C^*-extreme Maps and Nest Algebras

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To

Mammi, Papa, Didi and Chhotu

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Introduction

The classical notion of convexity has played a very important role in understanding the structure of various objects in mathematical analysis as well as in the foundation of quantum physics. In modern mathematics however, in order to capture higher order convexity structure, certain quantization (or noncommutative analogue) of this notion is required. Perhaps the defining moment in quantization of functional analysis came through the work of von Neumann and Murray on rings of operators in late 1930s and early 1940s, where noncommutative probability and integration theory was formulated by replacing functions with operators. Gelfand-Naimark-Segal construction for C^*-algebras, the introduction of theory of matrix ordered spaces by Choi-Effros [14], matrix normed spaces by Ruan [70], operator theoretic states by Stinespring [75] and Hahn-Banach type theorems by Arveson [5] were some of the notable developments in this direction. At the same time, a growing need to establish noncommutative analogue of convexity theory in linear spaces gained momentum among operator algebraists. Several attempts have been made to introduce operator version of convex analysis, most notably being matrix convexity by Wittstock [79] and Effros-Winkler [23], C^*-convexity by Loebl-Paulsen [49] and Farenick-Morenz [28], CP-convexity by Fujimoto [31], and nc-convexity by Davidson-Kennedy [18].

A widely studied notion which fits very naturally in the framework of noncommutative convexity is the concept of C^*-convexity. The prominent idea here is to replace scalar-valued convex combinations $\sum_{i=1}^n \lambda_i x_i$ for $\lambda_i \in [0,1]$ satisfying $\sum_{i=1}^n \lambda_i = 1$, by *$C^*$-convex combinations* of the form $\sum_{i=1}^n \alpha_i^* x_i \alpha_i$ for *C^*-convex coefficients* α_i in a unital C^*-algebra satisfying $\sum_{i=1}^n \alpha_i^* \alpha_i = 1$. Subsequently, one defines a notion of *C^*-convex sets* and an appropriate notion of extreme points, called *C^*-extreme points* (see Definition 2.1.2). Loebl and Paulsen [49] while trying to understand the generalized numerical range of an operator introduced the notion of C^*-convexity and C^*-extreme points for subsets of C^*-algebras. Subsequent study followed from the work of Hopenwasser-Moore-Paulsen [41] and Farenick-Morenz [25–28, 55]. The notion of C^*-convexity in due course got generalized on similar lines in different contexts; for subsets of bimodules over C^*-algebras [50–52], spaces of completely positive maps [28, 29, 33], and positive operator valued measures [24]. Our main interest in this thesis lies in the setting of C^*-convexity structure of spaces of unital completely positive maps on unital C^*-algebras as well as spaces of normalized positive operator valued measures.

Ever since Stinespring [75] introduced the notion of completely positive (CP) maps on C^*-

algebras and their dilation theory, there has been strong interest in their study. Arveson's seminal paper [5] provided a systematic and deep structure theory of CP maps, whereas its applications among many others to multivariate operator theory followed in his successive work. Since then CP maps have found a number of important applications in various contexts of operator algebras. For example CP maps play a central role in understanding the structure of nuclear and injective C^*-algebras, while Markov maps and trace preserving CP maps are among the most fundamental tools in quantum probability and quantum information theory respectively. Often an important approach that many researchers adopt while analyzing these objects is via the study of the convexity (both classical as well as operator theoretic) structure of the spaces of CP maps and their subclasses (see [5, 7, 13, 23, 28, 31, 41, 49, 52, 59, 60] for some general references.)

Given a unital C^*-algebra $\mathcal{A}$ and a complex separable Hilbert space $\mathcal{H}$, the *generalized state space* $S_{\mathcal{H}}(\mathcal{A})$ is the set of all unital completely positive (UCP) maps from $\mathcal{A}$ to the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$. Note that $S_{\mathbb{C}}(\mathcal{A})$ is the usual state space, so the generalized state spaces are perceived as the quantization of usual state spaces. Motivated by the ideas of Loebl and Paulsen, the notion of C^*-convexity and C^*-extreme points of $S_{\mathcal{H}}(\mathcal{A})$ was initiated and studied by Farenick and Morenz [28] (occasionally we will also use the term *C^*-extreme maps* for C^*-extreme points of $S_{\mathcal{H}}(\mathcal{A})$). Initial and some later developments in the theory by Farenick et al [24,28,29] remained limited under the assumption of finite dimensionality of Hilbert spaces. Although an abstract characterization of C^*-extreme points of $S_{\mathcal{H}}(\mathcal{A})$ due to Farenick-Zhou [29] and a sufficient condition for C^*-extreme maps on commutative C^*-algebras due to Gregg [33] did appear in general Hilbert space settings, so far a proper and systematic study on the structure of C^*-extreme points of $S_{\mathcal{H}}(\mathcal{A})$ for $\mathcal{H}$ infinite dimensional has been missing in literature. The main objective of this thesis is to investigate the structure of C^*-extreme points of $S_{\mathcal{H}}(\mathcal{A})$ for arbitrary C^*-algebra $\mathcal{A}$ and arbitrary dimensional Hilbert space $\mathcal{H}$, along with a wide variety of concepts associated with them.

We now outline the main contents of this thesis. Chapter 1 is devoted to the review of necessary underlying concepts in functional analysis on which most of the material is based. Specifically we recollect basic concepts from the theory of C^*-algebras and von Neumann algebras, and describe CP maps defined on them along with their dilation theory. The notion of positive operator valued measures and their correspondence with CP maps on commutative C^*-algebras are outlined. Finally we comment on factorization property of non-selfadjoint subalgebras of C^*-algebras, particularly the results involving nest algebras.

In Chapter 2, we begin by formally defining the notion of C^*-convexity and C^*-extreme points of the generalized state space $S_{\mathcal{H}}(\mathcal{A})$, and study some general properties. As often is the case, in the context of UCP maps it is the structure theorem of Stinespring through which we explore our results. As a byproduct of a result from Farenick-Zhou [29], we provide a slightly different but powerful abstract characterization of C^*-extreme points of $S_{\mathcal{H}}(\mathcal{A})$. This in essence implies that if ϕ is a C^*-extreme point of $S_{\mathcal{H}}(\mathcal{A})$ with minimal Stinespring triple $(\pi, V, \mathcal{H}_\pi)$, then the algebra $\mathcal{M} = \{T \in \pi(\mathcal{A})'; TVV^* = VV^*TVV^*\}$ *has factorization* in the von Neumann algebra $\pi(\mathcal{A})'$ (see Definition 1.5.1). We exploit this criteria in our study of direct sums of pure UCP maps, where a complete description is given for such maps to be C^*-extreme. This significantly extends a result of [29] from finite to infinite dimensional Hilbert space settings.

In the course of our research of certain C^*-extreme maps, we are naturally led to the study of nests of subspaces and associated nest algebras. We discover a strong mathematical link between the theory of C^*-extreme maps and factorization property of nest algebras. The theory of nest algebras and triangular form for operators began in late 1950's and early 1960's with the work of Gohberg-Krein [32], Ringrose [69] and Kadison-Singer [46]. Since then its literature has grown immensely. The similarity problem and its close relation to factorization property of nest algebras have attracted considerable interest from several researchers [1,2,17,32,47,48,64–66,69]. Some of these factorization results play very important roles in the development of our theory.

The aim of Chapter 3 is to analyze normal C^*-extreme maps of the set $S_{\mathcal{H}}(\mathcal{A})$ for a von Neumann algebra $\mathcal{A}$, more specifically for $\mathcal{A}$ of the form $\mathcal{B}(\mathcal{G})$ for some Hilbert space $\mathcal{G}$. Making use of the connection between C^*-extreme maps and factorization property of the algebra $\mathcal{M}$ (as mentioned above), we give a necessary and sufficient criterion for a normal C^*-extreme map in $S_{\mathcal{H}}(\mathcal{B}(\mathcal{G}))$ to be direct sum of pure normal UCP maps. Somewhat surprisingly, this criteria pertains to reflexivity of the algebra $\mathcal{M}$ in a type I factor and the lattices of its invariant subspaces. More precisely, it is shown that the lattice of invariant subspaces under an algebra with factorization property in $\mathcal{B}(\mathcal{H})$, is a complete countable and atomic nest. The proof of this assertion is relegated to Chapter 6, where we study a more general notion called logmodular algebras and their lattices.

An in-depth study has been carried out in Chapter 4 on the structure of C^*-extreme points of the space $\mathcal{P}_{\mathcal{H}}(X)$ of $\mathcal{B}(\mathcal{H})$-valued normalized positive operator valued measures (POVMs) on a measurable space $(X, \mathcal{O}(X))$. Our motivation to independently examine spaces of POVMs and their C^*-convexity structure stems from their eventual applications to the study of C^*-extreme maps on the commutative C^*-algebra $C(X)$ of all continuous functions on a compact Hausdorff space X. POVMs are called generalized measurements in quantum mechanics and are basic mathematical tools in quantum information theory. The notions of C^*-convexity and C^*-extreme points have natural extensions to POVMs. We first describe some abstract characterization of C^*-extreme POVMs parallel to those of UCP maps. An important path that we adopt is via decomposing a POVM into a sum of atomic and non-atomic POVMs, and analyze them separately. The main result of this chapter shows that all atomic C^*-extreme points of $\mathcal{P}_{\mathcal{H}}(X)$ are spectral measures. As a special case, it follows that C^*-extreme points of $\mathcal{P}_{\mathcal{H}}(X)$, when X is countable, are spectral. Moreover this says that in order to completely characterize C^*-extreme POVMs, it will be enough just to understand the behaviour of non-atomic C^*-extreme POVMs. We also discuss the notions of mutually singular POVMs and measure isomorphic POVMs, and their implications to C^*-convexity.

The main theme of Chapter 5 is to analyze C^*-extreme UCP maps on the commutative C^*-algebra $C(X)$ for a compact Hausdorff space X. Like the extremal points of usual state space on $C(X)$, it is known that C^*-extreme points of $S_{\mathbb{C}^n}(C(X))$ are $*$-homomorphism as well. In contrast, there exist non-homomorphic C^*-extreme points in $S_{\mathcal{H}}(C(X))$ for infinite dimensional Hilbert space $\mathcal{H}$. Nevertheless it is seen here that in a lot of cases, C^*-extreme points of

$S_{\mathcal{H}}(C(X))$ are $*$-homomorphisms even when $\mathcal{H}$ is infinite dimensional. The well-known correspondence of (unital) CP maps on $C(X)$ and (normalized) regular POVMs on Borel σ-algebra of X is very crucial in our approach. The theory developed in the previous chapter regarding POVMs is developed further here, where we examine regular POVMs on general topological Hausdorff spaces, only to be applied back to the study of C^*-extreme maps on commutative C^*-algebras. Our result on POVMs on countable spaces translates to saying that C^*-extreme points of $S_{\mathcal{H}}(C(X))$ for X countable (in particular for $S_{\mathcal{H}}(\mathbb{C}^n)$) are $*$-homomorphisms. **The problem of characterizing C^*-extreme points of $S_{\mathcal{H}}(\mathbb{C}^n)$ for infinite dimensional Hilbert space $\mathcal{H}$ has been open for over two decades, and we have settled it here.**

One of the most fundamental results in classical convexity theory is Krein-Milman theorem for compact convex sets in locally convex topological vector spaces, where one can extract information about the points of the convex sets using their extreme points. Naturally an analogue of Krein-Milman theorem is expected as well in non-commutative convex spaces under some appropriate topology. Several researchers have been quite successful in reaching this goal in varying set-ups, particularly when the operator-valued coefficients are taken from finite dimensional C^*-algebras: see for example, for compact C^*-convex subsets of M_n [55], for compact matrix convex sets in locally convex spaces [78], and for weak*-compact C^*-convex sets in hyperfinite factors [50]. However there are instances where such theorems fail to hold. In fact Magajna [52] produced an example of a weak*-compact C^*-convex subset of an operator $\mathcal{B}$-bimodule over a commutative von Neumann algebra $\mathcal{B}$ which does not even possess any C^*-extreme point. Nevertheless, for the C^*-convex spaces of UCP maps equipped with bounded weak topology, some promising results have appeared in restricted cases. More specifically a generalized Krein-Milman theorem is known to be true for C^*-convexity of the space $S_{\mathbb{C}^n}(\mathcal{A})$ when $\mathcal{A}$ is an arbitrary C^*-algebra [28]. We provide some significant new results in this line of research by showing a Krein-Milman type theorem for C^*-convexity of $S_{\mathcal{H}}(\mathcal{A})$ (for $\mathcal{H}$ separably infinite dimensional) in the following three cases, spread across different chapters: (1) $\mathcal{A}$ is a separable C^*-algebra (Theorem 2.4.3), (2) $\mathcal{A}$ is a type I factor (Theorem 3.2.2), and (3) $\mathcal{A}$ is a commutative C^*-algebra (Theorem 5.4.10). Whether the same holds for $S_{\mathcal{H}}(\mathcal{A})$ in full generality is a question which remains open.

The primary goal of Chapter 6 is to prove the aforementioned result in Chapter 3 about lattices of algebras having factorization property. Such algebras are special case of a notion called *logmodular algebra*, and here we undertake an independent study of logmodular algebras in von Neumann algebras. The main result shows that the lattice of projections in a factor von Neumann algebra $\mathcal{M}$ whose ranges are invariant under a logmodular algebra in $\mathcal{M}$, is always a nest. As a special case, it follows that all reflexive (in particular, completely distributive CSL) logmodular subalgebras of type I factors are nest algebras. **This answers in the affirmative a conjecture by Paulsen and Raghupathi [62].** Moreover this assertion when applied to algebras having factorization strengthens a well-known result about factorization property of nest algebras due to Larson [47]. We also explore some criteria under which a logmodular algebra is automatically reflexive and a nest algebra.

Chapter 1

Preliminaries

We briefly review some of the fundamental results in operator algebra literature on which the content of this thesis hinges upon. This will also help us in fixing notations and terminologies to be followed.

We will stick to the following convention throughout. All Hilbert spaces considered here are complex and separable, where the inner product is linear in second variable while antilinear in the first variable. We denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. If $\mathcal{H}$ and $\mathcal{K}$ are two Hilbert spaces, then $\mathcal{B}(\mathcal{H},\mathcal{K})$ denotes the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. By subspaces, projections and operators, we mean closed subspaces, orthogonal projections and bounded operators respectively. For any subset E of $\mathcal{H}$, $[E]$ denotes the closed subspace of $\mathcal{H}$ generated by E. The orthogonal complement of a subspace F in a subspace E will be denoted by $E \ominus F$, whereas $E^{\perp}$ denotes the orthogonal complement of E in $\mathcal{H}$ i.e. $E^{\perp} = \mathcal{H} \ominus E$. For any subspace E, we denote by P_E the projection onto E. All algebras (self-adjoint or non self-adjoint) considered are subalgebras of $\mathcal{B}(\mathcal{H})$, and are assumed to contain identity of $\mathcal{B}(\mathcal{H})$ which is denoted by 1 or $I_{\mathcal{H}}$. For other notations, we refer the readers to 'List of Symbols' at the end.

1.1 C^*-algebras

The concept of C^*-algebras plays a fundamental role in the study of noncommutative functional analysis. A thorough treatment on the theory of C^*-algebras can be found in introductory textbooks such as Arveson [3], Conway [15,16], Douglas [22], Kadison-Ringrose [45], and Takesaki [77].

A *Banach algebra* is an algebra $\mathcal{A}$ over $\mathbb{C}$ which is also a Banach space such that $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathcal{A}$.

Definition 1.1.1. A *C^*-algebra* $\mathcal{A}$ is a Banach algebra with a map $a \mapsto a^*$ from $\mathcal{A}$ to $\mathcal{A}$ (called involution) such that

(i) $(\lambda a + \gamma b)^* = \bar{\lambda} a^* + \bar{\gamma} b^*$, $(ab)^* = b^* a^*$, $(a^*)^* = a$,

(ii) $\|a^* a\| = \|a\|^2$

for all $\lambda, \gamma \in \mathbb{C}$ and $a, b \in \mathcal{A}$. A C^*-algebra is called *unital* if it contains an identity, which is denoted by 1.

Let $\mathcal{A}$ and $\mathcal{B}$ be two unital C^*-algebras. A linear map $\phi : \mathcal{A} \to \mathcal{B}$ is *unital* if $\phi(1) = 1$. A linear map $\phi : \mathcal{A} \to \mathcal{B}$ is called a $*$-*homomorphism* if $\phi(ab) = \phi(a)\phi(b)$ and $\phi(a^*) = \phi(a)^*$ for all $a, b \in \mathcal{A}$. A bijective $*$-homomorphism between two C^*-algebras is called $*$-*isomorphism*. Two C^*-algebras are *isomorphic* if there is a $*$-isomorphism between them.

Remark 1.1.2. A well-known fact says that any injective $*$-homomorphism between two C^*-algebras is isometric. In particular, there is exactly one norm on an algebra with involution which makes it into a C^*-algebra.

Example 1.1.3. Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Then $\mathcal{B}(\mathcal{H})$ is a unital C^*-algebra endowed with the operator norm. Further, any subalgebra of $\mathcal{B}(\mathcal{H})$ which preserves $*$-operation and is closed in norm topology, is a C^*-algebra equipped with the inherited operator norm.

A C^*-algebra $\mathcal{A}$ is called *commutative* or *abelian* if $ab = ba$ for all $a, b \in \mathcal{A}$. Following is a general example of commutative C^*-algebras.

Example 1.1.4. For any compact Hausdorff space X, let $C(X)$ denote the the space of all continuous functions from X to $\mathbb{C}$. Then $C(X)$ is a unital commutative C^*-algebra, where multiplication and scalar multiplication are pointwise, while involution is given by $f^*(x) = \overline{f(x)}$ for all $f \in C(X), x \in X$. The norm is the sup norm given by $\|f\| = \sup_{x\in X} |f(x)|$ for $f \in C(X)$.

The following theorem due to Gelfand says that Example 1.1.4 provides all unital commutative C^*-algebras upto isomorphism.

Theorem 1.1.5. *If $\mathcal{A}$ is a unital commutative C^*-algebra, then $\mathcal{A}$ is isomorphic to $C(X)$ for some compact Hausdorff space X.*

The topological space X in above theorem is nothing but the maximal ideal space of all non-zero complex homomorphisms on $\mathcal{A}$ equipped with weak*-topology. The space X is called the *spectrum* of the commutative algebra $\mathcal{A}$.

Definition 1.1.6. Let $\mathcal{A}$ be a unital C^*-algebra. An element $a \in \mathcal{A}$ is called

(i) *self-adjoint (or hermitian)* if $a^* = a$,
(ii) *normal* if $a^*a = aa^*$,
(iii) *projection* if $a^2 = a$ and $a^* = a$,
(iv) *isometry* if $a^*a = 1$,
(v) *co-isometry* if $aa^* = 1$,
(vi) *unitary* if $a^*a = aa^* = 1$.

Now let $\mathcal{A}$ be a unital C^*-algebra, and let $a \in \mathcal{A}$. For any scalar $\lambda \in \mathbb{C}$, we will always denote the element $\lambda \cdot 1_{\mathcal{A}}$ of $\mathcal{A}$ by λ only. The *spectrum* of a, denoted $\sigma(a)$, is defined by

$$\sigma(a) = \{\lambda \in \mathbb{C}; a - \lambda \text{ is invertible in } \mathcal{A}\}.$$

The spectrum of an element is always non-empty and compact.

Definition 1.1.7. A self-adjoint element a in a C^*-algebra $\mathcal{A}$ is called *positive* if $a = b^*b$ for some $b \in \mathcal{A}$; equivalently, $\sigma(a) \subseteq [0, \infty)$.

Proposition 1.1.8. *Let a be a positive element in a C^*-algebra, then there exists a unique positive element b (called* square root*) such that $b^2 = a$.*

For any element a in a C^*-algebra $\mathcal{A}$, we denote by $C^*(a)$ the smallest unital C^*-subalgebra of $\mathcal{A}$ generated by a and 1. Note that if a is normal, then $C^*(a)$ is commutative.

Theorem 1.1.9 (Continuous functional calculus)**.** *Let a be a normal element in a C^*-algebra. Then there is a unique $*$-isomorphism $\rho : f \mapsto f(a)$ from $C(\sigma(a))$ to $C^*(a)$ such that $\rho(z) = a$.*

Theorem 1.1.10 (Spectral Mapping Theorem)**.** *If a is a normal element in a C^*-algebra, then*

$$\sigma(f(a)) = f(\sigma(a))$$

for any continuous function $f : \sigma(a) \to \mathbb{C}$.

Also see Section 1.4 below for discussions on Borel functional calculus and the corresponding Spectral mapping theorem.

We now state the classical Gelfand-Naimark-Segal (GNS) theorem, which says that every C^*-algebra can be concretely realized as a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ as in Example 1.1.3.

Theorem 1.1.11 (GNS Theorem)**.** *Every C^*-algebra is isometrically isomorphic to a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

We now consider the notion of minimal tensor products between two C^*-algebras. In order to construct the minimal C^*-cross-norm, we recall the theory of tensor products of Hilbert spaces. Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, the assignment

$$\langle h_1 \otimes k_1, h_2 \otimes k_2 \rangle = \langle h_1, h_2 \rangle \langle k_1, k_2 \rangle$$

extends linearly to define an inner product on the algebraic tensor product $\mathcal{H} \otimes \mathcal{K}$. The completion of $\mathcal{H} \otimes \mathcal{K}$ with respect to this inner product is a Hilbert space, which we still denote by $\mathcal{H} \otimes \mathcal{K}$. If T and S are bounded operators on $\mathcal{H}$ and $\mathcal{K}$ respectively, then setting

$$T \otimes S(h \otimes k) = Th \otimes Sk$$

extends to define a bounded, linear operator on $\mathcal{H} \otimes \mathcal{K}$ satisfying $\|T \otimes S\| = \|T\|\|S\|$. It is easy to check that

$$(T_1 \otimes S_1)(T_2 \otimes S_2) = T_1T_2 \otimes S_1S_2 \quad \text{and} \quad (T \otimes S)^* = T^* \otimes S^*.$$

We refer the readers to Paulsen [61], Pisier [63] and Takesaki [77] for more details for the notions of tensor products.

Definition 1.1.12. Given two C^*-algebras $\mathcal{A}_i \subseteq \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$ we define the *minimal tensor product* of $\mathcal{A}_1$ and $\mathcal{A}_2$, denoted $\mathcal{A}_1 \otimes \mathcal{A}_2$, as the C^*-subalgebra of $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ generated by the operators $T_1 \otimes T_2$ for $T_i \in \mathcal{A}_i$, $i = 1, 2$.

It is a fact that the definition of minimal tensor product does not depend on the representing Hilbert spaces on which the C^*-algebras act.

von Neumann algebras

We now move to a special kind of C^*-algebras called von Neumann algebras. This notion was originally introduced by von Neumann and Murray motivated by their study of ergodic theory and quantum mechanics in a series of papers written under the name rings of operators in 1930s and 1940s. For basic development of the theory, one may refer to Conway [16], Dixmier [20], Kadison-Ringrose [45], and Takesaki [77].

We first define three very important topologies on $\mathcal{B}(\mathcal{H})$. The *weak operator topology (WOT)* is the smallest topology on $\mathcal{B}(\mathcal{H})$ such that the seminorms $p_{h,k} : \mathcal{B}(\mathcal{H}) \to [0, \infty)$ given by

$$p_{h,k}(T) = |\langle h, Tk \rangle|, \quad T \in \mathcal{B}(\mathcal{H})$$

is continuous for all $h, k \in \mathcal{H}$. The *strong operator topology (SOT)* is the smallest topology on $\mathcal{B}(\mathcal{H})$ such that the seminorms $p_h : \mathcal{B}(\mathcal{H}) \to [0, \infty)$ defined by

$$p_h(T) = \|Th\|, \quad T \in \mathcal{B}(\mathcal{H})$$

is continuous for all $h \in \mathcal{H}$. It is known that WOT and SOT closure of a convex subset of $\mathcal{B}(\mathcal{H})$ coincide. Further, the σ*-weak* or *ultraweak* topology is the smallest topology on $\mathcal{B}(\mathcal{H})$ such that the seminorms $p_S : \mathcal{B}(\mathcal{H}) \to [0, \infty)$ defined by

$$p_S(T) = |\operatorname{Tr}(TS)|, \quad T \in \mathcal{B}(\mathcal{H})$$

is continuous for all trace class operators S on $\mathcal{H}$. Here and elsewhere, Tr denotes the trace of a trace-class operator.

A subset $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ is called *self-adjoint* if $\mathcal{S}$ is closed under $*$-operation i.e. $\mathcal{S}^* = \mathcal{S}$, where

$$\mathcal{S}^* = \{x^*; x \in \mathcal{S}\}. \tag{1.1.1}$$

A self-adjoint subalgebra is also called $*$*-subalgebra*.

Definition 1.1.13. A *von Neumann algebra* is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed in WOT.

For any subset $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$, the *commutant* of $\mathcal{S}$ is defined by

$$\mathcal{S}' = \{T \in \mathcal{B}(\mathcal{H}); TS = ST \text{ for all } S \in \mathcal{S}\}. \tag{1.1.2}$$

The *double commutant* of $\mathcal{S}$ is defined by $\mathcal{S}'' = (\mathcal{S}')'$. The following theorem provides a relationship between algebraic and analytic structure of $*$-subalgebras of $\mathcal{B}(\mathcal{H})$. This also gives other equivalent definitions of von Neumann algebras.

Theorem 1.1.14 (Double commutant theorem)**.** *Let $\mathcal{B}$ be a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$. The following are equivalent:*

(i) $\mathcal{B}$ is a von Neumann algebra i.e. $\mathcal{B}$ is closed in WOT,

(ii) $\mathcal{B}$ is closed in SOT,

(iii) $\mathcal{B}$ is σ-weakly closed,

(iv) $\mathcal{B}'' = \mathcal{B}$.

Definition 1.1.15. A von Neumann algebra $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ is called *factor* if its center $\mathcal{B} \cap \mathcal{B}'$ consists of the scalar multiples of the identity.

For any projection $p \in \mathcal{B}$, we denote by $p\mathcal{B}p$ the algebra

$$p\mathcal{B}p = \{pxp; x \in \mathcal{B}\}. \tag{1.1.3}$$

Proposition 1.1.16. *$p\mathcal{B}p$ is a von Neumann algebra which is $*$-isomorphic to a von Neumann subalgebra of $\mathcal{B}(\mathcal{K})$, where $\mathcal{K}$ is the range subspace of p.*

A *partial isometry* is an operator W on a Hilbert space $\mathcal{H}$ such that $\|Wh\| = \|h\|$ for all $h \in (\ker W)^{\perp}$. The space $(\ker W)^{\perp}$ is called *initial space* and $\mathcal{R}(W)$ is called *final space* for W. Here $\ker W$ and $\mathcal{R}(W)$ denote respectively kernel and range of the operator W.

Theorem 1.1.17 (Polar decomposition)**.** *Let $T \in \mathcal{B}(\mathcal{H})$. Then there is a partial isometry W with initial space $(\ker T)^{\perp}$ and final space $\overline{\mathcal{R}(T)}$ such that $T = W|T|$, where $|T|$ is the square root of T^*T. Moreover, W belongs the von Neumann algebra generated by T.*

We recall some notions on projections in von Neumann algebras. Below and elsewhere $\leq$ denotes the usual order of self-adjoint operators i.e $A \leq B$ if $A - B$ is positive. And $<$ will denote the strict order.

Definition 1.1.18. Two projections p and q in a von Neumann algebra $\mathcal{B}$ are said to be *(Murray-von Neumann) equivalent*, denoted $p \sim q$, if there exists a partial isometry $v \in \mathcal{M}$ such that $v^*v = p$ and $vv^* = q$. We say $p \preceq q$ if there is a projection $q_1 \in \mathcal{B}$ such that $q_1 \leq q$ and $p \sim q_1$.

See Corollary 47.9, [16] for proof of the following comparison result for projections in a factor.

Theorem 1.1.19. *If $\mathcal{B}$ is a factor and p, q are two non-zero projections in $\mathcal{B}$, then either $p \preceq q$ or $q \preceq p$ i.e. there is a non-zero partial isometry $v \in \mathcal{B}$ such that $v^*v \leq p$ and $vv^* \leq q$.*

A projection $p \in \mathcal{M}$ is called *finite* if the only projection q in $\mathcal{M}$ such that $q \leq p$ and $q \sim p$ is p.

Definition 1.1.20. A von Neumann algebra $\mathcal{B}$ is called *finite* if the projection $1 \in \mathcal{B}$ is finite.

A bounded linear map $\phi : \mathcal{A} \to \mathcal{B}$ between two von Neumann algebras is called a *trace* if $\phi(ab) = \phi(ba)$ for all $a, b \in \mathcal{A}$. The following theorem tells of existence of tracial states on finite von Neumann algebras (see Corollary 50.13, [16]).

Theorem 1.1.21. *If $\mathcal{B}$ is a finite von Neumann algebra, then there exists a faithful tracial state $\tau : \mathcal{B} \to \mathbb{C}$ i.e. $\tau(1) = 1$, $\tau(a^*a) \geq 0$, and $\tau(a^*a) = 0$ implies $a = 0$.*

We have already seen above the notion of minimal tensor products of two C^*-algebras. Given two von Neumann algebras $\mathcal{B}_i \subseteq \mathcal{B}(\mathcal{H}_i), i = 1, 2$, we denote by $\mathcal{B}_1 \overline{\otimes} \mathcal{B}_2$ the von Neumann algebra generated by $\mathcal{B}_1 \otimes \mathcal{B}_2$ in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. The following theorem talks about commutant of such tensor products (see Theorem IV.5.9, [77] for proof).

Theorem 1.1.22. *If $\mathcal{B}_1$ and $\mathcal{B}_2$ are two von Neumann algebras, then $(\mathcal{B}_1 \otimes \mathcal{B}_2)' = \mathcal{B}_1' \overline{\otimes} B_2'$.*

1.2 Completely positive maps

The concept of completely positive (CP) maps on C^*-algebras and their dilation originates from the work of Stinespring [75]. A significant development in the theory and applications of CP maps came from Arveson in his seminal paper [5]. Over the years, it has attracted fair amount of attention in the context of quantum probability and quantum information theory. See Paulsen [61] for a detailed exposition of the theory of CP maps and related topics.

Let $\mathcal{A}$ be a unital C^*-algebra. Then by GNS theorem, $\mathcal{A}$ can be realized as a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Consider the set $M_n(\mathcal{A})$ of all $n \times n$ matrices with entries from $\mathcal{A}$. Then $M_n(\mathcal{A})$ is a unital $*$-algebra with usual matrix multiplication, involution, pointwise addition and scalar multiplication. Since $\mathcal{A}$ acts on $\mathcal{H}$, $M_n(\mathcal{A})$ acts on the Hilbert space $\mathcal{H}^{(n)}$ of n-times direct sum of $\mathcal{H}$. Hence $M_n(\mathcal{A})$ is a $*$-subalgebra of $\mathcal{B}(\mathcal{H}^{(n)})$, so that it inherits the operator norm from $B(\mathcal{H}^{(n)})$. It is straightforward to verify that with respect to this norm, $M_n(\mathcal{A})$ is closed and hence a unital C^*-algebra.

Let $\mathcal{A}$ and $\mathcal{B}$ be two unital C^*-algebras. For any linear map $\phi : \mathcal{A} \to \mathcal{B}$ and $n \geq 1$, we define the map $\phi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$, called *ampliation*, by

$$\phi_n([a_{ij}]) = [\phi(a_{ij})], \quad \text{for all} \quad [a_{ij}] \in M_n(\mathcal{A}). \tag{1.2.1}$$

Definition 1.2.1. A linear map $\phi : \mathcal{A} \to \mathcal{B}$ is called *positive* if $\phi(a) \geq 0$ in $\mathcal{B}$, for all $a \geq 0$ in $\mathcal{A}$. The map ϕ is called *completely positive (CP)* if ϕ_n is positive for all $n \geq 1$.

Remark 1.2.2. Throughout the thesis, we shall be using the abbreviated form 'CP' for completely positive, while 'UCP' will be used for unital completely positive. The readers are cautioned here that both the terminologies will keep appearing, and hence proper attention should be paid.

A positive map is automatically bounded. In fact we have the following (see Proposition 3.6, [61]).

Proposition 1.2.3. *Let ϕ be a positive map between unital C^*-algebras. Then $\|\phi\| = \|\phi(1)\|$ (moreover, if ϕ is CP, then $\sup_{n\geq 1} \|\phi_n\| = \|\phi(1)\|$). Conversely, if ϕ is a linear map such that $\phi(1) = 1$ and $\|\phi\| \leq 1$, then ϕ is positive.*

Definition 1.2.4. Let $\mathcal{A}$ be a unital C^*-algebra. A *representation* of $\mathcal{A}$ is a pair $(\mathcal{H}, \pi)$ consisting of a Hilbert space $\mathcal{H}$ and a unital $*$-homomorphism from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$.

Example 1.2.5. Any representation is a UCP map. Further, if $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is a representation and V is a bounded map from $\mathcal{H}$ to $\mathcal{K}$, then the map $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ defined by $\phi(a) = V^*\pi(a)V$ for all $a \in \mathcal{A}$, is a CP map. Moreover, ϕ is unital if and only if V is an isometry.

Conversely, Stinespring [75] showed that all CP maps are of the form as in Example 1.2.5. The proof (see Theorem 4.1, [61]) follows the usual GNS construction method.

Theorem 1.2.6 (Stinespring dilation theorem)**.** *Let $\mathcal{A}$ be a unital C^*-algebra, and let $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a CP map. Then there exists a triple $(\pi, V, \mathcal{K})$, where $\mathcal{K}$ is a Hilbert space, $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is a representation and $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that*

$$\phi(a) = V^*\pi(a)V \tag{1.2.2}$$

for all $a \in \mathcal{A}$, and satisfies the minimality condition: $\mathcal{K} = [\pi(\mathcal{A})V\mathcal{H}]$. Moreover any such triple is unique up to unitary equivalence i.e. if $(\pi_1, V_1, \mathcal{K}_1)$ is another such triple, then there is a unitary $U : \mathcal{K} \to \mathcal{K}_1$ such that $UV = V_1$ and $\pi_1(a)U = U\pi(a)$ for all $a \in \mathcal{A}$.

The triple $(\pi, V, \mathcal{H}_\pi)$ in Stinespring dilation theorem above is called the *minimal Stinespring triple* for the CP map ϕ.

Remark 1.2.7. For our purposes, all CP maps will be acting on separable Hilbert spaces. But note that the Hilbert space $\mathcal{H}_\pi$ in the minimal Stinespring triple $(\pi, V, \mathcal{H}_\pi)$ for a CP map $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ need not be separable. But one can easily verify (through the proof of above theorem) that whenever the C^*-algebra $\mathcal{A}$ is separable and $\mathcal{H}$ is separable, then the Hilbert space $\mathcal{H}_\pi$ is also separable.

In the same paper [75], Stinespring showed that positive maps on commutative C^*-algebras are automatically CP (see Theorem 4.11, [61] for proof).

Proposition 1.2.8. *Let X be a compact Hausdorff space, and let $\mathcal{A}$ be a unital C^*-algebra. Then any positive map $\phi : C(X) \to \mathcal{A}$ is CP.*

The notion of compression of operators is very common in functional analysis. The same can be defined for CP maps as follows:

Definition 1.2.9. Let $\phi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$ be two CP maps. We say ϕ_2 is a *compression* of ϕ_1 if there exists an isometry $W : \mathcal{H}_2 \to \mathcal{H}_1$ such that $\phi_2(a) = W^*\phi_1(a)W$ for all $a \in \mathcal{A}$.

Stinespring dilation theorem says that any unital completely positive (UCP) map is a compression of a representation.

The following proposition talks about elements of something called multiplicative domains of a UCP map. See Theorem 3.18, [61].

Proposition 1.2.10. *Let $(\pi, V, \mathcal{H}_\pi)$ be the minimal Stinespring triple for a UCP map $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Then for any $a \in \mathcal{A}$, $\phi(a)^*\phi(a) = \phi(a^*a)$ if and only if $V\phi(a) = \pi(a)V$.*

Proof. Since V is an isometry, we first note that

$$\begin{aligned}[(I_{\mathcal{H}_\pi} - VV^*)\pi(a)V]^*[(I_{\mathcal{H}_\pi} - VV^*)\pi(a)V] &= V^*\pi(a^*)(I_{\mathcal{H}_\pi} - VV^*)\pi(a)V \\ &= \phi(a^*a) - \phi(a)^*\phi(a).\end{aligned}$$

Hence $\phi(a^*a) = \phi(a)^*\phi(a)$ if and only if $(I_{\mathcal{H}_\pi} - VV^*)\pi(a)V = 0$ i.e. $\pi(a)V = V\phi(a)$. $\square$

As usually is the case, our study of the structure of C^*-extreme UCP maps in this thesis is upto unitarily equivalence of two UCP maps, which we formally define below:

Definition 1.2.11. Two UCP maps $\phi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$, are called *unitarily equivalent* if there is a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $\phi_1(a) = U^*\phi_2(a)U$ for all $a \in \mathcal{A}$.

Remark 1.2.12. If $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a UCP map with minimal Stinespring triple $(\pi, V, \mathcal{H}_\pi)$, then it is easy to verify that ϕ is unitarily equivalent to the UCP map $a \mapsto P_{\mathcal{R}(V)}\pi(a)_{|\mathcal{R}(V)}$ from $\mathcal{A}$ to $\mathcal{B}(\mathcal{R}(V))$.

We now consider tensor products of CP maps on minimal tensor products of C^*-algebras. See Theorem 12.3, [61] for proof of the following.

Theorem 1.2.13. *Let $\mathcal{A}_i$ be C^*-algebras and $\mathcal{H}_i$ be Hilbert spaces for $i = 1, 2$. If $\phi_i : \mathcal{A}_i \to \mathcal{B}(\mathcal{H}_i)$ are (unital) CP maps, then the assignment $a_1 \otimes a_2 \mapsto \phi_1(a_1) \otimes \phi(a_2)$ extends to a (unital) CP map from $\mathcal{A}_1 \otimes \mathcal{A}_2$ to $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.*

Disjoint representations

Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi)$ be a representation, and let $\mathcal{K}$ be a closed subspace of $\mathcal{H}_\pi$ such that $\mathcal{K}$ is invariant under $\pi(a)$ (i.e. $\pi(a)\mathcal{K} \subseteq \mathcal{K}$) for all $a \in \mathcal{A}$. Then the mapping $a \mapsto \pi(a)_{|\mathcal{K}}$ gives rise to another representation from $\mathcal{A}$ to $\mathcal{B}(\mathcal{K})$, called a *sub-representation* of π.

Definition 1.2.14. Two representations $\pi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_{\pi_i})$, $i = 1, 2$ are said to be *disjoint* if no non-zero sub-representation of π_1 is unitarily equivalent to any sub-representation of π_2.

The following proposition provides an equivalent criterion for disjoint representations (see Proposition 2.1.4, [3] for proof).

Proposition 1.2.15. *Let $\pi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_{\pi_i})$, $i = 1, 2$ be two representations. Then π_1 and π_2 are disjoint if and only if for any $S \in \mathcal{B}(\mathcal{H}_{\pi_1}, \mathcal{H}_{\pi_2})$ satisfying $S\pi_1(a) = \pi_2(a)S$ for all $a \in \mathcal{A}$, implies $S = 0$.*

Definition 1.2.16. A representation $(\pi, \mathcal{K})$ on $\mathcal{A}$ is called *irreducible* if π has no non-zero proper sub-representation i.e. $\pi(\mathcal{A})' = \mathbb{C} \cdot I_{\mathcal{H}_\pi}$.

Remark 1.2.17. Given two irreducible representations, they are either unitarily equivalent or disjoint. Also if π_1 and π_2 are two non-unitarily equivalent irreducible representations, then the representations $\pi_1(\cdot) \otimes I_{\mathcal{K}_1}$ and $\pi_2(\cdot) \otimes I_{\mathcal{K}_2}$ are disjoint (for any Hilbert spaces $\mathcal{K}_1$ and $\mathcal{K}_2$).

Radon-Nikodym type Theorem

Some very important theorems on CP maps proved by Arveson [5] are listed below. To this end, we consider some relevant terminologies.

Definition 1.2.18. For any two CP maps $\phi, \psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, we say ψ is *dominated* by ϕ, denoted $\psi \leq \phi$, if $\phi - \psi$ is CP.

Inspired from Radon-Nikodym theorem from measure theory, Arveson proved the following version of the theorem for comparison of two CP maps (Theorem 1.4.2, [5]):

Theorem 1.2.19 (Radon-Nikodym type theorem)**.** *Let $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a CP map with minimal Stinespring triple $(\pi, V, \mathcal{H}_\pi)$. Then a CP map $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ satisfies $\psi \leq \phi$ if and only if there is a positive contraction $T \in \pi(\mathcal{A})'$ such that $\psi(a) = V^*T\pi(a)V$ for all $a \in \mathcal{A}$.*

A large part of our major results revolves around the notion of pure UCP maps which we define below.

Definition 1.2.20. A CP map ϕ is called *pure* if whenever ψ is a CP map with $\psi \leq \phi$, then $\psi = \lambda\phi$ for some $\lambda \in [0, 1]$.

The following proposition (see Corollary 1.4.3, [5]) characterizes pure CP maps in terms of their Stinespring decomposition, which follows directly from Radon-Nikodym type theorem.

Proposition 1.2.21. *If ϕ is a CP map with minimal Stinespring triple $(\pi, V, \mathcal{H}_\pi)$, then ϕ is pure if and only if π is irreducible.*

Extreme point condition

The classical convexity structure of spaces of UCP maps and their subclasses has garnered considerable attention. In this thesis, we are not focusing much on extreme UCP maps. Nevertheless, we provide some results for the sake of comparison with C^*-extreme points (which is our main theme).

Recall that a subset $\mathcal{C}$ of a vector space (or an affine space) is called a *convex set* if whenever $x_i \in \mathcal{C}$ and $t_i \in [0, 1]$ with $\sum_{i=1}^n t_i = 1$, then $\sum_{i=1}^n t_i x_i \in \mathcal{C}$. A point x in a convex set $\mathcal{C}$ is called an *extreme point* if whenever

$$x = \sum_{i=1}^{n} t_i x_i$$

for $x_i \in \mathcal{C}$ and $t_i \in (0, 1]$ with $\sum_{i=1}^n t_i = 1$, then $x = x_i$ for every i.

We fix the following notation to be followed throughout the thesis. This set is our main focus for its convexity (and its quantum variant) structure.

Notation. We denote by $S_{\mathcal{H}}(\mathcal{A})$ the collection of all UCP maps from a unital C^*-algebra $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$.

The set $S_{\mathcal{H}}(\mathcal{A})$ is called *generalized state space* on the C^*-algebra $\mathcal{A}$ taking values in $\mathcal{B}(\mathcal{H})$. Note that $S_{\mathbb{C}}(\mathcal{A})$ is the usual state space of $\mathcal{A}$.

Note that $S_{\mathcal{H}}(\mathcal{A})$ is a convex set because if $\phi_i \in S_{\mathcal{H}}(\mathcal{A})$ and $t_i \in [0,1]$ with $\sum_{i=1}^n t_i = 1$, then $\sum_{i=1}^n t_i\phi_i \in S_{\mathcal{H}}(\mathcal{A})$. The following characterization of extreme points of $S_{\mathcal{H}}(\mathcal{A})$ is due to Arveson (Theorem 1.4.6, [5]).

Theorem 1.2.22 (Extreme point condition)**.** *Let $\phi \in S_{\mathcal{H}}(\mathcal{A})$, and let $(\pi, V, \mathcal{H}_\pi)$ be its minimal Stinespring triple. Then ϕ is extreme in $S_{\mathcal{H}}(\mathcal{A})$ if and only if the map $T \mapsto V^*TV$ from $\pi(\mathcal{A})'$ to $\mathcal{B}(\mathcal{H})$ is injective.*

BW (bounded weak) topology

We now describe a topology on the generalized state space $S_{\mathcal{H}}(\mathcal{A})$, whose definition is inspired from the weak*-topology on usual state spaces. We define the topology using convergence of nets in $S_{\mathcal{H}}(\mathcal{A})$. This topology first appeared in Arveson [5].

Definition 1.2.23. A net $\{\phi_i\}$ converges to ϕ in $S_{\mathcal{H}}(\mathcal{A})$ in *bounded weak (BW)* topology if

$$\phi_i(a) \to \phi(a) \quad \text{in WOT}$$

for all $a \in \mathcal{A}$.

Note that for any $\phi_0 \in S_{\mathcal{H}}(\mathcal{A})$, the sets of the form

$$\{\phi \in S_{\mathcal{H}}(\mathcal{A}); |\langle(\phi(a) - \phi_0(a_i))h_i, k_i\rangle| < \epsilon\},$$

for $h_i, k_i \in \mathcal{H}$, $a_i \in \mathcal{A}$, $1 \le i \le k$ and $\epsilon > 0$, forms a basis for BW-topology on $S_{\mathcal{H}}(\mathcal{A})$. As one would expect, the generalized state space $S_{\mathcal{H}}(\mathcal{A})$ is compact in BW-topology, just like usual state spaces are compact in weak*-topology. See Theorem 7.4, [61] for proof of the following.

Theorem 1.2.24. *Let $\mathcal{A}$ be a unital C^*-algebra, and $\mathcal{H}$ a Hilbert space. Then the set $S_{\mathcal{H}}(\mathcal{A})$ of all UCP maps on $\mathcal{A}$ is compact in BW-topology.*

Normal UCP maps

We now discuss structure of normal UCP maps on von Neumann algebras.

Definition 1.2.25. Let $\mathcal{A}, \mathcal{B}$ be two von Neumann algebras. A positive linear map $\phi : \mathcal{A} \to \mathcal{B}$ is called *normal* if whenever $\{X_i\}$ is a net of increasing self-adjoint operators (i.e. $X_i \le X_j$ for $i \le j$) converging to X in SOT, then $\phi(X_i) \to \phi(X)$ in SOT.

The following theorem describes the general structure of normal UCP maps on $\mathcal{B}(\mathcal{G})$ for some separable Hilbert space $\mathcal{G}$ via Stinespring dilation (see Theorem 1.41, [63]).

Theorem 1.2.26. *Let $\mathcal{G}$ and $\mathcal{H}$ be separable Hilbert spaces, and let $\phi : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ be a normal UCP map. Then there exist a Hilbert space $\mathcal{K}$ and an isometry $V : \mathcal{H} \to \mathcal{G} \otimes \mathcal{K}$ such that*

$$\phi(X) = V^*(X \otimes I_{\mathcal{K}})V \quad \textit{for all } X \in \mathcal{B}(\mathcal{G}),$$

and satisfies the minimality condition: $\mathcal{G} \otimes \mathcal{K} = \overline{\text{span}}\,\{(X \otimes I_{\mathcal{K}})Vh; h \in \mathcal{H}, X \in \mathcal{B}(\mathcal{G})\}$.

In above Theorem if we recognize the Hilbert space $\mathcal{G} \otimes \mathcal{K}$ with $\dim \mathcal{K}$-times direct sum of $\mathcal{G}$, then we get the following structure of normal UCP maps (see Theorem 2.3, [19]).

Corollary 1.2.27. *Let $\phi : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ be a normal UCP map. Then there exists a finite or countable sequence $\{V_n\}_{n\geq 1}$ of operators in $\mathcal{B}(\mathcal{H}, \mathcal{G})$ such that*

$$\phi(X) = \sum_{n\geq 1} V_n^* X V_n \quad \textit{in SOT}, \tag{1.2.3}$$

for all $X \in \mathcal{B}(\mathcal{G})$.

Remark 1.2.28. Note that the commutator of the set $\{X \otimes I_{\mathcal{K}}; X \in \mathcal{B}(\mathcal{G})\}$ in $\mathcal{B}(\mathcal{G} \otimes \mathcal{K})$ is the algebra $\{I_{\mathcal{G}} \otimes T; T \in \mathcal{B}(\mathcal{K})\}$. So ϕ is a normal pure UCP map on $\mathcal{B}(\mathcal{G})$ if and only if $\dim \mathcal{K} = 1$ i.e. $\phi(X) = V^* X V$ for some isometry V from $\mathcal{H}$ to $\mathcal{G}$.

1.3 Positive operator valued measures

In this section, we review the theory of positive operator valued measures (POVMs). POVMs play integral role in various areas of mathematics including quantum computing, quantum information theory and operator algebras. Some references on POVMs are Davies [19], Holevo [40], Schroeck [71], Paulsen [61] and Han-Larson-Liu-Liu [36].

Unless stated otherwise, X is a non-empty set and $\mathcal{O}(X)$ denotes a σ-algebra of subsets of X. The pair $(X, \mathcal{O}(X))$ is called a *measurable space* and the elements of $\mathcal{O}(X)$ are called *measurable subsets*. We shall simply call X a measurable space without always mentioning the underlying σ-algebra $\mathcal{O}(X)$, if there is no point of confusion.

Definition 1.3.1. Let $(X, \mathcal{O}(X))$ be a measurable space and let $\mathcal{H}$ be a Hilbert space. A *positive operator valued measure (POVM)* on X with values in $\mathcal{B}(\mathcal{H})$ is a map $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ satisfying the following:

- $\mu(A) \geq 0$ in $\mathcal{B}(\mathcal{H})$ for all $A \in \mathcal{O}(X)$, and
- for every $h, k \in \mathcal{H}$, the map $\mu_{h,k} : \mathcal{O}(X) \to \mathbb{C}$ defined by

  $$\mu_{h,k}(A) = \langle h, \mu(A)k \rangle \quad \text{for all } A \in \mathcal{O}(X), \tag{1.3.1}$$

  is a complex measure.

Moreover, a POVM μ is called

(i) *normalized* if $\mu(X) = I_{\mathcal{H}}$, the identity operator on $\mathcal{H}$.
(ii) *projection valued measure (PVM)* if $\mu(A)$ is a projection for each $A \in \mathcal{O}(X)$.
(iii) *spectral measure* if μ is a PVM and is normalized.

Notation. Let $\mathcal{P}_{\mathcal{H}}(X)$ denote the collection of all normalized POVMs from $\mathcal{O}(X)$ to $\mathcal{B}(\mathcal{H})$.

It follows from the definition of a POVM μ that, for any increasing (resp. decreasing) sequence $\{A_n\}$ of measurable subsets converging to A i.e. $A_n \subseteq A_{n+1}$ and $\cup_n A_n = A$ (resp.

$A_n \supseteq A_{n+1}$ and $\cap A_n = A$), we have $\mu(A_n) \to \mu(A)$ in weak operator topology (WOT) in $\mathcal{B}(\mathcal{H})$. Since convergence of an increasing or decreasing sequence of bounded operators is equivalent for both weak operator topology and strong operator topology (SOT), it follows that $\mu(A_n) \to \mu(A)$ in SOT. Also since on bounded subsets of $\mathcal{B}(\mathcal{H})$, WOT and σ-weak topology agree, we infer that $\mu(A_n) \to \mu(A)$ in σ-weak topology. Therefore, in the countable additivity of POVM:

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n), \quad B_n \in \mathcal{O}(X), B_n \cap B_m = \emptyset \text{ for } n \neq m,$$

the convergence of the series holds in WOT, SOT and σ-weak topologies. So for POVMs such sums can be considered in any of the three topologies.

Remark 1.3.2. For any POVM μ, by $\mu_{h,k}$ we would mean the complex measures defined in (1.3.1). It is clear that a POVM μ is determined by its associated family of complex measures $\{\mu_{h,k} : h, k \in \mathcal{H}\}$.

The following proposition gives an equivalent criteria for a POVM to be PVM. See page 34, [71] for a proof.

Proposition 1.3.3. *For a POVM μ, $\mu(A)$ is a projection for all $A \in \mathcal{O}(X)$ (i.e. μ is a PVM) if and only if $\mu(B \cap C) = \mu(B)\mu(C)$ for all $B, C \in \mathcal{O}(X)$*

Naimark's dilation theorem

The classical dilation theorem of Naimark [56] shows that POVMs can be dilated to spectral measures. This result is often considered the beginning of dilation theory. Naimark showed the result in more general set up for finitely additive POVMs on measurable spaces. Compare the following with Stinespring dilation theorem (Theorem 1.2.6).

Theorem 1.3.4 (Naimark dilation theorem)**.** *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM. Then there exists a triple $(\pi, V, \mathcal{H}_\pi)$ where $\mathcal{H}_\pi$ is a Hilbert space, $\pi : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_\pi)$ is a spectral measure and $V \in \mathcal{B}(\mathcal{H}, \mathcal{H}_\pi)$ such that*

$$\mu(A) = V^*\pi(A)V, \quad \text{for all } A \in \mathcal{O}(X) \tag{1.3.2}$$

and the minimality condition: $\mathcal{H}_\pi = [\pi(\mathcal{O}(X))V\mathcal{H}]$ is satisfied. Moreover such a dilation is unique up to unitary equivalence i.e. if $(\pi_1, V_1, \mathcal{H}_{\pi_1})$ is another such triple, then there is a unitary $U : \mathcal{H}_\pi \to \mathcal{H}_{\pi_1}$ such that $UV = V_1$ and $U\pi(A) = \pi_1(A)U$ for all $A \in \mathcal{O}(X)$.

The triple $(\pi, V, \mathcal{H}_\pi)$ is called the *minimal Naimark triple* of μ. Since π is spectral, we note that V is an isometry if and only if μ is a normalized POVM.

Naimark's theorem is text book material. The proof generally uses the usual GNS construction method. Some possible references are (Theorem II.11.F, [71]) and (Theorem 2.1.1, [40]). A proof using the Stinespring dilation theorem for CP maps is also well-known (Theorem 4.6, [61]), but then POVMs under consideration will have to be assumed to be regular on the Borel σ-algebra of some compact Hausdorff space. As an immediate application of Naimark's dilation theorem we have the following result.

Proposition 1.3.5. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a normalized POVM and $\mu(E)$ a projection for some $E \in \mathcal{O}(X)$. Then $\mu(E \cap A) = \mu(E)\mu(A) = \mu(A)\mu(E)$ for every $A \in \mathcal{O}(X)$. In particular, $\mu(E)$ and $\mu(F)$ have orthogonal ranges for any $F \in \mathcal{O}(X)$ with $E \cap F = \emptyset$.*

Proof. Let $(\pi, V, \mathcal{H}_\pi)$ be the minimal Naimark triple of μ. As noticed earlier, since μ is normalized and π is spectral, it follows that V is an isometry. Then for any $A \in \mathcal{O}(X)$, we have

$$\begin{aligned}[V\mu(A) - \pi(A)V]^* \cdot [V\mu(A) - \pi(A)V] &= [\mu(A)V^* - V^*\pi(A)] \cdot [V\mu(A) - \pi(A)V] \\ &= \mu(A)^2 - \mu(A)^2 - \mu(A)^2 + \mu(A) \\ &= \mu(A)^2 - \mu(A).\end{aligned}$$

In particular, since $\mu(E)$ is a projection, we get $V\mu(E) = \pi(E)V$. For any $A \in \mathcal{O}(X)$, therefore

$$\mu(A)\mu(E) = V^*\pi(A)V\mu(E) = V^*\pi(A)\pi(E)V = V^*\pi(A \cap E)V = \mu(A \cap E).$$

Similarly or by taking adjoint of the last equation we get $\mu(E)\mu(A) = \mu(E \cap A)$. □

Definition 1.3.6. A POVM μ is *concentrated* on a measurable subset E if $\mu(A) = \mu(A \cap E)$ for all $A \in \mathcal{O}(X)$.

Note that a POVM μ being concentrated on a subset E just means that $\mu(X \setminus E) = 0$. This is not same as saying that E is the support of μ. In fact when X is a topological space, the *support* of μ is defined as the smallest closed subset C such that $\mu(C) = \mu(X)$.

Proposition 1.3.7. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM with the minimal Naimark triple $(\pi, V, \mathcal{H}_\pi)$. Then for any $A \in \mathcal{O}(X)$, $\mu(A) = 0$ if and only if $\pi(A) = 0$. In particular, μ is concentrated on $E \in \mathcal{O}(X)$ if and only if π is concentrated on E.*

Proof. Let $\mu(A) = 0$. Then for any $B \in \mathcal{O}(X)$ and $h \in \mathcal{H}$, we get

$$\langle \pi(A)\pi(B)Vh, \pi(B)Vh \rangle = \langle V^*\pi(B \cap A)Vh, h \rangle = \langle \mu(B \cap A)h, h \rangle \leq \langle \mu(A)h, h \rangle = 0.$$

Since $\{\pi(B)Vh; h \in \mathcal{H}, B \in \mathcal{O}(X)\}$ is total in $\mathcal{H}_\pi$ by the minimality condition, we conclude that $\pi(A) = 0$. The converse is obvious. The second assertion follows from the first. □

Radon-Nikodym type theorem

In classical measure theory, the Radon-Nikodym derivative of a (σ-finite) positive measure absolutely continuous with respect to another (σ-finite) positive measure is a well-established fact. There have been several attempts to generalize it to the case of absolutely continuous POVMs (which is defined in a similar way as usual positive measures), especially for finite dimensional Hilbert spaces; see for example [24], [54]. In this thesis however, we consider a different notion of comparison of POVMs.

Definition 1.3.8. We say a POVM ν is *dominated* by another POVM μ, denoted by $\nu \leq \mu$, if $\mu - \nu$ is a POVM.

Here also a Radon-Nikodym type of theorem is known and is well studied. It is analogous to the Radon-Nikodym type theorem for CP maps by Arveson (see Theorem 1.2.19). First consider the following lemma:

Lemma 1.3.9. *If $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ is a POVM, then $[\mu(A_i \cap A_j)]$ is a positive matrix in $M_n(\mathcal{B}(\mathcal{H}))$ for any finite collection $A_1, \ldots, A_n \in \mathcal{O}(X)$.*

Proof. Let $(\pi, V, \mathcal{H}_\pi)$ be the minimal Naimark triple for μ. Then

$$[\mu(A_i \cap A_j)] = [V^*\pi(A_i \cap A_j)V] = [V^*\pi(A_i)\pi(A_j)V],$$

which is of the form $[{T_i}^* T_j]$ for $T_i = \pi(A_i)V$, and hence it is clearly positive. □

For readers convenience we present an outline of the proof as we couldn't trace a proper citation. Here the operator D can be thought of as the Radon-Nikodym derivative of ν with respect to μ.

Theorem 1.3.10 (Radon-Nikodym type theorem)**.** *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM with the minimal Naimark triple $(\pi, V, \mathcal{H}_\pi)$. Then for a POVM $\nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$, $\nu \leq \mu$ if and only if there exists a positive contraction $D \in \pi(\mathcal{O}(X))'$ such that $\nu(A) = V^* D\pi(A)V$ for all $A \in \mathcal{O}(X)$.*

Proof. The proof of 'if' part is obvious. For the converse, assume that $\mu - \nu$ is a POVM. Let $(\rho, W, \mathcal{H}_\rho)$ be the minimal Naimark triple for ν. Define an operator $T : \mathcal{H}_\pi \to \mathcal{H}_\rho$ as follows: first define T on the subspace $\operatorname{span}\{\pi(A)Vh; A \in \mathcal{O}(X), h \in \mathcal{H}\}$ of $\mathcal{H}_\pi$ by

$$T\left(\pi(A)Vh\right) = \rho(A)Wh, \quad \text{for all} \quad A \in \mathcal{O}(X), h \in \mathcal{H}$$

and extend it linearly. Then since $\mu - \nu$ is a POVM, we note from Lemma 1.3.9 that

$$\begin{aligned}
\|\sum_{i=1}^n T\left(\pi(A_i)Vh_i\right)\|^2 &= \sum_{i,j=1}^n \langle \rho(A_j)Wh_j, \rho(A_i)Wh_i\rangle = \sum_{i,j=1}^n \langle W^*\rho(A_i \cap A_j)Wh_j, h_i\rangle \\
&= \sum_{i,j=1}^n \langle \nu(A_i \cap A_j)h_j, h_i\rangle \leq \sum_{i,j=1}^n \langle \mu(A_i \cap A_j)h_j, h_i\rangle = \|\sum_{i=1}^n \pi(A_i)Vh_i\|^2
\end{aligned}$$

for any $A_i \in \mathcal{O}(X)$, $h_i \in \mathcal{H}$, $1 \leq i \leq n$. This shows that T is a well-defined operator which extends to a contraction on $[\pi(\mathcal{O}(X))V\mathcal{H}] = \mathcal{H}_\pi$. Set

$$D = T^*T.$$

Then D is a positive contraction. Since for all $A \in \mathcal{O}(X)$, we have $T\pi(A) = \rho(A)T$, it is immediate that $D\pi(A) = \pi(A)D$; hence $D \in \pi(\mathcal{O}(X))'$. Also it is easy to verify that $\nu(A) = V^*D\pi(A)V$ for all $A \in \mathcal{O}(X)$. □

Extreme POVMs

The set $\mathcal{P}_{\mathcal{H}}(X)$, which is the collection of all normalized POVMs on X with values in $\mathcal{B}(\mathcal{H})$ is clearly a convex set. Extreme points of this set are well studied especially when X is a finite set or a compact Hausdorff space, and $\mathcal{H}$ is a finite dimensional Hilbert space (see [12,24,37,60]). The following abstract characterization of extreme points of $\mathcal{P}_{\mathcal{H}}(X)$ is again inspired by Arveson's corresponding result (see Theorem 1.2.22) on UCP maps. This must have been noted by several researchers for the case of POVMs and so we just outline the proof.

Theorem 1.3.11 (Extreme point condition)**.** *Suppose that $\mu \in \mathcal{P}_{\mathcal{H}}(X)$ has the minimal Naimark triple $(\pi, V, \mathcal{H}_\pi)$. Then a necessary and sufficient criterion for μ to be extreme in $\mathcal{P}_{\mathcal{H}}(X)$ is that the map $D \mapsto V^*DV$ from $\pi(\mathcal{O}(X))'$ to $\mathcal{B}(\mathcal{H})$ is injective.*

Proof. First assume that μ is extreme in $\mathcal{P}_{\mathcal{H}}(X)$. Let $V^*DV = 0$ for some $D \in \pi\left(\mathcal{O}(X)\right)'$. Without loss of generality, we can assume that $-I_{\mathcal{H}_\pi} \leq D \leq I_{\mathcal{H}_\pi}$. Write $\mu = (\mu^+ + \mu^-)/2$ where

$$\mu^{\pm}(\cdot) = V^*(I_{\mathcal{H}_\pi} \pm D)\pi(\cdot)V.$$

Then as μ is extreme in $\mathcal{P}_{\mathcal{H}}(X)$, we must have $\mu = \mu^+$. Hence $V^*D\pi(\cdot)V = 0$, which implies $D = 0$.

For the converse, assume the injectivity of the map $D \mapsto V^*DV$, and let $\mu = (\mu_1 + \mu_2)/2$ for $\mu_1, \mu_2 \in \mathcal{P}_{\mathcal{H}}(X)$. By Radon-Nikodym type theorem (Theorem 1.3.10), there are positive contractions $D_i \in \pi\left(\mathcal{O}(X)\right)'$, $i = 1, 2$ such that

$$\mu_i(\cdot)/2 = V^*D_i\pi(\cdot)V.$$

But as μ_i is normalized, we have $V^*(2D_i - I_{\mathcal{H}_\pi})V = 0$ and hence the hypothesis implies $2D_i = I_{\mathcal{H}_\pi}$. Thus we get $\mu_i(\cdot) = V^*\pi(\cdot)V = \mu(\cdot)$ for $i = 1, 2$, which proves that μ is extreme in $\mathcal{P}_{\mathcal{H}}(X)$. ☐

The following is an immediate corollary of this theorem. It can also be seen directly, as projections are extremal in the set of positive contractions.

Corollary 1.3.12. *Every spectral measure is extreme in $\mathcal{P}_{\mathcal{H}}(X)$.*

Atomic and non-atomic POVMs

One of the approaches that we take in this thesis for exploring C^*-extreme POVMs is via the decomposition of POVMs into atomic and non-atomic POVMs and analyzing them separately. So we recall here the definitions and give some of their properties. These notions have been widely studied in classical measure theory. See [43] for a very general exposition.

Definition 1.3.13. Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM. A subset $A \in \mathcal{O}(X)$ is called an *atom* for μ if $\mu(A) \neq 0$ and whenever $B \subseteq A$ in $\mathcal{O}(X)$,

$$\text{either} \quad \mu(B) = 0 \;\text{ or }\; \mu(B) = \mu(A).$$

A POVM μ is called *atomic* if every $A \in \mathcal{O}(X)$ with $\mu(A) \neq 0$ contains an atom. A POVM μ is called *non-atomic* if it has no atom.

Remark 1.3.14. If A is an atom for a POVM μ then it is easy to verify that for any $B \in \mathcal{O}(X)$ with $B \subseteq A$, either $\mu(B) = 0$ or $A \cap B$ is an atom for μ.

The following proposition gives an equivalent condition for a POVM to be atomic.

Proposition 1.3.15. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM. Then μ is atomic if and only if there is a countable collection $\{B_n\}_{n\geq 1}$ of mutually disjoint atoms such that $\mu(A) = \sum_{n\geq 1} \mu(A \cap B_n)$ for all $A \in \mathcal{O}(X)$.*

Proof. Firstly assume that $\mu(A) = \sum_{n\geq 1} \mu(A \cap B_n), A \in \mathcal{O}(X)$, for a collection of mutually disjoint atoms $\{B_n\}_{n\geq 1}$. Let $A \in \mathcal{O}(X)$ be such that $\mu(A) \neq 0$. Then $\mu(A \cap B_n) \neq 0$ for some n. But then it follows from Remark 1.3.14 that $A \cap B_n$ is an atom, which is contained in A. Since A is arbitrary, this shows that μ is atomic.

Conversely let μ be atomic, and let $\{B_i\}_{i\in\Lambda}$ be a maximal family of mutually disjoint atoms of μ, which exists due to Zorn's Lemma. Clearly Λ is non-empty, as μ is atomic. Also we have $\mu(B_i) \neq 0$ for all $i \in \Lambda$, so it follows from Proposition 1.3.19 (see below) that Λ is countable. Now for any $A \in \mathcal{O}(X)$, we note that

$$\mu(A \setminus (\cup_{i\in\Lambda}(A \cap B_i))) = 0,$$

otherwise there is an atom, say $A_1 \subseteq A \setminus (\cup_{i\in\Lambda}(A \cap B_i))$ for μ, but then $\{B_i\}_{i\in\Lambda} \cup \{A_1\}$ is a family of mutually disjoint atoms, violating the maximality of the collection $\{B_i\}_{i\in\Lambda}$. This shows that

$$\mu(A) = \mu(\cup_{i\in\Lambda}(A \cap B_i)) = \sum_{i\in\Lambda} \mu(A \cap B_i)$$

for all $A \in \mathcal{O}(X)$. □

It is a well-known fact that every finite (more generally σ-finite) positive measure decomposes uniquely as a sum of an atomic positive measure and a non-atomic positive measure. We have a similar decomposition for POVMs as well. Even though the proof in [54] (which itself is inspired from the classical case) is for POVMs on locally compact Hausdorff spaces, the same proof will work for general measurable spaces (see the proof of Theorem 4.4.10 below). We state it here.

Theorem 1.3.16 (Theorem 3.10, [54]). *Every POVM decomposes uniquely as a sum of an atomic POVM and a non-atomic POVM.*

We make an useful observation on atoms of POVMs which shall be frequently used.

Proposition 1.3.17. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM with the minimal Naimark triple $(\pi, V, \mathcal{H}_\pi)$. Then a subset $A \in \mathcal{O}(X)$ is an atom for μ if and only if A is an atom for π. In particular, μ is atomic (non-atomic) if and only if π is atomic (non-atomic).*

Proof. For any subset $A \in \mathcal{O}(X)$, A is an atom for μ if and only if $\mu(A) \neq 0$ and for each $A' \subseteq A$ in $\mathcal{O}(X)$, we have either $\mu(A') = 0$ or $\mu(A \setminus A') = 0$. Equivalently $\pi(A) \neq 0$ and we have either $\pi(A') = 0$ or $\pi(A \setminus A') = 0$ from Proposition 1.3.7, which in turn is same as saying that A is an atom for π. The second assertion easily follows from the first. □

Positive measures induced from POVMs

Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM, and let $S \in \mathcal{B}(\mathcal{H})$ be a strictly positive density operator i.e. S is a positive trace class operator such that for any positive operator T, we have $\mathrm{Tr}(ST) = 0$ if and only if $T = 0$ (where Tr denotes the *trace* of an operator). Such S can always be found; for example choose a countable orthonormal basis $\{e_n\}_{n\geq 1}$ of $\mathcal{H}$ (as $\mathcal{H}$ is separable), and define $S \in \mathcal{B}(\mathcal{H})$ by

$$Sh = \sum_{n\geq 1} \frac{1}{2^n}\langle e_n, h\rangle e_n, \quad \text{for all } h \in \mathcal{H}.$$

Now consider the positive measure $\mu_S : \mathcal{O}(X) \to [0, \infty)$ defined by

$$\mu_S(A) = \mathrm{Tr}(\mu(A)S), \quad \text{for all} \ \ A \in \mathcal{O}(X). \tag{1.3.3}$$

The way S has been chosen, we note that for any $A \in \mathcal{O}(X)$, $\mu_S(A) = 0$ if and only if $\mu(A) = 0$. The following proposition then follows easily from this observation, which compares the properties of μ and μ_S.

Proposition 1.3.18. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM, and let $S \in \mathcal{B}(\mathcal{H})$ be a strictly positive density operator. Then,*

(i) for any $A \in \mathcal{O}(X)$, μ is concentrated on A if and only if μ_S is concentrated on A.

(ii) atoms of μ and μ_S are same.

(iii) μ is atomic (resp. non-atomic) if and only if μ_S is atomic (resp. non-atomic).

We make a useful observation regarding measures of mutually disjoint subsets, which would be crucial.

Proposition 1.3.19 (Lemma 3.1, [21])**.** *Let $(X, \mathcal{O}(X))$ be a measurable space and $\mathcal{H}$ a separable Hilbert space. If $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ is a POVM and $\{B_i\}_{i\in\Lambda}$ is a collection of mutually disjoint measurable subsets such that $\mu(B_i) \neq 0$ for each $i \in \Lambda$, then Λ is countable.*

Proof. Consider the positive measure $\mu_S : \mathcal{O}(X) \to [0, \infty)$ as in (1.3.3). Since $\mu_S(B_i) \neq 0$ for all $i \in \Lambda$ by Proposition 1.3.18, and since $\sum_{i\in\Lambda} \mu_S(B_i) \leq \mu_S(\cup_{i\in\Lambda} B_i) < \infty$, we conclude that Λ is countable. □

Regular POVMs on Topological Spaces

We now discuss POVMs on topological spaces. Let X be a Hausdorff topological space, and let $\mathcal{O}(X)$ denote the Borel σ-algebra on X. In this case, an additional property of a POVM that can be studied on X is that of regularity. Recall that a positive measure λ is *regular* if it is inner regular (or tight) with respect to compact subsets and outer regular with respect to open subsets:

$$\begin{aligned} \lambda(A) &= \sup\{\lambda(E) : E \text{ compact with } E \subseteq A\} \\ &= \inf\{\lambda(G) : G \text{ open with } A \subseteq G\}, \end{aligned}$$

for every $A \in \mathcal{O}(X)$.

Definition 1.3.20. A POVM $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ on a topological space X is said to be *regular* if $\mu_{h,h}$ as defined in equation (1.3.1), is a regular positive measure for each $h \in \mathcal{H}$.

Remark 1.3.21. The issue of regularity does not arise for complete separable metric spaces (Theorem 3.2, [57]), as all Borel measures are automatically regular.

The following results discuss the preservation of regularity under various operations.

Proposition 1.3.22. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM with the minimal Naimark triple $(\pi, V, \mathcal{H}_\pi)$. Then μ is regular if and only if π is regular.*

Proof. If π is regular then, since $\mu_{h,h} = \pi_{Vh,Vh}$ for each $h \in \mathcal{H}$, it is clear that μ is regular. For the converse, assume that μ is regular. First note that, if $k = \pi(B)Vh$ for some $B \in \mathcal{O}(X), h \in \mathcal{H}$, then for any $A \in \mathcal{O}(X)$, we have

$$\pi_{k,k}(A) = \langle \pi(B)Vh, \pi(A)\pi(B)Vh\rangle = \langle h, V^*\pi(A)\pi(B)Vh\rangle = \mu_{h,h}(A \cap B).$$

Since $A \mapsto \mu_{h,h}(A \cap B)$ is regular, it follows that $\pi_{\pi(B)Vh,\pi(B)Vh}$ is regular. Consequently, $\pi_{k,k}$ is regular for all $k \in \operatorname{span}\{\pi(A)Vh : A \in \mathcal{O}(X), h \in \mathcal{H}\}$. Now fix $\epsilon > 0$ and $B \in \mathcal{O}(X)$. Then for general $k \in \mathcal{H}_\pi$, let $\{k_0\}$ be in $\operatorname{span}\{\pi(A)Vh : A \in \mathcal{O}(X), h \in \mathcal{H}\}$ such that

$$\|k - k_0\| < \sqrt{\epsilon}/2.$$

Since π_{k_0,k_0} is regular as shown above, there is a compact subset C and an open subset O with $C \subseteq B \subseteq O$ such that

$$\langle k_0, \pi(O \setminus C)k_0\rangle < \epsilon/4.$$

Thus

$$\begin{aligned}\langle k, \pi(O \setminus C)k\rangle = \|\pi(O \setminus C)^{1/2}k\|^2 &\leq 2\|\pi(O \setminus C)^{1/2}k_0\|^2 + 2\|\pi(O \setminus C)^{1/2}(k_0 - k)\|^2 \\ &\leq 2\langle k_0, \pi(O \setminus C)k_0\rangle + 2\|k_0 - k\|^2 < 2\,(\epsilon/4 + \epsilon/4) = \epsilon.\end{aligned}$$

Since ϵ and B are arbitrary, we conclude that $\pi_{k,k}$ is regular. □

Proposition 1.3.23. *If $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ is a regular POVM, then*

(i) $T^*\mu(\cdot)T$ is regular for any $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

(ii) ν is regular for any POVM $\nu \leq \mu$.

Proof. (i) Let $\nu = T^*\mu(\cdot)T$. Then for any $k \in \mathcal{K}$, we note that $\nu_{k,k} = \mu_{Tk,Tk}$, which is clearly regular.

(ii) Since $\nu \leq \mu$, we have $\nu_{h,h} \leq \mu_{h,h}$ for $h \in \mathcal{H}$. As $\mu_{h,h}$ is regular, it follows that $\nu_{h,h}$ is regular for all $h \in \mathcal{H}$; hence ν is a regular POVM. □

1.4 Correspondence between CP maps and POVMs

Let X be a compact Hausdorff space and $\mathcal{H}$ a Hilbert space. In this case, $\mathcal{O}(X)$ will denote the Borel σ-algebra of X. Let $C(X)$ be the commutative C^*-algebra of continuous functions on X. We review the well-known correspondence between regular $\mathcal{B}(\mathcal{H})$-valued POVMs on $\mathcal{O}(X)$ and $\mathcal{B}(\mathcal{H})$-valued CP maps on $C(X)$. See Paulsen [61] and Hadwin [34] for discussions on this topic. This correspondence in a way generalizes the classical Riesz-Markov theorem (Theorem III.5.7, [15]), which we state below.

Theorem 1.4.1 (Riesz-Markov representation theorem). *Let X be a compact Hausdorff space. Then a map $\phi : C(X) \to \mathbb{C}$ is linear and bounded if and only if there exists a unique regular Borel complex measure μ_ϕ on X such that $\phi(f) = \int_X f d\mu_\phi$ for all $f \in C(X)$, and $\|\phi\| = \|\mu_\phi\|$. Moreover, ϕ is a positive functional if and only if μ_ϕ is a positive measure.*

We now describe the detailed procedure of the correspondence between CP maps on $C(X)$ and regular POVMs on X (see Definition 1.3.20). Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a regular POVM. For any $f \in C(X)$, consider the map $B_f : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ defined by

$$B_f(h, k) = \int_X f d\mu_{h,k} \text{ for all } h, k \in \mathcal{H},$$

where $\mu_{h,k}$ denotes the complex measures as in (1.3.1). It is straightforward to check that B_f is a sesquilinear form satisfying $\|B_f\| \leq \|f\|\|\mu(X)\|$; hence by Riesz Theorem (Theorem II.2.2, [15]), we obtain a unique bounded operator, call it $\phi_\mu(f) \in \mathcal{B}(\mathcal{H})$, satisfying $B_f(h, k) = \langle h, \phi_\mu(f)k\rangle$. Note that $\phi_\mu(f) \geq 0$ in $\mathcal{B}(\mathcal{H})$, whenever $f \geq 0$ in $C(X)$. Hence, the induced map $\phi_\mu : C(X) \to \mathcal{B}(\mathcal{H})$ defines a CP map via the assignment

$$\langle h, \phi_\mu(f)k\rangle = \int_X f d\mu_{h,k}, \quad \text{for all } f \in C(X) \text{ and } h, k \in \mathcal{H}. \tag{1.4.1}$$

Conversely, let $\phi : C(X) \to \mathcal{B}(\mathcal{H})$ be a CP map. For each $h, k \in \mathcal{H}$, consider the bounded linear functional on $C(X) := f \mapsto \langle h, \phi(f)k\rangle$. Then by Theorem 1.4.1, we obtain a unique regular Borel measure $\nu_{h,k}$ satisfying $\|\nu_{h,k}\| \leq \|\phi\|\|h\|\|k\|$ and

$$\langle h, \phi(f)k\rangle = \int_X f d\nu_{h,k} \text{ for all } f \in C(X).$$

Now for each bounded Borel measurable function g, consider the map: $(h, k) \mapsto \int_X g d\nu_{h,k}$ from $\mathcal{H} \times \mathcal{H}$ to $\mathbb{C}$, which is sesquilinear as above and bounded by $\|\phi\|\|g\|$. Hence again by Riesz Theorem, we obtain a unique bounded operator $\tilde{\phi}(g) \in \mathcal{B}(\mathcal{H})$ satisfying

$$\langle h, \tilde{\phi}(g)k\rangle = \int_X g d\nu_{h,k} \text{ for all } h, k \in \mathcal{H}. \tag{1.4.2}$$

Note that $\tilde{\phi}(g) \geq 0$ in $\mathcal{B}(\mathcal{H})$ whenever $g \geq 0$ in $B(X)$. Here $B(X)$ denotes the C^*-algebra of all bounded Borel measurable functions on X. In particular for $A \in \mathcal{O}(X)$, if we set

$$\mu_\phi(A) = \tilde{\phi}(\chi_A), \tag{1.4.3}$$

where $\chi_A \in B(X)$ is the characteristic function of the subset A, then $\mu_\phi(A)$ is a positive operator in $\mathcal{B}(\mathcal{H})$ and satisfies

$$\nu_{h,k}(A) = \langle h, \mu_\phi(A)k\rangle, \quad \text{for all } h, k \in \mathcal{H}.$$

Note that for any countable collection $\{A_n\}_{n\geq 1}$ of disjoint measurable subsets, we have

$$\langle h, \mu_\phi(\cup_{n\geq 1}A_n)k\rangle = \langle h, \tilde{\phi}(\chi_{\cup_{n\geq 1}A_n})k\rangle = \nu_{h,k}(\cup_{n\geq 1}A_n) = \sum_{n\geq 1}\nu_{h,k}(A_n) = \sum_{n\geq 1}\langle h, \mu_\phi(A_n)k\rangle$$

for all $h, k \in \mathcal{H}$. Because $\nu_{h,h}$ is a regular Borel positive measure for $h \in \mathcal{H}$, it is immediate that μ_ϕ defines a POVM which satisfies the equality $\mu_\phi(X) = \phi(1)$, where 1 denotes the constant function 1 on X.

Notation. For a POVM $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$, and any bounded Borel function f on X, we denote by $\int_X f d\mu$ the operator on $\mathcal{B}(\mathcal{H})$ satisfying

$$\langle h, \left(\int_X f d\mu\right) k\rangle = \int_X f d\mu_{h,k}, \tag{1.4.4}$$

for all $h, k \in \mathcal{H}$.

The following theorem summarises some basic properties of this correspondence (see Proposition 4.5, [61]).

Theorem 1.4.2. *Let X be a compact Hausdorff space, and let $\mathcal{H}$ be a Hilbert space. Then the correspondence described above between POVMs on X and CP maps on $C(X)$ taking values in $\mathcal{B}(\mathcal{H})$, satisfies the following:*

(i) $\phi_{\mu_\phi} = \phi$ *and* $\mu_{\phi_\mu} = \mu$.

(ii) $\phi(1) = \mu_\phi(X)$.

(iii) *μ is a projection valued measure (resp. spectral measure) if and only if ϕ_μ is a $*$-homomorphism (resp. representation).*

(iv) $\phi_{\mu_1+\mu_2} = \phi_{\mu_1} + \phi_{\mu_2}$ *and* $\mu_{\phi_1+\phi_2} = \mu_{\phi_1} + \mu_{\phi_2}$.

(v) *$\phi_{T^*\mu(\cdot)T} = T^*\phi_\mu(\cdot)T$ and $\mu_{T^*\phi(\cdot)T} = T^*\mu_\phi(\cdot)T$ for any $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, where $\mathcal{K}$ is a Hilbert space.*

Proof. (i) This is just uniqueness of the correspondence.

(ii) This follows from the discussion above.

(iii) Assume that ϕ_μ is a $*$-homomorphism. Then for all $f, g \in C(X)$ and $h, k \in \mathcal{H}$, we have

$$\int_X fgd\mu_{h,k} = \langle h, \phi_\mu(fg)k\rangle = \langle h, \phi_\mu(f)\phi_\mu(g)k\rangle = \int_X fd\mu_{h,\phi_\mu(g)k}.$$

Since $f \in C(X)$ is arbitrary, it follows from uniqueness of regular Borel measures in Riesz-Markov theorem that $gd\mu_{h,k} = d\mu_{h,\phi_\mu(g)k}$ as complex measures. Equivalently for any $A \in \mathcal{O}(X)$, we have

$$\int_A gd\mu_{h,k} = \mu_{h,\phi_\mu(g)k}(A),$$

that is
$$\int_X g\chi_A d\mu_{h,k} = \langle h, \mu(A)\phi_\mu(g)k\rangle = \langle \mu(A)h, \phi_\mu(g)k\rangle = \int_X g d\mu_{\mu(A)h,k}.$$
Again, since $g \in C(X)$ is arbitrary, we conclude that $\chi_A d\mu_{h,k} = d\mu_{\mu(A)h,k}$ as complex measures. Equivalently for any $B \in \mathcal{O}(X)$, we get

$$\int_X \chi_{A\cap B} d\mu_{h,k} = \int_X \chi_A\chi_B d\mu_{h,k} = \mu_{\mu(A)h,k}(B) = \langle \mu(A)h, \mu(B)k\rangle,$$

which further implies
$$\langle h, \mu(A\cap B)k\rangle = \langle h, \mu(A)\mu(B)k\rangle.$$
Since $h, k \in \mathcal{H}$ are arbitrary, we conclude that

$$\mu(A \cap B) = \mu(A)\mu(B) \text{ for all } A, B \in \mathcal{O}(X),$$

that is, μ is a projection valued measure. The converse of the statement follows just by reversing of the argument above.

(iv) This directly follows from the assignment in (1.4.1).

(v) Let $T \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, and set $\nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{K})$ defined by $\nu(A) = T^*\mu(A)T$ for all $A \in \mathcal{O}(X)$. Clearly ν is a POVM. Now for any $h, k \in \mathcal{K}$ and $B \in \mathcal{O}(X)$, we have

$$\langle h, \nu(B)k\rangle = \langle h, T^*\mu(B)Tk\rangle = \langle Th, \mu(B)Tk\rangle,$$

which equivalently says $\nu_{h,k} = \mu_{Th,Tk}$. Therefore for any $f \in C(X)$, we get

$$\langle h, \phi_\nu(f)k\rangle = \int_X f d\nu_{h,k} = \int_X f d\mu_{Th,Tk} = \langle Th, \phi_\mu(f)Tk\rangle = \langle h, T^*\phi_\mu(f)Tk\rangle,$$

which proves that $\phi_\nu(\cdot) = T^*\phi_\mu(\cdot)T$. The other equality follows similarly. □

Remark 1.4.3. Note that for a compact Hausdorff space X, if μ is a regular POVM with a Naimark dilation $(\pi, V, \mathcal{H}_\pi)$ then $(\phi_\pi, V, \mathcal{H}_\pi)$ is a Stinespring dilation for the corresponding CP map ϕ_μ (follows directly from part (v) of Theorem 1.4.2). Further, minimality conditions match:

$$[\pi(\mathcal{O}(X))V\mathcal{H}] = [\phi_\pi(C(X))V\mathcal{H}] \tag{1.4.5}$$

and therefore, the Stinespring dilation $\phi_\mu = V^*\phi_\pi(\cdot)V$ is minimal if and only if the Naimark dilation $\mu = V^*\pi(\cdot)V$ is minimal.

Here we have some additional technical properties of this correspondence which are quite useful for us.

Proposition 1.4.4. *Let X be a compact Hausdorff space and $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ a regular POVM. Then $\mu(\mathcal{O}(X))' = \phi_\mu(C(X))'$. Moreover, $\mu(A) \in$ WOT-$\overline{\phi_\mu(C(X))}$ and $\phi_\mu(f) \in$ WOT-$\overline{\operatorname{span}}\,\mu(\mathcal{O}(X))$ for all $A \in \mathcal{O}(X)$ and $f \in C(X)$ and in particular, WOT-$\overline{\phi_\mu(C(X))}$=WOT-$\overline{\operatorname{span}}\mu(\mathcal{O}(X))$.*

Proof. First assume $T \in \mu(\mathcal{O}(X))'$. Then $\mu(A)T = T\mu(A)$ for all $A \in \mathcal{O}(X)$ and hence

$$\langle T^*h, \mu(A)k\rangle = \langle h, T\mu(A)k\rangle = \langle h, \mu(A)Tk\rangle,$$

for all $h, k \in \mathcal{H}$, which is equivalent to $\mu_{T^*h,k} = \mu_{h,Tk}$ as complex measures. Therefore for all $f \in C(X)$, it follows that

$$\langle T^*h, \phi_\mu(f)k\rangle = \int_X f d\mu_{T^*h,k} = \int_X f d\mu_{h,Tk} = \langle h, \phi_\mu(f)Tk\rangle.$$

Since $h, k \in \mathcal{H}$ are arbitrary, we conclude that

$$T\phi_\mu(f) = \phi_\mu(f)T \ \text{ for all } f \in C(X),$$

which implies $T \in \phi_\mu(C(X))'$. Thus we have proved the inclusion $\mu(\mathcal{O}(X))' \subseteq \phi_\mu(C(X))'$. The other way of the inclusion is similarly proved just by reversing the implications above.

Now let $(\pi, V, \mathcal{H}_\pi)$ be the minimal Naimark triple for μ. To show that $\mu(A) \in$ WOT-$\overline{\phi_\mu(C(X))}$ for $A \in \mathcal{O}(X)$, firstly note that

$$\pi(\mathcal{O}(X))'' = \phi_\pi(C(X))'',$$

the double commutant of the respective sets in $\mathcal{B}(\mathcal{H}_\pi)$, which follows from first part of the proof. Therefore, since $\pi(A) \in \pi(\mathcal{O}(X))$ and $\pi(\mathcal{O}(X)) \subseteq \pi(\mathcal{O}(X))'' = \phi_\pi(C(X))''$, it follows from Double commutant theorem (Theorem 1.1.14) for the $*$-algebra $\phi_\pi(C(X))$, that there is a net $\{f_i\}$ in $C(X)$ such that

$$\phi_\pi(f_i) \to \pi(A) \quad \text{in WOT.}$$

This implies

$$\phi_\mu(f_i) = V^*\phi_\pi(f_i)V \to V^*\pi(A)V = \mu(A) \quad \text{in WOT}$$

and so we conclude that $\mu(A) \in$ WOT-$\overline{\phi_\mu(C(X))}$. Other assertions follow similarly. $\square$

Spectral Theory

Finally in this section, we recall the Spectral theorem and Borel functional calculus of normal operators on Hilbert spaces.

Theorem 1.4.5 (Spectral theorem)**.** *Let N be a normal operator on a Hilbert space $\mathcal{H}$. Then there is a spectral measure μ on the Borel σ-algebra $\mathcal{O}(\sigma(N))$ of the spectrum of N such that*

$$N = \int_{\sigma(N)} z d\mu.$$

For any normal operator N on $\mathcal{H}$ with spectral measure μ, and any bounded Borel function f on $\sigma(N)$, we define the operator $f(N)$ by

$$f(N) = \int_{\sigma(N)} f d\mu. \tag{1.4.6}$$

For any operator T, let $W^*(T)$ denote the smallest von Neumann algebra generated by T. Note that $W^*(N)$ is a commutative von Neumann algebra if N is a normal operator.

Theorem 1.4.6 (Borel functional calculus)**.** *If N is a normal operator with spectral measure μ, and $B(\sigma(N))$ is the C^*-algebra of bounded Borel functions on $\sigma(N)$, then the map $\rho : f \mapsto f(N)$ from $B(\sigma(N))$ to $W^*(N)$ is a representation, extending the continuous functional calculus from $C(\sigma(N))$.*

Theorem 1.4.7 (Spectral mapping theorem)**.** *Let N be a normal operator with spectral measure μ. Then for any bounded Borel function f on $\sigma(N)$, the spectrum of $f(N)$ is the essential range of f with respect to μ i.e.*

$$\sigma(f(N)) = \{\lambda \in \mathbb{C}; \mu(\{\gamma \in \sigma(N); |f(\gamma) - \lambda| < \epsilon\}) \neq 0 \quad \textit{for all } \epsilon > 0\}.$$

1.5 Nest algebras and factorization property

We digress from earlier discussion on $*$-algebras, and review the theory of certain non self-adjoint subalgebras of $\mathcal{B}(\mathcal{H})$. We are mainly concerned about factorization property of non self-adjoint algebras, particularly nest algebras. The study of non self-adjoint algebras goes back to Gohberg-Krein [32], Kadison-Singer [46], Ringrose [69] and Arveson [4]. A thorough treatment to the theory of nest algebras are given in the beautiful book by Davidson [17].

All algebras considered here will be norm closed subalgebras of $\mathcal{B}(\mathcal{H})$ containing the identity. For any subalgebra $\mathcal{M}$ in a C^*-algebra $\mathcal{A}$, denote by $\mathcal{M}^{-1}$ the set

$$\mathcal{M}^{-1} = \{A \in \mathcal{A}; A \text{ is invertible with } A, A^{-1} \in \mathcal{M}\}.$$

Definition 1.5.1. Let $\mathcal{M}$ be a subalgebra of a C^*-algebra $\mathcal{A}$. Then $\mathcal{M}$ *has factorization* in $\mathcal{A}$ if for any positive and invertible element D in $\mathcal{A}$, there is an invertible element S such that $S \in \mathcal{M}^{-1}$ and $D = S^*S$.

We shall consider this notion in a more general form in Chapter 6, where one will find a number of interesting examples of such algebras. We now consider some equivalent properties of algebras having factorization in von Neumann algebras. Compare the following result with Proposition 6.1.3. Also see Lemma 1.2 in Larson [47].

Proposition 1.5.2. *Let $\mathcal{M}$ be a closed subalgebra of a von Neumann algebra $\mathcal{B}$. The following are equivalent:*

(i) *$\mathcal{M}$ has factorization in $\mathcal{B}$.*

(ii) *For every invertible $T \in \mathcal{B}$, there exists a unitary $U \in \mathcal{B}$ such that $UT \in \mathcal{M}^{-1}$.*

(iii) *For every invertible $T \in \mathcal{B}$, there exists a unitary $U \in \mathcal{B}$ such that $TU \in \mathcal{M}^{-1}$.*

(iv) *$\mathcal{M}^*$ has factorization in $\mathcal{B}$ i.e. for each positive and invertible operator $D \in \mathcal{B}$, there exists an invertible operator S such that $S \in \mathcal{M}^{-1}$ and $D = SS^*$.*

(v) *For every positive and invertible operator $D \in \mathcal{B}$, there exist invertible operators S and T such that $S, T \in \mathcal{M}^{-1}$ and $D = ST^*$.*

Proof. (i) $\Longrightarrow$ (ii). Let T be an invertible operator in $\mathcal{B}$. Then T^*T is positive and invertible; hence there exists an operator $A \in \mathcal{M}^{-1}$ such that $T^*T = A^*A$. By polar decomposition, there is a unitary $U \in \mathcal{B}$ such that $A = UT$. In particular, $UT \in \mathcal{M}^{-1}$.

(ii) $\Longrightarrow$ (iii) If T is an invertible operator, then there is a unitary $V \in \mathcal{B}$ such that $VT^{-1} \in \mathcal{M}^{-1}$. If $U = V^*$, then $TU = (VT^{-1})^{-1} \in \mathcal{M}^{-1}$.

(iii) $\Longrightarrow$ (iv). Let D be a positive and invertible operator. Then there is a unitary U such that $D^{1/2}U \in \mathcal{M}^{-1}$. If we set $S = D^{1/2}U$, then we have $S \in \mathcal{M}^{-1}$ and $D = SS^*$.

(iv) $\Longrightarrow$ (v). This is obvious.

(v) $\Longrightarrow$ (i). Let D be a positive and invertible operator. Then D^{-1} is positive and invertible, hence we get $S, T \in \mathcal{M}^{-1}$ such that $D^{-1} = ST^*$. Since D is self adjoint, we have $ST^* = TS^*$, that is, $T^*S^{*-1} = S^{-1}T$. Set $A = S^{-1}T$. Clearly then $A \in \mathcal{M}^{-1}$ and A is self adjoint. Also $SA = T$ which implies that $SAS^* = ST^* = D^{-1}$. Thus we get $A = S^{-1}D^{-1}(S^{-1})^*$, which says that A is positive; hence we have $A \in \mathcal{M} \cap \mathcal{M}^*$. But $\mathcal{M} \cap \mathcal{M}^*$ is a C^*-algebra (as $\mathcal{M}$ is closed), which implies that $A^{1/2}$ and $A^{-1/2} \in \mathcal{M} \cap \mathcal{M}^* \subseteq \mathcal{M}$. Now if we set $C = A^{-1/2}S^{-1}$, then $C \in \mathcal{M}^{-1}$ and we get $C^*C = S^{*-1}A^{-1}S^{-1} = D$. $\square$

We are mostly concerned with factorization property of nest algebras. To this end, we review the theory of nests of subspaces and associated nest algebras. See [17, 47, 48, 65, 69] for many interesting results.

Definition 1.5.3. A *nest* $\mathcal{E}$ is a family of closed subspaces of a Hilbert space $\mathcal{H}$, which is totally ordered with respect to inclusion i.e. $E \subseteq F$ or $F \subseteq E$ for any $E, F \in \mathcal{E}$. A nest $\mathcal{E}$ is called *complete* if $0, \mathcal{H} \in \mathcal{E}$ and

$$\bigvee_{E \in \mathcal{E}_0} E \quad \text{and} \quad \bigwedge_{E \in \mathcal{E}_0} E \in \mathcal{E}_0$$

for any subfamily $\mathcal{E}_0$ of $\mathcal{E}$.

If $\mathcal{E}$ is a nest in $\mathcal{H}$, then there is a maximal nest containing $\mathcal{E}$. Note that a maximal nest must be complete, so there exists at least one complete nest containing $\mathcal{E}$.

Definition 1.5.4. The smallest complete nest containing a nest $\mathcal{E}$, denoted $\bar{\mathcal{E}}$, is called the *completion* of $\mathcal{E}$.

Lemma 1.5.5 (Lemma 2.2, [69])**.** *Let $\mathcal{E}$ be a nest in $\mathcal{H}$. Then the members of the completion $\bar{\mathcal{E}}$ are $\{0\}, \mathcal{H}$ and all subspaces of the form*

$$\bigwedge_{E \in \mathcal{E}_0} E, \quad \bigvee_{E \in \mathcal{E}_0} E$$

for some arbitrary subfamily $\mathcal{E}_0$ of $\mathcal{E}$.

Definition 1.5.6. A *nest algebra* associated with a nest $\mathcal{E}$ on $\mathcal{H}$, denoted $\operatorname{Alg}\mathcal{E}$, is the subalgebra of $\mathcal{B}(\mathcal{H})$ of all operators which leave subspaces of $\mathcal{E}$ invariant i.e.

$$\operatorname{Alg}\mathcal{E} = \{T \in \mathcal{B}(\mathcal{H}); T(E) \subseteq E \ \text{ for all } E \in \mathcal{E}\}.$$

Remark 1.5.7. For any nest $\mathcal{E}$, we note that $\operatorname{Alg}\mathcal{E} = \operatorname{Alg}\overline{\mathcal{E}}$. Also a nest algebra is always unital and WOT closed.

Example 1.5.8. Let $\mathcal{H}$ be a Hilbert space, and let $\{E_n\}_{n\geq 1}$ be an increasing sequence of finite dimensional subspaces whose union is dense in $\mathcal{H}$. Then $\mathcal{E} = \{\{0\}, \mathcal{H}, E_n; n \geq 1\}$ is a complete nest.

If $\{e_n\}_{n\geq 1}$ is an orthonormal basis for $\mathcal{H}$, and $E_n = \overline{\text{span}}\{e_m; m \leq n\}$, then we note that $\operatorname{Alg}\mathcal{E}$ is the nest algebra of upper triangular matrices in $\mathcal{B}(\mathcal{H})$ with respect to the basis $\{e_n\}_{n\geq 1}$.

Example 1.5.9. Let $\mathcal{H} = L^2([0,1])$ with Lebesgue measure. For each $t \in [0,1]$, let $E_t = L^2([0,t])$ considered as a subspace of $\mathcal{H}$. Then $\{E_t; t \in [0,1]\}$ is a complete nest in $\mathcal{H}$.

Example 1.5.10. Let $\{e_q\}_{q\in\mathbb{Q}}$ be an orthonormal basis for a Hilbert space, where $\mathbb{Q}$ is the set of rational numbers. For each $q \in \mathbb{Q}$, set

$$E_q = \overline{\text{span}}\{e_p; p \leq q\}.$$

Then $\{E_q\}_{q\in\mathbb{Q}}$ is a countable nest, which is *not* complete. In fact, its completion is given by the uncountable nest $\{\{0\}, \mathcal{H}, E_q, F_r; q \in \mathbb{Q}, r \in \mathbb{R}\}$, where $F_r = \overline{\text{span}}\{e_p; p < r\}$ for $r \in \mathbb{R}$.

Notation. For any nest $\mathcal{E}$, we denote by $\mathcal{E}^\perp$ the nest $\{E^\perp; E \in \mathcal{E}\}$.

It is easy to verify that $\operatorname{Alg}\mathcal{E}^\perp = (\operatorname{Alg}\mathcal{E})^*$. The following is immediate from Proposition 1.5.2.

Proposition 1.5.11 (Lemma 1.1, [47])**.** *Let $\mathcal{E}$ be a nest of subspaces of $\mathcal{H}$, and let T be an invertible operator in $\mathcal{B}(\mathcal{H})$. The following are equivalent:*

(i) *$T^*T = A^*A$ for some $A \in (\operatorname{Alg}\mathcal{E})^{-1}$.*
(ii) *There exists a unitary $U \in \mathcal{B}(\mathcal{H})$ such that $T(E) = U(E)$ for all $E \in \mathcal{E}$.*
(iii) *There exists a unitary $U \in \mathcal{B}(\mathcal{H})$ such that $UT \in (\operatorname{Alg}\mathcal{E})^{-1}$.*

Proposition 1.5.12. *If $\mathcal{E}$ is a nest in $\mathcal{H}$, then $\operatorname{Alg}\mathcal{E}$ has factorization in $\mathcal{B}(\mathcal{H})$ if and only if $\operatorname{Alg}\mathcal{E}^\perp$ has factorization in $\mathcal{B}(\mathcal{H})$.*

The following theorem is a deep result due to Larson [47] which gives a complete characterization of those nest algebras which have factorization in $\mathcal{B}(\mathcal{H})$. We have cleverly used the countability criteria of this result in the proof of Theorem 2.3.10, which is one of the main results of this thesis.

Theorem 1.5.13 (Theorem 4.6, [47])**.** *Let $\mathcal{E}$ be a nest in a separable Hilbert space $\mathcal{H}$. Then $\operatorname{Alg}\mathcal{E}$ has factorization in $\mathcal{B}(\mathcal{H})$ if and only if the completion $\overline{\mathcal{E}}$ of the nest $\mathcal{E}$ is countable.*

Let $\mathcal{E}$ be a complete nest on a separable Hilbert space $\mathcal{H}$. For any $E \in \mathcal{E}$, define

$$E_- = \vee\{F \in \mathcal{E};\ F \subsetneq E\} \quad \text{and} \quad E_+ = \wedge\{F \in \mathcal{E};\ E \subsetneq F\}.$$

Definition 1.5.14. Let $\mathcal{E}$ be a complete nest on a Hilbert space $\mathcal{H}$. An *atom* of $\mathcal{E}$ is a subspace of the form $E \ominus E_-$, for some $E \in \mathcal{E}$ with $E \neq E_-$. The nest $\mathcal{E}$ of $\mathcal{H}$ is called *atomic* if there is a countable collection of atoms $\{\mathcal{H}_n\}_{n\geq 1}$ of $\mathcal{E}$ such that $\mathcal{H} = \oplus_{n\geq 1}\mathcal{H}_n$.

Nest algebras belong to an important class of algebras called reflexive algebras. Let $\mathcal{M}$ be a subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Then its *lattice* $\operatorname{Lat}\mathcal{M}$ is defined by

$$\operatorname{Lat}\mathcal{M} = \{E \subseteq \mathcal{H}; E \text{ is a subspace such that } T(E) \subseteq E \text{ for all } T \in \mathcal{M}\}.$$

Dually, for any collection $\mathcal{E}$ of subspaces of $\mathcal{H}$, consider the unital closed algebra $\operatorname{Alg}\mathcal{E}$ defined by

$$\operatorname{Alg}\mathcal{E} = \{T \in \mathcal{B}(\mathcal{K});\ T(E) \subseteq E \text{ for all } E \in \mathcal{E}\}.$$

It is clear that $\mathcal{M} \subseteq \operatorname{Alg}\operatorname{Lat}\mathcal{M}$. This inclusion may be strict (see Radjavi-Rosenthal [68] for more details).

Definition 1.5.15. A subalgebra $\mathcal{M}$ is called *reflexive* if $\mathcal{M} = \operatorname{Alg}\operatorname{Lat}\mathcal{M}$.

Example 1.5.16. If $\mathcal{E}$ is a collection of subspaces in a Hilbert space, then $\operatorname{Alg}\mathcal{E}$ is a reflexive algebra. This easily follows from the fact that $\operatorname{Lat}\operatorname{Alg}\mathcal{E}$ is the smallest complete lattice, say $\mathcal{F}$, containing $\mathcal{E}$. But then $\operatorname{Alg}\mathcal{F} = \operatorname{Alg}\mathcal{E}$. In particular, any nest algebra is reflexive.

Remark 1.5.17. We shall revisit most of the terminologies of this section in Chapter 6, albeit in the language of projections rather than subspaces.

Chapter 2

C^*-convexity Structure of Generalized State Spaces

The set $S_{\mathcal{H}}(\mathcal{A})$ of all unital completely positive (UCP) maps from a unital C^*-algebra $\mathcal{A}$ to the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ is called *generalized state space*, as UCP maps taking values in $\mathcal{B}(\mathcal{H})$ with $\mathcal{H}$ one dimensional are just usual states. The study of convexity structure and extremal points of the set $S_{\mathcal{H}}(\mathcal{A})$ and its subclasses is classical, which began with the seminal paper by Arveson [5] and subsequently several studies followed in [7, 13, 59, 60, 76]. The natural operator version of convexity of the set $S_{\mathcal{H}}(\mathcal{A})$ has equally attracted fair amount of attention as well (see [18, 23, 28, 29, 31, 33, 49, 51, 80]), with particular emphasis coming from C^*-convexity theory.

Let us first consider an abstract context of occurrence of C^*-convexity. Let Y be a non-empty set, and let $\mathcal{H}$ be a Hilbert space. Consider the vector space $V_{\mathcal{H}}(Y)$ of all functions from Y to $\mathcal{B}(\mathcal{H})$ (with pointwise addition and scalar multiplication). The space $V_{\mathcal{H}}(Y)$ has a natural $\mathcal{B}(\mathcal{H})$-bimodule structure via the action $T_1\phi(\cdot)T_2 : y \mapsto T_1\phi(y)T_2$, for any $\phi \in V_{\mathcal{H}}(Y)$ and $T_1, T_2 \in \mathcal{B}(\mathcal{H})$. A subset $\mathcal{C}$ of $V_{\mathcal{H}}(Y)$ is called a *C^*-convex set* if

$$\sum_{i=1}^{n} {T_i}^* \phi_i(\cdot) T_i \in \mathcal{C}$$

for any $\phi_i \in \mathcal{C}$ and $T_i \in \mathcal{B}(\mathcal{H})$ with $\sum_{i=1}^{n} {T_i}^* T_i = I_{\mathcal{H}}$. We then define an appropriate notion of extreme point for a given C^*-convex set, which we call *C^*-extreme point* (see Definition 2.1.2). Our main interest in this thesis lies in the following two settings: (1) Y is a unital C^*-algebra and $\mathcal{C}$ is the set of all UCP maps, (2) Y is a σ-algebra of subsets of a set and $\mathcal{C}$ is the set of normalized positive operator valued measures. The theme of this chapter and Chapter 3 follows the first setting, while that of Chapter 4 and Chapter 5 follows the second. We shall see a strong connection between the two scenarios in Chapter 5.

We now return to the case of the generalized state space $S_{\mathcal{H}}(\mathcal{A})$ and its C^*-convexity structure. Taking cue from the ideas of Loebl-Paulsen [49], the notion of C^*-extreme points of the space $S_{\mathcal{H}}(\mathcal{A})$ was defined and studied by Farenick and Morenz [28]. We give a brief history of some of the well-known results on C^*-extreme points that exist in literature. Most of the research so far have focused on the case when $\mathcal{H}$ is a finite dimensional Hilbert space i.e. the case when $\mathcal{H} = \mathbb{C}^n, n \in \mathbb{N}$. In [28], one can see some general properties and a complete description of C^*-extreme points of $S_{\mathbb{C}^n}(\mathcal{A})$, $n \in \mathbb{N}$, whenever $\mathcal{A}$ is a commutative C^*-algebra or a finite-dimensional matrix algebra. All C^*-extreme points of $S_{\mathbb{C}^n}(\mathcal{A})$ (for $\mathcal{A}$ an arbitrary C^*-algebra) are shown to be extreme in the usual sense. A Krein-Milman type theorem for C^*-convexity of the space $S_{\mathbb{C}^n}(\mathcal{A})$ equipped with bounded weak-topology is also established.

Following this, Farenick and Zhou [29] came up with an abstract characterization of C^*-extreme points via Stinespring decomposition, while assessing the structure of C^*-extreme points of $S_{\mathbb{C}^n}(\mathcal{A})$ for an arbitrary C^*-algebra $\mathcal{A}$. In particular, it is shown that all such maps are direct sums of pure UCP maps satisfying some 'nested' properties. Further, Gregg [33] studied sufficient conditions for UCP maps on a commutative C^*-algebra $C(X)$ to be C^*-extreme in $S_{\mathcal{H}}(C(X))$ for $\mathcal{H}$ arbitrary dimensional, where the techniques of positive operator valued measures on X are exploited. In Farenick et al. [24], one can also see the study of C^*-extreme points of positive operator valued measures and its application to UCP maps.

In this chapter, we present a systematic study of the structure of C^*-extreme points of $S_{\mathcal{H}}(\mathcal{A})$ for arbitrary C^*-algebra $\mathcal{A}$ and infinite dimensional separable Hilbert space $\mathcal{H}$. Firstly, we begin with definitions and describe some abstract characterizations of C^*-extreme points. A connection between C^*-extreme maps and factorization property of an associated algebra is established (Corollary 2.2.9). We then discuss direct sums of pure UCP maps and their C^*-extremity conditions. The main result (Theorem 2.3.10) determines conditions for such maps to be C^*-extreme, which inevitably involves the notions of nests of subspaces. The theory of factorization property of associated nest algebras are very crucial to the study of such maps. One of the main applications of our results on direct sums of pure UCP maps can be seen in the proof of Krein-Milman type theorem for C^*-convexity of $S_{\mathcal{H}}(\mathcal{A})$ equipped with BW-topology, whenever $\mathcal{A}$ is a separable C^*-algebra (Theorem 2.4.3). Finally we see a number of examples of C^*-extreme maps and their applications.

2.1 Definitions and general properties

Throughout this thesis, $\mathcal{A}$ denotes a unital C^*-algebra and $\mathcal{H}$ a complex and separable Hilbert space. The *generalized state space* $S_{\mathcal{H}}(\mathcal{A})$ denotes the collection of all unital completely positive (UCP) maps from $\mathcal{A}$ to the algebra $\mathcal{B}(\mathcal{H})$. We begin by formally defining the C^*-convexity notions of the set $S_{\mathcal{H}}(\mathcal{A})$.

Definition 2.1.1. For $\phi_i \in S_{\mathcal{H}}(\mathcal{A})$ and $T_i \in \mathcal{B}(\mathcal{H})$ for $1 \leq i \leq n$ with $\sum_{i=1}^n {T_i}^* T_i = I_{\mathcal{H}}$, a sum of the form

$$\phi(\cdot) := \sum_{i=1}^{n} {T_i}^* \phi_i(\cdot) T_i$$

is called C^**-convex combination* for ϕ. The operators T_i's are called C^**-coefficients* . When T_i's are invertible, the sum is called *proper* C^**-convex combination* for ϕ.

Observe that the space $S_{\mathcal{H}}(\mathcal{A})$ is a C^**-convex set* in the sense that it is closed under C^*-convex combinations. Following Farenick-Morenz [28], we consider the following definition:

Definition 2.1.2. A UCP map $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is called C^**-extreme point* of $S_{\mathcal{H}}(\mathcal{A})$ if whenever

$$\phi(\cdot) := \sum_{i=1}^{n} T_i^* \phi_i(\cdot) T_i,$$

is a proper C^*-convex combination of ϕ, then ϕ_i is unitarily equivalent to ϕ for each i i.e. there is a unitary $U_i \in \mathcal{B}(\mathcal{H})$ such that $\phi_i(\cdot) = U_i^* \phi(\cdot) U_i$.

It is clear that every map unitarily equivalent to a C^*-extreme point is also C^*-extreme. The aim of this thesis is to understand the behaviour of C^*-extreme points of $S_{\mathcal{H}}(\mathcal{A})$, upto unitary equivalence.

Remark 2.1.3. We will also use the term 'C^*-extreme maps' for C^*-extreme points of the generalized state space $S_{\mathcal{H}}(\mathcal{A})$.

Below we list some of known examples of C^*-extreme points of $S_{\mathcal{H}}(\mathcal{A})$.

Theorem 2.1.4 (Proposition 1.2, [28])**.** *Let $\mathcal{A}$ be a unital C^*-algebra, and $\mathcal{H}$ a Hilbert space. Then a UCP map $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is C^*-extreme (as well as extreme) in $S_{\mathcal{H}}(\mathcal{A})$ in the following cases:*

(i) ϕ is a $*$-homomorphism.

(ii) ϕ is the inflation of a pure state. i.e. $\phi(a) = \psi(a) I_{\mathcal{H}}$, $a \in \mathcal{A}$, for some pure state $\psi : \mathcal{A} \to \mathbb{C}$.

(iii) ϕ is pure.

(iv) $V\mathcal{H}$ is invariant under $\pi(\mathcal{A})'$, where $(\pi, V, \mathcal{H}_\pi)$ is the minimal Stinespring triple for ϕ.

Moreover, it follows from condition (ii) and the existence of pure states on $\mathcal{A}$ that $S_{\mathcal{H}}(\mathcal{A})$ always has C^*-extreme points.

Example 2.1.5 (Example 2, [28])**.** Let $\mathbb{T}$ be the unit circle in the complex plane, and let $L^2(\mathbb{T})$ be the Hilbert space of square integrable functions on $\mathbb{T}$ with one-dimensional Lebesgue measure. Let $H^2 = H^2(\mathbb{T})$ denote the Hardy space on $\mathbb{T}$. i.e.

$$H^2 = \{f \in L^2(\mathbb{T});\ \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} d\theta = 0 \text{ for all } n < 0\},$$

Let $C(\mathbb{T})$ be the space of continuous functions on $\mathbb{T}$. Define the map $\phi : C(\mathbb{T}) \to \mathcal{B}(H^2)$ by

$$\phi(f) = P_{H^2} M_{f|_{H^2}}, \quad \text{for all } f \in C(\mathbb{T}). \tag{2.1.1}$$

Here M_f is the multiplication operator on $L^2(\mathbb{T})$ given by $M_f(g) = fg$ for all $g \in L^2(\mathbb{T})$. Then ϕ is a C^*-extreme point in $S_{H^2}(C(\mathbb{T}))$ (also see Corollary 2.5.7 below).

Below and elsewhere, M_n denotes the algebra of all $n \times n$ complex matrices.

Example 2.1.6 (Example 1, [28]). Define $\phi : M_2 \oplus \mathbb{C} \to M_2$ by

$$\phi\left(\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \oplus x_{33}\right) = \frac{1}{2}\begin{bmatrix} x_{11}+x_{33} & x_{11}-x_{33} \\ x_{11}-x_{33} & x_{11}+x_{33} \end{bmatrix}.$$

Then ϕ is a C^*-extreme point in $S_{\mathbb{C}^2}(M_2 \oplus \mathbb{C})$.

Example 2.1.7 (Example 4, [28]). Let $K(\mathcal{H})$ denote the space of all compact operators on a Hilbert spaces $\mathcal{H}$, and let ξ be a unit vector. Consider the UCP map $\phi : K(\mathcal{H}) + \mathbb{C}I_{\mathcal{H}} \to M_2$ defined by

$$\phi(X) = \begin{bmatrix} \alpha & 0 \\ 0 & \langle \xi, X\xi\rangle \end{bmatrix}$$

for $X = T + \alpha I_{\mathcal{H}} \in K(\mathcal{H}) + \mathbb{C}I_{\mathcal{H}}$. Then ϕ is C^*-extreme in $S_{\mathbb{C}^2}(K(\mathcal{H}) + \mathbb{C}I_{\mathcal{H}})$.

Example 2.1.8 (Theorem 3.3, [28]). Let $\xi \in \mathbb{C}^n$ be a unit vector. Then the UCP map $\phi : M_n \to M_n \oplus \mathbb{C} \subseteq M_{n+1}$ defined by

$$\phi(X) = X \oplus \langle \xi, X\xi\rangle, \quad X \in M_n$$

is a C^*-extreme point in $S_{\mathbb{C}^{n+1}}(M_n)$.

Example 2.1.9 (Example 1, [28]). The map $\phi : M_2 \to M_2$ defined by

$$\phi(X) = \begin{bmatrix} x_{11} & 0 \\ 0 & x_{22} \end{bmatrix}, \quad \text{for} \quad X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \in M_2$$

is an example of a UCP map, which is *not* a C^*-extreme point in $S_{\mathbb{C}^2}(M_2)$.

We shall provide many more examples of C^*-extreme maps in Section 2.5 and Section 3.3.

2.2 Abstract characterizations of C^*-extreme maps

A key ingredient in our approach is a result by Farenick-Zhou [29], who taking cue from Arveson's Extreme point condition provided an abstract characterization of C^*-extreme points of $S_{\mathcal{H}}(\mathcal{A})$ via their minimal Stinespring decomposition. However their proof seems to have an incomplete argument. Therefore we restate their result with minor modifications in our notation and give an outline of the proof. Here $\mathcal{R}(T)$ denotes the range of an operator T.

Theorem 2.2.1 (Theorem 3.1, [29]). *Let $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a UCP map with the minimal Stinespring triple $(\pi, V, \mathcal{H}_\pi)$. Then ϕ is C^*-extreme in $S_{\mathcal{H}}(\mathcal{A})$ if and only if for any positive operator $D \in \pi(\mathcal{A})'$ with V^*DV invertible, there exist a partial isometry $U \in \pi(\mathcal{A})'$ with $\mathcal{R}(U^*) = \mathcal{R}(U^*U) = \overline{\mathcal{R}(D^{1/2})}$ and an invertible $Z \in \mathcal{B}(\mathcal{H})$ such that $UD^{1/2}V = VZ$.*

Proof. $\Longrightarrow$ Let ϕ be C^*-extreme in $S_{\mathcal{H}}(\mathcal{A})$, and let $D \in \pi(\mathcal{A})'$ be positive with V^*DV invertible. Choose $\alpha > 0$ small enough so that $\|\alpha D\| < 1$. Then $\|\alpha V^*DV\| < 1$, which ensures that $I_{\mathcal{H}} - \alpha V^*DV$ is positive and invertible. Set

$$T_1 = (\alpha V^*DV)^{\frac{1}{2}} \text{ and } T_2 = (I_{\mathcal{H}} - \alpha V^*DV)^{1/2}.$$

Then both T_1 and T_2 are invertible such that $T_1^*T_1 + T_2^*T_2 = V^*V = I_{\mathcal{H}}$. Now we define $\phi_1, \phi_2 : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by

$$\phi_1(a) = T_1^{-1}(\alpha V^* D\pi(a)V)T_1^{-1} \quad \text{and} \quad \phi_2(a) = T_2^{-1}V^*(I_{\mathcal{H}_\pi} - \alpha D)\pi(a)VT_2^{-1}$$

for all $a \in \mathcal{A}$. Clearly, ϕ_1 and ϕ_2 are UCP maps such that

$$\phi(a) = T_1^*\phi_1(a)T_1 + T_2^*\phi_2(a)T_2, \quad \text{for all } a \in \mathcal{A}.$$

Since ϕ is a C^*-extreme point in $S_{\mathcal{H}}(\mathcal{A})$, there exists a unitary $W \in \mathcal{B}(\mathcal{H})$ such that for all $a \in \mathcal{A}$, we have $\phi(a) = W^*\phi_1(a)W$, that is,

$$\phi(a) = W^*T_1^{-1}(\alpha V^*D\pi(a)V)T_1^{-1}W = (\sqrt{\alpha}D^{1/2}VT_1^{-1}W)^*\pi(a)(\sqrt{\alpha}D^{1/2}VT_1^{-1}W) = X^*\pi(a)X,$$

where $X = \sqrt{\alpha}D^{1/2}VT_1^{-1}W$. Since T, W are onto and $[\pi(\mathcal{A})V\mathcal{H}] = \mathcal{H}_\pi$, we note that

$$[\pi(\mathcal{A})X(\mathcal{H})] = [\pi(\mathcal{A})D^{1/2}VT_1^{-1}W(\mathcal{H})] = [D^{1/2}\pi(\mathcal{A})V\mathcal{H}] = \overline{\mathcal{R}(D^{1/2})}.$$

Thus if we set $\mathcal{K} = \overline{\mathcal{R}(D^{1/2})}$, then $\mathcal{K}$ is an invariant subspace for $\pi(\mathcal{A})$. And if we think X as an operator from $\mathcal{H}$ to $\mathcal{K}$, then we note that $(\pi(\cdot)_{|\mathcal{K}}, X, \mathcal{K})$ is another minimal Stinespring triple for ϕ; hence by uniqueness of minimal Stinespring triple, there exists a unitary operator $\widetilde{U} : \mathcal{K} \to \mathcal{H}_\pi$ such that

$$\widetilde{U}X = V, \qquad \text{and} \qquad \pi(a)\widetilde{U} = \widetilde{U}\pi(a)_{|\mathcal{K}} \quad \text{for all } a \in \mathcal{A}.$$

Extend $\widetilde{U}$ to $\mathcal{H}_\pi$ by assigning 0 on the complement of $\mathcal{K}$, and call this map U. Then U is a partial isometry (in fact, a co-isometry) with $\mathcal{R}(U^*) = \mathcal{K}$. Also, we note that $UX = V$ and $\pi(a)U = U\pi(a)$ for all $a \in \mathcal{A}$, so $U \in \pi(\mathcal{A})'$. Further, we set $Z = \frac{1}{\sqrt{\alpha}}W^*T_1 \in \mathcal{B}(\mathcal{H})$, then Z is invertible and satisfies

$$VZ = UXZ = U\sqrt{\alpha}D^{1/2}VT_1^{-1}W\sqrt{\alpha}^{-1}W^*T_1 = UD^{1/2}V.$$

$\Longleftarrow$ Assume the 'only if' condition, and let $\phi(\cdot) = \sum_{i=1}^n {T_i}^*\phi_i(\cdot)T_i$ be a proper C^*-convex combination of ϕ. Then for each i, we have ${T_i}^*\phi_i(\cdot)T_i \leq \phi(\cdot)$, so by Radon-Nikodym type theorem (Theorem 1.2.19) there exists $D_i \in \pi(\mathcal{A})'$ with $0 \leq D_i \leq I_{\mathcal{H}_\pi}$ such that

$${T_i}^*\phi_i(a)T_i = V^*D_i\pi(a)V, \quad \text{for all } a \in \mathcal{A}.$$

Note that $V^*D_iV = {T_i}^*T_i$, so V^*D_iV is invertible; hence by our hypothesis, there exist a partial isometry $U_i \in \pi(\mathcal{A})'$ with $\mathcal{R}({U_i}^*U_i) = \overline{\mathcal{R}({D_i}^{1/2})}$ and an invertible $Z_i \in \mathcal{B}(\mathcal{H})$ such that $U_i{D_i}^{1/2}V = VZ_i$. Note that ${U_i}^*U_i{D_i}^{1/2} = {D_i}^{1/2}$; hence for all $a \in \mathcal{A}$, we have

$$\begin{aligned}
{T_i}^*\phi_i(a)T_i &= V^*D_i\pi(a)V = V^*{D_i}^{1/2}\pi(a){D_i}^{1/2}V = V^*{D_i}^{1/2}\pi(a){U_i}^*U_i{D_i}^{1/2}V \\
&= V^*{D_i}^{1/2}{U_i}^*\pi(a)U_i{D_i}^{1/2}V = (U_i{D_i}^{1/2}V)^*\pi(a)(U_i{D_i}^{1/2}V) \\
&= (VZ_i)^*\pi(a)(VZ_i) = {Z_i}^*(V^*\pi(a)V)A_i = {Z_i}^*\phi(a)Z_i,
\end{aligned}$$

which in other words says $\phi_i(a) = {T_i}^{*-1}{Z_i}^*\phi(a)Z_i{T_i}^{-1} = {W_i}^*\pi(a)W_i$, where $W_i = Z_i{T_i}^{-1}$. Note that ${W_i}^*W_i = \phi_i(1) = I_{\mathcal{H}}$, and since W_i is invertible, it follows that W_i is unitary. Thus ϕ_i is unitarily equivalent to ϕ for each i, which concludes that ϕ is a C^*-extreme point in $S_{\mathcal{H}}(\mathcal{A})$. $\square$

Remark 2.2.2. In the statement of Theorem 2.2.1, U is a partial isometry. At this point, it is not clear whether U can be chosen to be unitary as claimed in [29]. Of course this is automatic if $\mathcal{H}_\pi$ is finite dimensional.

Remark 2.2.3. We again emphasize here that although we consider mainly UCP maps, statements and proofs of some theorems may include CP maps which are not unital. In such situations, the readers should pay proper attention to the use of the terms 'UCP' and 'CP'.

The following corollary is a characterization of C^*-extreme points provided by Zhou [80]. The proof follows directly from Theorem 2.2.1 and Radon-Nikodym type theorem (Theorem 1.2.19). However, the statement as written in [80] has a minor error (see Example 2.2.5 below), so we reproduce the proof here.

Corollary 2.2.4 (Theorem 3.1.5, [80])**.** *Let $\phi \in S_{\mathcal{H}}(\mathcal{A})$. Then ϕ is C^*-extreme in $S_{\mathcal{H}}(\mathcal{A})$ if and only if for any CP map ψ satisfying $\psi \leq \phi$ with $\psi(1)$ invertible, there exists an invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that $\psi(a) = S^*\phi(a)S$ for all $a \in \mathcal{A}$.*

Proof. First assume that ϕ is a C^*-extreme point in $S_{\mathcal{H}}(\mathcal{A})$. Let $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a CP map such that $\psi \leq \phi$ and $\psi(1)$ is invertible. Let $(\pi, V, \mathcal{H}_\pi)$ be the minimal Stinespring triple for ϕ. By Radon-Nikodym type theorem (Theorem 1.2.19), there exists a positive contraction $D \in \pi(\mathcal{A})'$ such that

$$\psi(a) = V^*D\pi(a)V, \quad \text{for all } a \in \mathcal{A}.$$

Since $V^*DV = \psi(1)$ and $\psi(1)$ is invertible, it follows that V^*DV is invertible. Therefore, by Theorem 2.2.1 there exist a partial isometry $U \in \pi(\mathcal{A})'$ satisfying $U^*UD^{1/2} = D^{1/2}$ and an invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that $UD^{1/2}V = VS$. So for any $a \in \mathcal{A}$, we get

$$\begin{aligned}\psi(a) &= V^*D\pi(a)V = V^*D^{1/2}\pi(a)D^{1/2}V = V^*D^{1/2}\pi(a)U^*UD^{1/2}V \\ &= \left(UD^{1/2}V\right)^* \pi(a) \left(UD^{1/2}V\right) = (VS)^*\pi(a)(VS) = S^*\phi(a)S.\end{aligned}$$

Conversely, assume the given statement in the 'only if' part is true. Let $\phi = \sum_{i=1}^n {T_i}^*\phi_i(\cdot)T_i$ be a proper C^*-convex combination. Then ${T_i}^*\phi_i(\cdot)T_i \leq \phi$ for each i. Also, since ${T_i}^*\phi_i(1)T_i = {T_i}^*T_i$ and T_i is invertible, it follows that ${T_i}^*\phi_i(1)T_i$ is invertible. Hence using hypothesis, there exists an invertible operator $S_i \in \mathcal{B}(\mathcal{H})$ such that for all $a \in \mathcal{A}$, we have ${T_i}^*\phi_i(a)T_i = {S_i}^*\phi(a)S_i$, which when put differently yields

$$\phi_i(a) = {U_i}^*\phi(a)U_i,$$

where $U_i = S_i{T_i}^{-1}$. But, since ${U_i}^*U_i = {U_i}^*\phi(1)U_i = \phi_i(1) = I_{\mathcal{H}}$ and U_i is invertible, it follows that U_i is a unitary. This shows that ϕ_i is unitarily equivalent to ϕ, as was required. ☐

We wish to mention that the condition of $\psi(1)$ being invertible in Corollary 2.2.4 cannot be dropped. The original statement (Theorem 3.1.5, [80]) is somewhat ambiguous about the invertibility requirement in the characterization. But it is crucial as the following example shows.

Example 2.2.5. Consider the C^*-extreme map $\phi : C(\mathbb{T}) \to \mathcal{B}(H^2)$ as in Example 2.1.5:

$$\phi(f) = P_{H^2} M_{f|_{H^2}} = T_f, \quad \text{for all } f \in C(\mathbb{T}).$$

Let $g, h : \mathbb{T} \to [0,1]$ be two real-valued non-zero continuous functions such that $g(z)h(z) = 0$ for all $z \in \mathbb{T}$ (such functions can always be obtained). Note that the sets of zeros of both g and h have positive Lebesgue measure. Consider the map $\psi : C(\mathbb{T}) \to \mathcal{B}(H^2)$ defined by

$$\psi(f) = P_{H^2} M_{gf|_{H^2}} = T_{gf}, \quad \text{for all } f \in C(\mathbb{T}).$$

It is clear that ψ is a CP map and $\psi \leq \phi$ (since $0 \leq g \leq 1$). Also $\psi(1) = T_g$ is not invertible (as zero-set of g has positive measure). We claim that there is no operator $S \in \mathcal{B}(H^2)$ such that $\psi(\cdot) = S^*\phi(\cdot)S$.

Suppose this is not the case and S is one such operator such that $\psi(\cdot) = S^*\phi(\cdot)S$. Then $S^*S = \psi(1) = T_g$. Since $T_g^* = T_g$ and $T_h^* = T_h$, and g, h are non-zero, it follows from a fact due to Coburn that T_g and T_h are one-one operators (see Proposition 7.24, [22]). Hence S^*S is one-one, which further implies that S is one-one. In particular, T_hS is one-one. But on the other hand, since $gh = 0$, we have

$$\left(T_h^{1/2}S\right)^* \left(T_h^{1/2}S\right) = S^*T_hS = S^*\phi(h)S = \psi(h) = T_{gh} = 0,$$

which implies

$$T_h^{1/2}S = 0,$$

and hence $T_hS = 0$, contradicting the fact that T_hS is one-one.

We now give another abstract characterization of C^*-extreme points, whose proof follows from a direct application of Theorem 2.2.1 and polar decomposition of operators. This powerful characterization turns out to be the most useful for our purpose.

Corollary 2.2.6. *Let $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a UCP map with minimal Stinespring triple $(\pi, V, \mathcal{H}_\pi)$. Then ϕ is C^*-extreme in $S_{\mathcal{H}}(\mathcal{A})$ if and only if for any positive operator $D \in \pi(\mathcal{A})'$ with V^*DV invertible, there exists $S \in \pi(\mathcal{A})'$ such that $D = S^*S$, $SVV^* = VV^*SVV^*$ and V^*SV is invertible (i.e. $S(V\mathcal{H}) \subseteq V\mathcal{H}$ and $S_{|V\mathcal{H}}$ is invertible).*

Proof. $\Longrightarrow$ We use the equivalent conditions for C^*-extreme points as in Theorem 2.2.1. Assume first that ϕ is a C^*-extreme point in $S_{\mathcal{H}}(\mathcal{A})$. Let $D \in \pi(\mathcal{A})'$ be a positive operator such that V^*DV is invertible. By Theorem 2.2.1, there exist a partial isometry $U \in \pi(\mathcal{A})'$ with $U^*UD^{1/2} = D^{1/2}$ and an invertible $Z \in \mathcal{B}(\mathcal{H})$ such that $UD^{1/2}V = VZ$. Set $S = UD^{1/2}$. Then

$$S^*S = D^{1/2}U^*UD^{1/2} = D \quad \text{and} \quad V^*SV = V^*UD^{1/2}V = V^*VZ = Z.$$

Thus V^*SV is invertible, and we get

$$SVV^* = UD^{1/2}VV^* = (VZ)V^* = VV^*(VZ)V^* = VV^*(UD^{1/2}V)V^* = VV^*SVV^*.$$

$\Longleftarrow$ Assume the 'only if' conditions. To show that ϕ is C^*-extreme in $S_{\mathcal{H}}(\mathcal{A})$, let $D \in \pi(\mathcal{A})'$ be positive with V^*DV invertible. By hypothesis, there exists $S \in \pi(\mathcal{A})'$ such that $D = S^*S$, $SVV^* = VV^*SVV^*$ and V^*SV is invertible. Let $S = UD^{1/2}$ be the polar decomposition of S, where U is a partial isometry with initial space $\overline{\mathcal{R}(D^{1/2})}$ i.e. $\mathcal{R}(U^*) = \overline{\mathcal{R}(D^{1/2})}$. Since $S \in \pi(\mathcal{A})'$, and $\pi(\mathcal{A})'$ is a von Neumann algebra, it follows that $U \in \pi(\mathcal{A})'$. Further, we have

$$UD^{1/2}V = SV = (SVV^*)V = (VV^*SVV^*)V = VV^*SV = VZ,$$

where $Z = V^*SV \in \mathcal{B}(\mathcal{H})$, which is invertible. That ϕ is C^*-extreme in $S_{\mathcal{H}}(\mathcal{A})$ now follows from the equivalent criteria of Theorem 2.2.1. This completes the proof. $\square$

In the corollary above, we cannot drop the assumption that V^*DV is invertible as the following example shows.

Example 2.2.7. Consider the C^*-extreme map $\phi : C(\mathbb{T}) \to \mathcal{B}(H^2)$ as in Example 2.1.5 by

$$\phi(f) = P_{H^2} M_{f|_{H^2}} = T_f, \quad \text{for all } f \in C(\mathbb{T}). \tag{2.2.1}$$

Note that ϕ is already in minimal Stinespring form with the representation $\pi : C(\mathbb{T}) \to \mathcal{B}(L^2(\mathbb{T}))$ given by $\pi(f) = M_f$. Then it is well-known that $\pi(C(\mathbb{T}))' = \{M_f; f \in L^\infty(\mathbb{T})\} \subseteq \mathcal{B}(L^2(\mathbb{T}))$ (Theorem 52.8, [16]). Now let $d \in L^\infty(\mathbb{T})$ be such that $d \geq 0$ a.e. and the subset $\{x \in \mathbb{T}; d(x) = 0\}$ has positive one-dimensional Lebesgue measure. It is then clear that M_d is not invertible which is equivalent to saying that $P_{H^2} M_{d|_{H^2}}$ is not invertible. Now let if possible, there exists $s \in L^\infty(\mathbb{T})$ such that $d = \bar{s}s$ and $M_s(H^2) \subseteq H^2$. This implies that $s \in H^\infty(\mathbb{T})$. But then the zero set of any function in h^∞ (in particular, s) has zero measure (Theorem 25.3, [16]). This contradicts the assumption that zero set of the function d has positive measure.

Observation 1. Let ϕ be a C^*-extreme point in $S_{\mathcal{H}}(\mathcal{A})$ with minimal Stinespring triple $(\pi, V, \mathcal{H}_\pi)$. Then for any positive $D \in \pi(\mathcal{A})'$ with V^*DV invertible, we observe the following from the proof of Theorem 2.2.1:

(i) There is a co-isometry U with $\mathcal{R}(U^*) = \overline{\mathcal{R}(D^{1/2})}$ and invertible Z such that $UD^{1/2}V = VZ$. In particular if D is one-one (equivalently, D has dense range), then U is unitary.

(ii) If $S = UD^{1/2}$, then S^* is one-one.

(iii) Also V^*SV is invertible such that $\|(V^*SV)^{-1}\|^2 = \|(V^*DV)^{-1}\|$.

The next result provides a bridge between the theory of C^*-extreme maps and factorization property of associated algebras. For the reader's convenience, we restate below the definition of factorization of subalgebras in C^*-algebras (Definition 1.5.1). See Section 1.5 for more details on such algebras.

Definition 2.2.8. A subalgebra $\mathcal{M}$ of a C^*-algebra $\mathcal{A}$ has *factorization* in $\mathcal{A}$ if for any positive and invertible element $D \in \mathcal{A}$, there is an invertible element S such that $S, S^{-1} \in \mathcal{M}$ and $D = S^*S$.

The following corollary turns out to be very crucial in our subsequent results on pure UCP maps and determining nature of C^*-extremity of UCP maps in general.

Corollary 2.2.9. *Let ϕ be a C^*-extreme point in $S_{\mathcal{H}}(\mathcal{A})$, and let $(\pi, V, \mathcal{H}_\pi)$ be its minimal Stinespring triple. If D is any positive and invertible operator in $\pi(\mathcal{A})'$, then there exists an invertible operator $S \in \pi(\mathcal{A})'$ such that $D = S^*S$, $SVV^* = VV^*SVV^*$, and V^*SV is invertible with inverse $V^*S^{-1}V$. In particular, the algebra*

$$\mathcal{M} = \{T \in \pi(\mathcal{A})'; TVV^* = VV^*TVV^*\} \tag{2.2.2}$$

has factorization in $\pi(\mathcal{A})'$.

Proof. Let D be a positive and invertible operator in $\pi(\mathcal{A})'$. Clearly V^*DV is invertible; hence by Theorem 2.2.1 and Observation 1, we get a co-isometry $U \in \pi(\mathcal{A})'$ with initial space $\overline{\mathcal{R}(D^{1/2})}$ and an invertible $Z \in \mathcal{B}(\mathcal{H})$ such that $UD^{1/2}V = VZ$. Note that $\overline{\mathcal{R}(D^{1/2})} = \mathcal{H}_\pi$ as D is invertible; so U is unitary. Set $S = UD^{1/2}$. Then $S \in \pi(\mathcal{A})'$ and S is invertible. Also $D = S^*S$ and $SVV^* = VV^*SVV^*$ with V^*SV invertible. Note that

$$(V^*S^{-1}V)(V^*SV) = V^*S^{-1}(VV^*SVV^*)V = V^*S^{-1}(SVV^*)V = V^*(S^{-1}S)V = I_{\mathcal{H}},$$

and since V^*SV is invertible, it follows that $(V^*SV)^{-1} = V^*S^{-1}V$. Further

$$\begin{aligned}[][(I_{\mathcal{H}_\pi} - VV^*)S^{-1}V](V^*SV) &= (I_{\mathcal{H}_\pi} - VV^*)S^{-1}(VV^*SVV^*)V = (I_{\mathcal{H}_\pi} - VV^*)S^{-1}(SVV^*)V \\ &= (I_{\mathcal{H}_\pi} - VV^*)(SS^{-1}VV^*V) = (I_{\mathcal{H}_\pi} - VV^*)VV^*V = 0.\end{aligned}$$

Since V^*SV is invertible, it follows that $(I_{\mathcal{H}_\pi} - VV^*)S^{-1}V = 0$; hence $S^{-1}VV^* = VV^*S^{-1}VV^*$. In particular, $S, S^{-1} \in \mathcal{M}$, so we conclude that $\mathcal{M}$ has factorization in $\pi(\mathcal{A})'$. □

With these abstract characterizations of C^*-extreme maps in our hand, we are now ready to explore some concrete structure of C^*-extreme maps. We end this section by an immediate application of this.

We consider the question of when a C^*-extreme point is also extreme, and vice versa. If $\mathcal{H}$ is a finite dimensional Hilbert space, then it was shown in [28] that every C^*-extreme point of $S_{\mathcal{H}}(\mathcal{A})$ is extreme as well. Whether this is true for infinite dimensional Hilbert spaces is not known. Conversely, there are examples where an extreme point in $S_{\mathcal{H}}(\mathcal{A})$ is not C^*-extreme (see pg. 1470 in [29]). We discuss some sufficient criteria under which condition of C^*-extremity automatically implies extremity. Also see Corollary 2.3.18 below.

Proposition 2.2.10. *Let $\phi \in S_{\mathcal{H}}(\mathcal{A})$ with minimal Stinespring triple $(\pi, V, \mathcal{H}_\pi)$ such that π is multiplicity-free (i.e. $\pi(\mathcal{A})'$ is commutative). If ϕ is C^*-extreme in $S_{\mathcal{H}}(\mathcal{A})$, then ϕ is extreme in $S_{\mathcal{H}}(\mathcal{A})$.*

Proof. To show ϕ is extreme in $S_{\mathcal{H}}(\mathcal{A})$, we use Arveson's extreme point condition (Theorem 1.2.22). Let D be a self-adjoint operator in $\pi(\mathcal{A})'$ such that $V^*DV = 0$. By multiplying by small enough scalar, we assume without loss of generality that $-\frac{1}{2}I_{\mathcal{H}_\pi} \leq D \leq \frac{1}{2}I_{\mathcal{H}_\pi}$. Then $D + I_{\mathcal{H}_\pi}$ is positive and invertible. By Corollary 2.2.9, there exists an invertible $S \in \pi(\mathcal{A})'$ satisfying $SVV^* = VV^*SVV^*$ with V^*SV invertible such that $D + I_{\mathcal{H}_\pi} = S^*S$. Thus we have

$$(V^*SV)^*(V^*SV) = V^*S^*(VV^*SVV^*)V = V^*S^*(SVV^*)V = V^*S^*SV = V^*DV + V^*V = I_{\mathcal{H}},$$

and since V^*SV is invertible, it follows that V^*SV is unitary, that is, $V^*SVV^*S^*V = I_{\mathcal{H}}$. Further as $\pi(\mathcal{A})'$ is commutative by hypothesis, we have $SS^* = S^*S = D + I_{\mathcal{H}_\pi}$; hence $V^*SS^*V = V^*(D + I_{\mathcal{H}_\pi})V = I_{\mathcal{H}}$. Therefore we get

$$[V^*S(I_{\mathcal{H}_\pi} - VV^*)][V^*S(I_{\mathcal{H}_\pi} - VV^*)]^* = V^*S(I_{\mathcal{H}_\pi} - VV^*)S^*V = V^*SS^*V - V^*SVV^*S^*V = 0.$$

This implies $V^*S(I_{\mathcal{H}_\pi} - VV^*) = 0$, which further yields

$$VV^*S = VV^*SVV^* = SVV^*.$$

In other words, S commutes with VV^* which also implies that S^* commutes with VV^*; hence D commutes with VV^*. Therefore, we have $DV = DVV^*V = VV^*DV = 0$. But then $D\pi(\mathcal{A})V = \pi(\mathcal{A})DV = 0$ and since $\pi(\mathcal{A})V\mathcal{H}$ is dense in $\mathcal{H}_\pi$, we conclude that $D = 0$. Since D is arbitrary, this proves that ϕ is extreme in $S_{\mathcal{H}}(\mathcal{A})$. $\square$

2.3 Direct sums of pure UCP maps

The question of whether the direct sum of two C^*-extreme points is also C^*-extreme is very natural. For the case when the Hilbert space is finite dimensional, a necessary and sufficient criterion for the validity of the assertion is known due to Farenick-Zhou [29]. In fact if $\mathcal{A}$ is a unital C^*-algebra and $n \in \mathbb{N}$, then every C^*-extreme point in $S_{\mathbb{C}^n}(\mathcal{A})$ is a direct sum of pure UCP maps (Theorem 2.1, [28]), so in this case the question reduces to finding conditions under which direct sums of pure UCP maps are C^*-extreme (which was exploited in [29]). Before we talk about a similar result in infinite dimensional Hilbert space setting, we first formally define the notion of direct sums of UCP maps.

Remark 2.3.1. In the rest of the thesis, Λ will usually be a *countable* indexing set for a family of maps or subspaces.

Definition 2.3.2. For any family $\{\phi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_i)\}_{i\in\Lambda}$ of UCP maps, their *direct sum* $\oplus_{i\in\Lambda}\phi_i$ is the UCP map from $\mathcal{A}$ to $\mathcal{B}(\oplus_{i\in\Lambda}\mathcal{H}_i)$ defined by $(\oplus_{i\in\Lambda}\phi_i)(a) = \oplus_{i\in\Lambda}\phi_i(a)$ for all $a \in \mathcal{A}$.

The following remark records the minimal Stinespring triple for a direct sum of UCP maps, which is easy to verify.

Remark 2.3.3. Let $\phi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_i)$, $i \in \Lambda$, be a collection of UCP maps with respective minimal Stinespring triple $(\pi_i, V_i, \mathcal{K}_i)$. Then the minimal Stinespring triple for $\oplus_{i\in\Lambda}\phi_i$ is given by $(\pi, V, \mathcal{K})$, where $\mathcal{K} = \oplus_{i\in\Lambda}\mathcal{K}_i$, $V = \oplus_{i\in\Lambda}V_i$ and $\pi = \oplus_{i\in\Lambda}\pi_i$.

We now state the aforementioned result from [29] in the language of nests of subspaces (see Definition 1.5.3), which provides a characterization of C^*-extreme points in $S_{\mathbb{C}^n}(\mathcal{A})$ in terms of direct sums of pure UCP maps. See Section 1.5 for more details on nests.

Theorem 2.3.4 (Theorem 2.1, [29])**.** *Let $\mathcal{A}$ be a unital C^*-algebra, and $\phi \in S_{\mathbb{C}^n}(\mathcal{A})$ for $n \in \mathbb{N}$. Then ϕ is C^*-extreme in $S_{\mathbb{C}^n}(\mathcal{A})$ if and only if there exists finitely many pairwise non unitarily equivalent irreducible representations $\pi_1, \pi_2, \ldots, \pi_k$ and pure UCP maps ${\phi_j}^i$ of the form ${\phi_j}^i = {V_j}^i\pi_i(\cdot){V_j}^i$ for $1 \leq j \leq n_i, 1 \leq i \leq k$ such that* *(i)* $\{\mathcal{R}(V_j{}^i); 1 \leq j \leq n_i\}$ *forms a nest of subspaces for each* i*, and*

(ii) ϕ *is unitarily equivalent to the UCP map* $\oplus_{i=1}^{k} \oplus_{j=1}^{n_i} \phi_j{}^i$.

When the Hilbert space $\mathcal{H}$ is infinite dimensional, it is no longer the case that a C^*-extreme point of $S_{\mathcal{H}}(\mathcal{A})$ is a direct sum of pure UCP maps (see Example 2.1.5). Nevertheless, finding criteria for a direct sum of pure UCP maps to be C^*-extreme is interesting in its own right. In this section, we provide a complete characterization for such maps to be C^*-extreme. To this end, we consider some general properties of C^*-extremity under direct sums. Firstly, we emphasize the following:

Remark 2.3.5. For a family of Hilbert spaces $\{\mathcal{H}_i\}_{i\in\Lambda}$, an operator T in $\mathcal{B}(\oplus_{i\in\Lambda}\mathcal{H}_i)$ can also be expressed in the matrix form $[T_{ij}]$, where the entries $T_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$ is given by $T_{ij} = P_{\mathcal{H}_i} T_{|\mathcal{H}_j}$ for $i, j \in \Lambda$. Here $P_{\mathcal{H}_i}$ denotes the projection from $\oplus_{i\in\Lambda}\mathcal{H}_i$ onto the subspace $\mathcal{H}_i$.

We start with the following simple lemma about commutant of direct sum of disjoint representations. We refer the readers to Section 1.2 for the notions and their properties.

Lemma 2.3.6. *Let* $\pi_i : \mathcal{A} \to \mathcal{B}(\mathcal{K}_i)$*,* $i \in \Lambda$*, be a collection of mutually disjoint representations. If* $\pi = \oplus_{i\in\Lambda}\pi_i$*, then* $\pi(\mathcal{A})' = \{\oplus_{i\in\Lambda} T_i;\ T_i \in \pi_i(\mathcal{A})'\}$.

Proof. Let $S \in \pi(\mathcal{A})' \subseteq \mathcal{B}(\oplus_{i\in\Lambda}\mathcal{K}_i)$. Then $S = [S_{ij}]$ for some $S_{ij} \in \mathcal{B}(\mathcal{K}_j, \mathcal{K}_i)$, such that for all $a \in \mathcal{A}$, we have $[S_{ij}](\oplus_{i\in\Lambda}\pi_i(a)) = (\oplus_{i\in\Lambda}\pi_i(a))[S_{ij}]$, that is $[S_{ij}\pi_j(a)] = [\pi_i(a))S_{ij}]$; hence

$$S_{ij}\pi_j(a) = \pi_i(a)S_{ij} \ \text{ for all } \ i, j.$$

For $i \neq j$, since π_i is disjoint to π_j, it follows (see Proposition 1.2.15) that $S_{ij} = 0$. Also for each i, $S_{ii}\pi_i(a) = \pi_i(a)S_{ii}$ for $a \in \mathcal{A}$, implies that $S_{ii} \in \pi_i(\mathcal{A})'$. Thus $S = \oplus_{i\in\Lambda}S_{ii}$, where $S_{ii} \in \pi_i(\mathcal{A})'$. This shows that $\pi(\mathcal{A})' \subseteq \{\oplus_{i\in\Lambda}T_i; T_i \in \pi_i(\mathcal{A})'\}$. The other inclusion is obvious. □

Inspired from the notion of disjointness of representations (Definition 1.2.14), we define the same for CP maps as follows:

Definition 2.3.7. For any two UCP maps $\phi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$ with respective minimal Stinespring triple $(\pi_i, V_i, \mathcal{H}_{\pi_i})$, we say ϕ_1 is *disjoint* to ϕ_2 if π_1 and π_2 are disjoint representations.

The major results of this thesis deal with finding conditions under which direct sums of mutually disjoint UCP maps (especially, pure maps) are C^*-extreme. The next proposition is the first step in this direction.

Proposition 2.3.8. *Let* $\{\phi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_i)\}_{i\in\Lambda}$ *be a collection of mutually disjoint UCP maps. Then* $\phi = \oplus_{i\in\Lambda}\phi_i$ *is* C^**-extreme (resp. extreme) in* $S_{\oplus_{i\in\Lambda}\mathcal{H}_i}(\mathcal{A})$ *if and only if each* ϕ_i *is* C^**-extreme (resp. extreme) in* $S_{\mathcal{H}_i}(\mathcal{A})$.

Proof. Let $(\pi_i, V_i, \mathcal{K}_i)$ be the minimal Stinespring triple for $\phi_i, i \in \Lambda$. Then as noted in Remark 2.3.3, $(\pi, \mathcal{K}, V)$ is the minimal Stinespring triple for ϕ, where $\mathcal{K} = \oplus_{i\in\Lambda}\mathcal{K}_i$, $\pi = \oplus_{i\in\Lambda}\pi_i$, and $V = \oplus_{i\in\Lambda}V_i$. Since π_i is disjoint to π_j for $i \neq j$, it follows from Lemma 2.3.6 that

$$\pi(\mathcal{A})' = \{\oplus_{i\in\Lambda}T_i; T_i \in \pi_i(\mathcal{A})'\} \subseteq \mathcal{B}(\oplus_{i\in\Lambda}\mathcal{K}_i). \tag{2.3.1}$$

To prove the equivalent criteria for C^*-extremity, we shall use Corollary 2.2.6. Assume first that each ϕ_i is C^*-extreme in $S_{\mathcal{H}_i}(\mathcal{A})$. Let $D \in \pi(\mathcal{A})'$ be positive such that V^*DV is invertible. Then it follows from (2.3.1) that $D = \oplus_{i\in\Lambda} D_i$ for some $D_i \in \pi_i(\mathcal{A})'$, and hence $V^*DV = \oplus_{i\in\Lambda} {V_i}^* D_i V_i$. Clearly each D_i is positive such that $V_i^* D_i V_i$ is invertible satisfying $\sup_{i\in\Lambda} \|(V_i^* D_i V_i)^{-1}\| = \|(V^*DV)^{-1}\|$. Since each ϕ_i is C^*-extreme, there exists an operator $S_i \in \pi_i(\mathcal{A})'$ such that $D_i = {S_i}^* S_i$, $S_i V_i V_i^* = V_i {V_i}^* S_i V_i V_i^*$ and ${V_i}^* S_i V_i$ is invertible. Set $S = \oplus_{i\in\Lambda} S_i$. It is then immediate that $S \in \pi(\mathcal{A})'$, $D = S^*S$ and $SVV^* = VV^*SVV^*$. Also from Observation 1, it follows that

$$\sup_{i\in\Lambda} \|({V_i}^* S_i V_i)^{-1}\|^2 = \sup_{i\in\Lambda} \|(V_i D_i V_i)^{-1}\| = \|(V^*DV)^{-1}\| < \infty,$$

which implies that $V^*SV = \oplus_{i\in\Lambda} {V_i}^* S_i V_i$ is invertible. Since D is arbitrary, it follows that $\oplus_{i\in\Lambda}\phi_i$ is C^*-extreme.

Conversely, let $\oplus_{i\in\Lambda}\phi_i$ be C^*-extreme. Fix $j \in \Lambda$, and let $D_j \in \pi_j(\mathcal{A})'$ be a positive operator such that ${V_j}^* D_j V_j$ is invertible. For $i \neq j$, let $D_i = I_{\mathcal{K}_i}$ and set

$$D = \oplus_{i\in\Lambda} D_i.$$

It is clear that $D \in \pi(\mathcal{A})'$. Also D is positive and V^*DV is invertible, as each ${V_i}^* D_i V_i$ is invertible whose inverse is uniformly bounded. Since $\oplus_{i\in\Lambda}\phi_i$ is C^*-extreme, there is an operator $S \in \pi(\mathcal{A})'$ such that $D = S^*S$, $SVV^* = VV^*SVV^*$ and V^*SV is invertible. Again from (2.3.1), we have $S = \oplus_{i\in\Lambda} S_i$ for some $S_i \in \pi_i(\mathcal{A})'$. Then the expressions $D = S^*S$ and $SVV^* = VV^*SVV^*$ imply respectively that

$$D_j = {S_j}^* S_j \quad \text{and} \quad S_j V_j {V_j}^* = V_j {V_j}^* S_j V_j {V_j}^*.$$

Also invertibility of V^*SV implies that ${V_j}^* S_j V_j$ is invertible. Since D_j is arbitrary, we conclude that ϕ_j is C^*-extreme in $S_{\mathcal{H}_j}(\mathcal{A})$. The case of equivalence of extreme points can be proved in a similar fashion using Arveson's extreme point criterion (Theorem 1.2.22). $\square$

We are now ready to prove the main result of this section regarding direct sums of pure UCP maps. For doing so, we observe the following property about compression of pure UCP maps (see Definition 1.2.9).

Remark 2.3.9. If ϕ is a pure UCP map with the minimal Stinespring triple $(\pi, V, \mathcal{H}_\pi)$, and $\psi = W^*\phi(\cdot)W$ is a compression of ϕ for some isometry W, then $(\pi, VW, \mathcal{H}_\pi)$ is the minimal Stinespring triple for ψ, and so ψ is pure. This follows from the fact that $\pi(\mathcal{A})' = \mathbb{C} \cdot I_{\mathcal{H}_\pi}$, so that $\pi(\mathcal{A})'' = \mathcal{B}(\mathcal{H}_\pi)$, which further yields

$$[\pi(\mathcal{A})VW\mathcal{H}] = [\pi(\mathcal{A})''VW\mathcal{H}] = [\mathcal{B}(\mathcal{H}_\pi)VW\mathcal{H}] = \mathcal{H}_\pi.$$

Moreover, if $(\pi, V_i, \mathcal{H}_\pi)$ is the minimal Stinespring triple of UCP map ϕ_i, $i = 1, 2$ (i.e. both ϕ_1, ϕ_2 are compression of the same representation π), then one can easily show that ϕ_2 is a compression of ϕ_1 if and only if $V_2 V_2^* \leq V_1 V_1^*$ i.e. $\mathcal{R}(V_2) \subseteq \mathcal{R}(V_1)$.

Note that if $\phi, \psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ are two pure UCP maps, then either ϕ and ψ are mutually disjoint or they are compression of the same irreducible representation. Therefore in view of Proposition 2.3.8, in order to give criteria for C^*-extremity of direct sums of pure UCP maps, it suffices to consider direct sum of only those pure UCP maps which are compression of the same irreducible representation (i.e. those pure maps which are not mutually disjoint), as done in the next theorem.

In order to prove the following theorem, we invoke a deep result of Larson [47] about factorization property of nest algebras associated with countable complete nests (see Theorem 1.5.13). We refer the readers to Section 1.5 for more details on these notions. Larson's result is one of the most fundamental results in the theory of nest algebras and triangular forms of operators. It has had huge impact on the later study of the subject and related fields. We have used the countability criteria of the 'completion of nests' (see Definition 1.5.4) several times in this thesis in our analysis of C^*-extreme points (see Section 3.3 for some examples based on this idea). The countability criterion plays a central role in the proof of the following theorem, mainly in dealing with Condition (ii). This condition is inevitable in infinite-dimensional case, and reveals a stark contrast to the finite-dimensional situation.

Theorem 2.3.10. *Let $\psi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_i)$, $i \in \Lambda$, be a countable family of non-unitarily equivalent pure UCP maps with respective minimal Stinespring triple $(\pi, V_i, \mathcal{H}_\pi)$, where π is a fixed representation of $\mathcal{A}$, and let $\phi_i = \psi_i(\cdot) \otimes I_{\mathcal{K}_i}$ for some Hilbert space $\mathcal{K}_i$. Set $\mathcal{H} = \oplus_{i\in\Lambda}(\mathcal{H}_i \otimes \mathcal{K}_i)$, and $\phi = \oplus_{i\in\Lambda}\phi_i \in S_{\mathcal{H}}(\mathcal{A})$. Then ϕ is C^*-extreme in $S_{\mathcal{H}}(\mathcal{A})$ if and only if the following holds:*

(i) the family $\{\mathcal{R}(V_i)\}_{i\in\Lambda}$ of subspaces forms a nest in $\mathcal{H}_\pi$, which induces an order on Λ and

(ii) if $\mathcal{L}_i = \oplus_{j\leq i}\mathcal{K}_j$ for $i \in \Lambda$, then completion of the nest $\{\mathcal{L}_i\}_{i\in\Lambda}$ in $\oplus_{i\in\Lambda}\mathcal{K}_i$ is countable.

Proof. We know that each ψ_i is unitarily equivalent to the UCP map $a \mapsto P_{\mathcal{R}(V_i)}\pi(a)|_{\mathcal{R}(V_i)}$, $a \in \mathcal{A}$ (see Remark 1.2.12). So the fact from the hypothesis that ψ_i and ψ_j are not unitarily equivalent for $i \neq j$ then implies that $\mathcal{R}(V_i) \neq \mathcal{R}(V_j)$, that is,

$$V_iV_i^* \neq V_jV_j^*, \quad \text{for all } i \neq j. \tag{2.3.2}$$

Now set $\mathcal{H}_\rho = \oplus_{i\in\Lambda}(\mathcal{H}_\pi \otimes \mathcal{K}_i)$, and consider the representation $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\rho)$ defined by

$$\rho(a) = \oplus_{i\in\Lambda}(\pi(a) \otimes I_{\mathcal{K}_i}) \text{ for all } a \in \mathcal{A},$$

and the isometry $V \in \mathcal{B}(\mathcal{H}, \mathcal{H}_\rho)$ given by

$$V = \oplus_{i\in\Lambda}(V_i \otimes I_{\mathcal{K}_i}).$$

It is clear that $(\rho, V, \mathcal{H}_\rho)$ is the minimal Stinespring triple for ϕ. We identify the Hilbert space $\mathcal{H}_\rho = \oplus_{i\in\Lambda}(\mathcal{H}_\pi \otimes \mathcal{K}_i)$ with the Hilbert space $\mathcal{H}_\pi \otimes (\oplus_{i\in\Lambda}\mathcal{K}_i)$; so the representation ρ is given by

$$\rho(a) = \pi(a) \otimes (\oplus_{i\in\Lambda}I_{\mathcal{K}_i}) = \pi(a) \otimes I_{\oplus_{i\in\Lambda}\mathcal{K}_i}.$$

Since π is irreducible, $\pi(\mathcal{A})' = \mathbb{C} \cdot I_{\mathcal{H}_\pi}$; hence if we consider the operators on the Hilbert space $\mathcal{K} = \oplus_{i\in\Lambda}\mathcal{K}_i$ in matrix form, then $\rho(\mathcal{A})'$ is given by

$$\rho(\mathcal{A})' = (\pi(\mathcal{A}) \otimes I_{\mathcal{K}})' = I_{\mathcal{H}_\pi} \otimes \mathcal{B}(\mathcal{K}) = \{I_{\mathcal{H}_\pi} \otimes [T_{ij}];\ T_{ij} \in \mathcal{B}(\mathcal{K}_j, \mathcal{K}_i)\} \subseteq \mathcal{B}\left(\mathcal{H}_\pi \otimes (\oplus_{i\in\Lambda}\mathcal{K}_i)\right).$$

$\Longrightarrow$ Assume now that $\oplus_{i\in\Lambda}\phi_i$ is a C^*-extreme point in $S_{\mathcal{H}}(\mathcal{A})$. First we show that $\{\mathcal{R}(V_i)\}_{i\in\Lambda}$ is a nest in $\mathcal{H}_\pi$. Consider the subalgebra $\mathcal{M}$ of $\mathcal{B}(\oplus_{i\in\Lambda}\mathcal{K}_i)$ given by

$$\begin{aligned}\mathcal{M} &= \{[T_{ij}] \in \mathcal{B}(\oplus_{i\in\Lambda}\mathcal{K}_i);\ (I_{\mathcal{H}_\pi}\otimes[T_{ij}])VV^* = VV^*(I_{\mathcal{H}_\pi}\otimes[T_{ij}])VV^*\} \\ &= \{[T_{ij}] \in \mathcal{B}(\oplus_{i\in\Lambda}\mathcal{K}_i);\ V_jV_j{}^*\otimes T_{ij} = V_iV_i{}^*V_jV_j{}^*\otimes T_{ij}\ \forall\ i,j\in\Lambda\}.\end{aligned} \tag{2.3.3}$$

Since $\oplus_{i\in\Lambda}\phi_i$ is C^*-extreme, it follows from Corollary 2.2.9 that $I_{\mathcal{H}_\pi}\otimes\mathcal{M}$ has factorization in $\rho(\mathcal{A})' = I_{\mathcal{H}_\pi}\otimes\mathcal{B}(\mathcal{K})$, which is to say that $\mathcal{M}$ has factorization in $\mathcal{B}(\mathcal{K})$.

Note that if there is an operator $[T_{ij}]\in\mathcal{M}$ such that $T_{mn}\neq 0$ for some $m,n\in\Lambda$, then since $V_nV_n^*\otimes T_{mn} = V_mV_m^*V_nV_n^*\otimes T_{mn}$, it will follow that $V_nV_n^* = V_mV_m^*V_nV_n^*$, which further implies $V_mV_m^*\geq V_nV_n^*$. In other words, we have the following:

$$\text{If } V_mV_m^*\ngeq V_nV_n^* \text{ for some } m,n\in\Lambda,\ \text{then } T_{mn}=0 \text{ for all } [T_{ij}]\in\mathcal{M}. \tag{2.3.4}$$

For the remainder of this implication, we fix $m,n\in\Lambda$ with $m\neq n$. We shall prove that $V_mV_m^*\geq V_nV_n^*$ or $V_nV_n^*\geq V_mV_m^*$. Assume to the contrary that this is not the case. Then it follow from (2.3.4) that

$$T_{mn}=0 \text{ and } T_{nm}=0,\quad \text{for all } [T_{ij}]\in\mathcal{M}. \tag{2.3.5}$$

If Λ is a two point set, that is, $\Lambda=\{m,n\}$, then $\mathcal{K}=\mathcal{K}_m\oplus\mathcal{K}_n$, and with respect to this decomposition, (2.3.5) implies that each element T in $\mathcal{M}$ has the form

$$\begin{bmatrix} T_1 & 0\\ 0 & T_2\end{bmatrix},\quad \text{for } T_1\in\mathcal{B}(\mathcal{K}_m) \text{ and } T_2\in\mathcal{B}(\mathcal{K}_n).$$

But if we choose a positive and invertible operator D in $\mathcal{B}(\mathcal{K})$ of the form $\begin{bmatrix} I_{\mathcal{K}_m} & D_1\\ D_1^* & I_{\mathcal{K}_n}\end{bmatrix}$ with $D_1\in\mathcal{B}(\mathcal{K}_n,\mathcal{K}_m)$ non-zero, then we cannot find any operator T in $\mathcal{M}$ such that $D=T^*T$. This will contradict the fact that $\mathcal{M}$ has factorization in $\mathcal{B}(\mathcal{K})$.

Therefore we assume for the rest of the implication that $\Lambda\neq\{m,n\}$. Now consider the sets

$$\Lambda_1 = \{l\in\Lambda\setminus\{m,n\};\ T_{lm}=0 \text{ and } T_{ln}=0\ \text{ for all }\ [T_{ij}]\in\mathcal{M}\}, \tag{2.3.6}$$

and

$$\Lambda_2 = \Lambda\setminus(\Lambda_1\cup\{m,n\}).$$

Note that

$$\Lambda_1\cap\Lambda_2=\emptyset \ \text{ and }\ \Lambda_1\sqcup\Lambda_2\sqcup\{m,n\}=\Lambda.$$

Consider the following decomposition:

$$\mathcal{K}=\oplus_{i\in\Lambda}\mathcal{K}_i=\mathcal{K}_m\oplus\mathcal{K}_n\oplus(\oplus_{i\in\Lambda_1}\mathcal{K}_i)\oplus(\oplus_{i\in\Lambda_2}\mathcal{K}_i)=\mathcal{Q}_1\oplus\mathcal{Q}_2\oplus\mathcal{Q}_3\oplus\mathcal{Q}_4, \tag{2.3.7}$$

where

$$\mathcal{Q}_1=\mathcal{K}_m,\ \mathcal{Q}_2=\mathcal{K}_n,\ \mathcal{Q}_3=\oplus_{i\in\Lambda_1}\mathcal{K}_i,\ \text{ and }\ \mathcal{Q}_4=\oplus_{i\in\Lambda_2}\mathcal{K}_i.$$

We shall show that $\mathcal{Q}_3 \neq \{0\}$ and $\mathcal{Q}_4 \neq \{0\}$ (that is, Λ_1 and Λ_2 are non-empty), and that with respect to decomposition in (2.3.7), each T in $\mathcal{M}$ has the following form:

$$T = \begin{bmatrix} T_1 & 0 & A_1 & 0 \\ 0 & T_2 & A_2 & 0 \\ 0 & 0 & X_1 & X_2 \\ B_1 & B_2 & X_3 & X_4 \end{bmatrix}, \tag{2.3.8}$$

for appropriate operators $T_1, T_2, ..$ etc. For that, we first claim the following: If for some $l \neq m, n$, there exists an operator $[S_{ij}] \in \mathcal{M}$ such that $S_{lm} \neq 0$ or $S_{ln} \neq 0$, then

$$T_{ml} = 0 \text{ and } T_{nl} = 0, \quad \forall\ [T_{ij}] \in \mathcal{M}. \tag{2.3.9}$$

To prove the claim in (2.3.9), assume that $S_{lm} \neq 0$, and let $[T_{ij}]$ be an arbitrary operator in $\mathcal{M}$. Then it follows from (2.3.4) that $V_l V_l^* \geq V_m V_m^*$. Since $V_l V_l^* \neq V_m V_m^*$ from (2.3.2), it follows that $V_m V_m^* \ngeq V_l V_l^*$; again from (2.3.4), we get $T_{ml} = 0$. Further, we note that $V_n V_n^* \ngeq V_l V_l^*$ (otherwise we would have $V_n V_n^* \geq V_l V_l^* \geq V_m V_m^*$, and so $V_n V_n^* \geq V_m V_m^*$ which is against our assumption). This in turn implies by (2.3.4) that $T_{nl} = 0$. Similarly or by symmetry, the condition $S_{ln} \neq 0$ will imply the required claim in (2.3.9).

We now show that Λ_1 is a non-empty set. Assume otherwise that $\Lambda_1 = \emptyset$. Then for each $l \in \Lambda \setminus \{m, n\}$, we have $l \notin \Lambda_1$, so there exists $[S_{ij}] \in \mathcal{M}$ such that either $S_{lm} \neq 0$ or $S_{ln} \neq 0$. In either case, (2.3.9) implies that for all $T = [T_{ij}] \in \mathcal{M}$, we have $T_{ml} = 0$ and $T_{nl} = 0$; hence the (m, n) entry of the matrix TT^* satisfies

$$\sum_{l \in \Lambda} T_{ml} T_{nl}^* = T_{mm} T_{nm}^* + T_{mn} T_{nn}^* + \sum_{l \neq m,n} T_{ml} T_{nl}^* = 0,$$

as $T_{mn} = 0$ and $T_{nm}^* = 0$ from (2.3.5), where the sum is in WOT. Thus for any positive and invertible $D = [D_{ij}] \in \mathcal{B}(\mathcal{K})$ with $D_{mn} \neq 0$, we cannot find $T \in \mathcal{M}$ such that $D = TT^*$. We can always get such positive and invertible operator D (see the operator in (2.3.11) below). This violates the fact that $\mathcal{M}^*$ and hence $\mathcal{M}$ has factorization in $\mathcal{B}(\mathcal{K})$. Thus our claim that $\Lambda_1 \neq \emptyset$ is true.

We next show that Λ_2 is non-empty. Let if possible, $\Lambda_2 = \emptyset$. Then for each $l \in \Lambda$ with $l \neq m, n$, it follows that $l \in \Lambda_1$; hence for all $T = [T_{ij}] \in \mathcal{M}$, we have $T_{lm} = 0$ and $T_{ln} = 0$, so that (m, n) entry of T^*T satisfies

$$\sum_{l \in \Lambda} T_{lm}^* T_{ln} = 0,$$

as $T_{mn} = 0$ and $T_{nm}^* = 0$. Again for a positive and invertible operator $D = [D_{ij}]$ in $\mathcal{B}(\mathcal{K})$ with $D_{mn} \neq 0$, we can't find any $T \in \mathcal{M}$ such that $D = T^*T$, violating the fact that $\mathcal{M}$ has factorization in $\mathcal{B}(\mathcal{K})$. This shows our claim that $\Lambda_2 \neq \emptyset$.

Further we note that if $l \in \Lambda_2$, then $l \notin \Lambda_1$, so $S_{lm} \neq 0$ or $S_{ln} \neq 0$ for some $[S_{ij}] \in \mathcal{M}$; hence it follows from (2.3.9) that $T_{ml} = 0$ and $T_{nl} = 0$ for all $[T_{ij}] \in \mathcal{M}$. Thus we have

$$\Lambda_2 \subseteq \{l \in \Lambda \setminus \{m, n\};\ T_{ml} = 0 \ \text{ and } \ T_{nl} = 0 \ \text{ for all } \ [T_{ij}] \in \mathcal{M}\}. \tag{2.3.10}$$

Now let $T = [T_{ij}] \in \mathcal{M}$, then since $T_{lm} = 0$ and $T_{ln} = 0$ for all $l \in \Lambda_1$, it follows that

$$P_{\mathcal{Q}_3} T_{|\mathcal{Q}_1} = \sum_{l \in \Lambda_1} P_{\mathcal{K}_l} T_{|\mathcal{K}_m} = \sum_{l \in \Lambda_1} T_{lm} = 0, \quad \text{and} \quad P_{\mathcal{Q}_3} T_{|\mathcal{Q}_2} = \sum_{l \in \Lambda_1} P_{\mathcal{K}_l} T_{|\mathcal{K}_n} = \sum_{l \in \Lambda_1} T_{ln} = 0.$$

The sum above is in strong operator topology. Similarly from (2.3.10), since $T_{ml} = 0$ and $T_{nl} = 0$ for all $l \in \Lambda_2$, it follows that $P_{\mathcal{Q}_1} T_{|\mathcal{Q}_4} = 0$ and $P_{\mathcal{Q}_2} T_{|\mathcal{Q}_4} = 0$. These observations along with (2.3.5) prove our claim that every operator $T \in \mathcal{M}$ has the form as in (2.3.8).

Now with respect to the decomposition in (2.3.7), consider the operator D in $\mathcal{B}(\mathcal{K})$ given by

$$D = \begin{bmatrix} I_{\mathcal{Q}_1} & D_1 & 0 & 0 \\ D_1^* & I_{\mathcal{Q}_2} & 0 & 0 \\ 0 & 0 & I_{\mathcal{Q}_3} & 0 \\ 0 & 0 & 0 & I_{\mathcal{Q}_4} \end{bmatrix} \tag{2.3.11}$$

where $D_1 \in \mathcal{B}(\mathcal{Q}_2, \mathcal{Q}_1)$ satisfies $0 < \|D_1\| < 1$. It is then clear that D is a positive and invertible operator in $\mathcal{B}(\mathcal{K})$. Since $\mathcal{M}$ has factorization in $\mathcal{B}(\mathcal{K})$, there is an invertible operator $S \in \mathcal{M}$ with $S^{-1} \in \mathcal{M}$ such that $D = S^* S$. Then from (2.3.8), S and S^{-1} look like

$$S = \begin{bmatrix} S_1 & 0 & A_1 & 0 \\ 0 & S_2 & A_2 & 0 \\ 0 & 0 & X_1 & X_2 \\ B_1 & B_2 & X_3 & X_4 \end{bmatrix} \quad \text{and} \quad S^{-1} = \begin{bmatrix} T_1 & 0 & C_1 & 0 \\ 0 & T_2 & C_2 & 0 \\ 0 & 0 & Y_1 & Y_2 \\ E_1 & E_2 & Y_3 & Y_4 \end{bmatrix}.$$

Now

$$SS^{-1} = \begin{bmatrix} S_1 T_1 & 0 & S_1 C_1 + A_1 Y_1 & A_1 Y_2 \\ 0 & S_2 T_2 & S_2 C_2 + A_2 Y_1 & A_2 Y_2 \\ X_2 E_1 & X_2 E_2 & X_1 Y_1 + X_2 Y_3 & X_1 Y_2 + X_2 Y_4 \\ B_1 T_1 + X_4 E_1 & B_2 T_2 + X_4 E_2 & B_1 C_1 + B_2 C_2 + X_3 Y_1 + X_4 Y_3 & X_3 Y_2 + X_4 Y_4 \end{bmatrix}.$$

Thus we get $S_1 T_1 = I_{\mathcal{Q}_1}$ and $S_2 T_2 = I_{\mathcal{Q}_2}$. Similarly from the expression $S^{-1} S = I_{\mathcal{K}}$, we get $T_1 S_1 = I_{\mathcal{Q}_1}$ and $T_2 S_2 = I_{\mathcal{Q}_2}$. This shows that T_1 and T_2 are invertible. Further, from $(4,1)$ entry of SS^{-1}, we have $B_1 T_1 + X_4 E_1 = 0$, which yields

$$B_1 = -X_4 E_1 T_1^{-1} = X_4 F_1,$$

where $F_1 = -E_1 T_1^{-1}$. Also, from $(4,2)$ entry of SS^{-1}, we have $B_2 T_2 + X_4 E_2 = 0$, that is,

$$B_2 = -X_4 E_2 T_2^{-1} = X_4 F_2,$$

where $F_2 = -E_2 T_2^{-1}$. Next we note that $(1,2)$ entry of $S^* S$ is $B_1^* B_2$, and $(1,4)$ entry of $S^* S$ is $B_1^* X_4$. By substituting $B_1 = X_4 F_1$ and $B_2 = X_4 F_2$, and equating the corresponding entries of D, we get

$$F_1^* X_4^* X_4 F_2 = B_1^* B_2 = D_1 \quad \text{and} \quad F_1^* X_4^* X_4 = B_1^* X_4 = 0.$$

This implies that $D_1 = 0$, which is a contradiction. This again violates the fact that $\mathcal{M}$ has factorization in $\mathcal{B}(\mathcal{K})$. Thus we have shown our claim that $V_n V_n^* \geq V_m V_m^*$ or $V_m V_m^* \geq V_n V_n^*$, which is to say that $\mathcal{R}(V_n) \supseteq \mathcal{R}(V_m)$ or $\mathcal{R}(V_m) \supseteq \mathcal{R}(V_n)$. Since $m, n \in \Lambda$ are arbitrary, we conclude that $\mathcal{E} = \{\mathcal{R}(V_i)\}_{i\in\Lambda}$ is a nest.

Now we define an order on Λ by assigning

$$i \leq j \text{ if and only if } V_iV_i^* \leq V_jV_j^*, \tag{2.3.12}$$

for any $i, j \in \Lambda$. Since $V_iV_i^* \neq V_jV_j^*$ whenever $i \neq j$, the order on Λ is well-defined. Also Λ is a totally-ordered set, as $\{\mathcal{R}(V_i)\}_{i\in\Lambda}$ forms a nest of subspaces. For each $i \in \Lambda$, consider the subspace $\mathcal{L}_i$ of $\mathcal{K} = \oplus_{i\in\Lambda}\mathcal{K}_i$ given by

$$\mathcal{L}_i = \bigoplus_{j\leq i} \mathcal{K}_j. \tag{2.3.13}$$

Then it is clear that the collection $\mathcal{L} = \{\mathcal{L}_i; i \in \Lambda\}$ forms a nest in $\mathcal{K}$ such that $\mathcal{L}_i \subsetneq \mathcal{L}_j$ if and only if $i < j$. We have to show that the completion $\overline{\mathcal{L}}$ of the nest $\mathcal{L}$ is countable. We claim that

$$\mathcal{M} = (\operatorname{Alg}\mathcal{L})^*. \tag{2.3.14}$$

Since $\mathcal{M}$ has factorization in $\mathcal{B}(\mathcal{K})$, it will then follow from the claim and Proposition 1.5.2 that $\operatorname{Alg}\mathcal{L}$ has factorization in $\mathcal{B}(\mathcal{K})$, which further will imply our requirement using Theorem 1.5.13 that $\overline{\mathcal{L}}$ is countable (as $\mathcal{K}$ is separable).

To show the claim in (2.3.14), we first note that if an operator $S = [S_{ij}]$ in $\mathcal{B}(\mathcal{K})$ leaves all subspaces $\{\mathcal{L}_i\}$ invariant, then $S_{ij} = 0$ for all $i > j$; hence $\operatorname{Alg}\mathcal{L} = \{[S_{ij}] \in \mathcal{B}(\mathcal{K}), S_{ij} = 0 \text{ for } i > j\}$, that is,

$$(\operatorname{Alg}\mathcal{L})^* = \{[S_{ij}] \in \mathcal{B}(\mathcal{K}); S_{ij} = 0 \text{ for } i < j\}. \tag{2.3.15}$$

Now let $[S_{ij}] \in \mathcal{M}$. Then $V_jV_j^* \otimes S_{ij} = V_iV_i^*V_jV_j^* \otimes S_{ij}$ for all $i, j \in \Lambda$. For $i < j$, since $V_iV_i^*V_jV_j^* = V_iV_i^*$ and $V_iV_i^* \neq V_jV_j^*$, it forces that $S_{ij} = 0$. This shows that $[S_{ij}] \in (\operatorname{Alg}\mathcal{L})^*$. Thus $\mathcal{M} \subseteq (\operatorname{Alg}\mathcal{L})^*$. Conversely, if $[S_{ij}] \in (\operatorname{Alg}\mathcal{L})^*$, then $S_{ij} = 0$ for $i < j$; hence $V_jV_j^* \otimes S_{ij} = 0 = V_iV_i^*V_jV_j^* \otimes S_{ij}$ for $i < j$. On the other hand, for $i \geq j$, we have $V_iV_i^* \geq V_jV_j^*$, so that $V_iV_i^*V_jV_j^* \otimes S_{ij} = V_jV_j^* \otimes S_{ij}$. This shows that $V_iV_i^*V_jV_j^* \otimes S_{ij} = V_jV_j^* \otimes S_{ij}$ for all $i, j \in \Lambda$, which is to say that $[S_{ij}] \in \mathcal{M}$. Thus we have shown our claim that $\mathcal{M} = (\operatorname{Alg}\mathcal{L})^*$.

$\Longleftarrow$ To prove the converse implication, assume that the collection $\{\mathcal{R}(V_i)\}_{i\in\Lambda}$ is a nest (hence Λ is a totally ordered set) such that completion $\overline{\mathcal{L}}$ of the nest $\mathcal{L} = \{\mathcal{L}_i; i \in \Lambda\}$ as in (2.3.13) is countable. Similar to the claim in (2.3.14), we note that $\mathcal{M} = (\operatorname{Alg}\mathcal{L})^*$. Since $\overline{\mathcal{L}}$ is countable, it follows from Theorem 1.5.13 that $\operatorname{Alg}\mathcal{L}$ has factorization in $\mathcal{B}(\mathcal{K})$, which is to say that $\mathcal{M}$ has factorization in $\mathcal{B}(\mathcal{K})$.

Now to show that $\oplus_{i\in\Lambda}\phi_i$ is C^*-extreme, we use Corollary 2.2.6. Let $\widetilde{D} = I_{\mathcal{H}_\pi} \otimes [D_{ij}]$ be a positive operator in $\rho(\mathcal{A})'$ such that $V^*\widetilde{D}V$ is invertible. We claim that $[D_{ij}]$ is invertible. Since $V^*\widetilde{D}V$ is invertible, there exists $\beta > 0$ such that $V^*\widetilde{D}V \geq \beta V^*V$. Then we have

$$0 \leq V^*\widetilde{D}V - \beta V^*V = [V_i^*V_j \otimes D_{ij}] - \beta[V_i^*V_i \otimes \delta_{ij}I_{\mathcal{K}_i}] = [V_i^*V_j \otimes (D_{ij} - \delta_{ij}\beta I_{\mathcal{K}_i})],$$

where δ_{ij} denotes the Kronecker delta. In particular, for every finite subset $\Lambda_0 \subseteq \Lambda$, we have

$$[V_i{}^* V_j \otimes (D_{ij} - \delta_{ij}\beta I_{\mathcal{K}_i})]_{i,j\in\Lambda_0} \geq 0. \tag{2.3.16}$$

Now fix a finite subset $\Lambda_0 \subseteq \Lambda$, and let $h_{\Lambda_0} \in \cap_{i\in\Lambda_0} \mathcal{R}(V_i)$ be a unit vector (which exists because the set $\{\mathcal{R}(V_i)\}_{i\in\Lambda_0}$ is finite and is totally ordered). Then there exist unit vectors $h_i \in \mathcal{H}_i$ such that $V_i h_i = h_{\Lambda_0}$ for each $i \in \Lambda_0$. So for any vector $k_i \in \mathcal{K}_i$, $i \in \Lambda_0$, it follows from (2.3.16) that

$$\begin{aligned}
0 &\leq \sum_{i,j\in\Lambda_0} \langle (V_i{}^* V_j \otimes (D_{ij} - \delta_{ij}\beta I_{\mathcal{K}_i})(h_j \otimes k_j), (h_i \otimes k_i)\rangle \\
&= \sum_{i,j\in\Lambda_0} \langle (V_i{}^* V_j h_j, h_i\rangle \langle (D_{ij} - \delta_{ij}\beta I_{\mathcal{K}_i})k_j, k_i\rangle = \sum_{i,j\in\Lambda_0} \langle V_j h_j, V_i h_i\rangle \langle (D_{ij} - \delta_{ij}\beta I_{\mathcal{K}_i})k_j, k_i\rangle \\
&= \sum_{i,j\in\Lambda_0} \langle h_{\Lambda_0}, h_{\Lambda_0}\rangle \langle (D_{ij} - \delta_{ij}\beta I_{\mathcal{K}_i})k_j, k_i\rangle = \sum_{i,j\in\Lambda_0} \langle (D_{ij} - \delta_{ij}\beta I_{\mathcal{K}_i})k_j, k_i\rangle .
\end{aligned}$$

Since $k_i \in \mathcal{K}_i$ for $i \in \Lambda_0$, is arbitrary, we conclude that $[(D_{ij} - \delta_{ij}\beta I_{\mathcal{K}_i})]_{i,j\in\Lambda_0} \geq 0$. Also since Λ_0 is an arbitrary finite subset of Λ, it follows that $[(D_{ij} - \delta_{ij}\beta I_{\mathcal{K}_i})] \geq 0$ in $\mathcal{B}(\mathcal{K})$; hence $[D_{ij}] \geq \beta I_{\mathcal{K}}$ proving our claim that $D = [D_{ij}]$ is invertible.

Therefore, as $\mathcal{M}$ has factorization in $\mathcal{B}(\mathcal{K})$, there is an invertible operator $S \in \mathcal{B}(\mathcal{K})$ such that $S, S^{-1} \in \mathcal{M}$ and $D = S^*S$. Set $\widetilde{S} = I_{\mathcal{H}_\pi} \otimes S$. Clearly $\widetilde{S} \in \rho(\mathcal{A})'$ and $\widetilde{D} = \widetilde{S}^*\widetilde{S}$. Since $S^{-1} \in \mathcal{M}$, it follows that $\widetilde{S}^{-1}VV^* = VV^*\widetilde{S}^{-1}VV^*$; hence we have

$$(V^*\widetilde{S}V)(V^*\widetilde{S}^{-1}V) = V^*\widetilde{S}(VV^*\widetilde{S}^{-1}VV^*)V = V^*\widetilde{S}(\widetilde{S}^{-1}VV^*)V = V^*(\widetilde{S}\widetilde{S}^{-1})(VV^*V) = V^*V = I_{\mathcal{H}}.$$

Likewise we get $(V^*\widetilde{S}^{-1}V)(V^*\widetilde{S}V) = V^*V = I_{\mathcal{H}}$. This shows that $V^*\widetilde{S}V$ is invertible. Thus for a given $\widetilde{D} \in \rho(\mathcal{A})'$ with $V^*\widetilde{D}V$ invertible, we have got $\widetilde{S} \in \rho(\mathcal{A})'$ such that $\widetilde{D} = \widetilde{S}^*\widetilde{S}$, $\widetilde{S}VV^* = VV^*\widetilde{S}VV^*$ and $V^*\widetilde{S}V$ is invertible. We now conclude from Corollary 2.2.6 that $\phi = \oplus_{i\in\Lambda}\phi_i$ is a C^*-extreme point in $S_{\mathcal{H}}(\mathcal{A})$. ☐

Combining Theorem 2.3.10 and Proposition 2.3.8 we have the following complete characterization of those C^*-extreme points which are direct sums of pure UCP maps.

Theorem 2.3.11. *Let ϕ be a direct sum of pure UCP maps in $S_{\mathcal{H}}(\mathcal{A})$, so that ϕ is unitarily equivalent to $\bigoplus_{\alpha\in\Gamma}\bigoplus_{i\in\Lambda_\alpha}\psi^i_\alpha(\cdot)\otimes I_{\mathcal{K}^i_\alpha}$, where $\mathcal{K}^i_\alpha$ is a Hilbert space and ψ^i_α is a pure UCP map with minimal Stinespring triple $(\pi_\alpha, V^i_\alpha, \mathcal{H}_{\pi_\alpha})$ such that ψ^i_α is non-unitarily equivalent to ψ^j_α for each $i \neq j$ in Λ_α, $\alpha \in \Gamma$, and π_α is disjoint to π_β for $\alpha \neq \beta$. Then ϕ is C^*-extreme in $S_{\mathcal{H}}(\mathcal{A})$ if and only if the following holds for each $\alpha \in \Gamma$:*

(i) $\{\mathcal{R}(V^i_\alpha)\}_{i\in\Lambda_\alpha}$ is a nest in $\mathcal{H}_{\pi_\alpha}$, which makes Λ_α a totally ordered set, and

(ii) if $\mathcal{L}^i_\alpha = \oplus_{j\leq i}\mathcal{K}^j_\alpha$ for $i \in \Lambda_\alpha$, then the completion of the nest $\{\mathcal{L}^i_\alpha\}_{i\in\Lambda_\alpha}$ in $\oplus_{i\in\Lambda_\alpha}\mathcal{K}^i_\alpha$ is countable.

Remark 2.3.12. Based on their results for finite dimensions, Farenick and Zhou in their remarks towards the end of [29] suggest that Condition (i) in Theorem 2.3.11 is perhaps sufficient, even in infinite dimensions, for a direct sum of pure UCP maps to be C^*-extreme. Here in this Theorem we observe that Condition (i) is to be supplemented with Condition (ii), which is a somewhat more delicate restriction and is a purely infinite dimensional phenomenon (see Example 3.3.2). It has no role to play in finite dimensions.

Some straightforward corollaries of Theorems 2.3.10 and Theorem 2.3.11 are given below.

Corollary 2.3.13. *Let $\phi = \oplus_{i\in\Lambda}\phi_i$ be a direct sum of pure UCP maps ϕ_i. If ϕ is C^*-extreme, then for each $i, j \in \Lambda$, either ϕ_i and ϕ_j are disjoint or one of $\{\phi_i, \phi_j\}$ is a compression of the other.*

Corollary 2.3.14. *Let $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a direct sum of pure UCP maps. Then $\phi \oplus \phi$ is a C^*-extreme point in $S_{\mathcal{H}\oplus\mathcal{H}}(\mathcal{A})$ if and only if ϕ is a C^*-extreme point in $S_{\mathcal{H}}(\mathcal{A})$.*

Remark 2.3.15. It is not known in general whether $\phi \oplus \phi$ is a C^*-extreme map if ϕ is a C^*-extreme map.

Since a finite nest containing $\{0\}, \mathcal{H}$ is always complete, we recover Theorem 2.3.4 of Farenick-Zhou using our result in Theorem 2.3.11 and the fact that all C^*-extreme maps in $S_{\mathbb{C}^n}(\mathcal{A})$ decompose as direct sums of pure UCP maps (Theorem 2.1, [28]).

If Λ is a subset of the set of integers $\mathbb{Z}$, and if $\mathcal{E} = \{E_n\}_{n\in\Lambda}$ is a nest in a Hilbert space $\mathcal{K}$ with the property that $E_n \subseteq E_m$ for $n < m$, then the completion of $\mathcal{E}$ is given by the nest $\mathcal{E} \cup \{0, \mathcal{K}, \vee_{n\in\Lambda}E_n, \wedge_{n\in\Lambda}E_n\}$, which is already countable. Thus the following corollary is immediate from Theorem 2.3.10.

Corollary 2.3.16. *Let $\Lambda = \mathbb{N}$ or $\mathbb{Z}$ or $\mathbb{Z}_-$, or $\{1, 2, \ldots, m\}$ for some $m \in \mathbb{N}$, and let $\phi_n : \mathcal{A} \to \mathcal{B}(\mathcal{H}_n)$ be a pure UCP map for $n \in \Lambda$. If ϕ_n is a compression of ϕ_{n+1} for each n with $n, n+1 \in \Lambda$, then the direct sum $\phi = \oplus_{n\in\Lambda}\phi_n$ is a C^*-extreme point in $S_{\mathcal{H}}(\mathcal{A})$, where $\mathcal{H} = \oplus_{n\in\Lambda}\mathcal{H}_n$.*

We end this section by giving a necessary and sufficient criteria for a direct sum of pure UCP maps to be extreme. Note that in view of Proposition 2.3.8, it is enough to consider direct sums of only those pure UCP maps which are compression of the same irreducible representation.

Proposition 2.3.17. *Let $\phi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_i)$, $i \in \Lambda$, be a family of pure UCP maps with respective minimal Stinespring triple $(\pi, V_i, \mathcal{H}_\pi)$. Then $\phi = \oplus_{i\in\Lambda}\phi_i$ is extreme in $S_{\oplus_{i\in\Lambda}\mathcal{H}_i}(\mathcal{A})$ if and only if $V_i{}^*V_j \neq 0$ for all $i, j \in \Lambda$.*

Proof. Set $\mathcal{H} = \oplus_{i\in\Lambda}\mathcal{H}_i$. Note that $(\rho, V, \mathcal{H}_\rho)$ is the minimal Stinespring triple for ϕ, where $\mathcal{H}_\rho = \oplus_{i\in\Lambda}\mathcal{H}_\pi$, $\rho = \oplus_{i\in\Lambda}\pi$ and $V = \oplus_{i\in\Lambda}V_i$. Since π is irreducible, $\pi(\mathcal{A})' = \mathbb{C}\cdot I_{\mathcal{H}_\pi}$; so it follows that

$$\rho(\mathcal{A})' = \{[\lambda_{ij} I_{\mathcal{H}_\pi}];\ \lambda_{ij} \in \mathbb{C}\} \subseteq \mathcal{B}(\oplus_{i\in\Lambda}\mathcal{H}_\pi).$$

First assume that ϕ is extreme in $S_{\mathcal{H}}(\mathcal{A})$, and fix $m, n \in \Lambda$. Let $\lambda \neq 0$ in $\mathbb{C}$. Consider the operator $T = [\lambda_{ij} I_{\mathcal{H}_\pi}] \in \rho(\mathcal{A})'$, where $\lambda_{mn} = \lambda$ and $\lambda_{ij} = 0$ otherwise. Then $T \neq 0$. Since ϕ is extreme, it follow from Arveson's extreme point condition (Theorem 1.2.22) that $V^*TV \neq 0$. But $V^*TV = [\lambda_{ij} V_i{}^*V_j]$, and since $\lambda_{ij} V_i{}^*V_j = 0$ for all $(i, j) \neq (m, n)$, it follows that $\lambda V_m^* V_n \neq 0$, showing that $V_m^* V_n \neq 0$.

Conversely, let $V_i{}^*V_j \neq 0$ for all $i, j \in \Lambda$. Let $T = [\lambda_{ij} I_{\mathcal{H}_\pi}] \in \rho(\mathcal{A})'$, $\lambda_{ij} \in \mathbb{C}$, be such that $V^*TV = 0$. Then for each $i, j \in \Lambda$, we have $\lambda_{ij} V_i{}^*V_j = 0$, which yields $\lambda_{ij} = 0$; hence $T = 0$. Again by extreme point condition of Arveson, we conclude that ϕ is extreme in $S_{\mathcal{H}}(\mathcal{A})$. ∎

The following corollary is another condition (along which Proposition 2.2.10) under which a C^*-extreme map is also extreme.

Corollary 2.3.18. *Let $\phi \in S_{\mathcal{H}}(\mathcal{A})$ decompose as a direct sum of pure UCP maps. If ϕ is a C^*-extreme point in $S_{\mathcal{H}}(\mathcal{A})$, then ϕ is also an extreme point in $S_{\mathcal{H}}(\mathcal{A})$.*

Proof. Let $\phi = \oplus_{i\in\Lambda}\phi_i$ for some pure UCP maps $\phi_i, i \in \Lambda$. By separating out disjoint UCP maps and then invoking Proposition 2.3.8 if needed, we assume without loss of generality that each ϕ_i is a compression of the same irreducible representation, say π. Let $(\pi, V_i, \mathcal{H}_\pi)$ be the minimal Stinespring triple for ϕ_i. Since ϕ is C^*-extreme, it follows from Theorem 2.3.10 that either $V_iV_i^* \geq V_jV_j^*$ or $V_jV_j^* \geq V_iV_i^*$ for all $i, j \in \Lambda$. In either case, it is immediate that ${V_i}^*V_j \neq 0$ for $i, j \in \Lambda$. The required assertion now follows from Proposition 2.3.17. □

2.4 Krein-Milman type theorem for UCP maps on separable C^*-algebras

The Krein-Milman theorem is a very important result in classical functional analysis, which says that in a locally convex space, a convex compact subset is closure of the convex hull of its extreme points. So it is desired to have an analogue of this theorem for C^*-convexity in the space $S_{\mathcal{H}}(\mathcal{A})$ equipped with an appropriate topology. We equip the set $S_{\mathcal{H}}(\mathcal{A})$ with bounded weak (BW) topology (see Definition 1.2.23). We know that $S_{\mathcal{H}}(\mathcal{A})$ is compact in BW-topology (Theorem 1.2.24).

Definition 2.4.1. The C^**-convex hull* of any subset $\mathcal{S}$ of $S_{\mathcal{H}}(\mathcal{A})$ is given by

$$\left\{\sum_{i=1}^{n} {T_i}^*\phi_i(\cdot)T_i;\ \phi_i \in \mathcal{S}, T_i \in \mathcal{B}(\mathcal{H}) \text{ with } \sum_{i=1}^{n} {T_i}^*T_i = I_{\mathcal{H}}\right\}. \tag{2.4.1}$$

So a generalized Krein-Milman theorem for $S_{\mathcal{H}}(\mathcal{A})$ would be to ask whether $S_{\mathcal{H}}(\mathcal{A})$ is the closure of the C^*-convex hull of its C^*-extreme points in BW-topology. In this section, we prove such Krein-Milman type theorem for $S_{\mathcal{H}}(\mathcal{A})$, whenever $\mathcal{A}$ is a separable C^*-algebra and $\mathcal{H}$ a separable Hilbert space. Also see the similar results for the case when $\mathcal{A}$ is of the form $\mathcal{B}(\mathcal{G})$ for some Hilbert space $\mathcal{G}$ (Theorem 3.2.2) or $\mathcal{A}$ is a commutative C^*-algebra (Theorem 5.4.10). The proof of these three cases are different. Note that $\mathcal{B}(\mathcal{G})$ is not separable, when $\mathcal{G}$ is infinite dimensional and $C(X)$ is non-separable when X is non metrizable compact space. We still don't know the result in full generality (i.e. for non-separable C^*-algebras).

We mention here that such theorem can be found in [28] in the case when $\mathcal{H}$ is a finite-dimensional Hilbert space and $\mathcal{A}$ is an arbitrary C^*-algebra. Thus our result provides an important development towards this theorem in infinite dimensional Hilbert space settings.

Lemma 2.4.2. *Let $\phi \in S_{\mathcal{H}}(\mathcal{A})$ be such that $\phi(a) = \sum_{n\geq 1}\phi_n(a)$ in WOT, for all $a \in \mathcal{A}$, where $\{\phi_n : \mathcal{A} \to \mathcal{B}(\mathcal{H})\}_{n\geq 1}$ is a countable family of pure CP maps. Then ϕ is in the BW-closure of C^*-convex hull of C^*-extreme points of $S_{\mathcal{H}}(\mathcal{A})$.*

Proof. We assume that the collection $\{\phi_n\}_{n\geq 1}$ in the sum of ϕ is countably infinite. The finite case follows similarly and easily. For each $n \geq 1$, let $(\pi_n, V_n, \mathcal{K}_n)$ be the minimal Stinespring triple for ϕ_n. Then each π_n is irreducible, as ϕ_n is pure by hypothesis. Note that

$$\sum_{n\geq 1} V_n^* V_n = \sum_{n\geq 1} \phi_n(1) = \phi(1) = I_{\mathcal{H}}, \quad \text{in WOT.}$$

Set $A_n = V_n^* V_n \in \mathcal{B}(\mathcal{H})$, and let $V_n = W_n A_n^{1/2}$ be the polar decomposition of V_n. Here $W_n \in \mathcal{B}(\mathcal{H}, \mathcal{K}_n)$ is the partial isometry with initial space $\overline{\mathcal{R}(A_n^{1/2})}$ and final space $\overline{\mathcal{R}(V_n)}$. Define the map $\zeta_n : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by

$$\zeta_n(a) = W_n^* \pi_n(a) W_n \quad \text{for all} \quad a \in \mathcal{A}.$$

It is immediate to verify that ζ_n is a completely positive map with the minimal Stinespring triple $(\pi_n, W_n, \mathcal{K}_n)$. Let $\theta_n : \mathcal{A} \to \mathbb{C}$ be a pure state that is a compression of ζ_n (e.g. take a unit vector $e_n \in \mathcal{R}(W_n)$ and define $\theta_n(a) = \langle e_n, \pi_n(a) e_n \rangle$ for all $a \in \mathcal{A}$). Now we define $\xi_n : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by

$$\xi_n = \zeta_n + (1 - P_n)\theta_n,$$

where $P_n = W_n^* W_n$ is the projection from $\mathcal{H}$ onto $\overline{\mathcal{R}(A_n^{1/2})}$. Note that ξ_n is a UCP map from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$. If we set $U_n = W_{n|_{\mathcal{R}(P_n)}}$ (so that U_n is an isometry from $\mathcal{R}(P_n)$ to $\mathcal{K}_n$), then it is straightforward to verify that ξ_n is unitarily equivalent to the UCP map $\widetilde{\xi}_n : \mathcal{A} \to \mathcal{B}(\mathcal{R}(P_n) \oplus \mathcal{R}(P_n^\perp))$ given by

$$\widetilde{\xi}_n(a) = U_n^* \pi_n(a) U_n \oplus \theta(a) I_{\mathcal{R}(P_n^\perp)}, \quad \text{for all} \quad a \in \mathcal{A}.$$

Since θ_n is a compression of the map $a \mapsto U_n^* \pi_n(a) U_n$ (which is pure, as π_n is irreducible), it follows from Theorem 2.3.10 that $\widetilde{\xi}_n$ is C^*-extreme in $S_{\mathcal{R}(P_n)\oplus\mathcal{R}(P_n^\perp)}(\mathcal{A})$; hence ξ_n is C^*-extreme in $S_{\mathcal{H}}(\mathcal{A})$.

Now set $B_n = I_{\mathcal{H}} - \sum_{j=1}^n A_j$. Since $\sum_{n\geq 1} A_n = \sum_{n\geq 1} V_n^* V_n = I_{\mathcal{H}}$ in WOT; it follows that $B_n \geq 0$, and $B_n \to 0$ in WOT as $n \to \infty$. Now fix a C^*-extreme point ξ in $S_{\mathcal{H}}(\mathcal{A})$ and define the map $\psi_n : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by

$$\psi_n(a) = B_n^{1/2} \xi(a) B_n^{1/2} + \sum_{j=1}^n {A_j}^{1/2} \xi_j(a) {A_j}^{1/2}, \quad \text{for all} \quad a \in \mathcal{A}.$$

It is clear that each ψ_n is a UCP map such that ψ_n is a C^*-convex combination of C^*-extreme points of $S_{\mathcal{H}}(\mathcal{A})$. Since $B_n \to 0$ in WOT, it follows that $B_n^{1/2} \to 0$ in SOT; hence $B_n^{1/2}\xi(a)B_n^{1/2} \to 0$ in WOT for all $a \in \mathcal{A}$. This implies that

$$\lim_{n\to\infty} \psi_n(a) = \sum_{j=1}^\infty {A_j}^{1/2} \xi_j(a) {A_j}^{1/2} \quad \text{in WOT, for all } a \in \mathcal{A}.$$

Note that ${A_j}^{1/2}(I - P_j) = 0$ for all j. Hence for all $a \in \mathcal{A}$, we get ${A_j}^{1/2}\xi_j(a){A_j}^{1/2} = {A_j}^{1/2}\zeta_j(a){A_j}^{1/2}$, which further yields in WOT convergence

$$\lim_{n\to\infty} \psi_n(a) = \sum_{j=1}^\infty {A_j}^{1/2}\zeta_j(a){A_j}^{1/2} = \sum_{j=1}^\infty {A_j}^{1/2}{W_j}^*\pi_j(a)W_j{A_j}^{1/2} = \sum_{j=1}^\infty {V_j}^*\pi_j(a)V_j = \sum_{j=1}^\infty \phi_j(a) = \phi(a).$$

In other words, $\psi_n \to \phi$ in BW-topology. Thus we have approximated ϕ in BW-topology by a sequence ψ_n belonging to the C^*convex hull of C^*-extreme points of $S_{\mathcal{H}}(\mathcal{A})$. $\square$

The following is a Krein-Milman type theorem for UCP maps on separable C^*-algebras.

Theorem 2.4.3. *Let $\mathcal{A}$ be a separable C^*-algebra, and $\mathcal{H}$ a separable Hilbert space. Then $S_{\mathcal{H}}(\mathcal{A})$ is BW-closure of C^*-convex hull of its C^*-extreme points.*

Proof. Let $\phi \in S_{\mathcal{H}}(\mathcal{A})$, and let $(\pi, V, \mathcal{H}_\pi)$ be its minimal Stinespring triple. Since both $\mathcal{A}$ and $\mathcal{H}$ are separable, the Hilbert space $\mathcal{H}_\pi$ is also separable (see Remark 1.2.7). By a corollary of Voiculescu's theorem (see Theorem 42.1, [16]), there exists a sequence $\{U_n\}$ of unitaries on $\mathcal{H}_\pi$ and a representation $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi)$ such that ρ is a direct sum of irreducible representations and

$$\pi(a) = \lim_{n\to\infty} U_n^* \rho(a) U_n \quad \text{in WOT},$$

for all $a \in \mathcal{A}$. Therefore if we set $W_n = U_n V$, then each W_n is an isometry, and $\phi(a) = \lim_{n\to\infty} W_n^* \rho(a) W_n$ in WOT for all $a \in \mathcal{A}$. In other words, ϕ is approximated in BW-topology by UCP maps, all of which are compression of the representation ρ that is a direct sum of irreducible representations. Thus without loss of generality, we assume that π itself is a direct sum of a finite or countable irreducible representations, say,

$$\pi = \oplus_{n\geq 1} \pi_n, \tag{2.4.2}$$

where $\pi_n : \mathcal{A} \to \mathcal{B}(\mathcal{K}_n)$ is an irreducible representation on some Hilbert space $\mathcal{K}_n$. Now for each $n \geq 1$, let Q_n denote the projection of $\mathcal{H}_\pi$ onto $\mathcal{K}_n$, and let $V_n = Q_n V \in \mathcal{B}(\mathcal{H}, \mathcal{K}_n)$. Consider the completely positive map $\phi_n : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ defined by $\phi_n(a) = V_n^* \pi_n(a) V_n$ for all $a \in \mathcal{A}$. Since π_n is irreducible, each ϕ_n is a pure CP map. Also note that in WOT convergence, we have

$$\begin{aligned}\sum_{n\geq 1} \phi_n(a) &= \sum_{n\geq 1} V^* Q_n \pi_n(a) Q_n V = V^*(\sum_{n\geq 1} Q_n \pi_n(a) Q_n) V \\ &= V^*(\oplus_{n\geq 1} \pi_n(a)) V = V^* \pi(a) V = \phi(a),\end{aligned}$$

for all $a \in \mathcal{A}$. The required assertion that ϕ is in BW-closure of C^*-convex hull of C^*-extreme points of $S_{\mathcal{H}}(\mathcal{A})$ now follows from Lemma 2.4.2. $\square$

2.5 Examples and applications

In the final section, we discuss a number of examples of UCP maps with their C^*-extremity properties. We shall also see an application to a well-known result from classical functional analysis about factorization property of Hardy algebras. We believe that the connection between C^*-extreme points and factorization property of the algebra $\mathcal{M}$ as in Corollary 2.2.9 will produce many more examples and applications. Also see Section 3.3 for more examples of C^*-extreme maps.

First we look into the question of when tensor products of two C^*-extreme maps are C^*-extreme. This will help us in producing more C^*-extreme maps out of the existing ones. The tensor product in question is minimal tensor product. See Definition 1.1.12 for the notion of minimal tensor products of C^*-algebras and Definition 1.2.13 for tensor product of UCP maps.

The following proposition talks about C^*-extremity of tensor products, where one of the components is pure.

Proposition 2.5.1. *Let $\phi_i : \mathcal{A}_i \to \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$, be two UCP maps, and let ϕ_2 be pure. Then ϕ_1 is C^*-extreme (resp. extreme) in $S_{\mathcal{H}_1}(\mathcal{A}_1)$ if and only if $\phi_1 \otimes \phi_2$ is C^*-extreme (resp. extreme) in $S_{\mathcal{H}_1 \otimes \mathcal{H}_2}(\mathcal{A}_1 \otimes \mathcal{A}_2)$.*

Proof. Let $(\pi_i, V_i, \mathcal{K}_i)$ be the minimal Stinespring triple of ϕ_i for $i = 1, 2$. Then it is immediate that $(\pi_1 \otimes \pi_2, V_1 \otimes V_2, \mathcal{K}_1 \otimes \mathcal{K}_2)$ is the minimal Stinespring triple for $\phi_1 \otimes \phi_2$. Set $\pi = \pi_1 \otimes \pi_2$. Note that since $\pi_2(\mathcal{A}_2)' = \mathbb{C} \cdot I_{\mathcal{K}_2}$ (as ϕ_2 is pure), it follows from Theorem 1.1.22 that

$$\pi(\mathcal{A})' = (\pi_1(\mathcal{A}_1) \otimes \pi_2(\mathcal{A}_2))' = \pi_1(\mathcal{A})' \overline{\otimes} I_{\mathcal{K}_2} = \pi_1(\mathcal{A})' \otimes I_{\mathcal{K}_2}.$$

Now for any operator $D = D_1 \otimes I_{\mathcal{K}_2} \in \pi(\mathcal{A})'$, we note that D_1 is positive and $V_1^* D_1 V_1$ is invertible if and only if $D_1 \otimes I_{\mathcal{K}_2}$ is positive and $(V_1 \otimes V_2)^*(D_1 \otimes I_{\mathcal{K}_2})(V_1 \otimes V_2)$ is invertible. Also $D_1(V_1\mathcal{H}_1) \subseteq V_1\mathcal{H}_1$ if and only if $(D_1 \otimes I_{\mathcal{K}_2})(V_1 \otimes V_2)(\mathcal{H}_1 \otimes \mathcal{H}_2) \subseteq (V_1 \otimes V_2)(\mathcal{H}_1 \otimes \mathcal{H}_2)$. The assertion about equivalence of C^*-extreme points now follows from equivalent criteria in Corollary 2.2.6. The assertions about extreme points follow similarly using Extreme point condition (Theorem 1.2.22). ☐

Since the identity representation $\mathrm{id}_n : M_n \to M_n$ is pure, the following corollary about ampliation of a C^*-extreme map is immediate.

Corollary 2.5.2. *Let ϕ be a C^*-extreme point in $S_{\mathcal{H}}(\mathcal{A})$. Then the map $\phi \otimes \mathrm{id}_n : \mathcal{A} \otimes M_n \to \mathcal{B}(\mathcal{H} \otimes \mathbb{C}^n)$ is C^*-extreme in $S_{\mathcal{H} \otimes \mathbb{C}^n}(\mathcal{A} \otimes M_n)$, for each $n \in \mathbb{N}$.*

For the next result, we set up some notations. Let X be a countable set. For any Hilbert space $\mathcal{H}$ and a von Neumann algebra $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$, we consider the Hilbert space $\ell^2_{\mathcal{H}}(X)$ and von Neumann algebra $\ell^\infty_{\mathcal{B}}(X)$ given by

$$\ell^2_{\mathcal{H}}(X) = \{f : X \to \mathcal{H}; \Sigma_{x \in X} \|f(x)\|^2 < \infty\}, \quad \text{and} \quad \ell^\infty_{\mathcal{B}}(X) = \{F : X \to \mathcal{B}; F \text{ is bounded}\}.$$

Then $\ell^\infty_{\mathcal{B}}(X)$ acts on the Hilbert space $\ell^2_{\mathcal{H}}(X)$ via the operator $M_F, F \in \ell^\infty_{\mathcal{B}}(X)$, defined by

$$M_F f(x) = F(x) f(x) \quad \text{for} \quad f \in \ell^2_{\mathcal{H}}(X) \quad \text{and} \quad x \in X.$$

We write $\ell^2_{\mathbb{C}}(X)$ and $\ell^\infty_{\mathbb{C}}(X)$ simply by $\ell^2(X)$ and $\ell^\infty(X)$ respectively. Further we identify the Hilbert space $\ell^2(X) \otimes \mathcal{H}$ with $\ell^2_{\mathcal{H}}(X)$ via the map $f \otimes h \mapsto (x \mapsto f(x)h)$ for $f \in \ell^2(X)$ and $h \in \mathcal{H}$. Also the algebra $\ell^\infty(X) \overline{\otimes} \mathcal{B}$ is $*$-isomorphic to $\ell^\infty_{\mathcal{B}}(X)$ with isomorphism given by $f \otimes T \mapsto (x \mapsto f(x)T)$ for $f \in \ell^\infty(X)$ and $T \in \mathcal{B}$ (here $\mathcal{B}_1 \overline{\otimes} \mathcal{B}_2$ denotes the von Neumann algebra generated by the minimal tensor product $\mathcal{B}_1 \otimes \mathcal{B}_2$; see Section 1.1). If there is no possibility of confusion, we shall drop X from $\ell^2(X), \ell^2_{\mathcal{H}}(X)$ etc.

Proposition 2.5.3. *Let ϕ be a C^*-extreme point in $S_{\mathcal{H}}(\mathcal{A})$, and let $i : \ell^\infty(X) \to \mathcal{B}(\ell^2(X))$ be the natural inclusion map for some countable set X. Then $i \otimes \phi$ is C^*-extreme in $S_{\ell^2 \otimes \mathcal{H}}(\ell^\infty \otimes \mathcal{A})$.*

Proof. Let $(\pi, V, \mathcal{H}_\pi)$ be the minimal Stinespring triple for ϕ. Then $(\rho, U, \mathcal{H}_\rho)$ is the minimal Stinespring triple for $i \otimes \phi$, where

$$\mathcal{H}_\rho = \ell^2 \otimes \mathcal{H}_\pi = \ell^2_{\mathcal{H}_\pi}, \; U = i \otimes V : \ell^2 \otimes \mathcal{H} \to \ell^2 \otimes \mathcal{H}_\pi, \quad \text{and} \quad \rho = i \otimes \pi.$$

We know from Theorem 1.1.22 that

$$\rho(\ell^\infty \otimes \mathcal{A})' = (\ell^\infty \otimes \pi(\mathcal{A}))' = \ell^\infty \overline{\otimes} \pi(\mathcal{A})' = \ell^\infty_{\pi(\mathcal{A})'}.$$

Now let $M_D \in \ell^\infty_{\pi(\mathcal{A})'}$ be a positive operator such that $U^* M_D U$ is invertible. Then there exists $\alpha > 0$ such that $U^* M_D U \geq \alpha U^* U$. Note that for any $f \in \ell^2_{\mathcal{H}}$ and $x \in X$, we have

$$U^* M_D U f(x) = (V^* D(x) V) f(x).$$

Therefore for any unit vectors $g \in \ell^2$ and $h \in \mathcal{H}$, we have

$$\alpha \leq \langle U^* M_D U(g \otimes h), g \otimes h\rangle = \sum_{x \in X} \langle (V^* D(x) V) g(x) h, g(x) h\rangle = \sum_{x \in X} \langle (V^* D(x) V) h, h\rangle \, |g(x)|^2,$$

and since $g \in \ell^2$ varies over all unit vectors, it follows (by choosing g to be the canonical basis elements of ℓ^2) that $\langle (V^* D(x) V) h, h\rangle \geq \alpha$ for all $x \in X$. Again since $h \in \mathcal{H}$ is arbitrary, it follows that $V^* D(x) V \geq \alpha$ for all $x \in X$, i.e. $V^* D(x) V$ is invertible in $\mathcal{B}(\mathcal{H})$. Since ϕ is C^*-extreme in $S_{\mathcal{H}}(\mathcal{A})$, there exists an operator $S(x) \in \pi(\mathcal{A})'$ for each $x \in X$, such that

$$D(x) = S(x)^* S(x), \quad S(x) V V^* = V V^* S(x) V V^*$$

and $V^* S(x) V$ is invertible. Also note that $\|(V^* S(x) V)^{-1}\|^2 = \|(V^* D(x) V)^{-1}\| \leq 1/\alpha$. If S denotes the map $x \mapsto S(x)$ from X to $\pi(\mathcal{A})'$, then it is immediate to verify that $S \in \ell^\infty_{\pi(\mathcal{A})'}$ such that

$$M_D = M_S^* M_S \ \text{ and } \ M_S U U^* = U U^* M_S U U^*.$$

Also since $\sup_{x \in X} \|(V^* S(x) V)^{-1}\| \leq 1/\alpha$, it follows that $U^* M_S U$ is invertible. Since M_D is arbitrary, we conclude that $i \otimes \phi$ is C^*-extreme. □

If the set X in Proposition 2.5.3 is a two point set, then we get the following (note that the map ψ in the following corollary is different than $\phi \oplus \phi$):

Corollary 2.5.4. *Let ϕ be a C^*-extreme point in $S_{\mathcal{H}}(\mathcal{A})$. Then the map $\psi : \mathcal{A} \oplus \mathcal{A} \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ defined by $\psi(a \oplus b) = \phi(a) \oplus \phi(b)$, for all $a, b \in \mathcal{A}$, is a C^*-extreme point in $S_{\mathcal{H} \oplus \mathcal{H}}(\mathcal{A} \oplus \mathcal{A})$.*

The next result provides a family of C^*-extreme points, which can be thought as a generalization of Example 2.1.5, and whose proof follows almost the same lines. We give the proof for the sake of completeness. For doing so, we need some facts from C^*-convexity of unit ball of $\mathcal{B}(\mathcal{H})$ which we recall below. See [41, 49] for more details on these topics.

We say a contraction $S \in \mathcal{B}(\mathcal{H})$ is a *C^*-extreme point* of the closed unit ball of $\mathcal{B}(\mathcal{H})$ if, whenever $S = \sum_{i=1}^n {T_i}^* S_i T_i$ for contractions S_i and invertibles T_i in $\mathcal{B}(\mathcal{H})$, $1 \leq i \leq n$, with $\sum_{i=1}^n {T_i}^* T_i = I_{\mathcal{H}}$, then there are unitaries $U_i \in \mathcal{B}(\mathcal{H})$ such that $S = {U_i}^* S_i U_i$. The following result is very crucial for our purpose:

Proposition 2.5.5 (Theorem 1.1, [41])**.** *All isometries and co-isometries are C^*-extreme points of closed unit ball of $\mathcal{B}(\mathcal{H})$.*

Recall that $C^*(T)$ denotes the unital C^*-algebra generated by an operator T. The following proposition provides a number of C^*-extreme UCP maps on C^*-algebras generated by unitaries.

Proposition 2.5.6. *Let S be a unitary, and let $\phi : C^*(S) \to \mathcal{B}(\mathcal{H})$ be a UCP map such that $\phi(S)$ is an isometry or a co-isometry. Then ϕ is C^*-extreme as well as extreme in $S_{\mathcal{H}}(C^*(S))$.*

Proof. We assume that $\phi(S)$ is an isometry. The case of $\phi(S)$ a co-isometry follows similarly. Let $(\pi, V, \mathcal{H}_\pi)$ be the minimal Stinespring triple for ϕ. Since $\phi(S)$ is an isometry, we have $\phi(S)^*\phi(S) = I_{\mathcal{H}} = \phi(1) = \phi(S^*S)$, so it follows from Proposition 1.2.10 that $V\phi(S) = \pi(S)V$. This in particular implies for each $n \in \mathbb{N}$ that $V\phi(S)^n = \pi(S)^n V$, which yields

$$\phi(S)^n = V^*\pi(S)^n V = V^*\pi(S^n)V = \phi(S^n). \tag{2.5.1}$$

Now to prove that ϕ is C^*-extreme in $S_{\mathcal{H}}(C^*(S))$, let $\phi = \sum_{i=1}^n {T_i}^*\phi_i(\cdot)T_i$ be a proper C^*-convex combination for some UCP maps ϕ_i and invertible operators $T_i \in \mathcal{B}(\mathcal{H})$ with $\sum_{i=1}^n {T_i}^*T_i = I_{\mathcal{H}}$. Since $\phi(S)$ is an isometry, it is a C^*-extreme point in the closed unit ball of $\mathcal{B}(\mathcal{H})$ (Proposition 2.5.5); hence for each i, there exists a unitary $U_i \in \mathcal{B}(\mathcal{H})$ satisfying $\phi(S) = {U_i}^*\phi_i(S)U_i$. This implies that each $\phi_i(S)$ is an isometry, and in a similar fashion as in (2.5.1), we get

$$\phi_i(S)^n = \phi_i(S^n) \quad \text{for all } n \in \mathbb{N}. \tag{2.5.2}$$

Thus for each $n \in \mathbb{N}$, we use (2.5.1) and (2.5.2) to get

$$\phi(S^n) = \phi(S)^n = ({U_i}^*\phi_i(S)U_i)^n = {U_i}^*\phi_i(S)^n U_i = {U_i}^*\phi_i(S^n)U_i.$$

By taking adjoint both the sides, we also have $\phi(S^{*n}) = {U_i}^*\phi_i(S^{*n})U_i$. Since S is unitary, it follows that $\overline{\text{span}}\{S^n, S^{*m}; n, m \in \mathbb{N}\} = C^*(S)$. Thus we conclude that $\phi(T) = {U_i}^*\phi_i(T)U_i$ for every $T \in C^*(S)$ i.e. ϕ is unitarily equivalent to ϕ_i. The case of ϕ being extreme follows on similar lines, as isometries and co-isometries are extreme points of the closed unit ball of $\mathcal{B}(\mathcal{H})$. $\square$

As a special case of Proposition 2.5.6, we have the following result. Here $z \in C(\mathbb{T})$ is the function on the unit circle $\mathbb{T}$ given by $z(e^{i\theta}) = e^{i\theta}$ for $\theta \in \mathbb{R}$.

Corollary 2.5.7. *Let $\phi : C(\mathbb{T}) \to \mathcal{B}(\mathcal{H})$ be a UCP map such that $\phi(z)$ is an isometry or a co-isometry. Then ϕ is C^*-extreme as well as extreme in $S_{\mathcal{H}}(C(\mathbb{T}))$.*

As an application of Proposition 2.5.7, we give a new proof of a classical result of Szegö and its operator-valued analogue about factorization property of Hardy algebras. Let $\mathcal{K}$ be a Hilbert space (possibly infinite dimensional), and let $L^2_{\mathcal{K}}(\mathbb{T})$ denote the Hilbert space of $\mathcal{K}$-valued square integrable functions on $\mathbb{T}$ with respect to one-dimensional Lebesgue measure i.e.

$$L^2_{\mathcal{K}}(\mathbb{T}) = \{f : \mathbb{T} \to \mathcal{K}; f \text{ is measurable and } \int_0^{2\pi} \|f(e^{i\theta})\|^2 d\theta < \infty\}.$$

Note that $L^2_{\mathcal{K}}(\mathbb{T})$ is isomorphic to $L^2(\mathbb{T}) \otimes \mathcal{K}$. Let $H^2_{\mathcal{K}}(\mathbb{T})$ denote the vector-valued Hardy subspace of $L^2_{\mathcal{K}}(\mathbb{T})$ given by

$$H^2_{\mathcal{K}}(\mathbb{T}) = \{f \in L^2_{\mathcal{K}}(\mathbb{T}); \int_0^{2\pi} f(e^{i\theta})e^{-in\theta} d\theta = 0 \text{ for all } n < 0\}.$$

Consider the von Neumann algebra of all essentially bounded measurable functions i.e.

$$L^\infty_{\mathcal{B}(\mathcal{K})}(\mathbb{T}) = \{F : \mathbb{T} \to \mathcal{B}(\mathcal{K}); z \mapsto \|F(z)\| \text{ is essentially bounded}\},$$

which acts on $L^2_{\mathcal{K}}(\mathbb{T})$ by left multiplication i.e. for $F \in L^\infty_{\mathcal{B}(\mathcal{K})}(\mathbb{T})$, the operator $M_F : L^2_{\mathcal{K}}(\mathbb{T}) \to L^2_{\mathcal{K}}(\mathbb{T})$ is defined by

$$M_F f(x) = F(x)f(x), \quad \text{for all } f \in L^2_{\mathcal{K}}(\mathbb{T}), x \in \mathbb{T}.$$

Let $H^\infty_{\mathcal{B}(\mathcal{K})}(\mathbb{T})$ be its subalgebra defined by

$$H^\infty_{\mathcal{B}(\mathcal{K})}(\mathbb{T}) = \{F \in L^\infty_{\mathcal{B}(\mathcal{K})}(\mathbb{T}); \int_0^{2\pi} F(e^{i\theta})e^{-in\theta} d\theta = 0 \text{ for all } n < 0\}.$$

Note that $C(\mathbb{T}) \subseteq L^\infty_{\mathcal{B}(\mathcal{K})}(\mathbb{T})$. We have the following factorization property of $H^\infty_{\mathcal{B}(\mathcal{K})}(\mathbb{T})$ in $L^\infty_{\mathcal{B}(\mathcal{K})}(\mathbb{T})$.

Corollary 2.5.8. *For any positive and invertible $D \in L^\infty_{\mathcal{B}(\mathcal{K})}(\mathbb{T})$, there exists an invertible S with $S, S^{-1} \in H^\infty_{\mathcal{B}(\mathcal{K})}(\mathbb{T})$ such that $D = S^*S$. That is, $H^\infty_{\mathcal{B}(\mathcal{K})}(\mathbb{T})$ has factorization in $L^\infty_{\mathcal{B}(\mathcal{K})}(\mathbb{T})$.*

Proof. Consider the UCP map $\phi : C(\mathbb{T}) \to \mathcal{B}(H^2_{\mathcal{K}}(\mathbb{T}))$ defined by

$$\phi(f) = P_{H^2_{\mathcal{K}}(\mathbb{T})} M_f|_{H^2_{\mathcal{K}}(\mathbb{T})}, \quad \text{for all } f \in C(\mathbb{T}). \tag{2.5.3}$$

Clearly $\phi(z)$ is an isometry, so it follows from Corollary 2.5.7 that ϕ is a C^*-extreme point in $S_{H^2_{\mathcal{K}}(\mathbb{T})}(\mathbb{C}(\mathbb{T}))$. Note that the map ϕ is already in minimal Stinespring form, where the representation π acts on the Hilbert space $L^2_{\mathcal{K}}(\mathbb{T})$ by $\pi(f) = M_f$, for all $f \in C(\mathbb{T})$. It is well-known that $\pi(C(\mathbb{T}))' = L^\infty_{\mathcal{B}(\mathcal{K})}(\mathbb{T})$ (Theorem 52.8, [16]), and it is easy to verify that

$$H^\infty_{\mathcal{B}(\mathcal{K})}(\mathbb{T}) = \{F \in L^\infty_{\mathcal{B}(\mathcal{K})}(\mathbb{T}); M_F(H^2_{\mathcal{K}}(\mathbb{T})) \subseteq H^2_{\mathcal{K}}(\mathbb{T})\}.$$

The required assertion now follows from Corollary 2.2.9. □

Example 2.5.9. Let $T \in \mathcal{B}(\mathcal{H})$ be an isometry or a co-isometry. Consider the linear map $\phi : \mathbb{C}(\mathbb{T}) \to \mathcal{B}(\mathcal{H})$ satisfying $\phi(p + \bar{q}) = p(T) + q(T)^*$ for polynomials p and q. Then ϕ extends to a UCP map on $C(\mathbb{T})$ (Theorem 2.6, [61]), and it follows from Proposition 2.5.7 that ϕ is a C^*-extreme point in $S_{\mathcal{H}}(C(\mathbb{T}))$.

Following is an example ϕ of a C^*-extreme map of $S_{\mathcal{H}}(C(\mathbb{T}))$ which says that $\phi(z)$ need not be an isometry or a co-isometry.

Example 2.5.10. Let $g : \mathbb{T} \to \mathbb{T}$ be a homeomorphism, and let $\phi : C(\mathbb{T}) \to \mathcal{B}(\mathcal{H})$ be a UCP map. Set $\psi : C(\mathbb{T}) \to \mathcal{B}(\mathcal{H})$ by $\psi(f) = \phi(f \circ g)$ for all $f \in C(\mathbb{T})$. Then it is easy to verify that ϕ is C^*-extreme in $S_{\mathcal{H}}(C(\mathbb{T}))$ if and only if ψ is C^*-extreme in $S_{\mathcal{H}}(C(\mathbb{T}))$. Moreover one can choose a homeomorphism f such that $\phi(z)$ is an isometry but $\psi(z)$ is neither an isometry nor a co-isometry.

Chapter 3

Normal C^*-extreme Maps

Our attention now shifts towards the study of structure of normal C^*-extreme maps on von Neumann algebras, specifically on type I factors (i.e. $\mathcal{B}(\mathcal{G})$ for some Hilbert space $\mathcal{G}$). Normal UCP maps play an integral part in understanding various objects in von Neumann algebra theory. As is well-known, normal representations are nothing but multiplicities of the identity representation, so normal UCP maps are compression of such maps. We exploit this special form in studying their C^*-extremity conditions.

In this chapter, we first see some basic properties and examples of normal C^*-extreme maps. The set of normal UCP maps itself forms a C^*-convex set and hence its C^*-extreme points can similarly be defined and studied. However it is observed below that this is same as analysing normal C^*-extreme maps of the set $S_{\mathcal{H}}(\mathcal{B}(\mathcal{G}))$ of all UCP maps on $\mathcal{B}(\mathcal{G})$. One useful observation that we come across is that the conditions of C^*-extremity of normal UCP maps can be translated to certain properties of subspaces of tensor products of two Hilbert spaces.

All the examples of normal C^*-extreme maps on $\mathcal{B}(\mathcal{G})$ that we know are direct sums of pure normal UCP maps. The main result (Theorem 3.1.6) determines necessary and sufficient criteria for normal C^*-extreme maps on $\mathcal{B}(\mathcal{G})$ to be direct sum of normal pure UCP maps. This criteria surprisingly involves the notion of reflexivity of associated algebras of type I factors and their factorization properties. The study of algebras satisfying factorization property and their lattices of invariant subspaces has an independent interest of its own. We undertake a detailed investigation in Chapter 6 of such algebras through more general notion called logmodular algebras. Further we prove a Krein-Milman type theorem for UCP maps on type I factors, continuing our previous result for separable C^*-algebras. We also provide some examples of normal C^*-extreme maps.

3.1 Normal C^*-extreme maps on type I factors

Let $\mathcal{B} \subseteq \mathcal{B}(\mathcal{G})$ be a von Neumann algebra, and let $NS_{\mathcal{H}}(\mathcal{B})$ denote the collection of all normal UCP maps from $\mathcal{B}$ to $\mathcal{B}(\mathcal{H})$. See Section 1.2 for definitions and structure of normal UCP maps.

We note that if $\phi : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ is a normal CP map, then $T^*\phi(\cdot)T$ is also normal for any

$T \in \mathcal{B}(\mathcal{H})$. It then follows that $NS_{\mathcal{H}}(\mathcal{B})$ itself is a C^*-convex set i.e.

$$\sum_{i=1}^{n} T_i{}^* \phi_i(\cdot) T_i \in NS_{\mathcal{H}}(\mathcal{B})$$

whenever $\phi_i \in NS_{\mathcal{H}}(\mathcal{B})$ and $T_i \in \mathcal{B}(\mathcal{H})$, $1 \leq i \leq n$, with $\sum_{i=1}^{n} T_i{}^* T_i = I_{\mathcal{H}}$. Therefore, one can define and study C^*-extreme points of $NS_{\mathcal{H}}(\mathcal{B})$ on the same lines of Definition 2.1.2, and look into its structure.

Having said that, we however see below (Proposition 3.1.2) that any normal UCP map on $\mathcal{B}$ is C^*-extreme in $NS_{\mathcal{H}}(\mathcal{B})$ if and only if it is C^*-extreme in $S_{\mathcal{H}}(\mathcal{B})$. Therefore, it does not matter whether we explore C^*-extremity condition in the set $NS_{\mathcal{H}}(\mathcal{B})$ or the set $S_{\mathcal{H}}(\mathcal{B})$.

Lemma 3.1.1. *Let $\phi, \psi : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ be two CP maps such that $\psi \leq \phi$. If ϕ is normal, then ψ is normal.*

Proof. Let $\{X_i\}$ be a net of decreasing positive elements in $\mathcal{B}$ such that $X_i \downarrow 0$ in SOT. Then $\phi(X_i) \to 0$ in SOT, as ϕ is normal. As ψ is positive, we note that $\{\psi(X_i)\}$ is a decreasing net of positive elements; hence $\psi(X_i) \to Y$ in SOT for some positive operator $Y \in \mathcal{B}(\mathcal{H})$. But since $\psi(X_i) \leq \phi(X_i)$ for all i, it follows by taking limit in SOT that $Y \leq 0$; hence $Y = 0$. $\square$

Proposition 3.1.2. *A normal UCP map $\phi : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ is C^*-extreme in $NS_{\mathcal{H}}(\mathcal{B})$ if and only if it is C^*-extreme in $S_{\mathcal{H}}(\mathcal{B})$.*

Proof. Since $NS_{\mathcal{H}}(\mathcal{B}) \subseteq S_{\mathcal{H}}(\mathcal{B})$, it is immediate that every normal C^*-extreme point of $S_{\mathcal{H}}(\mathcal{B})$ is also a C^*-extreme point of $NS_{\mathcal{H}}(\mathcal{B})$. Conversely, let ϕ be a C^*-extreme point of $NS_{\mathcal{H}}(\mathcal{B})$. Let $\phi = \sum_{i=1}^{n} T_i{}^* \phi_i(\cdot) T_i$ be a proper C^*-convex combination in $S_{\mathcal{H}}(\mathcal{B})$ for some $\phi_i \in S_{\mathcal{H}}(\mathcal{B})$. Then for each i, we have $T_i{}^* \phi_i(\cdot) T_i \leq \phi(\cdot)$, so it follows from Lemma 3.1.1 that $T_i{}^* \phi_i(\cdot) T_i$ is normal; hence ϕ_i is normal. Since ϕ is C^*-extreme in $NS_{\mathcal{H}}(\mathcal{B})$, there is a unitary $U_i \in \mathcal{B}(\mathcal{H})$ such that $\phi_i(\cdot) = U_i{}^* \phi(\cdot) U_i$, as required to prove that ϕ is C^*-extreme in $S_{\mathcal{H}}(\mathcal{B})$. $\square$

For the rest of the chapter, we mainly deal with the von Neumann algebras of the form $\mathcal{B}(\mathcal{G})$ for some separable Hilbert space $\mathcal{G}$. The C^*-extreme condition (Corollary 2.2.6) for normal C^*-extreme points translates as follows. See the structure of normal UCP maps in Theorem 1.2.26, and also see Remark 1.2.28.

Theorem 3.1.3. *Let $\phi : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ be a normal UCP map with minimal Stinespring form $\phi(X) = V^*(X \otimes I_{\mathcal{K}})V$, for some Hilbert space $\mathcal{K}$. Then ϕ is C^*-extreme in $S_{\mathcal{H}}(\mathcal{B}(\mathcal{G}))$ if and only if for any positive operator $D \in \mathcal{B}(\mathcal{K})$ with $V^*(I_{\mathcal{G}} \otimes D)V$ invertible, there exists $S \in \mathcal{B}(\mathcal{K})$ such that $D = S^*S$, $(I_{\mathcal{G}} \otimes S)VV^* = VV^*(I_{\mathcal{G}} \otimes S)VV^*$ and $V^*(I_{\mathcal{G}} \otimes S)V$ is invertible.*

Remark 3.1.4. Let $\phi : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ be a normal UCP map with minimal Stinespring form $\phi(X) = V^*(X \otimes I_{\mathcal{K}})V$. We identify the subspace $V\mathcal{H}$ with $\mathcal{H}$, so that $\mathcal{H}$ is a subspace of $\mathcal{G} \otimes \mathcal{K}$. It then follows from Theorem 3.1.3 that ϕ is a C^*-extreme point in $S_{\mathcal{H}}(\mathcal{B}(\mathcal{G}))$ if and only if the subspace $\mathcal{H}$ of $\mathcal{G} \otimes \mathcal{K}$ satisfies the following *factorization property*:

(†) for any positive operator $D \in \mathcal{B}(\mathcal{K})$ with $P_{\mathcal{H}}(I_{\mathcal{G}} \otimes D)_{|_{\mathcal{H}}}$ invertible, there exists $S \in \mathcal{B}(\mathcal{K})$ satisfying $D = S^*S$, $(I_{\mathcal{G}} \otimes S)(\mathcal{H}) \subseteq \mathcal{H}$ and $(I_{\mathcal{G}} \otimes S)_{|_{\mathcal{H}}}$ is invertible.

Therefore, in order to understand the structure of normal C^*-extreme maps, one can characterize subspaces of $\mathcal{G} \otimes \mathcal{K}$ with factorization property (†).

We shall provide a number of examples of subspaces with factorization property (†) in Section 3.3.

We now state one of the major results of this thesis involving factorization property of algebras in $\mathcal{B}(\mathcal{H})$, and whose proof is postponed until Chapter 6 (see Corollary 6.2.7 therein). We shall rather first see its consequences in the study of normal C^*-extreme maps. For the notion of factorization property of algebras and atomic nests, we refer the readers to Section 1.5.

Theorem 3.1.5. *Let $\mathcal{M}$ be an algebra having factorization in $\mathcal{B}(\mathcal{H})$. Then* $\operatorname{Lat}\mathcal{M}$ *is a complete, countable and atomic nest.*

We are now ready to prove the main result of this section, which provides a necessary and sufficient criterion for a normal C^*-extreme UCP map to be direct sum of normal pure UCP maps.

Theorem 3.1.6. *Let $\phi : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ be a normal C^*-extreme map with minimal Stinespring form $\phi(X) = V^*(X \otimes I_{\mathcal{K}})V$, for some Hilbert space $\mathcal{K}$. Then ϕ is unitarily equivalent to a direct sum of normal pure UCP maps if and only if the algebra*

$$\mathcal{M} = \{T \in \mathcal{B}(\mathcal{K}); (I_{\mathcal{G}} \otimes T)(V\mathcal{H}) \subseteq V\mathcal{H}\}$$

is reflexive.

Proof. By identifying the Hilbert space $\mathcal{H}$ with $V\mathcal{H}$, we assume that $\mathcal{H}$ is a subspace of $\mathcal{G} \otimes \mathcal{K}$, so that

$$\phi(X) = P_{\mathcal{H}}(X \otimes I_{\mathcal{K}})_{|_{\mathcal{H}}} \quad \text{for} \quad X \in \mathcal{B}(\mathcal{G}),$$

and

$$\mathcal{M} = \{T \in \mathcal{B}(\mathcal{K}); (I_{\mathcal{G}} \otimes T)\mathcal{H} \subseteq \mathcal{H}\}.$$

First we assume that the algebra $\mathcal{M}$ is reflexive. Since ϕ is C^*-extreme in $S_{\mathcal{H}}(\mathcal{B}(\mathcal{G}))$, it follows from Corollary 2.2.9 that $I_{\mathcal{G}} \otimes \mathcal{M}$ has factorization in $I_{\mathcal{G}} \otimes \mathcal{B}(\mathcal{K})$, which is to say that $\mathcal{M}$ has factorization in $\mathcal{B}(\mathcal{K})$. It then follows from Theorem 3.1.5 that $\operatorname{Lat}\mathcal{M}$ is an atomic nest. Therefore by definition of atomic nests (see Definition 1.5.14), there exists an orthonormal basis $\{e_n\}_{n\geq 1}$ of $\mathcal{K}$ such that each e_n is contained in one of the atoms of $\operatorname{Lat}\mathcal{M}$. Now for all $n \geq 1$, consider the subspace $\mathcal{G}_n$ of $\mathcal{G}$ given by

$$\mathcal{G}_n = \{g \in \mathcal{G}; g \otimes e_n \in \mathcal{H}\}.$$

We claim that

$$\mathcal{H} = \bigoplus_{n\geq 1}(\mathcal{G}_n \otimes e_n). \tag{3.1.1}$$

Clearly, $\mathcal{G}_n \otimes e_n \subseteq \mathcal{H}$ for all $n \geq 1$; hence $\oplus_{n\geq 1}(\mathcal{G}_n \otimes e_n) \subseteq \mathcal{H}$. Conversely, let $h \in \mathcal{H}$. Then as $\{e_n\}_{n\geq 1}$ is an orthonormal basis of $\mathcal{K}$, there exists a sequence $\{g_n\}_{n\geq 1}$ of vectors in $\mathcal{G}$ such that

$$h = \sum_{n\geq 1} g_n \otimes e_n.$$

Now for any unit vector $e \in \mathcal{K}$, we denote by $|e\rangle\langle e|$ the rank one projection on $\mathcal{K}$ defined by

$$|e\rangle\langle e|(k) = e\langle e, k\rangle \quad \text{for all } k \in \mathcal{K}.$$

We claim that for all $n \geq 1$ that $|e_n\rangle\langle e_n| \in \operatorname{Alg}\operatorname{Lat}\mathcal{M}$. Indeed, if $E \ominus E_-$ is an atom of $\operatorname{Lat}\mathcal{M}$ and $e \in E \ominus E_-$ is a unit vector, then

$$|e\rangle\langle e|(F) = 0 \subseteq F \text{ for } \ F \subseteq E_-, \quad \text{ and } \quad |e\rangle\langle e|(F) = \mathbb{C}\cdot e \subseteq F \ \text{ for } \ F \supseteq E,$$

which shows that $|e\rangle\langle e| \in \operatorname{Alg}\operatorname{Lat}\mathcal{M}$. This proves our claim that $|e_n\rangle\langle e_n| \in \operatorname{Alg}\operatorname{Lat}\mathcal{M}$. Since $\mathcal{M}$ is reflexive, it then follows that $|e_n\rangle\langle e_n| \in \mathcal{M}$; hence $(I_{\mathcal{G}} \otimes |e_n\rangle\langle e_n|)\mathcal{H} \subseteq \mathcal{H}$, which implies

$$(I_{\mathcal{G}} \otimes |e_n\rangle\langle e_n|)h = g_n \otimes e_n \in \mathcal{H}.$$

In particular, $g_n \in \mathcal{G}_n$ and hence $g_n \otimes e_n \in \mathcal{G}_n \otimes e_n$. This shows that

$$h = \sum_{n\geq 1} g_n \otimes e_n \in \bigoplus_{n\geq 1} \mathcal{G}_n \otimes e_n.$$

Since $h \in \mathcal{H}$ is arbitrary, we conclude our claim that $\mathcal{H} = \oplus_{n\geq 1}(\mathcal{G}_n \otimes e_n)$. Now for each $n \geq 1$, define the map $\phi_n : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{G}_n)$ by

$$\phi_n(X) = P_{\mathcal{G}_n} X_{|_{\mathcal{G}_n}}, \quad \text{for all } \ X \in \mathcal{B}(\mathcal{G}).$$

If $\mathcal{G}_n$ is a zero subspace, then we ignore the map ϕ_n. Then it is clear that ϕ_n is a normal pure UCP map, and for all $X \in \mathcal{B}(\mathcal{G})$ we have

$$\phi(X) = P_{\mathcal{H}}(X \otimes I_{\mathcal{K}})_{|_{\mathcal{H}}} = \sum_{n\geq 1} P_{\mathcal{G}_n} X_{|_{\mathcal{G}_n}} \otimes |e_n\rangle\langle e_n| = \bigoplus_{n\geq 1} \phi_n(X) \otimes |e_n\rangle\langle e_n|.$$

This proves the required assertion that ϕ is unitarily equivalent to a direct sum of normal pure UCP maps ϕ_n.

To prove the converse, let ϕ be a direct sum of normal pure UCP maps. Then for some countable indexing set J, there is a collection $\{\mathcal{G}_i\}_{i\in J}$ of distinct subspaces of $\mathcal{G}$ and a collection $\{\mathcal{K}_i\}_{i\in J}$ of mutually orthogonal subspaces of $\mathcal{K}$ such that ϕ is unitarily equivalent to the map $\oplus_{i\in J} P_{\mathcal{G}_i} X_{|_{\mathcal{G}_i}} \otimes I_{\mathcal{K}_i}$. So without loss of generality we assume that

$$\mathcal{H} = \oplus_{i\in J}(\mathcal{G}_i \otimes \mathcal{K}_i).$$

Since ϕ is C^*-extreme in $S_{\mathcal{H}}(\mathcal{B}(\mathcal{G}))$, the collection $\{\mathcal{G}_i\}_{i\in J}$ is a nest by Theorem 2.3.10. This nest induces an order on J making it a totally ordered set. If we set $\mathcal{L}_i = \oplus_{j\geq i}\mathcal{K}_j$ for $i \in J$, then $\{\mathcal{L}_i\}_{i\in J}$ is a nest, and it is easy to verify that

$$\mathcal{M} = \{T \in \mathcal{B}(\mathcal{K}); (I_{\mathcal{G}} \otimes T)(\mathcal{H}) \subseteq \mathcal{H}\} = \operatorname{Alg}\{\mathcal{L}_i; i \in J\}$$

(to show this, one can follow the same argument as in (2.3.14) in the proof of Theorem 2.3.10). But then any algebra of the form $\operatorname{Alg}\mathcal{E}$ is reflexive (see Example 1.5.16). Thus we conclude that $\mathcal{M}$ is reflexive. $\square$

It is a known fact due to Juschenko [44] that any subalgebra having factorization in the finite dimensional matrix algebra M_n is automatically reflexive and unitarily equivalent to an algebra of block upper triangular matrices (Theorem 2.6, [44]). Also see Corollary 6.4.7 below for an alternate proof of this fact. Thus the following corollary is immediate from Theorem 3.1.6 and Theorem 2.3.10.

Corollary 3.1.7. *Let $\mathcal{H}$ be a subspace of $\mathcal{G} \otimes \mathcal{K}$, where $\mathcal{K}$ is a finite dimensional Hilbert space, such that the normal UCP map $\phi : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ given by $\phi(X) = P_{\mathcal{H}}(X \otimes I_{\mathcal{K}})_{|_{\mathcal{H}}}$, for $X \in \mathcal{B}(\mathcal{G})$, is in minimal Stinespring form. Then ϕ is C^*-extreme in $S_{\mathcal{H}}(\mathcal{B}(\mathcal{G}))$ if and only if ϕ is unitarily equivalent to a direct sum of a finite sequence of normal pure UCP maps $\{\phi_i\}_{i=1}^n$ such that ϕ_i is a compression of ϕ_{i+1}.*

As a consequence of Corollary 3.1.7, we recover the result of Farenick-Morenz [28] on the structure of C^*-extreme points form M_n to M_r, which we state below. Their proof was given through rather tedious matrix computations. Here we have provided a more conceptual approach using nest algebra theory.

Corollary 3.1.8 (Theorem 4.1, [28])**.** *A UCP map $\phi : M_n \to \mathcal{M}_r$ is C^*-extreme in $S_{\mathbb{C}^r}(M_n)$ if and only if there exists a finite sequence $\{\phi_i\}_{i=1}^k$ of pure UCP maps on M_n such that ϕ_i is a compression of ϕ_{i+1} and ϕ is unitarily equivalent to $\oplus_{i=1}^n \phi_i$.*

Corollary 3.1.7 suggests that perhaps the algebra $\mathcal{M}$ in Theorem 3.1.6 is always reflexive when ϕ is C^*-extreme. But we are not able to prove it. If this turns out to be true, then Theorem 3.1.6 along with Theorem 2.3.10 would characterize all normal C^*-extreme maps on $\mathcal{B}(\mathcal{G})$. Thus we propose the following conjecture:

Conjecture 3.1.9. *Every normal C^*-extreme map on a type I factor is a direct sum of normal pure UCP maps.*

3.2 Krein-Milman type theorem for UCP maps on type I factors

We have already seen a Krein-Milman type theorem for C^*-convexity of the set $S_{\mathcal{H}}(\mathcal{A})$ equipped with BW-topology, for the case when $\mathcal{A}$ is a separable C^*-algebra (Theorem 2.4.3). In this section, we prove a Krein-Milman type theorem for the set $S_{\mathcal{H}}(\mathcal{B}(\mathcal{G}))$. Note that $\mathcal{B}(\mathcal{G})$ is not a separable C^*-algebra when $\mathcal{G}$ is an infinite dimensional Hilbert space. So the proof presented in Theorem 2.4.3 is no longer valid in the case of $S_{\mathcal{H}}(\mathcal{B}(\mathcal{G}))$.

We begin with the following proposition, which seems to be a well-known result. However we could trace the proof only when $\mathcal{H}$ is a finite dimensional Hilbert space. So we outline a proof in general case for the sake of completeness.

Proposition 3.2.1. *Let $\mathcal{B}$ be a von Neumann algebra, and let $\phi : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ be a UCP map. Then there exists a sequence $\phi_n : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ of normal UCP maps such that $\phi_n(a) \to \phi(a)$ in SOT for all $a \in \mathcal{B}$. In particular, the set $NS_{\mathcal{H}}(\mathcal{B})$ of normal generalized states is dense in the set $S_{\mathcal{H}}(\mathcal{B})$ of all generalized states in BW-topology.*

Proof. If $\mathcal{H}$ is finite dimensional, then the assertion is proved in (Corollary 1.6.3, [11]). So assume that $\mathcal{H}$ is infinite dimensional. Let $\{P_n\}_{n\geq 1}$ be an increasing sequence of projections on $\mathcal{H}$ with finite dimensional ranges such that $P_n \to I_{\mathcal{H}}$ in SOT. Fix a normal UCP map $\psi : \mathcal{B} \to \mathcal{B}(\mathcal{H})$, and for each $n \geq 1$, consider the map $\phi_n : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ given by

$$\phi_n(a) = P_n\phi(a)P_n + (1 - P_n)\psi(a)(1 - P_n), \quad \text{for all } a \in \mathcal{B}.$$

Since $P_n \to I_{\mathcal{H}}$ in SOT, we note that $\phi_n(a) \to \phi(a)$ in SOT for all $a \in \mathcal{B}$. Also the second term in the above sum is normal, as ψ is normal. So it suffices to approximate the map $P_n\phi(\cdot)P_n$ by normal CP maps. The problem now reduces to approximation of (unital) CP maps by normal (unital) CP maps acting on finite dimensional Hilbert spaces, which is possible as already noted. □

The following is Krein-Milman type theorem for C^*-convexity of the set $S_{\mathcal{H}}(\mathcal{B}(\mathcal{G}))$.

Theorem 3.2.2. *Let $\mathcal{G}$ and $\mathcal{H}$ be separable Hilbert spaces. Then $S_{\mathcal{H}}(\mathcal{B}(\mathcal{G}))$ is BW-closure of C^*-convex hull of its (normal) C^*-extreme points.*

Proof. In view of Proposition 3.2.1, it suffices to approximate a normal UCP map by C^*-convex combinations of C^*-extreme points of $S_{\mathcal{H}}(\mathcal{B}(\mathcal{G}))$. Let $\phi : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ be a normal UCP map. Then by Corollary 1.2.27, there exists a finite or countable sequence of contractions $\{V_n\}_{n\geq 1}$ in $\mathcal{B}(\mathcal{H}, \mathcal{G})$ such that

$$\phi(X) = \sum_{n\geq 1} V_n^* X V_n \quad \text{for all} \quad X \in \mathcal{B}(\mathcal{G}), \quad \text{(WOT Convergence)}. \tag{3.2.1}$$

Note that the maps $X \mapsto V_n^* X V_n$ from $\mathcal{B}(\mathcal{G})$ to $\mathcal{B}(\mathcal{H})$ are pure maps. We now invoke Lemma 2.4.2 from Section 2.4 to conclude the required assertion. □

3.3 Examples of normal C^*-extreme maps

In this section, we consider examples of normal UCP maps some of which are C^*-extreme and some are not. Factorization property of algebras plays very important role in determining C^*-extremity conditions.

The following proposition provides a family of examples of subspaces in $\mathcal{G} \otimes \mathcal{K}$ satisfying factorization property (†), which further produces examples of normal C^*-extreme UCP maps (see Remark 3.1.4).

Proposition 3.3.1. *Let $\mathcal{H} = \bigvee_{i\in\Lambda} \mathcal{G}_i \otimes \mathcal{K}_i$ be a subspace of $\mathcal{G} \otimes \mathcal{K}$, for some family $\{\mathcal{G}_i\}_{i\in\Lambda}$ and $\{\mathcal{K}_i\}_{i\in\Lambda}$ of subspaces of $\mathcal{G}$ and $\mathcal{K}$ respectively, such that $\mathcal{G} \otimes \mathcal{K} = \overline{\text{span}}\{(X \otimes I_{\mathcal{K}})h; h \in \mathcal{H}, X \in \mathcal{B}(\mathcal{G})\}$. If either of the following is true:*

(i) $\mathcal{G}_i \perp \mathcal{G}_j$ for all $i \neq j$ and $\{\mathcal{K}_i\}_{i\in\Lambda}$ is a nest whose completion is countable,

(ii) $\{\mathcal{G}_i\}_{i\in\Lambda}$ is a nest and $\mathcal{K}_i \perp \mathcal{K}_j$ for $i \neq j$ such that the completion of the nest $\{\oplus_{i\leq n}\mathcal{K}_i\}_{n\in\Lambda}$ is countable,

then $\mathcal{H}$ satisfies factorization property (†).

Proof. (1) Firstly it is easy to verify that

$$\mathcal{K} = \bigvee_{i\in\Lambda} \mathcal{K}_i.$$

Indeed, if $k \in \mathcal{K} \ominus \bigvee_{i\in\Lambda} \mathcal{K}_i$, then for any non-zero $g \in \mathcal{G}$, we will have $g \otimes k \perp \{(X \otimes I_{\mathcal{K}})h; h \in \mathcal{H}, X \in \mathcal{B}(\mathcal{G})\}$, which will yield $g \otimes k = 0$.

Let $D \in \mathcal{B}(\mathcal{K})$ be a positive operator such that $P_{\mathcal{H}}(I_{\mathcal{G}} \otimes D)_{|_{\mathcal{H}}}$ is invertible. We claim that D is invertible. Let $\beta > 0$ be such that $P_{\mathcal{H}}(I_{\mathcal{G}} \otimes D)_{|_{\mathcal{H}}} \geq \beta I_{\mathcal{H}}$. Since $g_i \otimes k_i \in \mathcal{H}$, for any $0 \neq g_i \in \mathcal{G}_i$ and $k_i \in \mathcal{K}_i$, we get

$$\|g_i\|^2 \langle Dk_i, k_i\rangle = \langle (I_{\mathcal{G}} \otimes D)(g_i \otimes k_i), g_i \otimes k_i\rangle \geq \beta\ \langle g_i \otimes k_i, g_i \otimes k_i\rangle = \beta\ \|g_i\|^2\ \langle k_i, k_i\rangle,$$

which implies that $\langle Dk_i, k_i\rangle \geq \beta\langle k_i, k_i\rangle$. Since $\bigcup_{i\in\Lambda} \mathcal{K}_i$ is dense in $\mathcal{K}$, we conclude that

$$\langle Dk, k\rangle \geq \beta\langle k, k\rangle$$

for all $k \in \mathcal{K}$; hence D is invertible. Since the nest $\{\mathcal{K}_i\}_{i\in\Lambda}$ has a countable completion, by Theorem 1.5.13 there exists an invertible operator $S \in \mathcal{B}(\mathcal{K})$ satisfying $D = S^*S$ and $S(\mathcal{K}_i) \subseteq \mathcal{K}_i$, $S^{-1}(\mathcal{K}_i) \subseteq \mathcal{K}_i$ for all $i \in \Lambda$. Clearly then $(I_{\mathcal{G}} \otimes S)(\mathcal{H}) \subseteq \mathcal{H}$. Note that

$$(S^{-1})_{|_{\mathcal{K}_i}} = (S_{|_{\mathcal{K}_i}})^{-1} \in \mathcal{B}(\mathcal{K}_i) \ \text{ for each } \ i \in \Lambda \ \text{ and } \ \sup_{i\in\Lambda} \|(S_{|_{\mathcal{K}_i}})^{-1}\| = \|S^{-1}\| < \infty.$$

Hence $\oplus_{i\in\Lambda} I_{\mathcal{G}_i} \otimes (S_{|_{\mathcal{K}_i}})^{-1}$ is a bounded operator on $\mathcal{H}$ and

$$(I_{\mathcal{G}} \otimes S)_{|_{\mathcal{H}}}(\oplus_{i\in\Lambda} I_{\mathcal{G}_i} \otimes (S_{|_{\mathcal{K}_i}})^{-1}) = (\oplus_{i\in\Lambda} I_{\mathcal{G}_i} \otimes S_{|_{\mathcal{K}_i}})(\oplus_{i\in\Lambda} I_{\mathcal{G}_i} \otimes (S_{|_{\mathcal{K}_i}})^{-1}) = \oplus_{i\in\Lambda} I_{\mathcal{G}_i} \otimes I_{\mathcal{K}_i} = I_{\mathcal{H}}.$$

Similarly, $(\oplus_{i\in\Lambda} I_{\mathcal{G}_i} \otimes (S_{|_{\mathcal{K}_i}})^{-1})(I_{\mathcal{G}} \otimes S)_{|_{\mathcal{H}}} = I_{\mathcal{H}}$. This proves that $(I_{\mathcal{G}} \otimes S)_{|_{\mathcal{H}}}$ is invertible. Since $D \in \mathcal{B}(\mathcal{K})$ is arbitrary, we have shown that $\mathcal{H}$ satisfies factorization property (†).

(2) This assertion follows from Theorem 2.3.10, as the map

$$\phi(X) = P_{\mathcal{H}}(X \otimes I_{\mathcal{K}})_{|_{\mathcal{H}}} = \oplus_{i\in\Lambda}(P_{\mathcal{G}_i} X_{|_{\mathcal{G}_i}} \otimes I_{\mathcal{K}_i})$$

from $\mathcal{B}(\mathcal{G})$ to $\mathcal{B}(\mathcal{H})$ satisfies the equivalent criteria for it to be C^*-extreme in $S_{\mathcal{H}}(\mathcal{B}(\mathcal{G}))$. ☐

At this point, we are not sure if we can write subspaces of Part (1) in Proposition 3.3.1 in the form of subspaces in Part (2), and vice versa. However one can easily verify that if the concerned nests are already complete, then the two parts produce the same set of subspaces (we leave the details to the readers as we don't want to digress from our main theme).

The following are two examples of normal UCP maps which are not C^*-extreme points. In order to show this, we use the fact that nest algebras associated with uncountable nests do not have factorization.

Example 3.3.2. Let $\mathcal{K}$ be a Hilbert space, and let $\{\mathcal{K}_q\}_{q\in\mathbb{Q}}$ be a nest of subspaces indexed by rationals $\mathbb{Q}$ such that $\mathcal{K}_q \subsetneq \mathcal{K}_{q'}$ if $q < q'$, and $\mathcal{K} = \vee_{q\in\mathbb{Q}}\mathcal{K}_q$. Let $\mathcal{G}$ be a Hilbert space, and let $\{\mathcal{G}_q\}_{q\in\mathbb{Q}}$ be any collection of mutually orthogonal subspaces of $\mathcal{G}$. Consider the subspace

$$\mathcal{H} = \oplus_{q\in\mathbb{Q}} \mathcal{G}_q \otimes \mathcal{K}_q$$

of $\mathcal{G} \otimes \mathcal{K}$, and the map $\phi : \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ defined by

$$\phi(X) = P_{\mathcal{H}}(X \otimes I_{\mathcal{K}})_{|_{\mathcal{H}}}, \quad \text{for all } X \in \mathcal{B}(\mathcal{G}).$$

Note that the algebra $\mathcal{M} = \{T \in \mathcal{B}(\mathcal{K}); (I_{\mathcal{G}} \otimes T)(\mathcal{H}) \subseteq \mathcal{H}\}$ is nothing but $\operatorname{Alg}\mathcal{E}$, where $\mathcal{E}$ is the nest $\mathcal{E} = \{\mathcal{K}_q\}_{q\in\mathbb{Q}}$. Even though the nest $\mathcal{E}$ is countable, its completion is not a countable nest (indeed, completion of $\mathcal{E}$ is given by $\{0, \mathcal{K}, \mathcal{K}_q, \mathcal{L}_r; q \in \mathbb{Q}, r \in \mathbb{R}\}$ where $\mathcal{L}_r = \bigvee_{p<r} \mathcal{K}_p$; see Example 1.5.10). So it follows from Theorem 1.5.13 that $\mathcal{M}$ does not have factorization in $\mathcal{B}(\mathcal{K})$. Consequently, $I_{\mathcal{G}} \otimes \mathcal{M}$ does not have factorization in $I_{\mathcal{G}} \otimes \mathcal{B}(\mathcal{K}) = \pi(\mathcal{A})'$, where $\pi(X) = X \otimes I_{\mathcal{K}}$ is the minimal Stinespring representation for ϕ. Thus we conclude from Corollary 2.2.9 that ϕ is not a C^*-extreme point in $S_{\mathcal{H}}(\mathcal{B}(\mathcal{G}))$.

Example 3.3.3. Let $\mathcal{K} = L^2([0,1])$ with respect to Lebesgue measure, and let

$$\mathcal{H} = \{\chi_\Delta f; f \in L^2([0,1] \times [0,1])\} \subseteq \mathcal{K} \otimes \mathcal{K},$$

where $\Delta = \{(s,t); s,t \in [0,1], 0 \le s \le t \le 1\} \subseteq [0,1] \times [0,1]$. Here χ_Δ denotes the characteristic function on the set Δ. Define $\phi : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})$ by

$$\phi(X) = P_{\mathcal{H}}(X \otimes I_{\mathcal{K}})_{|_{\mathcal{H}}} \text{ for all } X \in \mathcal{B}(\mathcal{K}).$$

We claim that ϕ is not a C^*-extreme point in $S_{\mathcal{H}}(\mathcal{B}(\mathcal{K}))$. First consider the following observations, which are straightforward to verify:

(i) $\mathcal{H} = \overline{\operatorname{span}}\{\chi_{[0,t]} f \otimes \chi_{[t,1]} g; t \in [0,1], f, g \in \mathcal{K}\}$.
(ii) $\mathcal{H}^\perp = \overline{\operatorname{span}}\{\chi_{[s,1]} f \otimes \chi_{[0,s]} g; s \in [0,1], f, g \in \mathcal{K}\}$.
(iii) $\mathcal{K} \otimes \mathcal{K} = \overline{\operatorname{span}}\{(X \otimes I_{\mathcal{K}})h; h \in \mathcal{H}, X \in \mathcal{B}(\mathcal{K})\}$.
(iv) $\phi(X) = P_{\mathcal{H}}\pi(X)_{|_{\mathcal{H}}}$ is the minimal Stinespring dilation for ϕ where $\pi : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K} \otimes \mathcal{K})$ is defined by $\pi(X) = X \otimes I_{\mathcal{K}}$, $X \in \mathcal{B}(\mathcal{K})$.
(v) $\pi(\mathcal{B}(\mathcal{K}))' = \{I_{\mathcal{K}} \otimes S; S \in \mathcal{B}(\mathcal{K})\}$.

Let $\mathcal{M} = \{S \in \mathcal{B}(\mathcal{K}); (I_{\mathcal{K}} \otimes S)(\mathcal{H}) \subseteq \mathcal{H}\}$. We claim that $\mathcal{M} \subseteq \operatorname{Alg}\mathcal{E}$, for the complete nest $\mathcal{E} = \{E_t; t \in [0,1]\}$, where

$$E_t = \{\chi_{[t,1]} f; f \in \mathcal{K}\}, \text{ for } t \in [0,1].$$

Since $\mathcal{E}$ is uncountable, it will follow from Theorem 1.5.13 that $\operatorname{Alg}\mathcal{E}$ does not have factorization in $\mathcal{B}(\mathcal{K})$; hence $\mathcal{M}$ does not have factorization in $\mathcal{B}(\mathcal{K})$, that is, $I_{\mathcal{K}} \otimes \mathcal{M}$ does not have factorization in $I_{\mathcal{K}} \otimes \mathcal{B}(\mathcal{K}) = \pi(\mathcal{B}(\mathcal{K}))'$. This will imply from Corollary 2.2.9 that ϕ is not C^*-extreme in $S_{\mathcal{H}}(\mathcal{B}(\mathcal{K}))$. Now let $S \in \mathcal{M}$, so that $(I_{\mathcal{K}} \otimes S)(\mathcal{H}) \subseteq \mathcal{H}$. Fix $t \in (0,1]$, and let $0 < s < t$. Note that

$${E_s}^\perp = \{\chi_{[0,s]} f; f \in \mathcal{K}\}.$$

Now for any $f, g \in \mathcal{K}$, we note from above observations that $\chi_{[0,t]} \otimes \chi_{[t,1]} g \in \mathcal{H}$ (so that $(I_{\mathcal{K}} \otimes S)(\chi_{[0,t]} \otimes \chi_{[t,1]} g) \in \mathcal{H}$) and $\chi_{[s,1]} \otimes \chi_{[0,s]} f \in \mathcal{H}^\perp$; hence

$$\begin{aligned} 0 &= \left\langle (I_{\mathcal{K}} \otimes S)(\chi_{[0,t]} \otimes \chi_{[t,1]} g), \chi_{[s,1]} \otimes \chi_{[0,s]} f \right\rangle = \left\langle \chi_{[0,t]} \otimes S(\chi_{[t,1]} g), \chi_{[s,1]} \otimes \chi_{[0,s]} f \right\rangle \\ &= \langle \chi_{[0,t]}, \chi_{[s,1]} \rangle \langle S(\chi_{[t,1]} g), \chi_{[0,s]} f \rangle = (t-s) \langle S(\chi_{[t,1]} g), \chi_{[0,s]} f \rangle. \end{aligned}$$

Since $t - s \neq 0$, it follows that

$$\langle S(\chi_{[t,1]}g), \chi_{[0,s]}f\rangle = 0.$$

This shows that $S(\chi_{[t,1]}g) \perp E_s{}^{\perp}$, which is to say $S(\chi_{[t,1]}g) \in E_s$. Since $g \in \mathcal{K}$ is arbitrary, it follows that $S(E_t) \subseteq E_s$. Since $s < t$ is arbitrary, we conclude that

$$S(E_t) \subseteq \bigcap_{0<s<t} E_s = E_t.$$

This shows that $S \in \operatorname{Alg} \mathcal{E}$; thus we conclude our claim that $\mathcal{M} \subseteq \operatorname{Alg} \mathcal{E}$.

Inspired from the example of C^*-extreme point as in (2.1.5), we now consider its noncommutative analogue. For a C^*-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{K})$ and a subspace $\mathcal{H}$ of $\mathcal{K}$, consider the UCP map $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ given by

$$\phi(X) = P_{\mathcal{H}} X_{|_{\mathcal{H}}} \quad \text{for} \quad X \in \mathcal{A}.$$

If $\mathcal{A} = \mathcal{B}(\mathcal{K})$, then clearly ϕ is a pure map, so that ϕ is C^*-extreme in $S_{\mathcal{H}}(\mathcal{A})$. An example of C^*-extreme point of this form (when $\mathcal{A} \neq \mathcal{B}(\mathcal{K})$) is the map in (2.5.3). But for arbitrary $\mathcal{A}$, we do not know if ϕ is always C^*-extreme in $S_{\mathcal{H}}(\mathcal{A})$.

Let $\mathcal{B}$ be a finite von Neumann algebra with a distinguished faithful trace $\tau : \mathcal{B} \to \mathbb{C}$ (see Section 1.1 for the definition and existence of a trace). Let $L^2(\tau)$ denote the Hilbert space induced by τ, which is the closure of $\mathcal{B}$ with respect to the inner product on $\mathcal{B}$ defined by

$$\langle x, y\rangle = \tau(x^*y) \quad \text{for } x, y \in \mathcal{B}.$$

Then the left regular representation $\pi : \mathcal{B} \to \mathcal{B}(L^2(\tau))$ defined by $\pi(x) = L_x$ for all $x \in \mathcal{B}$, is cyclic with cyclic vector $\delta = 1$, where $L_x : L^2(\tau) \to L^2(\tau)$ is given by

$$L_x(y) = xy \quad \text{for all} \quad y \in \mathcal{B}.$$

Now let $\mathcal{M}$ be a subalgebra of $\mathcal{B}$ such that $\mathcal{M}$ has factorization in $\mathcal{B}$ (as defined in 1.5.1). Examples of such algebras are finite maximal subdiagonal algebras introduced by Arveson [4], which also include nest subalgebras (see Example 6.1.11 and Example 6.4.10). Consider the subspace

$$H^2 = [\mathcal{M}] \subseteq L^2(\tau)$$

(called noncommutative Hardy space), and let $\phi : \mathcal{B} \to \mathcal{B}(H^2)$ be the map defined by

$$\phi(x) = P_{H^2} L_{x|_{H^2}},$$

for $x \in \mathcal{B}$. It is clear that ϕ is a UCP map. We have the following:

Proposition 3.3.4. *For $\mathcal{B}, \mathcal{M}$ and ϕ as above, ϕ is a C^*-extreme point in $S_{H^2}(\mathcal{B})$.*

Proof. Note that $(\pi, V, L^2(\tau))$ is the minimal Stinespring triple, where V is the inclusion map from H^2 to $L^2(\tau)$. It is a well-known fact (see Proposition 11.16, [63]) that

$$\pi(\mathcal{B})' = \{R_x ; x \in \mathcal{B}\},$$

where $R_x \in \mathcal{B}(L^2(\tau))$ is the right multiplication operator defined by

$$R_x(y) = yx \quad \text{for all} \quad y \in \mathcal{B}.$$

Now to show that ϕ is C^*-extreme in $S_{H^2}(\mathcal{B})$, we let R_x to be a positive operator in $\pi(\mathcal{B})'$ for some $x \in \mathcal{B}$ such that $P_{H^2}R_{x|H^2}$ is invertible. Clearly $x \geq 0$ in $\mathcal{B}$. We claim that x is invertible in $\mathcal{B}$. Since $P_{H^2}R_{x|_{H^2}}$ is invertible, there is an $\alpha > 0$ such that $P_{H^2}R_{x|_{H^2}} \geq \alpha I_{H^2}$. Hence for all $z \in \mathcal{M}$, we have

$$\langle zx, z\rangle = \langle R_x z, z\rangle \geq \alpha\langle z, z\rangle$$

that is,

$$\tau((x-\alpha)z^*z) = \langle z(x-\alpha), z\rangle \geq 0.$$

Since $\{z^*z; z \in \mathcal{M}\}$ is dense in the set of all positive elements of $\mathcal{B}$ (as $\mathcal{M}$ has factorization in $\mathcal{B}$), it follows that $\tau((x-\alpha)y) \geq 0$, for all $y \geq 0$ in $\mathcal{B}$. Hence for all $a \in \mathcal{B}$, we get using the tracial property of τ that

$$\langle (x-\alpha)a, a\rangle = \tau(a^*(x-\alpha)a) = \tau((x-\alpha)aa^*) \geq 0,$$

which is to say that $x - \alpha \geq 0$ in $\mathcal{B}$. This shows that x is invertible. Therefore by factorization of $\mathcal{M}$ in $\mathcal{B}$, there exists an invertible element z with $z, z^{-1} \in \mathcal{M}$ such that $x = zz^*$; thus

$$R_x = R_{zz^*} = R_{z^*}R_z = R_z^*R_z.$$

Further, since $z \in \mathcal{M}$, it follows that $R_z(\mathcal{M}) \subseteq \mathcal{M}$ and hence $R_z(H^2) \subseteq H^2$. Also since $z^{-1} \in \mathcal{M}$, we have $R_z^{-1}(H^2) = R_{z^{-1}}(H^2) \subseteq H^2$, which in particular implies that $R_{z|_{H^2}}$ is invertible. Since R_x is arbitrary in $\pi(\mathcal{A})'$, we conclude that ϕ is a C^*-extreme point in $S_{H^2}(\mathcal{B})$. $\square$

Chapter 4

C^*-extreme Positive Operator Valued Measures

We digress from our earlier discussions on C^*-extreme UCP maps and instead consider C^*-extremity conditions of positive operator valued measures (POVMs). The correspondence between POVMs on a compact Hausdorff space X and UCP maps on the commutative C^*-algebra $C(X)$ is a folklore. Many authors while studying UCP maps on commutative C^*-algebras exploit this relationship. We follow the same approach and for the purpose study POVMs independently. Through this correspondence, the theory developed here will then be applied in the next chapter to the study of C^*-extreme UCP maps on commutative C^*-algebras.

The notions of C^*-convexity and C^*-extreme points have natural extensions to POVMs (see Definition 4.1.1 and 4.1.2). Here we study C^*-convexity of POVMs on a measurable space $(X, \mathcal{O}(X))$, where $\mathcal{O}(X)$ is a σ-algebra of subsets of a set X. The problem of identifying C^*-extreme points of POVMs has been open for several decades even for finite sets. The result from 1997 of Farenick and Morenz [28] translates to saying that C^*-extreme positive *matrix* valued measures on a finite set X are spectral measures. We generalize the result of [28] considerably, as we allow general POVMs on all countable spaces and still all the C^*-extreme points are spectral (Theorem 4.3.2). This is important because it is in stark contrast with classical convexity. Extreme points of POVMs under classical convexity are not necessarily spectral measures and are hard to describe even for finite sets, though abstract characterizations are available. C^*-extreme POVMs being spectral measures have physical significance as they relate to classical measurements. Our result reinforces the idea that C^*-convexity is perhaps the suitable notion of convexity in the quantum setting. One can see the study of C^*-convexity structure of POVMs in Farenick et. al. [24] and Gregg [33].

This chapter is organized as follows. We first translate the notions of C^*-convexity and C^*-extreme points in the setting of POVMs, and state the corresponding abstract characterizations for C^*-extreme POVMs. Inspired from classical case, we decompose a POVM as a sum of atomic and non-atomic POVMs and study their C^*-extremity conditions separately. In Section 4.2 and Section 4.3, we present some of our main results on C^*-extreme POVMs. The most crucial technical step is in the proof of Theorem 4.2.1. Heinosaari and Pellonpää [37] have shown that extreme points of POVMs with commutative ranges are spectral. The same conclusion holds under C^*-convexity (Theorem 4.2.2) as well. Most importantly all atomic C^*-extreme POVMs are also seen to be spectral (Theorem 4.3.2). This also helps us in proving that C^*-extreme POVMs are spectral for finite dimensional Hilbert spaces, which we prove in full generality.

Next we introduce a notion of disjoint spectral measures and compare it with the notion of singularity. We also see behaviour of C^*-extremity under the direct sum of mutually singular POVMs. Finally basic properties like C^*-convexity, atomicity etc are explored under a notion of measure isomorphism of POVMs.

4.1 General Properties of C^*-extreme POVMs

Throughout this chapter, X is a non-empty set and $\mathcal{O}(X)$ denotes a σ-algebra of subsets of X. The pair $(X, \mathcal{O}(X))$ is called a *measurable space* and the elements of $\mathcal{O}(X)$ are called *measurable subsets*. We shall simply call X a measurable space without mentioning the underlying σ-algebra $\mathcal{O}(X)$. To avoid some unnecessary complications in presentation, we assume that all singleton subsets of X are measurable. When X is a topological space, we shall assume $\mathcal{O}(X)$ to be the Borel σ-algebra on X. All topological spaces under consideration would be Hausdorff.

We refer the readers to Section 1.3 for the basics of positive operator valued measures (POVMs) and their dilation theory. As mentioned there, we fix the following notation:

Notation. We denote by $\mathcal{P}_{\mathcal{H}}(X)$ the collection of all normalized POVMs from $\mathcal{O}(X)$ to $\mathcal{B}(\mathcal{H})$.

We now describe the notions of C^*-convexity and C^*-extreme points of the set $\mathcal{P}_{\mathcal{H}}(X)$.

Definition 4.1.1. For any $\mu_i \in \mathcal{P}_{\mathcal{H}}(X)$ and $T_i \in \mathcal{B}(\mathcal{H})$, $1 \leq i \leq n$ with $\sum_{i=1}^n T_i^* T_i = I_{\mathcal{H}}$, a sum of the form

$$\mu(\cdot) = \sum_{i=1}^{n} T_i^* \mu_i(\cdot) T_i \tag{4.1.1}$$

is called a *C^*-convex combination* for μ. The operators T_i's here are called *C^*-coefficients*. When T_i's are invertible, the sum in (4.1.1) is called a *proper C^*-convex combination* for μ.

Observe that $\mathcal{P}_{\mathcal{H}}(X)$ is a *C^*-convex set* in the sense that it is closed under C^*-convex combinations i.e.

$$\sum_{i=1}^{n} T_i{}^* \mu_i(\cdot) T_i \in \mathcal{P}_{\mathcal{H}}(X),$$

whenever $\mu_i \in \mathcal{P}_{\mathcal{H}}(X)$ and $T_i \in \mathcal{B}(\mathcal{H})$ satisfying $\sum_{i=1}^n T_i{}^* T_i = I_{\mathcal{H}}$ for $1 \leq i \leq n$. The following definition of C^*-extreme points is the POVM analogue of Definition 2.1.2 for UCP maps.

Definition 4.1.2. A normalized POVM $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ is called a *C^*-extreme point* in $\mathcal{P}_{\mathcal{H}}(X)$ if, whenever

$$\sum_{i=1}^{n} T_i^* \mu_i(\cdot) T_i$$

is a proper C^*-convex combination of μ, then each μ_i is unitarily equivalent to μ i.e. there are unitary operators $U_i \in \mathcal{B}(\mathcal{H})$ such that $\mu_i(\cdot) = U_i^*\mu(\cdot)U_i$ for $1 \leq i \leq n$.

We now consider some abstract characterizations of C^*-extreme POVMs parallel to those of UCP maps. The characterization of C^*-extreme UCP maps (Theorem 2.2.1) due to Farenick-Zhou translates into the language of POVMs as follows and one obtains a characterization for C^*-extreme points of $\mathcal{P}_\mathcal{H}(X)$.

As we are dealing with the more general case of arbitrary measurable spaces, we are giving an outline of the proof here for completeness.

Theorem 4.1.3. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a normalized POVM with the minimal Naimark dilation $(\pi, V, \mathcal{H}_\pi)$. Then μ is a C^*-extreme point in $\mathcal{P}_\mathcal{H}(X)$ if and only if for any positive operator $D \in \pi(\mathcal{O}(X))'$ with V^*DV being invertible, there exists a co-isometry $U \in \pi(\mathcal{O}(X))'$ (i.e. $UU^* = I_{\mathcal{H}_\pi}$) satisfying $U^*UD^{1/2} = D^{1/2}$ and an invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that $UD^{1/2}V = VS$.*

Proof. First assume that μ is C^*-extreme in $\mathcal{P}_\mathcal{H}(X)$. Let $D \in \pi(\mathcal{O}(X))'$ be positive with V^*DV invertible. Choose $\alpha > 0$ small enough such that $I_{\mathcal{H}_\pi} - \alpha D$ is positive and invertible. Set

$$T_1 = (\alpha V^*DV)^{1/2} \quad \text{and} \quad T_2 = (I_\mathcal{H} - \alpha V^*DV)^{1/2}.$$

Then both T_1 and T_2 are invertible and $T_1^*T_1 + T_2^*T_2 = I_\mathcal{H}$. Now we define POVMs $\mu_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$, $i = 1, 2$ by

$$\mu_1(A) = T_1^{-1}\left(\alpha V^*D\pi(A)V\right)T_1^{-1} \text{ and } \mu_2(A) = T_2^{-1}\left(V^*(I_{\mathcal{H}_\pi} - \alpha D)\pi(A)V\right)T_2^{-1}, \tag{4.1.2}$$

for all $A \in \mathcal{O}(X)$. It is clear that μ_i is a POVM and $\mu_i(X) = I_\mathcal{H}$. Also,

$$T_1^*\mu_1(A)T_1 + T_2^*\mu_2(A)T_2 = V^*\pi(A)V = \mu(A) \quad \text{for all } A \in \mathcal{O}(X).$$

Since μ is C^*-extreme, there exists a unitary $W \in \mathcal{B}(\mathcal{H})$ such that $\mu(\cdot) = W^*\mu_1(\cdot)W$. This implies

$$\mu(\cdot) = \left(\sqrt{\alpha}W^*T_1^{-1}V^*D^{1/2}\right)\pi(\cdot)\left(\sqrt{\alpha}D^{1/2}VT_1^{-1}W\right) = V_1^*\pi(\cdot)V_1,$$

where $V_1 = \sqrt{\alpha}D^{1/2}VT_1^{-1}W \in \mathcal{B}(\mathcal{H}, \mathcal{H}_\pi)$. Now if we set $\mathcal{K} = [\pi\left(\mathcal{O}(X)\right)V_1\mathcal{H}] \subseteq \mathcal{H}_\pi$, then one can easily verify that $\mathcal{K} = \overline{\mathcal{R}(D^{1/2})}$, and the triple $(\pi(\cdot)_{|\mathcal{K}}, V_1, \mathcal{K})$ is the minimal Naimark dilation for μ. Therefore, by the uniqueness of minimal dilation (Theorem 1.3.2), there exists a unitary $\widetilde{U} : \mathcal{K} \to \mathcal{H}_\pi$ satisfying

$$\widetilde{U}V_1 = V \text{ and } \pi(A)\widetilde{U} = \widetilde{U}\pi(A)_{|\mathcal{K}} \quad \text{for all } A \in \mathcal{O}(X).$$

Extend $\widetilde{U}$ to the whole of $\mathcal{H}_\pi$ by assigning it to be 0 on $\mathcal{H}_\pi \ominus \mathcal{K}$, which we denote by U. Clearly then U is a co-isometry satisfying $U^*UD^{1/2} = D^{1/2}$. Also we have $\pi(A)U = U\pi(A)$ for all $A \in \mathcal{O}(X)$, and hence $U \in \pi\left(\mathcal{O}(X)\right)'$. Now if set $S = \sqrt{\alpha}^{-1}W^*T_1 \in \mathcal{B}(\mathcal{H})$. then S is invertible and, since $UV_1 = V$ and W is a unitary, we get

$$VS = UV_1S = \sqrt{\alpha}\sqrt{\alpha}^{-1}UD^{1/2}VT_1^{-1}WW^*T_1 = UD^{1/2}V.$$

For the converse, assume that the given statement in 'only if' part is true. Let $\mu = \sum_{i=1}^n T_i^*\mu_i(\cdot)T_i$ be a proper C^*-convex combination. Fix any $i \in \{1, \ldots, n\}$. Since $T_i^*\mu_i(\cdot)T_i \leq \mu$, it follows from Radon-Nikodym type Theorem (Theorem 1.3.10) that there exists a positive operator $D_i \in \pi(\mathcal{O}(X))'$ satisfying

$$T_i^*\mu_i(A)T_i = V^*D_i\pi(A)V \quad \text{for all } A \in \mathcal{O}(X).$$

Then $V^*D_iV = T_i^*T_i$ and since T_i is invertible, it follows that V^*D_iV is invertible. Hence the hypothesis ensures the existence of an operator $U_i \in \pi\left(\mathcal{O}(X)\right)'$ satisfying $U_i^*U_iD_i^{1/2} = D_i^{1/2}$ and an invertible operator $S_i \in \mathcal{B}(\mathcal{H})$ such that $U_iD_i^{1/2}V = VS_i$. Thus,

$$\begin{aligned} T_i^*\mu_i(\cdot)T_i &= V^*D_i\pi(\cdot)V = V^*D_i^{1/2}\pi(\cdot)D_i^{1/2}V = V^*D_i^{1/2}\pi(\cdot)U_i^*U_iD_i^{1/2}V \\ &= V^*D_i^{1/2}U_i^*\pi(\cdot)U_iD_i^{1/2}V = (VS_i)^*\pi(\cdot)(VS_i) = S_i^*\left(V^*\pi(\cdot)V\right)S_i = S_i^*\mu(\cdot)S_i, \end{aligned}$$

which implies $\mu_i = {T_i^*}^{-1}S_i^*\mu(\cdot)S_iT_i^{-1} = R_i^*\mu(\cdot)R_i$, where $R_i = S_iT_i^{-1}$. It is clear that R_i is invertible and since, $R_i^*R_i = \mu_i(X) = I_{\mathcal{H}}$, it follows that R_i is a unitary. This shows that μ_i is unitarily equivalent to μ, as required to conclude that μ is a C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$. $\square$

The following is an immediate corollary of Theorem 4.1.3. This is an analogue of C^*-extremity of $*$-homomorphisms in the spaces of UCP maps.

Corollary 4.1.4. *Every spectral measure is a C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$.*

Proof. If μ is a spectral measure then the minimal dilation for μ can be taken to be $(\mu, I_{\mathcal{H}}, \mathcal{H})$. For positive $D \in \mu(X)'$ with $D(= I_{\mathcal{H}}^*DI_{\mathcal{H}})$ invertible, we can take $U = I_{\mathcal{H}}$ and $S = D^{1/2}$ to satisfy the criterion. $\square$

Another abstract characterization of C^*-extreme points for UCP maps due to Zhou (Corollary 2.2.4) translates to POVM case as follows. Again as we are dealing with general measurable spaces, we provide a proof here.

Corollary 4.1.5. *Let $\mu \in \mathcal{P}_{\mathcal{H}}(X)$. Then μ is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X)$ if and only if for any POVM $\nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ with $\nu \leq \mu$ and $\nu(X)$ invertible, there exists an invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that $\nu(A) = S^*\mu(A)S$ for all $A \in \mathcal{O}(X)$.*

Proof. First assume that μ is a C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$. Let $\nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM such that $\nu \leq \mu$ and $\nu(X)$ is invertible. Let $(\pi, V, \mathcal{H}_\pi)$ be the minimal Naimark dilation for μ. By Theorem 1.3.10, there exists a positive operator $D \in \pi(\mathcal{O}(X))'$ such that

$$\nu(A) = V^*D\pi(A)V \quad \text{for all } A \in \mathcal{O}(X).$$

Since $V^*DV = \nu(X)$ and $\nu(X)$ is invertible, it follows that V^*DV is invertible. Therefore, by Theorem 4.1.3 there exists a co-isometry $U \in \pi(\mathcal{O}(X))'$ satisfying $U^*UD^{1/2} = D^{1/2}$ and an invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that $UD^{1/2}V = VS$. So for any $A \in \mathcal{O}(X)$, we get

$$\begin{aligned} \nu(A) &= V^*D\pi(A)V = V^*D^{1/2}\pi(A)D^{1/2}V = V^*D^{1/2}\pi(A)U^*UD^{1/2}V \\ &= \left(UD^{1/2}V\right)^*\pi(A)\left(UD^{1/2}V\right) = (VS)^*\pi(A)(VS) = S^*\mu(A)S. \end{aligned}$$

Conversely, assume the given statement in the 'only if' part is true. Let $\mu = \sum_{i=1}^n T_i^*\mu_i(\cdot)T_i$ be a proper C^*-convex combination. Then $T_i^*\mu_i(\cdot)T_i \leq \mu$ for each i. Also, since $T_i^*\mu_i(X)T_i = T_i^*T_i$ and T_i is invertible, it follows that $T_i^*\mu_i(X)T_i$ is invertible. Hence using hypothesis, there exists an invertible operator $S_i \in \mathcal{B}(\mathcal{H})$ such that for all $A \in \mathcal{O}(X)$, we have $T_i^*\mu_i(A)T_i = S_i^*\mu(A)S_i$ which when put differently yields

$$\mu_i(A) = U_i^*\mu(A)U_i,$$

where $U_i = S_iT_i^{-1}$. But, since $U_i^*U_i = U_i^*\mu(X)U_i = \mu_i(X) = I_\mathcal{H}$ and U_i is invertible, it follows that U_i is a unitary. This shows that μ_i is unitarily equivalent to μ, as was required. □

4.2 C^*-extreme POVMs with commutative ranges

With these two characterizations of C^*-extreme POVMs at our disposal, we are now ready to present the main results of this chapter. Gregg [33] shows that if a POVM μ is C^*-extreme in $\mathcal{P}_\mathcal{H}(X)$ (for a compact Hausdorff space X) then for any A in $\mathcal{O}(X)$, the spectrum of $\mu(A)$ is either contained in $\{0,1\}$ (so that $\mu(A)$ is a projection) or it is whole of the interval $[0,1]$. Our main observation is that the second situation can be avoided in a variety of cases. The proof uses straightforward Borel functional calculus, with a carefully chosen family of functions. These functions are necessarily discontinuous and so C^*-algebra setting and continuous functional calculus will not suffice.

Theorem 4.2.1. *Let μ be a C^*-extreme point in $\mathcal{P}_\mathcal{H}(X)$. If $E \in \mathcal{O}(X)$ is such that $\mu(A)\mu(E) = \mu(E)\mu(A)$ for all $A \subseteq E$ in $\mathcal{O}(X)$, then $\mu(E)$ is a projection. In particular if $\mu(E)$ commutes with all $\mu(B)$ for $B \in \mathcal{O}(X)$, then $\mu(E)$ is a projection.*

Proof. The second assertion is immediate from the first. So assume the hypothesis in the first statement. We claim that

$$\sigma(\mu(E)) \cap (r,s) = \emptyset \quad \text{for all} \quad 0 < r < s < 1,$$

where $\sigma(\mu(E))$ denotes the spectrum of the operator $\mu(E)$. As $\mu(E)$ is a positive contraction, it will follow that

$$\sigma(\mu(E)) \subseteq \{0,1\},$$

which in turn will imply that $\mu(E)$ is a projection. So fix $0 < r < s < 1$, and define the map $f := f_{r,s} : [0,1] \to [0,1]$ by

$$f_{r,s}(t) = \begin{cases} 1 & \text{if } t \notin [r,s], \\ \frac{r}{1-r}\left(\frac{1}{t} - 1\right) & \text{if } t \in [r,s]. \end{cases} \tag{4.2.1}$$

Clearly f is continuous except at one point namely s, and hence it is a Borel measurable function. So for any operator $0 \leq T \leq I_\mathcal{H}$, it follows from spectral theory that $f(T)$ is a well defined bounded operator (see (1.4.6)). Further we note for each $t \in [0,1]$, that

$$0 < \alpha := \left(\frac{r}{1-r}\right)\left(\frac{1-s}{s}\right) \leq f(t) \leq 1$$

and consequently,

$$\alpha I_{\mathcal{H}} \leq f(T) \leq I_{\mathcal{H}}. \tag{4.2.2}$$

Now consider the map $\nu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ defined by

$$\nu(B) = \mu(B \cap E) f(\mu(E)) + \mu(B \setminus E) \tag{4.2.3}$$

for any $B \in \mathcal{O}(X)$. We show that ν is a POVM by observing the following:

- For each $B \in \mathcal{O}(X)$, our hypothesis says that $\mu(B \cap E)$ and $\mu(E)$ commute and it then implies from spectral theory that $\mu(B \cap E)$ commutes with $f(\mu(E))$ (see Theorem 1.4.6). Therefore, as both $\mu(B \cap E)$ and $f(\mu(E))$ are positive operators, it follows that their product $\mu(E \cap B) f(\mu(E))$ is a positive operator, which amounts to saying that $\nu(B) \geq 0$ in $\mathcal{B}(\mathcal{H})$.

- If $B_1, B_2, \ldots$ is a countable collection of mutually disjoint measurable subsets of X and $B = \cup_n B_n$, then since μ is a POVM, we have in WOT convergence,

$$\begin{aligned} \nu(\cup_n B_n) &= \mu((\cup_n B_n) \cap E) f(\mu(E)) + \mu((\cup_n B_n) \setminus E) \\ &= \mu(\cup_n (B_n \cap E)) f(\mu(E)) + \mu(\cup_n (B_n \setminus E)) \\ &= \sum_n [\mu(B_n \cap E) f(\mu(E))] + \sum_n \mu(B_n \setminus E) \\ &= \sum_n \left[\mu(B_n \cap E) f(\mu(E)) + \mu(B_n \setminus E)\right] \\ &= \sum_n \nu(B_n). \end{aligned}$$

  This shows that μ is countably additive, which in particular implies that the function $B \mapsto \langle h, \nu(B) k \rangle$ is a complex measure on X for all $h, k \in \mathcal{H}$.

The observations above imply that ν is a POVM. Further since $f(\mu(E)) \leq I_{\mathcal{H}}$ from (4.2.2), it follows for each $B \in \mathcal{O}(X)$, that

$$\nu(B) = \mu(B \cap E) f(\mu(E)) + \mu(B \setminus E) \leq \mu(B \cap E) + \mu(B \setminus E) = \mu(B)$$

which is to say $\nu \leq \mu$. Also since $f(\mu(E)) \geq \alpha I_{\mathcal{H}}$ from (4.2.2), and $\mu(E) \leq I_{\mathcal{H}}$, we note that

$$\begin{aligned} \nu(X) &= \mu(E) f(\mu(E)) + \mu(X \setminus E) \\ &\geq \alpha \mu(E) + \mu(X \setminus E) \\ &= \alpha \mu(E) + I_{\mathcal{H}} - \mu(E) \\ &= I_{\mathcal{H}} - (1 - \alpha) \mu(E) \\ &\geq I_{\mathcal{H}} - (1 - \alpha) I_{\mathcal{H}} \\ &= \alpha I_{\mathcal{H}}, \end{aligned}$$

which is equivalent to saying that $\nu(X)$ is invertible. Therefore, as μ is a C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$, it follows from Corollary 4.1.5 that there exists an invertible operator $T \in \mathcal{B}(\mathcal{H})$ satisfying the condition

$$\nu(B) = T^*\mu(B)T \quad \text{for all } B \in \mathcal{O}(X). \tag{4.2.4}$$

We note that $\nu(X) = T^*T = |T|^2$ and hence,

$$|T| = \nu(X)^{1/2} = [\mu(E)f(\mu(E)) + I_{\mathcal{H}} - \mu(E)]^{1/2} \tag{4.2.5}$$

where $|T|$ denotes the square root of the positive operator T^*T. Set $S = \mu(E)$. By taking $B = E$ in (4.2.4) yields

$$T^*ST = T^*\mu(E)T = \nu(E) = \mu(E)f(\mu(E)) = Sf(S).$$

Let $T = U|T|$ be the polar decomposition of T. Then U is a unitary and $|T|$ is invertible, as T is invertible. Consequently,

$$U^*SU = |T|^{-1}Sf(S)|T|^{-1}. \tag{4.2.6}$$

Now let $g : [0,1] \to [0,1]$ be the map defined by

$$g(t) = \frac{tf(t)}{1 - t + tf(t)} = \begin{cases} t & \text{if } t \notin [r,s], \\ r & \text{if } t \in [r,s]. \end{cases}$$

Then $g(S)$ is a well-defined bounded operator and we get

$$g(S) = Sf(S)[I_{\mathcal{H}} - S + Sf(S)]^{-1}.$$

Hence (4.2.5) and (4.2.6) yield

$$U^*SU = g(S).$$

Therefore by Spectral mapping theorem (Theorem 1.4.7), spectrum of S satisfies the following:

$$\sigma(S) = \sigma(U^*SU) = \sigma(g(S)) \subseteq \operatorname{essran}(g),$$

where $\operatorname{essran}(g)$ is the essential range of g with respect to the spectral measure corresponding to the operator S. But,

$$\operatorname{essran}(g) \subseteq \overline{\mathcal{R}(g)} \subseteq [0,r] \cup [s,1],$$

where $\mathcal{R}(g)$ denotes the range of the function g. This implies that $\sigma(S) \subseteq [0,r] \cup [s,1]$, which is same as saying $\sigma(S) \cap (r,s) = \emptyset$. This is what we wanted to show. $\square$

A direct application of Theorem 4.2.1 is possible for C^*-extreme points with commutative ranges. We say a POVM μ is *commutative* if its range is commutative. It has been shown in [37] that a commutative normalized POVM is an extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ if and only if it is spectral.

A similar kind of result for C^*-extreme points holds true following the theorem above: if a C^*-extreme point μ in $\mathcal{P}_{\mathcal{H}}(X)$ is commutative, then it follows from Theorem 4.2.1 that $\mu(A)$ is projection for all $A \in \mathcal{O}(X)$ and hence μ is spectral. Thus we have arrived at the following theorem.

Theorem 4.2.2. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a commutative normalized POVM. Then μ is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X)$ if and only if it is a spectral measure.*

4.3 Atomic C^*-extreme POVMs

In this section, we examine atomic C^*-extreme POVMs and see their applications to POVMs on countable spaces and finite dimensional Hilbert spaces. See Definition 1.3.13 for relevant notions of atoms and atomic POVMs.

Theorem 4.2.1 is quite powerful. Here we have more applications of it. First consider the following lemma. Recall our assumption that singletons are measurable subsets.

Lemma 4.3.1. *Let μ be a C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$. Then $\mu(E)$ is a projection for every atom E for μ. In particular $\mu(\{x\})$ is a projection for all $x \in X$ and consequently $\mu(A)$ is a projection for every countable subset A of X.*

Proof. If E is an atom for μ then for each $B \subseteq E$ in $\mathcal{O}(X)$, either $\mu(B) = 0$ or $\mu(B) = \mu(E)$; hence $\mu(B)$ commutes with $\mu(E)$. Therefore Theorem 4.2.1 is applicable and it follows that $\mu(E)$ is a projection. This further implies that for each $x \in X$, since either $\mu(\{x\}) = 0$ or $\{x\}$ is an atom for μ, $\mu(\{x\})$ is a projection. Now let $x, y \in X$ be two distinct points and set

$$P = \mu(\{x\}) \quad \text{and } Q = \mu(\{y\}).$$

Note that

$$P + Q = \mu(\{x\}) + \mu(\{y\}) = \mu(\{x, y\}) \leq I_{\mathcal{H}}$$

and hence $P \leq I_{\mathcal{H}} - Q$. Because P and Q are projections as proved above, it follows that $P(I_{\mathcal{H}} - Q) = P$, which in turn yields

$$PQ = 0.$$

In other words, $\mu(\{x\})$ and $\mu(\{y\})$ are mutually orthogonal projections for any two distinct points x and y. Therefore, for any at most countable subset $A = \{x_1, x_2, \ldots\}$ of X, the collection $\{\mu(\{x_n\})\}$ consists of projections mutually orthogonal to one another and since

$$\mu(A) = \sum_{n \geq 1} \mu(\{x_n\}) \qquad \text{(in WOT)},$$

we conclude that $\mu(A)$ is a projection. □

The POVMs on finite sets have been natural settings for many applications in quantum theory. Several researchers have looked into the convexity structure in this set up and the structure of extreme points is very well studied. They are not always spectral measures. When it comes to C^*-convexity, it is shown in [24] that only spectral measures are C^*-extreme when $\mathcal{H}$ is finite dimensional. Here we show that it is true in full generality.

Following the results above, we now give a characterization of all atomic C^*-extreme points in $\mathcal{P}_{\mathcal{H}}(X)$. This in particular characterizes all C^*-extreme points in $\mathcal{P}_{\mathcal{H}}(X)$ whenever X is finite.

Theorem 4.3.2. *An atomic normalized POVM $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ is a C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ if and only if μ is spectral. In particular, if X is a countable measurable space then any C^*-extreme point of $\mathcal{P}_{\mathcal{H}}(X)$ is spectral.*

Proof. We have seen that spectral measures are always C^*-extreme. Conversely, assume that μ is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X)$. Since μ is atomic, it follows from Proposition 1.3.15 that there is a countable family $\{B_n\}_{n\geq 1}$ of mutually disjoint atoms for μ such that

$$\mu(A) = \sum_{n\geq 1} \mu(A \cap B_n), \quad \text{for all } A \in \mathcal{O}(X). \tag{4.3.1}$$

Now since B_n is an atom, we know that for any $A \in \mathcal{O}(X)$, either $\mu(A \cap B_n) = 0$ or $A \cap B_n$ is an atom for μ. Therefore since μ is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X)$, it follows from Lemma 4.3.1 that $\mu(A \cap B_n)$ is a projection for all $n \geq 1$. Since B_n's are mutually disjoint, Proposition 1.3.5 implies that the collection $\{\mu(A \cap B_n)\}_{n\geq 1}$ consists of mutually orthogonal projections. Consequently it follows from equation (4.3.1), that $\mu(A)$ is a projection. This proves that μ is spectral. Since any POVM on a countable measurable space is atomic, the second assertion follows. □

The question of when a C^*-extreme POVM is also extreme in $\mathcal{P}_{\mathcal{H}}(X)$ is very natural. When the Hilbert space is finite dimensional, this is always true as proved by Farenick et. al which we state below.

Lemma 4.3.3 (Proposition 2.1, [24])**.** *If $\mathcal{H}$ is a finite dimensional Hilbert space, then every C^*-extreme point of $\mathcal{P}_{\mathcal{H}}(X)$ is also extreme.*

However, the above result is not known in the case of infinite dimensional Hilbert spaces setting. Below we show this in a specific case of POVMs acting on countable spaces. Since all spectral measures are also extreme, the following corollary follows directly from Theorem 4.3.2.

Corollary 4.3.4. *If X is a countable (in particular, finite) measurable space, then every C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ is extreme.*

The case of finite dimensional Hilbert spaces

We end this section by recording the case of finite dimensional Hilbert spaces and general measurable spaces. This set up has been widely studied by several researchers. We recall that it is proved in [24] for a compact Hausdorff space X and a finite dimensional $\mathcal{H}$, that every C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ is spectral. We extend this result to full generality using Theorem 4.3.2.

Theorem 4.3.5. *Let $\mathcal{H}$ be a finite dimensional Hilbert space and X a measurable space. Then any C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ is spectral.*

Proof. Firstly, finite dimensionality of $\mathcal{H}$ ensures that every C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ is also extreme (Lemma 4.3.3). Now we show that every extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ is atomic (see Lemma 2, [12] for topological spaces) as follows: if μ is extreme in $\mathcal{P}_{\mathcal{H}}(X)$ and $(\pi, V, \mathcal{H}_\pi)$ is the minimal Naimark dilation for μ, then the map

$$D \mapsto V^* D V$$

from $\pi(\mathcal{O}(X))'$ to $\mathcal{B}(\mathcal{H})$ is one-to-one by Theorem 1.3.11. Since $\mathcal{H}$ is finite dimensional, $\mathcal{B}(\mathcal{H})$ is a finite dimensional algebra and hence $\pi(\mathcal{O}(X))'$ is a finite-dimensional algebra. Therefore, since

$$\pi(\mathcal{O}(X)) \subseteq \pi(\mathcal{O}(X))' \quad \text{and} \quad \mathcal{H}_\pi = [\pi(\mathcal{O}(X))\, V\mathcal{H}],$$

it follows that $\mathcal{H}_\pi$ is also finite-dimensional. Consequently $\{\pi(A) : A \in \mathcal{O}(X)\}$ is a commuting family of projections on a finite dimensional Hilbert space $\mathcal{H}_\pi$ and hence it is a finite set. This implies that π is atomic. Then by Proposition 1.3.17, μ is also atomic. Thus we have shown that every C^*-extreme point in $\mathcal{P}_\mathcal{H}(X)$ is atomic. The proof is complete in view of Theorem 4.3.2. □

Remark 4.3.6. In the theorem above, we noticed that any spectral measure acting on a finite dimensional Hilbert space is atomic.

4.4 Singular POVMs and their direct sums

The notion of mutual singularity of positive measures is very familiar from classical measure theory. We consider the similar notion of mutually singular POVMs. Our main aim here is to discuss the behaviour of C^*-extremity for direct sums of mutually singular POVMs. This helps us in characterization of C^*-extreme points, as we show that every C^*-extreme POVM can be decomposed into a direct sum of an atomic and a non-atomic normalized POVM.

Definition 4.4.1. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and $(X, \mathcal{O}(X))$ a measurable space. Two POVMs $\mu_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$, are called *mutually singular*, denoted $\mu_1 \perp \mu_2$, if there exist disjoint measurable subsets X_1 and X_2 of X such that $\mu_i(A) = \mu_i(A \cap X_i)$ for all $A \in \mathcal{O}(X)$.

The following proposition about singularity of atomic and non-atomic POVMs is very crucial for our subsequent results. It is a direct consequence of the classical case that an atomic finite positive measure is always mutually singular to a non-atomic positive measure (see Johnson [43]). We use it below.

Proposition 4.4.2. *Let $\mu_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_i)$, $i = 1, 2$ be two POVMs such that μ_1 is atomic and μ_2 is non-atomic. Then they are mutually singular.*

Proof. Consider strictly positive density operators S_i on $\mathcal{H}_i$ such that $T \mapsto \mathrm{Tr}(S_i T)$ (Tr denotes trace) are faithful normal states on $\mathcal{B}(\mathcal{H}_i)$ for $i = 1, 2$. Then $\lambda_i : \mathcal{O}(X) \to [0, \infty)$ defined by

$$\lambda_i(A) = \mathrm{Tr}(\mu_i(A) S_i) \quad \text{for all } A \in \mathcal{O}(X),$$

are positive measures which, for any $A \in \mathcal{O}(X)$ satisfy

$$\mu_i(A) = 0 \text{ if and only if } \lambda_i(A) = 0. \tag{4.4.1}$$

This in particular implies that λ_1 is atomic and λ_2 is non-atomic. Therefore, as noted above, λ_1 is mutually singular to λ_2 (see Theorem 2.5, [43]). This in turn implies due to (4.4.1) that μ_1 is mutually singular to μ_2. □

Disjoint spectral measures

Inspired by the notion of disjointness for representations of C^*-algebras (Definition 1.2.14), we introduce a similar notion for spectral measures. We do not know whether this concept has been studied before. It turns out that the concepts of singularity and disjointness of spectral measures are in fact same.

Let $\pi : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_\pi)$ be a spectral measure and let $\mathcal{H}$ be a closed subspace of $\mathcal{H}_\pi$ such that $\mathcal{H}$ is invariant (and hence reducing) under $\pi(A)$ for all $A \in \mathcal{O}(X)$. Then the mapping $A \mapsto \pi(A)_{|\mathcal{H}}$ gives rise to another spectral measure from $\mathcal{O}(X)$ to $\mathcal{B}(\mathcal{H})$, and is called a *sub-spectral measure* of π.

Definition 4.4.3. Two spectral measures $\pi_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_{\pi_i})$, $i = 1, 2$ are called *disjoint* if no non-zero sub-spectral measure of π_1 is unitarily equivalent to any sub-spectral measure of π_2.

Let $\lambda : \mathcal{O}(X) \to [0, \infty]$ be a σ-finite measure such that $L^2(\lambda)$ is a separable Hilbert space. Consider the map $\pi^\lambda : \mathcal{O}(X) \to \mathcal{B}(L^2(\lambda))$ defined by

$$\pi^\lambda(A) = M_{\chi_A} \quad \text{for all } A \in \mathcal{O}(X), \tag{4.4.2}$$

where M_{χ_A} is the multiplication operator by the characteristic function χ_A. It is straightforward to verify that π^λ is a spectral measure. Also $\pi^\lambda(A) = 0$ if and only if $\lambda(A) = 0$ for any $A \in \mathcal{O}(X)$. Such spectral measures are known as *canonical spectral measures*.

We first prove that the notion of singularity and disjointness are same in the case of canonical spectral measures, and then for general case. The proof here follows the same techniques which are usually employed for representations (see Theorem 2.2.2, [2]).

Lemma 4.4.4. *Let λ_1 and λ_2 be two σ-finite positive measures on X. Then λ_1 is mutually singular to λ_2 if and only if π^{λ_1} and π^{λ_2} are disjoint.*

Proof. Let π^{λ_1} and π^{λ_2} be disjoint spectral measures. Assume to the contrary that λ_1 and λ_2 are not mutually singular. Then by Lebesgue decomposition theorem, there is a non-zero σ-finite positive measure, say λ, such that λ is absolutely continuous with respect to both λ_1 and λ_2. Now for $i = 1, 2$, let

$$C_i = \{x \in X; \frac{d\lambda}{d\lambda_i}(x) > 0\}, \quad \text{and} \quad \mathcal{K}_i = \mathcal{R}(\pi^{\lambda_i}(C_i)) = \{\chi_{C_i} f; f \in L^2(\lambda_i)\} \subseteq L^2(\lambda_i),$$

where $\frac{d\lambda}{d\lambda_i}(x)$ is the Radon-Nikodym derivative of λ with respect to λ_i. Since $\lambda \neq 0$, we note that $\mathcal{K}_i \neq 0$. Now we define an operator $U_i : L^2(\lambda) \to \mathcal{K}_i$ by

$$U_i f = f\sqrt{\frac{d\lambda}{d\lambda_i}} = \chi_{C_i}\left(f\sqrt{\frac{d\lambda}{d\lambda_i}}\right), \quad f \in L^2(\lambda).$$

It is easy to see that U_i is a unitary operator such that $U_i\pi^\lambda(A) = \pi^{\lambda_i}(A)U_i$ for all $A \in \mathcal{O}(X)$. This shows that π^λ is unitarily equivalent to $\pi^{\lambda_i}(\cdot)_{|\mathcal{K}_i}$; hence the sub-spectral measures $\pi^{\lambda_i}(\cdot)_{|\mathcal{K}_i}$ are unitarily equivalent, which is a contradiction to the mutual disjointness of π^{λ_1} and π^{λ_2}. The proof of the converse is contained in the next theorem. $\square$

We use the familiar notion of direct sums of POVMs in the next theorem and in subsequent results. The *direct sum* of a collection $\{\mu_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_i)\}_{i\in\Lambda}$ of POVMs is the map $\oplus_i\mu_i : \mathcal{O}(X) \to \mathcal{B}(\oplus_i\mathcal{H}_i)$ defined by

$$(\oplus_i\mu_i)(A) = \oplus_i\mu_i(A) \quad \text{for all } A \in \mathcal{O}(X). \tag{4.4.3}$$

It is immediate that $\oplus_i\mu_i$ is a POVM. Further it is normalized if and only if each μ_i is normalized. Also $\oplus_i\mu_i$ is a spectral measure if and only if each μ_i is a spectral measure.

Remark 4.4.5. If $\{\mu_i\}_{i\in\Lambda}$ is a collection of POVMs with minimal Naimark triples $(\pi_i, V_i, \mathcal{K}_i)$, then it is immediate to verify that the minimal Naimark triple for $\oplus_i\mu_i$ is given by $(\pi, V, \mathcal{K})$, where $\mathcal{K} = \oplus_i\mathcal{K}_i$, $V = \oplus_i V_i$ and $\pi = \oplus_i\pi_i$.

We now give some equivalent conditions for disjoint spectral measures similar to those for disjointness of representations (Proposition 1.2.15). This also shows that the notions of singularity and disjointness are same.

Theorem 4.4.6. *Let $\pi_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_{\pi_i}), i = 1, 2$ be two spectral measures, where $\mathcal{H}_{\pi_i}$ are separable Hilbert spaces. Then the following are equivalent:*

(i) π_1 and π_2 are mutually singular.

(ii) π_1 is disjoint to π_2.

(iii) If for $T \in \mathcal{B}(\mathcal{H}_{\pi_1}, \mathcal{H}_{\pi_2})$, $T\pi_1(A) = \pi_2(A)T$ for all $A \in \mathcal{O}(X)$, then $T = 0$.

Proof. (i) $\implies$ (iii): Let π_1 and π_2 be mutually singular. Then there are disjoint measurable subsets C_1 and C_2 such that $\pi_i(A) = \pi_i(A\cap C_i)$ for all $A \in \mathcal{O}(X)$ and $i = 1, 2$. If $T \in \mathcal{B}(\mathcal{H}_{\pi_1}, \mathcal{H}_{\pi_2})$ satisfies $T\pi_1(A) = \pi_2(A)T$ for all $A \in \mathcal{O}(X)$, then since $\pi_1(C_1) = I_{\mathcal{H}_{\pi_1}}$ and $\pi_2(C_1) = 0$, it follows that

$$T = T\pi_1(C_1) = \pi_2(C_1)T = 0.$$

(iii) $\implies$ (ii): If π_1 and π_2 are not disjoint, then there are non-zero closed subspaces $\mathcal{K}_i$ of $\mathcal{H}_{\pi_i}$ invariant under $\pi_i(A)$ for all $A \in \mathcal{O}(X)$, and a unitary $U : \mathcal{K}_1 \to \mathcal{K}_2$ such that

$$U\pi_1(A)_{|\mathcal{K}_1} = \pi_2(A)_{|\mathcal{K}_2}U \quad \text{for all } A \in \mathcal{O}(X).$$

Extend U to $\mathcal{H}_{\pi_1}$ by assigning 0 on $\mathcal{H}_{\pi_1} \ominus \mathcal{K}_1$, which we call by $\widetilde{U}$. Then it is immediate that $\widetilde{U} \neq 0$ and $\widetilde{U}\pi_1(A) = \pi_2(A)\widetilde{U}$ for all $A \in \mathcal{O}(X)$, violating the condition in part (3).

(ii) $\implies$ (i): Let π_1 and π_2 be disjoint. We now invoke Hahn-Hellinger Theorem (see Theorem 7.6, [58]) to obtain a collection, say $\{\lambda_n^i\}_{n\in\mathbb{N}\cup\{\infty\}}$, of σ-finite positive measures (possibly zero measures) mutually singular to one another such that, upto unitary equivalence, we have

$$\pi_i = \bigoplus_{n\in\mathbb{N}\cup\{\infty\}} n \cdot \pi^{\lambda_n^i}$$

for $i = 1, 2$. Here $n \cdot \pi^{\lambda_n^i}$ denotes the direct sums of n copies of $\pi^{\lambda_n^i}$ (when $n = \infty$, the direct sum is countably infinite). Because π_1 and π_2 are disjoint, each $\pi^{\lambda_n^1}$ must be disjoint to $\pi^{\lambda_m^2}$

for $m, n \in \mathbb{N} \cup \{\infty\}$. It then follows from Lemma 4.4.4 that λ_n^1 is mutually singular to λ_m^2 as positive measures. Therefore for each n, m, there exist measurable subsets X_{nm}^1 and X_{nm}^2 satisfying $X_{nm}^1 \cap X_{nm}^2 = \emptyset$ and

$$\lambda_n^1(A) = \lambda_n^1(A \cap X_{nm}^1) \quad \text{and} \quad \lambda_m^2(A) = \lambda_m^2(A \cap X_{nm}^2),$$

for all $A \in \mathcal{O}(X)$. Set

$$X^1 = \cup_n \cap_m X_{nm}^1 \text{ and } X^2 = \cup_m \cap_n X_{nm}^2.$$

Then by usual set theory rules:

$$X^1 \cap X^2 = \Big(\cup_n \cap_m X_{nm}^1\Big) \cap \Big(\cup_k \cap_l X_{lk}^2\Big) = \cup_n \cup_k \Big[\Big(\cap_m X_{nm}^1\Big) \cap \Big(\cap_l X_{lk}^2\Big)\Big] \subseteq \cup_n \cup_k \Big(X_{nk}^1 \cap X_{nk}^2\Big) = \emptyset,$$

by using $X_{nk}^1 \cap X_{nk}^2 = \emptyset$. Further for any $A \in \mathcal{O}(X)$ and fixed n, since $\lambda_n^1(A \cap X_{nm}^1) = \lambda_n^1(A)$ for all m, we have

$$\lambda_n^1(A) \geq \lambda_n^1(A \cap X^1) \geq \lambda_n^1(\cap_m \Big(A \cap X_{nm}^1)\Big) = \lim_{l \to \infty} \lambda_n^1 \Big(\cap_{m=1}^l (A \cap X_{nm}^1)\Big) = \lambda_n^1(A),$$

where limit is taken in WOT. This implies

$$\lambda_n^1(A \cap X^1) = \lambda_n^1(A).$$

Similarly, we get

$$\lambda_m^2(A \cap X^2) = \lambda_m^2(A) \quad \text{for each} \quad m.$$

Put differently, we obtain $\pi^{\lambda_n^i}(A \cap X^i) = \pi^{\lambda_n^i}(A)$, which further implies that

$$\pi_i(A \cap X^i) = \bigoplus_{n \in \mathbb{N} \cup \{\infty\}} n \cdot \pi^{\lambda_n^i}(A \cap X^i) = \bigoplus_{n \in \mathbb{N} \cup \{\infty\}} n \cdot \pi^{\lambda_n^i}(A) = \pi_i(A),$$

for each $A \in \mathcal{O}(X)$ and $i = 1, 2$. Since X^1 and X^2 are disjoint, we conclude that π_1 is mutually singular to π_2. □

Remark 4.4.7. In Theorem 4.4.6, we assumed that the spectral measures act on separable Hilbert spaces. But the implication (1) $\implies$ (3) is true even for non-separable Hilbert spaces and the proof is similar. To see this, let $\pi_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{K}_i)$, $i = 1, 2$ be two mutually singular spectral measures concentrated on measurable subsets X_i with $X_1 \cap X_2 = \emptyset$. Here $\mathcal{K}_i$ need not be separable. Let $T \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ be such that $T\pi_1(A) = \pi_2(A)T$ for all $A \in \mathcal{O}(X)$. Then, since $\pi_1(X_1) = \pi_1(X) = I_{\mathcal{K}_1}$ and $\pi_2(X_1) = 0$, we have $T = T\pi_1(X_1) = \pi_2(X_1)T = 0$. We use this fact in the next theorem.

Direct sums and C^*-extreme points

We now explore the properties of being C^*-extreme (or extreme) under direct sums of mutually singular POVMs. Generally it is enough to look at individual components to obtain the same property for direct sums. The results and proof here are very similar to the case of direct sums of disjoint UCP maps (Proposition 2.3.8).

Theorem 4.4.8. *Let $\{\mu_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_i)\}_{i\in\Lambda}$ be a countable collection of normalized POVMs for some indexing set Λ such that μ_i and μ_j are mutually singular for $i \neq j$ in Λ. Then $\mu = \oplus_i \mu_i$ is C^*-extreme (resp. extreme) in $\mathcal{P}_{\oplus_i\mathcal{H}_i}(X)$ if and only if each μ_i is C^*-extreme (resp. extreme) in $\mathcal{P}_{\mathcal{H}_i}(X)$.*

Proof. For each $i \in \Lambda$, let $(\pi_i, V_i, \mathcal{H}_{\pi_i})$ be the minimal Naimark dilation for μ_i. Set

$$\mathcal{H} = \oplus_i\mathcal{H}_i, \quad \mathcal{H}_\pi = \oplus_i\mathcal{H}_{\pi_i}, \quad \pi = \oplus_i\pi_i \quad \text{and} \quad V = \oplus_i V_i.$$

Then $(\pi, V, \mathcal{H}_\pi)$ is the minimal Naimark dilation for μ (see Remark 4.4.5). Also for $i \neq j$ in Λ, since μ_i is mutually singular to μ_j, it follows from Proposition 1.3.7 that π_i is mutually singular to π_j. Now we claim (compare this with Lemma 2.3.6) that

$$\pi(\mathcal{O}(X))' = \oplus_i\pi_i(\mathcal{O}(X))' = \{\oplus_i S_i;\ S_i \in \pi_i(\mathcal{O}(X))'\}\,. \tag{4.4.4}$$

Let $S \in \pi(\mathcal{O}(X))' \subseteq \mathcal{B}(\oplus_i\mathcal{H}_{\pi_i})$. Then $S = [S_{ij}]$ for some $S_{ij} \in \mathcal{B}(\mathcal{H}_{\pi_j}, \mathcal{H}_{\pi_i})$. For any $A \in \mathcal{O}(X)$, therefore we have $[S_{ij}]\,(\oplus_i\pi_i(A)) = (\oplus_i\pi_i(A)[\,S_{ij}]$, that is, $[S_{ij}\pi_j(A)] = [\pi_i(A)S_{ij}]$; hence

$$S_{ij}\pi_j(A) = \pi_i(A)S_{ij} \ \text{ for all } i, j \in \Lambda.$$

In particular, this says that $S_{ii} \in \pi_i(\mathcal{O}(X))'$ for all $i \in \Lambda$. Also since π_i and π_j are mutually singular for $i \neq j$, it follows from Remark 4.4.7 that $S_{ij} = 0 \quad$ for $i \neq j$. Thus we get

$$S = [S_{ij}] = \oplus_i S_{ii} \in \oplus_i\pi_i(\mathcal{O}(X))'.$$

This proves that $\pi(\mathcal{O}(X))' \subseteq \oplus_i\pi_i(\mathcal{O}(X))'$. The other inclusion of our claim is obvious.

In order to prove the equivalent assertions of C^*-extremity, we use the claim above and the necessary and sufficient criterion of Theorem 4.1.3 throughout the proof without always mentioning them. First assume that μ is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X)$. Fix $j \in \Lambda$ and let $D_j \in \pi_j(\mathcal{O}(X))'$ be positive such that $V_j^* D_j V_j$ is invertible. Define

$$D = \oplus_i D_i$$

by assigning $D_i = I_{\mathcal{H}_{\pi_i}}$ for $i \neq j$. It is clear that D is positive and $D \in \pi(\mathcal{O}(X))'$. Since $V^*DV = \oplus_i V_i^* D_i V_i$, and $V_i^* D_i V_i$ is invertible for all i whose inverse is uniformly bounded, it follows that V^*DV is invertible. Therefore, as μ is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X)$, we get a co-isometry $U \in \pi(\mathcal{O}(X))'$ with $U^*UD^{1/2} = D^{1/2}$ and an invertible operator $T \in \mathcal{B}(\mathcal{H})$ such that $UD^{1/2}V = VT$. Then $T = [T_{ij}]$ for some $T_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$ and $U = \oplus_i U_i$ for $U_i \in \pi_i(\mathcal{O}(X))'$. Since U is a co-isometry, each U_i is a co-isometry. Also, since

$$\oplus_i U_i^* U_i D_i^{1/2} = (\oplus_i U_i^*)\,(\oplus_i U_i)\left(\oplus_i D_i^{1/2}\right) = U^*UD^{1/2} = D^{1/2} = \oplus_i D_i^{1/2},$$

it follows in particular that

$$U_j^* U_j D_j^{1/2} = D_j^{1/2}.$$

Further, since

$$\oplus_i U_i D_i^{1/2} V_i = UD^{1/2}V = VT = (\oplus_i V_i)[T_{ij}] = [V_i T_{ij}], \tag{4.4.5}$$

it follows for $i \neq j$ that, $V_i T_{ij} = 0$ and hence $T_{ij} = V_i^* V_i T_{ij} = 0$. This amounts to saying that $T = \oplus_i T_{ii}$ and its invertibility, in particular, implies that T_{jj} is invertible in $\mathcal{B}(\mathcal{H}_j)$. Also (4.4.5) yields

$$U_j D_j^{1/2} V_j = V_j T_{jj}.$$

As U_j is a co-isometry in $\pi_j(\mathcal{O}(X))'$ satisfying $U_j^* U_j D_j^{1/2} = D_j^{1/2}$ and T_{jj} is invertible in $\mathcal{B}(\mathcal{H}_j)$ such that $U_j D_j^{1/2} V_j = V_j T_{jj}$, we conclude that μ_j is C^*-extreme in $\mathcal{P}_{\mathcal{H}_j}(X)$.

Conversely, assume that each μ_i is C^*-extreme in $\mathcal{P}_{\mathcal{H}_i}(X)$. Let $D \in \pi(\mathcal{O}(X))'$ be positive such that $V^* D V$ is invertible. Then from (4.4.4), we have $D = \oplus_i D_i$ for some $D_i \in \pi_i(\mathcal{O}(X))'$. Clearly each D_i is positive. Since $V^* D V$ is invertible and $V^* D V = \oplus_i V_i^* D_i V_i$, it follows that $V_i^* D_i V_i$ is invertible for all $i \in \Lambda$. Again, as μ_i is C^*-extreme in $\mathcal{P}_{\mathcal{H}_i}(X)$, we obtain a co-isometry $U_i \in \pi_i(\mathcal{O}(X))'$ with $U_i^* U_i D_i^{1/2} = D_i^{1/2}$ and an invertible operator $T_i \in \mathcal{B}(\mathcal{H}_i)$ such that $U_i D_i^{1/2} V_i = V_i T_i$. Set

$$U = \oplus_i U_i \quad \text{and} \quad T = \oplus_i T_i.$$

Then $U \in \pi\left(\mathcal{O}(X)\right)'$ and U is a co-isometry, as each U_i is a co-isometry. Likewise T is invertible in $\mathcal{B}(\mathcal{H})$, since each T_i is invertible. Further we note that

$$U^* U D^{1/2} = \oplus_i U_i^* U_i D_i^{1/2} = \oplus_i D_i^{1/2} = D^{1/2}.$$

Similarly we get

$$U D^{1/2} V = \oplus_i U_i D_i^{1/2} V_i = \oplus_i V_i T_i = VT.$$

Thus we conclude that μ is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X)$.

The case of equivalent assertions of extremity can be proved similarly, using (4.4.4) and Extreme point condition (Theorem 1.3.11). □

The following corollary is just an explicit description of the theorem above.

Corollary 4.4.9. *Let $\mu \in \mathcal{P}_{\mathcal{H}}(X)$ and let $\{B_i\}_{i \in \Lambda}$ be a collection of disjoint measurable subsets such that $X = \cup_{i \in \Lambda} B_i$ and $\mu(B_i)$ is a projection for each i. Let $\mathcal{H}_i = \Re(\mu(B_i))$ and define $\mu_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_i)$ by*

$$\mu_i(A) = \mu(B_i \cap A)_{|\mathcal{H}_i} \quad \text{for all} \quad A \in \mathcal{O}(X).$$

Then μ is C^*-extreme (resp. extreme) in $\mathcal{P}_{\mathcal{H}}(X)$ if and only if each μ_i is C^*-extreme (resp. extreme) in $\mathcal{P}_{\mathcal{H}_i}(X)$.

Proof. Let $(\pi, V, \mathcal{H}_\pi)$ be the minimal Naimark dilation for μ. Since $\mu(B_i)$ is a projection for each i and B_i's are mutually disjoint, it follows from Proposition 1.3.5 that $\mu(B_i)$'s are mutually orthogonal projections. Also $\mu(B_i)$ commutes with $\mu(A)$ for each $A \in \mathcal{O}(X)$ by Proposition 1.3.5, which implies that each $\mathcal{H}_i$ is a reducing subspace for all $\mu(A), A \in \mathcal{O}(X)$ by Proposition 1.3.5 and hence μ_i is a well-defined normalized POVM. Further, since $X = \cup_i B_i$, we have

$$\mathcal{H} = \oplus_i \mathcal{H}_i \quad \text{and} \quad \mu = \oplus_i \mu_i.$$

The assertions now are direct consequence of Theorem 4.4.8. □

As we mentioned earlier in Theorem 1.3.16, every POVM decomposes uniquely as a sum of atomic and non-atomic POVMs. Additionally if μ is C^*-extreme then we show that this decomposition can be made into a direct sum of atomic and non-atomic POVMs such that each of the components is C^*-extreme. The following theorem effectively provides a proof of Theorem 1.3.16 and then discusses its role in identifying C^*-extreme POVMs. The proof here follows almost the same procedure which can be found in [54], [43].

Theorem 4.4.10. *Let μ be a C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$. Then $\mu = \mu_1 \oplus \mu_2$ where μ_1 is an atomic normalized POVM and μ_2 is a non-atomic normalized POVM and they are mutually singular. Such a decomposition is unique. Furthermore μ_1 and μ_2 are C^*-extreme and in particular μ_1 is spectral.*

Proof. Let $\{B_j\}_{j\in\Lambda}$ be a maximal collection of mutually disjoint atoms for μ, which exists due to Zorn's lemma. As in the proof of Theorem 4.3.2, since μ is C^*-extreme, we note using Lemma 4.3.1 that $\mu(B_j)$ is a projection for each j. Also $\{\mu(B_j)\}_{j\in\Lambda}$ are mutually orthogonal by Proposition 1.3.5. Since $\mathcal{H}$ is separable, it follows from Proposition 1.3.19 that Λ is countable. This further implies that if we set

$$X_1 = \cup_{j\in\Lambda} B_j,$$

then we have,

$$\mu(X_1) = \sum_{j\in\Lambda} \mu(B_j), \tag{4.4.6}$$

and hence $\mu(X_1)$ is a projection. Now set

$$X_2 = X \setminus X_1.$$

For $i = 1, 2$, let

$$\mathcal{H}_i = \mathcal{R}(\mu(X_i)),$$

and define the operator valued measures $\mu_i : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_i)$ by

$$\mu_i(A) = \mu(A \cap X_i)_{|\mathcal{H}_i} = \mu(A)_{|\mathcal{H}_i} \quad \text{for all } A \in \mathcal{O}(X).$$

It is clear that each μ_i is a normalized POVM. Also $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and

$$\mu = \mu_1 \oplus \mu_2.$$

Now we show that μ_1 is atomic. Assume that $\mu_1(A) \neq 0$ for some $A \in \mathcal{O}(X)$. Then $\mu(A\cap X_1) \neq 0$ and, since

$$\mu(A \cap X_1) = \sum_{j\in\Lambda} \mu(A \cap B_j),$$

it follows that $\mu(A\cap B_j) \neq 0$ for some j and hence $\mu_1(A\cap B_j) \neq 0$. Therefore, as B_j is an atom for μ, $A \cap B_j$ is an atom for μ. Consequently, as $\mu_1(A \cap B_j) \neq 0$, it follows that $A \cap B_j$ is an atom for μ_1. Thus we have got an atom contained in the subset A with $\mu_1(A) \neq 0$, which shows that μ_1 is atomic.

To prove that μ_2 is non-atomic, let if possible, A be an atom for μ_2. Since μ_2 is concentrated on X_2, $A \cap X_2$ is an atom for μ_2 and hence $A \cap X_2$ is an atom for μ. But then $\{B_j\}_{j\in\Lambda} \cup \{A \cap X_2\}$ is a collection of mutually disjoint atoms for μ, violating the maximality of the collection $\{B_j\}_{j\in\Lambda}$. Thus we conclude that μ_2 is non-atomic. It is clear that μ_1 and μ_2 are mutually singular.

To show the uniqueness, let $\nu_1 \oplus \nu_2$ be another such decomposition into a direct sum of atomic $\nu_1 \in \mathcal{P}_{\mathcal{K}_1}(X)$ and non-atomic $\nu_2 \in \mathcal{P}_{\mathcal{K}_2}(X)$ where $\mathcal{H} = \mathcal{K}_1 \oplus \mathcal{K}_2$. We shall show that $\mathcal{K}_i = \mathcal{H}_i$ and $\nu_i = \mu_i$ for $i = 1, 2$. Let Y_1 and Y_2 be disjoint measurable subsets such that

$$\nu_i(A) = \nu_i(A \cap Y_i) \ \text{ for all } \quad A \in \mathcal{O}(X).$$

We know from Proposition 4.4.2 that $\mu_1 \perp \nu_2$ and $\mu_2 \perp \nu_1$; hence Y_1 and Y_2 can be chosen so that $Y_1 \cap X_2 = Y_2 \cap X_1 = \emptyset$. Therefore for each $i = 1, 2$, since both μ_i and ν_i are concentrated on $X_i \cup Y_i$, and since $(X_1 \cup Y_1) \cap (X_2 \cup Y_2) = \emptyset$, we can assume without loss of generality, that $X_i = Y_i$ (just replace X_i, Y_i by $X_i \cup Y_i$). Further note that

$$I_{\mathcal{K}_i} = \nu_i(Y_i) = \mu(Y_i)_{|\mathcal{K}_i} = \mu(X_i)_{|\mathcal{K}_i} = P_{\mathcal{H}_i|_{\mathcal{K}_i}},$$

where $P_{\mathcal{H}_i}$ denotes the projection of $\mathcal{H}$ onto $\mathcal{H}_i$. This implies $\mathcal{K}_i \subseteq \mathcal{H}_i$. By symmetry, we have $\mathcal{H}_i \subseteq \mathcal{K}_i$. Hence $\mathcal{K}_i = \mathcal{H}_i$. Similarly for all $A \in \mathcal{O}(X)$, we get

$$\nu_i(A) = \nu_i(A \cap Y_i) = \mu(A \cap Y_i)_{|\mathcal{K}_i} = \mu(A \cap X_i)_{|\mathcal{H}_i} = \mu_i(A \cap X_i) = \mu_i(A)$$

showing that $\nu_i = \mu_i$. The last statement follows from Theorem 4.4.8 and Theorem 4.3.2. $\square$

Remark 4.4.11. In the theorem above, we cannot expect a similar kind of direct sum decomposition for a normalized POVM which is not C^*-extreme. To see an example, let λ_1 and λ_2 be two probability measures on some measurable space X such that λ_1 is atomic while λ_2 is non-atomic. Let $T \in \mathcal{B}(\mathcal{H})$ be a positive contraction which is not a projection. Consider the POVM $\mu \in \mathcal{P}_{\mathcal{H}}(X)$ defined by $\mu(\cdot) = \lambda_1(\cdot)T + \lambda_2(\cdot)(I_{\mathcal{H}} - T)$. One can easily verify that no decomposition of μ into a direct sum of atomic and non-atomic normalized POVMs exists.

One reason for us to study the notion of mutually singular POVMs is the following result. Its proof follows from Theorem 4.4.10 and Theorem 4.4.8. Since we have already characterized all atomic C^*-extreme points (Theorem 4.3.2), it says in particular that it is sufficient to look for the characterization of non-atomic C^*-extreme points to understand the general situation.

Corollary 4.4.12. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a normalized POVM and let $X_1 = \cup_{i\in\Lambda} B_i$ be the union of a maximal collection $\{B_i\}_{i\in\Lambda}$ of mutually disjoint atoms for μ. Let $X_2 = X \setminus X_1$. Then μ is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X)$ if and only if*

(i) the operators $\mu(X_1)$ and $\mu(X_2)$ are projections and,

(ii) $\mu = \mu_1 \oplus \mu_2$ such that μ_i is C^*-extreme in $\mathcal{P}_{\mathcal{H}_i}(X)$, where $\mathcal{H}_i = \mathcal{R}(\mu(X_i))$ and $\mu_i = \mu(\cdot)_{|\mathcal{H}_i}$ for $i = 1, 2$.

4.5 Measure Isomorphic POVMs

We digress a bit from the earlier discussions and explore C^*-extreme properties from the perspective of measure isomorphism. In classical measure theory, this notion has been studied extensively. The idea is to neglect measure zero subsets in considering isomorphisms. One consequence is that most questions about abstract measure spaces get reduced to questions about sub σ-algebras of the Borel σ-algebra of the unit interval $[0,1]$. In a sense this space is universal.

Measure isomorphism for POVMs seems to have been first studied in [21]. Our aim here is quite limited to investigate preservation of some natural properties of POVMs, especially C^*-extremity, under this isomorphism. Here too we see the role of the unit interval.

Let X be a measurable space and $\mathcal{H}$ a Hilbert space. Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM. For each $A \in \mathcal{O}(X)$, let $[A]_\mu$ denote the set

$$[A]_\mu := \{B \in \mathcal{O}(X); \mu(A \setminus B) = 0 = \mu(B \setminus A)\} = \{B \in \mathcal{O}(X); \mu(B) = \mu(A) = \mu(B \cap A)\}.$$

Consider

$$\Sigma(\mu) := \{[A]_\mu; A \in \mathcal{O}(X)\}.$$

Then $\Sigma(\mu)$ is a Boolean σ-algebra under the following operations:

$$\begin{aligned} [A]_\mu \setminus [B]_\mu &= [A \setminus B]_\mu \\ [A]_\mu \cap [B]_\mu &= [A \cap B]_\mu \end{aligned} \tag{4.5.1}$$

for any $A, B \in \mathcal{O}(X)$. Define $\tilde{\mu} : \Sigma(\mu) \to \mathcal{B}(\mathcal{H})$ by

$$\tilde{\mu}([A]_\mu) = \mu(A) \quad \text{for all } A \in \mathcal{O}(X),$$

which is well defined by virtue of the very definition of $[A]_\mu$. If there is no possibility of confusion, we shall still denote $\tilde{\mu}$ by μ only.

Definition 4.5.1. ([21]) For $i = 1, 2$, let X_i be two measurable spaces and let $\mathcal{H}$ be a Hilbert space. Two POVMs $\mu_i : \mathcal{O}(X_i) \to \mathcal{B}(\mathcal{H})$ are called *measure isomorphic*, and denoted $\mu_1 \cong \mu_2$, if there exists a Boolean isomorphism $\Phi : \Sigma(\mu_1) \to \Sigma(\mu_2)$ i.e. Φ is bijective and both Φ and Φ^{-1} preserve the operations in (4.5.1):

$$\begin{aligned} \Phi\left([A_1]_{\mu_1} \setminus [B_1]_{\mu_1}\right) &= \Phi([A_1]_{\mu_1}) \setminus \Phi([B_1]_{\mu_1}), \\ \Phi\left([A_1]_{\mu_1} \cap [B_1]_{\mu_1}\right) &= \Phi\left([A_1]_{\mu_1}\right) \cap \Phi\left([B_1]_{\mu_1}\right) \ \textit{etc.} \end{aligned} \tag{4.5.2}$$

such that $\mu_1(A_1) = \mu_2\left(\Phi([A_1]_{\mu_1})\right)$ for all $A_1, B_1 \in \mathcal{O}(X_1)$.

The following theorem compares some natural properties of POVMs under measure isomorphism.

Theorem 4.5.2. *Let $\mu_i : \mathcal{O}(X_i) \to \mathcal{B}(\mathcal{H})$, $i = 1, 2$ be two normalized POVMs such that they are measure isomorphic. Then we have the following:*

(i) μ_1 is a spectral measure if and only if μ_2 is a spectral measure.

(ii) μ_1 *is atomic (resp. non-atomic) if and only if* μ_2 *is atomic (resp. non-atomic).*

(iii) μ_1 *is* C^**-extreme (resp. extreme) in* $\mathcal{P}_{\mathcal{H}}(X_1)$ *if and only if* μ_2 *is* C^**-extreme (resp. extreme) in* $\mathcal{P}_{\mathcal{H}}(X_2)$.

Proof. Let $\Phi : \Sigma(\mu_1) \to \Sigma(\mu_2)$ be a Boolean isomorphism satisfying $\mu_1(A_1) = \mu_2(\Phi([A]_{\mu_1}))$ for all $A_1 \in \mathcal{O}(X_1)$. By symmetry, it is enough to prove the statements in just one direction.

(i) This is straightforward by isomorphism: If μ_2 is a spectral measure then for any $A_1 \in \mathcal{O}(X_1)$, $\mu_2(\Phi([A_1]_{\mu_1}))$ is a projection. Since $\mu_1(A_1) = \mu_2\left(\Phi([A_1]_{\mu_1})\right)$, it follows that $\mu_1(A_1)$ is a projection and hence μ_1 is a spectral measure.

(ii) Firstly we claim that if A_1 is an atom for μ_1, then A_2 is an atom for μ_2 for any $A_2 \in \Phi([A_1]_{\mu_1})$. To see this, first note that $\mu_2(A_2) = \mu_1(A_1) \neq 0$. Let $A_2' \subseteq A_2$ be a measurable subset. Then for any $A_1' \in \Phi^{-1}([A_2']_{\mu_2})$, we have

$$\Phi\left([A_1' \cap A_1]_{\mu_1}\right) = \Phi\left([A_1']_{\mu_1}\right) \cap \Phi\left([A_1]_{\mu_1}\right) = [A_2']_{\mu_2} \cap [A_2]_{\mu_2} = [A_2' \cap A_2]_{\mu_2} = [A_2']_{\mu_2} = \Phi([A_1']_{\mu_1})$$

and hence $[A_1' \cap A_1]_{\mu_1} = [A_1']_{\mu_1}$, which in turn implies

$$\mu_1(A_1' \cap A_1) = \mu_1(A_1'). \tag{4.5.3}$$

But since A_1 is atomic for μ_1, we have

$$\text{either } \mu_1(A_1' \cap A_1) = 0 \text{ or } \mu_1(A_1' \cap A_1) = \mu_1(A_1)$$

and therefore from (4.5.3),

$$\text{either } \mu_1(A_1') = 0 \text{ or } \mu_1(A_1') = \mu_1(A_1).$$

Since $A_1 \in \Phi^{-1}([A_2]_{\mu_2})$ and $A_1' \in \Phi^{-1}([A_2']_{\mu_2})$, it follows that

$$\text{either } \mu_2(A_2') = 0 \text{ or } \mu_2(A_2') = \mu_2(A_2).$$

This shows our claim that A_2 is an atom for μ_2. Now assume that μ_1 is atomic. To show that μ_2 is atomic, let $A_2 \in \mathcal{O}(X_2)$ be such that $\mu_2(A_2) \neq 0$. If $A_1 \in \Phi^{-1}([A_2]_{\mu_2})$, then

$$\mu_1(A_1) = \mu_2(A_2) \neq 0.$$

Since μ_1 is atomic, A_1 contains an atom for μ_1, say A_1'. Fix $A_2' \in \Phi([A_1']_{\mu_1})$. Then A_2' is an atom for μ_2 by the claim above. As above we show that

$$\mu_2(A_2' \cap A_2) = \mu_2(A_2'),$$

which implies that $A_2' \cap A_2$ is an atom for μ_2 contained in A_2. This proves that μ_2 is atomic. Similarly if μ_1 is non-atomic, then there is no atom for μ_1, and again it follows from the claim above that there is no atom for μ_2, which is equivalent to saying that μ_2 is non-atomic.

(iii) Assume that μ_2 is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X_2)$. To show that μ_1 is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X_1)$, let $\mu_1(\cdot) = \sum_{j=1}^n T_j^* \mu_1^j(\cdot) T_j$ be a proper C^*-convex combination in $\mathcal{P}_{\mathcal{H}}(X_1)$. For each j, define $\mu_2^j : \mathcal{O}(X_2) \to \mathcal{B}(\mathcal{H})$ by

$$\mu_2^j(A_2) = \mu_1^j\left(\Phi^{-1}\left([A_2]_{\mu_2}\right)\right) \quad \text{for all } A_2 \in \mathcal{O}(X_2).$$

For μ_2^j to be well defined, we need to show that

$$\mu_1^j(A_1) = \mu_1^j(A_1') \quad \text{for all } A_1, A_1' \in \Phi^{-1}([A_2]_{\mu_2}).$$

So fix $A_1, A_1' \in \Phi^{-1}([A_2]_{\mu_2})$. Then $[A_1]_{\mu_1} = [A_1']_{\mu_1}$ and hence, we get

$$\mu_1(A_1 \setminus A_1') = 0 = \mu_1(A_1' \setminus A_1).$$

Therefore, since $T_j^* \mu_1^j(\cdot) T_j \leq \mu_1(\cdot)$, it follows that

$$T_j^* \mu_1^j(A_1 \setminus A_1') T_j = 0 = T_j^* \mu_1^j(A_1' \setminus A_1) T_j$$

which, as T_j is invertible, yields

$$\mu_1^j(A_1 \setminus A_1') = 0 = \mu_1^j(A_1' \setminus A_1).$$

This implies the requirement for well-definedness of μ_2^j. Also note that

$$\mu_1^j(A_1) = \mu_1^j\left(\Phi^{-1}(\Phi([A_1]_{\mu_1})\right) = \mu_2^j(\Phi([A_1]_{\mu_1})), \tag{4.5.4}$$

for all $A_1 \in \mathcal{O}(X_1)$. Further for any $A_2 \in \mathcal{O}(X_2)$, we have

$$\sum_{j=1}^n T_j^* \mu_2^j(A_2) T_j = \sum_{j=1}^n T_j^* \mu_1^j\left(\Phi^{-1}([A_2]_{\mu_2})\right) T_j = \mu_1\left(\Phi^{-1}([A_2]_{\mu_2})\right) = \mu_2(A_2).$$

Subsequently, since μ_2 is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X_2)$, there exists an unitary operator $U_j \in \mathcal{B}(\mathcal{H})$ such that $\mu_2(\cdot) = U_j^* \mu_2^j(\cdot) U_j$ for each j. It then follows for all $A_1 \in \mathcal{O}(X_1)$, that

$$\mu_1(A_1) = \mu_2(\Phi([A_1]_{\mu_1})) = U_j^* \mu_2^j(\Phi([A_1]_{\mu_1})) U_j = U_j^* \mu_1^j(A_1) U_j,$$

where the last equality is due to (4.5.4). This proves that μ_1 is unitarily equivalent to each μ_1^j which consequently implies that μ_1 is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X_1)$. That μ_1 is extreme if and only if μ_2 is extreme follows similarly. □

In the proof of part (ii) of the theorem above, we observed the following:

Proposition 4.5.3. *Let $\mu_i : \mathcal{O}(X_i) \to \mathcal{B}(\mathcal{H})$, $i = 1, 2$ be two measure isomorphic POVMs with Boolean isomorphism $\Phi : \Sigma(\mu_1) \to \Sigma(\mu_2)$. Then A_1 is an atom for μ_1 if and only if any representative of $\Phi([A_1]_{\mu_1})$ is an atom for μ_2.*

Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a POVM. We say μ is *countably generated* if there exists a countable collection of subsets $\mathcal{F} \subseteq \mathcal{O}(X)$ such that for any $A \in \mathcal{O}(X)$, there exists $B \in \sigma(\mathcal{F})$ satisfying $[A]_\mu = [B]_\mu$. Here $\sigma(\mathcal{F})$ denotes the σ-algebra generated by $\mathcal{F}$. The following result has been borrowed from [6].

Theorem 4.5.4 (Proposition 59, [6])**.** *If $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ is a countably generated POVM, then μ is measure isomorphic to a POVM $\nu : \mathcal{O}([0,1]) \to \mathcal{B}(\mathcal{H})$.*

Recall that when X is a separable metric space, then $\mathcal{O}(X)$ is its Borel σ-algebra and in this case, any POVM on X is countably generated (in fact, the countable collection of open balls centered at elements of a countable dense set with radius of length rationals generates the Borel σ-algebra of X). What the theorem above basically says is that, to study C^*-extreme points in $\mathcal{P}_{\mathcal{H}}(X)$ for a separable metric space X, it is sufficient to just characterize the C^*-extreme points in $\mathcal{P}_{\mathcal{H}}([0,1])$ in view of Theorem 4.5.2. This result will also help us find an example (see Example 5.4.5) of a C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ which is not spectral, when $\mathcal{H}$ is infinite dimensional.

Now we consider measure isomorphism of POVMs induced from a bimeasurable map. Recall that for measurable spaces X_1 and X_2, a function $f : X_1 \to X_2$ is called *measurable* if $f^{-1}(A_2) \in \mathcal{O}(X_1)$ whenever $A_2 \in \mathcal{O}(X_2)$. Note that for any measurable space X and a measurable subset $Y \subseteq X$, Y itself inherits the natural measurable space structure from X with the σ algebra $\{A \cap Y; A \in \mathcal{O}(X)\}$.

Theorem 4.5.5. *For $i = 1, 2$, let X_i be two measurable spaces and let $Y_i \subseteq X_i$ be measurable subsets. Let $f : Y_1 \to Y_2$ be a bijective map such that both f and f^{-1} are measurable. Given a normalized POVM $\mu_1 : \mathcal{O}(X_1) \to \mathcal{B}(\mathcal{H})$ satisfying $\mu_1(A_1) = \mu_1(A_1 \cap Y_1)$ for all $A_1 \in \mathcal{O}(X_1)$, define $\mu_2 : \mathcal{O}(X_2) \to \mathcal{B}(\mathcal{H})$ by*

$$\mu_2(A_2) = \mu_1\left(f^{-1}(A_2 \cap Y_2)\right)$$

for all $A_2 \in \mathcal{O}(X_2)$. Then μ_1 and μ_2 are measure isomorphic.

Proof. We claim that the map $\Phi : \Sigma(\mu_1) \to \Sigma(\mu_2)$ defined by

$$\Phi([A_1]_{\mu_1}) = [f(A_1 \cap Y_1)]_{\mu_2} \quad \text{for all } A_1 \in \mathcal{O}(X_1), \tag{4.5.5}$$

is a Boolean isomorphism. First note that

$$\mu_1(A_1) = \mu_1(A_1 \cap Y_1) = \mu_1\left(f^{-1}\left(f(A_1 \cap Y_1)\right)\right) = \mu_2\left(f(A_1 \cap Y_1)\right) \tag{4.5.6}$$

for all $A_1 \in \mathcal{O}(X_1)$. This implies that $\mu_1(A_1) = 0$ if and only if $\mu_2(f(A_1 \cap Y_1)) = 0$ for any $A_1 \in \mathcal{O}(X_1)$. Therefore if $[A_1]_{\mu_1} = [A_1']_{\mu_1}$ for some $A_1, A_1' \in \mathcal{O}(X_1)$, then

$$[f(A_1 \cap Y_1)]_{\mu_2} = [f(A_1' \cap Y_1)]_{\mu_2}.$$

This proves the well-definedness of Φ. Similarly by symmetry, we prove that Φ is injective. That Φ is onto is straightforward by noting that

$$\Phi\left([f^{-1}(A_2 \cap Y_2)]_{\mu_1}\right) = [A_2 \cap Y_2]_{\mu_2} = [A_2]_{\mu_2}$$

for any $A_2 \in \mathcal{O}(X_2)$. This shows that Φ is a Boolean isomorphism as claimed. Further from (4.5.5) and (4.5.6), we have

$$\mu_2(\Phi([A_1]_{\mu_1})) = \mu_2(f(A_1 \cap Y_1)) = \mu_1(A_1)$$

for any $A_1 \in \mathcal{O}(X_1)$. Thus we conclude that μ_1 and μ_2 are measure isomorphic. $\square$

Now we apply these results to the study of C^*-extreme POVMs. Consider the map $g : [0,1) \to \mathbb{T}$ given by

$$g(t) = e^{2\pi it} \quad \text{for} \quad t \in [0,1),$$

where $\mathbb{T}$ is the unit circle. It is clear that g is a bijective map such that both g and g^{-1} are Borel measurable. Therefore for any Hilbert space $\mathcal{H}$, normalized POVMs $\mu \in \mathcal{P}_{\mathcal{H}}([0,1])$ with $\mu(\{1\}) = 0$, are in one-to-one correspondence with $\mathcal{P}_{\mathcal{H}}(\mathbb{T})$ through measure isomorphism, by Theorem 4.5.5. In particular, since singletons under non-atomic POVMs have zero measure, it follows that non-atomic POVMs in $\mathcal{P}_{\mathcal{H}}([0,1])$ are measure isomorphic to non-atomic POVMs in $\mathcal{P}_{\mathcal{H}}(\mathbb{T})$.

Next if X is a separable metric space, then non-atomic POVMs in $\mathcal{P}_{\mathcal{H}}(X)$ are measure isomorphic to non-atomic POVMs in $\mathcal{P}_{\mathcal{H}}([0,1])$ from Theorem 4.5.4 and Theorem 4.5.2, which in turn are measure isomorphic to non-atomic POVMs in $\mathcal{P}_{\mathcal{H}}(\mathbb{T})$ as seen above. Thus we conclude in view of Theorem 4.5.2 that, characterizing the non-atomic C^*-extreme points in $\mathcal{P}_{\mathcal{H}}(X)$ is equivalent to characterizing non-atomic C^*-extreme points in $\mathcal{P}_{\mathcal{H}}([0,1])$ or $\mathcal{P}_{\mathcal{H}}(\mathbb{T})$. Also we already know the structure of atomic C^*-extreme points from Theorem 4.3.2. Therefore what we observed from the discussion above and Corollary 4.4.12 is that, to characterize C^*-extreme points of $\mathcal{P}_{\mathcal{H}}(X)$, it is enough to understand the behaviour of C^*-extreme points of $\mathcal{P}_{\mathcal{H}}([0,1])$ or $\mathcal{P}_{\mathcal{H}}(\mathbb{T})$.

Remark 4.5.6. We still do not know a proper concrete structure of non-atomic C^*-extreme points of $\mathcal{P}_{\mathcal{H}}(\mathbb{T})$, and hence a complete characterization of C^*-extreme points of $\mathcal{P}_{\mathcal{H}}(\mathbb{T})$ is wide open.

Chapter 5

C^*-extreme Maps on Commutative C^*-algebras

We continue the investigation of structure of C^*-extreme points of POVMs here in this chapter, albeit we restrict ourselves to the special case of regular POVMs on topological Hausdorff spaces. Regular POVMs on compact Hausdorff spaces are the most natural frameworks for quantum information theorists and operator algebraists. As already mentioned, the correspondence between regular POVMs on a compact space X and UCP maps on the continuous function space $C(X)$ is essential in providing a bridge between the two theories. This relationship which quantizes the classical Riesz-Markov representation theorem also plays integral role in the study of several objects like positive definite functions and kernels. Our purpose here is to exploit this interplay in the study of behaviour of C^*-extremity in the two situations. The results developed in the previous chapter on POVMs will be crucial as well.

We begin with some general properties of regular POVMs on topological Hausdorff spaces. Similar to the classical case, a description of regular atomic and non-atomic POVMs are presented. We then discuss regular C^*-extreme POVMs, and in particular see their behaviour on discrete spaces. Taking cue from bounded-weak topology on UCP maps, a similar kind of topology is defined on the spaces of POVMs with respect to which a Krein-Milman type theorem is proved for C^*-convexity of the spaces of normalized POVMs. Finally, we bring back all the results from the case of regular POVMs on compact spaces as an application in the language of UCP maps on commutative C^*-algebras.

5.1 Regular atomic and non-atomic POVMs

Let X be a Hausdorff topological space. In this case, $\mathcal{O}(X)$ will always denote the Borel σ-algebra generated by open subsets of X. We begin by looking at the structure of atomic and non-atomic regular POVMs on X (see Section 1.3 and Definition 1.3.20 therein for the notion of regularity).

Similar to the case in classical measure theory, we show that every atom for a regular POVM is concentrated on a singleton up to a set of measure 0 and that every atomic regular POVM is concentrated on a countable subset. First step in that direction is the following lemma.

Lemma 5.1.1. *Let $\pi : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H}_\pi)$ be a regular spectral measure satisfying $\pi(A) = I_{\mathcal{H}_\pi}$ or 0 for each $A \in \mathcal{O}(X)$ (here $\mathcal{H}_\pi$ could be non-separable). Then there exists a unique $x \in X$ such that $\pi = \delta_x(\cdot)I_{\mathcal{H}_\pi}$, where δ_x denotes the Dirac measure concentrated at x.*

Proof. For each $A \in \mathcal{O}(X)$, let $\lambda(A) = 0$ or 1 accordingly so that $\pi(A) = \lambda(A)I_{\mathcal{H}_\pi}$. Clearly λ is a regular probability measure, as π is regular (e.g. $\lambda = \pi_{h,h}$ for any unit vector $h \in \mathcal{H}_\pi$). Whence by inner regularity, there is a compact subset $C \subseteq X$ such that $\lambda(C) > 0$ and thus, $\lambda(C) = 1$. We claim to find an element $x \in C$ such that

$$\lambda = \delta_x.$$

Suppose this is not the case, then $\lambda(\{x\}) = 0$ for each $x \in C$ (otherwise, $\lambda(\{x\}) = 1 = \lambda(C)$ for some x). Therefore it follows from outer regularity of λ, that there is an open subset E_x containing x such that $\lambda(E_x) < 1/2$ and thus,

$$\lambda(E_x) = 0.$$

Since $\{E_x\}_{x\in C}$ is an open cover for the compact subset C, there exist finitely many points $x_1, \ldots, x_n \in C$ such that $C \subseteq \cup_{i=1}^n E_{x_i}$. But then we have

$$\lambda(C) \leq \sum_{i=1}^n \lambda(E_{x_i}) = 0,$$

leading us to a contradiction. Thus $\lambda = \delta_x$ for some $x \in X$ and hence $\pi = \delta_x(\cdot)I_{\mathcal{H}_\pi}$. The uniqueness is obvious as $\lambda(X) = \lambda(\{x\}) = 1$. □

Remark 5.1.2. It is well-known that the lemma above fails to be true (even on compact Hausdorff spaces) for finite positive measures, if we drop the regularity assumption (see Example 7.1.3, [10]).

The following theorem and the subsequent corollary give characterization of all atomic and non-atomic regular POVMs.

Theorem 5.1.3. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be an atomic regular POVM. Then there exists a countable subset $\{x_n\}_{n\geq 1}$ of X and positive operators $\{T_n\}_{n\geq 1}$ in $\mathcal{B}(\mathcal{H})$ such that*

$$\mu(A) = \sum_{n\geq 1} \delta_{x_n}(A)T_n \tag{5.1.1}$$

for each $A \in \mathcal{O}(X)$.

Proof. Let $(\pi, V, \mathcal{H}_\pi)$ be the minimal Naimark triple for μ. Since μ is regular, π is regular by Proposition 1.3.22. Since μ is atomic, we know from Proposition 1.3.15 that there is a countable collection $\{B_n\}_{n\geq 1}$ of mutually orthogonal atoms for μ such that

$$\mu(A) = \sum_{n\geq 1} \mu(A \cap B_n), \quad \text{for all } A \in \mathcal{O}(X).$$

Now if we show that for any atom B for μ, there exists a unique element $x \in B$ such that

$$\mu(B) = \mu(\{x\}),$$

then we are done (because for each $n \geq 1$, there would exist $x_n \in B_n$ such that $\mu(B_n) = \mu(\{x_n\})$, and hence $\mu(A \cap B_n) = \delta_{x_n}(A)T_n$ for all $A \in \mathcal{O}(X)$, where $T_n = \mu(\{x_n\})$). So fix an atom B for μ. Then B is an atom for the spectral measure π as well by Proposition 1.3.17. Now set

$$\mathcal{K} = \mathcal{R}(\pi(B)),$$

which is an invariant subspace for $\pi(A)$ for all $A \in \mathcal{O}(X)$. Consider the spectral measure $\rho : \mathcal{O}(X) \to \mathcal{B}(\mathcal{K})$ defined by

$$\rho(A) = \pi(A \cap B)_{|\mathcal{K}}, \quad \text{for all } A \in \mathcal{O}(X).$$

It is clear that ρ is a regular spectral measure. Since B is an atom for π, it follows that either $\rho(A) = 0$ or $I_{\mathcal{K}}$, for all $A \in \mathcal{O}(X)$. Hence Lemma 5.1.1 implies that there is an element $x \in X$ such that

$$\rho(X) = \rho(\{x\}) = I_{\mathcal{K}}.$$

Note that $x \in B$, and we have $\pi(B) = \pi(\{x\})$; hence $\mu(B) = \mu(\{x\})$. This completes the proof. □

Remark 5.1.4. The proof of Theorem 5.1.3 could have been given in a shorter way as follows: one can consider the measure μ_S for any strictly positive density operator S as in (1.3.3). Then μ_S is atomic and we can invoke the classical result which says that atomic positive measures are concentrated on countable sets. We avoided this approach to have a self contained proof of the result using Naimark dilation theorem.

Corollary 5.1.5. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a regular POVM. Then*

(i) for any atom B for μ, there exists a (unique) $x \in B$ such that $\mu(B) = \mu(\{x\})$.

(ii) μ is atomic if and only if there exists a countable subset $Y \subseteq X$ such that $\mu(Y) = \mu(X)$.

(iii) μ is non-atomic if and only if $\mu(\{x\}) = 0$ for all $x \in X$.

Proof. (i) The proof of this is actually ingrained in the proof of Theorem 5.1.3.

(ii) First note that any POVM concentrated on a countable subset is atomic and hence the 'if' part follows. The converse follows from Theorem 5.1.3, by taking $Y = \{x_n\}$.

(iii) The 'only if' is trivial. To prove the 'if' part, since every atom is concentrated on a singleton by Part (i), the hypothesis implies that μ has no atom, which is equivalent to saying that μ is non-atomic. □

Corollary 5.1.6. *Let $\{\mu_n\}$ be a countable collection of regular POVMs and let $\mu = \oplus_n \mu_n$. Then μ is atomic (resp. non-atomic) if and only if each μ_n is atomic (resp. non-atomic).*

Proof. We use Corollary 5.1.5 to prove the assertions. If μ is atomic, then there is a countable subset Y such that $\mu(Y) = \mu(X)$. In particular $\mu_n(Y) = \mu_n(X)$ for each n, which implies that

μ_n is atomic. Conversely if each μ_n is atomic, then $\mu_n(Y_n) = \mu_n(X)$ for some countable subset Y_n. If $Y = \cup_n Y_n$, then Y is countable and $\mu(Y) = \mu(X)$, concluding that μ is atomic. The equivalence of non-atomicity follows similarly. □

5.2 Regular C^*-extreme POVMs

In this section, we come back to our original theme of C^*-convexity of POVMs. Let X be a Hausdorff topological space and $\mathcal{H}$ a separable Hilbert space. Again $\mathcal{O}(X)$ denotes the Borel σ-algebra on X.

Notation. We denote by $\mathcal{RP}_{\mathcal{H}}(X)$ the collection of all regular normalized POVMs from $\mathcal{O}(X)$ to $\mathcal{B}(\mathcal{H})$.

Note that $\mathcal{RP}_{\mathcal{H}}(X) \subseteq \mathcal{P}_{\mathcal{H}}(X)$ and $\mathcal{RP}_{\mathcal{H}}(X)$ is itself a C^*-convex set in the sense that

$$\sum_{i=1}^{n} {T_i}^* \mu_i(\cdot) T_i \in \mathcal{RP}_{\mathcal{H}}(X),$$

whenever $\mu_i \in \mathcal{RP}_{\mathcal{H}}(X)$ and T_i's are C^*-coefficients for $1 \leq i \leq n$. In a fashion similar to Definition 4.1.2, one can define C^*-extreme points of $\mathcal{RP}_{\mathcal{H}}(X)$. However the following proposition says that, for a regular normalized POVM μ, it does not matter whether we are considering C^*-extremity of μ in $\mathcal{RP}_{\mathcal{H}}(X)$ or in $\mathcal{P}_{\mathcal{H}}(X)$.

Proposition 5.2.1. *Let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a normalized regular POVM. Then μ is C^*-extreme (resp. extreme) in $\mathcal{P}_{\mathcal{H}}(X)$ if and only if μ is C^*-extreme (resp. extreme) in $\mathcal{RP}_{\mathcal{H}}(X)$.*

Proof. If we show that every proper C^*-convex combination of μ in $\mathcal{P}_{\mathcal{H}}(X)$ is also a proper C^*-convex combination in $\mathcal{RP}_{\mathcal{H}}(X)$ and vice versa, then we are done. So let $\mu(\cdot) = \sum_{i=1}^{n} {T_i}^* \mu_i(\cdot) T_i$ be a proper C^*-convex combination in $\mathcal{P}_{\mathcal{H}}(X)$ for $\mu_i \in \mathcal{P}_{\mathcal{H}}(X)$. Note that, since ${T_i}^* \mu_i(\cdot) T_i \leq \mu(\cdot)$ for each i, it follows from Proposition 1.3.23 that ${T_i}^* \mu_i(\cdot) T_i$ is regular. Again by the same Proposition, since

$$\mu_i(\cdot) = T_i^{*-1} \left({T_i}^* \mu_i(\cdot) T_i\right) {T_i}^{-1},$$

it follows that μ_i is regular. Thus $\mu_i \in \mathcal{RP}_{\mathcal{H}}(X)$, which shows that $\sum_{i=1}^{n} {T_i}^* \mu_i(\cdot) T_i$ is also a proper C^*-convex combination of μ in $\mathcal{RP}_{\mathcal{H}}(X)$. Since $\mathcal{RP}_{\mathcal{H}}(X) \subseteq \mathcal{P}_{\mathcal{H}}(X)$, the converse of the claim is immediate. The assertions about extreme points follow similarly. □

Remark 5.2.2. The purpose of writing Proposition 5.2.1 is that when we shall translate our results from regular POVMs on compact spaces to UCP maps on $C(X)$, we won't have to worry about the concerned C^*-convex sets $\mathcal{RP}_{\mathcal{H}}(X)$ or $\mathcal{P}_{\mathcal{H}}(X)$.

We now consider regular C^*-extreme POVMs on discrete spaces. We have already seen the following result for countable measurable spaces in Theorem 4.3.2 without the assumption of regularity. The extension to uncountable discrete spaces requires regularity in a crucial way.

Proposition 5.2.3. *Let X be a discrete (possibly uncountable) space. Then every regular POVM on X is atomic. Moreover, a normalized POVM in $\mathcal{RP}_{\mathcal{H}}(X)$ is C^*-extreme if and only if it is spectral.*

Proof. Firstly let λ be a regular Borel positive measure on X. By regularity of λ, for each $n \in \mathbb{N}$ there is a compact subset C_n such that $\lambda(X \setminus C_n) < 1/n$. Set $C = \cup_n C_n$. Since X is discrete, each of C_n is a finite subset and hence C is countable. Note that

$$\lambda(X \setminus C) \leq \lambda(X \setminus C_n) \leq 1/n,$$

for each n and hence, $\lambda(X \setminus C) = 0$. This says that every regular Borel positive measure on X is concentrated on a countable subset and so it is atomic.

Now let $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$ be a regular POVM. Let S be a strictly positive density operator in $\mathcal{B}(\mathcal{H})$, and let $\mu_S : \mathcal{O}(X) \to [0, \infty)$ be the positive measure (as in (1.3.3)) defined by

$$\mu_S(A) = \mathrm{Tr}(\mu(A)S), \quad A \in \mathcal{O}(X).$$

See Section 1.3 for more details on this measure. Since μ is regular, it is easy to verify that μ_S is regular; hence μ_S is concentrated on a countable set as observed above. It then follows from Part (i) in Proposition 1.3.18 that μ is concentrated on a countable set. This shows our requirement that μ is atomic. Thus if μ is a C^*-extreme point in $\mathcal{RP}_{\mathcal{H}}(X)$, then it is spectral by Theorem 4.3.2. □

The following corollary provides a family of examples of uncountable compact Hausdorff spaces where every regular C^*-extreme POVM is spectral.

Corollary 5.2.4. *Let $\widetilde{X} = X \cup \{\infty\}$ be the one-point compactification of a discrete space X and let $\mu : \mathcal{O}(\widetilde{X}) \to \mathcal{B}(\mathcal{H})$ be a regular C^*-extreme POVM. Then μ is spectral.*

Proof. Note that the restriction $\mu_{|_{\mathcal{O}(X)}}$ of μ to $\mathcal{O}(X)$ is also regular and hence concentrated on a countable subset, as seen in Proposition 5.2.3. In particular, μ itself is concentrated on a countable subset and hence is atomic. Therefore, we conclude from Theorem 4.3.2 that μ is spectral. □

5.3 Krein-Milman type theorem for $\mathcal{P}_{\mathcal{H}}(X)$

As earlier let X be a topological space and $\mathcal{H}$ a Hilbert space. Now we define a topology on the set $\mathcal{P}_{\mathcal{H}}(X)$ of all normalized POVMs.. We shall call this topology as 'bounded-weak' inspired from the topology defined on the collection of all UCP maps on a C^*-algebra with the same name (see Definition 1.2.23). The reason for this nomenclature will be apparent in the next section. Our aim here is to show a Krein-Milman type theorem for C^*-convexity in this topology on $\mathcal{P}_{\mathcal{H}}(X)$. This result continues our search for Krein-Milman type theorem for various C^*-convex sets.

Let $C_b(X)$ denote the space of all bounded continuous functions on X. Recall that $\mu_{h,k}$ is the complex measure as in (1.3.1) for any POVM μ. We define the topology by defining convergence of nets.

Definition 5.3.1. Given a net μ^i and μ in $\mathcal{P}_{\mathcal{H}}(X)$, we say $\mu^i \to \mu$ in $\mathcal{P}_{\mathcal{H}}(X)$ in *bounded weak (BW) topology* if

$$\int_X f d\mu^i_{h,k} \to \int_X f d\mu_{h,k}$$

for all $f \in C_b(X)$ and $h, k \in \mathcal{H}$.

Notice that the topology on $\mathcal{P}_{\mathcal{H}}(X)$ is the smallest topology which makes the maps:

$$\mu \mapsto \int_X f d\mu_{h,k}$$

from $\mathcal{P}_{\mathcal{H}}(X)$ to $\mathbb{C}$, continuous for all $f \in C_b(X)$ and $h, k \in \mathcal{H}$. It is then immediate to verify that, for a given $\mu \in \mathcal{P}_{\mathcal{H}}(X)$, sets of the form

$$O = \left\{ \nu \in \mathcal{P}_{\mathcal{H}}(X); \left| \int_X f_i d\nu_{h_i,k_i} - \int_X f_i d\mu_{h_i,k_i} \right| < \epsilon, 1 \leq i \leq n \right\}, \tag{5.3.1}$$

where $f_i \in C_b(X)$, $h_i, k_i \in \mathcal{H}$ for $1 \leq i \leq n$, $\epsilon > 0$, form a basis around μ in $\mathcal{P}_{\mathcal{H}}(X)$.

The definition here reminds us the weak topology considered in classical probability theory. Moreover, we shall see in Section 5.4 that this definition is directly connected to the bounded weak topology on the collection of UCP maps on a commutative C^*-algebra.

Remark 5.3.2. It should be added here that one can define a topology on $\mathcal{P}_{\mathcal{H}}(X)$ in several other ways. For example, for μ and a net μ_i of normalized POVMs, we could define the convergence $\mu_i \to \mu$ by saying that

$$\mu_i(A) \to \mu(A) \quad \text{in WOT (or } \sigma\text{-weak topology) for all} \quad A \in \mathcal{O}(X).$$

This topology is certainly stronger than the bounded weak topology defined above. This topology has been considered in [40]. We could have also defined a topology just by considering $C_c(X)$, the space of all compactly supported continuous functions, instead of $C_b(X)$ in the definition. In this case, we would get a weaker topology than we originally defined. Nevertheless in this case, one can show along the lines of classical probability theory that this topology agrees with bounded weak topology on $\mathcal{P}_{\mathcal{H}}(X)$ whenever X is a locally compact Hausdorff space.

We now return to our original topology on $\mathcal{P}_{\mathcal{H}}(X)$ as defined in 5.3.1. In general, the set $\mathcal{P}_{\mathcal{H}}(X)$ is not Hausdorff; for an example, one can consider the classically famous Dieudonné measure λ (which is not regular) on the compact Hausdorff space $X = [0, \omega_1]$ equipped with order topology, where ω_1 is the first uncountable ordinal (see Example 7.1.3, [10]). One can show that

$$\int_X f d\lambda = f(\omega_1) = \int_X f d\delta_{\omega_1} \quad \text{for all} \quad f \in C_b(X),$$

and hence the distinct elements $\lambda(\cdot)I_{\mathcal{H}}$ and $\delta_{\omega_1}(\cdot)I_{\mathcal{H}}$ in $\mathcal{P}_{\mathcal{H}}(X)$ are not separated by open subsets. However the topology restricted to $\mathcal{RP}_{\mathcal{H}}(X)$ is Hausdorff whenever X is a locally compact Hausdorff space, which is a consequence of uniqueness of regular Borel measures in Riesz-Markov theorem.

Remark 5.3.3. As in classical probability theory, for a locally compact Hausdorff space (more generally for completely regular space, see Lemma 8.9.2, [10]), the set $\{\delta_x(\cdot)I_{\mathcal{H}}; x \in X\}$ is closed in $\mathcal{RP}_{\mathcal{H}}(X)$ and it is homeomorphic to X. Using this or otherwise, one can show that $\mathcal{RP}_{\mathcal{H}}(X)$ is compact if and only if X is compact.

Now we move on to prove the main result of this section. We establish a Krein-Milman type theorem for C^*-convexity in the sense that $\mathcal{P}_{\mathcal{H}}(X)$ is the closure of C^*-convex hull of its C^*-extreme points. We mention here that a Krein-Milman type theorem for the set $\mathcal{P}_{\mathcal{H}}(X)$ was proved in [24] when X is a compact Hausdorff space and $\mathcal{H}$ is a finite dimensional Hilbert space. We generalize it to arbitrary topological spaces and arbitrary Hilbert spaces. Moreover, in our case the compactness of $\mathcal{P}_{\mathcal{H}}(X)$ in BW topology is not required. We first consider the following proposition, whose proof follows the same argument as normally used in classical measure theory.

Proposition 5.3.4. *Let X be a topological space and $\mathcal{H}$ a Hilbert space. Then the collection of all normalized POVMs concentrated on finite subsets is dense in $\mathcal{P}_{\mathcal{H}}(X)$ in BW topology.*

Proof. Let $\mu \in \mathcal{P}_{\mathcal{H}}(X)$, and E be a typical open set in $\mathcal{P}_{\mathcal{H}}(X)$ containing μ of the form

$$E = \left\{\nu \in \mathcal{P}_{\mathcal{H}}(X); \left|\int_X f_i d\nu_{h_i,k_i} - \int_X f_i d\mu_{h_i,k_i}\right| < \epsilon, 1 \leq i \leq n\right\},$$

for some fixed $f_i \in C_b(X)$, $h_i, k_i \in \mathcal{H}$, $i = 1, \ldots, n$ and $\epsilon > 0$. We shall obtain an element in E concentrated on a finite subset, which will imply the required result. Now for each $i \in \{1, \ldots, n\}$, get simple functions g_i on X satisfying

$$\sup_{x\in X} |f_i(x) - g_i(x)| < \epsilon/2M,$$

where M is a positive constant with $M > \sup_i \|h_i\|\|k_i\|$. Since g_i's are simple functions, there is a finite partition $\{A_{ij}\}$ of X and scalars $\{c_{ij}\} \subseteq \mathbb{C}$ (where j varies over some finite indexing set, say Λ_i for each $1 \leq i \leq n$) such that

$$g_i = \sum_{j\in\Lambda_i} c_{ij}\chi_{A_{ij}}$$

for each i. Pick $x_{ij} \in A_{ij}$ and set

$$\nu = \sum_{i=1}^{n} \sum_{j\in\Lambda_i} \delta_{x_{ij}}(\cdot)\mu(A_{ij}).$$

It is clear that ν is a POVM concentrated on the finite subset $\{x_{ij}\}$. Also we have

$$\nu(X) = \sum_{i=1}^{n} \sum_{j\in\Lambda_i} \mu(A_{ij}) = \mu(X) = I_{\mathcal{H}},$$

and hence ν is normalized. We claim that $\nu \in E$. Firstly note that

$$\int_X f d\nu = \sum_{i=1}^{n} \sum_{j\in\Lambda_i} \mu(A_{ij}) \int_X f\delta_{x_{ij}} = \sum_{i=1}^{n} \sum_{j\in\Lambda_i} f(x_{ij})\mu(A_{ij})$$

for any bounded Borel measurable function f on X (here $\int_X f d\nu \in \mathcal{B}(\mathcal{H})$ denotes the operator as defined in (1.4.6), which satisfies $\langle h, (\int_X f d\nu)\, k\rangle = \int_X f \nu_{h,k}$ for all $h, k \in \mathcal{H}$). Therefore for each $m \in \{1, \ldots, n\}$, we have

$$\int_X g_m d\nu = \sum_{i=1}^{n} \sum_{j\in\Lambda_i} g_m(x_{ij})\mu(A_{ij}) = \sum_{j\in\Lambda_m} c_{mj}\mu(A_{mj}) = \int_X g_m d\mu.$$

If we denote the total variation of a complex measure λ by $|\lambda|$, then we get the following:

$$\begin{aligned}\left|\int_X f_i d\nu_{h_i,k_i} - \int_X f_i d\mu_{h_i,k_i}\right| &\le \left|\int_X f_i d\nu_{h_i,k_i} - \int_X g_i d\nu_{h_i,k_i}\right| + \left|\int_X g_i d\nu_{h_i,k_i} - \int_X g_i d\mu_{h_i,k_i}\right| \\ &\quad + \left|\int_X g_i d\mu_{h_i,k_i} - \int_X f_i d\mu_{h_i,k_i}\right| \\ &\le \int_X |f_i - g_i|\; d|\nu_{h_i,k_i}| + \int_X |g_i - f_i|\; d|\mu_{h_i,k_i}| \\ &\le \left(\sup_{x\in X} |f_i(x) - g_i(x)|\right) (|\nu_{h_i,k_i}|(X) + |\mu_{h_i,k_i}|(X)) \\ &\le (\epsilon/2M)\,(2\|h_i\|\|k_i\|) \\ &< \epsilon\end{aligned}$$

for $i = 1, \ldots, n$. Here we have used the fact that $|\mu_{h_i,k_i}|(X) \le \|h_i\|\|k_i\|$, which is straightforward to verify. It then follows that $\nu \in E$, completing the proof. □

Definition 5.3.5. For a given subset $\mathcal{S}$ of $\mathcal{P}_{\mathcal{H}}(X)$, the C^**-convex hull* of $\mathcal{S}$ is the set defined by

$$\left\{\sum_{i=1}^{n} T_i{}^*\mu_i(\cdot)T_i :\; \mu_i \in \mathcal{S},\; T_i \in \mathcal{B}(\mathcal{H}) \;\text{ for } 1 \le i \le n \text{ such that } \sum_{i=1}^{n} T_i{}^*T_i = I_{\mathcal{H}}\right\}. \tag{5.3.2}$$

Below is a Krein-Milman type theorem for the spaces of normalized POVMs equipped with BW topology.

Theorem 5.3.6. *Let X be a Hausdorff topological space and $\mathcal{H}$ a Hilbert space. Then the C^*-convex hull of Dirac spectral measures (i.e. $\delta_x(\cdot)I_{\mathcal{H}}$ for $x \in X$) is dense in $\mathcal{P}_{\mathcal{H}}(X)$ in BW topology. In particular, the C^*-convex hull of all C^*-extreme points is dense in $\mathcal{P}_{\mathcal{H}}(X)$ in BW topology.*

Proof. Fix $\mu \in \mathcal{P}_{\mathcal{H}}(X)$. By Proposition 5.3.4, there is a net $\mu_i \in \mathcal{P}_{\mathcal{H}}(X)$ such that $\mu_i \to \mu$ in $\mathcal{P}_{\mathcal{H}}(X)$ and each μ_i is concentrated on a finite subset. Therefore if we show that each μ_i is in the C^*-convex hull of Dirac spectral measures, then we are done. So assume without loss of generality, that $\mu \in \mathcal{P}_{\mathcal{H}}(X)$ is concentrated on a finite subset, say $\{x_1, \ldots, x_n\}$. If $T_i = \mu(\{x_i\})$, then it is immediate that

$$\mu = \sum_{i=1}^{n} \delta_{x_i}(\cdot)T_i.$$

Set $S_i = T_i{}^{1/2} \in \mathcal{B}(\mathcal{H})$ for each i. Then

$$\sum_{i=1}^{n} S_i{}^*S_i = \sum_{i=1}^{n} T_i = \mu(X) = I_{\mathcal{H}}$$

and

$$\mu(\cdot) = \sum_{i=1}^{n} S_i{}^* \delta_{x_i}(\cdot) S_i,$$

which confirms that μ is a C^*-convex combination of Dirac spectral measures. $\square$

It is obvious that Dirac spectral measures are regular. Hence Theorem 5.3.6 along with Proposition 5.2.1 give us the following version of Krein-Milman theorem for regular POVMs. Its usefulness shall be apparent when we discuss UCP maps in the next section.

Corollary 5.3.7. *Let X be a Hausdorff topological space and $\mathcal{H}$ a Hilbert space. Then the C^*-convex hull of all regular spectral measures (in particular, regular C^*-extreme points) is dense in $\mathcal{RP}_{\mathcal{H}}(X)$ in subspace topology of BW topology.*

5.4 Applications to UCP Maps on $C(X)$

Finally, we return to our investigation of the structure of C^*-extreme points of generalized state spaces on commutative C^*-algebras. Here we apply the tools that we have developed for POVMs on compact Hausdorff spaces X, via their correspondence to UCP map on $C(X)$. See Section 1.4 for a detailed exposition of this correspondence.

Let X be a compact Hausdorff space, and $\mathcal{H}$ a separable Hilbert space. As mentioned in Section 1.4, given any regular (normalized) POVM $\mu : \mathcal{O}(X) \to \mathcal{B}(\mathcal{H})$, there is a unique (unital) CP map $\phi_\mu : C(X) \to \mathcal{B}(\mathcal{H})$ such that

$$\langle h, \phi_\mu(f)k\rangle = \int_X f d\mu_{h,k} \quad \text{for all} \quad f \in C(X),$$

where $\mu_{h,k}$ is the complex measure as in (1.3.1), and vice versa.

The correspondence $\mu \mapsto \phi_\mu$ of the set $\mathcal{RP}_{\mathcal{H}}(X)$ of normalized regular POVMs on X and the set $S_{\mathcal{H}}(C(X))$ of UCP maps on $C(X)$ described clearly preserves C^*-convexity and C^*-extreme points structures.

Theorem 5.4.1. *A normalized regular POVM μ is C^*-extreme in $\mathcal{RP}_{\mathcal{H}}(X)$ (or in $\mathcal{P}_{\mathcal{H}}(X)$) if and only if ϕ_μ is C^*-extreme in $S_{\mathcal{H}}(C(X))$.*

Proof. The proof follows from Part (iv) and Part (v) in Theorem 1.4.2, because C^*-convex combinations and unitary equivalences are preserved under the correspondence. $\square$

Following the discussions above, we are now ready to deduce some results for $S_{\mathcal{H}}(C(X))$. As noticed in Proposition 5.2.1, a regular normalized POVM μ is a C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ if and only if μ is a C^*-extreme point in $\mathcal{RP}_{\mathcal{H}}(X)$. Therefore, it follows from Theorem 5.4.1 that μ is C^*-extreme in $\mathcal{P}_{\mathcal{H}}(X)$ if and only if ϕ_μ is C^*-extreme in $S_{\mathcal{H}}(C(X))$. Thus, whenever X is a compact Hausdorff space, we have got freedom to bring back all the results on C^*-extreme points in $\mathcal{P}_{\mathcal{H}}(X)$ into the language of C^*-extreme points of $S_{\mathcal{H}}(C(X))$. We frequently make use of Theorem 1.4.2 and Theorem 5.4.1.

Firstly let X be a countable compact Hausdorff space. Then we saw in Theorem 4.3.2 that every C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ is spectral. Since spectral measures correspond to unital $*$-homomorphisms, here is the corresponding result.

Theorem 5.4.2. *Let $\mathcal{A}$ be a commutative unital C^*-algebra with countable spectrum and let $\phi \in S_{\mathcal{H}}(\mathcal{A})$. Then ϕ is C^*-extreme in $S_{\mathcal{H}}(\mathcal{A})$ if and only if ϕ is a $*$-homomorphism.*

In particular, when $\mathcal{A}$ is a finite dimensional commutative C^*-algebra i.e. $\mathcal{A} \cong \mathbb{C}^n$, we get the following corollary. This simple looking result had remained open for more than two decades, and we have settled it here.

Corollary 5.4.3. *Let $\phi : \mathbb{C}^n \to \mathcal{B}(\mathcal{H})$ be a UCP map. Then ϕ is C^*-extreme in $S_{\mathcal{H}}(\mathbb{C}^n)$ if and only if ϕ is a $*$-homomorphism.*

We apply Theorem 5.4.2 to certain UCP maps on C^*-algebra generated by a single normal operator with countable spectrum.

Example 5.4.4. Let $N \in \mathcal{B}(\mathcal{K})$ be a normal operator on a Hilbert space $\mathcal{K}$ with countable spectrum $\sigma(N)$ (in particular, when N is compact). It is known that for such a normal operator, a subspace $\mathcal{H} \subseteq \mathcal{K}$ is invariant for N if and only if it is reducing for N (Theorem 1.23, [68]). Let $C^*(N)$ be the unital C^*-algebra generated by N, and consider the UCP map $\phi_N : C^*(N) \to \mathcal{B}(\mathcal{H})$ defined by

$$\phi_N(T) = P_{\mathcal{H}} T_{|_{\mathcal{H}}} \quad \text{for all } T \in C^*(N).$$

It is easy to verify that ϕ_N is a $*$-homomorphism if and only if $\mathcal{H}$ is a reducing subspace for N. Thus since $C^*(N)$ is isomorphic to $C(\sigma(N))$ as C^*-algebra and $\sigma(N)$ is countable, the argument above along with Theorem 5.4.2 show that the following conditions are equivalent:

(i) ϕ_N is a C^*-extreme point in $S_{\mathcal{H}}(C^*(N))$.
(ii) ϕ_N is a $*$-homomorphism.
(iii) $\mathcal{H}$ is an invariant subspace of N.
(iv) $\mathcal{H}$ is a co-invariant subspace of N.
(v) $\mathcal{H}$ is a reducing subspace of N.

We momentarily go back to regular POVMs in order to produce a number of examples of C^*-extreme POVMs and hence C^*-extreme UCP maps. Using the results in Section 4.5, we provide here an example of a C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ which is not spectral, whenever X is an uncountable compact metric space and $\mathcal{H}$ an infinite dimensional Hilbert space.

Example 5.4.5. Consider the normalized POVM $\nu : \mathcal{O}(\mathbb{T}) \to \mathcal{B}(H^2)$ defined by

$$\nu(A) = P_{H^2} M_{\chi_A}{}_{|_{H^2}} \quad \text{for all } A \in \mathcal{O}(\mathbb{T}),$$

where H^2 denotes the Hardy space on the unit circle $\mathbb{T}$. Here M_f denotes the multiplication operator on $L^2(\mathbb{T})$ for any $f \in L^\infty(\mathbb{T})$. Then the corresponding UCP map $\phi_\nu : C(\mathbb{T}) \to \mathcal{B}(H^2)$ is given by

$$\phi_\nu(f) = P_{H^2} M_f{}_{|_{H^2}} \quad \text{for all } f \in C(\mathbb{T}).$$

We know from Example 2.1.5 that ϕ_ν is a C^*-extreme point in $S_{\mathcal{H}}(C(\mathbb{T}))$ and therefore, ν is C^*-extreme in $\mathcal{P}_{H^2}(\mathbb{T})$ by Theorem 5.4.1. Also note that ν is not spectral, since ϕ_ν is not a $*$-homomorphism.

Now let X be an uncountable compact metric space. Then by well-known theorems of Borel isomorphism (Theorem 2.12, [57]), there exists a Borel isomorphism $f : \mathbb{T} \to X$ (i.e. f is bijective such that f, f^{-1} are Borel measurable). Define the normalized POVM $\mu : \mathcal{O}(X) \to \mathcal{B}(H^2)$ by

$$\mu(A) = \nu(f^{-1}(A)) \quad \text{for all } A \in \mathcal{O}(X). \tag{5.4.1}$$

It is clear that μ is a regular normalized POVM. Then Theorem 4.5.5 along with Theorem 4.5.2 imply that μ is a C^*-extreme point in $\mathcal{P}_{H^2}(X)$ and is not spectral. Thus, since any infinite dimensional separable Hilbert space is isomorphic to H^2, what we have shown is that whenever X is an uncountable compact metric space and $\mathcal{H}$ an infinite dimensional Hilbert space, then $\mathcal{P}_{\mathcal{H}}(X)$ contains a C^*-extreme point which is not spectral. The assertion above can be applied to Polish spaces as well.

Let E be an uncountable compact subset of $\mathbb{C}$. Then E is a compact metric space. We consider the normalized POVM $\mu : \mathcal{O}(E) \to \mathcal{B}(H^2)$ constructed in Example 5.4.5, which is already in the minimal Naimark dilation form $\mu(\cdot) = V^*\pi(\cdot)V$. If

$$N = \int_E z d\pi \in \mathcal{B}(\mathcal{H}_\pi),$$

then N is a normal operator with spectrum E (see (1.4.6) for notation). Also the corresponding UCP map $\phi_\mu : C^*(N) \to \mathcal{B}(H^2)$ is of the form

$$\phi_\mu(T) = P_{H^2} T_{|_{H^2}}, \quad T \in C^*(N).$$

Thus we have got an example of a UCP map of the form ϕ_N as discussed in Example 5.4.4, which is C^*-extreme but not a $*$-homomorphism.

Now let $\mathcal{A}$ be a separable commutative unital C^*-algebra. Then its spectrum is a separable compact Hausdorff space (Theorem V.6.6, [15]) and hence metrizable, which is to say $\mathcal{A} = C(X)$ for a compact metric space X. Therefore, Example 5.4.5 and Theorem 5.4.1 give us the following result for a separable commutative unital C^*-algebra with uncountable spectrum.

Theorem 5.4.6. *Let $\mathcal{A}$ be a separable commutative unital C^*-algebra with uncountable spectrum and let $\mathcal{H}$ be an infinite dimensional separable Hilbert space. Then $S_{\mathcal{H}}(\mathcal{A})$ contains a C^*-extreme point which is not a $*$-homomorphism.*

The theorem above fails to be true if the separability assumption is removed, as we see below. If X is a discrete space and $\widetilde{X}$ denotes its one-point compactification, then we saw in Corollary 5.2.4 that every regular POVM in $\mathcal{P}_{\mathcal{H}}(\widetilde{X})$ is atomic, and hence every C^*-extreme point in $\mathcal{RP}_{\mathcal{H}}(\widetilde{X})$ is spectral. Equivalently, every C^*-extreme point in $S_{\mathcal{H}}(C(\widetilde{X}))$ is a $*$-homomorphism by Theorem 5.4.1. Note that, whenever X is an uncountable discrete space, then $\widetilde{X}$ is a non-separable compact Hausdorff space and in particular, $C(\widetilde{X})$ is a non separable C^*-algebra (Theorem V.6.6, [15]). Thus the assumption of separability of the C^*-algebra $\mathcal{A}$ in Theorem 5.4.6 is crucial. We have obtained the following:

Theorem 5.4.7. *Let $\mathcal{A}$ be a commutative unital C^*-algebra whose spectrum is a one-point compactification of a discrete space. Then every C^*-extreme point in $S_{\mathcal{H}}(\mathcal{A})$ is a $*$-homomorphism.*

For the next application, let $\phi : C(X) \to \mathcal{B}(\mathcal{H})$ be a UCP map such that $\phi(C(X))$ is commutative. Then WOT-$\overline{\phi(C(X))}$ is commutative. Since WOT-$\overline{\phi(C(X))}$ = WOT-$\overline{\text{span}}\mu_\phi(\mathcal{O}(X))$ by Proposition 1.4.4, it follows that WOT-$\overline{\text{span}}\mu_\phi(\mathcal{O}(X))$ is commutative. In particular, $\mu_\phi(\mathcal{O}(X))$ is commutative. Therefore if ϕ is a C^*-extreme point in $S_{\mathcal{H}}(C(X))$ with commutative range, then μ_ϕ is a C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ with commutative range. Then it follows from Theorem 4.2.2 that μ_ϕ is spectral, and hence ϕ is a $*$-homomorphism. Thus we have got the following result. A similar result for extreme points with commutative range in $S_{\mathcal{H}}(C(X))$ holds true (see Corollary 3.6, [76]).

Theorem 5.4.8. *Let $\mathcal{A}$ be a commutative unital C^*-algebra and $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ a UCP map with commutative range. Then ϕ is C^*-extreme in $S_{\mathcal{H}}(\mathcal{A})$ if and only if ϕ is a $*$-homomorphism.*

We now compare the topology on the spaces of normalized POVMs with BW-topology on UCP maps.

For a net μ^i and $\mu \in \mathcal{RP}_{\mathcal{H}}(X)$, since $\phi_\mu(f) = \int_X f d\mu$ for all $f \in C(X)$, it follows that $\mu^i \to \mu$ in $\mathcal{RP}_{\mathcal{H}}(X)$ if and only if $\phi_{\mu^i}(f) \to \phi_\mu(f)$ in WOT for all $f \in C(X)$. The following proposition is just a rephrasing of the definition of the topology on regular POVMs, which effectively says that $\mathcal{RP}_{\mathcal{H}}(X)$ and $S_{\mathcal{H}}(C(X))$ are topologically homeomorphic. Recall that by Riesz-Markov representation theorem, the space of all regular Borel complex measures $M(X)$ on X is Banach space dual of $C(X)$.

Proposition 5.4.9. *Let μ^i be a net in $\mathcal{RP}_{\mathcal{H}}(X)$ and $\mu \in \mathcal{RP}_{\mathcal{H}}(X)$. Then the following are equivalent:*

(i) $\mu^i \to \mu$ in $\mathcal{RP}_{\mathcal{H}}(X)$ (and, in $\mathcal{P}_{\mathcal{H}}(X)$) in BW topology.
(ii) $\phi_{\mu^i} \to \phi_\mu$ in BW topology in $S_{\mathcal{H}}(C(X))$.
(iii) $\mu^i_{h,k} \to \mu_{h,k}$ in weak-topology on $M(X)$ for all $h, k \in \mathcal{H}$.*

As a final application, we discuss the generalized Krein-Milman theorem for the space $S_{\mathcal{H}}(C(X))$ equipped with BW-topology. This follows from the corresponding result on regular POVMs in Corollary 5.3.7 and its homeomorphism with UCP maps via Proposition 5.4.9.

Note that when X is a non-metrizable compact Hausdorff space, then $C(X)$ is a non separable space (Theorem V.6.6, [15]). Therefore, proof of Theorem 2.4.3 for UCP maps on separable C^*-algebras is no longer valid in this setting. The following theorem completes a total of three different scenarios where Krein-Milman type theorem for the spaces of UCP maps have been proved.

Theorem 5.4.10. *Let $\mathcal{A}$ be a commutative unital C^*-algebra and $\mathcal{H}$ a Hilbert space. Then the C^*-convex hull of the collection of all unital $*$-homomorphisms (in particular, C^*-extreme points) is dense in $S_{\mathcal{H}}(\mathcal{A})$ with respect to bounded-weak topology.*

Chapter 6

Logmodular Algebras

The primary theme of this chapter revolves around algebras having factorization and logmodularity properties. The purpose here is to ultimately give the proof of the aforementioned result in Theorem 3.1.5 regarding lattices of invariant subspaces of algebras having factorization. The study of the factorization property of subalgebras of C^*-algebras is very classical. The well-known Cholesky theorem talks about the factorization property of upper-triangular matrices in M_n, the algebra of $n \times n$ complex matrices. More generally any algebra of block upper-triangular matrices has factorization in M_n. Conversely, Juschenko [44] shows that they are the only algebras in M_n which have factorization.

A classical result of Szegö says that the Hardy algebra $H^\infty(\mathbb{T})$ on the unit circle has factorization in $L^\infty(\mathbb{T})$. Some other function algebras like weak*-Dirichlet algebras introduced by Srinivasan and Wang [74] have factorization. Taking cue from analytic function algebras, Arveson [4] introduced the theory of finite maximal subdiagonal algebras as noncommutative variant and considered many results analogous to the classical Hardy space theory, showing in particular that they have factorization property. Later several authors have examined such algebras in different settings. For more about algebras with factorization see [2, 4, 17, 32, 47, 48, 64, 72], and for some closely related properties see [1, 42, 53, 65, 66, 73] to name a few.

Among other algebras, factorization property of nest algebras has attracted considerable amount of interest in recent decades, particularly through the works of Gohberg-Krein [32], Arveson [2] and Larson [47]. A deep result of [47] in particular says that all nest algebras associated with countable complete nests have factorization in $\mathcal{B}(\mathcal{H})$.

On the other hand, the converse question of what other algebras have factorization in $\mathcal{B}(\mathcal{H})$ is very natural. Here we answer this question with an additional assumption of reflexivity, where we show that all reflexive algebras with factorization must be nest algebras. The assumption of reflexivity is not far away from answering the raised question, considering the fact that all nest algebras are reflexive. Whether the assumption of reflexivity can be dropped remains open.

An algebra with factorization property is a particular case of *logmodular algebras*, and we explore such algebras more in depth. The notion of logmodularity was first introduced by Hoffman [38] for subalgebras of commutative C^*-algebras, whose main idea was to generalize some classical results of analytic function theory in the unit disc. Blecher and Labuschagne [8] extended this notion to subalgebras of non-commutative C^*-algebras. They studied completely contractive representations on such algebras and their extension properties. Paulsen and Raghupathi [62] also studied representations of logmodular algebras and explored conditions under which contractive representations are automatically completely contractive. In [44], Juschenko gave a complete characterization of all logmodular subalgebras of M_n. See [9] for a beautiful survey on logmodular algebras arising out of tracial subalgebras and their relation to finite subdiagonal algebras among others. They show how most results generalized in 1960's from the Hardy space on the unit disc to more general function algebras generalize further to the non-commutative situation, though more sophisticated proof techniques had to be developed for the purpose. We list some additional references on logmodular algebras in [8, 9, 30, 38, 39, 44, 62].

In this chapter, our aim is to understand the behaviour of lattices of subspaces (or projections) invariant under logmodular algebras, and use it to characterize reflexive logmodular algebras. The main result of this chapter answers a conjecture by Paulsen-Raghupathi [62] in the affirmative, which asks whether every completely distributive CSL logmodular algebra of $\mathcal{B}(\mathcal{H})$ is a nest algebra. In fact, we show more generally that the lattice of projections whose ranges are invariant under a logmodular algebra in a factor $\mathcal{B}$, is a nest, and hence any such $\mathcal{B}$-reflexive algebra is a nest subalgebra. As a special case, our promised result in Theorem 3.1.5 will follow (Corollary 6.2.7).

Moreover we explore some sufficient criteria under which an algebra with factorization is automatically reflexive and is a nest algebra. In particular it is proved that a subalgebra with factorization in $\mathcal{B}(\mathcal{H})$, whose lattice consists of finite dimensional atoms, is reflexive and so it is a nest algebra. Also we show that any subalgebra with factorization in a finite dimensional von Neumann algebra must be a nest subalgebra. Finally we give an example of a subalgebra in a von Neumann algebra (certainly infinite dimensional), which has factorization but it is not a nest subalgebra.

6.1 Definitions and examples

We caution the readers here that throughout this chapter we adopt a different convention than earlier for lattices of an algebra. Here it is defined in terms of projections rather than subspaces. Hence the notion of nests will also be considered in terms of projections. This is deliberately being done because here the projection on invariant subspaces of algebras belong to certain von Neumann algebras. To avoid confusion, we freshly define all the notions here which should strictly be followed only in this chapter. Nevertheless the readers can understand all the terminologies as defined in Section 1.5 in terms of projections just by replacing them over subspaces.

To define the notion of logmodular algebras, we recall some notations. Let $\mathcal{B}$ be a C^*-algebra, and let $\mathcal{M}$ be a subalgebra (not necessarily self-adjoint) of $\mathcal{B}$. Recall that we denote by $\mathcal{M}^{-1}$ the set

$$\mathcal{M}^{-1} = \{x \in \mathcal{M}; x \text{ is invertible with } x^{-1} \in \mathcal{M}\}. \tag{6.1.1}$$

Let $\mathcal{B}_+^{-1}$ denote the set of all positive and invertible elements of $\mathcal{B}$. Following [8, 38], we now consider the following definitions. We also restate the previously defined notion of algebras having factorization in order to compare the two.

Definition 6.1.1. Let $\mathcal{M}$ be a subalgebra of a C^*-algebra $\mathcal{B}$. Then

(i) $\mathcal{M}$ is called *logmodular or has logmodularity* in $\mathcal{B}$ if the set $\{a^*a; a \in \mathcal{M}^{-1}\}$ is norm dense in $\mathcal{B}_+^{-1}$.

(ii) $\mathcal{M}$ is said to have *factorization or strong logmodularity* in $\mathcal{B}$ if $\{a^*a; a \in \mathcal{M}^{-1}\} = \mathcal{B}_+^{-1}$.

It is clear that any algebra having factorization is logmodular. Below we collect some known and straightforward results about logmodular algebras whose proof is simple (see Proposition 4.6, [8]).

Proposition 6.1.2. *Let $\phi : \mathcal{B} \to \mathcal{A}$ be a $*$-isomorphism between two C^*-algebras, and let $\mathcal{M}$ be a subalgebra of $\mathcal{B}$. Then $\mathcal{M}$ has logmodularity (resp. factorization) in $\mathcal{B}$ if and only if $\phi(\mathcal{M})$ has logmodularity (resp. factorization) in $\mathcal{A}$. In particular if U is an appropriate unitary, then $U^*\mathcal{M}U$ has logmodularity (resp. factorization) in $U^*\mathcal{B}U$ if and only if $\mathcal{M}$ has logmodularity (resp. factorization) in $\mathcal{B}$.*

Proof. This is straightforward, as the map ϕ preserves positivity, invertibility, unitary etc. □

Recall that for any subset $\mathcal{M}$ of a C^*-algebra $\mathcal{B}$, we denote by $\mathcal{M}^*$ the set

$$\mathcal{M}^* = \{x \in \mathcal{B}; x^* \in \mathcal{M}\}.$$

The following results provide some equivalent criteria for logmodular algebras (compare this with Proposition 1.5.2 for algebras having factorization).

Proposition 6.1.3 (Proposition 4.1, [8])**.** *Let $\mathcal{M}$ be a closed subalgebra of a C^*-algebra $\mathcal{B}$. Then the following are equivalent:*

(i) $\mathcal{M}$ is logmodular in $\mathcal{B}$,

(ii) $\mathcal{M}^*$ is logmodular in $\mathcal{B}$,

(iii) for each invertible element $x \in \mathcal{B}$, there exist sequences $\{u_n\}, \{v_n\}$ of unitaries in $\mathcal{B}$ and invertible elements $\{a_n\}, \{b_n\}$ in $\mathcal{M}^{-1}$ such that $x = \lim_n u_n a_n = \lim_n b_n v_n$.

There are plenty of such algebras known in literature. The following are examples of logmodular algebras in commutative C^*-algebras.

Example 6.1.4. (Function algebras) A classical result of Szegö (see Corollary 25.12, [16]) says that the Hardy algebra $H^\infty(\mathbb{T})$ has factorization in $L^\infty(\mathbb{T}, \mu)$. Here $\mathbb{T}$ is the unit circle, μ is the one-dimensional Lebesgue measure on $\mathbb{T}$ and $H^\infty(\mathbb{T})$ is the algebra of all essentially bounded functions on $\mathbb{T}$ whose negative Fourier coefficients are zero.

More generally, let m be a probability measure, and let $\mathcal{M}$ be a unital subalgebra of $L^\infty(m)$ satisfying the following:

(i) $\int fg dm = \int f dm \int g dm$ for all $f, g \in \mathcal{M}$, and

(ii) if $h \in L^1(m)$ with $h \geq 0$ a.e. and $\int fhdm = \int fdm$ for all $f \in \mathcal{M}$, then $h = 1$ a.e..

Let $H^2(m)$ be the closure of $\mathcal{M}$ in the Hilbert space $L^2(m)$, and let

$$H^\infty(m) = H^2(m) \cap L^\infty(m).$$

Then the proof of Theorem 4 in [39] says that $H^\infty(m)$ has factorization in $L^\infty(m)$. The algebra $H^\infty(m)$ satisfies many other equivalent conditions analogues to the classical Hardy algebra (see Theorem 3.1, [74] for details). Also see [38, 39, 74] for more concrete examples of such measures and algebras.

Example 6.1.5. (Dirichlet algebras) A closed unital subalgebra $\mathcal{M}$ of a commutative C^*-algebra $C(X)$ is called *Dirichlet algebra* if $\mathcal{M} + \overline{\mathcal{M}}$ is uniformly dense in $C(X)$ (equivalently, $\operatorname{Re}\mathcal{M}$ is uniformly dense in $\operatorname{Re} C(X)$), where $\operatorname{Re}\mathcal{M}$ (resp. $\operatorname{Re} C(X)$) denotes the set of real parts of the functions in $\mathcal{M}$ (resp. $C(X)$). If $\mathcal{M}$ is a Dirichlet algebra, then since $\log|\mathcal{M}^{-1}| \subseteq \operatorname{Re}\mathcal{M}$, it is immediate that $\log|\mathcal{M}^{-1}|$ is dense in $\operatorname{Re} C(X)$; hence $\mathcal{M}$ is a logmodular algebra in $C(X)$.

But some Dirichlet algebras may not have factorization. For example, consider the algebra $A(\mathbb{D})$ of all continuous functions on the closed unit disc $\overline{\mathbb{D}}$ which is holomorphic on the open unit disc $\mathbb{D}$. Then $A(\mathbb{D})$ is a Dirichlet algebra when considered as the subalgebra of $C(\mathbb{T})$, which is a consequence of Fejér-Riesz Theorem on factorization of positive trigonometric polynomials, but $A(\mathbb{D})$ does not have factorization in $C(\mathbb{T})$. On the other hand, $H^\infty(\mathbb{T})$ is an example of an algebra which has factorization in $L^\infty(\mathbb{T})$, but which is not a Dirichlet algebra. See [38] for details of these facts and more concrete examples of Dirichlet algebras.

To see some examples and other properties of noncommutative algebras having factorization, we recall some notions to this end. We emphasize the following convention to be followed for the rest of the chapter.

Convention. All von Neumann algebras are assumed to be faithfully acting on separable Hilbert spaces (which is equivalent to saying that the von Neumann algebras have separable predual).

Notation. For any collection $\{p_i\}_{i\in\Lambda}$ of projections, $\vee_{i\in\Lambda} p_i$ denotes the projection onto the smallest subspace containing ranges of all $p_i's$, and $\wedge_{i\in\Lambda} p_i$ denotes the projection onto the intersection of ranges of all $p_i's$.

We recall that a collection $\mathcal{E}$ of projections in a von Neumann algebra $\mathcal{B}$ is called *lattice* if $p \wedge q$ and $p \vee q \in \mathcal{E}$ whenever $p, q \in \mathcal{E}$.

Let $\mathcal{M}$ be a subalgebra of a von Neumann algebra $\mathcal{B}$. Let $\operatorname{Lat}_{\mathcal{B}}\mathcal{M}$ denote the lattice of all projections in $\mathcal{B}$ whose ranges are invariant under every element of $\mathcal{M}$ i.e.

$$\operatorname{Lat}_{\mathcal{B}}\mathcal{M} = \{p \in \mathcal{B}; p = p^2 = p^* \text{ and } ap = pap \ \forall a \in \mathcal{M}\}.$$

If $\mathcal{B} = \mathcal{B}(\mathcal{H})$, we denote $\operatorname{Lat}_{\mathcal{B}}\mathcal{M}$ simply by $\operatorname{Lat}\mathcal{M}$. Note that if $\mathcal{M}$ is also considered as a subalgebra of $\mathcal{B}(\mathcal{H})$ (where $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$), then we have

$$\operatorname{Lat}_{\mathcal{B}}\mathcal{M} = \mathcal{B} \cap \operatorname{Lat}\mathcal{M}.$$

Also note that $0, 1 \in \operatorname{Lat}_{\mathcal{B}} \mathcal{M}$ and $\operatorname{Lat}_{\mathcal{B}} \mathcal{M}$ is closed under the operations $\vee$ and $\wedge$ of arbitrary sub-collection, as well as closed under weak operator topology (WOT).

Dually, let $\mathcal{E}$ be a collection of projections in $\mathcal{B}$ (which may not be a lattice), and let $\operatorname{Alg}_{\mathcal{B}} \mathcal{E}$ (or $\operatorname{Alg} \mathcal{E}$ when $\mathcal{B} = \mathcal{B}(\mathcal{H})$) denote the algebra of all operators in $\mathcal{B}$ which leave range of every projection of $\mathcal{E}$ invariant i.e.

$$\operatorname{Alg}_{\mathcal{B}} \mathcal{E} = \{x \in \mathcal{B}; xp = pxp \ \ \forall p \in \mathcal{E}\}.$$

Again we note that

$$\operatorname{Alg}_{\mathcal{B}} \mathcal{M} = \mathcal{B} \cap \operatorname{Alg} \mathcal{E}.$$

Also it is clear that $\operatorname{Alg}_{\mathcal{B}} \mathcal{E}$ is a unital subalgebra of $\mathcal{B}$, which is closed in WOT.

Definition 6.1.6. Let $\mathcal{E}$ be a lattice of projections in a von Neumann algebra $\mathcal{B}$. Then the lattice $\mathcal{E}$ is called

(i) a *nest* if $\mathcal{E}$ is totally ordered by usual operator ordering i.e. for any $p, q \in \mathcal{E}$, either $p \leq q$ or $q \leq p$ holds true.

(ii) a *commutative subspace lattice (CSL)* if the projections of $\mathcal{E}$ commute with one another.

(iii) *complete* if $0, 1 \in \mathcal{E}$, and $\vee_{i \in \Lambda} p_i$ and $\wedge_{i \in \Lambda} p_i \in \mathcal{E}$ for any arbitrary family $\{p_i\}_{i \in \Lambda}$ in $\mathcal{E}$.

Remark 6.1.7. Some authors assume a nest or a CSL to be always complete. This is not the case here.

Definition 6.1.8. Let $\mathcal{M}$ be a subalgebra of a von Neumann algebra $\mathcal{B}$. Then $\mathcal{M}$ is called

(i) a *nest subalgebra* of $\mathcal{B}$ (or *nest algebra* when $\mathcal{B} = \mathcal{B}(\mathcal{H})$) if $\mathcal{M} = \operatorname{Alg}_{\mathcal{B}} \mathcal{E}$ for a nest $\mathcal{E}$ in $\mathcal{B}$.

(ii) a *CSL subalgebra* (or *CSL algebra* when $\mathcal{B} = \mathcal{B}(\mathcal{H})$) if $\mathcal{M} = \operatorname{Alg}_{\mathcal{B}} \mathcal{E}$ for a CSL $\mathcal{E}$ in $\mathcal{B}$.

(iii) *$\mathcal{B}$-reflexive* (or *reflexive* when $\mathcal{B} = \mathcal{B}(\mathcal{H})$) if $\mathcal{M} = \operatorname{Alg}_{\mathcal{B}} \operatorname{Lat}_{\mathcal{B}} \mathcal{M}$.

It is clear that any nest is a CSL and hence all nest subalgebras are CSL subalgebras. Also one can easily verify that any subalgebra in $\mathcal{B}$ of the form $\operatorname{Alg}_{\mathcal{B}} \mathcal{E}$ for some collection $\mathcal{E}$ of projections in $\mathcal{B}$, is always $\mathcal{B}$-reflexive. In particular, a nest subalgebra or a CSL subalgebra of $\mathcal{B}$ is $\mathcal{B}$-reflexive. It should be noted here that if $\mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$, then a subalgebra of $\mathcal{B}$ can be reflexive in $\mathcal{B}(\mathcal{H})$, but it need not be $\mathcal{B}$-reflexive.

The following are some examples of algebras having factorization in noncommutative von Neumann algebras. We restate the well-known Larson's result about the factorization property of nest algebras associated with countable complete nests.

Theorem 6.1.9 (Theorem 4.6, [47])**.** *Let $\mathcal{E}$ be a complete nest of projection on a separable Hilbert space $\mathcal{H}$. Then $\operatorname{Alg} \mathcal{E}$ has factorization in $\mathcal{B}(\mathcal{H})$ if and only if $\mathcal{E}$ is countable.*

Example 6.1.10. (Nest subalgebras) As already mentioned in Theorem 6.1.9, $\operatorname{Alg} \mathcal{E}$ has factorization in $\mathcal{B}(\mathcal{H})$ for any countable complete nest $\mathcal{E}$ in $\mathcal{B}(\mathcal{H})$. More generally, Pitts proves that if $\mathcal{E}$ is a complete nest in a factor $\mathcal{B}$, then $\operatorname{Alg}_{\mathcal{B}} \mathcal{E}$ has factorization in $\mathcal{B}$ if and only if "certain" subnest $\mathcal{E}_r$ of $\mathcal{E}$ is countable (Theorem 6.4, [64]).

Moreover, if $\mathcal{E}$ is a nest (not necessarily countable) in a finite von Neumann algebra $\mathcal{B}$ (not necessarily a factor), then $\operatorname{Alg}_{\mathcal{B}} \mathcal{E}$ has factorization in $\mathcal{B}$ (Corollary 5.11, [64]).

Example 6.1.11. (Subdiagonal algebras) Let $\mathcal{M}$ be a unital subalgebra of a von Neumann algebra $\mathcal{B}$, and let $\phi : \mathcal{B} \to \mathcal{B}$ be a faithful unital positive linear map, which is an idempotent (i.e. $\phi \circ \phi = \phi$). Then $\mathcal{M}$ is called a *subdiagonal algebra* with respect to ϕ if it satisfies the following:

(i) $\mathcal{M} + \mathcal{M}^*$ is σ-weakly dense in $\mathcal{B}$,
(ii) $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in \mathcal{M}$, and
(iii) $\phi(\mathcal{M}) \subseteq \mathcal{M} \cap \mathcal{M}^*$.

If there is no larger subdiagonal algebra containing $\mathcal{M}$, then $\mathcal{M}$ is *maximal* with respect to ϕ. Moreover, if the von Neumann algebra $\mathcal{B}$ is finite with a distinguished trace τ, then the subdiagonal algebra $\mathcal{M}$ is called *finite* if $\tau \circ \phi = \tau$.

Arveson proves that if $\mathcal{M}$ is a maximal (with respect to ϕ) finite subdiagonal algebra of $\mathcal{B}$, then $\mathcal{M}$ has factorization in $\mathcal{B}$ (Theorem 4.2.1, [4]). A nest subalgebra of a finite von Neumann algebra is an example of maximal finite subdiagonal algebras (Corollary 3.1.2, [4]). See 6.4.10 for another concrete example of a finite subdiagonal algebra. There are other subdiagonal algebras (not necessarily finite) as well, which are known to have factorization. For example, all subdiagonal algebras arising out of periodic flow have factorization. See [72] for more details of these notions and Corollary 3.11 therein.

Remark 6.1.12. We believe that some known facts about subdiagonal algebras can also be deduced from our result. One such is Theorem 5.1 in [53], which follows directly from Corollary 6.2.2. However, we have not explored other possible consequences in depth.

Below we have some concrete examples of nest algebras which do not have factorization. We do not know at this point whether they are logmodular.

Example 6.1.13. Let $\mathcal{E}$ be the nest $\{p_t; t \in [0,1]\}$ of projections on $L^2([0,1])$, where p_t denotes the projection onto $L^2([0,t])$, considered as subspace of $L^2([0,1])$. Then $\mathcal{E}$ is complete and uncountable; hence $\operatorname{Alg}\mathcal{E}$ does not have factorization in $\mathcal{B}(L^2([0,1]))$ by Theorem 6.1.9.

Additionally let $\mathcal{F} = \{p_i; i \in \mathbb{Q}\}$ be the nest of projections on $\ell^2(\mathbb{Q})$, where p_i denotes the projection onto the subspace $\overline{\operatorname{span}}\{e_j; j \leq i\}$, for the canonical basis $\{e_i; i \in \mathbb{Q}\}$ of $\ell^2(\mathbb{Q})$. Although $\mathcal{F}$ is a countable nest, it is easy to verify that its completion is not countable (actually indexed by $\mathbb{R} \sqcup \mathbb{Q}$; see Example 1.5.10) and hence $\operatorname{Alg}\mathcal{F}$ does not have factorization in $\mathcal{B}(\ell^2(\mathbb{Q}))$. At this point, we do not know whether these algebras are logmodular.

6.2 Lattices of logmodular algebras

We are now ready to state the main result on logmodular algebras. This tells us the behaviour of lattices of projections whose ranges are invariant under them.

Theorem 6.2.1. *Let $\mathcal{M}$ be a logmodular algebra in a von Neumann algebra $\mathcal{B}$. Then $\operatorname{Lat}_{\mathcal{B}}\mathcal{M}$ is a commutative subspace lattice. Moreover if $\mathcal{B}$ is a factor, then $\operatorname{Lat}_{\mathcal{B}}\mathcal{M}$ is a nest.*

We postpone the proof of Theorem 6.2.1 to the next section, and instead look at some of its consequences first. Note that since any algebra having factorization is also logmodular, the following corollary is immediate.

Corollary 6.2.2. *Let $\mathcal{M}$ be an algebra having factorization in a von Neumann algebra $\mathcal{B}$. Then* $\operatorname{Lat}_{\mathcal{B}} \mathcal{M}$ *is a commutative subspace lattice. Moreover if $\mathcal{B}$ is a factor, then* $\operatorname{Lat}_{\mathcal{B}} \mathcal{M}$ *is a nest.*

Remark 6.2.3. If $\mathcal{B}$ is an arbitrary von Neumann algebra which is not a factor, and $\mathcal{M}$ is a subalgebra of $\mathcal{B}$, then we can never expect $\operatorname{Lat}_{\mathcal{B}} \mathcal{M}$ to be a nest irrespective of whether $\mathcal{M}$ is logmodular or has factorization. In fact if $\mathcal{P}_{\mathcal{Z}}$ denotes the lattice of all projections in the center $\mathcal{Z}$ of $\mathcal{B}$, then it is always true that $\mathcal{P}_{\mathcal{Z}} \subseteq \operatorname{Lat}_{\mathcal{B}} \mathcal{M}$. So $\operatorname{Lat}_{\mathcal{B}} \mathcal{M}$ can never be a nest if the center $\mathcal{Z}$ is non-trivial.

Now let $\mathcal{B}$ be a factor, and let $\mathcal{M}$ be a $\mathcal{B}$-reflexive subalgebra of $\mathcal{B}$. If $\mathcal{M}$ is logmodular in $\mathcal{B}$, then $\operatorname{Lat}_{\mathcal{B}} \mathcal{M}$ is a nest by Theorem 6.2.1. But since $\mathcal{M} = \operatorname{Alg}_{\mathcal{B}} \operatorname{Lat}_{\mathcal{B}} \mathcal{M}$, it follows that $\mathcal{M}$ is a nest subalgebra of $\mathcal{B}$.

We now answer an open question posed by Paulsen and Raghupathi (see pg. 2630, [62]) using above observations. They conjectured that every completely distributive CSL logmodular algebra in $\mathcal{B}(\mathcal{H})$ is a nest algebra. See [17] for more details on completely distributive CSL algebras. More importantly, any completely distributive CSL algebra is of the form $\operatorname{Alg} \mathcal{E}$, where $\mathcal{E}$ is a completely distributive CSL, and hence it is a special case of reflexive algebras. We have thus answered their question in affirmative, which we record below.

Corollary 6.2.4. *Any $\mathcal{B}$-reflexive logmodular algebra in a factor $\mathcal{B}$ is a nest subalgebra of $\mathcal{B}$. In particular, all reflexive (hence completely distributive CSL) logmodular algebras in $\mathcal{B}(\mathcal{H})$ are nest algebras.*

If an algebra $\mathcal{M}$ has factorization in $\mathcal{B}(\mathcal{H})$, then $\operatorname{Alg} \operatorname{Lat} \mathcal{M}$ also has factorization in $\mathcal{B}(\mathcal{H})$ as $\mathcal{M}$ is contained in $\operatorname{Alg} \operatorname{Lat} \mathcal{M}$. Since $\operatorname{Lat} \mathcal{M}$ is a complete nest, it then follows from Theorem 6.1.9 that $\operatorname{Lat} \mathcal{M}$ is a countable nest. In particular, if $\mathcal{M} = \operatorname{Alg} \mathcal{E}$ for a lattice $\mathcal{E}$ of projections in $\mathcal{H}$, then $\mathcal{E}$ is a countable nest because $\mathcal{E} \subseteq \operatorname{Lat} \mathcal{M}$. Thus we get the following corollary, which is a strengthening of Theorem 6.1.9 of Larson.

Corollary 6.2.5. *Let $\mathcal{E}$ be a complete lattice of projections on a separable Hilbert space $\mathcal{H}$. Then* $\operatorname{Alg} \mathcal{E}$ *has factorization in $\mathcal{B}(\mathcal{H})$ if and only if $\mathcal{E}$ is a countable nest.*

To understand the nest result, we recall some terminologies to this end. Let $\mathcal{B}$ be a von Neumann algebra, and let $\mathcal{E}$ be a complete nest in $\mathcal{B}$. For any projection $p \in \mathcal{E}$, let

$$p_- = \vee\{q \in \mathcal{E}; q < p\} \text{ and } p_+ = \wedge\{q \in \mathcal{E}; q > p\}.$$

Definition 6.2.6. An *atom* of a complete nest $\mathcal{E}$ is a nonzero projection of the form $p - p_-$ for some $p \in \mathcal{E}$ with $p \neq p_-$. The nest $\mathcal{E}$ is called *atomic* if there is a countable sequence $\{r_n\}_{n\geq 1}$ of atoms of $\mathcal{E}$ such that $\sum_{n\geq 1} r_n = 1$, where the sum converges in WOT.

Clearly two distinct atoms are always mutually orthogonal. Let $\mathcal{E}$ be a complete nest in $\mathcal{B}(\mathcal{H})$. Let $\{r_n\}_{n\geq 1}$ be the collection of all atoms of $\mathcal{E}$, and let $r = \sum_{n\geq 1} r_n$ in WOT convergence. If $r \neq 1$, then it is straightforward to check that the nest $\{p \wedge r^{\perp};\ p \in \mathcal{E}\}$ in $\mathcal{B}(\mathcal{R}(r^{\perp}))$ is complete and has no atom (such nests without any atom are called *continuous*). Here $r^{\perp} = 1 - r$. But then any continuous complete nest has to be uncountable (in fact indexed by $[0, 1]$; see Lemma 13.3 in [17]). In particular, if the nest $\mathcal{E}$ is countable, then $r = 1$ and hence $\mathcal{E}$ is atomic.

We thus get the following corollary, which is nothing but the aforementioned result as in Theorem 3.1.5.

Corollary 6.2.7. *Let an algebra $\mathcal{A}$ have factorization in $\mathcal{B}(\mathcal{H})$. Then* $\mathrm{Lat}\,\mathcal{A}$ *is an atomic and countable nest.*

Proof. Since $\mathcal{A}$ has factorization in $\mathcal{B}(\mathcal{H})$, $\mathrm{Alg}\,\mathrm{Lat}\,\mathcal{A}$ also has factorization in $\mathcal{B}(\mathcal{H})$ as it contains $\mathcal{A}$. Consequently $\mathrm{Lat}\,\mathcal{A}$ is a countable nest by Corollary 6.2.5, so it is atomic. □

6.3 Proof of the main result

This section is devoted to the proof of our main result (Theorem 6.2.1) on logmodular algebras. We first discuss some general ingredients required for this. A simple observation that we shall be using throughout the chapter is the following remark. Recall that $p^{\perp}$ denotes the projection $1 - p$ for any projection p.

Remark 6.3.1. For any subalgebra $\mathcal{M}$ of a von Neumann algebra $\mathcal{B}$, $p \in \mathrm{Lat}_{\mathcal{B}}\,\mathcal{M} \iff ap = pap\ \ \forall a \in \mathcal{M} \iff pa^* = pa^*p\ \forall a \in \mathcal{M} \iff a^*p^{\perp} = p^{\perp}a^*p^{\perp}\ \ \forall a \in \mathcal{M} \iff p^{\perp} \in \mathrm{Lat}_{\mathcal{B}}\,\mathcal{M}^*$.

The first step towards the proof is the following proposition which says that logmodularity and factorization are preserved under compression of algebras by appropriate projections. Here $p\mathcal{M}p$ denotes the subspace

$$p\mathcal{M}p = \{pap; a \in \mathcal{M}\}$$

for any projection p and an algebra $\mathcal{M}$. Note that $p\mathcal{M}p$ need not always be an algebra.

Proposition 6.3.2. *Let $\mathcal{M}$ be an algebra having logmodularity (resp. factorization) in a von Neumann algebra $\mathcal{B}$, and let $p, q \in \mathrm{Lat}_{\mathcal{B}}\,\mathcal{M}$ be such that $p \geq q$. Then the following statements are true:*

(i) $p\mathcal{M}p(= \mathcal{M}p)$ has logmodularity (resp. factorization) in $p\mathcal{B}p$.

(ii) $p^{\perp}\mathcal{M}p^{\perp}$ has logmodularity (resp. factorization) in $p^{\perp}\mathcal{B}p^{\perp}$.

(iii) $(p - q)\mathcal{M}(p - q)$ has logmodularity (resp. factorization) in $(p - q)\mathcal{B}(p - q)$.

Proof. We shall prove only part (iii). Part (i) follows from (iii) by taking $q = 0$, and (ii) follows from (iii) by taking $p = 1$ and $q = p$. Also we shall prove only the case of logmodularity. That of factorization follows similarly. So assume that $\mathcal{M}$ is logmodular in $\mathcal{B}$.

First we show that $(p - q)\mathcal{M}(p - q)$ is an algebra. For all $a \in \mathcal{M}$, since $ap = pap$ and $aq = qaq$, we note that

$$(p - q)aq = (p - q)qaq = 0,$$

and

$$pa(p-q) = pap - paq = ap - pqaq = ap - qaq = ap - aq = a(p-q). \tag{6.3.1}$$

Combining the two expressions above, it follows for all $a, b \in \mathcal{M}$ that

$$(p-q)a(p-q)b(p-q) = (p-q)apb(p-q) - (p-q)aqb(p-q) = (p-q)ab(p-q), \tag{6.3.2}$$

which shows that $(p-q)\mathcal{M}(p-q)$ is an algebra. Next to show that $(p-q)\mathcal{M}(p-q)$ is logmodular in $(p-q)\mathcal{B}(p-q)$, fix a positive and invertible element x in $(p-q)\mathcal{B}(p-q)$ and set

$$\tilde{x} = x + q + p^{\perp}.$$

Note that $x = (p-q)\tilde{x}(p-q)$. It is clear that $\tilde{x}$ is positive in $\mathcal{B}$. Since x is positive and invertible in $(p-q)\mathcal{B}(p-q)$, there is some $\alpha \in (0,1)$ such that $x \geq \alpha(p-q)$; from which we get

$$\tilde{x} = x + q + p^{\perp} \geq \alpha(p-q) + \alpha q + \alpha p^{\perp} = \alpha.$$

This shows that $\tilde{x}$ is invertible in $\mathcal{B}$. We then use logmodularity of $\mathcal{M}$ in $\mathcal{B}$ to get a sequence $\{\tilde{a}_n\}$ in $\mathcal{M}^{-1}$ such that

$$\tilde{x} = \lim_n \tilde{a}_n^* \tilde{a}_n.$$

So for each n, we have $\tilde{a}_n q = q\tilde{a}_n q$ and $\tilde{a}_n^{-1} q = q\tilde{a}_n^{-1} q$. It then follows that

$$(q\tilde{a}_n q)(q\tilde{a}_n^{-1} q) = q\tilde{a}_n \tilde{a}_n^{-1} q = q \quad \text{and} \quad (q\tilde{a}_n^{-1} q)(q\tilde{a}_n q) = q\tilde{a}_n^{-1}\tilde{a}_n q = q,$$

which is to say that $q\tilde{a}_n q$ is invertible in $q\mathcal{B}q$ with $(q\tilde{a}_n q)^{-1} = q\tilde{a}_n^{-1} q \in q\mathcal{M}q$. In particular, since the sequence $\{\tilde{a}_n^{-1}\}$ is bounded (as $\{(\tilde{a}_n^* \tilde{a}_n)^{-1}\}$ is a convergent sequence), it follows that the sequence $\{(q\tilde{a}_n q)^{-1}\}$ is bounded. Note that $q\tilde{x}(p-q) = 0$, and since $q\tilde{a}_n^* = q\tilde{a}_n^* q$ for all n, we have

$$0 = q\tilde{x}(p-q) = \lim_n q\tilde{a}_n^* \tilde{a}_n (p-q) = \lim_n (q\tilde{a}_n^* q)(q\tilde{a}_n(p-q)).$$

Multiplying to the left of the sequence by $(q\tilde{a}_n^* q)^{-1}$ (which is bounded) yields

$$\lim_n q\tilde{a}_n(p-q) = 0,$$

using which and the expression $\tilde{a}_n(p-q) = p\tilde{a}_n(p-q)$ from (6.3.1), we get the following:

$$\begin{aligned} x = (p-q)\tilde{x}(p-q) &= \lim_n (p-q)\tilde{a}_n^* \tilde{a}_n (p-q) = \lim_n (p-q)\tilde{a}_n^* [p\tilde{a}_n(p-q)] \\ &= \lim_n (p-q)\tilde{a}_n^* [q\tilde{a}_n(p-q)] + \lim_n (p-q)\tilde{a}_n^* [(p-q)\tilde{a}_n(p-q)] \\ &= \lim_n (p-q)\tilde{a}_n^* (p-q)\tilde{a}_n(p-q) = \lim_n a_n^* a_n, \end{aligned}$$

where $a_n = (p-q)\tilde{a}_n(p-q) \in (p-q)\mathcal{M}(p-q)$. Also for each n, we have from (6.3.2) that

$$p - q = (p-q)\tilde{a}_n^{-1}(p-q)\tilde{a}_n(p-q) = (p-q)\tilde{a}_n(p-q)\tilde{a}_n^{-1}(p-q),$$

which shows that $a_n = (p-q)\tilde{a}_n(p-q)$ is invertible with inverse $(p-q)\tilde{a}_n^{-1}(p-q)$ in $(p-q)\mathcal{M}(p-q)$. Thus we get a sequence $\{a_n\}$ of invertible elements with $a_n, a_n^{-1} \in (p-q)\mathcal{M}(p-q)$ for all n such that $x = \lim_n a_n^* a_n$. Since x is an arbitrary positive and invertible element, we conclude that $(p-q)\mathcal{M}(p-q)$ is logmodular in $(p-q)\mathcal{B}(p-q)$. □

At this point, we need to recall some basic facts about subspaces in a separable Hilbert space. Following Halmos [35], consider the following:

Definition 6.3.3. Two non-zero subspaces E and F of a Hilbert space are said to be in *generic position* if all the following subspaces

$$E \cap F,\ E \cap F^{\perp},\ E^{\perp} \cap F,\ E^{\perp} \cap F^{\perp}$$

are zero.

We are going to use the following characterization of subspaces in generic position. Recall that P_E denotes the projection onto a subspace E. Also recall that $\ker x$ denotes the kernel of any operator x.

Lemma 6.3.4 (Theorem 2, [35])**.** *Let E and F be two subspaces in generic position in a separable Hilbert space $\mathcal{H}$. Then there exist a Hilbert space $\mathcal{K}$, a unitary $U : \mathcal{H} \to \mathcal{K} \oplus \mathcal{K}$, and commuting positive contractions $x, y \in \mathcal{B}(\mathcal{K})$ such that $x^2 + y^2 = 1$, $\ker x = \ker y = 0$ and*

$$UP_EU^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \textit{and} \quad UP_FU^* = \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}.$$

Lemma 6.3.5. *Let E and F be two subspaces in a Hilbert space $\mathcal{H}$, and let $\mathcal{H}_1$ denote the subspace of $\mathcal{H}$ given by*

$$\mathcal{H}_1 = \mathcal{H} \ominus \left(E \cap F + E \cap F^{\perp} + E^{\perp} \cap F + E^{\perp} \cap F^{\perp}\right).$$

If $E_1 = E \cap \mathcal{H}_1$ and $F_1 = F \cap \mathcal{H}_1$, then exactly one of the following holds true:

(i) $E_1, F_1 = \{0\}$, and $\mathcal{H}_1 = \{0\}$.

(ii) E_1 and F_1 are non-zero, and they are in generic position as subspaces of $\mathcal{H}_1$.

Moreover, the projections P_E and P_F commute if and only if the first condition is satisfied (i.e. $\mathcal{H}_1 = \{0\}$).

Proof. Firstly note that if E_1 is non-zero, then the map $P_{F|_{E_1}} : E_1 \to F$ is one-one with range contained in F_1 and hence $F_1 \neq 0$. Similarly by symmetry, $F_1 \neq 0$ implies $E_1 \neq 0$. Therefore, either both E_1, F_1 are zero or both are non-zero.

First assume that E_1 and F_1 are zero. Then we have $E \cap \mathcal{H}_1 = \{0\} = F \cap \mathcal{H}_1$ i.e. $(E \cup F) \cap \mathcal{H}_1 = \{0\}$. By taking orthogonal complement both the sides, and using the facts that $(M \cap N)^{\perp} = M^{\perp} \vee N^{\perp}$ and $(M \cup N)^{\perp} = M^{\perp} \cap N^{\perp}$ for any subspaces M, N of $\mathcal{H}$, we get

$$\mathcal{H} = (E \cup F)^{\perp} \vee \mathcal{H}_1^{\perp} = (E^{\perp} \cap F^{\perp}) \vee \mathcal{H}_1^{\perp} = \mathcal{H}_1^{\perp},$$

which implies that $\mathcal{H}_1 = 0$. This proves the first assertion. Now let E_1 and F_1 are non-zero. Then we have the following:

$$\begin{aligned} &E_1 \cap F_1 = (E \cap F) \cap \mathcal{H}_1 = 0, \\ &E_1 \cap (\mathcal{H}_1 \ominus F_1) = E_1 \cap \mathcal{H}_1 \cap F_1^{\perp} = E \cap \mathcal{H}_1 \cap (F^{\perp} \vee \mathcal{H}_1^{\perp}) = E \cap F^{\perp} \cap \mathcal{H}_1 = 0. \end{aligned}$$

Similarly we have

$$(\mathcal{H}_1 \ominus E_1) \cap F_1 = (E^\perp \cap F) \cap \mathcal{H}_1 = 0,$$
$$(\mathcal{H}_1 \ominus E_1) \cap (\mathcal{H}_1 \ominus F_1) = \mathcal{H}_1 \cap E_1^\perp \cap F_1^\perp = (E^\perp \cap F^\perp) \cap \mathcal{H}_1 = 0.$$

This proves the second assertion that E_1 and F_1 are in generic position as subspaces of $\mathcal{H}_1$. □

The subspaces E_1 and F_1 as in Lemma 3.4 are called *generic part* of the subspaces E and F. The structure of two general subspaces can now be described in the following proposition. The proof directly follows from the two lemmas above, so it is left to the readers.

Proposition 6.3.6. *Let E and F be two subspaces in a separable Hilbert space $\mathcal{H}$. Then there is a Hilbert space $\mathcal{K}$ (could be zero), and commuting positive contractions $x, y \in \mathcal{B}(\mathcal{K})$ with $x^2 + y^2 = 1$ and $\ker x = \ker y = 0$ such that, upto unitary equivalence*

$$\mathcal{H} = E \cap F \oplus E \cap F^\perp \oplus E^\perp \cap F \oplus E^\perp \cap F^\perp \oplus \mathcal{K} \oplus \mathcal{K},$$

and

$$P_E = 1 \oplus 1 \oplus 0 \oplus 0 \oplus 1 \oplus 0 \quad \textit{and} \quad P_F = 1 \oplus 0 \oplus 1 \oplus 0 \oplus \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}.$$

Here any of the components in the decomposition could be 0. Moreover, $P_E P_F = P_F P_E = P_{E\cap F}$ if and only if $\mathcal{K} = \{0\}$.

We are now ready to give proof of our main result through a series of lemmas. The next two lemmas deal with factor von Neumann algebras only. We reiterate here that throughout, convergence of any sequence of operators is taken in norm topology unless stated otherwise.

Lemma 6.3.7. *Let $\mathcal{B}$ be a factor, and let p, q be mutually orthogonal projections in $\mathcal{B}$. Then $\mathrm{Alg}_{\mathcal{B}}\{p, q\}$ is not logmodular in $\mathcal{B}$.*

Proof. Since $\mathcal{B}$ is a factor and $p, q \in \mathcal{B}$ are non-zero, it follows from Theorem 1.1.19 that there is a non-zero partial isometry $v \in \mathcal{B}$ such that $v^*v \leq p$ and $vv^* \leq q$. In particular, we have

$$v = qv = vp. \tag{6.3.3}$$

Assume to the contrary that $\mathrm{Alg}_{\mathcal{B}}\{p, q\}$ is logmodular in $\mathcal{B}$. Let $x = 1 + \epsilon(v + v^*)$ for some $\epsilon > 0$, where ϵ is chosen small enough so that x is positive and invertible in $\mathcal{B}$. Then there exists a sequence $\{a_n\}$ of invertible elements in $\mathrm{Alg}_{\mathcal{B}}\{p, q\}$ such that

$$x = \lim_n a_n^* a_n.$$

Now since $pq = 0$, we note from (6.3.3) that $v^*p = (v^*q)p = 0$; hence we get $qxp = \epsilon qvp = \epsilon v$. We also have $a_n p = p a_n p$ and $q a_n^* = q a_n^* q$ for all n; thus it follows that

$$\epsilon v = qxp = \lim_n q a_n^* a_n p = \lim_n q a_n^* q p a_n p = 0,$$

which is a contradiction, as $v \neq 0$. □

We recall here a simple fact that if p and q are commuting projections, then pq is a projection such that $p \wedge q = pq$ and $p \vee q = p + q - pq$. The following lemma provides a proof of the second assertion of Theorem 6.2.1, once we assume the first.

Lemma 6.3.8. *Let $\mathcal{B}$ be a factor, and let $p, q \in \mathcal{B}$ be two commuting projections. If $\mathrm{Alg}_{\mathcal{B}}\{p.q\}$ is logmodular in $\mathcal{B}$, then either $p \leq q$ or $q \leq p$ holds true.*

Proof. Since p and q commuting projections, the operators $pq, pq^{\perp}$ and $p^{\perp}q$ are projections. We know from the given hypothesis and Lemma 6.3.7 that $pq \neq 0$. Note that the required assertion will follow by the following argument, once we show that either $pq^{\perp} = 0$ or $p^{\perp}q = 0$: say $pq^{\perp} = 0$, then $p = p(q + q^{\perp}) = pq$ which implies that $p \leq q$. Similarly, $p^{\perp}q = 0$ will imply $q \leq p$.

Assume opposite to our requirement that both the projections $pq^{\perp}$ and $p^{\perp}q$ are non-zero. Since $\mathcal{B}$ is a factor, it follows from Theorem 1.1.19 that there is a non-zero partial isometry $v \in \mathcal{B}$ such that $v^*v \leq pq^{\perp}$ and $vv^* \leq p^{\perp}q$; in particular we have,

$$v = vpq^{\perp} = p^{\perp}qv. \tag{6.3.4}$$

Now let $x = 1 + \epsilon(v + v^*)$ for $\epsilon > 0$, where we choose ϵ small enough so that x is positive and invertible in $\mathcal{B}$. Since $\mathrm{Alg}_{\mathcal{B}}\{p, q\}$ is logmodular in $\mathcal{B}$, there exists a sequence $\{a_n\}$ of invertible elements in $\mathcal{B}$ such that $a_n, a_n^{-1} \in \mathrm{Alg}_{\mathcal{B}}\{p, q\}$ for all n, and

$$x = \lim_n a_n^* a_n.$$

Note that $pqa_npq = a_npq$ and $pqa_n^{-1}pq = a_n^{-1}pq$; hence each pqa_npq is invertible in $pq\mathcal{B}pq$ with respective inverse $pqa_n^{-1}pq$. Consequently, the sequence $\{(pqa_npq)^{-1}\}$ is bounded, as the sequence $\{a_n^{-1}\}$ is bounded. Also note from (6.3.4) that $vpq = 0$ and $p^{\perp}qv^* = 0$; hence we get

$$p^{\perp}qxpq = p^{\perp}qpq + \epsilon p^{\perp}q(vpq) + \epsilon(p^{\perp}qv^*)pq = 0.$$

Thus we have

$$0 = p^{\perp}qxpq = \lim_n p^{\perp}qa_n^*a_npq = \lim_n (p^{\perp}qa_n^*pq)(pqa_npq),$$

where we multiply by $\{(pqa_npq)^{-1}\}$ to right side of sequence to get $\lim_n p^{\perp}qa_n^*pq = 0$; using which and the expressions $qa_n^* = qa_n^*q$ and $a_np = pa_np$ for all n, it follows that

$$p^{\perp}qxpq^{\perp} = \lim_n p^{\perp}qa_n^*a_npq^{\perp} = \lim_n (p^{\perp}qa_n^*qp)a_npq^{\perp} = 0.$$

On the other hand, we again use (6.3.4) and the condition $p^{\perp}qv^* = 0$ to get

$$p^{\perp}qxpq^{\perp} = p^{\perp}qpq^{\perp} + \epsilon p^{\perp}qvpq^{\perp} + \epsilon(p^{\perp}qv^*)pq^{\perp} = \epsilon p^{\perp}qvpq^{\perp} = \epsilon v \neq 0.$$

So we get a contradiction, which arose because we assumed that both $pq^{\perp}$ and $p^{\perp}q$ are non-zero. Thus one of them is zero and we have the required result. $\square$

We are going to use the following simple lemma very frequently.

Lemma 6.3.9. *Let $\{a_n\}$ be a sequence of invertible elements in a C^*-algebra such that $\lim_n a_n^* a_n = 1$. Then $\{a_n^{-1}\}$ is bounded and $\lim_n a_n a_n^* = 1$.*

Proof. Since $\lim_n a_n^* a_n = 1$, it follows that $\lim_n (a_n^* a_n)^{-1} = 1$ and so $\{(a_n^* a_n)^{-1}\}$ is bounded. This implies the first assertion that $\{a_n^{-1}\}$ is bounded. Further we have $\lim_n a_n^* a_n a_n^* a_n = 1$, and hence

$$0 = \lim_n (a_n^* a_n a_n^* a_n - a_n^* a_n) = \lim_n a_n^* (a_n a_n^* - 1) a_n.$$

Since the sequence $\{a_n^{-1}\}$ is bounded, it follows by multiplying $a_n^{*\,-1}$ to the left and a_n^{-1} to the right of the sequence that $\lim_n (a_n a_n^* - 1) = 0$, as to be proved. $\square$

Next we consider lattices of logmodular algebras in arbitrary von Neumann algebras, where our aim is to prove that the generic part of any two invariant subspaces is zero. Recall that $\mathcal{R}(x)$ denotes the range of an operator x.

Lemma 6.3.10. *Let $\mathcal{B}$ be a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ for some separable Hilbert $\mathcal{H}$, and let p, q be two non-zero projections in $\mathcal{B}$ such that $\mathcal{R}(p)$ and $\mathcal{R}(q)$ are in generic position in $\mathcal{H}$. Then $\mathrm{Alg}_{\mathcal{B}}\{p, q\}$ is not logmodular in $\mathcal{B}$.*

Proof. Assume contrary to the assertion that the algebra $\mathrm{Alg}_{\mathcal{B}}\{p, q\}$ is logmodular in $\mathcal{B}$. Since $\mathcal{R}(p)$ and $\mathcal{R}(q)$ are in generic position in $\mathcal{H}$, it follows from Lemma 6.3.4 that there exist a Hilbert space $\mathcal{K}$ and commuting positive contractions $x, y \in \mathcal{B}(\mathcal{K})$ satisfying

$$\ker x = 0, \ \ \ker y = 0 \ \text{ and } \ x^2 + y^2 = 1$$

such that upto unitary equivalence, we have $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$ and

$$p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \ \text{ and } \ q = \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}. \tag{6.3.5}$$

Since logmodularity is preserved under unitary equivalence by Proposition 6.1.2, we can assume without loss of generality that $\mathcal{B}$ is a von Neumann subalgebra of $\mathcal{B}(\mathcal{K} \oplus \mathcal{K})$, and p, q are of the form as in (6.3.5).

Now let S be an invertible operator such that $S, S^{-1} \in \mathrm{Alg}_{\mathcal{B}}\{p, q\}$. Then $Sp = pSp$ and $S^{-1}p = pS^{-1}p$, which imply that S and S^{-1} have the following form:

$$S = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \ \text{ and } \ S^{-1} = \begin{bmatrix} a' & b' \\ 0 & c' \end{bmatrix},$$

for some operators $a, b, c, a', b', c' \in \mathcal{B}(\mathcal{K})$. It is then clear from the expression $SS^{-1} = 1 = S^{-1}S$ that a and c are invertible in $\mathcal{B}(\mathcal{K})$ with respective inverses a' and c'. Now we have

$$Sq = \begin{bmatrix} ax^2 + bxy & axy + by^2 \\ cxy & cy^2 \end{bmatrix}$$

and

$$qSq = \begin{bmatrix} x^2ax^2 + x^2bxy + xycxy & x^2axy + x^2by^2 + xycy^2 \\ xyax^2 + xybxy + y^2cxy & xyaxy + xyby^2 + y^2cy^2 \end{bmatrix}.$$

Since $Sq = qSq$, we equate $(2,1)$ entries of the two matrices, and use the condition $1 - y^2 = x^2$ to get the expression $x^2cxy = xyax^2 + xybxy$; but x is injective (and hence x has dense range, as x is positive) and $xy = yx$, so x can be cancelled from both the sides to get the following:

$$xcy = yax + yby. \tag{6.3.6}$$

Now fix $\alpha \geq 1$, and let

$$Z = \begin{bmatrix} 1 & \alpha \\ \alpha & \alpha^2 + 1 \end{bmatrix} \in \mathcal{B}(\mathcal{K} \oplus \mathcal{K}).$$

It is clear that Z is a positive and invertible operator. We claim that $Z \in \mathcal{B}$. Since p and q are in $\mathcal{B}$, it follows that

$$\begin{bmatrix} x^2 & 0 \\ 0 & 0 \end{bmatrix} = pqp \in \mathcal{B}.$$

Similarly

$$\begin{bmatrix} 0 & 0 \\ 0 & y^2 \end{bmatrix} = p^\perp qp^\perp \in \mathcal{B}.$$

Thus

$$\begin{bmatrix} x^2 & 0 \\ 0 & y^2 \end{bmatrix} \in \mathcal{B} \text{ and hence } \begin{bmatrix} 0 & xy \\ xy & 0 \end{bmatrix} \in \mathcal{B}.$$

Set

$$T = \begin{bmatrix} 0 & xy \\ xy & 0 \end{bmatrix}$$

and let $T = U|T|$ be its polar decomposition, where $|T|$ denotes the square root of the operator T^*T. It is clear that T is one-one (as xy is one-one), so U is unitary. It is straightforward to check (using uniqueness of polar decomposition) that

$$|T| = \begin{bmatrix} xy & 0 \\ 0 & xy \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since $\mathcal{B}$ is a von Neumann algebra and $T \in \mathcal{B}$, it follows that $U \in \mathcal{B}$ and so

$$\begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} = \alpha U \in \mathcal{B}.$$

Also since

$$\begin{bmatrix} 1 & 0 \\ 0 & \alpha^2 + 1 \end{bmatrix} = p + (\alpha^2 + 1)p^\perp \in \mathcal{B},$$

we conclude that $Z \in \mathcal{B}$, as claimed. Thus by logmodularity of $\mathrm{Alg}_{\mathcal{B}}\{p, q\}$ in $\mathcal{B}$, we get a sequence $\{S_n\}$ of invertible operators with $S_n, S_n^{-1} \in \mathrm{Alg}_{\mathcal{B}}\{p, q\}$ for all n such that $Z = \lim_n S_n^* S_n$. It then follows from above discussion that each S_n is of the form:

$$S_n = \begin{bmatrix} a_n & b_n \\ 0 & c_n \end{bmatrix},$$

for some $a_n, b_n, c_n \in \mathcal{B}(\mathcal{K})$ such that a_n and c_n are invertible operators, and from (6.3.6) we have

$$xc_ny = ya_nx + yb_ny. \tag{6.3.7}$$

Now we have

$$\begin{bmatrix} 1 & \alpha \\ \alpha & \alpha^2+1 \end{bmatrix} = Z = \lim_n S_n^* S_n = \lim_n \begin{bmatrix} a_n^* a_n & a_n^* b_n \\ b_n^* a_n & b_n^* b_n + c_n^* c_n \end{bmatrix}. \tag{6.3.8}$$

So we get $\lim_n a_n^* a_n = 1$, and since each a_n is invertible, it follows from Lemma 6.3.9 that

$$\lim_n a_n a_n^* = 1. \tag{6.3.9}$$

We also get from (6.3.8) that $\lim_n a_n^* b_n = \alpha$, which further yields by multiplying a_n to the left side of the sequence and using (6.3.9) that

$$\lim_n (b_n - \alpha a_n) = 0. \tag{6.3.10}$$

Set $d_n = b_n - \alpha a_n$ for all n. Then $\lim_n d_n = 0$, and since $\lim_n a_n^* a_n = 1$ we have

$$\lim_n b_n^* b_n = \lim_n (d_n + \alpha a_n)^*(d_n + \alpha a_n) = \lim_n \alpha^2 a_n^* a_n = \alpha^2,$$

using which and the equation $\alpha^2 + 1 = \lim_n (b_n^* b_n + c_n^* c_n)$ from (6.3.8), we get $\lim_n c_n^* c_n = 1$. Again as each c_n is invertible, it follows from Lemma 6.3.9 that

$$\lim_n c_n c_n^* = 1. \tag{6.3.11}$$

Next we substitute $b_n = \alpha a_n + d_n$ in equation (6.3.7) to get

$$xc_ny = ya_nx + y(\alpha a_n + d_n)y = ya_n(x + \alpha y) + yd_ny = ya_nz + yd_ny,$$

where $z = x + \alpha y$. Since $\alpha \geq 1$, we note that z is positive and invertible (in fact $z^2 = 1 + (\alpha^2 - 1)y^2 + 2\alpha xy \geq 1$), and thus we get

$$ya_n = xc_nyz^{-1} - yd_nyz^{-1}. \tag{6.3.12}$$

Note that

$$z^2 = (x + \alpha y)^2 = x^2 + \alpha^2 y^2 + 2\alpha xy \geq \alpha^2 y^2,$$

and since y and z commutes, it follows that

$$y^2 z^{-2} \leq 1/\alpha^2. \tag{6.3.13}$$

Finally we combine the expression $\lim_n d_n = 0$ from (6.3.10), and equations in (6.3.9), (6.3.11), (6.3.12) and (6.3.13) to get the following:

$$\begin{aligned} y^2 &= \lim_n ya_na_n^*y = \lim_n (ya_n)(ya_n)^* \\ &= \lim_n (xc_nyz^{-1} - yd_nyz^{-1})(xc_nyz^{-1} - yd_nyz^{-1})^* \\ &= \lim_n (xc_nyz^{-1})(xc_nyz^{-1})^* = \lim_n xc_ny^2z^{-2}c_n^*x \\ &\leq \lim_n \frac{1}{\alpha^2} xc_nc_n^*x = \frac{1}{\alpha^2}x^2. \end{aligned}$$

Since $\alpha \geq 1$ is arbitrary, it follows by letting α tend to ∞ that $y = 0$, which is clearly not true. Thus our assumption that $\mathrm{Alg}_{\mathcal{B}}\{p, q\}$ is logmodular is false, completing the proof. $\square$

Finally we prove our main theorem in full generality, for which we need the following lemma.

Lemma 6.3.11. *Let an algebra $\mathcal{M}$ have logmodularity (resp. factorization) in a von Neumann algebra $\mathcal{B}$, and let $p, q \in \operatorname{Lat}_{\mathcal{B}} \mathcal{M}$. If $r = (p \wedge q) \vee (p^\perp \wedge q^\perp)$, then $r^\perp \mathcal{M} r^\perp$ has logmodularity (resp. factorization) in $r^\perp \mathcal{B} r^\perp$.*

Proof. Set $r_1 = p \wedge q$ and $r_2 = p^\perp \wedge q^\perp$. It is clear that $r_1 r_2 = 0$ and $r = r_1 + r_2$. Since $p, q \in \operatorname{Lat}_{\mathcal{B}} \mathcal{M}$, it follows that $r_1 \in \operatorname{Lat}_{\mathcal{B}} \mathcal{M}$. Also we note that $p^\perp, q^\perp \in \operatorname{Lat}_{\mathcal{B}} \mathcal{M}^*$ and hence $r_2 \in \operatorname{Lat}_{\mathcal{B}} \mathcal{M}^*$, which is to say that $r_2^\perp \in \operatorname{Lat}_{\mathcal{B}} \mathcal{M}$. Note that $r^\perp = 1 - r_2 - r_1 = r_2^\perp - r_1$, and so $r_1 \leq r_2^\perp$. Both the assertions about logmodularity and factorization now follow from part (3) of Proposition 6.3.2. $\square$

Proof of Theorem 6.2.1. Let $\mathcal{M}$ be a logmodular subalgebra of a von Neumann algebra $\mathcal{B}$, and let $p, q \in \operatorname{Lat}_{\mathcal{B}} \mathcal{M}$. We have to show that $pq = qp$. The second assertion that $p \leq q$ or $q \leq p$ whenever $\mathcal{B}$ is a factor, will then follow from Lemma 6.3.8. Set

$$r = (p \wedge q) \vee (p^\perp \wedge q^\perp).$$

Then $r^\perp \mathcal{M} r^\perp$ is a logmodular algebra in $r^\perp \mathcal{B} r^\perp$ by Lemma 6.3.11. Note that the projections p and q commute with r, and hence with $r^\perp$. So if we set

$$p' = r^\perp p r^\perp \quad \text{and} \quad q' = r^\perp q r^\perp,$$

then it is immediate that p', q' are projections in $r^\perp \mathcal{B} r^\perp$, and we have $p' = p \wedge r^\perp$ and $q' = q \wedge r^\perp$. Note that $pq(p \wedge q) = p \wedge q = qp(p \wedge q)$ and $pq(p^\perp \wedge q^\perp) = 0 = qp(p^\perp \wedge q^\perp)$; hence

$$pqr = p \wedge q = qpr,$$

which further yields

$$\begin{aligned} pq = pq(r + r^\perp) = pqr + pqr^\perp &= p \wedge q + (r^\perp p r^\perp)(r^\perp q r^\perp) = p \wedge q + p'q', \\ qp = qpr + qpr^\perp &= p \wedge q + (r^\perp q r^\perp)(r^\perp p r^\perp) = p \wedge q + q'p'. \end{aligned}$$

Therefore, in order to show the required assertion it is enough to prove that $p'q' = q'p'$. Also we note that

$$p' \wedge q' = p \wedge q \wedge r^\perp \leq r \wedge r^\perp = 0,$$

and

$$\begin{aligned} (r^\perp - p') \wedge (r^\perp - q') &= (r^\perp - pr^\perp) \wedge (r^\perp - qr^\perp) = p^\perp r^\perp \wedge q^\perp r^\perp \\ &= (p^\perp \wedge q^\perp) \wedge r^\perp \leq r \wedge r^\perp = 0. \end{aligned}$$

Here $r^\perp - p'$ and $r^\perp - q'$ are the orthogonal complement of the projections p' and q' in $r^\perp \mathcal{B} r^\perp$ respectively. Thus if necessary, by replacing the algebras $\mathcal{B}$ and $\mathcal{M}$ by $r^\perp \mathcal{B} r^\perp$ and $r^\perp \mathcal{M} r^\perp$ respectively, and the projections p, q by p', q' respectively, we assume without loss of generality that

$$p \wedge q = 0 = p^\perp \wedge q^\perp, \tag{6.3.14}$$

so that $r = 0$ and $\mathcal{B} = r^{\perp}\mathcal{B}r^{\perp}$. The purpose of reducing $\mathcal{B}$ to $r^{\perp}\mathcal{B}r^{\perp}$ is just to avoid multiple cases, and work with 4×4 matrices rather than 6×6 matrices, as we shall see.

Now assume that $pq \neq qp$, contrary to what we need to show. Then the generic part of $\mathcal{R}(p)$ and $\mathcal{R}(q)$ in $\mathcal{H}$ are non-zero by Proposition 6.3.6, where $\mathcal{H}$ is the separable Hilbert space on which the von Neumann algebra $\mathcal{B}$ acts. Further if both $p \wedge q^{\perp}$ and $p^{\perp} \wedge q$ are zero, then (as $p \wedge q = 0 = p^{\perp} \wedge q^{\perp}$) $\mathcal{R}(p)$ and $\mathcal{R}(q)$ would be in generic position in $\mathcal{H}$, which is not possible by Lemma 6.3.10, since $\mathrm{Alg}_{\mathcal{B}}\{p, q\}$ (which contains $\mathcal{M}$) is logmodular in $\mathcal{B}$. Therefore at least one of the projections $p \wedge q^{\perp}$ and $p^{\perp} \wedge q$ is non-zero.

For the remainder of the proof, we assume that both the projections $p \wedge q^{\perp}$ and $p^{\perp} \wedge q$ are non-zero (the proof for the case of exactly one of them being non-zero goes on the similar lines). It then follows from Proposition 6.3.6 that there exist a non-zero Hilbert space $\mathcal{K}$ and commuting positive contractions $x, y \in \mathcal{B}(\mathcal{K})$ satisfying

$$x^2 + y^2 = 1 \quad \text{and} \quad \ker x = 0 = \ker y$$

such that upto unitary unitary equivalence,

$$\mathcal{H} = \mathcal{R}(p \wedge q^{\perp}) \oplus \mathcal{K} \oplus \mathcal{K} \oplus \mathcal{R}(p^{\perp} \wedge q) \tag{6.3.15}$$

and

$$p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & x^2 & xy & 0 \\ 0 & xy & y^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{6.3.16}$$

Since logmodularity is preserved under unitary equivalence by Proposition 6.1.2, we assume without loss of generality that $\mathcal{B}$ is a von Neumann subalgebra of $\mathcal{B}(\mathcal{R}(p \wedge q^{\perp}) \oplus \mathcal{K} \oplus \mathcal{K} \oplus \mathcal{R}(p^{\perp} \wedge q))$, and p, q have the form as in (6.3.16). Now set

$$\widetilde{\mathcal{K}}_1 = \mathcal{R}(p \wedge q^{\perp}) \oplus \mathcal{K} \quad \text{and} \quad \widetilde{\mathcal{K}}_2 = \mathcal{K} \oplus \mathcal{R}(p^{\perp} \wedge q)$$

so that

$$\mathcal{H} = \widetilde{\mathcal{K}}_1 \oplus \widetilde{\mathcal{K}}_2. \tag{6.3.17}$$

Throughout the proof, we make use of both the decomposition of $\mathcal{H}$ in (6.3.15) and (6.3.17), which should be understood according to the context. Now fix $\alpha \geq 1$ and define the operator $Z \in \mathcal{B}(\mathcal{H})$ by

$$Z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \alpha & 0 \\ 0 & \alpha & \alpha^2 + 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =: \begin{bmatrix} 1 & Z_2 \\ Z_2^* & Z_3 \end{bmatrix}, \tag{6.3.18}$$

where

$$Z_2 = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix} \quad \text{and} \quad Z_3 = \begin{bmatrix} \alpha^2 + 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is clear that Z is a positive and invertible operator in $\mathcal{B}(\mathcal{H})$. In the similar fashion as in Lemma 6.3.10, it is easy to show, by using $p, q \in \mathcal{B}$, that $Z \in \mathcal{B}$. Since $\mathcal{M}$ is logmodular in $\mathcal{B}$, we then get a sequence $\{S_n\}$ of invertible operators in $\mathcal{M}^{-1}$ such that $Z = \lim_n S_n^* S_n$. Then for each n, we have $S_n p = p S_n p$ and $S_n^{-1} p = p S_n^{-1} p$; hence the operators S_n and S_n^{-1} have the form

$$S_n = \begin{bmatrix} a_n & b_n & r_n & s_n \\ c_n & d_n & t_n & u_n \\ 0 & 0 & e_n & f_n \\ 0 & 0 & g_n & h_n \end{bmatrix} =: \begin{bmatrix} A_n & B_n \\ 0 & C_n \end{bmatrix}$$

and

$$S_n^{-1} = \begin{bmatrix} a_n' & b_n' & r_n' & s_n' \\ c_n' & d_n' & t_n' & u_n' \\ 0 & 0 & e_n' & f_n' \\ 0 & 0 & g_n' & h_n' \end{bmatrix} =: \begin{bmatrix} A_n' & B_n' \\ 0 & C_n' \end{bmatrix},$$

for appropriate operators $a_n, b_n, .., a_n', b_n', ..$ etc. In particular, we have $A_n A_n' = 1 = A_n' A_n$ i.e. A_n is invertible in $\mathcal{B}(\widetilde{\mathcal{K}}_1)$. Similarly C_n is invertible in $\mathcal{B}(\widetilde{\mathcal{K}}_2)$. Now

$$\begin{bmatrix} 1 & Z_2 \\ Z_2^* & Z_3 \end{bmatrix} = Z = \lim_n S_n^* S_n = \lim_n \begin{bmatrix} A_n^* A_n & A_n^* B_n \\ B_n^* A_n & B_n^* B_n + C_n^* C_n \end{bmatrix}. \tag{6.3.19}$$

Then we have $\lim_n A_n^* A_n = 1$ and since A_n is invertible, it follows from Lemma 6.3.9 that

$$\lim_n A_n A_n^* = 1. \tag{6.3.20}$$

We also have $\lim_n A_n^* B_n = Z_2$, which after multiplied by A_n to left side of the sequence and using (6.3.20) yields $\lim_n (B_n - A_n Z_2) = 0$; but

$$B_n - A_n Z_2 = \begin{bmatrix} r_n & s_n \\ t_n & u_n \end{bmatrix} - \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix} = \begin{bmatrix} r_n - \alpha b_n & s_n \\ t_n - \alpha d_n & u_n \end{bmatrix},$$

and thus we get the following equations:

$$\lim_n (r_n - \alpha b_n) = 0, \tag{6.3.21}$$

$$\lim_n (t_n - \alpha d_n) = 0. \tag{6.3.22}$$

Also if $D_n = B_n - A_n Z_2$ for all n, then $\lim_n D_n = 0$ and since $\lim_n A_n^* A_n = 1$, we have

$$\lim_n B_n^* B_n = \lim_n (D_n + A_n Z_2)^* (D_n + A_n Z_2) = \lim_n Z_2^* A_n^* A_n Z_2 = Z_2^* Z_2.$$

This along with the expression $\lim_n (B_n^* B_n + C_n^* C_n) = Z_3$ from (6.3.19), further yield

$$\lim_n C_n^* C_n = Z_3 - Z_2^* Z_2 = \begin{bmatrix} \alpha^2 + 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \alpha^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{6.3.23}$$

Consequently, by computing entries of the matrices $C_n^* C_n$, we get $\lim_n (e_n^* e_n + g_n^* g_n) = 1$; hence there exists $m \in \mathbb{N}$ such that $\|e_n^* e_n\| \leq 2$, which in turn yields

$$e_n e_n^* \leq 2, \qquad \text{for all } n \geq m. \tag{6.3.24}$$

Now

$$S_nq = \begin{bmatrix} 0 & b_nx^2 + r_nxy & b_nxy + r_ny^2 & s_n \\ 0 & d_nx^2 + t_nxy & d_nxy + t_ny^2 & u_n \\ 0 & e_nxy & e_ny^2 & f_n \\ 0 & g_nxy & g_ny^2 & h_n \end{bmatrix}$$

and

$$qS_nq = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & x^2d_nx^2 + x^2t_nxy + xye_nxy & x^2d_nxy + x^2t_ny^2 + xye_ny^2 & x^2u_n + xyf_n \\ 0 & xyd_nx^2 + xyt_nxy + y^2e_nxy & xyd_nxy + xyt_ny^2 + y^2e_ny^2 & xyu_n + y^2f_n \\ 0 & g_nxy & g_ny^2 & h_n \end{bmatrix}.$$

Since $S_nq = qS_nq$ for each n, by equating $(3,2)$ entries of the respective matrices and then using $1-y^2 = x^2$, we get the expression $x^2e_nxy = xyd_nx^2 + xyt_nxy$; but x is one-one and hence x has dense range, so x cancels from both sides of the equation to yield

$$xe_ny = yd_nx + yt_ny.$$

If we set $v_n = t_n - \alpha d_n$ for all n, then above equation further implies

$$xe_ny = yd_nx + y(\alpha d_n + v_n)y = yd_n(x+\alpha y) + yv_ny,$$

which in other words says that

$$yd_n = xe_nyz^{-1} - yv_nyz^{-1}, \tag{6.3.25}$$

where $z = x + \alpha y$, which is clearly positive and invertible as $z^2 \geq 1$. In a similar vein as in (6.3.13) in Lemma 6.3.10, z and y commute and we get

$$y^2z^{-2} \leq 1/\alpha^2. \tag{6.3.26}$$

Also by equating $(1,2)$ entries of S_nq and qS_nq, we get $b_nx^2 + r_nxy = 0$; again since x has dense range, it follows that $b_nx + r_ny = 0$ for all n, so by using (6.3.21) we have

$$0 = \lim_n(b_nx + r_ny) = \lim_n b_n(x+\alpha y) + \lim_n(r_n - \alpha b_n)y = \lim_n b_n(x+\alpha y).$$

But $x + \alpha y$ is invertible as seen before, so the above equation yields

$$\lim_n b_n = 0. \tag{6.3.27}$$

Similarly since each S_n^{-1} also has all these properties, we have

$$\lim_n b'_n = 0. \tag{6.3.28}$$

Note that the $(2,2)$ entry of the matrix $S_nS_n^{-1}$ (with respect to the decomposition $\mathcal{R}(p\wedge q^\perp)\oplus \mathcal{K}\oplus\mathcal{K}\oplus\mathcal{R}(p^\perp\wedge q)$) is $c_nb'_n + d_nd'_n$; hence we have $c_nb'_n + d_nd'_n = 1$ for all n. Since $\lim_n b'_n = 0$

from (6.3.28), it follows that $\lim_n d_n d_n' = 1$. Hence there exists $n_0 \in \mathbb{N}$ such that $\|d_n d_n' - 1\| < 1$ for all $n \geq n_0$, which in particular says that $d_n d_n'$ is invertible for all $n \geq n_0$; thus

$$d_n d_n' (d_n d_n')^{-1} = 1,$$

which implies that d_n is right invertible for all $n \geq n_0$. Likewise, from $(2,2)$ entry of $S_n^{-1} S_n$ and using $\lim_n b_n = 0$ from (6.3.27), we get $\lim_n d_n' d_n = 1$. Again this implies that $d_n' d_n$ is invertible, and hence d_n is left invertible for large n. Thus we have shown that d_n is both left and right invertible, which is to say that d_n is invertible, for large n.

Now for each n, note that the $(2,2)$ entry of the matrix $S_n^* S_n$ (with respect to the decomposition $\mathcal{R}(p \wedge q^\perp) \oplus \mathcal{K} \oplus \mathcal{K} \oplus \mathcal{R}(p^\perp \wedge q))$ is $b_n^* b_n + d_n^* d_n$. Since $\lim_n S_n^* S_n = Z$, it then follows that $\lim_n (b_n^* b_n + d_n^* d_n) = 1$, and since $\lim_n b_n = 0$ from (6.3.27), we get $\lim_n d_n^* d_n = 1$. But d_n is invertible for large n, so it follows from Lemma 6.3.9 that

$$\lim_n d_n d_n^* = 1. \tag{6.3.29}$$

Now using $\lim_n v_n = 0$ from (6.3.22), and equations (6.3.24), (6.3.25), (6.3.26) and (6.3.29), we get the following:

$$\begin{aligned}
y^2 &= \lim_n y d_n d_n^* y = \lim_n (y d_n)(y d_n)^* \\
&= \lim_n (x e_n y z^{-1} - y v_n y z^{-1})(x e_n y z^{-1} - y v_n y z^{-1})^* \\
&= \lim_n (x e_n y z^{-1})(x e_n y z^{-1})^* = \lim_n x e_n y^2 z^{-2} e_n^* x \\
&\leq \frac{1}{\alpha^2} \lim_n x e_n e_n^* x \leq \frac{2}{\alpha^2} x^2.
\end{aligned}$$

Since $\alpha \geq 1$ is arbitrary, it follows by taking $\alpha \to \infty$ that $y = 0$, which is a contradiction. Thus our assumption that $pq \neq qp$ is false. The proof is now complete. $\square$

6.4 Reflexivity of algebras with factorization

One of the main results of this chapter says that the lattice of any algebra with factorization property in a factor is a nest. A natural question that arises is whether algebras having factorization are also nest subalgebras i.e. are they reflexive? Certainly, we cannot always expect automatic reflexivity of such algebras (see Example 6.4.10). But then what extra condition can be imposed in order to show that they are reflexive?

A result due to Radjavi and Rosenthal [67] says that a WOT closed algebra in $\mathcal{B}(\mathcal{H})$ whose lattice is a nest, is a nest algebra if and only if it contains a maximal abelian self-adjoint algebra (masa). See Kadison-Ringrose [45] or Takesaki [77] for more details on masa. In this section, we show that if the lattice of an algebra with factorization in $\mathcal{B}(\mathcal{H})$ has finite dimensional atoms, then it contains a masa and hence it is reflexive. This fact further helps us in characterizing all logmodular algebras in finite dimensional von Neumann algebras. We recall some terminologies to this end.

Definition 6.4.1. An algebra $\mathcal{M}$ in a von Neumann algebra $\mathcal{B}$ is called *$\mathcal{B}$-transitive* (simply *transitive* when $\mathcal{B} = \mathcal{B}(\mathcal{H})$) if $\mathrm{Lat}_{\mathcal{B}} \mathcal{M} = \{0, 1\}$.

Transitive algebras are very well studied objects and have attracted deep investigations over the decades. Our purpose here is limited upto an application of Burnside's theorem about transitive algebras in M_n. Interested readers can see Radjavi-Rosenthal [68] for history and some major unsolved open problems on this topic. We now consider the following simple lemma.

Lemma 6.4.2. *Let $\mathcal{M}$ be an algebra in a von Neumann algebra $\mathcal{B}$ such that $\operatorname{Lat}_{\mathcal{B}} \mathcal{M}$ is a nest, and let $p, q \in \operatorname{Lat}_{\mathcal{B}} \mathcal{M}$ with $p < q$. If $r = q - p$, then $\operatorname{Lat}_{r\mathcal{B}r}(r\mathcal{M}r) = \{s \in r\mathcal{B}r; p + s \in \operatorname{Lat}_{\mathcal{B}} \mathcal{M}\}$. In particular, if $p = q_-$ then $r\mathcal{M}r$ is $r\mathcal{B}r$-transitive.*

Proof. As seen in Proposition 6.3.2, $r\mathcal{M}r$ is a subalgebra of $r\mathcal{B}r$. Now let $s \in \operatorname{Lat}_{r\mathcal{B}r}(r\mathcal{M}r)$, and let $a \in \mathcal{M}$. Note that $(rar)s = s(rar)s$, and since $rs = s$, it follows that $ras = sas$, using which and the conditions $aq = qaq$ and $qs = s$, we have

$$as = aqs = qaqs = qas = pas + ras = pas + sas = (p + s)as. \tag{6.4.1}$$

Also since $sp = 0$ and $ap = pap$, we have $sap = spap = 0$, which along with (6.4.1) yield

$$(p + s)a(p + s) = pap + sap + (p + s)as = ap + as = a(p + s).$$

Since a is arbitrary in $\mathcal{M}$, it follows that $p + s \in \operatorname{Lat}_{\mathcal{B}} \mathcal{M}$. Conversely let $s \in r\mathcal{B}r$ be a projection such that $p + s \in \operatorname{Lat}_{\mathcal{B}} \mathcal{M}$, and fix $a \in \mathcal{M}$. Then $a(p + s) = (p + s)a(p + s)$, and since $ps = 0 = pr$ and $rs = s$, we have

$$(rar)s = ras = ra(p + s)s = r(p + s)a(p + s)s = s(rar)s.$$

Again as $a \in \mathcal{M}$ is arbitrary, we conclude that $s \in \operatorname{Lat}_{r\mathcal{B}r}(r\mathcal{M}r)$. Thus we have proved the first assertion. Note that if $p = q_-$ then for any $s \in r\mathcal{B}r$, $p + s \in \operatorname{Lat}_{\mathcal{B}} \mathcal{M}$ if and only if $s = 0$ or $s = r$. The second assertion then follows from the first. $\square$

The following proposition is the crux of this section.

Proposition 6.4.3. *Let $\mathcal{M}$ be a closed algebra having factorization in a von Neumann algebra $\mathcal{B}$, and let $p, q \in \operatorname{Lat}_{\mathcal{B}} \mathcal{M}$ such that $p < q$. If $q - p$ has finite dimensional range, then $q - p \in \mathcal{M}$. In particular, if either p or $p^\perp$ has finite dimensional range, then $p \in \mathcal{M}$.*

Proof. The second assertion clearly follows from the first. To prove the first assertion, set $r = q - p$. Let $\mathcal{B}$ be a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Note that

$$\mathcal{H} = \mathcal{R}(p) \oplus \mathcal{R}(r) \oplus \mathcal{R}(q^\perp),$$

and we consider operators of $\mathcal{B}(\mathcal{H})$ with respect to this decomposition. For each $n \in \mathbb{N}$, consider the operator

$$X_n = r + \frac{1}{n} r^\perp = \begin{bmatrix} 1/n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/n \end{bmatrix}.$$

It is clear that each X_n is a positive and invertible operator, and since $r \in \mathcal{B}$ it follows that $X_n \in \mathcal{B}$. So by factorization property of $\mathcal{M}$ in $\mathcal{B}$, there exists an invertible operator $S_n \in \mathcal{M}^{-1}$ such that $X_n = S_n^* S_n$. Then each S_n leaves $\mathcal{R}(p)$ and $\mathcal{R}(q)$ invariant, which equivalently says that S_n has the form

$$S_n = \begin{bmatrix} a_n & b_n & c_n \\ 0 & d_n & e_n \\ 0 & 0 & f_n \end{bmatrix}, \tag{6.4.2}$$

for appropriate operators a_n, b_n.. etc. We claim that the off-diagonal entries b_n, c_n, e_n are 0 for all n. Since each $S_n^{-1} \in \mathcal{M}$, S_n^{-1} leaves $\mathcal{R}(p)$ and $\mathcal{R}(q)$ invariant, meaning that S_n^{-1} is also upper triangular. Consequently, the diagonal entries a_n, d_n, f_n of S_n are invertible. Now for all n, we have

$$\begin{bmatrix} 1/n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/n \end{bmatrix} = X_n = S_n^* S_n = \begin{bmatrix} a_n^* a_n & a_n^* b_n & a_n^* c_n \\ b_n^* a_n & b_n^* b_n + d_n^* d_n & b_n^* c_n + d_n^* e_n \\ c_n^* a_n & c_n^* b_n + e_n^* d_n & c_n^* c_n + e_n^* e_n + f_n^* f_n \end{bmatrix}.$$

We now equate entries of the matrices above to get the expressions $a_n^* b_n = 0$ and $a_n^* c_n = 0$. Since a_n is invertible, it follows that

$$b_n = 0 \quad \text{and} \quad c_n = 0.$$

We also have $b_n^* c_n + d_n^* e_n = 0$, and since $b_n = 0$ and d_n is invertible, it follows that

$$e_n = 0.$$

This proves the claim that for all n, the operators b_n, c_n and e_n are 0. We further get $a_n^* a_n = 1/n$ and $c_n^* c_n + e_n^* e_n + f_n^* f_n = 1/n$ for all n, which imply that $\lim_n a_n = 0$ and $\lim_n f_n = 0$. Also $b_n^* b_n + d_n^* d_n = 1$; but $b_n = 0$, so we have

$$d_n^* d_n = 1.$$

Since $\mathcal{R}(r)$ is finite dimensional by hypothesis, it follows that d_n is a unitary for every n. By compactness of the unitary group in finite dimensions, we get a subsequence $\{d_{n_k}\}$ converging to a unitary d in $\mathcal{B}(\mathcal{R}(r))$. Thus we have $\lim_k S_{n_k} = S$, where

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since each $S_{n_k} \in \mathcal{M}$ and $\mathcal{M}$ is norm closed, it follows that $S \in \mathcal{M}$. Note that $\lim_k d_{n_k}^{-1} = \lim_k d_{n_k}^* = d^* = d^{-1}$, using which we have

$$\lim_k S_{n_k}^{-1} S = \lim_k \begin{bmatrix} a_{n_k}^{-1} & 0 & 0 \\ 0 & d_{n_k}^{-1} & 0 \\ 0 & 0 & f_{n_k}^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 0 \end{bmatrix} = \lim_k \begin{bmatrix} 0 & 0 & 0 \\ 0 & d_{n_k}^* d & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

that is, $\lim_k S_{n_k}^{-1} S = r$. Since $S_{n_k}^{-1} S \in \mathcal{M}$ (as $S_{n_k}^{-1}$ and $S \in \mathcal{M}$) for all k, we conclude that $r \in \mathcal{M}$, as required to prove. ☐

We now discuss a sufficient criterion imposed on dimension of atoms of the lattice to prove the reflexivity of an algebra having factorization in $\mathcal{B}(\mathcal{H})$. It is clearly not necessary as any nest algebra arising out of a countable nest has factorization and is reflexive.

Theorem 6.4.4. *Let $\mathcal{M}$ be a WOT closed algebra having factorization in $\mathcal{B}(\mathcal{H})$. If all the atoms of lattice* $\operatorname{Lat}\mathcal{M}$ *have finite dimensional range, then $\mathcal{M}$ is reflexive and hence $\mathcal{M}$ is a nest algebra.*

Proof. We shall show that $\mathcal{M}$ contains a masa. As noted above, this claim along with the fact that $\operatorname{Lat}\mathcal{M}$ is a nest (from Corollary 6.2.2) will imply the required assertion that $\mathcal{M}$ is reflexive and a nest algebra (see Theorem 9.24, [68]).

Let $\{r_i\}_{i\in\Lambda}$ be the collection of all the atoms of $\operatorname{Lat}\mathcal{M}$ for some finite or countable indexing set Λ. Since $\operatorname{Lat}\mathcal{M}$ is atomic from Corollary 6.2.7, it follows that $\sum_{i\in\Lambda} r_i = 1$ in WOT convergence; hence

$$\mathcal{H} = \oplus_{i\in\Lambda}\mathcal{H}_i,$$

where $\mathcal{H}_i = \mathcal{R}(r_i)$ which satisfies $\mathcal{H}_i \perp \mathcal{H}_j$ for all $i \neq j$. For each $i \in \Lambda$ since r_i is an atom, we note that $r_i = p_i - q_i$ for some $p_i, q_i \in \operatorname{Lat}\mathcal{M}$ (where $q_i = p_{i-}$), and since r_i has finite dimensional range by hypothesis, it follows from Proposition 6.4.3 that $r_i \in \mathcal{M}$.

Now recognize the von Neumann algebra $r_i\mathcal{B}(\mathcal{H})r_i$ with $\mathcal{B}(\mathcal{H}_i)$, for each $i \in \Lambda$. Since r_i is an atom, we know from Lemma 6.4.2 that $r_i\mathcal{M}r_i$ is a transitive subalgebra of $\mathcal{B}(\mathcal{H}_i)$. Therefore, as $\mathcal{H}_i$ is finite-dimensional, it follows from Burnside's Theorem (Corollary 8.6, [68]) that $r_i\mathcal{M}r_i = \mathcal{B}(\mathcal{H}_i)$. In other words, this says that $r_i\mathcal{B}(\mathcal{H})r_i = r_i\mathcal{M}r_i$, and since $r_i \in \mathcal{M}$, it follows that

$$r_i\mathcal{B}(\mathcal{H})r_i \subseteq \mathcal{M}. \tag{6.4.3}$$

Now for each i, let $\mathcal{L}_i$ be a masa in $\mathcal{B}(\mathcal{H}_i)$ (for example, $\mathcal{L}_i$ can be chosen to be the algebra of diagonal matrices in the finite dimensional algebra $\mathcal{B}(\mathcal{H}_i)$). Set

$$\mathcal{L} = \bigoplus_{i\in\Lambda}\mathcal{L}_i,$$

which is considered a subalgebra of $\mathcal{B}(\mathcal{H})$. It is clear that $\mathcal{L}$ is a masa in $\mathcal{B}(\mathcal{H})$. Note that $\mathcal{L}r_i = r_i\mathcal{L}$ for all $i \in \Lambda$. Also it follows from (6.4.3) that $r_i\mathcal{L}r_i \subseteq r_i\mathcal{B}(\mathcal{H})r_i \subseteq \mathcal{M}$, and since $\mathcal{M}$ is WOT closed we have

$$\mathcal{L} = \mathcal{L}\sum_{i\in\Lambda} r_i \subseteq \sum_{i\in\Lambda}\mathcal{L}r_i = \sum_{i\in\Lambda} r_i\mathcal{L}r_i \subseteq \mathcal{M},$$

where the sum above is in WOT. Thus we have shown our requirement that $\mathcal{M}$ contains a masa, completing the proof. □

A nest of projections on a Hilbert space is called *maximal or simple* if it is not contained in any larger nest. It is easy to verify that a nest $\mathcal{E}$ is maximal if and only if all atoms in $\mathcal{E}$ are one-dimensional (Lemma 2.1, [69]). Thus the following corollary is immediate from Theorem 6.4.4.

Corollary 6.4.5. *Let $\mathcal{M}$ be a WOT closed algebra have factorization in $\mathcal{B}(\mathcal{H})$, and let* $\operatorname{Lat}\mathcal{M}$ *be a maximal nest. Then $\mathcal{M}$ is reflexive, and so $\mathcal{M}$ is a nest algebra.*

We emphasize the importance of the above corollary in the following example.

Example 6.4.6. Consider the Hilbert space $\mathcal{H} = \ell^2(\Gamma)$, for $\Gamma = \mathbb{N}$ or $\mathbb{Z}$, and let $\mathcal{M}$ be the reflexive algebra of upper triangular matrices in $\mathcal{B}(\mathcal{H})$ with respect to the canonical basis $\{e_n\}_{n\in\Gamma}$. Note that $\operatorname{Lat}\mathcal{M} = \{p_n; n \in \Gamma\}$, where p_n is the projection onto the subspace $\overline{\operatorname{span}}\{e_m; m \leq n\}$. Clearly $\operatorname{Lat}\mathcal{M}$ is a maximal nest. So if $\mathcal{N}$ is any subalgebra of $\mathcal{M}$ with $\operatorname{Lat}\mathcal{N}$ a nest, then $\operatorname{Lat}\mathcal{M} \subseteq \operatorname{Lat}\mathcal{N}$, which implies by maximality that $\operatorname{Lat}\mathcal{M} = \operatorname{Lat}\mathcal{N}$. Thus it follows from Corollary 6.4.5 that the only subalgebra of $\mathcal{M}$ that has factorization in $\mathcal{B}(\mathcal{H})$ is $\mathcal{M}$.

Next we consider some consequences of the above results for subalgebras of finite dimensional von Neumann algebras.

Let $\mathcal{M}$ be a logmodular algebra in the algebra M_n of all $n \times n$ complex matrices. It can easily be verified using compactness of the closed unit ball of M_n that the algebra $\mathcal{M}$ has factorization in M_n as well. Since all atoms of $\operatorname{Lat}\mathcal{M}$ are clearly finite dimensional, it follows from Theorem 6.4.4 that $\mathcal{M}$ is a nest algebra in M_n. Thus we have shown that upto unitary equivalence, $\mathcal{M}$ is an algebra of block upper triangular matrices in M_n. This assertion was put as a conjecture in [62], and an affirmative answer was given in [44]. We have provided a different solution, and we state it below.

Corollary 6.4.7. *Let $\mathcal{M}$ be a logmodular algebra in M_n. Then $\mathcal{M}$ is an algebra of block upper triangular matrices upto unitary equivalence.*

Moreover, we have the following generalization of the corollary above:

Corollary 6.4.8. *Let $\mathcal{B}$ be a (possibly countably infinite) direct sum of finite dimensional von Neumann algebras, and let $\mathcal{M}$ be a WOT closed logmodular algebra in $\mathcal{B}$. Then $\mathcal{M}$ is a nest subalgebra of $\mathcal{B}$ and $\mathcal{M}$ is $\mathcal{B}$-reflexive.*

Proof. We know that every finite dimensional von Neumann algebra is $*$-isomorphic to a direct sum of matrix algebras of the form M_n (see Theorem I.11.2, [77]). In particular, $\mathcal{B}$ is $*$-isomorphic to a countable direct sum of matrix algebras. Therefore in view of Proposition 6.1.2, we assume without loss of generality that

$$\mathcal{B} = \oplus_{k\geq 1} M_{n_k},$$

which faithfully acts on the Hilbert space $\mathcal{H} = \oplus_{k\geq 1}\mathbb{C}^{n_k}$. Now for $k \geq 1$, let p_k denote the orthogonal projection of $\mathcal{H}$ onto the subspace $\mathbb{C}^{n_k}$ (considered as a subspace of $\mathcal{H}$), and let

$$\mathcal{M}_k = p_k \mathcal{M} p_k.$$

We claim that

$$\mathcal{M} = \oplus_{k\geq 1}\mathcal{M}_k.$$

Firstly note that $p_k \in \mathcal{B} \cap \mathcal{B}'$; hence $p_k \in \operatorname{Lat}_{\mathcal{B}} \mathcal{M}$. This in particular says that $\mathcal{M}_k$ is an algebra. Since p_k has finite dimensional range, it follows from Proposition 6.4.3 that $p_k \in \mathcal{M}$. This implies that $\mathcal{M}_k \subseteq \mathcal{M}$ for each k; hence $\oplus_{k\geq 1}\mathcal{M}_k \subseteq \mathcal{M}$. On the other hand, note that $\sum_{k\geq 1} p_k = 1$ in WOT, and since $\mathcal{M}$ is WOT closed, we get

$$\mathcal{M} = \mathcal{M} \sum_{k\geq 1} p_k \subseteq \sum_{k\geq 1} \mathcal{M} p_k = \sum_{k\geq 1} p_k \mathcal{M} p_k = \oplus_{k\geq 1}\mathcal{M}_k,$$

proving our claim that $\mathcal{M} = \oplus_{k\geq 1}\mathcal{M}_k$. Note that $M_{n_k} = p_k \mathcal{B} p_k$ for each k. So the algebra $\mathcal{M}_k$ is logmodular in M_{n_k} by Proposition 6.3.2. Then it follows from Corollary 6.4.7 that

$$\mathcal{M}_k = \operatorname{Alg}_{M_{n_k}} \mathcal{E}_k,$$

for the nest $\mathcal{E}_k = \operatorname{Lat}_{M_{n_k}} \mathcal{M}_k$ in M_{n_k}. Now consider the lattice

$$\mathcal{E} = \bigoplus_{k\geq 1} \mathcal{E}_k = \{\oplus_{k\geq 1} q_k;\ q_k \in \mathcal{E}_k\}$$

in $\mathcal{B}$. Since $\mathcal{E}_k = \operatorname{Lat}_{M_{n_k}} \mathcal{M}_k$, it is immediate that $\mathcal{E} = \operatorname{Lat}_{\mathcal{B}} \mathcal{M}$ which implies $\mathcal{M} \subseteq \operatorname{Alg}_{\mathcal{B}} \mathcal{E}$. Note that $\mathcal{E}$ is not a nest if $k \geq 2$. Now choose a sublattice, namely $\mathcal{F}$, of $\mathcal{E}$ such that $\mathcal{F}$ is a nest and each element q_k in $\mathcal{E}_k$ appears at least once as the kth coordinate of an element of $\mathcal{F}$. Such $\mathcal{F}$ can always be chosen: for example consider the nest $\mathcal{F}_k$, for each k, given by

$$\mathcal{F}_k = \{e_1 \oplus \ldots \oplus e_{k-1} \oplus q_k \oplus 0 \oplus 0 \oplus \ldots;\ q_k \in \mathcal{E}_k\} \subseteq \mathcal{E},$$

where e_k denotes the identity of M_{n_k}, and let $\mathcal{F} = \cup_{k\geq 1}\mathcal{F}_k$. Since each $\mathcal{E}_k$ is a nest and $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$ for all $k \geq 1$, it follows that the sublattice $\mathcal{F}$ is a nest in $\mathcal{B}$, and $\mathcal{F}$ fulfils the requirement. We now claim that

$$\mathcal{M} = \operatorname{Alg}_{\mathcal{B}} \mathcal{F},$$

which will prove that $\mathcal{M}$ is a nest subalgebra of $\mathcal{B}$. Clearly as $\mathcal{F} \subseteq \mathcal{E}$, we have $\mathcal{M} \subseteq \operatorname{Alg}_{\mathcal{B}} \mathcal{E} \subseteq \operatorname{Alg}_{\mathcal{B}} \mathcal{F}$. Conversely let $x \in \operatorname{Alg}_{\mathcal{B}} \mathcal{F}$, and let $x = \oplus_{k\geq 1} x_k$ for some $x_k \in M_{n_k}$. The way $\mathcal{F}$ has been chosen, each element of $\mathcal{E}_k$ appears as the kth coordinate of some element of $\mathcal{F}$, so it follows that $x_k q = q x_k q$ for all $q \in \mathcal{E}_k$ and $k \geq 1$. This shows that

$$x_k \in \operatorname{Alg}_{M_{n_k}} \mathcal{E}_k = \mathcal{M}_k,$$

hence $x \in \mathcal{M}$. Thus we conclude that $\operatorname{Alg}_{\mathcal{B}} \mathcal{F} \subseteq \mathcal{M}$ proving the claim that $\mathcal{M} = \operatorname{Alg}_{\mathcal{B}} \mathcal{F}$. Finally since $\mathcal{F} \subseteq \mathcal{E} = \operatorname{Lat}_{\mathcal{B}} \mathcal{M}$, it follows that $\operatorname{Alg}_{\mathcal{B}} \operatorname{Lat}_{\mathcal{B}} \mathcal{M} \subseteq \operatorname{Alg}_{\mathcal{B}} \mathcal{F} = \mathcal{M}$. Since the other inclusion is obvious, we have $\mathcal{M} = \operatorname{Alg}_{\mathcal{B}} \operatorname{Lat}_{\mathcal{B}} \mathcal{M}$ which is to say that $\mathcal{M}$ is $\mathcal{B}$-reflexive. $\square$

As a consequence of Corollary 6.4.8 and the fact that all nest subalgebras in a finite von Neumann algebra are logmodular (see Example 6.1.10), we have thus characterized all logmodular algebras in finite dimensional von Neumann algebras extending the result of Juschenko [44] from matrix algebras.

Corollary 6.4.9. *Let $\mathcal{M}$ be a subalgebra in a finite dimensional von Neumann algebra $\mathcal{B}$. Then $\mathcal{M}$ is logmodular in $\mathcal{B}$ if and only if $\mathcal{M}$ is a nest subalgebra of $\mathcal{B}$.*

In general, Corollary 6.4.9 fails to be true for algebras having factorization (or logmodularity) in infinite dimensional von Neumann algebras, as the following example suggests.

Example 6.4.10. Let $\mathcal{A}$ be an algebra having factorization in a von Neumann algebra $\mathcal{M}$ such that $\mathcal{A} \neq \mathcal{M}$, and $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^*$ is a factor. We claim that $\mathcal{A}$ is not $\mathcal{M}$-reflexive. Assume otherwise that $\mathcal{A} = \mathrm{Alg}_{\mathcal{M}} \mathrm{Lat}_{\mathcal{M}} \mathcal{A}$. Then note that since $\mathrm{Lat}_{\mathcal{M}} \mathcal{A}$ is commutative (by Corollary 6.2.2), we have $\mathrm{Lat}_{\mathcal{M}} \mathcal{A} \subseteq \mathcal{D}$. Also it is easy to verify that $\mathrm{Lat}_{\mathcal{M}} \mathcal{A} \subseteq \mathcal{D}'$ and thus we have $\mathrm{Lat}_{\mathcal{M}} \mathcal{A} \subseteq \mathcal{D} \cap \mathcal{D}' = \mathbb{C}$. It then follows that $\mathrm{Lat}_{\mathcal{M}} \mathcal{A} = \{0, 1\}$, so $\mathcal{A} = \mathrm{Alg}_{\mathcal{M}}\{0, 1\} = \mathcal{M}$ which is not true.

There are plenty of such algebras. To see one, let G be a countable discrete ordered group (i.e. there is a linear order $\leq$ on G such that $g_1 \leq g_2$ implies $hg_1 \leq hg_2$ for all $h, g_1, g_2 \in G$). Let

$$\ell^2(G) = \{f : G \to \mathbb{C}; \sum_{g\in G} |f(g)|^2 < \infty\},$$

and for each $g \in G$, let $U_g : \ell^2(G) \to \ell^2(G)$ be the unitary operator defined by $U_g f(g') = f(g^{-1}g')$ for $f \in \ell^2(G)$ and $g' \in G$. Let $\mathcal{B}$ be the finite von Neumann algebra in $\mathcal{B}(\ell^2(G))$ generated by the family $\{U_g\}_{g\in G}$, called the group von Neumann algebra of G. Note that each element X of $\mathcal{B}(\ell^2(G))$ has a matrix representation (x_{gh}) with respect to the canonical basis of $\ell^2(G)$. Let

$$\mathcal{M} = \{X = (x_{gh}) \in \mathcal{B}; x_{gh} = 0 \text{ for } g > h\}.$$

Then $\mathcal{M}$ is an example of a finite maximal subdiagonal algebra in $\mathcal{B}$ with respect to the expectation $\phi : \mathcal{B} \to \mathcal{B}$ given by

$$\phi((x_{gh})) = x_{ee}1 \quad \text{for} \quad (x_{gh}) \in \mathcal{B},$$

where e denotes the identity of G (see Example 3, [4]). In particular, $\mathcal{M}$ has factorization in $\mathcal{B}$ (Theorem 4.2.1, [4]). But note that

$$\mathcal{M} \cap \mathcal{M}^* = \mathbb{C}.$$

Indeed if $(x_{gh}) \in \mathcal{M} \cap \mathcal{M}^*$, then $x_{gh} = 0$ for all $g \neq h$ and $x_{gg} = x_{g'g'}$ for all $g, g' \in G$. So $\mathcal{M}$ cannot be $\mathcal{B}$-reflexive as discussed above. Moreover, we can choose the ordered group G to be countable with infinite conjugacy class property (e.g. $G = \mathbb{F}_2$, the free group on two generators), so that $\mathcal{B}$ is a factor. In this case although $\mathrm{Lat}_{\mathcal{B}} \mathcal{M}$ is a nest (Corollary 6.2.2), $\mathcal{M}$ cannot be a nest subalgebra of $\mathcal{B}$ (otherwise $\mathcal{M} \cap \mathcal{M}^*$ will contain the nest and so cannot be equal to $\mathbb{C}$).

Open Problems

To summarize our work, we have mainly undertaken the study of structure of C^*-extreme points of the spaces of UCP maps on C^*-algebras. The theory for UCP maps taking values in matrices (i.e. $\mathcal{B}(\mathcal{H})$ for finite dimensional Hilbert spaces) already had rich literature through the works of Farenick et al [24,28,29,80]. We have carried forward the investigation in infinite dimensional Hilbert space settings, where we have managed to prove some open problems in the process of generalizing a number of results to infinite dimensions for specific type of UCP maps via different methods. In the meantime, we came across a number of questions relevant to our studies which we were not able to answer. Some of them which we mention below, deserve more attention and whose solution may further give us more insight in development of the theory.

Firstly, we rewrite below the aforementioned conjecture about normal C^*-extreme maps on type I factors (See Conjecture 3.1.9).

Question 1. *Is every normal C^*-extreme map on a type I factor a direct sum of normal pure UCP maps?*

We know that any C^*-extreme point in the space $\mathcal{P}_{\mathcal{H}}(\mathbb{N})$ of normalized POVMs on the natural numbers $\mathbb{N}$ is spectral (Theorem 4.3.2). It is also known that any completely positive map on $\ell^\infty(=\ell^\infty(\mathbb{N}))$ corresponds to finitely additive positive operator valued measure on $\mathbb{N}$, whereas (countably additive) POVMs correspond to the normal completely positive maps on ℓ^∞ and hence all normal C^*-extreme points are $*$-homomorphic. It is not clear as of now how C^*-extreme points in the collection of all finitely additive POVMs behave (which can be defined and studied in a similar fashion). Approaching another way, the spectrum of ℓ^∞ is of course the Stone-Čech compactification of $\mathbb{N}$. Unfortunately this space is not metrizable and our result on existence of a non-homomorphic C^*-extreme point (Theorem 5.4.6) is not applicable and so we are left with the following question:

Question 2. *Are C^*-extreme UCP maps on the C^*-algebra ℓ^∞ always $*$-homomorphisms?*

We have seen characterization of all atomic C^*-extreme points in $\mathcal{P}_{\mathcal{H}}(X)$ (Theorem 4.3.2). Also any C^*-extreme point in $\mathcal{P}_{\mathcal{H}}(X)$ decomposes as a direct sum of atomic and non-atomic

C^*-extreme points. Therefore, it suffices to understand the structure of non-atomic C^*-extreme points in $\mathcal{P}_{\mathcal{H}}(X)$. In particular, we raise the following question which could be tractable.

Question 3. *Describe the structure of non-atomic C^*-extreme POVMs on the unit circle $\mathbb{T}$.*

In Chapter 6, we have discussed 'universal' or 'strong' factorization property for subalgebras of von Neumann algebras. But there are weaker notions of factorization which can also be explored. Say a subalgebra $\mathcal{A}$ has *weak factorization property (WFP)* in a von Neumann algebra $\mathcal{M}$ if for any positive element $x \in \mathcal{M}$, there is an element $a \in \mathcal{A}$ such that $x = a^*a$. Here the invertibility requirement on the elements is dropped.

Power [66] has studied WFP of nest algebras where he proved that if a nest $\mathcal{E}$ of projections on a Hilbert space $\mathcal{H}$ is *well-ordered* (i.e. $p \neq p_+ = \cap_{q>p} q$ for all $p \in \mathcal{E}$ with $p \neq 1$), then $\operatorname{Alg}\mathcal{E}$ has WFP in $\mathcal{B}(\mathcal{H})$. Inspired from our result on lattices of algebras with factorization, we may surmise that lattices of algebras with WFP in a factor should also be a nest. But it is not clear to us at this point. However, for a subalgebra in a finite von Neumann algebra we can certainly say so. We can follow the similar lines of proof along with the fact that any left (or right) invertible element in a finite von Neumann algebra is invertible. We record it here.

Theorem. *Let $\mathcal{A}$ be a subalgebra of a finite von Neumann algebra (resp. factor) $\mathcal{M}$ having WFP. Then $\operatorname{Lat}_{\mathcal{M}} \mathcal{A}$ is a commutative subspace lattice (resp. nest).*

So a natural question is the following:

Question 4. *Is the lattice of a subalgebra having WFP in a von Neumann algebra (resp. factor) is a commutative subspace lattice (resp. nest)?*

We conclude with a question of reflexivity of algebras with factorization. It was shown in Chapter 6 that a weakly closed algebra with factorization in $\mathcal{B}(\mathcal{H})$ has a masa and hence is reflexive, if we impose some dimensionality condition on the atoms of its lattice. But we still do not know whether every algebra with factorization in $\mathcal{B}(\mathcal{H})$ has a masa. Thus the following question related to the famous transitive algebra problem of Kadison remains open.

Question 5. *Is a weakly closed algebra having factorization in $\mathcal{B}(\mathcal{H})$ automatically reflexive? In particular, is a weakly closed transitive algebra with factorization equal to $\mathcal{B}(\mathcal{H})$?*

List of Publications

The material of this thesis is primarily based on the following three research articles:

(i) Tathagata Banerjee, B.V. Rajarama Bhat, and Manish Kumar,
C^*-extreme points of positive operator valued measures and unital completely positive maps,
Communications in Mathematical Physics 388 (2021), no. 3, 1235-1280.

(ii) B.V. Rajarama Bhat and Manish Kumar,
Lattices of logmodular algebras,
preprint, arXiv:2101.00782 (2021).

(iii) B.V. Rajarama Bhat and Manish Kumar,
C^*-extreme maps and nests,
preprint, arXiv:2103.09600 (2021).

The contents of Chapter 2 and Chapter 3 follow Paper (iii). The contents of Chapter 4 and Chapter 5 are partially borrowed from Paper (i), while Chapter 6 is entirely based on Paper (ii).

Bibliography

[1] M. Anoussis and E.G. Katsoulis, *Factorisation in nest algebras*, Proc. Amer. Math. Soc. **125** (1997), no. 1, 87–92.

[2] W. Arveson, *Interpolation problems in nest algebras*, J. Funct. Anal. **20** (1975), no. 3, 208–233.

[3] ______, *An invitation to C^*-algebras*, Graduate Texts in Mathematics, vol. 39, Springer-Verlag, New York-Heidelberg, 1976.

[4] W.B. Arveson, *Analyticity in operator algebras*, Amer. J. Math. **89** (1967), 578–642.

[5] ______, *Subalgebras of C^*-algebras*, Acta Math. **123** (1969), 141–224.

[6] R. Beukema, *Positive operator-valued measures and phase-space representations*, PhD Thesis, Technische Universiteit Eindhoven (The Netherlands), 2003.

[7] R. Bhat, V. Pati, and V.S. Sunder, *On some convex sets and their extreme points*, Math. Ann. **296** (1993), no. 4, 637–648.

[8] D.P. Blecher and L.E. Labuschagne, *Logmodularity and isometries of operator algebras*, Trans. Amer. Math. Soc. **355** (2003), no. 4, 1621–1646.

[9] ______, *Von neumann algebraic H^p theory*, Function spaces, Contemp. Math., vol. 435, Amer. Math. Soc., Providence, RI, 2007, p. 89–114.

[10] V.I. Bogachev, *Measure theory*, vol. II, Springer-Verlag, Berlin, 2007.

[11] N.P. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008.

[12] G. Chiribella, G.M. D'Ariano, and D. Schlingemann, *How continuous quantum measurements in finite dimensions are actually discrete*, Phys. Rev. Lett. **98** (2007), no. 19, 4 pp.

[13] M.D. Choi, *Completely positive linear maps on complex matrices*, Linear Algebra and Appl. **10** (1975), 285–290.

[14] M.D. Choi and E.G. Effros, *Injectivity and operator spaces*, J. Functional Analysis **24** (1977), no. 2, 156–209.

[15] J.B. Conway, *A course in functional analysis*, second ed., Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990.

[16] ———, *A course in operator theory*, Graduate Studies in Mathematics, vol. 21, American Mathematical Society, Providence, RI, 2000.

[17] K.R. Davidson, *Nest algebras*, Pitman Research Notes in Mathematics Series, vol. 191, Longman Scientific and Technical, Harlow, New York, 1988.

[18] K.R. Davidson and M. Kennedy, *Noncommutative choquet theory*, arXiv:1905.08436 (2019).

[19] E.B. Davies, *Quantum theory of open systems*, Academic Press, London, 1976.

[20] J. Dixmier, *von Neumann algebras*, North-Holland Mathematical Library, vol. 27, North-Holland Publishing Co., Amsterdam-New York, 1981.

[21] S.V. Dorofeev and J. de Graaf, *Some maximality results for effect-valued measures*, Indag. Math. (N.S.) **8** (1997), no. 3, 349–369.

[22] R.G. Douglas, *Banach algebra techniques in operator theory*, Graduate Texts in Mathematics, vol. 179, Springer-Verlag, New York, 1998.

[23] E.G. Effros and S. Winkler, *Matrix convexity: operator analogues of the bipolar and hahn-banach theorems*, J. Funct. Anal. **144** (1997), no. 1, 117–152.

[24] D. Farenick, S. Plosker, and J. Smith, *Classical and nonclassical randomness in quantum measurements*, J. Math. Phys. **52** (2011), no. 12, 26 pp.

[25] D.R. Farenick, *C^*-convexity and matricial ranges*, Canad. J. Math. **44** (1992), no. 2, 280–297.

[26] ———, *Krĕin-milman-type problems for compact matricially convex sets*, Linear Algebra Appl. **162/164** (1992), 325–334.

[27] D.R. Farenick and P.B. Morenz, *C^*-extreme points of some compact C^*-convex sets*, Proc. Amer. Math. Soc. **118** (1993), no. 3, 765–775.

[28] ———, *C^*-extreme points in the generalised state spaces of a C^*-algebra*, Trans. Amer. Math. Soc. **349** (1997), no. 5, 1725–1748.

[29] D.R. Farenick and H. Zhou, *The structure of C^*-extreme points in spaces of completely positive linear maps on C^*-algebras*, Proc. Amer. Math. Soc. **126** (1998), no. 5, 1467–1477.

[30] C. Foiaş and I. Suciu, *On operator representation of logmodular algebras*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **16** (1968), 505–509.

[31] I. Fujimoto, *CP-duality for C^*- and W^*-algebras*, J. Operator Theory **30** (1993), no. 2, 201–215.

[32] I.C. Gohberg and M.G. Krein, *Theory and applications of volterra operators in hilbert space*, Trans. Math. Monographs, vol. 24, Amer. Math. Soc. Providence, 1970.

[33] M.C. Gregg, *On C^*-extreme maps and $*$-homomorphisms of a commutative C^*-algebra*, Integral Equations Operator Theory **63** (2009), no. 3, 337–349.

[34] D.W. Hadwin, *Dilations and hahn decompositions for linear maps*, Canadian J. Math. **33** (1981), no. 4, 826–839.

[35] P.R. Halmos, *Two subspaces*, Trans. Amer. Math. Soc. **144** (1969), 381–389.

[36] D. Han, D.R. Larson, B. Liu, and R. Liu, *Operator-valued measures, dilations, and the theory of frames*, vol. 229, Mem. Amer. Math. Soc., 2014.

[37] T. Heinosaari and J.-P. Pellonpää, *Extreme commutative quantum observables are sharp*, J. Phys. A **44** (2011), no. 31, 4 pp.

[38] K. Hoffman, *Analytic functions and logmodular banach algebras*, Acta Math. **108** (1962), 271–317.

[39] K. Hoffman and H. Rossi, *Function theory and multiplicative linear functionals*, Trans. Amer. Math. Soc. **116** (1965), 536–543.

[40] A.S. Holevo, *Statistical structure of quantum theory*, Lecture Notes in Physics, Monographs, vol. 67, Springer-Verlag, Berlin, 2001.

[41] A. Hopenwasser, R.L. Moore, and V.I. Paulsen, *C^*-extreme points*, Trans. Amer. Math. Soc. **266** (1981), no. 1, 291–307.

[42] G. Ji and K.-S. Saito, *Factorization in subdiagonal algebras*, J. Funct. Anal. **159** (1998), no. 1, 191–202.

[43] R.A. Johnson, *Atomic and nonatomic measures*, Proc. Amer. Math. Soc. **25** (1970), 650–655.

[44] K. Juschenko, *Description of logmodular subalgebras in finite-dimensional C^*-algebras*, Indiana Univ. Math. J. **60** (2011), no. 4, 1171–1176.

[45] R.V. Kadison and J.R. Ringrose, *Fundamentals of the theory of operator algebras. Vol. I*, Graduate Studies in Mathematics, vol. 15, American Mathematical Society, Providence, RI, 1997.

[46] R.V. Kadison and I.M. Singer, *Triangular operator algebras. fundamentals and hyperreducible theory*, Amer. J. Math. **82** (1960), 227–259.

[47] D.R. Larson, *Nest algebras and similarity transformations*, Ann. of Math. **121** (1985), no. 2, 409–427.

[48] ______, *Triangularity in operator algebras*, Surveys of some recent results in operator theory, Vol. II, Pitman Res. Notes Math. Ser., vol. 192, Longman Sci. Tech., Harlow, 1988, p. 121–188.

[49] R.I. Loebl and V.I. Paulsen, *Some remarks on C^*-convexity*, Linear Algebra Appl. **35** (1981), 63–78.

[50] B. Magajna, *On C^*-extreme points*, Proc. Amer. Math. Soc. **129** (2001), no. 3, 771–780.

[51] ______, *C^*-convex sets and completely positive maps*, Integral Equations Operator Theory **85** (2016), no. 1, 37–62.

[52] ______, *Maps with the unique extension property and C^*-extreme points*, Complex Anal. Oper. Theory **12** (2018), no. 8, 1903–1927.

[53] M. McAsey, P.S. Muhly, and K.-S. Saito, *Nonselfadjoint crossed products (invariant subspaces and maximality)*, Trans. Amer. Math. Soc. **248** (1979), no. no. 2, 381–409.

[54] D. McLaren, S. Plosker, and C. Ramsey, *On operator valued measures*, Houston J. Math. **46** (2020), no. 1, 201–226.

[55] P.B. Morenz, *The structure of C^*-convex sets*, Canad. J. Math. **46** (1994), no. 5, 1007–1026.

[56] M.A. Neumark, *On a representation of additive operator set functions*, C.R. (Doklady) Acad. Sci. URSS (N.S.) **41** (1943), 359–361.

[57] K.R. Parthasarathy, *Probability measures on metric spaces*, Academic Press, New York, 1967.

[58] ______, *An introduction to quantum stochastic calculus*, Monographs in Mathematics, vol. 85, Birkhäuser Verlag, Basel, 1992.

[59] ______, *Extreme points of the convex set of stochastic maps on a C^*-algebra*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **1** (1998), no. 4, 599–609.

[60] ______, *Extremal decision rules in quantum hypothesis testing*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **2** (1999), no. 4, 557–568.

[61] V. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, vol. 78, Cambridge University Press, Cambridge, 2002.

[62] V.I. Paulsen and M. Raghupathi, *Representations of logmodular algebras*, Trans. Amer. Math. Soc. **363** (2011), no. 5, 2627–2640.

[63] G. Pisier, *Tensor products of C^*-algebras and operator spaces: The connes–kirchberg problem*, London Mathematical Society Student Texts, Cambridge University Press, Cambridge, 2020.

[64] D.R. Pitts, *Factorization problems for nests: factorization methods and characterizations of the universal factorization*, J. Funct. Anal. **79** (1988), no. 1, 57–90.

[65] S. Power, *Analysis in nest algebras*, Surveys of some recent results in operator theory, Vol. II, Pitman Res. Notes Math. Ser., vol. 192, Longman Sci. Tech., Harlow, 1988, p. 189–234.

[66] S.C. Power, *Factorization in analytic operator algebras*, J. Funct. Anal. **67** (1986), no. 3, 413–432.

[67] H. Radjavi and P. Rosenthal, *On invariant subspaces and reflexive algebras*, Amer. J. Math. **91** (1969), 683–692.

[68] ______, *Invariant subspaces*, Dover Publications, Mineola, NY, 2003.

[69] J.R. Ringrose, *On some algebras of operators*, Proc. London Math. Soc. (3) **15** (1965), 61–83.

[70] Z.-J. Ruan, *Subspaces of C^*-algebras*, J. Funct. Anal. **76** (1988), no. 1, 217–230.

[71] F.E. Schroeck Jr., *Quantum mechanics on phase space*, Kluwer Academic Publishers, 1996.

[72] B. Solel, *Analytic operator algebras (factorization and an expectation)*, Trans. Amer. Math. Soc. **287** (1985), no. 2, 799–817.

[73] ______, *Cocycles and factorization in analytic operator algebras*, J. Operator Theory **20** (1988), no. 2, 295–309.

[74] T.P. Srinivasan and J-K. Wang, *Weak*-dirichlet algebras*, Function Algebras, Proc. Internat. Sympos. on Function Algebras, Tulane Univ., 1965, Scott-Foresman, Chicago, 1966, p. 216–249.

[75] W.F. Stinespring, *Positive functions on C^*-algebras*, Proc. Amer. Math. Soc. **6** (1955), 211–216.

[76] E. Størmer, *Positive linear maps of operator algebras*, Springer Monographs in Mathematics, Springer, Heidelberg, 2013.

[77] M. Takesaki, *Theory of operator algebras. I*, Encyclopaedia of Mathematical Sciences, vol. 124, Springer-Verlag, Berlin, 2002.

[78] C. Webster and S. Winkler, *The krein-milman theorem in operator convexity*, Trans. Amer. Math. Soc. **351** (1999), no. 1, 307–322.

[79] G. Wittstock, *On matrix order and convexity*, Functional analysis: surveys and recent results, III, North-Holland Mathematics Studies, vol. 90, North-Holland, Amsterdam, 1984, p. 175–188.

[80] H. Zhou, *C*-extreme points in spaces of completely positive maps*, PhD thesis, University of Regina, 1998.

List of Symbols

$\emptyset$	Empty Set
$\mathbb{C}$	Set of complex numbers
$\mathbb{N}$	Set of natural numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{Q}$	Set of rational numbers
$\mathbb{T}$	Unit circle in the complex plane
M_n	Algebra of $n \times n$ complex matrices
$\mathcal{B}(\mathcal{H})$	Algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$
$\mathcal{B}(\mathcal{H}, \mathcal{K})$	Space of all bounded linear operators between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$
$S_{\mathcal{H}}(\mathcal{A})$	Set of all unital completely positive maps from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$
$\mathcal{O}(X)$	σ-algebra of subsets of a set X
$\mathcal{P}_{\mathcal{H}}(X)$	Set of all normalized $\mathcal{B}(\mathcal{H})$-valued positive operator valued measures on X
$\mathcal{RP}_{\mathcal{H}}(X)$	Set of all regular POVMs in $\mathcal{P}_{\mathcal{H}}(X)$
$\mathcal{R}(T)$	Range of an operator T
$\ker T$	Kernel of an operator T
Tr	Trace of a trace class operator
$\mathcal{M}'$	Commutant of a subalgebra $\mathcal{M}$ in $\mathcal{B}(\mathcal{H})$
$\mathcal{M}^{-1}$	Set of all invertible elements in an algebra $\mathcal{M}$ whose inverse is also in $\mathcal{M}$
$\mathcal{S}^*$	$\{S^*; S \in \mathcal{S}\}$ for any subset $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$

$[E]$	Closed subspace generated by a subset E in a Hilbert space
$E \ominus F$	Complement of the subspace F in a subspace E
$\bigwedge_{E\in\mathcal{E}} E$	Intersection of all subspaces in a collection $\mathcal{E}$
$\bigvee_{E\in\mathcal{E}} E$	Smallest closed subspace containing all subspaces in a collection $\mathcal{E}$
P_E	Projection onto a subspace E
$\wedge_i p_i$	Projection onto the intersection of the ranges of the projections p_i
$\vee_i p_i$	Projection onto the smallest subspace containing the ranges of the projections p_i
$\operatorname{Lat}\mathcal{M}$	Lattice of subspaces (or projections) invariant under an algebra $\mathcal{M}$ of operators
$\operatorname{Alg}\mathcal{E}$	Algebra of operators which leave invariant elements of a collection $\mathcal{E}$ of subspaces or projections
$\overline{\mathcal{E}}$	Completion of a nest $\mathcal{E}$
E_-	$\vee\{F \in \mathcal{E}; F \subseteq E\}$ for E in a nest $\mathcal{E}$ of subspaces
E_+	$\wedge\{F \in \mathcal{E}; F \supseteq E\}$ for E in a nest $\mathcal{E}$ of subspaces
$\mathcal{A}_1 \otimes \mathcal{A}_2$	Minimal tensor product of two C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$
$\mathcal{B}_1 \overline{\otimes} \mathcal{B}_2$	von Neumann algebra generated by the minimal tensor product $\mathcal{B}_1 \otimes \mathcal{B}_2$

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