *$\mathcal{A}_K(\dot{E})$ is a graded quotient algebra of $K(\widehat{E})$ with respect to the standard $\mathbb{Z}$-grading given by length of paths.*

Proof. The proof follows on similiar lines of [6, Lemma 1.2.12]. □

3.3 The categories BSG and tBSG

In this section, we introduce two categories **BSG** of bi-separated graphs and the category **tBSG** of tame bi-separated graphs. We study the functoriality and continuity of the functor $\mathcal{A}_K(\_)$ from **BSG** to **K-Alg**. We also show that each object of **tBSG** is a direct limit of sub-objects based on finite graphs, from which we obtain every Cohn-Leavitt path algebra of tame bi-separated graph as a direct limit of unital Cohn-Leavitt path algebras.

Definition 3.3.1. We define a category **BSG** of bi-separated graphs as follows: The objects of **BSG** are bi-separated graphs (with distinguished subsets) $\dot{E} = (E, (C, S), (D, T))$. A morphism $\phi : \dot{E} \to \dot{\widetilde{E}}$ in **BSG** is a graph morphism $\phi : E \to \widetilde{E}$ is a triple $\phi = (\phi_0, \phi_1, \phi_2)$ satisfying the following conditions:

1. $\phi_0 : E \to \widetilde{E}$ is a graph morphism such that ϕ_0^0 is injective.
2. For each $X \in C$ there is a (unique) $\widetilde{X} \in \widetilde{C}$ such that ϕ_0^1 restricts to an injective map $X \to \widetilde{X}$. The map $\phi_1 : C \to \widetilde{C}$ assigns $X \mapsto \widetilde{X}$ such that for all $v \in E^0$ and $X \in C_v$, we have $\phi_1(X) \in C_{\phi_0^0(v)}$.
3. $\phi_1(S) \subset \widetilde{S}$. Moreover $\phi_0^1|_X : X \to \widetilde{X}$ is a bijection, for every $X \in S$.
4. For each $Y \in D$ there is a (unique) $\widetilde{Y} \in \widetilde{D}$ such that ϕ_0^1 restricts to an injective map $Y \to \widetilde{Y}$. The map $\phi_2 : D \to \widetilde{D}$ assigns $Y \mapsto \widetilde{Y}$ such that for all $v \in E^0$ and $Y \in D_v$, we have $\phi_1(Y) \in D_{\phi_0^0(v)}$.
5. $\phi_2(T) \subset \widetilde{T}$. Moreover $\phi_0^1|_Y : Y \to \widetilde{Y}$ is a bijection, for every $Y \in T$.
6. If $X \in S, Y \in D$ and $X \cap Y = \emptyset$, then $\widetilde{X} \cap \widetilde{Y} = \emptyset$.
7. If $X \in C, Y \in T$ and $X \cap Y = \emptyset$, then $\widetilde{X} \cap \widetilde{Y} = \emptyset$.

Proposition 3.3.2. *The category* **BSG** *admits arbitrary direct limits.*

Proof. The proof is similar to [20, Proposition 3.3]. The only addition is that we have to define D and T analogous to the way we define C and S.

$\square$

Recall that a functor is *continuous* if it preserves direct limits.

Proposition 3.3.3. *The assignment $\dot{E} \rightsquigarrow \mathcal{A}_K(\dot{E})$ extends to a continuous covariant functor $\mathcal{A}_K$ from* **BSG** *to* **K-Alg**.

Proof. The proof is similar to [20, Proposition 3.6]. $\square$

Definition 3.3.4. We say a morphism $\phi : \dot{E} \to \dot{\widetilde{E}}$ in **BSG** is **complete** if $\phi_1^{-1}(\widetilde{S}) = S$ and $\phi_2^{-1}(\widetilde{T}) = T$.

Definition 3.3.5. Let $\dot{E}$ be an object in **BSG**. A **sub-object** of $\dot{E}$ is an object $\dot{E}' = (E', (C', S'), (D', T'))$ such that E' is a sub-graph of E and the following conditions hold:

$$\begin{aligned} C' &= \{X \cap (E')^1 \mid X \in C \setminus S,\ X \cap (E')^1 \neq \emptyset\} \sqcup \\ &\quad\ \{X \in S \mid X \cap (E')^1 \neq \emptyset\}. \\ S' &= \{X \in S \mid X \cap (E')^1 \neq \emptyset\}. \\ D' &= \{Y \cap (E')^1 \mid Y \in D \setminus T,\ D \cap (E')^1 \neq \emptyset\} \sqcup \\ &\quad\ \{D \in T \mid D \cap (E')^1 \neq \emptyset\}. \\ T' &= \{Y \in T \mid Y \cap (E')^1 \neq \emptyset\}. \end{aligned}$$

Definition 3.3.6. Let $\dot{E}$ be an object in **BSG**. A *complete sub-object* of $\dot{E}$ is a sub-object $\dot{E}'$ such that the inclusion morphism is complete.

Proposition 3.3.7. *Any object in* **BSG** *is a direct limit of countable complete sub-objects.*

Proof. Let $\dot{E}$ be an object in **BSG**. For a non-empty finite subset $A \subset E^0 \sqcup E^1$, let E_A be the graph generated by A, i.e.,

$$E_A^1 = A \cap E^1 \text{ and } E_A^0 = (A \cap E^0) \cup s_E(E_A^1) \cup r_E(E_A^1).$$

Take $v \in E_A^0$ and set

$$\mathcal{E}_{0v} = s_{E_A}^{-1}(v) \cup r_{E_A}^{-1}(v) \bigcup_{\substack{X \in S \cap C_v \\ X \cap A \neq \emptyset}} X \cup \bigcup_{\substack{Y \in T \cap D_v \\ Y \cap A \neq \emptyset}} Y, \text{ where } s_{E_A} := s_{|_{E_A^1}} \text{ and } r_{E_A} := r_{|_{E_A^1}}.$$

Let $\mathcal{E}_0$ be the graph generated by $E_A^0 \sqcup \bigsqcup_{v \in E_A^0} \mathcal{E}_{0v}$. Let

$$\begin{aligned}
\mathcal{C}_0 &= \{X \cap A \mid X \in C \backslash S, X \cap A \neq \emptyset\} \sqcup \\
&\quad \{X \in S \mid X \cap A \neq \emptyset\}. \\
\mathcal{S}_0 &= \{X \in S \mid X \cap A \neq \emptyset\}. \\
\mathcal{D}_0 &= \{Y \cap A \mid Y \in D \backslash T, Y \cap A \neq \emptyset\} \sqcup \\
&\quad \{Y \in T \mid Y \cap A \neq \emptyset\}. \\
\mathcal{T}_0 &= \{Y \in T \mid Y \cap A \neq \emptyset\}.
\end{aligned}$$

If $\dot{\mathcal{E}}_0$ is a complete sub-object of $\dot{E}$, we are done. If not, then for each $v \in \dot{\mathcal{E}}_0$, define

$$\mathcal{E}_{1v} \text{ to be } \mathcal{E}_{0v} \cup \bigcup_{\substack{X \in S \cap C_v \\ X \cap \mathcal{E}_{0v} \neq \emptyset}} X \cup \bigcup_{\substack{Y \in T \cap D_v \\ Y \cap \mathcal{E}_{0v} \neq \emptyset}} Y.$$

Now let $\mathcal{E}_1$ be the graph generated by $\mathcal{E}_0^0 \sqcup \bigsqcup_{v \in \mathcal{E}_0^0} \mathcal{E}_{0v}$. Let

$$\begin{aligned}
\mathcal{C}_1 &= \{X \cap \mathcal{E}_0^1 \mid X \in C \backslash S, X \cap \mathcal{E}_0^1 \neq \emptyset\} \sqcup \\
&\quad \{X \in S \mid X \cap \mathcal{E}_0^1 \neq \emptyset\}. \\
\mathcal{S}_1 &= \{X \in S \mid X \cap \mathcal{E}_0^1 \neq \emptyset\}. \\
\mathcal{D}_1 &= \{Y \cap \mathcal{E}_0^1 \mid Y \in D \backslash T, Y \cap \mathcal{E}_0^1 \neq \emptyset\} \sqcup \\
&\quad \{Y \in T \mid Y \cap \mathcal{E}_0^1 \neq \emptyset\}. \\
\mathcal{T}_1 &= \{Y \in T \mid Y \cap \mathcal{E}_0^1 \neq \emptyset\}.
\end{aligned}$$

If $\mathcal{E}_1$ is a complete sub-object of $\dot{E}$, we are done. If not, define $\mathcal{E}_{2v}$ similarly and continue this process.

This gives us a chain $\mathcal{E}_0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \ldots$, there by giving a directed system $\{\dot{\mathcal{E}}_i\}_{i \in \mathbb{N} \cup \{0\}}$. We claim that for each $i \geq 0$, the inclusion morphism $\dot{\mathcal{E}}_i \xrightarrow{\Psi_i} \dot{\mathcal{E}_{i+1}}$ is complete.

Suppose $X \in \Psi_i^{-1}(\mathcal{S}_{i+1})$. We want to prove $X \in \mathcal{S}_i$. If $X \in \mathcal{C}_i \backslash \mathcal{S}_i$, then by definition of $\mathcal{C}_i \backslash \mathcal{S}_i$, there exists $X_1 \in C \backslash S$ such that $X = X_1 \cap \mathcal{E}_{i-1}^1 \neq \emptyset$. So $X_1 \cap \mathcal{E}_i^1 \neq \emptyset$. Since $X_1 \in C \backslash S$, we have $X \subseteq X_1 \cap \mathcal{E}_i^1 \in \mathcal{C}_{i+1} \backslash \mathcal{S}_{i+1}$ which contradicts the fact that

$X \in \Psi_i^{-1}(\mathcal{S}_{i+1})$. Similar argument holds for $\mathcal{T}_i \to \mathcal{T}_{i+1}$. This proves that Ψ_i is a complete morphism.

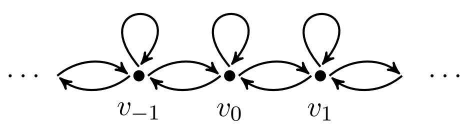
Let $\dot{\mathcal{E}} = (\mathcal{E}, (\mathcal{C}, \mathcal{S}), (\mathcal{D}, \mathcal{T}))$ be the direct limit of the directed system $\{\dot{\mathcal{E}}_i = (\mathcal{E}_i, (\mathcal{C}_i, \mathcal{S}_i), (\mathcal{D}_i, \mathcal{T}_i))\}$. We claim that $\dot{\mathcal{E}}$ is a complete sub-object of $\dot{E}$. Let $\Phi = (\Phi_0, \Phi_1, \Phi_2)$ be the natural morphism from $\dot{\mathcal{E}}$ to $\dot{E}$. If $X \in S$ such that $X \cap \mathcal{E}^1 \neq \emptyset$, then $X \cap \mathcal{E}_i^1 \neq \emptyset$ for some $i \geq 0$. So $X \in \mathcal{S}_{i+1}$. This implies $X \in \mathcal{S}$ (by the definition of direct limit). If $X \in C \backslash S$ and $X \cap \mathcal{E}^1 \neq \emptyset$, then again $X \cap \mathcal{E}_i^1 \neq \emptyset$ for some $i \geq 0$ and so $X \cap \mathcal{E}_i \in \mathcal{C}_{i+1} \backslash S_{i+1}$. Since the morphism $\dot{\mathcal{E}}_i \xrightarrow{\Psi_i} \dot{\mathcal{E}_{i+1}}$ is complete for each $i \geq 0$, we can conclude that the morphism $\dot{\mathcal{E}}_i \to \dot{\mathcal{E}}$ is also complete and so $X \cap \mathcal{E}_1 \in \mathcal{C} \backslash \mathcal{S}$. Similarly one can argue for $Y \in T$ or $Y \in D \backslash T$ with $Y \cap \mathcal{E}^1 \neq \emptyset$. This proves that $\dot{\mathcal{E}}$ is a sub-object of $\dot{\mathcal{E}}$. The completeness of Φ can be seen in exactly the same way as that of Ψ_i. This proves that $\dot{\mathcal{E}}$ is a complete sub-object of $\dot{E}$.

Since the vertex set and edge set of $\mathcal{E}$ are countable union of finite sets, it is a countable sub-graph of E. So, for each finite non-empty subset $A \subset E^0 \sqcup E^1$, there exists a countable complete sub-object $\dot{\mathcal{E}}$ of $\dot{E}$. Now by keeping the set of all finite subsets of $E^0 \sqcup E^1$ as the indexing set, we get a directed system of countable complete sub-objects whose direct limit is $\dot{E}$. □

We note that a general object in **BSG** cannot be written as a direct limit of finite complete sub-objects as the following example illustrates:

Example 3.3.8. Consider the following simple graph Γ_∞ on countably infinite vertices.

$\Gamma_\infty =$



Observe that Γ_∞ is a simple graph. Consider the standard bi-separation (C, D) on Γ_∞ and let $S = C_{\text{fin}} = C$ and $T = D_{\text{fin}} = D$. Then C_∞ cannot be written as direct limit of finite complete sub-objects. For, if there is a complete sub-object of Γ_∞ then by definition we are forced to include all the edges and so, it will no more be finite.

3.3.1 The category of tame bi-separated graphs

Notation 3.3.9. Let $\dot{E}$ be an object in **BSG**. Set

$$\begin{aligned} S_1 &:= \{X \in S \mid X \cap Y \neq \emptyset, \quad \text{for some} \quad Y \in T\}, \\ S_2 &:= S - S_1, \\ T_1 &:= \{Y \in T \mid X \cap Y \neq \emptyset, \quad \text{for some} \quad X \in S\}, \\ T_2 &:= T - T_1. \end{aligned}$$

We define a relation $\sim_T$ on S_1 as follows: For $X, X' \in S_1$, define $X \sim_T X'$ if there exists a finite sequence $X_0, Y_1, X_1, Y_2, X_2, \ldots, Y_{n-1}, X_{n-1}, Y_n, X_n$ such that for each $0 \leq i \leq n$, $X_i \in S, Y_i \in T$, with $X_0 = X$, $X_n = X'$, and $X_i \cap Y_{i+1} \neq \emptyset$ $Y_i \cap X_i \neq \emptyset$. It is not hard to see that $\sim_T$ is an equivalence relation on S_1. Let $S_1 = \bigsqcup_{\lambda \in \Lambda} \mathcal{X}_\lambda$ be the partition of S_1 induced by $\sim_T$.

Define $\sim_S$ on T_1 similarly and let $T_1 = \bigsqcup_{\lambda' \in \Lambda'} \mathcal{Y}_{\lambda'}$ be the partition induced by $\sim_S$.

We claim that the indexing sets Λ and Λ' are in bijection. To see this, start with $\lambda \in \Lambda$. Let $X \in \mathcal{X}_\lambda$ be an arbitrarily fixed element. This means, there exists a $\lambda' \in \Lambda'$ and $Y \in \mathcal{Y}_{\lambda'}$ such that $X \cap Y \neq \emptyset$. If $X' \neq X$ is another element of $\mathcal{X}_\lambda$, and if there is a $Y' \in T$ such that $X' \cap Y' \neq \emptyset$, then $Y' \sim_S Y$ because $X' \sim_T X$. So Y' belongs to the same $\mathcal{Y}_{\lambda'}$ as Y. Also if there is another element $Y_1 \in T$ such that $X \cap Y_1 \neq \emptyset$, then clearly $Y_1 \sim_S Y$ and so Y_1 also lies in same $\mathcal{Y}_{\lambda'}$. This implies that the map $\Lambda \to \Lambda'$ defined by $\lambda \mapsto \lambda'$ is well-defined. Similarly one can define a map $\Lambda' \to \Lambda$. It is not hard to see that these maps are inverses of each other which proves the claim. Therefore, we have the following proposition:

Proposition 3.3.10. *Let $\dot{E}$ be an object in* **BSG** *and let S_1, T_1 be as defined in notation 3.3.9. Then there exist canonical partitions $S_1 = \bigsqcup_{\lambda \in \Lambda} \mathcal{X}_\lambda$ and $T_1 = \bigsqcup_{\lambda' \in \Lambda'} \mathcal{Y}_{\lambda'}$ of S_1 and T_1 respectively such that the indexing sets Λ and Λ' are bijective.*

Remark 3.3.11. Because of the above proposition, we will denote the indexing sets of the canonical partitions of both S_1 and T_1 by Λ.

Definition 3.3.12. A bi-separated graph $\dot{E}$ is called **tame** if $|\mathcal{X}_\lambda| < \infty$ and $|\mathcal{Y}_\lambda| < \infty$, for each $\lambda \in \Lambda$. The tame bi-separated graphs along with complete morphisms form a category which we call a **tame (sub)category of bi-separated graphs**. It will be denoted by **tBSG**.

Note that any finite bi-separated graph is tame. Also, the classes of bi-separated graphs in examples 3.2.4, 3.2.5, 3.2.6, and 3.2.7 are all tame.

Proposition 3.3.13. *Let $\dot{E}$ be a tame bi-separated graph such that $S = C_{\text{fin}} = C$, $T = D_{\text{fin}} = D$ and $|E^0| = 1$. For $\lambda \in \Lambda$, let E_λ be the subgraph of E with edge set $\bigcup_{\substack{e \in X \\ X \in \mathcal{X}_\lambda}} \{e\}$ and consider the bi-separation $C_\lambda = \{X \in \mathcal{X}_\lambda\}$, and $D_\lambda = \{Y \in \mathcal{Y}_\lambda\}$. Then $\mathcal{A}_K(\dot{E})$ is isomorphic to the free-product of algebras $\mathcal{A}_K(\dot{E}_\lambda)$, where λ varies over the indexing set Λ.*

Corollary 3.3.14 ([20] Proposition 2.10)**.** *Let (E, C) be a separated graph with $|E^0| = 1$. Then $L_K(E, C)$ is isomorphic to the free-product of algebras $L_K(1, |X|)$, where $L_K(1, |X|)$ is the Leavitt algebra of type $(1, |X|)$ and X varies over C.*

Theorem 3.3.15. *Every object $\dot{E}$ in **tBSG** is a direct limit of finite (complete) sub-objects. Conversely, if $\dot{E}$ in **BSG** is a direct limit of finite complete sub-objects then it belongs to **tBSG**.*

Proof. Let $\dot{E}$ be an object in **tBSG**. By exactly same arguments as in Proposition 3.3.7, $\dot{E}$ is a direct limit of the directed system

$$\{\left(\dot{\mathcal{E}}_A = (\mathcal{E}, (\mathcal{C}, \mathcal{S}), (\mathcal{D}, \mathcal{T}))_A, \hookrightarrow\right) \mid A \quad \text{is a finite subset of} \quad E^0 \sqcup E^1\}.$$

It follows from the definition of tame bi-separated graphs that $\dot{\mathcal{E}}_A$ is finite, for each finite subset A of $E^0 \sqcup E^1$.

Conversely, let $\dot{E}$ be a direct limit of the directed system

$$\{\left(\dot{\mathcal{E}}_i = (\mathcal{E}_i, (\mathcal{C}_i, \mathcal{S}_i), (\mathcal{D}_i, \mathcal{T}_i)), \hookrightarrow\right) \mid i \in I\}$$

of finite complete sub-objects. We know that S and T can be partitioned as $S_1 \sqcup S_2$ and $T_1 \sqcup T_2$ respectively (see equations 1 to 4 in 3.3.9). If $S_1 = \emptyset$, then $T_1 = \emptyset$ and clearly $\dot{E}$ is tame.

Suppose $S_1 \neq \emptyset$. Then we have $S_1 = \bigsqcup_{\lambda \in \Lambda} \mathcal{X}_\lambda$ and $T_1 = \bigsqcup_{\lambda \in \Lambda} \mathcal{Y}_\lambda$. We claim that $|\mathcal{X}_\lambda|$ and $|\mathcal{Y}_\lambda|$ are finite for every λ.

Suppose $X \in \mathcal{X}_\lambda$ for some λ. This means that $X \in S$. So there exists $i \in I$ such that $X \in \mathcal{S}_i$. We claim $\mathcal{X}_\lambda \subseteq \mathcal{S}_i$ and $\mathcal{Y}_\lambda \subseteq \mathcal{T}_i$. If we assume this claim, then since $\dot{\mathcal{E}}_i$ is a finite sub-object, we can conclude that $|\mathcal{X}_\lambda|$ and $|\mathcal{Y}_\lambda|$ are both finite. So suppose $X' \in \mathcal{X}_\lambda$. Then $X' \sim_T X$. So there exists a sequence $X = X_0, Y_1, X_1, Y_2, \dots, Y_n, X_n = X'$ such that for $i \geq 0$, $X_i \cap Y_{i+1} \neq \emptyset$ and for $i \geq 1$, $Y_i \cap X_i \neq \emptyset$, where $X_i \in S$ and $Y_i \in T$. Now since $X \in \mathcal{S}_i$ and $X \cap Y_1 \neq \emptyset$, we have $Y_1 \cap \mathcal{E}_i^1 \neq \emptyset$, which means $Y_1 \in \mathcal{T}_i$ (because $\dot{\mathcal{E}}_i$ sub-object of $\dot{E}$). For the same reason, we can conclude that $X_i \in \mathcal{S}_i$ for each $1 \leq i \leq n$. This implies in particular that $X' \in \mathcal{S}_i$ which proves $\mathcal{X}_\lambda \subseteq \mathcal{S}_i$. Similarly one can show that $\mathcal{Y}_\lambda \subseteq \mathcal{T}_i$. This completes the proof. □

Corollary 3.3.16. *Let $\dot{E}$ be an object in* **tBSG**. *Then the Cohn-Leavitt path algebra $\mathcal{A}_K(\dot{E})$ is the direct limit of the directed system of unital algebras $\{\mathcal{A}_K(\dot{E}_i)\}_{i\in I}$ such that whenever $j \geq i$, the map $\mathcal{A}_K(\dot{E}_i) \to \mathcal{A}_K(\dot{E}_j)$ is a monomorphism, where $\{\dot{E}_i\}_{i\in I}$ is a directed system of finite complete sub-objects of $\dot{E}$ whose direct limit is $\dot{E}$.*

Proof. By the previous theorem, $\dot{E}$ is a direct limit of a directed system $\{\dot{E}_i\}_{i\in I}$ consisting of its finite complete sub-objects. Therefore, by proposition 3.3.3 and Theorem 3.4.5, $\mathcal{A}_K(\dot{E})$ is the direct limit of the directed system of algebras $\{\mathcal{A}_K(\dot{E}_i)\}_{i\in I}$. □

By corollary 3.3.16 (Cohn-)Leavitt path algebras of the classes of bi-separated graphs in examples 3.2.4-3.2.7 are direct limits of unital sub-(Cohn-)Leavitt path algebras of same type.

3.4 Normal forms and their applications

Definition 3.4.1. (i) Let $\dot{E}$ be an object in **tBSG** such that $S_1 \neq \emptyset$ (which automatically means $T_1 \neq \emptyset$). Suppose that for each $\lambda \in \Lambda$, there exists $X_\lambda \in \mathcal{X}_\lambda$ and $Y_\lambda \in \mathcal{Y}_\lambda$ such that $X_\lambda \cap Y \neq \emptyset$ for each $Y \in \mathcal{Y}_\lambda$ and $X \cap Y_\lambda \neq \emptyset$ for each $X \in \mathcal{X}_\lambda$. Then we call X_λ (resp. Y_λ) a **distinguished element** of $\mathcal{X}_\lambda$ (resp. $\mathcal{Y}_\lambda$).

(ii) An object $\dot{E}$ in **tBSG** is called a **docile object** if either $S_1 = \emptyset = T_1$, or, for each $\lambda \in \Lambda$, there exist distinguished elements $X_\lambda \in \mathcal{X}_\lambda$ and $Y_\lambda \in \mathcal{Y}_\lambda$.

Remark 3.4.2. (a) The docile objects in **tBSG** along with morphisms form a subcategory of **tBSG** which we call **docile category**. If we further insist that a morphism should map a distinguished element X_λ to a distinguished element X'_λ, then we need to assume it to be a complete morphism.

(b) It is not hard to see that the bi-separated graphs in examples refCuntz-Krieger bi-separation, 3.2.5-3.2.7 are all objects in docile category.

Definition 3.4.3. Let $\dot{E}$ be an object in docile category. If $S_1 \neq \emptyset$, then for every $\lambda \in \Lambda$, we fix a distinguished element $X_\lambda \in \mathcal{X}_\lambda$ and $Y_\lambda \in \mathcal{Y}_\lambda$.

1. For each pair $X, X' \in \mathcal{X}_\lambda$, we call the word $(XY_\lambda)(Y_\lambda X')^*$ a **forbidden word of type I**.

2. For each $Y, Y' \in \mathcal{Y}_\lambda$, we call the word $(YX_\lambda)^*(X_\lambda Y')$ a **forbidden word of type II**.

3. Suppose $S_2 \neq \emptyset$. For each pair $X, X' \in S_2$, if there exists $Y \in D$ such that $X \cap Y \neq \emptyset$ and $X' \cap Y \neq \emptyset$, then we fix one such Y (this may vary with X, X') and call $(XY)(YX')^*$ a **forbidden word of type III**.

4. Suppose $T_2 \neq \emptyset$. For each pair $Y, Y' \in T_2$, if there exists $X \in C$ such that $Y \cap X \neq \emptyset$ and $Y' \cap X \neq \emptyset$, then we fix one such X and call $(YX)^*(XY')$ a **forbidden word of type IV**.

Definition 3.4.4. A generalized path $\mu \in \widehat{E}^\star$ is called *normal* if it does not contain any forbidden sub-word of the types mentioned above. An element of $K(\widehat{E})$ is called *normal* if it lies in the K-linear span of generalized normal paths.

From now on throughout this section, we will work in the docile category . We show that given any such object $\dot{E}$, every element of $\mathcal{A}_K(\dot{E})$ has precisely one normal representative in $K(\widehat{E})$. For this, we need to use Bergman's diamond lemma. We refer the reader to subsection 1.3 or [25, pp. 180-182] for the statement of the lemma and basic terminologies.

Theorem 3.4.5. *Let $\dot{E}$ be an docile object in **tBSG** . Then $\mathcal{A}_K(\dot{E})$ has a basis consisting of normal generalized paths.*

Proof. In order to apply Bergman's diamond lemma, we replace the defining relations by the following:

$1'$: For any $v, w \in E^0$,

$$vw = \delta_{v,w} v.$$

$2'$: For any $v \in E^0, e \in E^1$,

$$ve = \delta_{v,s(e)} e,$$

$$ev = \delta_{v,r(e)} e,$$

$$ve^* = \delta_{v,r(e)} e^*,$$

$$e^* v = \delta_{v,s(e)} e^*.$$

$3'$: For any $e, f \in E^1$,

$$ef = 0, \quad \text{if} \quad r(e) \neq s(f),$$

$$e^* f = 0, \quad \text{if} \quad s(e) \neq s(f),$$

$$ef^* = 0, \quad \text{if} \quad r(e) \neq r(f),$$

$$e^* f^* = 0, \quad \text{if} \quad s(e) \neq r(f).$$

$\mathcal{A}'1$: For each $X, X' \in S$ for which there exists $Y \in D$ such that $X \cap Y \neq \emptyset$ and $X' \cap Y \neq \emptyset$,

$$e_X e_{X'}^* := (XY)(YX')^* = \delta_{X,X'} s(X) - \sum_{\substack{Y_1 \in D \\ Y_1 \neq Y}} (XY_1)(Y_1 X')^*.$$

$\mathcal{A}'2$: For each $Y, Y' \in T$ for which there exists $X \in C$ such that $X \cap Y \neq \emptyset$ and $X \cap Y' \neq \emptyset$,

$$e_Y^* e_{Y'} := (YX)(XY')^* = \delta_{Y,Y'} r(Y) - \sum_{\substack{X_1 \in C \\ X_1 \neq X}} (YX_1)^* (X_1 Y').$$

(i.e. In $\mathcal{A}'1$, LHS contains forbidden words of types I and III. In $\mathcal{A}'2$, LHS has forbidden words of types II and IV).

Denote by Σ the reduction system consisting of all pairs $\sigma = (w_\sigma, f_\sigma)$, where w_σ equals the LHS of an equation above and f_σ the corresponding RHS. Let $\langle \bar{P} \rangle$ be the monoid consisting of all words formed by letters in $E^0 \cup E^1 \cup \overline{E^1}$ and $\langle P \rangle$ be the semigroup obtained by removing the identity element of $\langle \bar{P} \rangle$. We define a partial order on $\langle P \rangle$ as follows:

Let $A = x_1 x_2 \ldots x_n \in \langle P \rangle$. Set $l(A) = n$ and

$$m(A) = |\{i \in \{1, 2, \ldots, (n-1)\} \mid x_i x_{i+1} \text{is of type I or type II}\}|.$$

Define a partial order $\leq$ on $\langle P \rangle$ by $A \leq B$ if and only if one of the following holds:

1. $A = B$,
2. $l(A) < l(B)$ or
3. $l(A) = l(B), \quad$ and for each $\quad G, H \in \langle \bar{P} \rangle, \quad m(GAH) < m(GBH).$

Clearly $\leq$ is a semigroup partial order on $\langle P \rangle$ compatible with Σ and also the descending chain condition is satisfied. It remains to show that all ambiguities of Σ are resolvable. Recall from Proposition 1.3.3 that $\widehat{E}^{\star}$ is a linear K-basis for $K(\widehat{E})$. Hence it is sufficient to show that the following ambiguities are resolvable:

$$e_X e^*_{X'} e_Y = (XY')(Y'X')^*(X'Y) \qquad (\mathcal{A}'1 - \mathcal{A}'2)$$

$$e^*_Y e_{Y'} e^*_X = (YX')^*(X'Y')(Y'X')^*. \qquad (\mathcal{A}'2 - \mathcal{A}'1)$$

We note that there are no inclusion ambiguities. We only show how to resolve ambiguity of type $\mathcal{A}'1 - \mathcal{A}'2$ and the other case follows similarly. Also suppose $X, X_1 \in S_2$, $Y_1 \in D$ and $(XY_1)(Y_1X_1)^*$ is a forbidden word of type III. Then any word of the form $(XY_1)(Y_1X_1)^*(X_1Y)$, where $Y \in D$ will not result in an overlap ambiguity. This is because, X, X_1, being elements of S_2 will not intersect with elements of T and so $(Y_1X_1)^*(X_1Y)$ will not be a forbidden word. The same argument is true for forbidden words of type IV. So we only have to resolve those overlap ambiguities which involve forbidden words of types I and II which we exhibit in the diagram below. Here, we assume $X, X_1 \in \mathcal{X}_\lambda$, $Y, Y_1 \in \mathcal{Y}_\lambda$ and X_1 and Y_1 to be the fixed distinguished elements of $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ respectively, where λ is an arbitrarily fixed element of Λ, the indexing set (see remark 3.3.11).

$$
\begin{tikzcd}[column sep=tiny]
& (XY_1)(Y_1X_1)^*(X_1Y) \arrow[dl, bend right, "\mathcal{A}'1"'] \arrow[dr, bend left, "\mathcal{A}'2"] & \\
\Big[\delta_{X,X_1}s(X_1) - \sum\limits_{\substack{Y_2\in D\\ Y_2\neq Y_1}} (XY_2)(Y_2X_1)^*\Big](X_1Y) \arrow[d, "="'] & & (XY_1)\Big[\delta_{Y,Y_1}r(Y_1) - \sum\limits_{\substack{X_2\in C\\ X_2\neq X_1}} (Y_1X_2)^*(X_2Y)\Big] \arrow[d, "="] \\
\delta_{X,X_1}(X_1Y) - \sum\limits_{\substack{Y_2\in D\\ Y_2\neq Y_1}} (XY_2)(Y_2X_1)^*(X_1Y) \arrow[d, "\mathcal{A}'2"'] & & \delta_{Y,Y_1}(XY_1) - \sum\limits_{\substack{X_2\in C\\ X_2\neq X_1}} (XY_1)(Y_1X_2)^*(X_2Y) \arrow[d, "\mathcal{A}'1"] \\
\delta_{X,X_1}(X_1Y) - \sum\limits_{\substack{Y_2\in D\\ Y_2\neq Y_1}} (XY_2)\Big[\delta_{Y,Y_2}r(Y) - \sum\limits_{\substack{X_2\in C\\ X_2\neq X_1}} (Y_2X_2)^*(X_2Y)\Big] \arrow[d, "="'] & & \delta_{Y,Y_1}(XY_1) - \sum\limits_{\substack{X_2\in C\\ X_2\neq X_1}} \Big[\delta_{X,X_2}s(X) - \sum\limits_{\substack{Y_2\in D\\ Y_2\neq Y_1}} (XY_2)(Y_2X_2)^*\Big](X_2Y) \arrow[d, "="] \\
\delta_{X,X_1}(X_1Y) - \delta_{Y,Y_2}(XY_2) + \sum\limits_{\substack{Y_2\in D\\ Y_2\neq Y_1}} \sum\limits_{\substack{X_2\in C\\ X_2\neq X_1}} (XY_2)(Y_2X_2)^*(X_2Y) \arrow[dr, bend right, "="'] & & \delta_{Y,Y_1}(XY_1) - \delta_{X,X_2}(X_2Y) + \sum\limits_{\substack{X_2\in C\\ X_2\neq X_1}} \sum\limits_{\substack{Y_2\in D\\ Y_2\neq Y_1}} (XY_2)(Y_2X_2)^*(X_2Y) \arrow[dl, bend left, "="] \\
& \sum\limits_{\substack{X_2\in C\\ X_2\neq X_1}} \sum\limits_{\substack{Y_2\in D\\ Y_2\neq Y_1}} (XY_2)(Y_2X_2)^*(X_2Y) &
\end{tikzcd}
$$

This proves the confluence condition. The final expression written above is a finite sum as $X \in S$ and $Y \in T$. Also the final expression clearly does not involve X_1 and Y_1 which are the distinguished elements of $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ respectively and so does not contain any forbidden word. The result now follows from Bergman's diamond lemma. □

Corollary 3.4.6. *Let $\dot{E}$ be a docile object in* ***tBSG****. Then the natural homomorphism from the path algebra $K(E)$ to the algebra $\mathcal{A}_K(\dot{E})$ is an inclusion.*

Proof. From the theorem, it follows that each path $\mu \in E^\star$ is a part of a basis of $\mathcal{A}_K(\dot{E})$ as μ does not contain any forbidden word. $\square$

Remark 3.4.7. (a) The advantage of restricting our attention to the docile category is that the presence of distinguished elements in $\mathcal{X}_\lambda$ and $\mathcal{Y}_\lambda$ will help us in defining forbidden words in a 'canonical way'. With this definition of forbidden words, it is very easy to check the compatibility of the semigroup partial ordering $\leq$ defined in the theorem with Σ. Also, as mentioned in the remark 3.4.2(b), the graph-theoretic objects corresponding to various generalizations of Leavitt-path algebras are all objects in docile category. Therefore, the docile category itself provides a common platform for studying various generalizations of Leavitt-path algebras.

(b) One might compute the normal forms of algebras corresponding to some non-docile objects in **tBSG**. However, for an arbitrary object in **tBSG** there is no canonical way of defining forbidden words and this might make it very hard to check the compatibility of the partial ordering $\leq$ with Σ.

In the following subsections we give some applications of normal forms of Cohn-Leavitt path algebras. That is, we find 'bi-separated graph theoretic properties' that correspond to algebraic properties. We start by recalling some definitions and propositions from the theory of rings with enough idempotents. Then we give their applications to the case of Cohn-Leavitt path algebras. We note that the reasoning is very similar to that of [61]. Wherever some care is required we provide complete proofs, else the reader is refered to [61] for proofs.

3.4.1 Local valuations and their applications

Definition 3.4.8. Let (R, I) be a ring with enough idempotents. A **local valuation** on (R, I) is a map $nu : R \to \mathbb{Z}^+ \cup \{-\infty\}$ such that

1. $\nu(x) = -\infty$ if and only if $x = 0$
2. $\nu(x - y) \leq \max\{\nu(x), \nu(y)\}$ for any $x, y \in R$ and
3. $\nu(xy) = \nu(x) + \nu(y)$ for any $e \in I$, $x \in Re$ and $y \in eR$.

A local valuation ν on (R, I) is called trivial if $\nu(x) = 0$ for each $x \in R - \{0\}$.

Let R be a ring. A left ideal $\mathfrak{a}$ of R is called **essential** if $\mathfrak{a} \cap \mathfrak{b} = 0 \Rightarrow \mathfrak{b} = 0$ for any left ideal $\mathfrak{b}$ of R. For any $x \in R$, recall that the left annihilator ideal of x is $\mathrm{Ann}(x) := \{y \in R \mid yx = 0\}$. A ring R is called *left non-singular* if for any $x \in R$, $\mathrm{Ann}(x)$ is essential $\Leftrightarrow x = 0$. A right non-singular ring is defined similarly. A ring is *non-singular* is if it is both left and right non-singular.

Proposition 3.4.9. *[61, Proposition 37] Let (R, I) be a ring with enough idempotents which admits a local valuation. Then R is non-singular.*

A non-zero ring R is called a **prime ring** if $\mathfrak{a}\mathfrak{b} = 0 \Rightarrow \mathfrak{a} = 0$ or $\mathfrak{b} = 0$ for any ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R. A ring with enough idempotents (R, I) is **connected** if $eRf \neq 0$ for any $e, f \in I$.

Proposition 3.4.10. *[61, Proposition 38] Let (R, I) be a nonzero, connected ring with enough idempotents which admits a local valuation. Then R is a prime ring.*

A ring R is said to be **von Neumann regular** if for any $x \in R$ there exists $y \in R$ such that $xyx = x$.

Proposition 3.4.11. *[61, Proposition 39] Let (R, I) be a ring with enough idempotents that has a nontrivial local valuation. Then R is not von Neumann regular.*

Recall that the **Jacobson radical** of a ring R is the ideal consisting of those elements in R that annihilate all simple (right or left) R-modules. A ring is called **semiprimitive** if its Jacobson radical is the zero ideal.

Proposition 3.4.12. *[61, Proposition 40] Let (R, I) be a connected K-algebra with enough idempotents which admits a local valuation ν such that $\nu(x) = 0$ if and only if x is a nonzero K-linear combination of elements in I. Then R is semiprimitive.*

Now we find conditions on a bi-separated graph $\dot{E}$ for which the corresponding Cohn-Leavitt path algebra admits a local valuation.

Definition 3.4.13. A docile onject $\dot{E} = (E, (C, S), (D, T))$ in **tBSG** is said to satisfy **Condition LV** if $X \in S$ (resp. $Y \in T$) implies $|X| > 1$ (resp. $|Y| > 1$) and one of the following holds:

$(LV1)$: $|S| \leq 1$, $|T| \leq 1$.

$(LV2)$: $|S| > 1$ or $|T| > 1$, and the following two conditions are satisfied:

(a) For any distinct pair $X_1, X_2 \in S$, either there is no $Y \in D$ such that $X_1 \cap Y \neq \emptyset$ and $X_2 \cap Y \neq \emptyset$ or there are at least two distinct elements $Y_1, Y_2 \in D$, such that $X_i \cap Y_j \neq \emptyset$ for each $1 \leq i, j \leq 2$.

(b) For any distinct pair $Y_1, Y_2 \in T$, either there is no $X \in C$ such that $Y_1 \cap X \neq \emptyset$ and $Y_2 \cap X \neq \emptyset$ or there are at least two distinct elements $X_1, X_2 \in C$ such that $X_i \cap Y_j \neq \emptyset$ for each $1 \leq i, j \leq 2$.

We say $\dot{E}$ satisfies **Domain condition** if $|E^0| = 1$ and either $(LV1)$ or $(LV2)$ holds.

Remark 3.4.14. We emphasize the fact that if $\dot{E}$ satisfies domain condition, then $|X|$ (resp. $|Y|$) could be equal to 1 also for $X \in S$ (resp. for $Y \in T$) unlike the (LV) condition where for $X \in S$ (resp. for $Y \in T$), $|X|$ (resp. $|Y|$) has to be strictly greater than 1.

Proposition 3.4.15. *Let $\dot{E}$ be a docile object in **tBSG** and for any $a \in \mathcal{A}_K(\dot{E})$, let* $\operatorname{supp}(a)$ *denote the set of all normal generalized paths occuring in* $\operatorname{NF}(a)$ *with nonzero coefficients, where* $\operatorname{NF}(a)$ *is the unique normal representative of a. If $\dot{E}$ satisfies condition LV, then the map $\nu : \mathcal{A}_K(\dot{E}) \to \mathbb{Z}^+ \cup \{-\infty\}$ defined by*

$$0 \neq a \mapsto \max\{|p| \mid p \in \operatorname{supp}(a)\}$$

$$0 \mapsto -\infty$$

is a local valuation on $\mathcal{A}_K(\dot{E})$, where by $|p|$, we mean the length of the path p.

Proof. The first two conditions of a local valuation are obvious. It remains to show $\nu(ab) = \nu(a) + \nu(b)$, for any $v \in E^0$, $a \in \mathcal{A}_K(\dot{E})v$ and $b \in v\mathcal{A}_K(\dot{E})$. If one of $\nu(a)$ and $\nu(b)$ is 0 or $-\infty$, then the result is clear. Suppose now $\nu(a), \nu(b) \geq 1$. Since any reduction preserves or decreases the length of a generalized path, it follows that $\nu(ab) \leq \nu(a) + \nu(b)$. So it remains to show that $\nu(ab) \geq \nu(a) + \nu(b)$. Let

$$p_k = x_1^k \dots x_{\nu(a)}^k \ \ (1 \leq k \leq r)$$

be the elements of $\operatorname{supp}(a)$ with length $\nu(a)$ and

$$q_l = y_1^l \dots y_{\nu(b)}^l \ \ (1 \leq l \leq s)$$

be the elements of supp(b) with length $\nu(b)$. We assume that the p_k's are pairwise distinct and so are q_l's. Since NF is a linear map, we can make the following conclusions:

1. If $x^k_{\nu(a)}y^l_1$ is not a forbidden word of any type mentioned above, then
$$\mathrm{NF}(p_kq_l) = p_kq_l.$$

2. If $x^k_{\nu(a)}y^l_1$ is a forbidden word of type I or III, then there are $X, X' \in S$ and $Y \in D$ such that $x^k_{\nu(a)}y^l_1 = (XY)(X'Y)^*$ and $(XY)(X'Y)^*$ is forbidden. So
$$\begin{aligned}\mathrm{NF}(p_kq_l) &= \left[\delta_{XX'}x^k_1\dots x^k_{\nu(a)-1}y^l_2\dots y^l_{\nu(b)}\right]\\ &- \sum_{\substack{Y\in D\\ (XY)(X'Y)^*\neq x^k_{\nu(a)}y^l_1}} x^k_1\dots x^k_{\nu(a)-1}(XY)(X'Y)^*y^l_2\dots y^l_{\nu(b)}.\end{aligned}$$

3. If $x^k_{\nu(a)}y^l_{\nu(b)}$ is a forbidden word of type II or IV, then there are $Y, Y' \in T$ and $X \in C$ such that $x^k_{\nu(a)}y^l_{\nu(b)} = (XY)^*(XY')$ and $(XY)^*(XY')$ is forbidden. So
$$\begin{aligned}\mathrm{NF}(p_kq_l) &= \left[\delta_{YY'}x^k_1\dots x^k_{\nu(a)-1}y^l_2\dots y^l_{\nu(b)}\right]\\ &- \sum_{\substack{X\in C\\ (XY)^*(XY')\neq x^k_{\nu(a)}y^l_1}} x^k_1\dots x^k_{\nu(a)-1}(XY)^*(XY')y^l_2\dots y^l_{\nu(b)}.\end{aligned}$$

Case 1 : Assume that $x^k_{\nu(a)}y^l_1$ is not a forbidden word of any type for any k, l.
Then $p_kq_l \in$ supp(a), for any k, l. So $\nu(ab) \geq |p_kq_l| = \nu(a) + \nu(b)$.

Case 2 : Assume that there are k and l such that $x^k_{\nu(a)}y^l_{\nu(b)}$ is a forbidden word of type I or III.
Then there are $X, X' \in S$ and $Y \in D$ such that $x^k_{\nu(a)}y^l_1 = (XY)(X'Y)^*$ and $(XY)(X'Y)^*$ is forbidden. Since $\dot{E}$ is an LV-object, there is at least one more element $\widetilde{Y} \in D$ other than Y such that $X\widetilde{Y} \neq 0$ and $X'\widetilde{Y} \neq 0$.
Case 2.1 : Assume $p_{k'}q_{l'} \neq x^k_1\dots x^k_{\nu(a)-1}(X\widetilde{Y})(X'\widetilde{Y})^*y^l_2\dots y^l_{\nu(b)}$, for any k', l'.

Then $x^k_1\dots x^k_{\nu(a)-1}(X\widetilde{Y})(X'\widetilde{Y})^*y^l_2\dots y^l_{\nu(b)} \in$ supp(ab), since it does not cancel with any other term. So we are done.
Case 2.2 : Assume $p_{k'}q_{l'} = x^k_1\dots x^k_{\nu(a)-1}(X\widetilde{Y})(X'\widetilde{Y})^*y^l_2\dots y^l_{\nu(b)}$, for some k', l'. In this

case, $p_k q_{l'} = x_1^k \dots x_{\nu(a)-1}^k (XY)(X'\widetilde{Y})^* y_2^l \dots y_{\nu(b)}^l \in \mathrm{supp}(ab)$ and so we are done.

Case 3 : Assume that there are k and l such that $x_{\nu(a)}^k y_1^l$ is a forbidden word of type II or IV.

Then there are $Y, Y' \in T$ and $X \in C$ such that $x_{\nu(a)}^k y_1^l = (XY)^*(XY')$ and $(XY)^*(XY')$ is forbidden. Again since $\dot{E}$ is an LV-object, there is at least one more element $\widetilde{X} \in C$ other than X such that $\widetilde{X}Y \neq 0$ and $\widetilde{X}Y' \neq 0$. The proof now follows in exactly the same way as in **Cases 2.1** and **2.2**. □

Corollary 3.4.16. *Let $\dot{E}$ be a docile object in **tBSG** satifying condition LV. Then*

1. *$\mathcal{A}_K(\dot{E})$ is nonsingular.*
2. *$\dot{E}$ is connected implies $\mathcal{A}_K(\dot{E})$ is semiprimitive and prime.*
3. *$|E^1| \geq 1$ implies $\mathcal{A}_K(\dot{E})$ is not von Neumann regular.*

Theorem 3.4.17. *Let $\dot{E}$ be a docile object in **tBSG**. Then $\mathcal{A}_K(\dot{E})$ is a domain if and only if $\dot{E}$ satisfies domain condition.*

Proof. If $\dot{E}$ satisfies domain condition, then we consider the two following cases:

Case 1 : Assume that $X \in S \Rightarrow |X| > 1$ and $Y \in T \Rightarrow |Y| > 1$. If both S and T are empty, then $\mathcal{A}_K(\dot{E})$ is a free unital K-algebra and hence a domain (since K is a field). Otherwise, by the proposition 3.4.15, there is a local valuation on $\mathcal{A}_K(\dot{E})$. So if $ab = 0$ in $\mathcal{A}_K(\dot{E})$, then $\nu(ab) = -\infty$, which implies $\nu(a) + \nu(b) = -\infty$. This means that $\nu(a) = -\infty$ or $\nu(b) = -\infty$. Hence $a = 0$ or $b = 0$. Therefore $\mathcal{A}_K(\dot{E})$ is a domain.

Case 2 : The only remaining cases to be considered are when $S = \{X\}$ with $X = \{e\}$ or $T = \{Y\}$ with $Y = \{f\}$. In both these cases the relations imposed on $K(\widehat{E})$ are not of the form $ab = 0$.

For converse, if $|E^0| > 1$, then obviously $\mathcal{A}_K(\dot{E})$ is not a domain. Otherwise, we consider the following cases separately:

Case 1 : Assume that there are two distinct elements $X, X' \in S$ which have only one common $Y \in D$ such that $XY \neq 0$, $X'Y \neq 0$. Then $(XY)(X'Y)^* = \delta_{XX'}s(X) = 0$. So we are done.

Case 2 : Assume that there are two distinct elements $Y, Y' \in T$ which have only one common $X \in C$ such that $XY \neq 0$, $XY' \neq 0$. Then $(XY)^*(XY') = \delta_{YY'}r(Y) = 0$.

This completes the proof. □

3.4.2 The Gelfand-Kirillov dimension

We first recall some basic facts on the growth of algebras from [43]. Suppose B is a finitely generated K-algebra. Choose a finite generating set of B and let V be the K-subspace of B spanned by this chosen generating set. For each positive integer n, let V^n denote the K-subspace of B spanned by all words in V of length less than or equal to n. In particular, $V^1 = V$. Then we have an ascending chain

$$K \subseteq V^1 \subseteq V^2 \subseteq \ldots \subseteq V^n \subseteq \ldots$$

of finite dimensional K-subspaces of B such that $B = \bigcup_{n\in\mathbb{N}_0} V^n$, where, by convention, $V^0 = K$. Clearly, the sequence $\{\dim_K(V^n)\}$ is a montonically increasing sequence and the asymptotic behaviour (see the definition 3.4.18) of this sequence provides an invariant of the algebra B, called the *Gelfand-Kirillov dimension* of B, which is defined to be

$$\mathrm{GKdim\ B} = \overline{\lim}\frac{\log\ \dim_{\mathrm{K}}(V^n)}{\log\ \mathrm{n}}. \tag{3.4.1}$$

Definition 3.4.18. Given two eventually monotonically increasing functions $\phi, \psi : \mathbb{N} \to \mathbb{R}^+$, we say $\phi \preceq \psi$ if there are natural numbers a and b such that $\phi(n) \leq a\psi(bn)$, for almost all $n \in \mathbb{N}$. We say ϕ *is asymptotically equivalent to* ψ, if both $\phi \preceq \psi$ and $\psi \preceq \phi$. If ϕ and ψ are asymptotically equivalent, we write $\phi \sim \psi$.

Coming back to GK dimension of algebras, if a K-algebra B has two distinct finite generating sets, and if V and W are the finite dimensional subspaces of B spanned by these sets, then setting $\phi(n) = \dim_{\mathrm{K}}(V^n)$ and $\psi(n) = \dim_{\mathrm{K}}(W^n)$, one can show that $\phi \sim \psi$ [43, Lemma 1.1]. In this notation, if $\phi \preceq n^m$ for some $m \in \mathbb{N}$, then B is said to have *polynomial growth* and in this case GKdim(B)$\leq m$. If on the other hand, $\phi \sim a^n$ for some $a \in \mathbb{R}$ such that $a > 1$, then B is said to have *exponential growth* and in this case GKdim(B)$= \infty$.

Definition 3.4.19. A docile object $\dot{E}$ in $tBSG$ is said to satisfy **Condition** (A') if

$(A'1)$: $S = T = \emptyset$ implies either $|E^1| > 0$ or $|E^0| = \infty$.

$(A'2)$: $S \neq \emptyset$ or $T \neq \emptyset$ implies at least one of the following holds:

(a) $\exists X_1, X_2 \in S,\ X_1 \neq X_2,\ s(X_1) = s(X_2)$ and $Y \in D$ such that for $i \in \{1, 2\}$, $X_i \cap Y \neq \emptyset$ and $(X_iY),\ (X_iY)^*$ are not part of any forbidden word.

(b) $\exists Y_1, Y_2 \in T,\ Y_1 \neq Y_2,\ r(Y_1) = r(Y_2)$ and $X \in C$ such that for $i \in \{1, 2\}$, $Y_i \cap X \neq \emptyset$ and $(Y_iX),\ (Y_iX)^*$ are not part of any forbidden word.

(c) $\exists X \in S,\ Y \in D$ such that $X \cap Y \neq \emptyset$, $s(X) = r(Y)$ and (XY),$(XY)^*$ are not part of any forbidden word.

(d) $\exists Y \in T,\ X \in C$ such that $X \cap Y \neq \emptyset$, $s(X) = r(Y)$ and (XY),$(XY)^*$ are not part of any forbidden word.

Proposition 3.4.20. *If $\dot{E}$ is a finite docile object in **tBSG** and satisfies condition A' then $\mathcal{A}_K(\dot{E})$ has exponential growth.*

Definition 3.4.21. [61, Definition 20,21] Let $\dot{E}$ be any docile object in **tBSG**. A **quasi-cycle** is a normal generalized path p in $\widehat{E}$ such that p^2 is normal and none of the sub-words of p^2 of length less than $|p|$ is normal. A quasi-cycle p is called **self-connected** if there is a normal path o in $\widehat{E}$ such that p is not a prefix of o and pop is normal.

Theorem 3.4.22. *Let $\dot{E}$ be a finite docile object in **tBSG**. Then $\mathcal{A}_K(\dot{E})$ has exponential growth if and only if there is a self-connected quasi-cycle.*

Remark 3.4.23. Let $\dot{E}$ be any docile object in **tBSG** and suppose that $\{\dot{E}_i \mid i \in I\}$ is a directed system of all finite complete sub-objects of $\dot{E}$. By results of [53, Section 3], we have $\text{GKdim}(\mathcal{A}_K(\dot{E})) = \sup_{i \in I} \text{GKdim}(\mathcal{A}_K(\dot{E}_i))$.

3.4.3 Additional applications of Linear bases

In this subsection we fix the following notations. Let (R, I) be a K-algebra with enough idempotents. An element $a \in R$ is called **homogeneous** if $a \in vRw$ for some $v, w \in I$.

Let B denote a K-basis for R which consists of homogeneous elements and contains I. Let $l : B \to \mathbb{Z}^+$ be a map such that $l(b) = 0 \Leftrightarrow b \in I$.

Definition 3.4.24. An element $b \in B \cap vRw$ is called **left adhesive** if $ab \in B$ for any $a \in B \cap Rv$ and **right adhesive** if $bc \in B$ for any $c \in B \cap wR$. A *left valuative basis element* is a left adhesive element $b \in B \cap eR$ such that $l(ab) = l(a) + l(b)$ for any $a \in B \cap Rv$. A **right valuative basis element** is defined similarly. A **valuative basis element** is an adhesive element $b \in B \cap vRw$ such that $l(abc) = l(a) + l(b) + l(c)$ for any $a \in B \cap Rv$ and $c \in B \cap wR$.

Proposition 3.4.25. *[61, Proposition 53] Suppose there exists a valuative basis element $b \in (B - I) \cap vRv$. Then $\dim_K(R) = \infty$, R is not simple, neither left nor right Artinian and not von Newmann regular.*

Definition 3.4.26. A docile object $\dot{E}$ in **tBSG** is said to satisfy **Condition** (A) if

$(A1)$: $S = T = \emptyset$ implies $|E^0| = \infty$ or $|E^1| > 0$.

$(A2)$: $S \neq \emptyset$ or $T \neq \emptyset$ implies at least one of the following holds:

(a) $\exists\ X \in S,\ Y \in D$ such that $X \cap Y \neq \emptyset$, $(XY)(XY)^*$ and $(XY)^*(XY)$ are not forbidden words.

(b) $\exists\ Y \in T,\ X \in C$ such that $X \cap Y \neq \emptyset$, $(XY)(XY)^*$ and $(XY)^*(XY)$ are not forbidden words.

Let B denote the set of all normal generalized paths of $\mathcal{A}_K(\dot{E})$. Let $l : B \to \mathbb{Z}^+$ denote the map which maps a path to its length. If $\dot{E}$ satisfies Condition $(A2)$ then we can choose either $X \in S$, $Y \in D$ or $Y \in T$, $X \in C$ such that $X \cap Y \neq \emptyset$ and $(XY)(XY)^*$ is not forbidden. Set $b = (XY)(XY)^*$, then b is a valuative basis element. Hence we have the following corollary:

Corollary 3.4.27. *Let $\dot{E}$ be a docile object in **tBSG** that satisfies Condition (A). Then $\dim_K(\mathcal{A}_K(\dot{E})) = \infty$, $\mathcal{A}_K(\dot{E})$ is not simple, neither left nor right Artinian and not von Neumann regular.*

Definition 3.4.28. Let $b \in B \cap vRw$ and $b' \in B \cap v'Rw'$. We say that b and b' have **no common left multiple** if there is no $a \in B \cap Rv$ and $a' \in B \cap Rv'$ such that $ab = a'b'$.

We say that b and b' have **no common right multiple** if there is no $c \in B \cap wR$ and $c' \in B \cap w'R$ such that $bc = b'c'$.

An element $b \in B \cap vRw$ is called **right cancellative** if $ab = cb \Rightarrow a = c$ for any $a, c \in B \cap Rv$ and **left cancellative** if $ba' = bc' \Rightarrow a' = c'$ for any $a', c' \in B \cap wR$.

Proposition 3.4.29. *[61, Proposition 56] If there are elements $b, b' \in B \cap vRv$ such that b is adhesive and right cancellative, b' is left adhesive and b and b' have no common left multiple, then R is not left Noetherian.*

If there are elements $c, c' \in B \cap vRv$ such that c is adhesive and left cancellative, c' is right adhesive and c and c' have no common left multiple, then R is not right Noetherian.

We have the following corollary which gives a necessary condition for $\mathcal{A}_K(\dot{E})$ to be a left or right Noetherian in terms of $\dot{E}$.

Corollary 3.4.30. *Let $\dot{E}$ be a docile object in **tBSG** that satisfies Condition (A'). Then $\mathcal{A}_K(\dot{E})$ is neither left nor right Noetherian.*

Proof. We prove the statement only for conditions $(A'2)(a)$ and $(A'2)(c)$ leaving the other simple cases to the reader.

Suppose there exist $X_1, X_2 \in S$, $X_1 \neq X_2$, $s(X_1) = s(X_2) = v$ and $Y \in D$ such that for $i \in \{1, 2\}$, $X_i \cap Y \neq \emptyset$ and (X_iY), $(X_iY)^*$ are part of forbidden words. Then set $b_1 = (X_1Y)(X_1Y)^*$, $b_2 = (X_1Y)(X_2Y)^*$ and $b_3 = (X_2Y)(X_1Y)^*$. Then $b_1, b_2, b_3 \in (B - E^0) \cap v\mathcal{A}_K(\dot{E})v$. It is easy to check that b_1 is adhesive and both left and right cancellative, b_2 is left adhesive, b_3 is right adhesive, $b_1,\ b_2$ have no common left multiple and $b_1,\ b_3$ have no common right multiple. Thus $\mathcal{A}_K(\dot{E})$ is neither left nor right Noetherian.

Now suppose that there exist $X \in S$ and $Y \in D$ such that $X \cap Y \neq \emptyset$, $s(X) = r(Y) = v$ and (XY), $(XY)^*$ are not part of any forbidden word. Then both (XY), $(XY)^*$ are in $(B - E^0) \cap v\mathcal{A}_K(\dot{E})v$, they are adhesive, both left and right cancellative and have neither left nor right common multiple. Therefore $\mathcal{A}_K(\dot{E})$ is neither left nor right Noetherian. □

Chapter 4

Cohn-Leavitt path algebras of semi-regular hypergraphs

In this chapter we specialize our attention to hypergraphs and study their Cohn-Leavitt path algebras.

4.1 Semi-regular hypergraphs and their H-monoids

We begin by recalling the definition of hypergraphs introduced in [61] (See Definition 3.1.8).

Definition 4.1.1. A **hypergraph** is a quadruple $\mathcal{H} = (\mathcal{H}^0, \mathcal{H}^1, s, r)$ where $\mathcal{H}^0$ and $\mathcal{H}^1$ are sets called the set of vertices and the set of hyperedges respectively. For each $h \in \mathcal{H}^1$ there exists a pair of non-empty indexing sets I_h, J_h such that $s(h) : I_h \to \mathcal{H}^0$, and $r(h) : J_h \to \mathcal{H}^0$ are families of vertices.

Let $\mathcal{H}$ be a hypergraph. A hyperedge $h \in \mathcal{H}^1$ is called **source regular** (resp. **range regular**) if I_h is finite (resp. J_h is finite). The set of all source regular hyperedges of $\mathcal{H}$ is denoted by $\mathcal{H}^1_{\text{sreg}}$ and the set of all range regular hyperedges of $\mathcal{H}$ is denoted by $\mathcal{H}^1_{\text{rreg}}$. The hypergraph $\mathcal{H}$ is said to be **regular** if $\mathcal{H}^1 = \mathcal{H}^1_{\text{sreg}} = \mathcal{H}^1_{\text{sreg}}$.

The **Leavitt path algebra** $L_K(\mathcal{H})$ **of the hypergraph** $\mathcal{H}$ is the K-algebra presented by the generating set $\{v, h_{ij}, h^*_{ij} \mid v \in \mathcal{H}^0, h \in \mathcal{H}^1, i \in I_h, j \in J_h\}$ and the relations

1. $uv = \delta_{u,v}u$, for every $u, v \in \mathcal{H}^0$,

2. $s(h)_i h_{ij} = h_{ij} = h_{ij} r(h)_j$ and $r(h)_j h^*_{ij} = h^*_{ij} = h^*_{ij} s(h)_i$, for every $h \in \mathcal{H}^1$, $i \in I_h$, and $j \in J_h$,

3. $\sum\limits_{j \in J_h} h_{ij} h^*_{kj} = \delta_{ik} s(h)_i$, for every $h \in \mathcal{H}^1_{\text{rreg}}$ and $i, k \in I_h$,

4. $\sum\limits_{i \in I_h} h^*_{ij} h_{ik} = \delta_{jk} r(h)_j$, for every $h \in \mathcal{H}^1_{\text{sreg}}$ and $j, k \in J_h$.

In the above definition, a hyperedge h gives rise to a matrix $[h]$ of order $|I_h| \times |J_h|$, whose $(i, j)^{\text{th}}$ entry is h_{ij}. Thus relation (3) and (4) can be re written as follows:

3. For every $h \in \mathcal{H}^1_{\text{rreg}}$, the matrix equation $[h][h]^* = D^s_h$ holds, where D^s_h is the diagonal matrix of order $|I_h| \times |I_h|$ whose $(i, i)^{\text{th}}$ entry is $s(h)_i$.

4. For every $h \in \mathcal{H}^1_{\text{sreg}}$, the matrix equation $[h]^*[h] = D^r_h$ holds, where D^r_h is the diagonal matrix of order $|J_h| \times |J_h|$ whose $(j, j)^{\text{th}}$ entry is $r(h)_j$.

First note that a hyperedge h, which is neither source regular nor range regular (that is both I_h and J_h are infinite), does not contribute to the defining relations of the Leavitt path algebras. Thus we could consider the class of 'semi-regular' hypergraphs: A hypergraph $\mathcal{H}$ in which for each $h \in \mathcal{H}^1$, either I_h is finite or J_h is finite. In other words, its only hyperedges are either source regular or range regular.

We would like to recast the definition of semi-regular hypergraphs in terms of bi-separated graphs (with distinguished subsets) so that we can study their Cohn-Leavitt path algebras.

Definition 4.1.2. A **semi-regular hypergraph** is a pair $(\dot{E}, \Lambda)$, where $\dot{E} = (E, (C, S), (D, T))$ is a bi-separated graph and Λ is a nonempty indexing set, called the *hyperedges*, such that for each $\lambda \in \Lambda$ there exists $\mathcal{X}_\lambda \subseteq C$ and $\mathcal{Y}_\lambda \subseteq D$ which further satisfy the following conditions:

1. $\Lambda = \Lambda_T^S \sqcup \Lambda_{\text{fin}}^S \sqcup \Lambda_\infty^S \sqcup \Lambda_T^{\text{fin}} \sqcup \Lambda_T^\infty$, and

$$\begin{aligned}
S &= \bigsqcup_{\lambda \in \Lambda^S} \mathcal{X}_\lambda, \quad \text{where} \quad \Lambda^S = \Lambda_T^S \sqcup \Lambda_{\text{fin}}^S \sqcup \Lambda_\infty^S,\\
T &= \bigsqcup_{\lambda \in \Lambda_T} \mathcal{Y}_\lambda, \quad \text{where} \quad \Lambda_T = \Lambda_T^S \sqcup \Lambda_T^{\text{fin}} \sqcup \Lambda_T^\infty,\\
C_{\text{fin}} - S &= \bigsqcup_{\lambda \in \Lambda_T^{\text{fin}}} \mathcal{X}_\lambda,\\
D_{\text{fin}} - T &= \bigsqcup_{\lambda \in \Lambda_{\text{fin}}^S} \mathcal{Y}_\lambda,\\
C - C_{\text{fin}} &= \bigsqcup_{\lambda \in \Lambda_T^\infty} \mathcal{X}_\lambda,\\
D - D_{\text{fin}} &= \bigsqcup_{\lambda \in \Lambda_\infty^S} \mathcal{Y}_\lambda.
\end{aligned}$$

2. $X \notin S$ and $Y \notin T \implies X \cap Y = \emptyset$,

3. for any $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$, $X \in \mathcal{X}_\alpha$ and $Y \in \mathcal{Y}_\beta \implies X \cap Y = \emptyset$,

4. for any $\lambda \in \Lambda$, $X \in \mathcal{X}_\lambda$ and $Y \in \mathcal{Y}_\lambda \implies X \cap Y \neq \emptyset$,

In order to make the above definition more transparent, let us make a few remarks.

Remark 4.1.3. First note that the hyperedge $\lambda \in \Lambda$ corresponds to the matrix $[\lambda]$, whose set of rows is $\mathcal{X}_\lambda$ and the set of columns is $\mathcal{Y}_\lambda$. Further, for $X \in \mathcal{X}_\lambda$ and $Y \in \mathcal{Y}_\lambda$, the $(X,Y)^{\text{th}}$ entry is $X \cap Y$. Thus the condition (1) in the above definition says that set of all hyperedges can be partitioned according to their corresponding matrices in the following way:

i. Λ_T^S is the set of all hyperedges whose rows and columns are in S and T respectively.

ii. Λ_{fin}^S is the set of all hyperedges whose the rows are in S and columns finite.

iii. Λ_∞^S is the set of all hyperedges whose rows are in S and columns infinite.

iv. Λ_T^{fin} is the set of all hyperedges whose rows are finite and columns are in T.

v. Λ_T^∞ is the set of all hyperedges whose rows are infinite and columns are in T.

It is easy to see that we have avoided those hyperedges whose rows finite but not in S and columns finite but not in T. This is because, in the definition of Cohn-Leavitt path algebras the relations are contributed from rows and columns belonging to the distinguished sets S and T respectively.

Remark 4.1.4.

1. It is easy to check that semi-regular hypergraphs are tame, docile and that semi-regular hypergraphs, along with complete morphisms, form a category. This category will be denoted by **BHG**.
2. Note that given a semi-regular hypergraph $(\dot{E}, \Lambda)$ with $S = C_{\text{fin}}$ and $T = D_{\text{fin}}$ we can identify $(\dot{E}, \Lambda)$ with a hypergraph $\mathcal{H}$ as follows: $\mathcal{H}^0 = E^0$, $\mathcal{H}^1 = \Lambda$, and for each $\lambda \in \Lambda$ $s(\lambda) = (s(X))_{X \in \mathcal{X}_\lambda}$ and $r(\lambda) = (r(Y))_{Y \in \mathcal{Y}_\lambda}$.

Notation 4.1.5. For $\lambda \in \Lambda_T$, set

$$\begin{aligned} Q_\lambda &= \{q_Z \mid Z \subseteq \mathcal{Y}_\lambda,\ 0 < |Z| < \infty\} \text{ and} \\ Q &= \bigsqcup_{\lambda \in \Lambda_T} Q_\lambda. \end{aligned}$$

For $\lambda \in \Lambda^S$, set

$$\begin{aligned} P_\lambda &= \{p_W \mid W \subseteq \mathcal{X}_\lambda,\ 0 < |W| < \infty\} \text{ and} \\ P &= \bigsqcup_{\lambda \in \Lambda^S} P_\lambda. \end{aligned}$$

Definition 4.1.6. Given a semi-regular hypergraph $(\dot{E}, \Lambda)$, its H-monoid $H(\dot{E}, \Lambda)$ is defined as the abelian monoid generated by $E^0 \sqcup Q \sqcup P$ modulo the following relations:

1. For $\lambda \in \Lambda_T$ and $q_Z \in Q_\lambda$,

$$\sum_{X \in \mathcal{X}_\lambda} s(X) = \sum_{Y \in Z} r(Y) + q_Z,$$

2. For $\lambda \in \Lambda^S$ and $p_W \in P_\lambda$,

$$\sum_{Y \in \mathcal{Y}_\lambda} r(Y) = \sum_{X \in W} s(X) + p_W,$$

3. For $\lambda \in \Lambda_T$ and $q_{Z_1}, q_{Z_2} \in Q_\lambda$ with $Z_1 \subsetneq Z_2$

$$q_{Z_1} = q_{Z_2} + \sum_{Y \in Z_2 - Z_1} r(Y),$$

4. For $\lambda \in \Lambda^S$ and $p_{W_1}, p_{W_2} \in P_\lambda$ with $W_1 \subsetneq W_2$

$$p_{W_1} = p_{W_2} + \sum_{X \in W_2 - W_1} s(X),$$

5. for $\lambda \in \Lambda^S_T$,

$$q_{\mathcal{Y}_\lambda} = 0 = p_{\mathcal{X}_\lambda}.$$

If $(\dot{E}, \Lambda)$ is semi-regular hypergraph then $H(\dot{E}, \Lambda)$ is a conical monoid. This is easy to see from the relations defining $H(\dot{E}, \Lambda)$ because these relations ensure that $(x+y) \neq 0$ whenever $x \neq 0$ or $y \neq 0$, for $x, y \in H(\dot{E}, \Lambda)$.

Definition 4.1.7. Let R be a ring, and let $M_\infty(R)$ denote the set of all $\omega \times \omega$ matrices over R with finitely many nonzero entries, where ω varies over $\mathbb{N}$. For idempotents $e, f \in M_\infty(R)$, the **Murray-von Neumann equivalence** $\sim$ is defined as follows: $e \sim f$ if and only if there exists $x, y \in M_\infty(R)$ such that $xy = e$ and $yx = f$.

Let $\mathcal{V}(R)$ denote the set of all equivalence classes $[e]$, for idempotents $e \in M_\infty(R)$. Define $[e]+[f] = [e \oplus f]$, where $e \oplus f$ denotes the block diagonal matrix $\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$. Under this operation, $\mathcal{V}(R)$ is an abelian monoid, and it is conical, that is, $a + b = 0$ in $\mathcal{V}(R)$ implies $a = b = 0$. Also $\mathcal{V}(\_)$:**Rings**$\to$**Mon** is a continuous functor.

Let R be a unital ring and let $\mathcal{U}(R)$ be the set of all isomorphic classes of finitely generated projective left R-modules, endowed with direct sum as binary operation. Then $(\mathcal{U}(R), \oplus)$ is an abelian monoid. We also have $\mathcal{U}(R) \cong \mathcal{V}(R)$.

Theorem 4.1.8. *There is an isomorphism* $\Gamma : H \to \mathcal{V} \circ \mathcal{A}_K$ *of funtors* **BHG** $\to$ **Mon**.

Proof. We first define the map Γ as follows: For each object $\dot{E}$ in **BHG**,

$$\Gamma(\dot{E}, \Lambda) : H(\dot{E}, \Lambda) \to \mathcal{V} \circ \mathcal{A}_K(\dot{E}, \Lambda)$$

is the monoid homomorphism sending

$$v \mapsto [v],$$

$$q_Z \mapsto [\mathrm{diag}(s(X)) - BB^*]$$

and

$$p_W \mapsto [\mathrm{diag}(r(Y)) - N^*N],$$

where $v \in E^0$, Z is any non-empty finite subset of $\mathcal{Y}_\lambda$, $\mathrm{diag}(s(X))$ is the diagonal matrix of order $|\mathcal{X}_\lambda|$ with diagonal entries coming from the set $s(\mathcal{X}_\lambda)$ in any order (without repetition), $\mathrm{diag}(r(Y))$ is the diagonal matrix of order $|\mathcal{Y}_\lambda|$ with diagonal entries coming from the set $r(\mathcal{Y}_\lambda)$ in any order (without repetition), B is the $|\mathcal{X}_\lambda| \times |Z|$ matrix whose columns are precisely the ones in Z and whose i^{th} row consists elements of X if and only if the diagonal entry in the i^{th} row of $\mathrm{diag}(s(X))$ is $s(X)$, and N is the $|W| \times |\mathcal{Y}_\lambda|$ matrix whose rows are precisely the ones in W and whose j^{th} column has elements from Y if and only if the diagonal entry in the j^{th} column of $\mathrm{diag}(r(Y))$ is $r(Y)$.

It is not hard to see that the above map is well defined. Also the fact that every element in **BHG** is a direct sum of its finite complete sub-objects and the continuity of the functors involved will suggest that it is enough to prove the results for finite sub-objects. For the finite case, we use induction on $|\Lambda|$. For $\Lambda = \emptyset$, the result is trivial. So, suppose that the result holds for all finite objects $(\dot{F}, \Lambda_{\dot{F}})$ in **BHG** with $|\Lambda_{\dot{F}}| \leq (n-1)$, for some $n \geq 1$. Let $(\dot{E}, \Lambda_{\dot{E}})$ be a finite object with $|\Lambda_{\dot{E}}| = n$. Fix an element $\lambda \in \Lambda_{\dot{E}}$. We can now apply induction to the object $\dot{F}$ obtained from the $\dot{E}$ by deleting all the edges in λ and leaving the remaining structure as it is, keeping $F^0 = E^0$.

First suppose that $\lambda \in \Lambda_T^S$. Then $H(\dot{E}, \Lambda_{\dot{E}})$ is obtained from $H(\dot{F}, \Lambda_{\dot{F}})$ by going modulo the relation $\sum\limits_{X \in \mathcal{X}_\lambda} s(X) = \sum\limits_{Y \in \mathcal{Y}_\lambda} r(Y)$. Also, the algebra $\mathcal{A}_K(\dot{E}, \Lambda_{\dot{E}})$ is the Bergman algebra obtained from $\mathcal{A}_K(\dot{F}, \Lambda_{\dot{F}})$ by adjoining a universal isomorphism between the finitely generated projective modules $\bigoplus\limits_{X \in \mathcal{X}_\lambda} \mathcal{A}_K(\dot{F}, \Lambda_{\dot{F}})s(X)$ and $\bigoplus\limits_{Y \in \mathcal{Y}_\lambda} \mathcal{A}_K(\dot{F}, \Lambda_{\dot{F}})r(Y)$. So by [24, Theorem 5.2], $\mathcal{V}(\mathcal{A}_K(\dot{E}, \Lambda_{\dot{E}}))$ is the quotient of $\mathcal{V}(\mathcal{A}_K(\dot{F}, \Lambda_{\dot{F}}))$ modulo the relation

$$[\mathrm{diag}(s(X))] = [\mathrm{diag}(r(Y))].$$

Since the map

$$\Gamma(\dot{F}, \Lambda_{\dot{F}}) : H(\dot{F}, \Lambda_{\dot{F}}) \to \mathcal{V} \circ \mathcal{A}_K(\dot{F}, \Lambda_{\dot{F}})$$

is an isomorphism by induction hypothesis, the desired result follows.

Now suppose λ does not belong to Λ_T^S. Then it is either in Λ_T^∞ or in Λ_∞^S. Let us first assume that $\lambda \in \Lambda_T^\infty$. In this case, $H(\dot{E}, \Lambda_{\dot{E}})$ is obtained from $H(\dot{F}, \Lambda_{\dot{F}})$ by adjoining a new generator $q_{\mathcal{Y}_\lambda}$ and going modulo the relation

$$\sum_{X\in\mathcal{X}_\lambda} s(X) = \sum_{Y\in\mathcal{Y}_\lambda} r(Y) + q_{\mathcal{Y}_\lambda}.$$

On the algebra side, $\mathcal{A}_K(\dot{E}, \Lambda_{\dot{E}})$ is obtained from $\mathcal{A}_K(\dot{F}, \Lambda_{\dot{F}})$ in two steps by

1. first adjoining the mutually perpendicular idempotents $\operatorname{diag}(s(X)) - BB^*$ and $q'_{\mathcal{Y}_\lambda}$, and going modulo the relation

$$[\operatorname{diag}(s(X))] = [BB^*] + q'_{\mathcal{Y}_\lambda},$$

thereby, getting a new algebra R and then

2. adjoining a universal isomorphism between the left module corresponding to $[BB^*]$ and the left module $\bigoplus_{Y\in\mathcal{Y}_\lambda} Rr(Y)$.

So, by [24, Theorems 5.1, 5.2], $\mathcal{V}(\mathcal{A}_K(\dot{E}, \Lambda_{\dot{E}}))$ is obtained from $\mathcal{V}(\mathcal{A}_K(\dot{F}, \Lambda_{\dot{F}}))$ by adjoining a new generator $q''_{\mathcal{Y}_\lambda}$ and going modulo the relation

$$[\operatorname{diag}(s(X))] = [\operatorname{diag}(r(Y))] + q''_{\mathcal{Y}_\lambda}.$$

This, along with the induction hypothesis, proves the theorem for the considered case.

Finally suppose $\lambda \in \Lambda^S_\infty$. Again $H(\dot{E}, \Lambda_{\dot{E}})$ is obtained from $H(\dot{F}, \Lambda_{\dot{F}})$ by adjoining a new generator $p_{\mathcal{X}_\lambda}$ and going modulo the relation

$$\sum_{Y\in\mathcal{Y}_\lambda} r(Y) = \sum_{X\in\mathcal{X}_\lambda} s(X) + p_{\mathcal{X}_\lambda}.$$

On the other hand, analogous to the previous case, the algebra $\mathcal{A}_K(\dot{E}, \Lambda_{\dot{E}})$ is obtained from $\mathcal{A}_K(\dot{F}, \Lambda_{\dot{F}})$ in two steps by

1. first adjoining the mutually perpendicular idempotents $\operatorname{diag}(r(Y)) - N^*N$ and $p'_{\mathcal{X}_\lambda}$, and going modulo the relation

$$[\operatorname{diag}(r(Y))] = [N^*N] + p'_{\mathcal{X}_\lambda},$$

thereby, getting a new algebra R' and then

2. adjoining a universal isomorphism between the left module corresponding to $[N^*N]$ and the left module $\bigoplus_{X\in\mathcal{X}_\lambda} R's(X)$.

So, by [24, Theorems 5.1, 5.2], $\mathcal{V}(\mathcal{A}_K(\dot{E},\Lambda_{\dot{E}}))$ is obtained from $\mathcal{V}(\mathcal{A}_K(\dot{F},\Lambda_{\dot{F}}))$ by adjoining a new generator $p''_{\mathcal{X}_\lambda}$ and going modulo the relation

$$[\mathrm{diag}(r(Y))] = [\mathrm{diag}(s(X))] + p''_{\mathcal{X}_\lambda},$$

thereby completing the proof (using induction hypothesis).

□

Remark 4.1.9. We note that if M is any conical abelian monoid then there exists a semi-regular hypergraph $(\dot{E},\Lambda_{\dot{E}})$ such that $M \cong H(\dot{E},\Lambda) \cong \mathcal{V}(\mathcal{A}_K(\dot{E},\Lambda_{\dot{E}}))$. For two different proofs of this fact, we refer the reader to [20, Proposition 4.4] or [61, Proposition 62].

4.2 Ideal lattices and Simplicity

In this section, $(\dot{E},\Lambda)$ always denotes a semi-regular hypergraph. Throughout this section, we use the following notation: For $\lambda\in\Lambda$,

$$s(\lambda) := \bigcup_{X\in\mathcal{X}_\lambda} s(X) \text{ and } r(\lambda) := \bigcup_{Y\in\mathcal{Y}_\lambda} r(X).$$

4.2.1 The lattice of admissible triples in $(\dot{E},\Lambda)$

Definition 4.2.1. A subset V of E^0 is called **bisaturated** if for each $\lambda\in\Lambda_T^S$,

$$s(\lambda)\subseteq V \iff r(\lambda)\subseteq V.$$

The set of all bisaturated subsets of E^0 is denoted by $\mathrm{BS}(\dot{E},\Lambda)$.

Note that empty set and E^0 are always elements of $\mathrm{BS}(\dot{E},\Lambda)$. It is easy to check that $\mathrm{BS}(\dot{E},\Lambda)$ is closed under arbitrary intersections.

If V is a subset of E^0, the **bisaturated closure of** V, denoted $\overline{V}$, is the smallest bisaturated subset of E^0 containing V. Since the intersection of bisaturated subsets of E^0 is again bisaturated, $\overline{V}$ is well defined.

For $V \subseteq E^0$, $\overline{V}$ can be explicitly constructed as follows: Define $V_0 = V$. If n is an odd positive integer, define

$$V_n = V_{n-1} \cup \{r(Y) \mid Y \in \mathcal{Y}_\lambda, \lambda \in \Lambda_T^S, \text{ and } s(\lambda) \subseteq V_{n-1}\},$$

and if n is an even positive integer, define

$$V_n = V_{n-1} \cup \{s(X) \mid X \in \mathcal{X}_\lambda, \lambda \in \Lambda_T^S, \text{ and } r(\lambda) \subseteq V_{n-1}\}.$$

Then $\overline{V} = \bigcup_{n \geq 0} V_n$.

Definition 4.2.2. Let $V \subseteq E^0$ be bisaturated and for any $\lambda \in \Lambda$, set

$$\mathcal{X}_{\lambda/V} = \{X \in \mathcal{X}_\lambda \mid s(X) \notin V\} \text{ and } \mathcal{Y}_{\lambda/V} = \{Y \in \mathcal{Y}_\lambda \mid r(Y) \notin V\}.$$

Then set

$$\Lambda/V = \Lambda_T^S/V \sqcup \Lambda_{\text{fin}}^S/V \sqcup \Lambda_\infty^S/V \sqcup \Lambda_T^{\text{fin}}/V \sqcup \Lambda_T^\infty/V,$$

where

$$\begin{aligned}
\Lambda_T^S/V &:= \{\lambda \in \Lambda_T^S \mid 0 < |\mathcal{X}_{\lambda/V}|\} = \{\lambda \in \Lambda_T^S \mid 0 < |\mathcal{Y}_{\lambda/V}|\},\\
\Lambda_{\text{fin}}^S/V &:= \{\lambda \in \Lambda_{\text{fin}}^S, \mid 0 < |\mathcal{X}_{\lambda/V}|\},\\
\Lambda_\infty^S/V &:= \{\lambda \in \Lambda_\infty^S \mid 0 < |\mathcal{X}_{\lambda/V}| < \infty\},\\
\Lambda_T^{\text{fin}}/V &:= \{\lambda \in \Lambda_T^{\text{fin}} \mid 0 < |\mathcal{Y}_{\lambda/V}|\},\\
\Lambda_T^\infty/V &:= \{\lambda \in \Lambda_T^\infty \mid 0 < |\mathcal{Y}_{\lambda/V}| < \infty\}.
\end{aligned}$$

Let $V \subseteq E^0$ be a bisaturated set, $\Sigma \subseteq \Lambda_{\text{fin}}^S/V \sqcup \Lambda_\infty^S/V$ and $\Theta \subseteq \Lambda_T^{\text{fin}}/V \sqcup \Lambda_T^\infty/V$. A triple (V, Σ, Θ) is called an **admissible triple** and the set of all admissible triples in $(\dot{E}, \Lambda)$ is denoted by $\mathrm{AT}(\dot{E}, \Lambda)$.

We define a relation $\leq$ in $\mathrm{AT}(\dot{E}, \Lambda)$ as follows: $(V_1, \Sigma_1, \Theta_1) \leq (V_2, \Sigma_2, \Theta_2)$ if

$$V_1 \subseteq V_2,$$

$$\Sigma_1 \subseteq \Sigma_2 \sqcup \Lambda^S(V), \text{ where } \Lambda^S(V) = \Lambda^S - \Lambda/V,$$

$$\Theta_1 \subseteq \Theta_2 \sqcup \Lambda_T(V), \text{ where } \Lambda_T(V) = \Lambda_T - \Lambda/V.$$

We note that $(E^0, \emptyset, \emptyset)$ is the maximum and $(\emptyset, \emptyset, \emptyset)$ is the minimum in $\mathrm{AT}(\dot{E}, \Lambda)$.

Definition 4.2.3. Let V be a bisaturated subset of E^0, $\Sigma \subseteq \Lambda^S(V) \sqcup \Lambda^S_{\text{fin}}/V \sqcup \Lambda^S_{\infty}/V$, and $\Theta \subseteq \Lambda_T(V) \sqcup \Lambda_T^{\text{fin}}/V \sqcup \Lambda_T^{\infty}/V$. The (Σ, Θ)**-bisaturation of** V is defined as the smallest bisaturated subset $\overline{V}(\Sigma, \Theta)$ of E^0 containing H such that

1. If $\lambda \in \Sigma$ and $s(\lambda) \subseteq \overline{V}(\Sigma, \Theta)$, then $r(\lambda) \subseteq \overline{V}(\Sigma, \Theta)$ and
2. If $\lambda \in \Theta$ and $r(\lambda) \subseteq \overline{V}(\Sigma, \Theta)$, then $s(\lambda) \subseteq \overline{V}(\Sigma, \Theta)$.

We can construct (Σ, Θ)-bisaturation of V as follows- Define $\overline{V}_0(\Sigma, \Theta) = V$. If n is odd positive integer, define

$$\overline{V}_n(\Sigma, \Theta) = \overline{V}_{n-1}(\Sigma, \Theta) \cup \{r(Y) \in E^0 - \overline{V}_{n-1}(\Sigma, \Theta) \mid Y \in \mathcal{Y}_\lambda, \lambda \in \Lambda^S_{\text{fin}} \cup \Sigma \text{ and } s(\lambda) \subseteq \overline{V}_{n-1}(\Sigma, \Theta)\},$$

and if n is an even positive integer, define

$$\overline{V}_n(\Sigma, \Theta) = \overline{V}_{n-1}(\Sigma, \Theta) \cup \{s(X) \in E^0 - \overline{V}_{n-1}(\Sigma, \Theta) \mid X \in \mathcal{X}_\lambda, \lambda \in \Lambda_T^{\text{fin}} \cup \Theta \text{ and } r(\lambda) \subseteq \overline{V}_{n-1}(\Sigma, \Theta)\}.$$

Then $\overline{V}(\Sigma, \Theta) = \bigcup_{n \geq 0} \overline{V}_n(\Sigma, \Theta)$.

Proposition 4.2.4. $(\mathrm{AT}(\dot{E}, \Lambda), \leq)$ *is a lattice, with supremum* $\vee$ *and infimum* $\wedge$ *given by*

$$(V_1, \Sigma_1, \Theta_1) \vee (V_2, \Sigma_2, \Theta_2) = (\widetilde{V}, \widetilde{\Sigma}, \widetilde{\Theta}),$$

where

$$\widetilde{V} = \overline{V_1 \cup V_2}(\Sigma_1 \cup \Sigma_2, \Theta_1 \cup \Theta_2),$$

$$\widetilde{\Sigma} = (\Sigma_1 \cup \Sigma_2) - \Lambda^S(\widetilde{V}),$$

$$\widetilde{\Theta} = (\Theta_1 \cup \Theta_2) - \Lambda_T(\widetilde{V}),$$

and

$$(V_1, \Sigma_1, \Theta_1) \wedge (V_2, \Sigma_2, \Theta_2) = (\widehat{V}, \widehat{\Sigma}, \widehat{\Theta}),$$

where

$$\widehat{V} = (V_1 \cap V_2),$$

$$\widehat{\Sigma} = (\Sigma_1 \cup \Lambda^S(V)) \cap (\Sigma_2 \cup \Lambda^S(V)) \cap (\Lambda^S_{\text{fin}}/V \sqcup \Lambda^S_\infty/V),$$

$$\widehat{\Theta} = (\Theta_1 \cup \Lambda_T(V)) \cap (\Theta_2 \cup \Lambda_T(V)) \cap (\Lambda_T^{\text{fin}}/V \sqcup \Lambda_T^\infty/V).$$

Proof. Clearly, $(\widetilde{V}, \widetilde{\Sigma}, \widetilde{\Theta}) \in \mathrm{AT}(\dot{E}, \Lambda)$ and is greater than $(V_i, \Sigma_i, \Theta_i)$ for $i = 1, 2$. Now let $(V, \Sigma, \Theta) \in \mathrm{AT}(\dot{E}, \Lambda)$ such that $(V_i, \Sigma_i, \Theta_i) \leq (V, \Sigma, \Theta)$ for $i = 1, 2$. It is enough to prove that $\widetilde{V} \subseteq V$ for all $n \in \mathbb{Z}^+$. We do this inductively. Define $\widetilde{V}_n = \overline{(V_1 \cup V_2)}_n(\Sigma_1 \cup \Sigma_2, \Theta_1 \cup \Theta_1)$. For $n = 0$ the claim is clear by assumption. Now assume that $n \geq 1$ and that $\widetilde{V}_{n-1} \subseteq V$. Let $v \in \widetilde{V}_n$. If $v \in s(\lambda)$ or $v \in r(\lambda)$ for $\lambda \in \Lambda^S_T$, then $v \in V$ because V is bisaturated. Now suppose $v \in s(\lambda)$ for $\lambda \in \Theta_1 \cup \Theta_2$. By definition and the induction hypothesis, we have $r(\lambda) \subseteq \widetilde{V}_m \subseteq V$, where m is largest even integer less than n. In particular, this implies that $\lambda \notin \Theta$. Since $\lambda \in \Theta_1 \cup \Theta_2 \subseteq \Lambda_T(V) \cup \Theta$ we conclude that $v \in H$, which completes the induction step. The inclusion $(\Theta_1 \cup \Theta_2) - \Lambda_T(\widetilde{V}) \subseteq \Theta$ follows. Similar arguments shows that if $v \in r(\lambda)$ for $\lambda \in \Sigma_1 \cup \Sigma_2$, then $v \in V$ and $(\Theta_1 \cup \Theta_2) - \Lambda_T(\widetilde{V}) \subseteq \Theta$.

It is clear that $(\widehat{V}, \widehat{\Sigma}, \widehat{T}) \in \mathrm{AT}(\dot{E}, \Lambda)$ and $(\widehat{V}, \widehat{\Sigma}, \widehat{T}) \leq (V_i, \Sigma_i, \Theta_i)$ for $i = 1, 2$. If $(V, \Sigma, \Theta) \in \mathrm{AT}(\dot{E}, \Lambda)$ such that $(V, \Sigma, \Theta) \leq (V_i, \Sigma_i, \Theta_i)$ for $i = 1, 2$, then clearly $V \subseteq \widehat{V}$. Consider $\lambda \in \Theta - \Lambda_T(\widehat{V})$. Then there exists $v \in s(\lambda) - \widehat{V}$, so $v \notin V_j$ for some $j \in \{1, 2\}$, and $\lambda \notin \Lambda_T(V_j)$. Let us fix $j = 1$. Since $(V, \Sigma, \Theta) \leq (V_1, \Sigma_1, \Theta_1)$, it follows that $\lambda \in \Theta_1$. Hence, $\mathcal{Y}_{\lambda/V_1}$ is nonempty and $\mathcal{Y}_{\lambda/\widehat{V}}$ is nonempty. On the other hand, $\lambda \in \Theta$ implies that $\mathcal{Y}_{\lambda/V}$ is finite, hence $\mathcal{Y}_{\lambda/\widehat{V}}$ is finite. Thus $\lambda \in \Lambda_T^{\text{fin}}/\widehat{V} \sqcup \Lambda_T^\infty/\widehat{V}$. We also have $\lambda \in \Theta_i \sqcup \Lambda_T(V_i)$ for $i = 1, 2$, because $(V, \Sigma, \Theta) \leq (V_i, \Sigma_i, \Theta_i)$ for $i = 1, 2$, and consequently $\lambda \in \widehat{\Theta}$. This shows $\Theta \subseteq \widehat{\Theta} \sqcup \Lambda_T(V)$. Similarly we can show that $\Sigma \subseteq \widehat{\Sigma} \sqcup \Lambda^S(V)$ proving that $(V, \Sigma, \Theta) \leq (\widehat{V}, \widehat{\Sigma}, \widehat{\Theta})$. This shows that $(\widehat{V}, \widehat{\Sigma}, \widehat{\Theta})$ is the infimum required.

Hence $\mathrm{AT}(\dot{E}\Lambda)$ is a lattice. □

4.2.2 The lattice of order-ideals in $H(\dot{E}, \Lambda)$

Definition 4.2.5. An **order-ideal** of a monoid M is a submonoid I of M such that $x + y \in I$ for some $x, y \in M$ implies that both x and y belong to I.

Every monoid M is equipped with a pre-order $\leq$ as follows: for $x, y \in M$, $x \leq y$ if and only if there exists $z \in M$ such that $x + z = y$. Hence an equivalent definition of an order-ideal I is as follows: For each $x, y \in M$, if $x \leq y$ and $y \in I$ then $x \in I$.

Let $\mathcal{L}(M)$ denote the set of all order-ideals of M. We note that $\mathcal{L}(M)$ is closed under arbitrary intersections. For a submonoid J of M, let $\langle J \rangle$ consists of those elements $x \in M$ such that $x \leq y$ for some $y \in J$. Then $\langle J \rangle$ denotes the order-ideal generated by J. Then $\mathcal{L}(M)$ can be shown to be a complete lattice with respect to inclusion. For, given an arbitrary family $\{I_i\}$ of order-ideals of M, the supremum is given by $\langle \sum I_i \rangle$.

We want to study the lattice of order-ideals of $H(\dot{E}, \Lambda)$. For convenience, we modify some notations of the previous section as follows:

Notation 4.2.6.

$$\text{For } \lambda \in \Lambda_T, \quad \mathbf{s}(\lambda) := \sum_{X \in \mathcal{X}_\lambda} s(X).$$

$$\text{For } \lambda \in \Lambda^S, \quad \mathbf{r}(\lambda) := \sum_{Y \in \mathcal{Y}_\lambda} r(Y).$$

Note that the above sums are finite.

$$\text{For } \lambda \in \Lambda_T^{\text{fin}}, \quad q_\lambda := q_{\mathcal{Y}_\lambda}.$$

$$\text{For } \lambda \in \Lambda^S_{\text{fin}}, \quad p_\lambda := p_{\mathcal{X}_\lambda}.$$

Also,

$$\text{for } \lambda \in \Lambda_T^\infty, \text{ set } \mathcal{Z}_\lambda = \{Z \mid Z \subseteq \mathcal{Y}_\lambda, 0 < |Z| < \infty\},$$

$$\text{for } \lambda \in \Lambda^S_\infty, \text{ set } \mathcal{W}_\lambda = \{W \mid W \subseteq \mathcal{X}_\lambda, 0 < |W| < \infty\}.$$

$$\mathcal{Z} := \bigsqcup_{\lambda \in \Lambda_T^\infty} \mathcal{Z}_\lambda \text{ and } \mathcal{W} := \bigsqcup_{\lambda \in \Lambda^S_\infty} \mathcal{W}_\lambda.$$

Finally set

$$Q^0 = \{q_\lambda \mid \lambda \in \Lambda_T^{\text{fin}}\} \sqcup \{q_Z \mid Z \in \mathcal{Z}\},$$

$$P^0 = \{p_\lambda \mid \lambda \in \Lambda^S_{\text{fin}}\} \sqcup \{p_W \mid W \in \mathcal{W}\}.$$

Definition 4.2.7. Let F be the free abelian monoid on $E^0 \sqcup Q^0 \sqcup P^0$. We identify $H(\dot{E}, \Lambda)$ with $F/\sim$, where $\sim$ is the congruence on F generated by the relations

$$\mathbf{s}(\lambda) \sim \begin{cases} \mathbf{r}(\lambda) & \text{if } \lambda \in \Lambda_T^S, \\ \mathbf{r}(\lambda) + q_\lambda & \text{if } \lambda \in \Lambda_T^{\text{fin}}, \text{and} \\ \mathbf{r}(\lambda) + q_Z & \text{if } \lambda \in \Lambda_T^\infty \text{ and } Z \in \mathcal{Z}_\lambda, \end{cases}$$

and

$$\mathbf{r}(\lambda) \sim \begin{cases} \mathbf{s}(\lambda) + p_\lambda & \text{if } \lambda \in \Lambda_{\text{fin}}^S, \\ \mathbf{s}(\lambda) + p_W & \text{if } \lambda \in \Lambda_\infty^S \text{ and } W \in \mathcal{W}_\lambda, \end{cases}$$

for $Z_1, Z_2 \in \mathcal{Z}$ with $Z_1 \subsetneq Z_2$, and $q_{Z_1} \sim q_{Z_2} + \mathbf{r}(Z_2 - Z_1)$, and for $W_1, W_2 \in \mathcal{W}$ with $W_1 \subsetneq W_2$, and $p_{W_1} \sim p_{W_2} + \mathbf{s}(W_2 - W_1)$.

Lemma 4.2.8. *If I is an oder-ideal of $H(\dot{E}, \Lambda)$, then the set $V = I \cap E^0$ is bisaturated.*

Proof. Let $\lambda \in \Lambda_T^S$ and $r(\lambda) \subseteq V$, then $\mathbf{r}(\lambda) = \mathbf{s}(\lambda) \in I$. Since I is order-ideal, and $s(X) \leq \mathbf{s}(\lambda)$ for each $X \in \mathcal{X}_\lambda$, we have $s(X) \in I$ for each $X \in \mathcal{X}_\lambda$, and hence $s(\lambda) \subseteq V$. Converse follows similarly. $\square$

Definition 4.2.9. Let V be a bisaturated subset of E^0.

$$\text{For } \lambda \in \Lambda_T^\infty/V, \text{ if } 0 < |\mathcal{Y}_{\lambda/V}| < \infty, \quad q_{\lambda/V} := q_{\mathcal{X}_{\lambda/V}}.$$

$$\text{For } \lambda \in \Lambda_\infty^S/V, \text{ if } 0 < |\mathcal{X}_{\lambda/V}| < \infty, \quad p_{\lambda/V} := p_{\mathcal{X}_{\lambda/V}}$$

If I is an order-ideal of $V(\dot{E}, \Lambda)$, set $\psi(I) = (V, \Sigma, \Theta)$, where

$$\begin{aligned} V &:= I \cap E^0, \\ \Sigma &:= \{\lambda \in \Lambda_{\text{fin}}^S/V \mid p_\lambda \in I\} \sqcup \{\lambda \in \Lambda_\infty^S/V \mid p_{\lambda/V} \in I\}, \text{ and} \\ \Theta &:= \{\lambda \in \Lambda_T^{\text{fin}}/V \mid q_\lambda \in I\} \sqcup \{\lambda \in \Lambda_T^\infty/V \mid q_{\lambda/V} \in I\}. \end{aligned}$$

Conversely, for any $(V, \Sigma, \Theta) \in AT(\dot{E}, \Lambda)$, let $I(V, \Sigma, \Theta)$ denote the submonoid of $H(\dot{E}, \Lambda)$ generated by the set $V \sqcup Q(\Theta) \sqcup P(\Sigma)$, where

$$\begin{aligned} Q(\Theta) &= \{q_\lambda \mid \lambda \in \Lambda_T^{\text{fin}}/V \cap \Theta\} \sqcup \{q_{\lambda/V} \mid \lambda \in \Lambda_T^{\infty}/V \cap \Theta\}, \\ P(\Sigma) &= \{p_\lambda \mid \lambda \in \Lambda_{\text{fin}}^{S}/V \cap \Sigma\} \sqcup \{p_{\lambda/V} \mid \lambda \in \Lambda_{\infty}^{S}/V \cap \Sigma\}, \end{aligned}$$

and $\langle I(V, \Sigma, \Theta)\rangle$ be the order-ideal generated by $I(V, \Sigma, \Theta)$. Set $\phi(V, \Sigma, \Theta) = \langle I(V, \Sigma, \Theta)\rangle$.

Lemma 4.2.10. *If I is any order-ideal of $H(\dot{E}, \Lambda)$, then $I = \phi\psi(I)$.*

Proof. Let $\psi(I) = (V, \Sigma, \Theta)$ and $I(V, \Sigma, \Theta) = J$ so that $\phi\psi(I) = \langle J\rangle$. It is clear that $J \subseteq I$ and therefore $\langle J\rangle \subseteq I$. For converse, consider a nonzero element $x \in I$. Then $x = \sum_i v_i + \sum_j q_{\alpha_j} + \sum_k p_{\beta_k} + \sum_l q_{Z_l} + \sum_m p_{W_m}$ for some $v_i \in E^0$, $\alpha_j \in \Lambda_T^{\text{fin}}$, $\beta_k \in \Lambda_{\text{fin}}^S$, $Z_l \in \mathcal{Z}$, and $W_m \in \mathcal{W}$. Since I is an order ideal, $v_i, q_{\alpha_j}, p_{\beta_k}, q_{Z_l}, p_{W_m} \in I$, and so to prove that $x \in \langle J\rangle$, it is enough to show that $v, q_\alpha, p_\beta, q_Z, p_W$ for all $v \in E^0, \alpha \in \Lambda_T^{\text{fin}}, \beta \in \Lambda_{\text{fin}}^S, Z \in \mathcal{Z}$ and $W \in \mathcal{W}$.

Case 1 If $v \in E^0 \cap I$, then $v \in V$ by definition of H, hence $v \in J$.

Case 2 Let $\alpha \in \Lambda_T^{\text{fin}}$ such that $q_\alpha \in I$.

Subcase 2.1 If $r(\alpha) \subseteq V$, then $\mathbf{r}(\alpha) \in I$ and so $\mathbf{s}(\alpha) = \mathbf{r}(\alpha) + q_\alpha \in I$. Hence $s(X) \in V$ for each $X \in \mathcal{X}_\alpha$, and so $\mathbf{s}(\alpha) \in J$. Since $q_\alpha \leq \mathbf{s}(\alpha)$, it follows that $q_\alpha \in \langle J\rangle$.

Subcase 2.2 If $r(\alpha) \nsubseteq V$, then by definition $\alpha \in \Theta \cap \Lambda_T^{\text{fin}}/V$. Hence $q_\alpha \in J$.

Case 3 Let $\lambda \in \Lambda_T^{\infty}$ and $Z \in \mathcal{Z}_\lambda$ such that $q_Z \in I$.

Subcase 3.1 $\mathcal{Y}_{\lambda/V} = \emptyset$. This is equivalent to $r(\lambda) \subseteq V$ and the argument follows similar to subcase 2.1.

Subcase 3.2 $0 < |\mathcal{Y}_{\lambda/V}| < \infty$. In this case, we have

$$\mathbf{r}(\mathcal{Y}_{\lambda/V} - Z) = \mathbf{r}([\mathcal{Y}_{\lambda/V} \cup Z] - Z) \leq q_{(\mathcal{Y}_{\lambda/V} \cup Z)} + \mathbf{r}([\mathcal{Y}_{\lambda/V} \cup Z] - Z) = q_Z \in I.$$

It follows that $\mathbf{r}(\mathcal{Y}_{\lambda/V} - Z) \in I$ and so $r(\mathcal{Y}_{\lambda/V} - Z) \subseteq H$. Hence $\mathcal{Y}_{\lambda/V} \subseteq Z$. Since $r(Z - \mathcal{Y}_{\lambda/V}) \subseteq H$, we get $q_{\lambda/V} = \mathbf{r}(Z - \mathcal{Y}_{\lambda/V}) + q_Z \in I$, so that $\lambda \in \Theta$ by definition, and since $q_Z \leq q_{\lambda/V}$, we get $q_Z \in \langle J\rangle$.

Subcase 3.3 $|\mathcal{Y}_{\lambda/V}| = \infty$. Then there exists $Y \in \mathcal{Y}_{\lambda/V} - Z$, and we have

$$r(Y) \leq \mathbf{r}(Z \sqcup \{Y\} - Z) + q_{\{Y\}} = q_Z \in I.$$

But this implies that $r(Y) \in I$ and so $r(Y) \in V$, which contradicts $Y \in \mathcal{Y}_{\lambda/H}$. Thus $q_Z \in \langle J \rangle$.

The remaining cases are proved analogously. $\square$

Construction 4.2.11. Let $(\dot{E}, \Lambda)$ be a semi-regular hypergraph and $(V, \Sigma, \Theta) \in AT(\dot{E}, \Lambda)$. For $A \subseteq E^1$, define

$$A_r(V) = A \cap r^{-1}(V) \text{ and } A_s(V) = A \cap s^{-1}(V).$$

We define the quotient semi-regular hypergraph $(\dot{\widetilde{E}}, \widetilde{\Lambda})$ as follows: $\dot{\widetilde{E}}$ is given by

$$\widetilde{E}^0 = E^0 - V \quad \text{and} \quad \widetilde{E}^1 = E^1_r(V) \cup E^1_s(V).$$

$r_{\widetilde{E}}$ and $s_{\widetilde{E}}$ are restriction maps of r_E and s_E respectively.

For $v \in \widetilde{E}^0$, set

$$\begin{aligned}
\widetilde{C}_v &= \{X_r(V) \mid X \in C_v \text{ and } X_r(V) \neq \emptyset\} \text{ and } \widetilde{C} = \bigsqcup_{v \in \widetilde{E}^0} \widetilde{C}_v, \\
\widetilde{D}_v &= \{Y_s(V) \mid Y \in D_v \text{ and } Y_s(V) \neq \emptyset\} \text{ and } \widetilde{D} = \bigsqcup_{v \in \widetilde{E}^0} \widetilde{D}_v, \\
\widetilde{S} &= \{X_r(V) \mid X \in S \text{ and } X_r(V) \neq \emptyset\} \sqcup \{X_r(V) \mid X \in \mathcal{X}_\lambda, \lambda \in \Sigma\}, \text{ and} \\
\widetilde{T} &= \{Y_s(V) \mid Y \in S \text{ and } Y_s(V) \neq \emptyset\} \sqcup \{Y_s(V) \mid Y \in \mathcal{Y}_\lambda, \lambda \in \Theta\}.
\end{aligned}$$

Let $\widetilde{\Lambda}$ be defined as follows:

$$\begin{aligned}
\widetilde{\Lambda}^{\widetilde{S}}_{\widetilde{T}} &= \{\widetilde{\lambda} \mid \lambda \in \Lambda^S_T/V \sqcup \Sigma \sqcup \Theta\}, \\
\widetilde{\Lambda}^{\widetilde{S}}_{\text{fin}} &= \{\widetilde{\lambda} \mid \lambda \in \Lambda^S_{\text{fin}}/V - \Sigma\}, \\
\widetilde{\Lambda}^{\widetilde{S}}_{\infty} &= \{\widetilde{\lambda} \mid \lambda \in \Lambda^S_{\infty}/V - \Sigma\}, \\
\widetilde{\Lambda}^{\text{fin}}_{\widetilde{T}} &= \{\widetilde{\lambda} \mid \lambda \in \Lambda^{\text{fin}}_T/V - \Theta\}, \text{ and} \\
\widetilde{\Lambda}^{\infty}_{\widetilde{T}} &= \{\widetilde{\lambda} \mid \lambda \in \Lambda^{\infty}_T/V - \Theta\}.
\end{aligned}$$

We note that if $\pi : M_1 \to M_2$ is a monoid homomorphism and M_2 is conical, then $\ker \pi := \pi^{-1}(0)$ is an order-ideal of M_1.

Theorem 4.2.12. *Let $(\dot{E}, \Lambda)$ be a semi-regular hypergraph, $(V, \Sigma, \Theta) \in \mathrm{AT}(\dot{E}, \Lambda)$ and $(\dot{\widetilde{E}}, \widetilde{\Lambda})$ is the corresponding quotient semi-regular hypergraph. Suppose that $I := \langle I(V, \Sigma, \Theta) \rangle$ is the order ideal in $M := H(\dot{E}, \Lambda)$. Then there exists a monoid homomorphism $\pi : M \to \widetilde{M} := M(\dot{\widetilde{E}})$ such that $I = \ker \pi$.*

Proof. We begin by defining $\widetilde{v}, \widetilde{q}_\alpha, \widetilde{p}_\beta, \widetilde{q}_Z, \widetilde{p}_W \in \widetilde{M}$ for $v \in E^0$, $\alpha \in \Lambda_T^{\mathrm{fin}}$, $\beta \in \Lambda_{\mathrm{fin}}^S$, $Z \in \mathcal{Z}$, and $W \in \mathcal{W}$. For $v \in E^0$, set

$$\widetilde{v} = \begin{cases} v & \text{if } v \notin V \text{ and} \\ 0 & \text{if } v \in V. \end{cases}$$

For $\alpha \in \Lambda_T^{\mathrm{fin}}$, we define $\widetilde{q}_\alpha$ as follows:

1. If $s(\alpha) \subseteq V$ or $r(\alpha) \subseteq V$, $\widetilde{q}_\alpha = 0$.
2. If $s(\alpha) \not\subseteq V$ and $r(\alpha) \not\subseteq V$, $\widetilde{q}_\alpha = \begin{cases} 0 & \text{if } \alpha \in \Theta \text{ and} \\ q_{\widetilde{\alpha}} & \text{if } \alpha \notin \Theta. \end{cases}$

For $\beta \in \Lambda_{\mathrm{fin}}^S$, we define $\widetilde{p}_\beta$ as follows:

1. If $s(\beta) \subseteq V$ or $r(\beta) \subseteq V$, $\widetilde{p}_\beta = 0$.
2. If $s(\beta) \not\subseteq V$ and $r(\beta) \not\subseteq V$, $\widetilde{p}_\beta = \begin{cases} 0 & \text{if } \beta \in \Sigma \text{ and} \\ p_{\widetilde{\beta}} & \text{if } \beta \notin \Sigma. \end{cases}$

For $\lambda \in \Lambda_T^\infty$, and $Z \in \mathcal{Z}_\lambda$, we define $\widetilde{q}_Z$ as follows:

1. If $s(\lambda) \subseteq V$, $\widetilde{q}_Z = 0$.
2. If $s(\lambda) \not\subseteq V$, set $\widetilde{Z} = \{Y_s(V) \in \mathcal{Y}_{\widetilde{\lambda}} \mid Y \in Z\}$.
   (a) If $\lambda \in \Theta$ $\widetilde{q}_Z = \mathbf{r}(\mathcal{Y}_{\widetilde{\lambda}} - \widetilde{Z})$.
   (b) If $\lambda \notin \Theta$ and $r(Z) \subseteq H$, $\widetilde{q}_Z = \mathbf{s}(\lambda)$.
   (c) If $\lambda \notin \Theta$, $r(Z) \not\subseteq V$ and $\lambda \notin \Lambda_T^\infty/V$, $\widetilde{q}_Z = q_{\widetilde{Z}}$.

(d) If $\lambda \notin \Theta$, $r(Z) \not\subseteq V$, and $\lambda \in \Lambda_T^\infty/V$. $\widetilde{q}_Z = q_{\widetilde{\lambda}} + \mathbf{r}(\mathcal{Y}_{\widetilde{\lambda}} - \widetilde{Z})$.

For $\lambda \in \Lambda_\infty^S$, and $W \in \mathcal{W}_\lambda$, we define $\widetilde{p}_W$ as follows:

1. If $r(\lambda) \subseteq V$, $\widetilde{p}_W = 0$.
2. If $r(\lambda) \not\subseteq V$ set $\widetilde{W} = \{X_r(V) \in \mathcal{X}_{\widetilde{\lambda}} \mid X \in W\}$.
   (a) If $\lambda \in \Sigma$, $\widetilde{p}_W = \mathbf{s}(\mathcal{X}_{\widetilde{\lambda}} - \widetilde{W})$.
   (b) If $r(\lambda) \not\subseteq V$, $\lambda \notin \Sigma$ and $s(W) \subseteq V$, $\widetilde{p}_W = \mathbf{r}(\lambda)$.
   (c) If $r(\lambda) \not\subseteq V$, $\lambda \notin \Sigma$, $s(W) \not\subseteq V$ and $\lambda \notin \Lambda_\infty^S/V$, $\widetilde{p}_W = p_{\widetilde{W}}$.
   (d) If $r(\lambda) \not\subseteq V$, $\lambda \notin \Sigma$, $s(W) \not\subseteq V$, and $\lambda \in \Lambda_\infty^S/V$, $\widetilde{p}_W = p_{\widetilde{\lambda}} + \mathbf{s}(\mathcal{X}_{\widetilde{\lambda}} - \widetilde{W})$.

We define $\pi : M \to \widetilde{M}$ by mapping generators $v \mapsto \widetilde{v}$, for all $v \in E^0$, $q_\alpha \mapsto \widetilde{q}_\alpha$, for all $\alpha \in \Lambda_T^{\text{fin}}$, $p_\beta \mapsto \widetilde{p}_\beta$ for all $\beta \in \Lambda_{\text{fin}}^S$, $q_Z \mapsto \widetilde{q}_Z$ for all $Z \in \mathcal{Z}$, and $p_W \mapsto \widetilde{p}_W$ for all $W \in \mathcal{W}$. To show that π defines a homomorphism we need to verify that images of the generators satisfy all the defining relations. Here we only show for $\lambda \in \Lambda_T$ and the argument follows analogously for $\lambda \in \Lambda_S$.

Let $\lambda \in \Lambda_T$. We introduce a new notation

$$\begin{aligned}
\widetilde{\mathbf{s}}(W) &:= \sum_{X \in W} \widetilde{s(X)} \text{ for subsets } W \subseteq \mathcal{X}_\lambda \text{ and} \\
\widetilde{\mathbf{r}}(Z) &:= \sum_{Y \in Z} \widetilde{r(Y)} \text{ for subsets } Z \subseteq \mathcal{Y}_\lambda.
\end{aligned}$$

Suppose that $\lambda \in \Lambda_T^S$. If $s(\lambda) \subseteq V$, then $r(\lambda) \subseteq V$, and we get

$$\widetilde{\mathbf{s}}(\lambda) = 0 = \widetilde{\mathbf{r}}(\lambda).$$

If $s(\lambda) \not\subseteq V$ then $r(\lambda) \not\subseteq V$, and we get

$$\widetilde{\mathbf{s}}(\lambda) = \mathbf{s}(\mathcal{X}_{\lambda/V}) = \mathbf{s}(\widetilde{\lambda}) = \mathbf{r}(\widetilde{\lambda}) = \mathbf{r}(\mathcal{Y}_{\lambda/V}) = \widetilde{\mathbf{r}}(\lambda).$$

Suppose that $\lambda \in \Lambda_T^{\text{fin}}$. If $s(\lambda) \subseteq V$, then $\mathcal{Y}_{\lambda/V} = \emptyset$. If $s(\lambda) \not\subseteq V$, and $r(\lambda) \subseteq V$, then $\mathcal{X}_{\lambda/V} = \emptyset$. In both of the above cases we have

$$\widetilde{\mathbf{s}}(\lambda) = 0 = \widetilde{\mathbf{r}}(\lambda) + \widetilde{q}_\lambda.$$

So let $s(\lambda) \nsubseteq V$ and $r(\lambda) \nsubseteq V$. If $\lambda \in \Theta$ then

$$\widetilde{\mathbf{s}}(\lambda) = \mathbf{s}(\mathcal{X}_{\lambda/V}) = \mathbf{s}(\widetilde{\lambda}) = \mathbf{r}(\widetilde{\lambda}) = \mathbf{r}(\mathcal{Y}_{\lambda/V}) + 0 = \widetilde{\mathbf{r}}(\lambda) + \widetilde{q}_{\lambda}.$$

If $\lambda \notin \Theta$ then

$$\widetilde{\mathbf{s}}(\lambda) = \mathbf{s}(\mathcal{X}_{\lambda/V}) = \mathbf{s}(\widetilde{\lambda}) = \mathbf{r}(\widetilde{\lambda}) = \mathbf{r}(\mathcal{Y}_{\lambda/V}) + q_{\mathcal{Y}_{\lambda/V}} = \widetilde{\mathbf{r}}(\lambda) + \widetilde{q}_{\lambda}.$$

Now suppose that $\lambda \in \Lambda_T^{\infty}$, and $Z \in \mathcal{Z}_{\lambda}$. If $s(\lambda) \subseteq V$, then $\mathcal{Y}_{\lambda/V} = \emptyset$, and hence $r(Z) = \emptyset$. Then we have

$$\widetilde{\mathbf{s}}(\lambda) = 0 = \widetilde{\mathbf{r}}(Z) + \widetilde{q}_Z.$$

Hence we assume that $s(\lambda) \nsubseteq V$ for rest of the step. If $\lambda \in \Theta$ then $\widetilde{\lambda} \in \widetilde{\Lambda}_{\widetilde{T}}^{\widetilde{S}}$ and we have

$$\widetilde{\mathbf{s}}(\lambda) = \mathbf{s}(\widetilde{\lambda}) = \mathbf{r}(\widetilde{\lambda}) = \mathbf{r}(\mathcal{Y}_{\widetilde{\lambda}}) = \mathbf{r}(\widetilde{Z}) + \mathbf{r}(\mathcal{Y}_{\widetilde{\lambda}} - \widetilde{Z}) = \mathbf{r}(\widetilde{Z}) + \widetilde{q}_Z = \widetilde{\mathbf{r}}(Z) + \widetilde{q}_Z.$$

If $\lambda \notin \Theta$ and $r(Z) \subseteq V$ then we have

$$\widetilde{\mathbf{s}}(\lambda) = \mathbf{s}(\widetilde{\lambda}) = \widetilde{q}_Z = \widetilde{\mathbf{r}}(Z) + \widetilde{q}_Z.$$

If $\lambda \notin \Theta$, $r(Z) \nsubseteq V$ and $\lambda \notin \Lambda_T^{\infty}/V$ then we have

$$\widetilde{\mathbf{s}}(\lambda) = \mathbf{s}(\widetilde{\lambda}) = \mathbf{r}(\widetilde{Z}) + q_{\widetilde{Z}} = \widetilde{\mathbf{r}}(Z) + \widetilde{q}_Z.$$

If $\lambda \notin \Theta$, $r(Z) \nsubseteq V$ and $\lambda \in \Lambda_T^{\infty}/V$ then we have

$$\widetilde{\mathbf{s}}(\lambda) = \mathbf{s}(\widetilde{\lambda}) = \mathbf{r}(\widetilde{\lambda}) + q_{\widetilde{\lambda}} = \mathbf{r}(\mathcal{Y}_{\widetilde{\lambda}} - \widetilde{Z}) + \mathbf{r}(\widetilde{Z}) + q_{\widetilde{\lambda}} = \widetilde{\mathbf{r}}(Z) + \widetilde{q}_Z.$$

Now assume that for $\lambda \in \Lambda_T^{\infty}$ let $Z_1, Z_2 \in \mathcal{Z}_{\lambda}$ and $Z_1 \subsetneq Z_2$. If $s(\lambda) \subseteq V$ then we have

$$\widetilde{q}_{Z_1} = 0 = \widetilde{\mathbf{r}}(Z_2 - Z_1) + \widetilde{q}_{Z_2}.$$

So we may assume that $s(\lambda) \nsubseteq V$. If $\lambda \in \Theta$,

$$\widetilde{q}_{Z_1} = \mathbf{r}(\mathcal{Y}_{\widetilde{\lambda}}) - \widetilde{Z}_1 = \mathbf{r}(\widetilde{Z}_2 - \widetilde{Z}_1) + \mathbf{r}(\mathcal{Y}_{\widetilde{\lambda}}) = \widetilde{\mathbf{r}}(Z_2 - Z_1) + \widetilde{q}_{Z_2}.$$

Only remaining case is when $\lambda \notin \Theta$. If $\lambda \in \Lambda_T^\infty/V - \Theta$, then

$$\widetilde{q}_{Z_1} = q_{\widetilde{\lambda}} + \mathbf{r}(\mathcal{Y}_{\widetilde{\lambda}} - Z_1) = q_{\widetilde{\lambda}} + \mathbf{r}(\mathcal{Y}_{\widetilde{\lambda}} - Z_2) + \mathbf{r}(\widetilde{Z_2} - \widetilde{Z_1}) = \widetilde{q}_{Z_2} + \widetilde{\mathbf{r}}(Z_2 - Z_1).$$

Hence, we may assume that $\lambda \notin \Lambda_T^\infty/V$. If $r(Z_2) \subseteq V$ then we have

$$\widetilde{q}_{Z_1} = \mathbf{s}(\lambda) = \mathbf{r}(\widetilde{Z}_2 - \widetilde{Z}_1) + \mathbf{s}(\lambda) = \widetilde{\mathbf{r}}(Z_2 - Z_1) + \widetilde{q}_{Z_2}.$$

If $r(Z_1) \subseteq V$ but $r(Z_2) \nsubseteq V$, we have

$$\widetilde{q}_{Z_1} = \mathbf{s}(\lambda) = \mathbf{r}(\widetilde{Z_2}) + q_{\widetilde{Z}} = \widetilde{\mathbf{r}}(Z_2 - Z_1) + \widetilde{q}_{Z_2}.$$

Finally, if $r(Z_1) \nsubseteq V$, then we have

$$\widetilde{q}_{Z_1} = q_{\widetilde{Z}_1} = \mathbf{r}(\widetilde{Z}_2 - \widetilde{Z}_1) + q_{\widetilde{Z}_2} = \widetilde{\mathbf{r}}(Z_2 - Z_1) + \widetilde{q}_{Z_2}.$$

Thus we have shown that π is a monoid homomorphism.

Now we show that $I \subseteq \ker\pi$. Since $\ker \pi$ is an order-ideal, it suffices to show that $I(V, \Sigma, \Theta) \subseteq \ker\pi$. For $v \in H$ we have $\pi(v) = \widetilde{v} = 0$. For $\lambda \in \Theta \cap \Lambda_T^{\text{fin}}$, we have $\pi(q_\lambda) = \widetilde{q}_\lambda = 0$. If $\lambda \in \Theta \cap \Lambda_T^\infty$, then $\pi(q_{\lambda/V}) = \widetilde{q}_{\lambda/V} = 0$. Similarly we can verify that if $\lambda \in \Sigma \cap \Lambda_{\text{fin}}^S$ then $\pi(q_\lambda) = 0$ and if $\lambda \in \Sigma \cap \Lambda_\infty^S$ then $\pi(q_{\lambda/V}) = 0$.

We claim that $\psi(\ker \pi) = (V, \Sigma, \Theta)$. For, let $\psi(\ker \pi) = (\widetilde{V}, \widetilde{\Sigma}, \widetilde{\Theta})$. It follows from definition that $\widetilde{V} = I \cap E^0 = V$ and by the previous paragraph $\Sigma \subseteq \widetilde{\Sigma}$ and $\Theta \subseteq \widetilde{\Theta}$. Consider $\lambda \in \Lambda_T^{\text{fin}}/V \sqcup \Lambda_T^\infty/V$. If $\mathcal{Y}_\lambda$ is finite and $\lambda \notin \Theta$, then $\pi(q_\lambda) = \widetilde{q}_\lambda \neq 0$. Hence $q_\lambda \notin \ker \pi$ and so $\lambda \notin \widetilde{\Theta}$. If $\mathcal{Y}_\lambda$ is infinite and $\lambda \notin \Theta$, then $\pi(q_{\mathcal{Y}_{\lambda/V}}) = \widetilde{q}_{\mathcal{Y}_{\lambda/V}} \neq 0$. Thus $\lambda \notin \widetilde{\Theta}$. Hence $\Theta = \widetilde{\Theta}$. Similarly $\Sigma = \widetilde{\Sigma}$.

Finally, since $\psi(\ker \pi) = (V, \Sigma, \Theta)$ and $I = \phi \circ \psi(I)$, we have that $\ker \pi = \langle I(V, \Sigma, \Theta) \rangle = I$. □

Corollary 4.2.13. *If* $(V, \Sigma, \Theta) \in \mathrm{AT}(\dot{E}, \Lambda)$, *then* $(V, \Sigma, \Theta) = \psi \circ \phi(V, \Sigma, \Theta)$.

Theorem 4.2.14. *Let $\dot{E}$ be a semi-regular hypergraph. Then there are mutually inverse lattice isomorphisms*

$$\phi : \mathrm{AT}(\dot{E}, \Lambda) \to \mathcal{L}(H(\dot{E}, \Lambda)) \text{ and } \psi : \mathcal{L}(H(\dot{E}, \Lambda)) \to \mathrm{AT}(\dot{E}, \Lambda),$$

where $\phi(V,\Sigma,\Theta) = \langle I(V,\Sigma,\Theta)\rangle$ *for* $(V,\Sigma,\Theta) \in \mathrm{AT}(\dot{E},\Lambda)$ *and* ψ *is defined as in definition 4.2.9.*

Proof. The maps ψ and ϕ are well defined by definition. By Lemma 4.2.10, $\phi \circ \psi$ is the identity map on $\mathcal{L}(H(\dot{E},\Lambda))$, and by Corollary 4.2.13 $\psi \circ \phi$ is the identity map on $\mathrm{AT}(\dot{E},\Lambda)$. We only to have to show that ψ and ϕ are order-preserving.

Suppose $I_1 \subseteq I_2$ are order-ideals of $H(\dot{E},\Lambda)$ and $(V_j,\Sigma_j,\Theta_j) = \psi(I_j)$ for $j = 1,2$. Clearly $V_1 \subseteq V_2$. We only show that $\Theta_1 \subseteq \Theta_2 \sqcup \Lambda_T(V_2)$. Let $\lambda \in \Theta_1$. First suppose that $\lambda \in \Lambda_T^{\mathrm{fin}}/V_1$ and $q_\lambda \in I_1$. If $\lambda \in \Lambda_T^{\mathrm{fin}}/V_2$, then $\lambda \in \Theta_2$. Otherwise, $r(\lambda) \subseteq V_2$ and so $\mathbf{s}(\lambda) \in I_2$, which implies $s(\lambda) \in V_2$ and $\lambda \in \Lambda_T(V_2)$. Now suppose that $\lambda \in \Lambda_Y^{\infty}/V_2$ and $q_{\lambda/V_1} \in I_1$. If $\lambda \in \Lambda_T^{\infty}/V_2$, then q_{λ/V_2} is defined and also

$$q_{\lambda/V_2} = \mathbf{r}(\{Y \in \mathcal{Y}_\lambda \mid r(Y) \in V_2 - V_1\}) + q_{\lambda/V_1} \in I_2.$$

So $\lambda \in \Theta_2$. Otherwise, $r(\lambda) \subseteq V_2$ and so $\mathbf{r}(\lambda/V_1) \in I_2$, hence $\mathbf{s}(\lambda) \in I_2$, again giving $\lambda \in \Lambda_T(V_2)$. $\Sigma_1 \subseteq \Sigma_2 \sqcup \Lambda^S(V_2)$ follows on similar lines.

Finally, let (V_1,Σ_1,Θ_1) and (V_2,Σ_2,Θ_2) be elements of $\mathrm{AT}(\dot{E},\Lambda)$ such that $(V_1,\Sigma_1,\Theta_1) \leq (V_2,\Sigma_2,\Theta_2)$. Clearly $V_1 \subseteq I(V_2,\Sigma_2,\Theta_2)$. Consider $\lambda \in \Theta_1 \cap \Lambda_T^{\mathrm{fin}}$. If $\lambda \in \Theta_2$, then $q_\lambda \in I(V_2,\Sigma_2,\Theta_2)$ by definition of $I(V_2,\Sigma_2,\Theta_2)$. If $\lambda \in \Lambda_T(V)$, then

$$q_\lambda \leq q_\lambda + \mathbf{r}(\lambda) = \mathbf{s}(\lambda) \in I(V_2,\Sigma_2,\Theta_2)$$

and so $q_\lambda \in \langle I(V_2,\Sigma_2,\Theta_2)\rangle$. Now consider $\lambda \in \Theta_1 \cap \Lambda_T^{\infty}$. If $\lambda \in \Theta_2$, then $q_{\lambda/V_2} \in I(V_2,\Sigma_2,\Theta_2)$ and since

$$q_{\lambda/V_1} \leq q_{\lambda/V_1} + \mathbf{r}(\mathcal{Y}_{\lambda/V_2} - \mathcal{Y}_{\lambda/V_1}) = q_{\lambda/V_2},$$

it follows that $q_{\lambda/V_1} \in \langle I(V_2,\Sigma_2,\Theta_2)\rangle$. If $\lambda \in \Lambda(V)$, then

$$q_{\lambda/V_1} \leq q_{\lambda/V_1} + \mathbf{r}(\mathcal{Y}_{\lambda/V_1}) = \mathbf{s}(\lambda) \in I(V_2,\Sigma_2,\Theta_2),$$

and again $q_{\lambda/V_1} \in \langle I(V_2,\Sigma_2,\Theta_2)\rangle$. A similar arguments shows that $\Sigma_1 \subseteq \Sigma_2 \sqcup \Lambda^S(V)$. Therefore all the generators of $I(V_1,\Sigma_1,\Theta_1$ lie in $\phi(V_2,\Sigma_2,\Theta_2)$, and we conclude that $\phi(V_1,\Sigma_1,\Theta_1) \subseteq \phi(V_2,\Sigma_2,\Theta_2)$. Hence ϕ is order-preserving.

□

4.2.3 The lattice of trace-ideals in $\mathcal{A}_K(\dot{E}, \Lambda)$

Definition 4.2.15. Let R be an arbitrary ring and $\mathrm{Idem}(M_\infty(R))$ denote the set of idempotents in $M_\infty(R)$. An ideal I of R is called a **trace-ideal** provided I can be generated by the entries of the matrices in some subset of $\mathrm{Idem}(M_\infty(R))$. We denote by $\mathrm{Tr}(R)$ the set of all trace ideals of R. Since $\mathrm{Tr}(R)$ is closed under arbitrary sums and arbitrary intersections, it forms a complete lattice with respect to inclusion.

Proposition 4.2.16. *[20, Proposition 10.10] For any ring R there are mutually inverse lattice isomorphisms*

$$\Phi : \mathcal{L}(\mathcal{V}(R)) \to \mathrm{Tr}(R) \quad \textit{and} \quad \Psi : \mathrm{Tr}(R) \to \mathcal{L}(\mathcal{V}(R))$$

given by

$$\Phi(I) = \langle \textit{entries of } e \mid e \in \ \mathrm{Idem}(M_\infty(R)) \ \textit{and} \ [e] \in I \rangle \ \textit{and}$$

$$\Psi(J) = \{[e] \in \mathcal{V}(R) \mid e \in \mathrm{Idem}(M_\infty(J))\}.$$

Lemma 4.2.17. *Let $(\dot{E}, \Lambda)$ be a semi-regular hypergraph. Then the trace ideals of $A := \mathcal{A}_K(\dot{E}, \Lambda)$ are precisely the idempotent generated ideals and the lattice isomorphism $\Phi : \mathcal{L}(\mathcal{V}(A)) \to \mathrm{Tr}(A)$ is expressed as*

$$\Phi(I) = \langle \textit{idempotents } e \in A \mid [e] \in I \rangle.$$

Proof. The proof goes exactly similar to [20, Proposition 6.2], except that in the present case, the V-monoid $\mathcal{V}(\mathcal{A}_K(\dot{E}))$ is generated by

$$\begin{aligned} \{[v] \mid v \in E^0\} \ \cup \ & \{[q'_Z] \mid Z \subseteq \mathcal{Y}_\lambda, \ 0 < |Z| < \infty \text{ and } \lambda \notin \Lambda_S\} \\ \cup \ & \{[p'_W] \mid W \subseteq \mathcal{X}_\lambda, \ 0 < |W| < \infty \text{ and } \lambda \notin \Lambda_T\}. \end{aligned}$$

(Here, because of Theorem 4.1.8, we are using the notation $[q'_Z]$ and $[p'_W]$ for the generators of $\mathcal{V}(\mathcal{A}_K(\dot{E}))$. Strictly speaking, $[q'_z]$ and $[p'_W]$ stand for images of q_Z and p_W respectively under the map Γ defined in Theorem 4.1.8). □

Theorem 4.2.18. *Let $(\dot{E}.\Lambda)$ be a semi-regular hypergraph and $A = \mathcal{A}_K(\dot{E},\Lambda)$. Then there exist mutually inverse lattice isomorphisms*

$$\xi : \mathrm{AT}(\dot{E},\Lambda) \to \mathrm{Tr}(A) \textit{ and } \zeta : \mathrm{Tr}(A) \to \mathrm{AT}(\dot{E},\Lambda).$$

Proof. Set $M := H(\dot{E},\Lambda)$. Let $\Gamma : M \to \mathcal{V}(A)$ be the monoid isomorphism. By abuse of notation, we also use Γ to denote the induced lattice isomorphism $\mathcal{L}(M) \to \mathcal{L}(\mathcal{V}(A))$. Due to Theorem 4.2.14 and Proposition 4.2.16, we have mutually inverse lattice isomorphisms

$$\Phi\Gamma\phi : \mathrm{AT}(\dot{E},\Lambda) \to \mathrm{Tr}(A) \quad \text{and} \quad \psi\Gamma^{-1}\Psi : \mathrm{Tr}(A) \to \mathrm{AT}(\dot{E},\Lambda).$$

More explicitly, if $J \in \mathit{Tr}(A)$, then $\zeta(J) = (H,\Sigma,\Theta)$, where

$$H = E^0 \cap J,$$

$$\Sigma = \{\lambda \in \Lambda^S_{\text{fin}}/H \mid p_\lambda \in J\} \sqcup \{\lambda \in \Lambda^S_\infty/H \mid p_{\lambda/H} \in J\},$$

$$\Theta = \{\lambda \in \Lambda^{\text{fin}}_T/H \mid q_\lambda \in J\} \sqcup \{\lambda \in \Lambda^\infty_T/H \mid q_{\lambda/H} \in J\}.$$

For converse, let $(H,\Sigma,\Theta) \in \mathit{AT}(\dot{E},\Lambda)$. First define $\xi(H,\Sigma,\Theta) = \langle H \sqcup P(\Sigma) \sqcup Q(\Theta)\rangle$, where $P(\Sigma)$ and $Q(\Theta)$ are defined as in Definition 4.2.9. Then define $J(H,\Sigma,\Theta)$ to be the order-ideal of $\mathcal{V}(A)$ generated by the set $H' \sqcup P'(\Sigma) \sqcup Q'(\Sigma)$, where

$$H' = \{[v] \mid v \in H\},$$

$$P'(\Sigma) = \{[p'_\lambda] \mid \lambda \in \Sigma \cap \Lambda^S_{\text{fin}}/H\} \sqcup \{[p'_{\lambda/H}] \mid \lambda \in \Sigma \cap \Lambda^S_\infty\},$$

$$Q'(\Theta) = \{[q'_\lambda] \mid \lambda \in \Theta \cap \Lambda^{\text{fin}}_T/H\} \sqcup \{[q'_{\lambda/H}] \mid^\lambda \in \Theta \cap \Lambda^\infty_T\}.$$

By Lemma 4.2.17, it follows that

$$\Phi\Gamma\phi(H,\Sigma,\Theta) = \langle \text{idempotents } e \in A \mid [e] \in J(H,\Sigma,\Theta)\rangle.$$

It is clear that $\xi(H,\Sigma,\Theta) \subseteq \Phi\Gamma\phi(H,\Sigma,\Theta)$.

If e is an idempotent in A such that $[e] \in J(H, \Sigma, \Theta)$, then

$$[e] \leq \sum_{i=1}^{n_1}[v_i] + \sum_{j=1}^{n_2}[p'_{\alpha_j}] + \sum_{k=1}^{n_3}[p'_{\beta_k/H}] + \sum_{l=1}^{n_4}[q'_{\gamma_l}] + \sum_{m=1}^{n_5}[q'_{\delta_m/H}]$$

where $v_i \in H$, $\alpha_j \in \Sigma \cap \Lambda^S_{\text{fin}}$, $\beta_k \in \Sigma \cap \Lambda^S_{\infty}$, $\gamma_l \in \Theta \cap \Lambda^{\text{fin}}_T$ and $\delta_m \in \Theta \cap \Lambda^{\infty}_T$. Therefore e is equivalent to some idempotent $e' \leq \mathcal{D}$ where $\mathcal{D}$ is a diagonal matrix with entries $v_i, p_{\alpha_j}, p_{\beta_k/H}, q_{\gamma_l}$, and $q_{\delta_m/H}$.Thus it follows that e lies in these $v_i, p_{\alpha_j}, p_{\beta_k/H}, q_{\gamma_l}$, and $q_{\delta_m/H}$. Hence $\Phi\Gamma\phi = \xi$. □

4.2.4 Simplicity

A non-zero conical monoid M is **simple** if its only order-ideals are $\{0\}$ and M.

Theorem 4.2.19. *Let $(\dot{E}, \Lambda)$ be a semi-regular hypergraph. Then the following conditions are equivalent*

1. *The only trace ideals of $\mathcal{A}_K(\dot{E}, \Lambda)$ are 0 and $\mathcal{A}_K(\dot{E}, \Lambda)$.*
2. *$H(\dot{E}, \Lambda)$ is a simple monoid.*
3. *$S = C_{\text{fin}}$, $T = D_{\text{fin}}$ and the only bisaturated subsets of E^0 are $\emptyset$ and E^0.*

Proof. From Proposition 4.2.16 it follows that (1) ⇔ (2).

(2) ⇒ (3) : Observe that $(\Lambda^S_{\text{fin}}/\emptyset) \sqcup (\Lambda^S_{\infty}/\emptyset) = C_{\text{fin}} - S$ and $(\Lambda^{\text{fin}}_T/\emptyset) \sqcup (\Lambda^{\infty}_T/\emptyset) = D_{\text{fin}} - T$. Similarly, $(\Lambda^S_{\text{fin}}/E^0) \sqcup (\Lambda^S_{\infty}/E^0) = \emptyset$ and $(\Lambda^{\text{fin}}_T/E^0) \sqcup (\Lambda^{\infty}_T/E^0) = \emptyset$. By Theorem 4.2.14, the only members of $\mathrm{AT}(\dot{E}, \Lambda)$ are $(\emptyset, \emptyset, \emptyset)$ and $(E^0, \emptyset, \emptyset)$. If $\lambda \in \Lambda^S_{\text{fin}}$, then $(\emptyset, \{\lambda\}, \emptyset) \in \mathrm{AT}(\dot{E}, \Lambda)$. This proves that $S = C_{\text{fin}}$. Similarly $T = D_{\text{fin}}$. If H is any bisaturated subset of E^0, then $(H, \emptyset, \emptyset) \in \mathrm{AT}(\dot{E}, \Lambda)$ and hence the only bisaturated subsets of E^0 are E^0 and $\emptyset$.

(3) ⇒ (2) : In this case $\mathrm{AT}(\dot{E}, \Lambda) = \{(E^0, \emptyset, \emptyset), (\emptyset, \emptyset, \emptyset)\}$. The result follows at once from Theorem 4.2.14. □

4.3 Representations of Leavitt path algebras of regular hypergraphs

Given a graph E, the **category of quiver representations** of E is the category of functors from the path category $\mathcal{C}_E$ to the category of K-vector spaces. A morphism of quiver representations is a natural transformation between two such functors. In other words, a quiver representation ρ assigns a (possibly infinite dimensional) K-vector space $\rho(v)$ to each $v \in E^0$ and a linear transformation $\rho(e) : \rho(s(e)) \to \rho(r(e))$ to each $e \in E^1$. A morphism of quiver representations $\phi : \rho \to \rho'$ is a family of linear transformations $\{\phi_v \mid \rho(v) \to \rho'(v)\}_{v \in E^0}$ such that for each $e \in E^1$ the following diagram commutes:

$$\begin{array}{ccc} \rho(s(e)) & \xrightarrow{\rho(e)} & \rho(r(e)) \\ {\scriptstyle \phi_{s(e)}}\big\downarrow & & \big\downarrow{\scriptstyle \phi_{r(e)}} \\ \rho'(s(e)) & \xrightarrow{\rho'(e)} & \rho'(r(e)) \end{array}$$

This section generalizes the results of [41]. Throughout this section by a hypergraph we always mean a regular hypergraph. In this section, we will work in the category $\mathfrak{M}_L$ of unital (right) modules over $L := L_K(\dot{E}, \Lambda)$ where $(\dot{E}, \Lambda)$ is a hypergraph. The category $\mathfrak{M}_L$ is closed under taking quotients, submodules, extensions and arbitrary sums and hence it is an abelian category with sums. Note however that its is not closed under infinite product if E^0 is infinite.

Lemma 4.3.1. *Let M be a right L-module. Then $\bigoplus_{X \in \mathcal{X}_\lambda} Ms(X)$ is isomorphic (as vector space) to $\bigoplus_{Y \in \mathcal{Y}_\lambda} Mr(Y)$ for every $\lambda \in \Lambda$.*

Proof. For each $\lambda \in \Lambda$, let $[\lambda]$ be the rectangular matrix of size $|\mathcal{Y}_\lambda| \times |\mathcal{X}_\lambda|$ whose entry in Y^{th} row and X^{th} column is the edge YX. Then $[\lambda] : \bigoplus_{X \in \mathcal{X}_\lambda} Ms(X) \to \bigoplus_{Y \in \mathcal{Y}_\lambda} Mr(Y)$, given by

$$(m_X)_{X \in \mathcal{X}_\lambda}[\lambda] = \left(\sum_{X' \in \mathcal{X}_\lambda} m_X \mu_{(X'Y)} \right)_{Y \in \mathcal{Y}_\lambda},$$

is a well defined linear map, where $\mu_{(XY)}$ is right multiplication by the edge XY. We show that $[\lambda]$ is an isomorphism with the inverse $[\lambda]^*$, which is the adjoint transpose matrix of $[\lambda]$. Note that $[\lambda]^* : \bigoplus\limits_{Y \in \mathcal{Y}_\lambda} Mr(Y) \to \bigoplus\limits_{X \in \mathcal{X}_\lambda} Ms(X)$ is given by

$$(m_Y)_{Y \in \mathcal{Y}_\lambda} [\lambda]^* = \left(\sum_{Y' \in \mathcal{Y}_\lambda} m_Y \mu_{(Y'X)^*} \right)_{X \in \mathcal{X}_\lambda}.$$

We check their compositions:

$$\begin{aligned}
(m_Y)_{Y \in \mathcal{Y}_\lambda} [\lambda]^* [\lambda] &= \left(\sum_{Y' \in \mathcal{Y}_\lambda} m_Y \mu_{(Y'X)^*} \right)_{X \in \mathcal{X}_\lambda} [\lambda] \\
&= \left(\sum_{X \in \mathcal{X}_\lambda} \left(\sum_{Y' \in \mathcal{Y}_\lambda} m_Y \mu_{(Y'X)^*} \right) \mu_{(XY)} \right)_{Y \in \mathcal{Y}_\lambda} \\
&= \left(\sum_{X \in \mathcal{X}_\lambda} \left(\sum_{Y' \in \mathcal{Y}_\lambda} m_Y \mu_{(Y'X)^*} \mu_{(XY)} \right) \right)_{Y \in \mathcal{Y}_\lambda} \\
&= \left(\sum_{X \in \mathcal{X}_\lambda} \sum_{Y' \in \mathcal{Y}_\lambda} m_Y \mu_{(Y'X)^*(XY)} \right)_{Y \in \mathcal{Y}_\lambda} \\
&= \left(\sum_{Y' \in \mathcal{Y}_\lambda} \sum_{X \in \mathcal{X}_\lambda} m_Y \mu_{(Y'X)^*(XY)} \right)_{Y \in \mathcal{Y}_\lambda} \\
&= \left(\sum_{Y' \in \mathcal{Y}_\lambda} m_Y \mu_{\sum\limits_{X \in \mathcal{X}_\lambda} (Y'X)^*(XY)} \right)_{Y \in \mathcal{Y}_\lambda} \\
&= \left(\sum_{Y' \in \mathcal{Y}_\lambda} m_Y \mu_{\delta_{Y'Y} r(Y')} \right)_{Y \in \mathcal{Y}_\lambda} \qquad \text{(by } L2\text{)} \\
&= (m_Y)_{Y \in \mathcal{Y}_\lambda}.
\end{aligned}$$

Similarly by $L1$, we get

$$(m_X)_{X \in \mathcal{X}_\lambda} [\lambda][\lambda]^* = (m_X)_{X \in \mathcal{X}_\lambda}$$

which establishes the result. □

Remark 4.3.2. If M is unital, then for any $m \in M$, we have $m = \sum_{k=1}^{l} m_k v_k$ for some vertices $v_k \in E^0$. Hence $M = \sum\limits_{v \in E^0} Mv$. When considered as paths, the vertices of E form a set of orthogonal idempotents, hence the above sum is direct. Therefore we have

$$M = \bigoplus_{v \in E^0} Mv.$$

Theorem 4.3.3. *The category $\mathfrak{M}_L$ is equivalent to the full subcategory of quiver representations ρ of E satisfying:*

$$\textit{For all } \lambda \in \Lambda,\ [\rho(\lambda)] : \bigoplus_{X \in \mathcal{X}_\lambda} \rho(s(X)) \to \bigoplus_{Y \in \mathcal{Y}_\lambda} \rho(r(Y)) \textit{ is an isomorphism.} \tag{H}$$

Proof. Let M be a right L-module. We define a quiver representation ρ_M as follows: $\rho_M(v) = Mv$ for each $v \in E^0$ and for the map $\rho_M(e) : Ms(e) \to Mr(e)$, $ms(e)\rho_M(e) = ms(e)e = me = mr(e)$. By Lemma 4.3.1, (H) is satisfied. If $\varphi : M \to N$ is an L-module homomorphism then φ_v is the linear transformation making the following diagram commutative:

$$\begin{tikzcd} \rho_M(v) = Mv \arrow[r, hook] \arrow[d, "\varphi_v"'] & M \arrow[d, "\varphi"] \\ \rho_N(v) = Nv \arrow[r, hook] & N \end{tikzcd}$$

Since right multiplication by an edge e commutes with φ, this defines a homomorphism of quiver representations.

Given a quiver representation ρ, we define the correspoding module $M_\rho := \bigoplus_{v \in E^0} \rho(v)$. To get an L-module structure on M_ρ, we define the following projections and inclusions: For each $v \in E^0$, define

$$p_v : M_\rho \twoheadrightarrow \rho(v) \quad ; \quad i_v : \rho(v) \hookrightarrow M_\rho,$$

and for each $\lambda \in \Lambda$, $X \in \mathcal{X}_\lambda$ and $Y \in \mathcal{Y}_\lambda$, define

$$p_X : \bigoplus_{X \in \mathcal{X}_\lambda} \rho(s(X)) \twoheadrightarrow \rho(s(X)) \quad ; \quad i_X : \rho(s(X)) \hookrightarrow \bigoplus_{X \in \mathcal{X}_\lambda} \rho(s(X)).$$

$$p_Y : \bigoplus_{Y \in \mathcal{Y}_\lambda} \rho(r(Y)) \twoheadrightarrow \rho(r(Y)) \quad ; \quad i_Y : \rho(r(Y)) \hookrightarrow \bigoplus_{Y \in \mathcal{Y}_\lambda} \rho(r(Y)).$$

Now let $mv := mp_v i_v$, $m(XY) := mp_{s(X)} i_X [\rho(\lambda)] p_Y i_{r(Y)}$, and $m(YX)^* := mp_{r(Y)} i_Y [\rho(\lambda)]^* p_X i_{s(X)}$. To keep track of the last defining relations, we draw the following diagram:

$$\begin{tikzcd}
& \rho(s(X)) \arrow[r, "i_X"] & \bigoplus\limits_{X\in\mathcal{X}_\lambda}\rho(s(X)) \arrow[r, "{[\lambda]}"] & \bigoplus\limits_{Y\in\mathcal{Y}_\lambda}\rho(r(Y)) \arrow[r, "p_Y"] & \rho(r(Y)) \arrow[dr, bend left, "i_{r(Y)}"] & \\
\bigoplus\limits_{w\in E^0}\rho(w) \arrow[ur, bend left, "p_{s(X)}"] & & & & & \bigoplus\limits_{w\in E^0}\rho(w) \arrow[dl, bend left, "p_{r(Y)}"] \\
& \rho(s(X)) \arrow[ul, bend left, "i_{s(X)}"] & \bigoplus\limits_{X\in\mathcal{X}_\lambda}\rho(s(X)) \arrow[l, "p_X"] & \bigoplus\limits_{Y\in\mathcal{Y}_\lambda}\rho(r(Y)) \arrow[l, "{[\lambda]^*}"] & \rho(r(Y)) \arrow[l, "i_Y"] &
\end{tikzcd}$$

Here the composition of the upper arrows correspond to right multiplication by (XY) and the composition by lower arrows correspond to right multiplication by $(YX)^*$. Verifying that the above defining relations satisfy defining relations of L is left to the reader.

Now we show that the above constructions give equivalance of categories. By Remark 4.3.2, we have $M_{\rho_M} = \bigoplus\limits_{v\in E^0} Mv = M$ and their L-module structures also match. Given a module homomorphism $\varphi : M \to N$, we have $\varphi = \bigoplus\limits_{v\in E^0}\varphi_v : \bigoplus\limits_{v\in E^0} Mv \to \bigoplus\limits_{v\in E^0} N$.

For the composition in the other order $\rho_{M_\rho}(v) = M_\rho v = \left(\bigoplus\limits_{w\in E^0}\rho(w)\right) v = \rho(v)$ and $\rho(e) = \rho_{M_\rho}(e) : M_\rho s(e) \to M_\rho r(e)$. For, let $e = (XY)$ for some $X \in \mathcal{X}_\lambda$ and $Y \in \mathcal{Y}_\lambda$, then the following diagram commutes.

$$\begin{tikzcd}
M_\rho s(e) = \rho(s(e)) \arrow[r, hook, "i_{s(e)}"] \arrow[d, "\rho(e)"'] & M_\rho = \bigoplus\limits_{w\in E^0}\rho(w) \arrow[d, "p_{s(e)} i_X [\rho(\lambda)] p_Y i_{r(e)}"] \\
M_\rho r(e) = \rho(r(e)) \arrow[r, hook, "i_{r(e)}"] & M_\rho = \bigoplus\limits_{w\in E^0}\rho(w)
\end{tikzcd}$$

Finally, for any homomorphism $\{\varphi_v : \rho(v) \to \sigma(v)\}_{v\in E^0}$ from ρ to σ, the v-component of $\bigoplus\limits_{w\in E^0}\varphi_w$ is $\varphi_v : \rho_{M_\rho}(v) = \rho(v) \to \sigma_{M_\sigma}(v) = \sigma(v)$. $\square$

Remark 4.3.4. We note that the full subcategory of graded quiver representations with respect to standard $\mathbb{Z}$-grading satisfying condition (H) is equivalent to the category of graded unital L-modules. The proof follows on similar lines of the proof of Theorem 4.3.3.

Theorem 4.3.5. *The composition of the forgetful functor from $\mathfrak{M}_L$ to $\mathfrak{M}_E$ with $\_\otimes_{K(E)} L$ from $\mathfrak{M}_E$ to $\mathfrak{M}_L$ is naturally equivalent to the identity functor on $\mathfrak{M}_L$.*

Proof. We note that both forgetful functor and $\_\otimes_{K(E)} L$ send unital modules to unital modules. Let the composition of forgetful functor with $\_\otimes_{K(E)} L$ be denoted by $\mathcal{F}$

and the identity functor on $\mathfrak{M}_L$ be denoted by $\mathcal{I}$. If M is an L-module, the L-module homomorphism $M \otimes_{K(E)} L \to M$ given by $m \otimes a \mapsto ma$ defines an natural transformation from $\mathcal{F}$ to $\mathcal{I}$. To see that this is an isomorphism, we define its inverse $M \to M \otimes_{K(E)} L$ by $m \mapsto \sum_{\substack{v \in E^0 \\ mv \neq 0}} m \otimes v$. Observe that this sum is finite since M is unital.

To check that the above inverse defines an L-linear map, we need to check on generators. For every $w \in E^0$ and $m \in M$, we have $\sum mu \otimes v = m \otimes u = \left(\sum m \otimes v\right) u$, since E^0 is a set of orthogonal idempotents. For all $\lambda \in \Lambda$, $X \in \mathcal{X}_\lambda$, $Y \in \mathcal{Y}_\lambda$ and $m \in M$ we have

$$\begin{aligned}
\sum m(XY) \otimes v &= m(XY) \otimes r(Y), && \text{since } ev = 0 \text{ iff } r(e) \neq v \\
&= m(XY) \otimes \left(\sum_{X' \in \mathcal{X}_\lambda} (YX')^*(X'Y) \right) && \text{by } (L2) \\
&= \sum_{X' \in \mathcal{X}_\lambda} m(XY)(YX')^* \otimes (X'Y) \\
&= ms(X) \otimes (XY) && \text{by } (L1) \\
&= m \otimes (XY) \\
&= \left(\sum m \otimes v \right)(XY).
\end{aligned}$$

Similarly $\sum m(YX)^* \otimes v = (\sum m \otimes v)(YX)^*$.

The composition $m \mapsto \sum m \otimes v \mapsto \sum mv = m$. Since elements of the form $m \otimes v$ with $m \in Mv$ generate $M \otimes L$ as an L-module and for such elements we have $m \otimes v \mapsto mv \mapsto mv \otimes v = m \otimes v$, the other composition is also identity. $\square$

Recall that the **universal localization** $\Sigma^{-1}A$ of an algebra A with respect to a set $\Sigma = \{\sigma : P_\sigma \to Q_\sigma\}$ of homomorphisms between finitely generated projective A-modules, is an initial object among algebra homomorphisms $f : A \to B$ such that $\sigma \otimes id_B : P_\sigma \otimes_A B \to Q_\sigma \otimes_A B$ is an isomorphism for every $\sigma \in \Sigma$.

Theorem 4.3.6. *L is the universal localization of $K(E)$ with respect to*

$$\left\{ \sigma_\lambda : \bigoplus_{Y \in \mathcal{Y}_\lambda} (r(Y))K(E) \longrightarrow \bigoplus_{X \in \mathcal{X}_\lambda} (s(X))K(E) \right\}_{\lambda \in \Lambda},$$

$$(a_Y)_{Y\in\mathcal{Y}_\lambda} \overset{\sigma_\lambda}{\longmapsto} \left(\sum_{Y\in\mathcal{Y}_\lambda}(XY)a_Y\right)_{X\in\mathcal{X}_\lambda}.$$

Proof. Since $v \in E^0$ is an idempotent, the cyclic module $vK(E)$ is projective. For each $\lambda \in \Lambda$, $\sigma_\lambda \otimes id_L$ is an isomorphism with inverse σ^*_λ, where $(a_X)_{X\in\mathcal{X}_\lambda} \overset{\sigma^*_\lambda}{\longmapsto} \left(\sum_{X\in\mathcal{X}_\lambda}(YX)^*a_X\right)_{Y\in\mathcal{Y}_\lambda}$.
If $f : K(E) \to B$ is an algebra homomorphism, then $f(v)^2 = f(v)$ and $vK(E)\otimes_{K(E)} B \cong f(v)B$ by $a \otimes b \mapsto f(a)b$ and $b \mapsto v \otimes b$.

Let $f : K(E) \to B$ be an algebra homomorphism such that $\sigma_\lambda \otimes id_B$ is an isomorphism for all $\lambda \in \Lambda$. Then the composition $f(s(X))B \cong (s(X))K(E) \otimes_{K(E)} B \xrightarrow{i_{s(X)}\otimes id_B} \left(\bigoplus_{X\in\mathcal{X}_\lambda} s(X)K(E)\right)\otimes_{K(E)}B \xrightarrow{\sigma^*_\lambda\otimes\ id_B} \left(\bigoplus_{Y\in\mathcal{Y}_\lambda} r(Y)K(E)\right)\otimes_{K(E)}B \xrightarrow{p_{r(Y)}\otimes id_B} r(Y)K(E)\otimes_{K(E)} B \cong f(r(Y))B$ is uniquely and completely determined, which we call $f((YX)^*)$. Now $\tilde{f}(v) := f(v)$ for all $v \in E^0$, $\tilde{f}((XY)) = f((XY))$ for all $\lambda \in \Lambda$, $X \in \mathcal{X}_\lambda$ and $Y \in \mathcal{Y}_\lambda$ defines the unique homomorphism $\tilde{f} : L \to B$ factoring f through $K(E) \to L$. □

Proposition 4.3.7. *Let $(\dot{E}, \Lambda)$ be hypergraph. If $d : E^0 \to \mathbb{N} \cup \{\infty\}$ satisfies*

$$\sum_{X\in\mathcal{X}_\lambda} d(s(X)) = \sum_{Y\in\mathcal{Y}_\lambda} d(r(Y)) \text{ for all } \lambda \in \Lambda,$$

then there is an L-module M with $\dim_K(Mv) = d(v)$.

Proof. Define the quiver representation ρ by $\rho(v) = K^{d(v)}$ if $d(v) < \infty$ and $\rho(v) := K^{(\mathbb{N})}$ otherwise. Then by definition of d we can find isomorphism $\theta_\lambda : \bigoplus_{X\in\mathcal{X}_\lambda} \rho(s(X)) \to \bigoplus_{Y\in\mathcal{Y}-\lambda} \rho(r(Y))$ for all $\lambda \in \Lambda$. Let $\rho(XY) := i_{s(X)}\theta_\lambda p_{r(Y)}$ for all $\lambda \in \Lambda$, $X \in \mathcal{X}_\lambda$ and $Y \in \mathcal{Y}_\lambda$. Condition (H) is satisfied by construction and the corresponding L-module M of Theorem 4.3.3 has $\dim_K(Mv) = \dim_K\rho(v) = d(v)$. □

Definition 4.3.8. A **dimension function** of a hypergraph $(\dot{E}, \Lambda)$ is a function $d : E^0 \to \mathbb{N}$ satisfying $\sum_{X\in\mathcal{X}_\lambda} s(X) = \sum_{Y\in\mathcal{Y}_\lambda} r(Y)$ for all $\lambda \in \Lambda$.

Remark 4.3.9. If the L-module M is finitary, i.e, $\dim(Mv) < \infty$ for all $v \in E^0$ then by Lemma 4.3.1, $d(v) := \dim(Mv)$ is a dimension function. By Proposition 4.3.7, the converse also holds. That is, every dimension function is realizable. Moreover, since $\dim(M) = \sum_{v\in E^0} \dim(Mv)$, $d(v) := \dim(Mv)$ has finite support if M is finite dimensional.

4.3.1 Support subgraphs and the hypergraph monoid

Let $\mathcal{H} = (\dot{E}, \Lambda)$ be a hypergraph and E' be a full subgraph of E. Then there is a natural (hyper) biseparation induced on E' from $\mathcal{H}$ as follows: For each $X \in C$, if $s(X) \in (E')^0$, define $X' := X \bigcap (E')^1$ and similarly for each $Y \in D$, if $r(Y) \in (E')^0$, define $Y' := Y \bigcap (E')^0$. Also for each $\lambda \in \Lambda$, define λ' using the following data: $\mathcal{X}_{\lambda'} := \{X' \mid X \in \mathcal{X}_\lambda\}$ and $\mathcal{Y}_{\lambda'} := \{Y' \mid Y \in \mathcal{Y}_\lambda\}$. Finally define $\Lambda' = \{\lambda' \mid \lambda \in \Lambda, \mathcal{X}_\lambda \neq \emptyset \text{ and } \lambda \neq \emptyset\}$. We call $\mathcal{H}' = (\dot{E}', \Lambda')$ a **full sub-hypergraph** of $\mathcal{H}$ (hyper-induced from E').

Definition 4.3.10. Let $\mathcal{H} = (\dot{E}, \Lambda)$ be a hypergraph. A full sub-hypergraph $\mathcal{H}' = (\dot{E}', \Lambda')$ is called **co-bisaturated** if the following conditions are satisfied: For every $\lambda' \in \Lambda'$,

1. if $s(X) \in (E')^0$, then $X \cap (E')^1 \neq \emptyset$, where $X \in \mathcal{X}_{\lambda'}$
2. if $r(Y) \in (E')^0$, then $Y \cap (E')^1 \neq \emptyset$, where $Y \in \mathcal{Y}_{\lambda'}$.

We note that a full sub-hypergraph $\dot{E}'$ of $\dot{E}$ is co-bisaturated if and only if $E^0 - (E')^0$ is bisaturated subset of E^0.

Let $\mathcal{H} = (\dot{E}, \Lambda)$ be a hypergraph and M be a right $L(\mathcal{H})$-module. The **support subgraph** of M, denoted by E_M, is the full subgraph of E induced on $V_M := \{v \in E^0 \mid Mv \neq 0\}$ and the hypergraph $\mathcal{H}_M = (\dot{E}_M, \Lambda_M)$, which is the full sub-hypergraph of $\mathcal{H}$ hyper-induced from E_M, is called the **support sub-hypergraph** of M.

Lemma 4.3.11. *Let $\mathcal{H} = (\dot{E}, \Lambda)$ be a hypergraph and $\mathcal{H}' = (\dot{E}', \Lambda')$ be a full sub-hypergraph of $\mathcal{H}$. Then the following are equivalent.*

1. *$\mathcal{H}' = \mathcal{H}_M$, is the support sub-hypergraph of a unital $L(\mathcal{H})$-module M.*
2. *$\mathcal{H}'$ is co-bisaturated.*
3. *The map $\theta : L(\mathcal{H}) \to L(\mathcal{H}')$ defined (on generators) by*

$$\theta(x) := \begin{cases} x, & \text{if } x \in (E')^0 \sqcup (E')^1 \sqcup \overline{(E')^1}, \\ 0, & \text{otherwise,} \end{cases}$$

extends to an onto algebra homomorphism.

Proof. (1) $\Rightarrow$ (2): Let $\lambda' \in \Lambda'$ and $X \in \mathcal{X}_{\lambda'}$. Assume $s(X) \in (E_M)^0$, then $0 \neq Ms(X) \hookrightarrow \bigoplus_{X\in\mathcal{X}_\lambda} Ms(X) \cong \bigoplus_{Y\in\mathcal{Y}_\lambda} Mr(Y)$ implies that there exists $Y \in \mathcal{Y}_\lambda$ such that $X \cap Y \neq \emptyset$ which is equivalent to $X \cap (E')^1 \neq \emptyset$. Similarly, if $Y \in \mathcal{Y}_{\lambda'}$ and $r(Y) \in (E_M)^0$, then $Y \cap (E')^1 \neq \emptyset$.

(2) $\Rightarrow$ (3): We check that θ preserves the defining relations of $L(\mathcal{H})$. It is direct that path algebra relations are satisfied. Let $\lambda' \in \Lambda'$, $X_1, X_2 \in \mathcal{X}_{\lambda'}$ and $Y_1, Y_2 \in \mathcal{Y}_{\lambda'}$. If $s(X_i), r(Y_i) \in (E')^0$ then $X_i \cap (E')^1 \neq \emptyset$ and $Y_1 \cap (E')^1 \neq \emptyset$. Hence the image of $\sum_{X\in\mathcal{X}_\lambda} (Y_1X)^*(XY_2) = \delta_{Y_1Y_2} r(Y)$ is $\sum_{X\in\mathcal{X}_{\lambda'}} (Y_1X)^*(XY_2) = \delta_{Y_1Y_2} r(Y)$. Similarly, the image of $\sum_{Y\in\mathcal{Y}_\lambda} (X_1Y)(YX_2)^* = \delta_{X_1X_2} s(X)$ is $\sum_{Y\in\mathcal{Y}_{\lambda'}} (X_1Y)(YX_2)^* = \delta_{X_1X_2} s(X)$.

(3) $\Rightarrow$ (1): Let $M := L(\mathcal{H}') \cong L(\mathcal{H})/\mathrm{Ker}\theta$. Now $v \in (E')^0$ if and only if $\theta(v) \neq 0$ and $Mv = L(\mathcal{H}')v \neq 0$. Hence the vertex set of E_M is $(E')^0$. It is routine to check that $\mathcal{H}'$ is full sub-hypergraph and hence $\mathcal{H}_M = \mathcal{H}'$. $\square$

Proposition 4.3.12. *If M is a unital $L(\mathcal{H})$-module then M also has the structure of a unital $L(\mathcal{H}_M)$-module induced through the epimorphism $\theta : L(\mathcal{H}) \to L(\mathcal{H}_M)$. Moreover,* $\mathrm{Ker}\theta$ *is generated by $E^0 - V_M = \{v \in E^0 \mid Mv = 0\}$ and* $\mathrm{Ker}\theta \subseteq \mathrm{Ann}M$.

Proof. Let ρ_M be the quiver representation of E corresponding to M as defined in the proof of Theorem 4.3.3. We claim that the restriction of ρ_M to E_M satisfies (H). That is, if $\rho' := \rho_M|_{E_M}$, then for all $\lambda' \in \Lambda_M$,

$$[\rho'] : \bigoplus_{X\in\mathcal{X}_{\lambda'}} \rho'(s(X)) \to \bigoplus_{Y\in\mathcal{Y}_{\lambda'}} \rho'(r(Y)) \text{ is an isomorphism.}$$

For,

$$\begin{aligned}
\bigoplus_{X\in\mathcal{X}_{\lambda'}} \rho'(s(X)) &= \bigoplus_{X\in\mathcal{X}_{\lambda'}} Ms(X) \\
&= \bigoplus_{X\in\mathcal{X}_{\lambda}} Ms(X) \quad \text{since } Ms(X) = 0 \text{ for } X \notin \mathcal{X}_{\lambda'} \\
&\cong \bigoplus_{Y\in\mathcal{Y}_{\lambda}} Mr(Y) \\
&= \bigoplus_{Y\in\mathcal{Y}_{\lambda'}} Mr(Y) \quad \text{since } Mr(Y) = 0 \text{ for } Y \notin \mathcal{Y}_{\lambda'} \\
&= \bigoplus_{Y\in\mathcal{Y}_{\lambda'}} \rho'(r(Y)).
\end{aligned}$$

Let M' be the unital $L(\mathcal{H}_M)$-module corresponding to ρ'. Now M' is also an $L(\mathcal{H})$-module via $\theta : L(\mathcal{H}) \to L(\mathcal{H}_M)$. As vector spaces $M' = \bigoplus_{v \in V_M} Mv \cong \bigoplus v \in E^0 Mv = M$. We can define an $L(\mathcal{H}_M)$-module structure on M via this isomorphism. The action of the generators on M and M' is compatible with this isomorphism, so $M \cong M'$ as $L(\mathcal{H})$-modules. Thus the $L(\mathcal{H})$-module structure of M is induced from the $L(\mathcal{H}_M)$-module structure via θ.

For the second part, Let I_M be the ideal generated by $E^0 - V_M$. We show that $L(\mathcal{H}_M) \cong L(\mathcal{H})/\mathrm{Ker}\theta$ and $L(\mathcal{H})/I_M$ are isomorphic. Since, $E^0 - V_M \subseteq \mathrm{Ker}\theta$, we have a surjection from $L(\mathcal{H})/I_M$ to $L(\mathcal{H}_M)$. Let $\varphi : L(\mathcal{H}_M) \to L(\mathcal{H})/I$ be defined on generators by $x \mapsto x + I$, where $x \in E^0 \sqcup E^1 \sqcup \overline{E^1}$. It is not hard to show that φ is a homomorphism and the inverse of the above surjection. $\square$

Recall that given a hypergraph $\mathcal{H} = (\dot{E}, \Lambda)$, its hypergraph monoid $H(\mathcal{H})$ is defined as the additive monoid generated by E^0 modulo the following relations:

$$\sum_{X \in \mathcal{X}_\lambda} s(X) = \sum_{Y \in \mathcal{Y}_\lambda} r(Y) \quad \text{for all } \lambda \in \Lambda.$$

Therefore, dimension functions of $\mathcal{H}$ correspond exactly to monoid homomorphisms from $H(\mathcal{H})$ to $\mathbb{N}$.

Since $H(\mathcal{H})$ is isomorphic to the monoid $\mathcal{V}(L(\mathcal{H}))$ the generator v of $H(\mathcal{H})$ corresponds to the (right) projective $L(\mathcal{H})$-module $vL(\mathcal{H})$. The corresponding relations among the isomorphism class of the cyclic projective modules was shown to hold in the proof of Theorem 4.3.6. We can now reinterpret the existence of a nonzero finite dimensional represenatation in terms of the nonstable K-theory of $L(\mathcal{H})$.

Theorem 4.3.13. *$L(\mathcal{H})$ has a nonzero finite dimensional representation if and only if $\dot{E}$ has a finite, full co-bisaturated sub-hypergraph $\mathcal{G}$ with a nonzero monoid homomorphism from $\mathcal{V}(L(\mathcal{G}))$ to $\mathbb{N}$.*

Proof. By Remark 4.3.9, $L(\mathcal{H})$ has a nonzero finite dimensional representation if and only if $\mathcal{H}$ has a nonzero dimension function of finite support. The support of such dimension function defines a finite, full, co-bisaturated sub-hypergraph $\mathcal{G}$ and its restriction gives a nonzero dimension function on $\mathcal{G}$ and thus a nonzero monoid homomorphism from $\mathcal{V}(L(\mathcal{G}))$ to $\mathbb{N}$ as well. Conversely, since $\mathcal{G}$ is co-bisaturated, any nonzero dimension function on $\mathcal{G}$ can be extended by 0 to a dimension function on $\mathcal{H}$ and this gives a nonzero dimension function of finite support on $\mathcal{H}$. □

4.4 Some remarks on Cohn-Leavitt path algebras of semi-regular hypergraphs with Invariant Basis Number

Let $\mathcal{H} = (\dot{E}, \Lambda)$ be a finite semi-regular hypergraph and let $H := H(\mathcal{H})$ be the H-monoid of $\mathcal{H}$. Let the Cohn-Leavitt path algebra $\mathcal{A}_K(\mathcal{H})$ be denoted simply by L and its Grothendieck group by $\mathcal{K}_0(L)$. Let $\mathcal{U}(L)$ denote the submonoid of the $\mathcal{V}$-monoid $\mathcal{V}(L)$ generated by the element $[L] \in \mathcal{V}(L)$. Then L has IBN property if and only if $\mathcal{U}(L)$ has infinite order. Now suppose $G(\mathcal{U}(L))$ denotes the Grothendieck group of $\mathcal{U}(L)$. Then one can show that the natural map $G(\mathcal{U}(L)) \to \mathcal{K}_0(L)$ induced by the inclusion $\mathcal{U}(L) \hookrightarrow \mathcal{V}(L)$ is an embedding (see [39, Proposition 7]). So $[L]$, treated as an element in the group $\mathcal{K}_0(L)$, has infinite order. This means that the element $[L] \otimes 1$ in $\mathcal{K}_0(L) \otimes \mathbb{Q}$ is nonzero. We know that $[L] \in \mathcal{V}(L)$ corresponds to the element $[\sum_{v \in E^0} v] \in H$ under the isomorphism of functors proved in Theorem 4.1.8. So, if $G(H)$ denotes the Grothendieck group of H, from the above arguments we can conclude that L has IBN if and only if $\sum_{v \in E^0} v$ is nonzero in $G(H) \otimes \mathbb{Q}$, which is equivalent saying that $\sum_{v \in E^0} v$ is not in the $\mathbb{Q}-$ linear span of the elements of R in $\mathbb{Q}\Omega$ (cf. [39, Theorem 13]), where Ω is the set $E^0 \sqcup Q \sqcup P$ (Q and P are as in the Definition 4.1.6), and

$$\begin{aligned} R \ := \ & \bigcup_{\lambda \in \Lambda_S^T} \left[\sum_{X \in \mathcal{X}_\lambda} s(X) - \sum_{Y \in \mathcal{Y}_\lambda} r(Y) \right] \\ \sqcup \ & \bigcup_{\lambda \in \Lambda_T^{\text{fin}}} \left[\sum_{X \in \mathcal{X}_\lambda} s(X) - \sum_{Y \in \mathcal{Y}_\lambda} r(Y) - q_{\mathcal{Y}_\lambda} \right] \\ \sqcup \ & \bigcup_{\lambda \in \Lambda_S^{\text{fin}}} \left[\sum_{Y \in \mathcal{Y}_\lambda} r(Y) - \sum_{X \in \mathcal{X}_\lambda} s(X) - p_{\mathcal{X}_\lambda} \right]. \end{aligned}$$

4.4.1 Matrix criteria for Leavitt path algebra of a finite hypergraph having IBN

In this subsection, we generalize the main result of [55, Section 3]. Let $(\dot{E}, \Lambda)$ be a finite hypergraph such that $|\Lambda| = h$ and $E^0 = n$. Then by theorem 4.1.8, the $\mathcal{V}$-monoid of $L_K(\dot{E})$ is generated by the set E^0 modulo h relations of the form

$$\sum_{t=1}^{m} s(X_t) = \sum_{u=1}^{l} r(Y_u), \tag{4.4.1}$$

one corresponding to each element of Λ. Let A and B be the coefficient matrices corresponding to the LHS and RHS respectively of the h relations (4.4.1). Then it is clear that both A and B are $h \times n$ matrices with entries as non-negative integers. Let T be a free abelian monoid on the set of all vertices. For each element $x \in T$, and for each i such that $1 \leq i \leq h$, let $M_i(x)$ denote the element of T which results by applying to x the relation (4.4.1) corresponding to the element $\lambda_i \in \Lambda$. For any sequence σ taken from the set $\{1, 2, \ldots, h\}$, and any $x \in T$, let $\Delta_\sigma(x) \in T$ be the element obtained by applying M_i operations in the order specified by σ.

Definition 4.4.1. Suppose for each pair $x, y \in T$, $[x] = [y]$ in the $\mathcal{V}$-monoid if and only if there are two sequences σ and σ' taken from the set $\{1, 2, \ldots, h\}$ such that $\Delta_\sigma(x) = \Delta_{\sigma'}(y)$ in T. Then we say that the **confluence condition** holds in T.

Theorem 4.4.2. *Let $(\dot{E}, \Lambda)$ be a finite hypergraph such that $|\Lambda| = h$ and $E^0 = n$. Suppose A and B are the coefficient matrices corresponding to LHS and RHS respectively of the h relations of the $\mathcal{V}$-monoid of $L_K(\dot{E})$. Also suppose that the confluence condition holds in T, the free abelian monoid on E^0. Then $L_K(\dot{E})$ has Invariant Basis Number if and only if rank$(B^t - A^t) <$ rank$([B^t - A^t\ c])$, where c is the column matrix of order $n \times 1$ with all its entries equal to 1.*

Proof. Suppose that $\text{rank}(B^t - A^t) < \text{rank}(B^t - A^t\ c)$. We prove that if m and p are positive integers such that

$$m[\sum_{i=1}^{n} v_i] = p[\sum_{i=1}^{n} v_i] \tag{4.4.2}$$

in the $\mathcal{V}$-monoid, then $m = p$. So let us assume that the equation (4.4.2) holds for some positive integers m and p. Since the confluence condition holds in T, there are two sequences σ and σ' taken from $\{1, 2, \ldots, h\}$ such that $\Delta_\sigma(m\sum_{i=1}^n v_i) = \Delta_{\sigma'}(p\sum_{i=1}^n v_i) = \gamma$(say) in T. Now suppose M_j is invoked k_j times in Δ_σ and k'_j times in $\Delta_{\sigma'}$. Then we have

$$\begin{aligned}
\gamma &= \Delta_\sigma(m\sum_{i=1}^n v_i) \\
&= [m + k_1(b_{11} - a_{11}) + \cdots + k_h(b_{h1} - a_{h1})]v_1 \\
&+ [m + k_1(b_{12} - a_{12}) + \cdots + k_h(b_{h2} - a_{h2})]v_2 \\
&+ \ldots\ldots\ldots \\
&+ [m + k_1(b_{1z} - a_{1z}) + \cdots + k_h(b_{hz} - a_{hz})]v_z \\
&+ [m + k_1(b_{1(z+1)}) + \cdots + k_h(b_{h(z+1)})]v_{z+1} \\
&+ \ldots\ldots\ldots \\
&+ [m + k_1(b_{1n}) + \cdots + k_h(b_{hn})]v_n.
\end{aligned}$$

Also

$$\begin{aligned}
\gamma &= \Delta_{\sigma'}(p\sum_{i=1}^n v_i) \\
&= [p + k'_1(b_{11} - a_{11}) + \cdots + k'_h(b_{h1} - a_{h1})]v_1 \\
&+ [p + k'_1(b_{12} - a_{12}) + \cdots + k'_h(b_{h2} - a_{h2})]v_2 \\
&+ \ldots\ldots\ldots \\
&+ [p + k'_1(b_{1z} - a_{1z}) + \cdots + k'_h(b_{hz} - a_{hz})]v_z \\
&+ [p + k'_1(b_{1(z+1)}) + \cdots + k'_h(b_{h(z+1)})]v_{z+1} \\
&+ \ldots\ldots\ldots \\
&+ [p + k'_1(b_{1n}) + \cdots + k'_h(b_{hn})]v_n.
\end{aligned}$$

Let $m_i = (k_i' - k_i)$ for $i = 1, \ldots, h$. From the above two equations, we have the following system of equations-

$$\begin{aligned}
(m-p) &= m_1(b_{11} - a_{11}) + \cdots + m_h(b_{h1} - a_{h1}) \\
(m-p) &= m_1(b_{12} - a_{12}) + \cdots + m_h(b_{h2} - a_{h2}) \\
&\vdots \\
(m-p) &= m_1(b_{1z} - a_{1z}) + \cdots + m_h(b_{hz} - a_{hz}) \\
&\vdots \\
(m-p) &= m_1(b_{1n} - a_{1n}) + \cdots + m_h(b_{hn} - a_{hn}).
\end{aligned}$$

So $(m_1, \ldots, m_h) \in \mathbb{Z}^h$ is a solution of the linear system $(B^t - A^t)x = (m-p)c$, where $x = (x_1, \ldots, x_h)^t$ and c is the column matrix mentioned in the statement of the theorem. This means $\text{rank}(B^t - A^t) = \text{rank}(B^t - A^t \;\; (m-p)c)$. We know that if $m \neq p$, then $\text{rank}(B^t - A^t \;\; (m-p)c) = \text{rank}(B^t - A^t \;\; c)$. This would mean that $\text{rank}(B^t - A^t) = \text{rank}(B^t - A^t \;\; c)$ whenever $m \neq p$, contrary to our initial assumption. This proves the first part.

Conversely, assume that $\text{rank}(B^t - A^t) = \text{rank}(B^t - A^t \;\; c) := r$. We will prove that there exists a pair of distinct positive integers m and p such that $m[\sum_{i=1}^n v_i] = p[\sum_{i=1}^n v_i]$ in the $\mathcal{V}$-monoid of $\mathcal{A}_K(\dot{E})$.

The fact that $\text{rank}(B^t - A^t \;\; c) = r$ means that after finite number of elementary row operations, $(B^t - A^t \;\; c)$ can be brought to the form

$$\begin{pmatrix}
0 & \ldots & d_{1j_1} & \ldots & d_{1j_2-1} & 0 & d_{1j_2+1} & \ldots & d_{1j_r-1} & 0 & \ldots & c_1 \\
0 & \ldots & 0 & \ldots & 0 & d_{2j_2} & d_{2j_2+1} & \ldots & d_{2j_r-1} & 0 & \ldots & c_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\
0 & \ldots & \ldots & \ldots & 0 & \ldots & \ldots & 0 & 0 & d_{rj_r} & \ldots & c_r \\
0 & \ldots & \ldots & \ldots & 0 & \ldots & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots \\
0 & \ldots & \ldots & \ldots & 0 & \ldots & \ldots & 0 & 0 & 0 & \ldots & 0
\end{pmatrix}$$

where the entries are integers, $d_{1j_1} d_{2j_2} \ldots d_{rj_r} \neq 0$ and $\sum_{i=1}^r c_i^2 \neq 0$. So it is clear that one particular solution for the linear system $(B^t - A^t)x = c$ is the column vector $(\frac{c_1}{d_{1j_1}}, \frac{c_2}{d_{2j_2}}, \ldots, \frac{c_r}{d_{rj_r}}, 0, \ldots, 0)^t$.

Now let

$$m_j := \begin{cases} \frac{c_i|d_{1j_1}d_{2j_2}\dots d_{rj_r}|}{d_{ij_i}}, & \text{if } j = j_i \ (1 \leq i \leq r) \\ 0, & \text{otherwise,} \end{cases}$$

$$p := \max\{|m_j| \mid j = 1, \dots, h\}, \qquad m := |d_{1j_1}d_{2j_2}\dots d_{rj_r}| + p \text{ and}$$

$$(k'_j, k_j) := \begin{cases} (0,0), & \text{if } m_j = 0, \\ (m_j, 0), & \text{if } m_j > 0, \\ (0, -m_j), & \text{if } m_j < 0. \end{cases}$$

From the above definitions, it is clear that $(m-p) > 0$. So the h-tuple $(m_1, m_2, \dots, m_h)$ is a solution for the linear system $(B^t - A^t)x = (m-p)c$. This, from the first part of the proof, is equivalent to showing that $m[\sum_{i=1}^n v_i] = p[\sum_{i=1}^n v_i]$. This means that $\mathcal{A}_K(\dot{E})$ does not have Invariant Basis Number, thereby completing the proof. $\square$

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List of Publications

1. Mohan. R and B.N. Suhas:
   Cohn-Leavitt path algebras of bi-separated graphs.
   Preprint, 2019. arXiv:1907.00159

2. Mohan. R:
   Leavitt path algebras of weighted Cayley graphs $C_n(S, w)$.
   Accepted for publication in *Proceedings - Mathematical Sciences*.
   arXiv:1810.09866