Essays in Auctions and Robust Bilateral Trading

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To my parents and teachers

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Contents

Chapter 1

Introduction

This thesis is divided into four essays. Each essay deals with the allocation or trading mechanisms of private goods. The first and last essays deal with private good allocation problems. In the first essay, we consider combinatorial auctions where the preferences of buyers need not be quasilinear, thus departing from standard models of auction theory.

The next two essays of the thesis are about robust mechanism design. We consider a model of bilateral trade with private values. The value of the buyer and the cost of the seller are jointly distributed but the true joint distribution is unknown to the designer. However, the marginal distributions of the value and the cost are known to the designer.

In the last chapter of the thesis, we consider endogenous entry in single object procurement auctions with asymmetric suppliers and characterise the optimal auction.

1.1 Pareto efficient combinatorial auctions: dichotomous preferences without quasilinearity

We consider a combinatorial auction model where preferences of agents over bundles of objects and payments need not be quasilinear. However, we restrict the preferences of agents to be dichotomous. An agent with dichotomous preference partitions the set of bundles of objects as *acceptable* and *unacceptable*, and at the same payment level, she is indifferent between bundles in each class but strictly prefers acceptable to unacceptable bundles.

We show that there is no Pareto efficient, dominant strategy incentive compatible (DSIC), individually rational (IR) mechanism satisfying no subsidy if the domain of preferences includes *all* dichotomous preferences. However, a generalization of the VCG mechanism is Pareto efficient, DSIC, IR and satisfies no subsidy if the domain of preferences contains only *positive income effect* dichotomous preferences. We show tightness of this result: adding any non-dichotomous preference (satisfying some natural properties) to the domain of quasilinear dichotomous preferences brings back the impossibility result.

1.2 An equivalence result in bilateral trading: robust BIC and DSIC mechanisms

We consider a model of bilateral trade with private values. The value of the buyer and the cost of the seller are jointly distributed but the true joint distribution is unknown to the designer. However, the marginal distributions of the value and the cost are known to the designer. The designer wants to find a trading mechanism that is robustly Bayesian incentive compatible, robustly individually rational, budget-balanced, and maximizes the expected gains from trade over all such mechanisms. We refer to such a mechanism as an *optimal robust mechanism*.

We establish equivalence between Bayesian incentive compatible mechanisms (BIC) and dominant strategy mechanisms (DSIC). The equivalence result holds for robust efficiency gains for robust BIC and DSIC block mechanisms along with the additional constraints on budget balancedness and individual rationality. The result implies it does not make a difference to the designer whether the joint distribution of valuations is known or unknown to the agents.

It is an important result as it simplifies the problem of the designer significantly. Hagerty and Rogerson (1987) shows that a block mechanism implementable in dominant strategy, budget balanced and individually rational mechanism are implemented by "posted-price" mechanism. This result allows us to focus on posted price mechanisms to find an optimal mechanism.

1.3 Optimal robust mechanism in bilateral trading

We consider the same model of bilateral trading as Chapter 3 and use its result on equivalence to find an optimal robust mechanism. We characterise the worst distribution for a given mechanism and use this characterisation to find an optimal robust mechanism. We show that there is an optimal robust mechanism that is deterministic (posted-price), dominant strategy incentive compatible and ex-post individually rational. We also derive an explicit expression of the posted-price of such an optimal robust mechanism.

We show the equivalence between the efficiency gains from the optimal robust mechanism (max-min problem) and guaranteed efficiency gains if the designer could choose the mechanism after observing the true joint distribution (min-max problem).

1.4 Asymmetric auctions with entry

In this chapter, we consider endogenous entry in single object procurement auctions with asymmetric suppliers. The potential suppliers decide to enter the auction before realizing their per-unit cost. Suppliers incur a fixed cost for entry into the auction.

We characterize the optimal procurement auction in such an environment. The second price auction with supplier-specific non-negative participation fee is an optimal auction.

We then study a two-period model, where a single object is procured in every period from the same potential set of suppliers. In the first period, suppliers are symmetric but suppliers who win the contract in the first period get cost (distribution) advantage in the second period. We apply our result to derive sufficient conditions (on how cost distribution changes in the second period) under which single-sourcing is not optimal in the first period.

Chapter 2

Pareto efficient combinatorial auctions: dichotomous preferences without quasilinearity

2.1 Introduction

The Vickrey-Clarke-Groves (VCG) mechanism (Vickrey, 1961; Clarke, 1971; Groves, 1973) occupies a central role in mechanism design theory (specially, with private values). It satisfies two fundamental desiderata: it is dominant strategy incentive compatible (DSIC) and Pareto efficient. We study a model of combinatorial auctions, where multiple objects are sold to agents simultaneously, who may buy any bundle of objects. For such combinatorial auction models, the VCG mechanism and its indirect implementations (like ascending price auctions) have been popular. The VCG mechanism is also individually rational (IR) and satisfies no subsidy (i.e., does not subsidize any agent) in these models.

Unfortunately, these desirable properties of the VCG mechanism critically rely on the fact that agents have quasilinear preferences. While analytically convenient and a good approximation of actual preferences when payments involved are low, quasilinearity is a debatable assumption in practice. For instance, consider an agent participating in a combinatorial auction for spectrum licenses, where agents often borrow from various investors at non-negligible interest rates. Such borrowing naturally leads to a preference which is not quasilinear. Further, income effects are ubiquitous in settings with non-negligible payments. For instance, a bidder in a spectrum auction often needs to invest in telecom infrastrastructure to realize the full value of spectrum. Higher payment in the auction will lead to lower investments in infrastructure, and hence, a lower value for the spectrum.

This has initiated a small literature in mechanism design theory (discussed later in this section and again in Section 2.4), where the quasilinearity assumption is relaxed to allow any *classical* preference of the agent over consumption bundles: (bundle of objects, payment) pairs. [1] The main research question addressed in this literature is the following:

> *In combinatorial auction models, if agents have classical preferences, is it possible to construct a "desirable" mechanism: a mechanism which inherits the DSIC, Pareto efficiency, IR, and no subsidy properties of the VCG mechanism?*

2.1.1 Dichotomous preferences

This paper contributes to this literature, focusing on the particular case in which agents' preferences belong to a class of preferences, which we call *dichotomous*. If an agent has a dichotomous preference, she partitions the set of bundles of objects into *acceptable* and *unacceptable*. If the payments for all the bundles of objects are the same, then an agent is indifferent between her acceptable bundles of objects; she is also indifferent between unacceptable bundles of objects; but she prefers every acceptable bundle to every unacceptable bundle.

Such preferences, though restrictive, are found in many settings of interest. For instance, consider the recent "incentive auction" done by the US Government (Leyton-Brown et al., 2017). It involved a "reverse auction" phase where the broadcast licenses from existing broadcasters were bought; a "forward auction" phase where buyers bought broadcast licenses; and a clearing phase. The auction resulted in billions of dollars in revenue for US treasury (Leyton-Brown et al., 2017). The theoretical analysis of the reverse auction phase was done by Milgrom and Segal (2020), where they assume quasilinear preferences with "single-

[1]Classical preferences assume mild continuity and monotonicity (in money and bundles of objects) properties of preferences.

minded" bidders, a specific kind of dichotomous preference where the bidder has a *unique* acceptable bundle (a broadcast band in this case). In these auctions, a broadcaster had some feasible frequency bands in which it can operate. Any of those feasible frequency bands were "acceptable" and it was indifferent between them (since any of these frequencies allowed the broadcaster to realize its full value of broadcast). This resulted in dichotomous preferences of agents. [2] Milgrom and Segal (2020) argue that the VCG mechanism is computationally challenging in this setting and propose a simpler mechanism.

The assumption of dichotomous preferences seems natural in settings where a bidder is acquiring some resources, and finds any bundle acceptable if it satisfies some requirements. For instance, consider the following examples.

- Consider a scheduling problem, where a certain set of jobs (say, flights at the take-off slots of an airport) need to be scheduled on a server. There are certain intervals where each job is available and can be processed and other intervals are not acceptable. For instance, a supplier bidding to supply to a firm's production schedule can do so only on some fixed interval of dates. So, certain dates are acceptable to it and others are not acceptable. A traveller is buying tickets between a pair of cities but find certain dates acceptable for travel and realize value only on those dates.

- Consider a seller who is selling land to different buyers. The lands differ in size but are homogeneous otherwise. Each buyer only demands a land of a fixed size. For instance, suppose the Government is allocating land to firms to set up factories in a region, and each firm needs a land of a fixed size to set up its factory. This means all the bundles of land exceeding the size requirement are acceptable to a firm.

- Consider firms (data providers) buying paths on (data) networks (Babaioff et al., 2009) - a firm is interested in sending data from node x to node y on a directed graph whose edges are up for sale, and as long as a bundle of edges contain a path from x to y, it is acceptable to the firm.

[2] Quoting Milgrom and Segal (2017), "Milgrom and Segal (2015) (hereafter MS) offer a theoretical analysis which assumes that all bidders are single-station owners who know their station values and are "single-minded", that is, willing to bid only for a single option. This assumption is reasonable for commercial UHF broadcasters that view VHF bands as ill-suited for their operations and for non-profit broadcasters that are willing to move for compensation to a particular VHF band but that view going off-air as incompatible with their mission."

In all the examples above, if the payment involved are high, we can expect income effects, which will mean that agents do not have quasilinear preferences. One may also consider the dichotomous preference restriction as a behavioural assumption, where the agent does not consider computing values for each of the exponential number of bundles but classifies the bundles as acceptable and unacceptable. Hence, they are easy to elicit even in combinatorial auction setting. Even with quasilinear preferences, the dichotomous restriction poses interesting combinatorial challenges for computing the VCG outcome. This has led to a large literature in computer science for looking at *approximately desirable* VCG-style mechanisms (Babaioff et al., 2005, 2009; Lehmann et al., 2002; Ledyard, 2007; Milgrom and Segal, 2014). Also related is the literature in matching and social choice theory (models without payments), where dichotomous preferences have been widely studied (Bogomolnaia and Moulin, 2004; Bogomolnaia et al., 2005; Bade, 2015).

2.1.2 Summary and intuition of results

We show that if the domain of preferences contains *all* dichotomous classical preferences, there is no desirable mechanism. However, a natural generalization of the VCG mechanism to classical preferences, which we call the *generalized VCG* (GVCG) mechanism, is desirable if the domain contains *only positive income effect* dichotomous preferences. In other words, when *normal* goods are sold, the GVCG mechanism is desirable. Further, the GVCG mechanism is the *unique* desirable mechanism in any domain of positive income effect dichotomous preferences if it contains the quasilinear dichotomous preferences. The GVCG mechanism allocates the goods in a way such that the collective *willingness to pay* of all the bidders is maximized. Classical preferences imply that willingness to pay for a bundle of objects depends on the payment level. Thus, it is not clear what the counterpart of "valuation" of a bundle of objects is in this setting. Our generalized VCG is defined by treating the willingness to pay at *zero* payment as the "valuation" of a bundle and then defining the VCG outcome with respect to these valuations, i.e., the allocation maximizes the sum of agents' valuations and each agent pays her externality.

The intuition for these results is the following. The GVCG mechanism allocates the goods in a way that maximizes the collective willingness to pay of all the bidders. In fact, with enough richness in the domain, every desirable mechanism must allocate objects like the

GVCG mechanism at certain profiles. Individual rationality implies that winning bidders pay an amount less than their willingness to pay. So, winning makes a winning bidder wealthier. With dichotomous preferences, the payments in the GVCG mechanism can be quite low. If bidders have negative income effect, then their *willingness to sell* (i.e., the compensating amount needed to make a winning bidder lose her bundle of objects) is lower than their willingness to pay. This creates ex-post trading opportunities and the GVCG mechanism is no longer efficient. On the other hand, with positive income effect, the willingness to sell of winning bidders is higher than their willingness to pay and the GVCG mechanism is efficient.

Our positive result is tight: we get back impossibility in any domain containing quasilinear dichotomous preferences and at least one more positive income effect non-dichotomous preference (satisfying some extra reasonable conditions). Such an additional preference may be a unit-demand preference, where the agent is interested in at most one object (Demange and Gale, 1985). To get an intuition for this result, suppose we consider a domain which contains all quasilinear dichotomous preferences and one unit-demand positive income effect preference. We know that the GVCG mechanism may not be strategy-proof in the domain of unit-demand preferences if agents have income effects (Morimoto and Serizawa, 2015). But, we know that in the quasilinear domain with dichotomous preferences, the GVCG mechanism is the unique desirable mechanism. With two agents having positive income effect unit-demand preference and others having quasilinear dichotomous preference, we show that the outcome in a desirable mechanism, if it existed, would still have to be the outcome of the GVCG mechanism. As a result, the agents with positive income effect unit-demand preferences could manipulate at such preference profiles. This negative result not only establishes the tightness of our positive result, but also helps to illuminate the bigger picture of possibility and impossibility domains without quasilinearity.

We briefly connect our results to some relevant results from the literature. A detailed literature survey is given in Section 2.4. Saitoh and Serizawa (2008) was the first paper to define the generalized VCG mechanism for the single object auction model. They show that the generalized VCG mechanism is desirable in their model even if preferences have *negative income effect*. This is in contrast to our model, where we get impossibility with negative income effect preferences but the generalized VCG mechanism is desirable with positive income effect.

When we go from single object to multiple object combinatorial auctions, the generalized VCG may fail to be DSIC. For instance, Demange and Gale (1985) consider a combinatorial auction model where multiple heterogenous objects are sold but each agent demands at most one object. In this model, the generalized VCG is no longer DSIC. However, Demange and Gale (1985) propose a different mechanism (based on the idea of market-clearing prices), which is desirable.

When agents can demand more than one object in a combinatorial auction model with multiple heterogeneous objects, Kazumura and Serizawa (2016) show that a desirable mechanism may not exist - this result requires certain richness of the domain of preferences which is violated by our dichotomous preference model. Similarly, Baisa (2020) shows that in the homogeneous objects sale case, if agents demand multiple units, then a desirable mechanism may not exist – he requires slightly different axioms than our desirability axioms.

These results point to a conjecture that when agents demand multiple objects in a combinatorial auction model, a desirable mechanism may not exist. Since ours is a combinatorial auction model where agents can consume multiple objects, an impossibility result might not seem surprising. However, dichotomous preferences are somewhat close to the single object model preference. So, it is not clear which intuition dominates. Our impossibility result with dichotomous preferences complement the earlier impossibility results, showing that the multi-demand intuition goes through if we include all possible dichotomous preferences. However, what is surprising is that we recover the desirability of the generalized VCG mechanism with positive income effect dichotomous preferences. This shows that *not all* multi-demand combinatorial auction models without quasilinearity are impossibility domains.

2.2 Preliminaries

Let $N = \{1, \ldots, n\}$ be the set of agents and M be a set of m objects. Let $\mathcal{B}$ be the set of all subsets of M. We will refer to elements in $\mathcal{B}$ as **bundles** (of objects). A seller (or a planner) is selling/allocating bundles from $\mathcal{B}$ to agents in N using payments. We introduce the notion of classical preferences and type spaces corresponding to them below.

2.2.1 Classical Preferences

Each agent has preference over possible *outcomes*, which are pairs of the form (A, t), where $A \in \mathcal{B}$ is a bundle and $t \in \mathbb{R}$ is the amount paid by the agent. Let $\mathcal{Z} = \mathcal{B} \times \mathbb{R}$ denote the set of all outcomes. A preference R_i of agent i over $\mathcal{Z}$ is a complete transitive preference relation with strict part denoted by P_i and indifference part denoted by I_i. This formulation of preference is very general and can capture wealth effects. For instance, varying levels of transfers will correspond to varying levels of wealth and this can be captured by our preference over $\mathcal{Z}$.

We restrict attention to the following class of preferences.

Definition 1 *Preference R_i of agent i over $\mathcal{Z}$ is* **classical** *if it satisfies*

1. **Monotonicity**. *for each $A, A' \in \mathcal{B}$ with $A' \subseteq A$ and for each $t, t' \in \mathbb{R}$ with $t' > t$, the following hold: (i) $(A, t)\ P_i\ (A, t')$ and (ii) $(A, t)\ R_i\ (A', t)$.*
2. **Continuity**. *for each $Z \in \mathcal{Z}$, the upper contour set $\{Z' \in \mathcal{Z} : Z'\ R_i\ Z\}$ and the lower contour set $\{Z' \in \mathcal{Z} : Z\ R_i\ Z'\}$ are closed.*
3. **Finiteness**. *for each $t \in \mathbb{R}$ and for each $A, A' \in \mathcal{B}$, there exist $t', t'' \in \mathbb{R}$ such that $(A', t')\ R_i\ (A, t)$ and $(A, t)\ R_i\ (A', t'')$.*

Restricting attention to such classical preferences is standard in mechanism design literature without quasilinearity (Demange and Gale, 1985; Baisa, 2020; Morimoto and Serizawa, 2015). The monotonicity conditions mentioned above are quite natural. The continuity and finiteness are technical conditions needed to ensure nice structure of the indifference vectors. A quasilinear preference is always classical, where *indifference vectors* are "parallel". Notice that the monotonicity condition requires a free-disposal property: at a fixed payment level, every bundle is weakly preferred to every other bundle which is a subset of it. All our results continue to hold even if we relax this free-disposal property to require that at a fixed payment level, every bundle be weakly preferred to the empty bundle only.

Given a classical preference R_i, the **willingness to pay (WP)** of agent i at t for bundle A is defined as the unique solution x to the following equation:

$$(A, t + x)\ I_i\ (\emptyset, t).$$

We denote this solution as $WP(A,t;R_i)$. The following fact is immediate from monotonicity, continuity, and finiteness.

Fact 1 *For every classical preference R_i, for every $A \in \mathcal{B}$ and for every $t \in \mathbb{R}$, $WP(A,t;R_i)$ is a unique non-negative real number.*

For quasilinear preference, $WP(A,t;R_i)$ is independent of t and represents the valuation for bundle A.

Another way to represent a classical preference is by a collection of indifference vectors. Fix a classical preference R_i. Then, by definition, for every $t \in \mathbb{R}$ and for every $A \in \mathcal{B}$, agent i with classical preference R_i will be indifferent between the following outcomes:

$$(\emptyset, t) \ I_i \ (A, t + WP(A,t;R_i)).$$

Figure 2.1 shows a representation of classical preference for three objects $\{a,b,c\}$. The horizontal lines correspond to payment levels for each of the bundles. Hence, these lines are the set of all outcomes Z – the space between these eight lines have no meaning and are kept only for ease of illustration. As we go to the right along any of these lines, the outcomes become worse since the payment (payment made by the agent) increases. Figure 2.1 shows eight points, each corresponding to a unique bundle and a payment level for that bundle. These points are joined to show that the agent is indifferent between these outcomes for a classical preference. Classical preference implies that all the points to the left of this indifference vector are better than these outcomes and all the points to the right of this indifference vector are worse than these outcomes. Indeed, every classical preference can be represented by a collection of an infinite number of such indifference vectors.

2.2.2 Domains and mechanisms

A *bundle allocation* is an ordered sequence of objects $(A_1, \ldots, A_n)$, where A_i denotes the bundle allocated to agent i, such that for each $A_i, A_j \in \mathcal{B}$, we have $A_i \cap A_j = \emptyset$ - note that A_i can be equal to $\emptyset$ for any i in an object allocation. Let $\mathcal{X}$ denote the set of all bundle allocations.

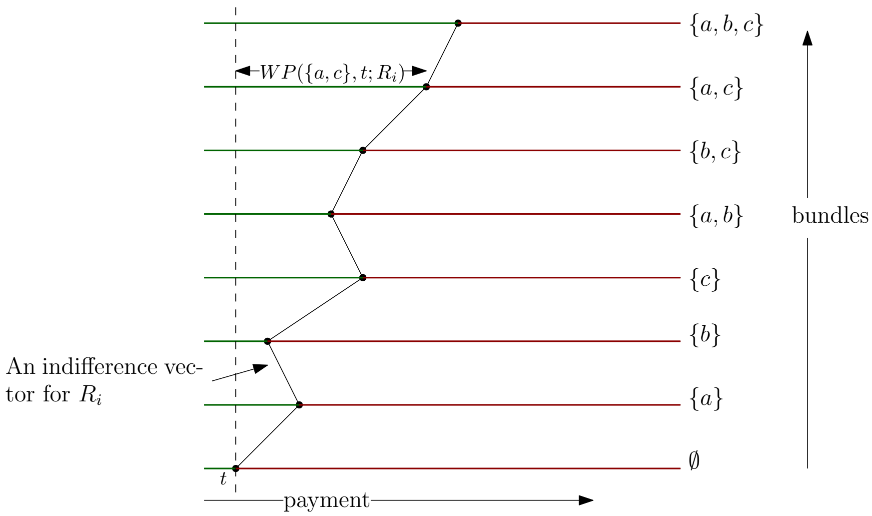


Figure 2.1: Representation of classical preferences

An outcome profile $((A_1,t_1),\ldots,(A_n,t_n))$ is a collection of n outcomes such that $(A_1,\ldots,A_n)$ is the bundle allocation and t_i denotes the payment made by agent i. An outcome profile $((A_1,t_1),\ldots,(A_n,t_n))$ is **Pareto efficient** at $R \equiv (R_1,\ldots,R_n)$, if there does not exist another outcome profile $((A'_1,t'_1),\ldots,(A'_n,t'_n))$ such that

1. for each $i \in N, (A'_i,t'_i)\ R_i\ (A_i,t_i)$,
2. $\displaystyle\sum_{i\in N} t'_i \geq \sum_{i\in N} t_i$,

with one of the inequalities strictly satisfied. The first relation says that each agent i prefers (A'_i,t'_i) to (A_i,t_i). The second relation requires that the seller is not spending money to make everyone better off. Without the second relation, we can always improve any outcome profile by subsidizing the agents.[3]

A **domain or type space** is any subset of classical preferences. A typical domain of preferences will be denoted by $\mathcal{T}$. A **mechanism** is a pair $(f,\mathbf{p})$, where $f:\mathcal{T}^n \to \mathcal{X}$ and $\mathbf{p} \equiv (p_1,\ldots,p_n)$ is a collection of payment rules with each $p_i:\mathcal{T}^n \to \mathbb{R}$. Here, f is the

[3]Our efficiency definition says that the agents and the designer cannot improve using an outcome profile, which may involve negative payments. Later, we impose no-subsidy as an axiom for our mechanism. The way to think about this is that Pareto efficient improvements are outside the mechanism and may involve one agent or the designer "buying" a bundle of objects from another agent by compensating (negative payment) her.

bundle allocation rule and p_i is the payment rule of agent i. We denote the bundle allocated to agent i at type profile R by $f_i(R) \in \mathcal{B}$ in the bundle allocation rule f.

We require the following properties from a mechanism, which we term desirable.

Definition 2 (Desirable mechanisms) *A mechanism* $(f, \mathbf{p})$ *is* **desirable** *if*

1. *it is* **dominant strategy incentive compatible (DSIC)***: for all* $i \in N$*, for all* $R_{-i} \in \mathcal{T}^{n-1}$*, and for all* $R_i, R'_i \in \mathcal{T}$*, we have*

$$\Big(f_i(R), p_i(R)\Big) \; R_i \; \Big(f_i(R'_i, R_{-i}), p_i(R'_i, R_{-i})\Big).$$

2. *it is* **Pareto efficient***:* $\Big((f_1(R), p_1(R)), \ldots, (f_n(R), p_n(R))\Big)$ *is Pareto efficient at* R*, for all* $R \in \mathcal{T}^n$*.*

3. *it is* **individually rational (IR)***: for all* $R \in \mathcal{T}^n$ *and for all* $i \in N$*,*

$$\Big(f_i(R), p_i(R)\Big) \; R_i \; (\emptyset, 0).$$

4. *satisfies* **no subsidy***: for all* $R \in \mathcal{T}^n$ *and for all* $i \in N$*,*

$$p_i(R) \geq 0.$$

We will explore domains where a desirable mechanism exists. DSIC, Pareto efficiency, and IR are standard constraints in mechanism design. No subsidy is debatable. Our motivation for considering it as desirable stems from the fact that most auction formats in practice and the VCG mechanism satisfy it. It also discourages *fake* buyers from participating in the mechanism.

2.2.3 A motivating example

In this section, we provide an example to give some intuition for one of our main results.

Example 1

Consider a setting with three agents $N = \{1, 2, 3\}$, and two objects $M = \{a, b\}$. We are interested in a preference profile where agents 2 and 3 have identical preference: $R_2 = R_3 =$

R_0. In particular, all non-empty bundles have the same willingness to pay according to R_0 and satisfy

$$WP(\{a,b\},t;R_0) = WP(\{a\},t;R_0) = WP(\{b\},t;R_0) = 2+3t,$$

for $t > -\frac{1}{2}$. We are silent about the willingness to pay below $-\frac{1}{2}$, but it can be taken to be 0.5. We will only consider payments $t > -\frac{1}{2}$ for this example. At preference R_0, we have

$$(\{a,b\}, 2+4t)\ I_0\ (\{b\}, 2+4t)\ I_0\ (\{a\}, 2+4t)\ I_0\ (\emptyset, t),$$

for all $t > -\frac{1}{2}$. Hence, as t increases, bundle $\{a\}$ (or $\{b\}$ or $\{a,b\}$) will require more payment to be indifferent to $(\emptyset, t)$. We term this *negative income effect*.

	$\{a\}$	$\{b\}$	$\{a,b\}$
$WP(\cdot,0;R_1)$	0	0	3.9
$WP(\cdot,0;R_2=R_0)$	2	2	2
$WP(\cdot,0;R_3=R_0)$	2	2	2

Table 2.1: A profiles of preferences with $M=\{a,b\}$, $N=\{1,2,3\}$.

Agent 1 has *quasilinear* preference with a value of 3.9 for bundle $\{a,b\}$; value zero (or, arbitrarily close to zero) for bundle $\{a\}$ and bundle $\{b\}$, and value of bundle $\emptyset$ is normalized to zero. We denote this preference as R_1. The willingness to pay at zero payment for these preferences are shown in Table 2.1.

Suppose $(f,\mathbf{p})$ is a desirable mechanism defined on a (rich enough) type space $\mathcal{T}$ containing the preference profile $R \equiv (R_1, R_2 = R_0, R_3 = R_0)$. Notice that the value of $\{a,b\}$ for agent 1 is 3.9 but $WP(\{a\},0;R_2) + WP(\{b\},0;R_3) = 4$. Hence, a consequence of Pareto efficiency, individual rationality, and no subsidy is that $f_1(R)=\emptyset$.[4] Then, without loss of generality, agent 2 gets bundle $\{a\}$ and agent 3 gets bundle $\{b\}$ due to Pareto efficiency.

Next, we can pin down the payments of agents at R. Since agent 1 gets $\emptyset$, her payment must be zero by IR and no subsidy. Now, pretend as if agents 2 and 3 have quasilinear preference with valuations equal to their willingness to pay at zero payment (see Table 2.1).

[4]This follows from the following reasoning. Individual rationality and no subsidy imply that agents who are not allocated any object pay zero. Hence, any outcome where agent 1 is given both the objects can be Pareto improved.

Then, the VCG mechanism would charge them their externalities, which is equal to 1.9 for both the agents. If the type space $\mathcal{T}$ is sufficiently rich (in a sense, we make precise later), DSIC will still require that $p_2(R) = p_3(R) = 1.9$ (a precise argument is given in the proof of Theorem 1).

The negative income effect of R_0 makes the Pareto improvement possible in this example. The maximum payment we can extract from agent 1 is 3.9. Hence, to collect more payment than the VCG outcome, we can pay a maximum of $0.1 (= 3.9 - 3.8)$ to agents 2 and 3. If the preference R_0 was quasilinear, agents 2 and 3 would have required a compensation of 0.1 each to be indifferent between not getting any objects and the VCG outcome. Due to negative income effect, agents 2 and 3 can be made to improve from their VCG outcome by paying them much lower amounts. This in turn enables us to Pareto dominate the VCG outcome.

To be precise, the following outcome vector Pareto dominates the outcome of the mechanism at R:

$$z_1 := (\{a, b\}, 3.9), \quad z_2 := (\emptyset, -0.025), \quad z_3 := (\emptyset, -0.025).$$

To see why, note that (a) sum of payments in z is $3.85 > p_2(R) + p_3(R) = 3.8$; (b) agent 1 is indifferent between z_1 and $(\emptyset, 0)$; (c) agents 2 and 3 are also indifferent between their outcomes in the mechanism and z since $(\emptyset, -0.025)\ I_0\ (\{a\}, 1.9)$ (because $WP(\{a\}, t; R_0) = 2 + 3t$ for all $t > -0.5$).

It is important to note that R_1 having high value on $\{a, b\}$ and (almost) zero value on all other bundles played a crucial role in determining payments of agents, and hence, in the impossibility. Indeed, if agent 1 also had equal willingness to pay on some smaller bundle, then the example will not work.[5] This motivates the class of preferences we study in the next section. $\diamondsuit$

[5] If the willingness to pay of agent 1 is 3.9 on $\{a\}$ or $\{b\}$, then her preference will satisfy the *unit demand* property (for a formal definition, see Section 2.3.3). Preference R_0 also satisfies the unit demand property. It is known that if agents have unit demand preferences, a desirable mechanism exists, even if such preferences have negative income effect (Demange and Gale, 1985).

2.2.4 Dichotomous preferences

We turn our focus on a subset of classical preferences which we call dichotomous. The dichotomous preferences can be described by: (a) a collection of bundles, which we call the *acceptable* bundles, and (b) a willingness to pay function, which only depends on the payment level. Formally, it is defined as follows.

Definition 3 *A classical preference R_i of agent i is* **dichotomous** *if there exists a non-empty set of bundles $\emptyset \neq \mathcal{S}_i \subseteq (\mathcal{B} \setminus \{\emptyset\})$ and a willingness to pay (WP) map $w_i : \mathbb{R} \to \mathbb{R}_{++}$ such that for every $t \in \mathbb{R}$,*

$$WP(A, t; R_i) = \begin{cases} w_i(t) & \forall\ A \in \mathcal{S}_i \\ 0 & \forall\ A \in \mathcal{B} \setminus \mathcal{S}_i. \end{cases}$$

In this case, we refer to $\mathcal{S}_i$ as the collection of **acceptable** *bundles.*

The interpretation of the dichotomous preference is that, given same price (payment) for all the bundles, the agent is indifferent between the bundles in $\mathcal{S}_i$. Similarly, she is indifferent between the bundles in $\mathcal{B} \setminus \mathcal{S}_i$, but it strictly prefers a bundle in $\mathcal{S}_i$ to a bundle outside it. Hence, a dichotomous preference can be succinctly represented by a pair $(w_i, \mathcal{S}_i)$, where $w_i : \mathbb{R} \to \mathbb{R}_{++}$ is a WP map and $\emptyset \neq \mathcal{S}_i \subseteq (\mathcal{B} \setminus \{\emptyset\})$ is the set of acceptable bundles.

By our monotonicity requirement (free-disposal) of classical preference, for every $S, T \in \mathcal{B}$, we have

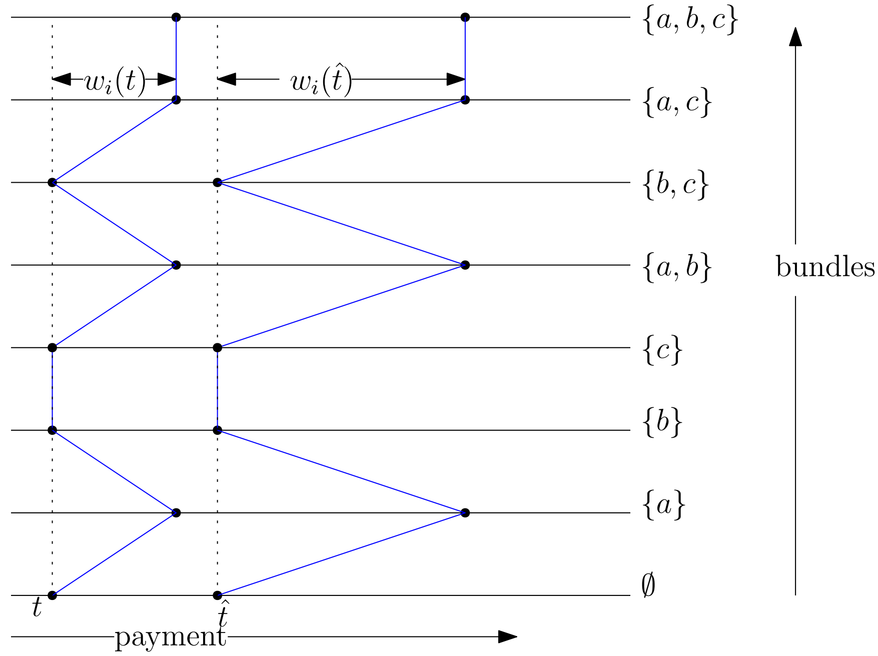
$$\Big[S \subseteq T, S \in \mathcal{S}_i\Big] \Rightarrow \Big[T \in \mathcal{S}_i\Big].$$

Hence, a dichotomous preference can be described by w_i and a *minimal* set of bundles $\mathcal{S}_i^{min}$ such that

$$\mathcal{S}_i := \{T \in \mathcal{B} : S \subseteq T \text{ for some } S \in \mathcal{S}_i^{min}\}.$$

Figure 2.2 shows two indifference vectors of a dichotomous preference. The figure shows that the bundles $\{a\}, \{a, c\}, \{a, b\}$ and $\{a, b, c\}$ are acceptable but others are not.

We will denote the domain of **all** dichotomous preferences as $\mathcal{D}$, where each preference in $\mathcal{D}$ for agent i is described by a w_i map and a collection of minimal bundles $\mathcal{S}_i^{min}$. A **dichotomous domain** is any subset of dichotomous preferences.



Two indifference vectors corresponding to a dichotomous classical preference

Acceptable bundles: $\{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}$.

Figure 2.2: A dichotomous preference

For some of our results, we will need a particular type of dichotomous preference.

Definition 4 *A dichotmous preference* $R_i \equiv (\mathcal{S}_i^{min}, w_i)$ *is called a* **single-minded preference** *if* $|\mathcal{S}_i^{min}| = 1$.

An agent having a single-minded dichotomous preference has *a unique* bundle of objects and all its supersets as acceptable bundles. Let $\mathcal{D}^{single}$ denote the set of all single-minded preferences. Single-minded preferences are well-studied in the algorithmic game theory literature (Lehmann et al., 2002; Babaioff et al., 2005, 2009). They were also central in the recent analysis of US incentive auction (Milgrom and Segal, 2020). Our main negative result will be for domains containing $\mathcal{D}^{single}$. Establishing a negative result on domains containing $\mathcal{D}^{single}$ implies a negative result on domains containing $\mathcal{D}$ since $\mathcal{D}^{single} \subsetneq \mathcal{D}$.

Before concluding this section, we briefly discuss how dichotomous preferences are similar to some other kinds of preferences in the literature. In the single object model, the preferences are clearly dichotomous, where there is no uncertainty about the acceptable bundles.

Similarly, consider the unit demand preferences studied in Demange and Gale (1985); Morimoto and Serizawa (2015). A preference R_i is a unit demand preference if for every $S \in \mathcal{B}$ and every $t \in \mathbb{R}$, we have $WP(S, t; R_i) = \max_{a \in S} WP(\{a\}, t; R_i)$. Now, suppose the objects are *homogeneous* in the following sense: $WP(\{a\}, t; R_i) = WP(\{b\}, t; R_i)$ for all $a, b \in M$ and for all $t \in \mathbb{R}$. It is clear that a unit demand preference R_i over homogeneous objects is a dichotomous preference, where $\mathcal{S}_i^{min}$ consists of singleton bundles. If the objects are not homogeneous, the unit demand preferences are not dichotmous since the willingness to pay of different objects may be different.

2.3 The results

We describe our main results in this section.

2.3.1 An impossibility result

We start with our main negative result: if the domain consists of *all* single-minded preferences, then there is no desirable mechanism. This generalizes the intuition we demonstrated in the example in Section 2.2.3.

Theorem 1 (Impossibility) *Suppose $\mathcal{T} \supseteq \mathcal{D}^{single}$ (i.e., the domain contains* all *single-minded preferences), $n \geq 3$, and $m \geq 2$. Then, no desirable mechanism exists in $\mathcal{T}^n$.*

The proof of this theorem and all other proofs are relegated to an appendix at the end. The proof formalizes the sketch given in the example in Section 2.2.3. The main idea of the proof is that if a desirable mechanism exists in $\mathcal{D}^{single}$, it has to define outcomes at *all* single-minded preference profiles, which includes an n-agent and m-object version of the preference profile discussed in Section 2.2.3. The challenge is to show that any desirable mechanism at that profile must coincide with the outcome of a *generalized* VCG mechanism (where agents pay their "externalities"). Once this is shown, the rest of the proof is similar to the discussion in Section 2.2.3.

As discussed in the introduction, Theorem 1 adds to a small list of papers that have established such negative results in other combinatorial auction problems. Notice that the domain $\mathcal{T}$ may contain preferences that are not dichotomous or it may be equal to $\mathcal{D}$, the set of all dichotomous preferences.

The conditions $m \geq 2$ and $n \geq 3$ are both necessary: if $m = 1$, we know that a desirable mechanism exists (Saitoh and Serizawa, 2008); if $n = 2$, the mechanism that we propose next is desirable – see Proposition 1 and discussions after it.

Definition 5 *The* **generalized Vickrey-Clarke-Groves mechanism with loser's payment** t_L **(GVCG-t_L)**, *denoted as* $(f^{vcg,t_L}, \mathbf{p}^{vcg,t_L})$, *is defined as follows: for every profile of preferences* R,

$$f^{vcg,t_L}(R) \in \arg\max_{A \in \mathcal{X}} \sum_{i \in N} WP(A_i, t_L; R_i)$$
$$p_i^{vcg,t_L}(R) = t_L + \max_{A \in \mathcal{X}} \sum_{j \neq i} WP(A_j, t_L; R_j) - \sum_{j \neq i} WP(f_j^{vcg,t_L}(R), t_L; R_j).$$

We refer to the GVCG-0 *mechanism as the* **GVCG** *mechanism.*

The GVCG class of mechanisms is a natural generalization of the VCG mechanism to our setting without quasilinearity. Note that the current definition does not use anything about dichotomous preferences. It computes the "externality" of every agent with respect to a reference transfer level t_L. This transfer level t_L corresponds to the payment by any agent who does not win any non-empty bundle of objects in the mechanism (such an agent has zero externality). The additional term t_L in the payment expression ensures that when we use t_L as the reference transfer level to compute externalities, we maintain incentive compatibility in the dichotomous domain. In the quasilinear domain, the reference transfer level does not matter as the willingness to pay does not change with reference transfer: $WP(S, t_L, R_i) = WP(S, 0, R_i)$ for each S, if R_i is a quasilinear preference.

Theorem 1 implies that the GVCG mechanism is not desirable. Indeed, no GVCG mechanism can be DSIC in an arbitrary combinatorial auction domain without quasilinearity. For instance, Morimoto and Serizawa (2015) show that there is a unique desirable mechanism in the domain of "unit-demand" (where agents have demand for at most one object) preferences, and it is **not** a GVCG mechanism. We show that the GVCG mechanism is DSIC, individually rational, and satisfies no subsidy in *any* dichotomous preference domain.

Proposition 1 *Consider the GVCG-t_L mechanism for some $t_L \in \mathbb{R}$, defined on an arbitrary dichotomous domain $\mathcal{T} \subseteq \mathcal{D}$. Then, the following are true.*

1. *The GVCG-t_L mechanism is DSIC.*
2. *The GVCG-t_L mechanism is individually rational if $t_L \leq 0$.*
3. *The GVCG-t_L mechanism satisfies individual rationality and no subsidy if $t_L = 0$.*
4. *The GVCG-t_L mechanism is Pareto efficient if $n = 2$.*
5. *The GVCG-t_L mechanism is not Pareto efficient if $n > 2, m > 1$, and $\mathcal{T} \supseteq \mathcal{D}^{single}$.*

We explain below why the GVCG class of mechanisms are compatible with Pareto efficiency when $n = 2$ but not compatible when $n > 2$. For simplicity, we assume that preferences of agents are single-minded, i.e., the domain is $\mathcal{D}^{single}$. We consider various cases.

One object ($m = 1$). It is well known that the GVCG mechanism is Pareto efficient if $m = 1$ (Saitoh and Serizawa, 2008). Note that for $m = 1$, every preference is single-minded. The GVCG mechanism allocates the object to an agent k with the highest WP at 0, i.e., $w_k(0) = \max_{i \in N} w_i(0)$. All agents except agent k pay zero and agent k pays $\max_{i \neq k} w_i(0)$. This outcome is always Pareto efficient. The main reason for this is that there is *only one* object, and any new outcome can only give this object to one agent (may be the same or another agent). Take any such outcome $z \equiv (z_1, \ldots, z_n)$ and assume for contradiction that it Pareto dominates the GVCG outcome. If agent k continues to get the object in z_k also, her payment cannot be more than $\max_{i \neq k} w_i(0)$. Further, payments of other agents cannot be more than zero. As a result, total payment cannot be more than $\max_{i \neq k} w_i(0)$. Similarly, if any other agent $j \neq k$ receives the object in z, then her payment cannot be more than $w_j(0)$ (else, she will prefer the GVCG outcome of getting nothing and paying zero). Further, in this case, since agent k does not receive the object in z, her payment will be non-positive. As a result, the total payment cannot be more than $\max_{i \neq k} w_i(0)$. In fact, the total payment in z in both the cases will be strictly less than the GVCG payments if any agent strictly improves, which is a contradiction.

Two agents ($n = 2$) but arbitrary m. Since preferences of agents are single-minded, at every preference profile the acceptable bundles of each agent i are supersets of some $S_i \in \mathcal{B}$. Since there are two agents, we have only two cases to consider: (i) $S_1 \cap S_2 = \emptyset$ and (ii) $S_1 \cap S_2 \neq \emptyset$. Intuitively, in the first case, the two agents are not competing against each other. Pareto efficiency requires us to allocate each agent $i \in \{1, 2\}$ her acceptable bundle S_i. The GVCG mechanism charges zero payment to the agents. Clearly, this cannot be Pareto dominated. In the second case, the two agents compete against each other like the single object case. This is because $S_1 \cap S_2 \neq \emptyset$ means exactly one agent can be assigned an acceptable bundle. In fact the allocation and payment in the GVCG mechanism for this case mirrors the single object case: the agent with the higher WP at 0 gets her acceptable bundle and pays the willingness to pay of the other agent. The fact that this outcome cannot be Pareto dominated follows an argument similar to the $m = 1$ case. Summarizing, if there are two agents, independent of the number of objects, the Pareto efficiency requirement is very similar to the single object case. Hence, the GVCG mechanism remains compatible with Pareto efficiency.

More than two agents and more than one object ($n > 2, m > 1$). With more than two agents and more than one object, the Pareto efficiency requirement is no longer like the single object case. To understand, let us consider Example 1 (see Table 2.1). The GVCG mechanism allocates objects a and b to agents 2 and 3 but charges them low payments (1.9 each). This is akin to low payments in the VCG mechanism as documented in Ausubel and Milgrom (2006).[6] In our example, even though agent 1 is not allocated any object, she has high enough willingness to pay for the bundle of objects – with one object, if the payment of the winning agent is low, then the willingness to pay of all losing agents is also low. With negative income effect, agents 2 and 3 feel "wealthier" after getting the objects at low payments. So, their "willingness to sell" amount is low. Hence, it is easier to compensate them. With agent 1 having a high enough willingness to pay (3.9), a Pareto improving trade is thus possible. Such a Pareto improving trade is not possible if agent 1 has positive income effect preferences. This is because with positive income effect, the "willingness to

[6]They point out that when there are at least two objects and at least three agents, the VCG mechanism outcome may not lie in the "core" of the associated game if objects are complements. This in turn results in low payments. The dichotomous preferences exhibit extreme form of complementarity.

sell" amount is higher than the willingness to pay.

2.3.2 Positive income effect and possibility

Proposition 1 and Theorem 1 point out that the GVCG is not Pareto efficient in the entire dichotomous domain. A closer look at the proof of Theorem 1 (and Example 1) reveals that the impossibility is driven by a particular kind of dichotomous preferences: the ones where the willingness to pay of an agent increases with payment. We term such preferences *negative income effect*.

A standard definition of positive income effect will say that as income rises, a preferred bundle becomes "more preferred". We do not model income explicitly, but our preferences implicitly account for income. So, if payment decreases from t to t', the income level of the agent increases implicitly. As a result, she is willing to pay more for his acceptable bundles at t' than at t. Thus, positive income effect captures a reasonable (and standard) restriction on preferences of the agents.

Definition 6 *A dichotomous preference $R_i \equiv (w_i, \mathcal{S}_i)$ satisfies* **positive income effect** *if for all $t > t'$, we have $w_i(t) \leq w_i(t')$.*

A dichotomous domain of preferences $\mathcal{T}$ satisfies positive income effect if every preference in $\mathcal{T}$ satisfies positive income effect.

As an illustration, the indifference vectors shown in Figure 2.2 cannot be part of a dichotomous preference satisfying positive income effect – we see that $\hat{t} > t$ but $w_i(\hat{t}) > w_i(t)$. The preference R_0 in Example 1 also violated positive income effect. A quasilinear preference (where $w_i(t) = w_i(t')$ for all t, t') always satisfies positive income effect, and the GVCG mechanism is known to be a desirable mechanism in this domain. We show below that the GVCG mechanism is Pareto efficient if the domain contains preferences that satisfy positive income effect. Before stating the result, let us reconsider Example 1 and see why the GVCG mechanism becomes desirable with positive income effect.

Example 2

We revisit Example 1 but with an important difference: the preferences of agents 2 and 3 now satisfy positive income effect. So, we have three agents $N = \{1, 2, 3\}$ and two objects

$M = \{a, b\}$. As in Example 1, agent 1 has single-minded quasilinear preference R_1 with valuation 3.9 on the unique acceptable bundle $\{a, b\}$. All the bundles are acceptable bundles for agents 2 and 3. But their preference is now $\widehat{R}_0$ which satisfies positive income effect. However, similar to Example 1, we have $\widehat{w}(0) = 2$. Figure 2.3 shows two indifference vectors of $\widehat{R}_0$. Since $\widehat{R}_0$ satisfies positive income effect, we have $\widehat{w}(t) > \widehat{w}(0)$, where $t < 0$.

The GVCG outcome does not change from Example 1 at this profile: agent 2 gets object a and agent 3 gets object b with payments $p_1^{vcg} = 0, p_2^{vcg} = p_3^{vcg} = 1.9$. To Pareto dominate this outcome, we need to give both the objects to agent 1.

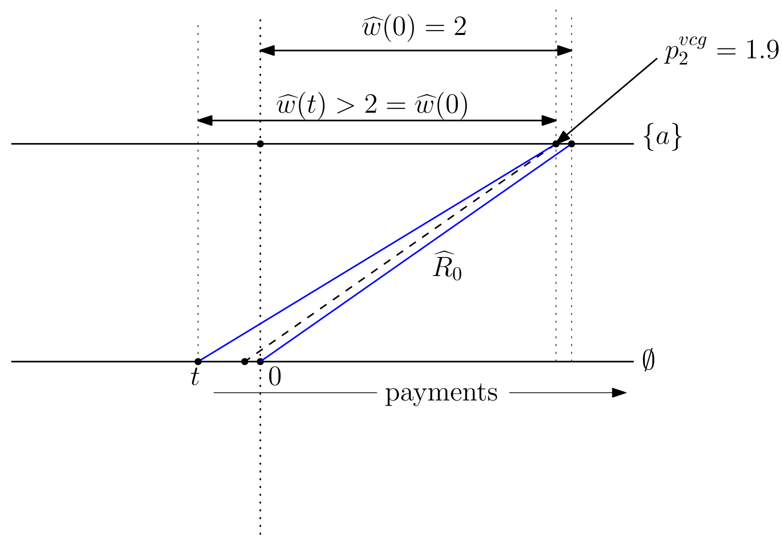


Figure 2.3: Possibility with positive income effect

	$\{a\}$	$\{b\}$	$\{a, b\}$
$WP(\cdot, 0; R_1)$	0	0	3.9
$WP(\cdot, 0; R_2 = \widehat{R}_0)$	2	2	2
$WP(\cdot, 0; R_3 = \widehat{R}_0)$	2	2	2

Table 2.2: A profiles of preferences with $M = \{a, b\}$, $N = \{1, 2, 3\}$.

Now, the GVCG outcome to agent 2 is $(\{a\}, 1.9)$ and, by Table 2.2 (see Figure 2.3 also), $(\{a\}, 2)\ \widehat{I}_0\ (\emptyset, 0)$. If $(\{a\}, 1.9)\ \widehat{I}_0\ (\emptyset, t)$, then by positive income effect $t < -0.1$. A pictorial description of the indifference vectors of $\widehat{R}_0$ for these transfer amounts are shown in Figure 2.3. This means that if agent 2 is not given any object, the total compensation required for her alone will be more than 0.1. Since agent 3 needs to be compensated too and the total revenue collected in the VCG outcome is 3.8, we need to charge more than 3.9 to agent 1 to Pareto dominate the VCG outcome. This is impossible since the value of agent 1 for both the objects is only 3.9. $\Diamond$

The intuition in this example generalizes. The main idea is that the GVCG mechanism allocates goods in a way that maximizes the collective willingness to pays (at zero) of the winning bidders. IR implies that the winning bidders pay a price less than their willingness to pay for their winning bundles. Thus, winning essentially makes the bidders feel "wealthier". Positive income effect then implies that their "willingness to sell" after the auction exceeds the willingness to pay before the auction. This rules out any Pareto improving trades[7].

Our next result says that the impossibility in Theorem 1 is overturned in any domain of dichotomous preferences satisfying positive income effect.

Theorem 2 (Possibility) *The GVCG mechanism is desirable on any dichotomous domain satisfying positive income effect.*

Theorem 2 can be interpreted to be a generalization of the well-known result that the VCG mechanism is desirable in the quasilinear domain. Indeed, we know that if the domain of preferences is the set of *all* quasilinear preferences, then standard revenue equivalence result (which holds in the quasilinear domain) implies that the VCG mechanism is the *only* desirable mechanism. Though we do not have a revenue equivalence result, we show below a similar uniqueness result of the GVCG mechanism. For this, we first remind ourselves of the definition of a quasilinear preference. A dichotomous preference $(w_i, \mathcal{S}_i)$ is **quasilinear** if for every $t, t' \in \mathbb{R}$, we have $w_i(t) = w_i(t')$. We denote by $\mathcal{D}^{QL}$ the set of **all** dichotomous quasilinear preferences. This leads to a characterization of the GVCG mechanism.

Theorem 3 (Uniqueness) *Suppose the domain of preferences $\mathcal{T}$ is a dichotomous domain satisfying positive income effect and contains $\mathcal{D}^{QL}$. Let $(f, \mathbf{p})$ be a mechanism defined on $\mathcal{T}^n$. Then, the following statements are equivalent.*

1. $(f, \mathbf{p})$ *is a desirable mechanism.*
2. $(f, \mathbf{p})$ *is the GVCG mechanism.*

[7] We are grateful to an anonymous referee for this intuition. Baisa and Burkett (2019) give similar intuition in a single object auction model to establish a mapping between non-quasilinear and quasilinear economies.

We reiterate that the GVCG is known to fail DSIC with non-quasilinear preferences if agents demand multiple objects. So, Theorems 2 and 3 show that under dichotomous classical preferences with positive income effect, we recover the desirability of the GVCG mechanism.

2.3.3 Tightness of results

In this section, we investigate if the positive results in the previous sections continue to hold if the domain includes (positive income effect) non-dichotomous preferences. In particular, we investigate the consequences of adding a non-dichotomous preference satisfying positive income effect and some other reasonable properties (we precisely define them later in the section). Both these conditions are natural properties to impose on preferences. Our results below can be summarized as follows: if we take the set of *all* quasilinear dichotomous preferences and add *any* non-dichotomous preference satisfying the above two conditions, then no desirable mechanism can exist in such a type space. As corollaries, we uncover new type spaces where no desirable mechanism can exist with non-quasilinear preferences, and establish the role of dichotomous preferences in such type spaces. Before we formally state the result, we give an example to show why we should expect such an impossibility result.

	$\{a\}$	$\{b\}$	$\{a,b\}$
$WP(\cdot, 0; R_1)$	0	0	5
$WP(\cdot, 0; R_2 = R_0)$	3	4	4
$WP(\cdot, 0; R_3 = R_0)$	3	4	4
$WP(\cdot, 0; R_2')$	0	4	4

Table 2.3: Two profiles of preferences with $M = \{a, b\}$, $N = \{1, 2, 3\}$.

Example 3

We consider an example with two object $M := \{a, b\}$ and three agents $N := \{1, 2, 3\}$. We will require the following preferences of the agents. The preference R_1 of agent 1 is quasilinear and the corresponding values for bundles of objects is shown in Table 2.3. It is clear that R_1 is a single-minded preference. We have two preferences of agent 2: $R_2 = R_0$ and R_2'. Preference R_0 is not quasilinear, but it satisfies positive income effect (decreasing prices by the same amount of two indifferent consumption bundles lead the agents to strictly prefer the costlier object): $(\{b\},4)\ I_0\ (\{a\},3)$ and $(\{b\},2)\ P_0\ (\{a\},1)$. This is shown in Figure 2.4, where we show some indifference vectors of R_0. Note that the other indifference vectors of R_0 can be constructed such that it satisfies the unit demand property and positive income effect. Preference R_2' is a quasilinear single-minded preference with $\{b\}$ and $\{a,b\}$ as acceptable bundles and value 4. Finally, preference R_3 of agent 3 is also R_0.

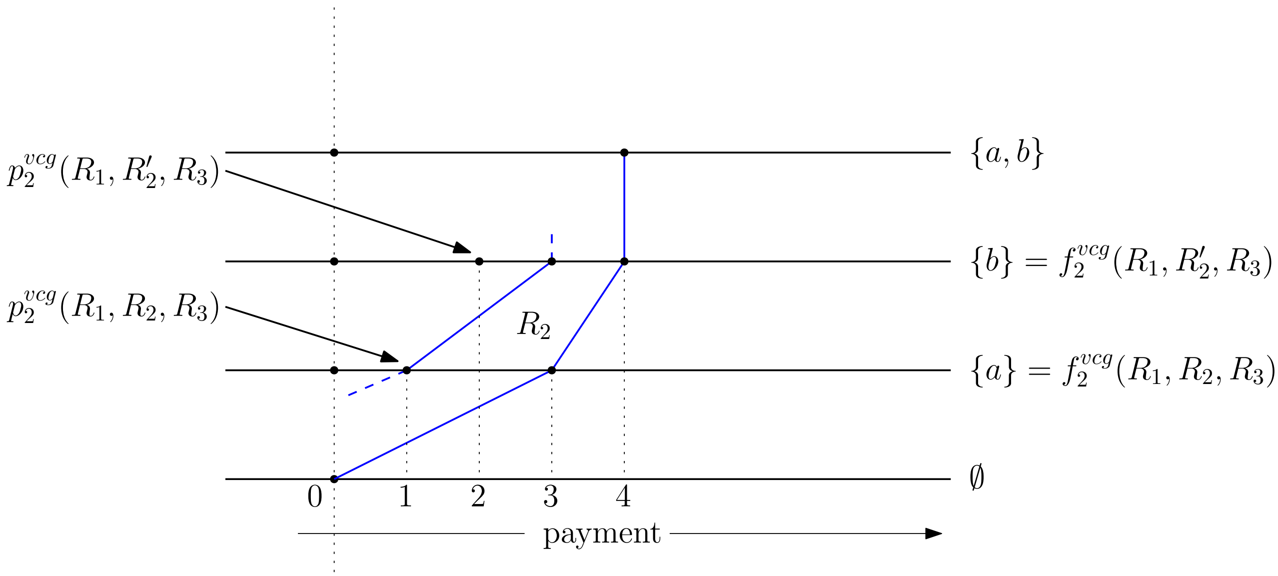


Figure 2.4: Positive income effect preference of agents 2 and 3.

We argue that the GVCG mechanism containing all quasilinear dichotomous preferences and R_0 is **not DSIC**. So, our domain is $\mathcal{T} = \mathcal{D}^{QL} \cup \{R_0\}$. We will look at two preference profiles: (R_1, R_2, R_3) and (R_1, R_2', R_3). At the preference profile (R_1, R_2, R_3), agents 2 and 3 should get objects from $\{a,b\}$ according to GVCG. Since they have identical preferences, we break the tie by giving object a to agent 2 and object b to agent 3: $f_1^{vcg}(R_1, R_2, R_3) = \{a\}, f_2^{vcg}(R_1, R_2, R_3) = \{b\}$.[8] The payment of agent 2 is $p_2^{vcg}(R_1, R_2, R_3) = 1$.

Now, consider the preference profile (R_1, R_2', R_3). Here, since agent 2 has only $\{b\}$ and $\{a,b\}$ in her acceptable bundle, her GVCG outcome changes: $f_2^{vcg}(R_1, R_2', R_3) = \{b\}$ and $p_2^{vcg}(R_1, R_2', R_3) = 2$. In other words, the externality of agent 2 changes from 1 at preference profile (R_1, R_2, R_3) to 2 at (R_1, R_2', R_3).

[8]The example can be modified to work if the tie is broken by giving object b to agent 2 and object a to agent 3.

If R_2 was a quasilinear preference, then agent 2 would have been indifferent between $(\{a\}, 1)$ and $(\{b\}, 2)$. But since $R_2 = R_0$ satisfies positive income effect (see Figure 2.4), $(\{b\}, 2)\ P_2\ (\{a\}, 1)$. This shows that with positive income effect, agent 2 can manipulate in the GVCG mechanism in this domain.

This is a general problem. We formalize this in Theorem 4. We show in the proof of Theorem 4 that any desirable mechanism in such a domain must have the GVCG outcomes at these profiles, and this will lead to manipulation by the agent having positive income effect.

It is crucial that $WP(\{a\}, 0; R_0) < WP(\{b\}, 0; R_0)$ for this manipulation to happen in this example. If $WP(\{a\}, 0; R_0) = WP(\{b\}, 0; R_0) = 4$, then R_0 can be a dichotomous preference (i.e., besides the indifference vector shown in Table 2.3, we can construct other indifference vectors such that it is a dichotomous preference). We know that the GVCG mechanism is DSIC in such domains. Indeed, in that case, the externality of agent 2 remains unchanged across profiles (R_1, R_2, R_3) and (R_1, R_2', R_3). In other words, we have $p_2^{vcg}(R_1, R_2, R_3) = p_2^{vcg}(R_1, R_2', R_3) = 1$. So, no manipulation is possible by agent 2 across these two preference profiles.[9]

$\diamondsuit$

We formalize the intuition in Example 3 now. We consider a preference where an agent can demand multiple heterogeneous objects. We require that at least two objects are heterogeneous in the following sense.

Definition 7 *A preference R_i satisfies* **heterogenous** *demand if there exists $a, b \in M$,*

$$WP(\{a\}, 0; R_i) \neq WP(\{b\}, 0; R_i).$$

Heterogeneous demand requires that for *some* pair of objects, the WP at 0 must be different for them. If objects are not the same (i.e., not homogeneous), then we should expect this condition to hold. We can provide an analogous tightness result if objects are homogeneous.[10]

Besides the heterogeneous demand, we will impose two natural conditions on preferences.

[9]This is true even if this preference does not satisfy positive income effect.
[10]The result is available on request.

The first condition is a mild form of substitutability condition.

Definition 8 *A preference R_i satisfies* **strict decreasing marginal WP** *if for every $a, b \in M$,*

$$WP(\{a\}, 0; R_i) + WP(\{b\}, 0; R_i) > WP(\{a, b\}, 0; R_i).$$

Strict decreasing marginal WP requires a minimal degree of submodularity: the marginal increase in WP (at 0) by adding $\{a\}$ to $\{b\}$ is less than adding $\{a\}$ to $\emptyset$. Notice that this substitutability requirement is *only* for bundles of size two. Hence, larger bundles may exhibit complementarity or substitutability. Because of free disposal, for every $a, b \in M$, we have

$$WP(\{a, b\}, 0; R_i) \geq \max(WP(\{a\}, 0; R_i), WP(\{b\}, 0; R_i)).$$

Hence, strict decreasing marginal WP implies that $WP(\{a\}, 0; R_i) > 0$ and $WP(\{b\}, 0; R_i) > 0$, i.e., each object is a *good* in a weak sense (getting an object is preferred to getting nothing at payment 0).

We point out that unit demand preferences (studied in (Demange and Gale, 1985; Morimoto and Serizawa, 2015)) satisfy strict decreasing marginal WP. A preference R_i is called a **unit demand** preference if for every S,

$$WP(S, t; R_i) = \max_{a \in S} WP(\{a\}, t; R_i) \ \forall \ t \in \mathbb{R}_+.$$

If R_i is a unit demand preference and objects are *goods*, then it satisfies strict decreasing marginal WP. To see this, call every object $a \in M$ **a real good** if $WP(\{a\}, 0; R_i) > 0$ at every R_i. If every object is a real good, then for every $a, b \in M$, we see that

$$WP(\{a\}, 0; R_i) + WP(\{b\}, 0; R_i) > \max_{x \in \{a,b\}} WP(\{x\}, 0; R_i) = WP(\{a, b\}, 0; R_i).$$

Besides the strict decreasing marginal WP condition, we will also be requiring strict positive income effect, but *only* for singleton bundles.

Definition 9 *A classical preference R_i satisfies* **strict positive income effect** *if for every $a, b \in M$ and for every t, t' with $t' > t$, the following holds for every $\delta > 0$:*

$$\Big[(\{b\}, t') \ I_i \ (\{a\}, t)\Big] \ \Rightarrow \ \Big[(\{b\}, t' - \delta) \ P_i \ (\{a\}, t - \delta)\Big].$$

This definition of strict positive income effect requires that if two objects are indifferent then decreasing their prices by the same amount makes the higher priced (lower income) object better. This is a generalization of the definition of positive income effect we had introduced for dichotomous preferences in Definition 6, but only restricted to singleton bundles.[11] This means that for larger bundles, we do not require positive income effect to hold.

We are ready to state the main tightness result with heterogeneous objects.

Theorem 4 *Suppose $n \geq 4, m \geq 2$. Let R_0 be a heterogeneous demand preference satisfying strict positive income effect and strict decreasing marginal WP. Consider any domain $\mathcal{T}$ containing $\mathcal{D}^{QL} \cup \{R_0\}$. Then, no desirable mechanism exists in $\mathcal{T}^n$.*

We make a quick remark about the statement of Theorem 4.

Remark 1. Though Theorem 4 requires $n \geq 4$, a careful look at its proof reveals that we only need $n \geq 4$ if $m > 2$. If there are only two objects, the impossibility result in Theorem 4 holds with $n \geq 3$. This was shown in Example 3 also.

The basic idea of the proof of Theorem 4 is similar to Example 3. With more than two object ($m > 2$), we will need at least four agents. The reason is slightly delicate. Notice that R_0 in the statement of Theorem 4 is an arbitrary preference. As in Example 3, the proof ensures that three agents compete for two objects, say $\{a, b\}$, out of which two agents have R_0 as their preference. With more than two objects, we need a way to ensure that $\{a, b\}$ are allocated among these three agents. In the absence of a fourth agent, it is not possible to ensure that the two agents having R_0 preference are not assigned objects outside of $\{a, b\}$. A fourth agent having arbitrarily large willingness to pay for the bundle $M \setminus \{a, b\}$ ensures that.

We do not know if the impossibility result holds for $n = 2$ or $n = 3$ when $m > 2$, but we conjecture that it does not. $\diamondsuit$

Unlike the negative result in Theorem 1, Theorem 4 does not require the existence of

[11] An alternate definition along the lines of Definition 6 using willingness to pay map is also possible. It will require *decreasing differences of willingness to pay*. Formally, a preference R_i satisfies strict positive income effect if for every $t' > t$ and for every $a, b \in M$, we have $WP(\{a\}, t'; R_i) > WP(\{b\}, t'; R_i)$ implies $WP(\{a\}, t'; R_i) - WP(\{b\}, t'; R_i) < WP(\{a\}, t; R_i) - WP(\{b\}, t; R_i)$.

negative income effect dichotomous preferences. It requires the domain to contain the set of quasilinear dichotomous preferences and one heterogeneous demand preference satisfying some reasonable conditions. This negative result parallels a result of Kazumura and Serizawa (2016) who show that adding *any* multi-demand preference to a class of *rich* unit demand preference gives rise to a similar impossibility. As was explained in Example 3, our proof exploits the fact that any desirable mechanism must coincide with the GVCG mechanism in the positive income effect dichotomous domain, and adding any strictly positive income effect preference to the domain leads to manipulation. In the case of Kazumura and Serizawa (2016), they add an arbitrary multi-demand preference (which may or may not satisfy positive income effect) to a domain of unit demand preferences, where the GVCG mechanism is *not* desirable. So, neither of the results imply the other and the proof strategies are different.

We now spell out an exact implication of Theorem 4 in a corollary below. Let $\mathcal{D}^+$ be the set of all *positive income effect* dichotomous preferences (note that $\mathcal{D}^{QL} \subsetneq \mathcal{D}^+$) and $\mathcal{U}^+$ be the set of all heterogeneous unit demand preferences satisfying positive income effect (as argued earlier, unit demand preferences satisfy strict decreasing marginal WP). Then, the following corollary is immediate from Theorem 4.

Corollary 1 *Suppose $\mathcal{T} = \mathcal{D}^+ \cup \mathcal{U}^+$. Then, no desirable mechanism exists on $\mathcal{T}^n$.*

Theorem 3 shows that the GVCG mechanism is the unique desirable mechanism on $\mathcal{D}^+$. Similarly, Demange and Gale (1985) have shown that a desirable mechanism exists in $\mathcal{U}^+$. This mechanism is called the *minimum Walrasian equilibrium price mechanism* and collapses to the VCG mechanism if preferences are quasilinear. Corollary 1 says that we lose these possibility results if we consider the unions of these two type spaces.

2.4 Related Literature

The quasilinearity assumption is at the heart of mechanism design literature with payments. Our formulation of classical preferences was studied in the context of single object auction by Saitoh and Serizawa (2008), who proposed the generalized VCG mechanism and axiomatized it for that setting. Other such axiomatizations include Sakai (2008, 2013). As discussed, Demange and Gale (1985) had shown that a mechanism different from the generalized VCG mechanism is desirable when multiple heterogeneous objects are sold to agents with unit demand. Characterizations of this mechanism have been given in Morimoto and Serizawa (2015), Zhou and Serizawa (2018) and Kazumura et al. (2020b). However, impossibility results for the existence of a desirable mechanism were shown (a) by Kazumura and Serizawa (2016) for multi-object auctions with multi-demand agents and (b) by Baisa (2020) for multiple homogeneous object model with multi-demand agents. Social choice problems with payments are studied with particular form of non-quasilinear preferences in Ma et al. (2016, 2018). These papers establish dictatorship results in this setting with non-quasilinear preferences.

Baisa (2016) considers non-quasilinear preferences with randomization in a single object auction environment. He proposes a randomized mechanism and establishes strategic properties of this mechanism. Dastidar (2015) considers a model where agents have same utility function but models income explicitly to allow for different incomes. He considers equilibria of standard auctions. Samuelson and Noldeke (2018) discuss an implementation duality without quasilinear preferences and apply it to matching and adverse selection problems. Kazumura et al. (2020a) discuss monotonicity based characterization of DSIC mechanisms in domains which admit non-quasilinear preferences. Baisa and Burkett (2019) discuss a model of single object allocation when bidders have interdependent values and non-quasilinear preferences with positive income effects. They give necessary and sufficient conditions for the existence of an ex-post implementable and Pareto efficient mechanism in two settings: (i) where the auctioneer is the seller; and (ii) the procurement setting, where the auctioneer is the buyer. In the former setting, their condition requires existence of an ex-post implementable and Pareto efficient mechanism in a corresponding quasilinear economy. In the latter setting, they show an impossibility result if the level of interdependence is strong.

The literature on auction design with budget constrained bidders models budget constraint such that if an agent has to pay more than budget, then his utility is minus infinity. This introduces non-quasilinear utility functions but it does not fit our model because of the hard budget constraint. For the multi-unit auction with such budget-constrained agents, Lavi and May (2012) establish that no desirable mechanism can exist – see an extension of this result in Dobzinski et al. (2012). They prove this result by considering two bidders each with publicly known budgets and two units. Their result shows an impossibility similar to ours as long as the public budgets of the bidders are not equal. Their paper also allows complementary preferences but not of the extreme form seen with dichotomous preferences.

For combinatorial auctions with single-minded and quasilinear preferences, Le (2018) shows that these impossibilities with budget-constrained agents can be overcome in a *generic* sense – he defines a "truncated" VCG mechanism and shows that it is desirable *almost everywhere*.

There is a literature in algorithmic mechanism design on combinatorial auctions with quasilinear but "single-minded" preferences. Apart from practical significance, the problem is of interest because computing a VCG outcome is computationally challenging but various "approximately" desirable mechanisms which are computationally tractable can be constructed (Babaioff et al., 2005, 2009; Lehmann et al., 2002; Milgrom and Segal, 2020). Rastegari et al. (2011) show that in this model, the revenue from the VCG mechanism (and any DSIC mechanism) may not satisfy monotonicity, i.e., adding an agent may *decrease* revenue. Our paper adds to this literature by illustrating the implications of non-quasilinear preferences.

2.5 Appendix

2.5.1 Proof of Theorem 1

The proof extends the intuition in Example 1.

Proof: We start by providing two useful lemmas.

Lemma 1 *Suppose $(f, \mathbf{p})$ is an individually rational mechanism satisfying no subsidy. Then for every agent $i \in N$ and every $R \in \mathcal{T}^n$, we have $p_i(R) = 0$ if $f_i(R) \notin \mathcal{S}_i$.*

Proof: Suppose R is a profile such that $f_i(R) \notin \mathcal{S}_i$ for agent i. By individual rationality, $(f_i(R), p_i(R))\ R_i\ (\emptyset, 0)$. But $f_i(R) \notin \mathcal{S}_i$ implies that $(\emptyset, p_i(R))\ I_i\ (f_i(R), p_i(R))\ R_i\ (\emptyset, 0)$. Hence, $p_i(R) \leq 0$. But no subsidy implies that $p_i(R) = 0$. ■

Lemma 2 *Suppose $(f, \mathbf{p})$ is an individually rational mechanism satisfying no subsidy. Then for every agent $i \in N$ and every $R \in \mathcal{T}^n$, we have $0 \leq p_i(R) \leq WP(f_i(R), 0; R_i)$.*

Proof: If $f_i(R) \notin \mathcal{S}_i$, then the claim follows from Lemma 1. Suppose $f_i(R) \in \mathcal{S}_i$. By individual rationality, $(f_i(R), p_i(R))\ R_i\ (\emptyset, 0)\ I_i\ (f_i(R), WP(f_i(R), 0; R_i))$. This implies that $p_i(R) \leq WP(f_i(R), 0; R_i)$. No subsidy implies that $p_i(R) \geq 0$. ■

Consider any three non-empty bundles S, S_1, S_2 such that $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \emptyset$. Consider a profile of single-minded preferences $R^* \in \left(\mathcal{D}^{single}\right)^n$ as follows. Since all the agents have dichotomous preferences, to describe any agent i's preference, we describe the *minimal* acceptable bundles $\mathcal{S}_i^{min}$ (i.e., the set of acceptable bundles $\mathcal{S}_i$ are derived by taking supersets of each element in $\mathcal{S}_i^{min}$) and the willingness to pay map w_i. Preference R_1^* of agent 1 is quasilinear:

$$\mathcal{S}_1^{min} = \{S\}, w_1(t) = 3.9\ \forall\ t \in \mathbb{R}.$$

Preference R_2^* of agent 2 is:

$$\mathcal{S}_2^{min} = \{S_1\}, w_2(t) = 2 + 3t\ \forall\ t > -\frac{1}{2} \text{ and } w_2(t) = \frac{1}{2} \text{ otherwise}$$

Preference R_3^* of agent 3 is:

$$\mathcal{S}_3^{min} = \{S_2\}, w_3(t) = 2 + 3t\ \forall\ t > -\frac{1}{2} \text{ and } w_3(t) = \frac{1}{2} \text{ otherwise}$$

Preference R_i^* of each agent $i \notin \{1, 2, 3\}$ is quasilinear:

$$\mathcal{S}_i^{min} = \{S\}, w_i(t) = \epsilon\ \forall\ t \in \mathbb{R},$$

where $\epsilon > 0$ but very close to zero.

Assume for contradiction that there exists a DSIC, Pareto efficient, individually rational mechanism $(f, \mathbf{p})$ satisfying no subsidy. We now do the proof in several steps.

Step 1. In this step, we show that at every preference profile R with $R_i = R_i^*$ for all $i \notin \{2, 3\}$, we must have $S \not\subseteq f_i(R)$ if $i \notin \{1, 2, 3\}$. We know that $\mathcal{S}_i^{min} = \{S\}$ for all $i \notin \{2, 3\}$. Assume for contradiction $S \subseteq f_k(R)$ for some $k \notin \{1, 2, 3\}$. Then, $S \not\subseteq f_1(R)$. By

Lemma 1, $p_1(R) = 0$. Consider the following outcome:

$$Z_1 = (S, \epsilon), Z_k = (\emptyset, p_k(R) - \epsilon), Z_j = (f_j(R), p_j(R)) \;\forall\; j \notin \{1, k\}.$$

Since preferences of agent 1 and agent k are quasilinear (note that $R_1 = R_1^*$ and $R_k = R_k^*$) and ϵ is very close to zero, we have

$$Z_1 \; P_1 \; (f_1(R), p_1(R) = 0), Z_k \; I_k \; (f_k(R), p_k(R)), Z_j \; I_j \; (f_j(R), p_j(R)) \;\forall\; j \notin \{1, k\}.$$

Also, the sum of payments in the outcome vector $Z \equiv (Z_1, \ldots, Z_n)$ is $\sum_{i \in N} p_i(R)$. This contradicts Pareto efficiency of $(f, \mathbf{p})$.

Step 2. Fix a preference $\hat{R}_2$ of agent 2 such that $\hat{\mathcal{S}}_2^{min} = \{S_1\}$ and $\hat{w}_2(0) > 1.9$. We show that at preference profile $\hat{R} = (\hat{R}_2, R^*_{-2})$, $S \not\subseteq f_1(\hat{R})$. Suppose $S \subseteq f_1(\hat{R})$. Then, $S_1 \not\subseteq f_2(\hat{R})$ and $S_2 \not\subseteq f_3(\hat{R})$. By Lemma 1, $p_2(\hat{R}) = 0, p_3(\hat{R}) = 0$. Consider a new outcome vector:

$$Z_1 = (\emptyset, p_1(\hat{R}) - 3.9), Z_2 = (S_1, \hat{w}_2(0)), Z_3 = (S_2, w_3(0)), Z_j = (f_j(\hat{R}), p_j(\hat{R})) \;\forall\; j \notin \{1, 2, 3\}.$$

By quasilinearity of R_1^*, we get $Z_1 \; I_1^* \; (f_1(\hat{R}), p_1(\hat{R}))$. By definition,

$$Z_2 \; \hat{I}_2 \; (\emptyset, 0) \; \hat{I}_2 \; (f_2(\hat{R}), p_2(\hat{R})).$$

Similarly, $Z_3 \; I_3^* \; (f_3(\hat{R}), p_3(\hat{R}))$. Further, the sum of payments in the outcome vector Z is

$$p_1(\hat{R}) - 3.9 + \hat{w}_2(0) + w_3(0) + \sum_{j \notin \{1,2,3\}} p_j(\hat{R}) > \sum_{j \in N} p_j(\hat{R}),$$

where the inequality used the fact that $p_2(\hat{R}) = p_3(\hat{R}) = 0$ and $\hat{w}_2(0) > 1.9, w_3(0) = 2$. This contradicts Pareto efficiency of $(f, \mathbf{p})$.

Step 3. Fix any quasilinear preference $\hat{R}_2$ of agent 2 such that $\hat{\mathcal{S}}_2^{min} = \{S_1\}$ and $\hat{w}_2(t) = 1.9 - \delta$, where $\delta \in (0, 1.9)$. We show that at preference profile $\hat{R} = (\hat{R}_2, R^*_{-2})$, we must have $S \subseteq f_1(\hat{R})$. If not, then by Step 1 and by Pareto efficiency, $S_1 \subseteq f_2(\hat{R})$ and $S_2 \subseteq f_3(\hat{R})$.

Now, consider the following outcome Z':

$$Z'_1 = (S, 3.9),\ Z'_2 = \Big(\emptyset, p_2(\hat{R}) - (1.9 - \frac{\delta}{2})\Big),\ Z'_3 = (\emptyset, p_3(\hat{R}) - 2),$$

$$Z'_j = (f_j(\hat{R}), p_j(\hat{R}))\ \forall\ j \notin \{1, 2, 3\}.$$

Note that by Lemma 1, $p_1(\hat{R}) = 0$. Hence, using quasilinearity of R^*_1, we get $(f_1(\hat{R}), p_1(\hat{R}) = 0)\ I^*_1\ (S, 3.9)$. Similarly, by quasilinearity of $\hat{R}_2$, we get $Z'_2\ \hat{P}_2\ (f_2(\hat{R}), p_2(\hat{R}))$. Also, the sum of payments in outcome Z' is

$$3.9 + p_2(\hat{R}) - (1.9 - \frac{\delta}{2}) + p_3(\hat{R}) - 2 + \sum_{j \notin \{1,2,3\}} p_j(\hat{R}) = \sum_{i \in N} p_i(\hat{R}) + \frac{\delta}{2} > \sum_{i \in N} p_i(\hat{R}),$$

where we used the fact that $p_1(\hat{R}) = 0$.

We now prove that $(\emptyset, p_3(\hat{R}) - 2)\ R^*_3\ (f_3(\hat{R}), p_3(\hat{R}))$. For this, let $t = p_3(\hat{R}) - 2$. Note that $w(t) \leq 2$ follows from the definition of w and the fact that $t \leq 0$ by Lemma 2. This implies $(\emptyset, t)\ R^*_3\ (f_3(\hat{R}), t + 2)$ i.e. $(\emptyset, p_3(\hat{R}) - 2)\ R^*_3\ (f_3(\hat{R}), p_3(\hat{R}))$

Hence, we get a contradiction to Pareto efficiency.

Step 4. In this step, we show that at preference profile R^*,

$$S_1 \subseteq f_2(R^*), S_2 \subseteq f_3(R^*),$$

and

$$p_2(R^*) = p_3(R^*) = 1.9.$$

Since $w_2(0) = 2$ in preference R^*_2, by Step 2, $S \not\subseteq f_1(R^*)$. By Step 1, $S \not\subseteq f_i(R^*)$ for all $i \notin \{1, 2, 3\}$. By Pareto efficiency, it must be

$$S_1 \subseteq f_2(R^*), S_2 \subseteq f_3(R^*).$$

Now, assume for contradiction $p_2(R^*) > 1.9$. Fix a preference $\hat{R}_2$ of agent 2 such that $\hat{\mathcal{S}}^{min}_2 = \{S_1\}$ and $p_2(R^*) > \hat{w}_2(0) > 1.9$. By Step 2, $S_1 \subseteq f_2(\hat{R}_2, R^*_{-2})$. By DSIC, $p_2(R^*) = p_2(\hat{R}_2, R^*_{-2})$. Hence, $p_2(\hat{R}_2, R^*_{-2}) > \hat{w}_2(0)$. This is a contradiction to Lemma 2.

Finally, assume for contradiction $p_2(R^*) < 1.9$. Then, consider any quasilinear preference $\hat{R}_2$ of agent 2 such that $\hat{\mathcal{S}}_2^{min} = \{S_1\}$ and $p_2(R^*) < \hat{w}_2(0) < 1.9$. By Step 3, $S_1 \nsubseteq f_2(\hat{R}_2, R^*_{-2})$ and by Lemma 1, $p_2(\hat{R}_2, R^*_{-2}) = 0$. But by reporting R^*_2, agent 2 gets S_1 at a payment less than $\hat{w}_2(0)$. By quasilinearity of $\hat{R}_2$ and the fact that $S_1 \nsubseteq f_2(\hat{R}_2, R^*_{-2})$, she prefers this outcome to outcome $(f_2(\hat{R}_2, R^*_{-2}), 0)$, which is a contradiction to DSIC.

This concludes the proof that $p_2(R^*) = 1.9$. A similar argument establishes (with Steps 2 and 3 applied to agent 3) that $p_3(R^*) = 1.9$.

Step 5. We now complete the proof. By Step 4, we know that the outcome at preference profile R^* satisfies:

$$S \nsubseteq f_1(R^*), S_1 \subseteq f_2(R^*), S_2 \subseteq f_3(R^*),$$

$$p_1(R^*) = 0, \; p_2(R^*) = p_3(R^*) = 1.9.$$

Note that by Lemma 1, $p_j(R^*) = 0$ for all $j \notin \{1, 2, 3\}$.
Now, consider the following outcome: $Z'_j = (f_j(R^*), p_j(R^*))$ for all $j \notin \{1, 2, 3\}$ and

$$Z'_1 = (S, 3.9), Z'_2 = (\{\emptyset\}, -0.025), \; Z'_3 = (\{\emptyset\}, -0.025).$$

Note that sum of payments in Z' is $3.85 > p_2(R^*) + p_3(R^*) = 3.8$.
Agent 1 is indifferent between Z'_1 and $(f_1(R^*), p_1(R^*))$. Agents 2 and 3 are also indifferent between Z'_i and $(f_i(R^*), p_i(R^*))$. This follows from the fact that $(-0.025) + w_2(-0.025) = (-0.025) + w_3(-0.025) = 1.9$.

This contradicts Pareto efficiency. ■

2.5.2 Proof of Proposition 1

Proof: Fix a dichotomous domain $\mathcal{T}$. For some $t_L \in \mathbb{R}$, consider the GVCG-t_L mechanism and denote it as $(f, \mathbf{p}) \equiv (f^{vcg,t_L}, \mathbf{p}^{vcg,t_L})$. We prove the following claim first.

Claim 1 *For every agent $i \in N$ and for every profile of preferences $R \in \mathcal{T}^n$, the following* *hold:*

$$(f_i(R), p_i(R)) \ R_i \ (\emptyset, t_L), \tag{2.1}$$
$$p_i(R) = t_L \qquad \text{if } f_i(R) \notin \mathcal{S}_i, \tag{2.2}$$

where $\mathcal{S}_i$ is the acceptable set of bundles of agent i at R_i.

Proof: The following inequalities follow straightforwardly.

$$\begin{aligned}
&\max_{A \in \mathcal{X}} \sum_{j \in N} WP(A_j, t_L; R_j) \geq \max_{A \in \mathcal{X}} \sum_{j \neq i} WP(A_j, t_L; R_j) \\
\Rightarrow &\sum_{j \in N} WP(f_j(R), t_L; R_j) \geq \max_{A \in \mathcal{X}} \sum_{j \neq i} WP(A_j, t_L; R_j) \\
\Rightarrow &\, WP(f_i(R), t_L; R_i) + t_L \geq \max_{A \in \mathcal{X}} \sum_{j \neq i} WP(A_j, t_L; R_j) - \sum_{j \neq i} WP(f_j(R), t_L; R_j) + t_L = p_i(R).
\end{aligned}$$

But this implies that

$$\Big(f_i(R), p_i(R)\Big) \ R_i \ \Big(f_i(R), WP(f_i(R), t_L; R_i) + t_L\Big) \ I_i \ (\emptyset, t_L),$$

where the second relation comes from the definition of WP.

Suppose $f_i(R)$ is not an acceptable bundle at R_i, then $(f_i(R), p_i(R)) \ I_i \ (\emptyset, p_i(R))$. Then, the relation (2.1) implies that $t_L \geq p_i(R)$. But by construction, $p_i(R) \geq t_L$. Hence, $p_i(R) = t_L$ if $f_i(R) \notin \mathcal{S}_i$. ■

Using Claim 1, we prove each assertion of the proposition.

PROOF OF (1). We prove that the GVCG-t_L is DSIC. Fix agent $i \in N$, $R_{-i} \in \mathcal{T}^{n-1}$, and $R_i, R'_i \in \mathcal{T}$. Let $A \equiv f(R_i, R_{-i})$ and $A' \equiv f(R'_i, R_{-i})$. We start with a simple lemma.

Lemma 3 *If A_i and A'_i belong to the acceptable bundle set at R_i, then*

$$p_i(R_i, R_{-i}) \leq p_i(R'_i, R_{-i}).$$

Proof: Note that

$$\begin{aligned}
p_i(R_i, R_{-i}) - p_i(R'_i, R_{-i}) &= \big[\max_{\hat{A}\in\mathcal{X}} \sum_{j\neq i} WP(\hat{A}_j, t_L; R_j) - \sum_{j\neq i} WP(A_j, t_L; R_j)\big] \\
&- \big[\max_{\hat{A}\in\mathcal{X}} \sum_{j\neq i} WP(\hat{A}_j, t_L; R_j) - \sum_{j\neq i} WP(A'_j, t_L; R_j)\big] \\
&= \sum_{j\neq i} WP(A'_j, t_L; R_j) - \sum_{j\neq i} WP(A_j, t_L; R_j) \\
&= WP(A'_i, t_L; R_i) + \sum_{j\neq i} WP(A'_j, t_L; R_j) \\
&- WP(A_i, t_L; R_i) - \sum_{j\neq i} WP(A_j, t_L; R_j) \\
&= \sum_{j\in N} WP(A'_j, t_L; R_j) - \sum_{j\in N} WP(A_j, t_L; R_j) \\
&\leq 0,
\end{aligned}$$

where the third equality follows from the fact that A_i, A'_i belong to the acceptable bundle set at R_i and the last inequality follows from the fact that $f(R) = A$. ■

Let $\mathcal{S}_i$ be the acceptable bundle set of agent i according to R_i. We consider two cases.

CASE 1. $A_i \in \mathcal{S}_i$. If $A'_i \in \mathcal{S}_i$, then Lemma 3 implies that

$$(A_i, p_i(R_i, R_{-i}))\ I_i\ (A'_i, p_i(R_i, R_{-i}))\ R_i\ (A'_i, p_i(R'_i, R_{-i})).$$

If $A'_i \notin \mathcal{S}_i$, then Equation (2.2) implies that $p_i(R'_i, R_{-i}) = t_L$. But, then Inequality (2.1) implies that

$$(A_i, p_i(R_i, R_{-i}))\ R_i\ (\emptyset, t_L)\ I_i\ (A'_i, t_L).$$

CASE 2. $A_i \notin \mathcal{S}_i$. By Equation 2.2, $p_i(R_i, R_{-i}) = t_L$. Now, note that since $A_i \notin \mathcal{S}_i$, we have $WP(A_i, t_L; R_i) = 0$, and hence,

$$\sum_{j\in N} WP(A_j, t_L; R_j) = \max_{\hat{A}\in\mathcal{X}} \sum_{j\neq i} WP(\hat{A}_j, t_L; R_j).$$

This implies that

$$\sum_{j\in N} WP(A'_j, t_L; R_j) \leq \sum_{j\in N} WP(A_j, t_L; R_j) = \max_{\hat{A}\in\mathcal{X}} \sum_{j\neq i} WP(\hat{A}_j, t_L; R_j),$$

where the first inequality followed from the definition of A. This implies that

$$WP(A'_i, t_L; R_i) \leq \max_{\hat{A}\in\mathcal{X}} \sum_{j\neq i} WP(\hat{A}_j, t_L; R_j) - \sum_{j\neq i} WP(A'_j, t_L; R_j) = p_i(R'_i, R_{-i}) - t_L.$$

This further implies that

$$\Big(A_i, p_i(R_i, R_{-i})\Big)\ I_i\ (\emptyset, t_L)\ I_i\ \Big(A'_i, WP(A'_i, t_L; R_i) + t_L\Big)\ R_i\ \Big(A'_i, p_i(R'_i, R_{-i})\Big).$$

Hence, in both cases, we see that agent i prefers his outcome $(A_i, p_i(R_i, R_{-i}))$ in the GVCG mechanism to the outcome obtained by reporting R'_i. This concludes the proof that the GVCG-t_L is strategy-proof.

Proofs of (2) and (3). By Inequality (2.1), for every $i \in N$ and for every R, we have $\Big(f_i(R), p_i(R)\Big)\ R_i\ (\emptyset, t_L)$. If $t_L \leq 0$, we get that $\Big(f_i(R), p_i(R)\Big)\ R_i\ (\emptyset, 0)$, which is individual rationality. (3) follows from (2).

Proof of (4). We now show that for $n = 2$, the GVCG-t_L mechanism (for any $t_L \in \mathbb{R}$) is Pareto efficient in any dichotomous domain. Let $N = \{1, 2\}$ and consider a preference profile $R \equiv (R_1, R_2)$ with $\mathcal{S}_1$ and $\mathcal{S}_2$ as the collection of acceptable bundles of agents 1 and 2 respectively. We consider two cases. As before, denote by $(f, \mathbf{p}) \equiv (f, \mathbf{p}^{vcg,t_L})$.

Case 1. There exists $S_1 \in \mathcal{S}_1$ and $S_2 \in \mathcal{S}_2$ such that $S_1 \cap S_2 = \emptyset$. Then, $f_1(R) \in \mathcal{S}_1$ and $f_2(R) \in \mathcal{S}_2$ and $p_1(R) = p_2(R) = t_L$. Denote $A_1^* := f_1(R)$ and $A_2^* := f_2(R)$. Assume for contradiction that there is an outcome profile $((A_1, p_1), (A_2, p_2))$ such that $p_1 + p_2 \geq 2t_L$, $(A_1, p_1)\ R_1\ (A_1^*, t_L)$, and $(A_2, p_2)\ R_2\ (A_2^*, t_L)$ with strict inequality holding for one of them. By the last two relations, it must be that $p_1 \leq t_L$ and $p_2 \leq t_L$ with strict inequality holding whenever these relations are strict, which means that $p_1 + p_2 \leq 2t_L$. But this means

$p_1 + p_2 = 2t_L$ since we assumed $p_1 + p_2 \geq 2t_L$. Hence, none of the relations can hold strict, a contradiction.

CASE 2. For every $S_1 \in \mathcal{S}_1$ and for every $S_2 \in \mathcal{S}_2$, we have $S_1 \cap S_2 \neq \emptyset$. Then, one of the agents in $\{1, 2\}$ will be assigned an acceptable bundle in f. Let this agent be 1. Hence, $f_1(R) \in \mathcal{S}_1$ and $f_2(R) = \emptyset$. Further, $p_1(R) = w_2(t_L) + t_L$, where $w_2(t_L)$ is the willingness to pay of agent 2 at t_L, and $p_2(R) = t_L$.

Denote $A_1^* := f_1(R)$ and assume for contradiction that there is an outcome profile $((A_1, p_1), (A_2, p_2))$ such that $p_1 + p_2 \geq w_2(t_L) + 2t_L$, $(A_1, p_1)\ R_1\ (A_1^*, w_2(t_L) + t_L)$, and $(A_2, p_2)\ R_2\ (\emptyset, t_L)$ with strict inequality holding for one of them. Consider the following two subcases - by our assumption that for every $S_1 \in \mathcal{S}_1$ and for every $S_2 \in \mathcal{S}_2$, we have $S_1 \cap S_2 \neq \emptyset$, only the following two subcases may happen.

- CASE 2A. Suppose $A_1 \in \mathcal{S}_1$ and $A_2 \notin \mathcal{S}_2$. Since $(A_1, p_1)\ R_1\ (A_1^*, w_2(t_L) + t_L)$ and $(A_2, p_2)\ R_2\ (\emptyset, t_L)$, we have $p_1 \leq w_2(t_L) + t_L$ and $p_2 \leq t_L$. Hence, we have $p_1 + p_2 \leq w_2(t_L) + 2t_L$.
- CASE 2B. Suppose $A_1 \notin \mathcal{S}_1$ and $A_2 \in \mathcal{S}_2$. Inequality (2.1) implies $(A_1, p_1)\ R_1\ (A_1^*, w_2(t_L) + t_L)\ R_1\ (\emptyset, t_L)$. Hence, $p_1 \leq t_L$. Similarly, Inequality (2.1) for agent 2 implies that $p_2 \leq w_2(t_L) + t_L$. Hence, again we have $p_1 + p_2 \leq w_2(t_L) + 2t_L$.

Both the cases imply that $p_1 + p_2 \leq w_2(t_L) + 2t_L$ with strict inequality holding if

$$(A_1, p_1)\ P_1\ \Big(A_1^*, w_2(t_L) + t_L\Big) \text{ or } (A_2, p_2)\ P_2\ (\emptyset, t_L).$$

But we are given that $p_1 + p_2 > w_2(t_L) + 2t_L$ or $(A_1, p_1)\ P_1\ (A_1^*, w_2(t_L))$ or $(A_2, p_2)\ P_2\ (\emptyset, t_L)$. This is a contradiction.

PROOF OF (5). We show the impossibility for $N = \{1, 2, 3\}$ and $M = \{a, b\}$. The impossibility can be extended easily to the case when $n > 3$ and $m > 2$ by (i) considering preference profiles where each agent i has minimal acceptable bundle set $\mathcal{S}_i^{min} \subseteq \{a, b\}$ and

(ii) every agent $i \notin \{1,2,3\}$ has arbitrarily small willingness to pay (at every transfer level) on acceptable bundles. This is similar as in the proof of Theorem 1.

Fix the GVCG-t_L mechanism for some $t_L \in \mathbb{R}$ and denote it as $(f, \mathbf{p}) \equiv (f^{vcg,t_L}, \mathbf{p}^{vcg,t_L})$. Consider the following single-minded preference profile (R_1, R_2, R_3) such that

$$\mathcal{S}_1^{min} = \{a\}, \mathcal{S}_2^{min} = \{b\}, \mathcal{S}_3^{min} = \{a,b\}.$$

The WP values at transfer level t_L are as follows:

$$WP(\{a\}, t_L; R_1) = w_1; WP(\{b\}, t_L; R_2) = w_2; WP(\{a,b\}, t_L; R_3) = w_3,$$

such that $w_1 + w_2 > w_3 > \max(w_1, w_2)$. Further, we require R_1 and R_2 to satisfy the following:

$$\big(\{a\}, w_3 - w_2 + t_L\big)\ I_1\ \big(\emptyset, t_L - \epsilon\big) \text{ and} \big(\{b\}, w_3 - w_1 + t_L\big)\ I_2\ \big(\emptyset, t_L - \epsilon\big).$$

Such dichotomous preferences R_1, R_2, R_3 are possible to construct. Figure 2.5 illustrates the some indifference vectors of R_1, R_2, and R_3.

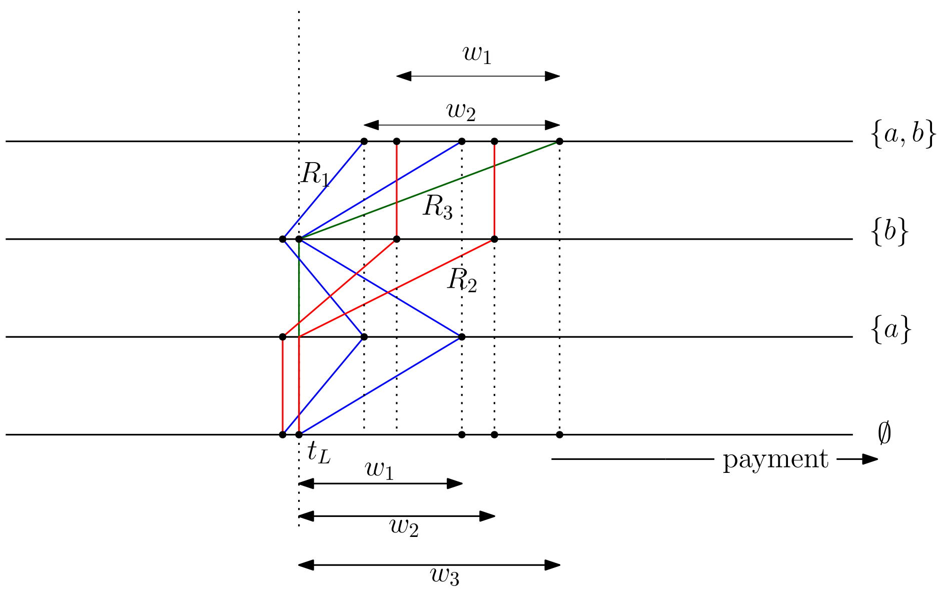


Figure 2.5: A profile of dichotomous preferences for $N = \{1,2,3\}$ and $M = \{a,b\}$.

Hence, the GVCG-t_L mechanism produces the following outcome:

$$f_1(R_1,R_2,R_3) = \{a\}, \quad f_2(R_1,R_2,R_3) = \{b\}, \quad f_3(R_1,R_2,R_3) = \emptyset;$$
$$p_1(R_1,R_2,R_3) = w_3 - w_2 + t_L, \quad p_2(R_1,R_2,R_3) = w_3 - w_1 + t_L, \quad p_3(R_1,R_2,R_3) = t_L.$$

Consider the following outcome profile

$$z_1 := (\emptyset, t_L - \epsilon); z_2 := (\emptyset, t_L - \epsilon); z_3 := (\{a,b\}, w_3 + t_L).$$

By construction (see Figure 2.5), each agent $i \in \{1,2,3\}$ is indifferent between z_i and $(f_i(R), p_i(R))$. Total transfers in the outcome profile z is: $w_3 + 3t_L - 2\epsilon$. Total transfers in the GVCG-t_L mechanism: $2w_3 - (w_1+w_2) + 3t_L < w_3 + 3t_L - \epsilon$, where the inequality follows since $w_3 < w_1 + w_2$ and $\epsilon > 0$ is arbitrarily close to zero. Hence, the GVCG-t_L mechanism is not Pareto efficient. ■

2.5.3 Proof of Theorem 2

Proof: By Proposition 1, the GVCG mechanism is DSIC, individually rational, and satisfies no subsidy. Now, we prove Pareto efficiency. Let $\mathcal{T}$ be a dichotomous domain satisfying positive income effect. Assume for contradiction that there exists a profile $R \in \mathcal{T}^n$ such that $(f^{vcg}(R), \mathbf{p}^{vcg}(R))$ is not Pareto efficient. As before, let $(\mathcal{S}_i, w_i)$ denote the dichotomous preference R_i of any agent i. Let $f^{vcg}(R) \equiv A$ and $\mathbf{p}^{vcg}(R) \equiv (p_1, \ldots, p_n)$. Then there exists, an outcome profile $((A'_1, p'_1), \ldots, (A'_n, p'_n))$ which Pareto dominates $((A_1, p_1), \ldots, (A_n, p_n))$.

We consider various cases to derive relationship between p_i and p'_i for each $i \in N$.

Case 1. Pick $i \in N$ such that $A_i, A'_i \in \mathcal{S}_i$ or $A_i, A'_i \notin \mathcal{S}_i$. Dichotomous preference implies that $(A'_i, p'_i)\ I_i\ (A_i, p'_i)$. But $(A'_i, p'_i)\ R_i\ (A_i, p_i)$ implies that $(A_i, p'_i)\ R_i\ (A_i, p_i)$. Hence, we get

$$p_i \geq p'_i \ \forall\ i \text{ such that } A_i, A'_i \in \mathcal{S}_i \text{ or } A_i, A'_i \notin \mathcal{S}_i. \tag{2.3}$$

Case 2. Pick $i \in N$ such that $A_i \notin \mathcal{S}_i$ but $A'_i \in \mathcal{S}_i$. This implies that $p_i = 0$ (by Lemma 1).

Hence, $(A_i', p_i')\ R_i\ (A_i, p_i)\ I_i\ (A_i, 0)\ I_i\ (\emptyset, 0)\ I_i\ (A_i', w_i(0))$. Thus,

$$w_i(0) + p_i \geq p_i' \ \forall\ i \text{ such that } A_i \notin \mathcal{S}_i, A_i' \in \mathcal{S}_i. \tag{2.4}$$

Case 3. Pick $i \in N$ such that $A_i \in \mathcal{S}_i$ but $A_i' \notin \mathcal{S}_i$. Since $A_i' \notin \mathcal{S}_i$, we can write $(A_i', p_i')\ I_i\ (\emptyset, p_i')\ I_i\ (A_i, p_i' + w_i(p_i'))$. But $(A_i', p_i')\ R_i\ (A_i, p_i)$ implies that

$$p_i \geq p_i' + w_i(p_i').$$

Also, $(\emptyset, p_i')\ I_i\ (A_i', p_i')\ R_i\ (A_i, p_i)\ R_i\ (\emptyset, 0)$, where the last inequality is due to individual rationality of the GVCG mechanism. Hence, $p_i' \leq 0$. But then, positive income effect implies that $w_i(p_i') \geq w_i(0)$. This gives us

$$p_i \geq p_i' + w_i(0) \ \forall\ i \text{ such that } A_i \in \mathcal{S}_i, A_i' \notin \mathcal{S}_i. \tag{2.5}$$

By summing over Inequalities 2.3, 2.4, and 2.5, we get

$$\begin{aligned}
\sum_{i \in N} p_i &\geq \sum_{i \in N} p_i' + \sum_{i: A_i \in \mathcal{S}_i, A_i' \notin \mathcal{S}_i} w_i(0) - \sum_{i: A_i \notin \mathcal{S}_i, A_i' \in \mathcal{S}_i} w_i(0). \\
&= \sum_{i \in N} p_i' + \sum_{i: A_i \in \mathcal{S}_i, A_i' \notin \mathcal{S}_i} w_i(0) + \sum_{i: A_i, A_i' \in \mathcal{S}_i} w_i(0) - \sum_{i: A_i, A_i' \in \mathcal{S}_i} w_i(0) - \sum_{i: A_i \notin \mathcal{S}_i, A_i' \in \mathcal{S}_i} w_i(0). \\
&= \sum_{i \in N} p_i' + \sum_{i \in N} WP(A_i, 0; R_i) - \sum_{i \in N} WP(A_i', 0; R_i) \\
&\geq \sum_{i \in N} p_i',
\end{aligned}$$

where the inequality follows from the definition of the GVCG mechanism. Also, note that the inequality above is strict if any of the Inequalities 2.3, 2.4, and 2.5 is strict. This contradicts the fact that the outcome $((A_1', p_1'), \ldots, (A_n', p_n'))$ Pareto dominates $((A_1, p_1), \ldots, (A_n, p_n))$. ■

2.5.4 Proof of Theorem 3

Proof: Let $(f, \mathbf{p})$ be a Pareto efficient, DSIC, IR mechanism satisfying no subsidy. The proof proceeds in two steps. We assume without loss of generality that at every preference profile R, if an agent $i \in N$ is assigned an acceptable bundle $f_i(R)$, then $f_i(R)$ is a *minimal* acceptable bundle at R_i, i.e., there does not exist another acceptable bundle $S_i \subsetneq f_i(R)$ at R_i. [12] We now proceed with the proof in two Steps.

Allocation is GVCG allocation. In this step, we argue that f must satisfy:

$$f(R) \in \arg\max_{A \in \mathcal{X}} \sum_{i \in N} WP(A_i, 0; R_i) \ \forall\ R \in \mathcal{T}^n$$

Assume for contradiction that for some $R \in \mathcal{T}^n$, we have

$$\sum_{i \in N} WP(f_i(R), 0; R_i) < \max_{A \in \mathcal{X}} \sum_{i \in N} WP(A_i, 0; R_i).$$

Before proceeding with the rest of the proof, we fix a generalized VCG mechanism (f^{vcg}, p^{vcg}) and introduce a notation. For every R', denote by

$$N_{0+}(R') := \Big\{ i \in N : \big[(f_i^{vcg}(R'), p_i^{vcg}(R'))\ I_i'\ (\emptyset, 0) \big] \text{ and } \big[(f_i(R'), p_i(R'))\ P_i'\ (\emptyset, 0) \big] \Big\}.$$

We now construct a sequence of preference profiles, starting with preference profile R, as follows. Let $R^0 := R$. Also, we will maintain a sequence of subsets of agents, which is initialized as $B^0 := \emptyset$. We will denote the preference profile constructed in step t of the sequence as R^t and the willingness to pay map at preference R_i^t as w_i^t for each $i \in N$.

S1. If $N_{0+}(R^t) \setminus B^t = \emptyset$, then stop. Else, go to the next step.

[12] This is without loss of generality for the following reason. For every Pareto efficient, DSIC, IR mechanism $(f, \mathbf{p})$ satisfying no subsidy, we can construct another mechanism $(f', \mathbf{p}')$ such that: for all R and for all $i \in N$, $f_i'(R) \subseteq f_i(R)$ and $f_i'(R)$ is a minimal acceptable bundle at R_i whenever $f_i(R)$ is an acceptable bundle at R_i and $f_i'(R) = f_i(R)$ otherwise. Further, $\mathbf{p}' = \mathbf{p}$. It is routine to verify that $(f', \mathbf{p}')$ is DSIC, IR, Pareto efficient and satisfies no subsidy. Finally, by construction, if $(f', \mathbf{p}')$ is a generalized VCG mechanism, then $(f, \mathbf{p})$ is also a generalized VCG mechanism.

S2. Choose $k^t \in N_{0+}(R^t) \setminus B^t$ and consider $R_{k^t}^{t+1}$ to be a quasilinear dichotomous preference with valuation $w_{k^t}^{t+1}(0) \in (p_{k^t}(R^t), w_{k^t}^t(0))$ and a unique minimal acceptable bundle $f_{k^t}(R^t)$ - such a quasilinear preference exists because $\mathcal{T} \supseteq \mathcal{D}^{QL}$. Let $R_j^{t+1} = R_j^t$ for all $j \neq k^t$.

S3. Set $B^{t+1} := B^t \cup \{k^t\}$ and $t := t+1$. Repeat from Step S1.

Because of finiteness of number of agents, this process will terminate finitely in some $T < \infty$ steps. We establish some claims about the preference profiles generated in this procedure.

Claim 2 *For every $t \in \{0, \ldots, T-1\}$, $f_{k^t}(R^{t+1}) = f_{k^t}(R^t)$ and $p_{k^t}(R^{t+1}) = p_{k^t}(R^t)$.*

Proof: Fix t and assume for contradiction $f_{k^t}(R^{t+1}) \neq f_{k^t}(R^t)$. Since $f_{k^t}(R^t)$ is the unique minimal acceptable bundle at $R_{k^t}^{t+1}$ and f only assigns a minimal acceptable bundle whenever it assigns acceptable bundles, it must be that $f_{k^t}(R^{t+1})$ is not an acceptable bundle at $R_{k^t}^{t+1}$. Then, by Lemma 1, we get $p_{k^t}(R^{t+1}) = 0$. Since $w_{k^t}^{t+1}(0) > p_{k^t}(R^t)$ and $f_{k^t}(R^t)$ is an acceptable bundle at $R_{k^t}^{t+1}$, we get

$$(f_{k^t}(R^t), p_{k^t}(R^t)) \ P_{k^t}^{t+1} \ (\emptyset, 0) \ I_{k^t}^{t+1}(f_{k^t}(R^{t+1}), p_{k^t}(R^{t+1})).$$

This contradicts DSIC. Finally, if $f_{k^t}(R^{t+1}) = f_{k^t}(R^t)$, we must have $p_{k^t}(R^{t+1}) = p_{k^t}(R^t)$ due to DSIC since acceptable bundle at $R_{k^t}^{t+1}$ is $f_{k^t}(R^t)$ and $f_{k^t}(R^t)$ is also an acceptable bundle at $R_{k^t}^t$. ■

The next claim establishes a useful inequality.

Claim 3 *For every $t \in \{0, \ldots, T\}$, the following holds:*

$$w_{k^t}^t(0) + \max_{A \in \mathcal{X}, A_{k^t} = f_{k^t}(R^t)} \sum_{j \neq k^t} WP_j(A_j, 0; R_j^t) \leq \max_{A \in \mathcal{X}} \sum_{j \neq k^t} WP_j(A_j, 0; R_j^t).$$

Proof: Pick some $t \in \{0, \ldots, T\}$ and suppose the above inequality does not hold. We complete the proof in two steps.

Step 1. In this step, we argue that $f^{vcg}_{k^t}$ must be an acceptable bundle for agent k^t at preference R^t. If this is not true, then we must have

$$\begin{aligned}
\sum_{j\in N} WP(f^{vcg}_j(R^t),0;R^t_j) &= \sum_{j\neq k^t} WP(f^{vcg}_j(R^t),0;R^t_j) \\
&\leq \max_{A\in\mathcal{X}} \sum_{j\neq k^t} WP(A_j,0;R^t_j) \\
&< w^t_{k^t}(0) + \max_{A\in\mathcal{X},A_{k^t}=f_{k^t}(R^t)} \sum_{j\neq k^t} WP(A_j,0;R^t_j) \\
&= WP(f_{k^t}(R^t),0;R^t_{k^t}) + \max_{A\in\mathcal{X},A_{k^t}=f_{k^t}(R^t)} \sum_{j\neq k^t} WP(A_j,0;R^t_j),
\end{aligned}$$

where the last inequality follows from our assumption that the claimed inequality does not hold and the last equality follows from the fact that $f_{k^t}(R^t)$ is an acceptable bundle of agent k^t at $R^t_{k^t}$. But, then the resulting inequality contradicts the definition of f^{vcg}.

Step 2. We complete the proof in this step. Notice that the payment of agent k^t in (f^{vcg},p^{vcg}) is defined as follows.

$$\begin{aligned}
p^{vcg}_{k^t}(R^t) &= \max_{A\in\mathcal{X}} \sum_{j\neq k^t} WP(A_j,0;R^t_j) - \sum_{j\neq k^t} WP(f^{vcg}_j(R^t),0;R^t_j) \\
&< w^t_{k^t}(0) + \max_{A\in\mathcal{X}:A_{k^t}=f_{k^t}(R^t)} \sum_{j\neq k^t} WP(A_j,0;R^t_j) - \sum_{j\neq k^t} WP(f^{vcg}_j(R^t),0;R^t_j) \\
&= w^t_{k^t}(0) + \max_{A\in\mathcal{X}:A_{k^t}=f_{k^t}(R^t)} \sum_{j\neq k^t} WP(A_j,0;R^t_j) \\
&- \sum_{j\in N} WP(f^{vcg}_j(R^t),0;R^t_j) + WP(f^{vcg}_{k^t}(R^t),0;R^t_{k^t}) \\
&= w^t_{k^t}(0) + \max_{A\in\mathcal{X},A_{k^t}=f_{k^t}(R^t)} \sum_{j\in N} WP(A_j,0;R^t_j) - \sum_{j\in N} WP(f^{vcg}_j(R^t),0;R^t_j) \\
&\leq w^t_{k^t}(0),
\end{aligned}$$

where the strict inequality followed from our assumption and the last equality follows from the fact both $f_{k^t}(R^t)$ and $f^{vcg}_{k^t}(R^t)$ are acceptable bundles for agent k^t at $R^t_{k^t}$ (Step 1). But, this implies that

$$(f^{vcg}_{k^t}(R^t), p^{vcg}_{k^t}(R^t)) \; P^t_{k^t} \; (f^{vcg}_{k^t}(R^t), w^t_{k^t}(0)) \; I^t_{k^t} \; (\emptyset, 0).$$

This is a contradiction to the fact that $k^t \in N_{0+}(R^t)$. This completes the proof. ■

We now establish an important claim regarding an inequality satisfied by the sequence of preferences generated.

Claim 4 *For every $t \in \{0, \ldots, T\}$,*

$$\sum_{j \in N} WP(f_j(R^t), 0; R^t_j) < \sum_{j \in N} WP(f^{vcg}_j(R^t), 0; R^t_j).$$

Proof: The inequality holds for $t = 0$ by assumption. We now use induction. Suppose the inequality holds for $t \in \{0, \ldots, \tau - 1\}$. We show that it holds for τ. To see this, denote $k \equiv k^{\tau-1}$. By Claim 2, we know that $f_k(R^{\tau-1}) = f_k(R^\tau)$. Further, by definition, $f_k(R^\tau)$ belongs to the acceptable bundle of k at R^τ_k and $R^{\tau-1}_k$. Now, observe the following:

$$\begin{aligned}
\sum_{j \in N} WP(f_j(R^\tau), 0; R^\tau_j) &= w^\tau_k(0) + \sum_{j \neq k} WP(f_j(R^\tau), 0; R^\tau_j) \qquad \text{(follows from definition of } k\text{)} \\
&\leq w^\tau_k(0) + \max_{A \in \mathcal{X}: A_k = f_k(R^{\tau-1}) = f_k(R^\tau)} \sum_{j \neq k} WP(A_j, 0; R^\tau_j) \\
&= w^\tau_k(0) + \max_{A \in \mathcal{X}: A_k = f_k(R^{\tau-1}) = f_k(R^\tau)} \sum_{j \neq k} WP(A_j, 0; R^{\tau-1}_j) \\
&\text{(using the fact that } R^\tau_j = R^{\tau-1}_j \text{ for all } j \neq k\text{)} \\
&\leq w^\tau_k(0) - w^{\tau-1}_k(0) + \max_{A \in \mathcal{X}} \sum_{j \neq k} WP(A_j, 0; R^{\tau-1}_j) \qquad \text{(using Claim 3)} \\
&< \max_{A \in \mathcal{X}} \sum_{j \neq k} WP(A_j, 0; R^{\tau-1}_j) \qquad \text{(using the fact that } w^\tau_k(0) < w^{\tau-1}_k(0)\text{)} \\
&= \max_{A \in \mathcal{X}} \sum_{j \neq k} WP(A_j, 0; R^\tau_j) \\
&\leq \max_{A \in \mathcal{X}} \sum_{j \in N} WP(A_j, 0; R^\tau_j).
\end{aligned}$$

■

We now complete our claim that the allocation is the same as in a GVCG mechanism. Let $R^T \equiv R'$. Let $f^{vcg}(R') = A^{vcg}$ and $f(R') = A'$. Partition the set of agents as follows.

$$
\begin{aligned}
N_{++} &:= \{i : WP_i(A_i^{vcg}, 0; R_i') = WP(A_i', 0; R_i') > 0\} \\
N_{+-} &:= \{i : WP_i(A_i^{vcg}, 0; R_i') > 0, WP(A_i', 0; R_i') = 0\} \\
N_{-+} &:= \{i : WP_i(A_i^{vcg}, 0; R_i') = 0, WP(A_i', 0; R_i') > 0\} \\
N_{--} &:= \{i : WP_i(A_i^{vcg}, 0; R_i') = WP(A_i', 0; R_i') = 0\}.
\end{aligned}
$$

Now, consider the following consumption bundle Z:

$$
Z_i := \begin{cases} (A_i^{vcg}, p_i(R')) & \text{if } i \in N_{++} \cup N_{--} \\ (A_i^{vcg}, p_i(R') - WP(A_i', 0; R_i')) & \text{if } i \in N_{-+} \\ (A_i^{vcg}, WP(A_i^{vcg}, 0; R_i')) & \text{if } i \in N_{+-} \end{cases}
$$

Notice that for each $i \in N_{++} \cup N_{--}$, we have $Z_i = (A_i^{vcg}, p_i(R'))\ I_i'\ (A_i', p_i(R'))$. For each $i \in N_{+-}$, we know that $WP(A_i', 0; R_i') = 0$ - this implies that A_i' is not an acceptable bundle at R_i'. Hence, for all $i \in N_{+-}$, we have $Z_i = (A_i^{vcg}, WP(A_i^{vcg}, 0; R_i'))\ I_i'\ (\emptyset, 0)\ I_i'\ (A_i', p_i(R'))$, where the last relation follows from Lemma 1. Finally, for all $i \in N_{-+}$, $WP_i(A_i^{vcg}, 0; R_i') = 0$ implies that $(A_i^{vcg}, p_i^{vcg}(R'))\ I_i'\ (\emptyset, 0)$. Then, for every $i \in N_{-+}$, either we have $(A_i', p_i(R'))\ I_i'\ (\emptyset, 0)$ or we have $i \in B^T$ (i.e., R_i' is a quasilinear preference). In the first case, $p_i(R') = WP(A_i', 0; R_i')$ implies

$$
(A_i^{vcg}, p_i(R') - WP(A_i', 0; R_i'))\ I_i'\ (A_i^{vcg}, 0)\ I_i'\ (\emptyset, 0)\ I_i'\ (A_i', p_i(R')).
$$

In the second case, quasilinearity of R_i' implies $(A_i^{vcg}, p_i(R') - WP(A_i', 0; R_i'))\ I_i'\ (A_i', p_i(R'))$. This completes the argument that $Z_i\ R_i'\ (A_i', p_i(R'))$ for every $i \in N$.

Now, observe the sum of payments across all agents in Z is:

$$\begin{aligned}
&\sum_{i \notin N_{+-}} p_i(R') - \sum_{i \in N_{-+}} WP(A'_i, 0; R'_i) + \sum_{i \in N_{+-}} WP(A_i^{vcg}, 0; R'_i) \\
&= \sum_{i \in N} p_i(R') - \sum_{i \in N_{-+}} WP(A'_i, 0; R'_i) + \sum_{i \in N_{+-}} WP(A_i^{vcg}, 0; R'_i) \\
&\text{(since } A'_i \text{ is not acceptable, Lemma 1 implies } p_i(R') = 0 \text{ for all } i \in N_{+-}) \\
&= \sum_{i \in N} p_i(R') + \sum_{i \in N} WP(A_i^{vcg}, 0; R'_i) - \sum_{i \in N} WP(A'_i, 0; R'_i) \\
&> \sum_{i \in N} p_i(R'),
\end{aligned}$$

where the last inequality follows from Claim 4.

Hence, Z Pareto dominates the outcome $(f(R'), p(R'))$, contradicting Pareto efficiency. We now proceed to the next step to show that the payment in $(f, \mathbf{p})$ must also coincide with the generalized VCG outcome.

Payment is GVCG payment. Fix a preference profile R. We now know that

$$f(R) \in \arg\max_{A \in \mathcal{X}} \sum_{i \in N} WP(A_i, 0; R_i).$$

By Lemma 1, for every $i \in N$, if $f_i(R) = f_i^{vcg}(R)$ is not acceptable for agent i, then $p_i(R) = p_i^{vcg}(R) = 0$ - here, we assume, without loss of generality, that $f(R') = f^{vcg}(R')$ for all R'.[13] We now consider two cases.

Case 1. Assume for contradiction that there exists $i \in N$ such that $f_i(R)$ is an acceptable bundle of agent i and

$$p_i(R) > \max_{A \in \mathcal{X}} \sum_{j \neq i} WP(A_j, 0; R_j) - \sum_{j \neq i} WP(f_j(R), 0; R_j). \tag{2.6}$$

Now consider R'_i with the set of acceptable bundles the same in R_i and R'_i but $WP(f_i(R), 0; R'_i) <$

[13]Depending on how we break ties for choosing a maximum in the maximization of sum of willingness to pay, we have a different generalized VCG mechanism. This assumption ensures that we pick the generalized VCG mechanism that breaks the ties the same way as f.

$p_i(R)$ but arbitrarily close to $p_i(R)$. Let $A' \equiv f(R'_i, R_{-i})$. We argue that A'_i is an acceptable bundle (at R'_i). If not, then

$$\max_{A \in \mathcal{X}} \sum_{j \neq i} WP(A_j, 0; R_j) \geq \sum_{j \neq i} WP(A'_j, 0; R_j) = WP(A'_i, 0; R'_i) + \sum_{j \neq i} WP(A'_j, 0, R_j),$$

where we used the fact that A'_i is not an acceptable bundle for i. But then, by construction of R'_i and Inequality (2.6), we get

$$WP(f_i(R), 0; R'_i) + \sum_{j \neq i} WP(f_j(R), 0; R_j) > \max_{A \in \mathcal{X}} \sum_{j \neq i} WP(A_j, 0; R_j) \geq WP(A'_i, 0; R'_i) + \sum_{j \neq i} WP(A'_j, 0, R_j),$$

which is a contradiction to our earlier step that f is the same allocation as in the GVCG mechanism. Hence, A'_i is an acceptable bundle at R'_i. But, then $p_i(R) = p_i(R'_i, R_{-i})$ by DSIC (since $f_i(R)$ is also an acceptable bundle at R_i and the set of acceptable bundles at R_i and R'_i are the same). Since $WP(A'_i, 0; R'_i) < p_i(R) = p_i(R'_i, R_{-i})$, we get a contradiction to individual rationality.

Case 2. Assume for contradiction that there exists $i \in N$ such that $f_i(R)$ is an acceptable bundle of agent i and

$$p_i(R) < p_i^{vcg}(R) = \max_{A \in \mathcal{X}} \sum_{j \neq i} WP(A_j, 0; R_j) - \sum_{j \neq i} WP(f_j(R), 0; R_j).$$

Pick R'_i such that the set of acceptable bundles at R'_i and R_i are the same but $WP(f_i(R), 0; R'_i) \in (p_i(R), p_i^{vcg}(R))$. Notice that if $f_i(R'_i, R_{-i})$ is not an acceptable bundle at R'_i, then his payment is zero (Lemma 1). In that case, $WP(f_i(R), 0; R'_i) > p_i(R)$ implies that

$$(f_i(R), p_i(R)) \ P'_i \ (\emptyset, 0) \ I'_i \ (f_i(R'_i, R_{-i}), p_i(R'_i, R_{-i})),$$

contradicting DSIC. Hence, $f_i(R'_i, R_{-i}) = f_i^{vcg}(R'_i, R_{-i})$ is an acceptable bundle at R'_i. This implies that $f_i^{vcg}(R'_i, R_{-i})$ is an acceptable bundle at R'_i. Since the generalized VCG is DSIC, we get that $p_i^{vcg}(R) = p_i^{vcg}(R'_i, R_{-i})$. But $WP(f_i^{vcg}(R'_i, R_{-i}), 0; R'_i) < p_i^{vcg}(R) = p_i^{vcg}(R'_i, R_{-i})$ is a contradiction to IR of the generalized VCG. This completes the proof. ■

2.5.5 Proof of Theorem 4

Proof: Assume for contradiction that $(f, \mathbf{p})$ is a desirable mechanism on $\mathcal{T}^n$. By heterogeneous demand, there exist objects a and b such that $0 < WP(a, 0; R_0) < WP(b, 0; R_0)$. Consider a preference profile $R \in \mathcal{T}^n$ as follows:

1. Agent 1 has quasilinear dichotomous preference with $\mathcal{S}_i^{min} = \{\{a, b\}\}$ and value $w_1(0)$ that satisfies

$$WP(\{a, b\}, 0; R_0) < w_1(0) < WP(\{a\}, 0; R_0) + WP(\{b\}, 0; R_0). \tag{2.7}$$

2. $R_i = R_0$ for all $i \in \{2, 3\}$.

3. If $m > 2$, agent 4 has quasilinear dichotomous preference with acceptable bundle $M \setminus \{a, b\}$ and value *very high*. If $m = 2$, agent 4 has quasilinear dichotomous preference with acceptable bundle M and value equals to ϵ, which is very close to zero.

4. For all $i > 4$, let R_i be a quasilinear dichotomous preference with $\mathcal{S}_i^{min} = \{M\}$ and value equals to ϵ, which is very close to zero.

We begin by a useful claim.

Claim 5 *Pick $k \in \{2, 3\}$ and $x \in \{a, b\}$. Let R' be a preference profile such that $R'_i = R_i$ for all $i \neq k$. Suppose R'_k is such that*

$$WP(\{x\}, 0; R'_k) + WP(\{a, b\} \setminus \{x\}, 0; R_0) > w_1(0) > WP(\{a, b\}, 0; R'_k). \tag{2.8}$$

Then, the following are true:

1. $f_1(R') = \emptyset$
2. $f_2(R') \cup f_3(R') = \{a, b\}$
3. $f_2(R') \neq \emptyset$ *and* $f_3(R') \neq \emptyset$.

Proof: It is without loss of generality (due to Pareto efficiency) that $f_i(R') = \emptyset$ or $f_i(R') \in \mathcal{S}_i^{min}$ for all i who has dichotomous preference. Since ϵ is very close to zero, Pareto efficiency implies that (a) if $m = 2$, $f_i(R') = \emptyset$ for all $i > 3$; and (b) if $m > 2$, since agent 4 has very high value for $M \setminus \{a, b\}$, $f_4(R') = M \setminus \{a, b\}$ and $f_i(R') = \emptyset$ for all $i > 4$. Hence, agents $1, 2$, and 3 will be allocated $\{a, b\}$ at R'. Denote $y \equiv \{a, b\} \setminus \{x\}$ and $\ell \equiv \{2, 3\} \setminus \{k\}$.

Proof of (1) and (2). Assume for contradiction $f_1(R') \neq \emptyset$. Pareto efficiency implies that $f_1(R') = \{a, b\}$ and $f_2(R') = f_3(R') = \emptyset$. Lemma 1 implies that $p_2(R') = p_3(R') = 0$. Then, consider the following outcome:

$$z_1 := \Big(\emptyset, p_1(R') - w_1(0)\Big),\ z_k := \Big(\{x\}, WP(\{x\}, 0; R'_k)\Big),\ z_\ell := \Big(\{y\}, WP(\{y\}, 0; R'_\ell)\Big),$$

$$z_i := \Big(f_i(R'), p_i(R')\Big)\ \forall\ i > 3.$$

By definition of willingness to pay, $z_i\ I_i\ (\emptyset, 0) \equiv \Big(f_i(R'), p_i(R')\Big)$ for all $i \in \{2, 3\}$. Since agent 1 has quasilinear preferences, she is also indifferent between z_1 and $\Big(\{a, b\}, p_1(R')\Big) \equiv \Big(f_1(R'), p_1(R')\Big)$. Thus, the difference in total payment between the outcome z and the payment in $(f, \mathbf{p})$ at R' is

$$WP(\{x\}, 0; R'_k) + WP(\{y\}, 0; R'_\ell) - w_1(0) = WP(\{x\}, 0; R'_k) + WP(\{y\}, 0; R_0) - w_1(0) > 0,$$

where the inequality follows from Inequality (2.8). This is a contradiction to Pareto efficiency of $(f, \mathbf{p})$. Hence, $f_1(R) = \emptyset$. By Pareto efficiency, $f_2(R') \cup f_3(R') = \{a, b\}$.

Proof of (3). Now, we show that $f_2(R') \neq \emptyset$ and $f_3(R') \neq \emptyset$. Suppose $f_3(R') = \emptyset$. Then, $f_2(R') = \{a, b\}$ and Lemma 1 implies that $p_3(R') = 0$. We first argue that $p_2(R') = WP(\{a, b\}, 0; R'_2)$. To see this, consider a quasilinear dichotomous preference $\tilde{R}_2$ with acceptable bundle $\{a, b\}$ and value equal to $WP(\{a, b\}, 0; R'_2)$. Notice that $w_1(0) > WP(\{a, b\}, 0; R'_2)$ - if $k = 2$, then this is true by Inequality (2.8) and if $\ell = 2$, then $R'_\ell = R_0$ satisfies $w_1(0) > WP(\{a, b\}, 0; R_0)$ by Inequality (2.7). Since agents 1 and 2 have the same acceptable bundle at $(\tilde{R}_2, R'_{-2})$ but $w_1(0) > WP(\{a, b\}, 0; R'_2)$, this implies that (due to Pareto efficiency), $f_2(\tilde{R}_2, R'_{-2}) = \emptyset$ and $p_2(\tilde{R}_2, R'_{-2}) = 0$ (Lemma 1). By DSIC, $(\emptyset, 0)\ \tilde{R}_2\ (\{a, b\}, p_2(R'))$. This implies that $WP(\{a, b\}, 0; R'_2) \leq p_2(R')$. IR of agent 2 at R' implies $WP(\{a, b\}, 0; R'_2) = p_2(R')$.

Next, consider the following outcome

$$z'_k := (\{x\}, WP(\{x\}, 0; R'_k)), z'_\ell := (\{y\}, WP(\{y\}, 0; R'_\ell)), z'_i := (f_i(R'), p_i(R')) \ \forall\ i \notin \{2, 3\}.$$

By definition, for every agent i, $z'_i \ I'_i \ (f_i(R'), p_i(R'))$. The difference between the sum of payments of agents in z' and $(f, \mathbf{p})$ at R is:

$$\begin{aligned} WP(\{x\}, 0; R'_k) + WP(\{y\}, 0; R'_\ell) - p_2(R') &= WP(\{x\}, 0; R'_k) + WP(\{y\}, 0; R_0) - WP(\{a, b\}, 0; R'_2) \\ &> w_1(0) - WP(\{a, b\}, 0; R'_2) \\ &> 0, \end{aligned}$$

where the first inequality follows from Inequality (2.8) and the second inequality follows from Inequality (2.8) if $k = 2$ and from Inequality (2.7) if $\ell = 2$. This contradicts Pareto efficiency of $(f, \mathbf{p})$. A similar proof shows that $f_2(R') \neq \emptyset$. ■

Now, pick any $k \in \{2, 3\}$ and set $R'_k = R_0$ in Claim 5. By Inequality (2.7), Inequality (2.8) holds for R_0. As a result, we get that $f_2(R) \neq \emptyset$, $f_3(R) \neq \emptyset$, and $f_2(R) \cup f_3(R) = \{a, b\}$. Hence, without loss of generality, assume that $f_2(R) = \{a\}$ and $f_3(R) = \{b\}$.[14] We now complete the proof in two steps.

STEP 1. We argue that $p_2(R) = w_1(0) - WP(\{b\}, 0; R_0)$ and $p_3(R) = w_1(0) - WP(\{a\}, 0; R_0)$. Suppose $p_2(R) > w_1(0) - WP(\{b\}, 0; R_0)$. Then, consider the quasilinear dichotomous preference R_2^Q such that the minimum acceptable bundle of agent 2 is $\{a\}$ and his value v satisfies

$$w_1(0) - WP(\{b\}, 0; R_0) < v < p_2(R). \tag{2.9}$$

Now, note that by IR of agent 2 at R, we have

$$p_2(R) \leq WP(\{a\}, 0; R_0) \leq WP(\{a, b\}, 0; R_0) < w_1(0),$$

[14]Since we have assumed $WP(\{b\}, 0; R_0) > WP(\{a\}, 0; R_0)$, this may appear to be with loss of generality. However, if we have $f_2(R) = \{b\}$ and $f_3(R) = \{a\}$, then we will swap 2 and 3 in the entire argument following this.

where the strict inequality followed from Inequality (2.7). Hence, $v < w_1(0)$ and $w_1(0) < v + WP(\{b\}, 0; R_0)$ by Inequality (2.9). Hence, choosing $k = 2$, $x = a$ and $R'_k = R_2^Q$, we can apply Claim 5 to conclude that $f_2(R_2^Q, R_{-2}) \cup f_3(R_2^Q, R_{-2}) = \{a, b\}$ and $f_2(R_2^Q, R_{-2}) \neq \emptyset$, $f_3(R_2^Q, R_{-2}) \neq \emptyset$. Since R_2^Q is a dichotomous preference with acceptable bundle $\{a\}$, Pareto efficiency implies that $f_2(R_2^Q) = \{a\} = f_2(R)$. By DSIC, $p_2(R) = p_2(R_2^Q, R_{-2})$. But Inequality (2.9) gives $v < p_2(R) = p_2(R_2^Q, R_{-2})$, and this contradicts individual rationality.

Next, suppose $p_2(R) < w_1(0) - WP(\{b\}, 0; R_0)$. Then, consider the quasilinear dichotomous preference $\hat{R}_2^Q$ such that the minimal acceptable bundle of agent 2 is $\{a\}$ and his value $\hat{v}$ satisfies

$$p_2(R) < \hat{v} < w_1(0) - WP(\{b\}, 0; R_0). \tag{2.10}$$

Now, consider the preference profile $\hat{R}$ such that $\hat{R}_2 = \hat{R}_2^Q$ and $\hat{R}_i = R_i$ for all $i \neq 2$. We first argue that $f_2(\hat{R}) = \emptyset$. Suppose not, then by Pareto efficiency, $f_2(\hat{R}) = \{a\}$. By Pareto efficiency, we have $f_3(\hat{R}) = \{b\}$ and $f_1(\hat{R}) = \emptyset$. By Lemma 1, $p_1(\hat{R}) = 0$. We argue that $p_3(\hat{R}) = WP(\{b\}, 0; R_0)$. To see this, consider a profile $\hat{R}'$ where $\hat{R}'_i = \hat{R}_i$ for all $i \neq 3$ and $\hat{R}'_3$ is a quasilinear dichotomous preferences with minimum acceptable bundle $\{b\}$ and value equal to $WP(\{b\}, 0; R_0)$ - notice that every agent in $\hat{R}'$ has quasilinear preference. As a result, Theorem 3 implies that the outcome of $(f, \mathbf{p})$ at $\hat{R}'$ must coincide with the GVCG mechanism. But $w_1(0) > \hat{v} + WP(\{b\}, 0; R_0)$ implies that $f_1(\hat{R}') = \{a, b\}$ and $f_2(\hat{R}') = f_3(\hat{R}') = \emptyset$. Then, DSIC implies that (incentive constraint of agent 3 from $\hat{R}'$ to $\hat{R}$) $0 \geq WP(\{b\}, 0; R_0) - p_3(\hat{R})$. By individual rationality of agent 3 at $\hat{R}$ we get, $p_3(\hat{R}) \leq WP(\{b\}, 0; R_0)$, and combining these we get $p_3(\hat{R}) = WP(\{b\}, 0; R_0)$.

Now, consider the following allocation vector $\hat{z}$:

$$\hat{z}_1 := \Big(\{a, b\}, w_1(0)\Big), \hat{z}_2 := \Big(\emptyset, p_2(\hat{R}) - \hat{v}\Big), \hat{z}_3 := \Big(\emptyset, 0\Big),$$

$$\hat{z}_i := \Big(f_i(\hat{R}), p_i(\hat{R})\Big) \ \forall\ i > 3.$$

By definition of $w_1(0)$, we get that $\hat{z}_1 \ \hat{I}_1 \ (\emptyset, 0)$. Also, since $\hat{R}_2$ is quasilinear with value $\hat{v}$, we get $(\emptyset, p_2(\hat{R}) - \hat{v}) \ \hat{I}_2 \ (\{a\}, p_2(\hat{R}))$. For agent 3, notice that $R_3 = R_0$ and by the definition of willingness to pay, we get $(\emptyset, 0) \ \hat{I}_3 \ \Big(\{b\}, WP(\{b\}, 0; R_0)\Big)$. For $i > 3$, each agent i gets the same outcome in $\hat{z}$ and $(f, \mathbf{p})$. Finally, the sum of payments of agents $1, 2$, and 3 (payments of other agents remain unchanged) in $\hat{z}$ is

$$w_1(0) + p_2(\hat{R}) - \hat{v} > p_2(\hat{R}) + p_3(\hat{R}),$$

where the strict inequality follows from Inequality (2.10) and the fact that $p_3(\hat{R}) = WP(\{b\}, 0; R_0)$. This contradicts the fact that $(f, \mathbf{p})$ is Pareto efficient.

Hence, we must have $f_2(\hat{R}) = \emptyset$. By Lemma 1, we have $p_2(\hat{R}) = 0$. But since $v > p_2(R)$, we get $(\{a\}, p_2(R))\ \hat{P}_2\ (\emptyset, 0)$. Hence, $(f_2(R), p_2(R))\ \hat{P}_2\ (f_2(\hat{R}), p_2(\hat{R}))$. This contradicts DSIC.

An identical argument establishes that $p_3(R) = w_1(0) - WP(\{a\}, 0; R_0)$.

Step 2. In this step, we show that agent 2 can manipulate at R, thus contradicting DSIC and completing the proof. Consider a quasilinear dichotomous preference $\bar{R}_2^Q$ where the minimum acceptable bundle of agent 2 is $\{b\}$ (note that $f_2(R) = \{a\}$) and his value $\bar{v}$ is $WP(\{b\}, 0; R_0)$. Consider the preference profile $\bar{R}$ where $\bar{R}_2 = \bar{R}_2^Q$ and $\bar{R}_i = R_i$ for all $i \neq 2$. Notice that if we let $k = 2$, $x = b$, and $R'_k = \bar{R}_2^Q$, Inequality (2.8) holds, and hence, Claim 5 implies that $f_2(\bar{R}) \neq \emptyset$ and $f_3(\bar{R}) \neq \emptyset$ but $f_2(\bar{R}) \cup f_3(\bar{R}) = \{a, b\}$. Hence, Pareto efficiency implies that $f_2(\bar{R}) = \{b\}$ and $f_3(\bar{R}) = \{a\}$. Then, we can mimic the argument in Step 1 to conclude that

$$p_2(\bar{R}) = w_1(0) - WP(\{a\}, 0; R_0).$$

Now, by the definition of willingness to pay,

$$\Big(\{b\}, WP(\{b\}, 0; R_0)\Big)\ I_0\ \Big(\{a\}, WP(\{a\}, 0; R_0)\Big)$$

and by our assumption, $WP(\{b\}, 0; R_0) > WP(\{a\}, 0; R_0)$. By subtracting $WP(\{a\}, 0; R_0) + WP(\{b\}, 0; R_0) - w_1(0)$ (which is positive by Inequality (2.7)) from payments on both sides, and using the fact that R_0 satisfies strict positive income effect, we get

$$\Big(\{b\}, w_1(0) - WP(\{a\}, 0; R_0)\Big)\ P_0\ \Big(\{a\}, w_1(0) - WP(\{b\}, 0; R_0)\Big).$$

Hence, $(f_2(\bar{R}), p_2(\bar{R}))\ P_2\ (f_2(R), p_2(R))$. This contradicts DSIC.

■

Chapter 3

An equivalence result in bilateral trading: robust BIC and DSIC mechanisms

3.1 Introduction

We consider a model of bilateral trading with private values. The valuation of the buyer and the cost of the seller are jointly distributed. The true joint distribution of valuation and cost is common knowledge among agents but is unknown to the designer. However designer knows the marginal distribution of valuation of the buyer and cost of the seller. Since the designer does not know the true joint distribution, she wants to design a mechanism that is robust to the joint distribution to valuations. Also, the mechanism must be implementable for all the possible joint distributions consistent with the marginal distribution of types as agents know the true joint distribution.

Consider an example of real estate market. The factors like quality of material used in the house, crime level in the area impact the valuation of both, the real estate manager (seller) and the buyer. Thus, we expect positive correlation between the cost of seller and valuation of buyer. Since it is relatively easier to get the data on the what the buyers bid and sellers ask price separately compared to getting information about the tuple of bid and ask price, an assumption of knowledge about just marginal distributions of valuation and cost is reasonable in this environment.

We establish equivalence between Bayesian incentive compatible mechanisms (BIC) and dominant strategy mechanisms (DSIC). The equivalence result holds for robust efficiency gains for BIC and DSIC mechanisms along with the additional constraints on budget balancedness and individual rationality. The result implies it does not make a difference to the designer whether the joint distribution of valuations is known or unknown to the agents. The implementation of mechanisms and additional constraints over all the "consistent" distributions is quite demanding and results in dominant strategy implementation with additional ex-post constraints.

It is an important result as it simplifies the problem of the designer significantly. Hagerty and Rogerson (1987) shows that a block mechanism implementable in dominant strategy, budget balanced and individually rational mechanism are implemented by "posted-price" mechanism. This result allows us to focus on this small class of mechanisms to find an optimal robust mechanism.

Consider a setting where the designer is a third party that is designing a platform for trade and its revenue is a function of efficiency gains. It is plausible that the third party does not have precise information about the joint distribution of value and cost. In such situations, it is natural to seek distributional robustness. The assumption about unknown joint distribution but knowledge about the marginal distribution of value and cost seems reasonable. A similar approach was adopted by Carroll (2017) in the monopoly setting with the buyer having multi-dimensional demand. He and Li (2020) takes a similar approach in the auction environment and shows the second price auctions with no reserve price are asymptotically optimal.

Game theory has a great advantage in explicitly analyzing the consequences of trading rules that presumably are really common knowledge; it is deficient to the extent it assumes other features to be common knowledge, such as one player's probability assessment about another's preferences or information.I foresee the progress of game theory as depending on successive reductions in the base of common knowledge required to conduct useful analyses of practical problems. Only by repeated weakening of common knowledge assumptions will the theory approximate reality- Wilson (1987). For practical implications of robustness, rich literature has developed on robustness.

A strand of literature looks at robustness with respect to information structure (Bergemann et al., 2017; Brooks and Du, 2020; Carroll, 2018). In those environments, there is a "fixed" prior distribution over the valuations that results from the distribution over state spaces and the associated joint distribution over the valuations. Then, there is an information structure that determines how the signals would be generated. The information structure affects the strategy of players as it affects the posterior beliefs about valuations. Our approach is different in the sense that only the information about priors is common knowledge, not the joint distribution. Secondly, they consider general information structures whereas we consider a particular information structure where the signal of each player reveals the true valuation of that player.

The main result on the equivalence between Bayesian strategy implementation and dominant strategy implementation is observed in many other settings. Bergemann and Morris (2005) finds environments in which the ex-post implementation is equivalent to interim implementation for all types; the equivalence holds for separable environments, for eg. implementation of social choice function, a quasi-linear environment with no restriction on transfers. It also shows that equivalence result does not hold for quasi-linear environments if budget balancedness is added as a requirement. Gershkov et al. (2013) and Chen et al. (2019) shows such equivalence in the social choice environment. Manelli and Vincent (2010) shows the equivalence in a single unit, private values auction environment. The analysis in our paper is different from the above literature in two aspects: Firstly, we consider "robustness" in Bayesian incentive compatibility. We show equivalence between the "robust" BIC and DSIC mechanism. Secondly, we also consider additional constraints of budget balancedness and individual rationality.

In section 3.2, we introduce the model and class of mechanisms that we are interested in. In section 3.3, we present the main results in the paper. The section 3.4 contains the proofs of main results.

3.2 Model

We consider the private values model of bilateral trading. There is a single object for trade, which the seller can produce and the buyer is willing to buy. The valuation of the buyer for the object and the cost of the seller for producing the object are jointly distributed according to a distribution H. The marginal distribution of valuation of the buyer is denoted by F and the marginal distribution of cost of the seller is denoted by G. Though the joint distribution H is common knowledge among agents (the buyer and the seller), the designer does not know H. However, she knows the marginal distributions F and G. We assume that the valuations and costs lie in $\theta = [0, 1]$ and the marginal distributions are continuous.[1] We define $\Theta := \theta \times \theta$.

We focus attention on the direct revelation mechanisms. A *(direct) mechanism* is a triplet (q, t_b, t_s), where $q : [0, 1]^2 \to [0, 1]$ and $t_i : [0, 1]^2 \to \mathbb{R}$ for each $i \in \{b, s\}$. Here, $q(v, c)$ denotes the probability of trade and $t_b(v, c)$ is the payment made *by* the buyer and $t_s(v, c)$ is the payment made *to* the seller at type profile (v, c).

3.2.1 Notions of incentive compatibility

We introduce two notions of incentive compatibility in this section. The first two notions are standard in the literature.

Definition 10 *A mechanism* (q, t_b, t_s) *is* **dominant strategy incentive compatible (DSIC)** *if for every* $(v, c) \in [0, 1]^2$

$$\begin{aligned} vq(v, c) - t_b(v, c) &\geq vq(v', c) - t_b(v', c) \qquad && \forall\ v' \in [0, 1] \\ t_s(v, c) - cq(v, c) &\geq t_s(v, c') - cq(v, c') \qquad && \forall\ c' \in [0, 1] \end{aligned}$$

While DSIC is a prior-free notion, the weaker requirement of *Bayesian incentive compatibility* is not. In our model, the designer only knows the marginal distributions of types of individual agents. Hence, we require a robust version of Bayesian incentive compatibility.

A joint probability distribution $\widehat{H}$ of (v, c) is *consistent* with (F, G) if the marginal

[1]The result extends to type space of the form $[0, \bar{\theta}]$ and different supports for marginal distributions of value of the buyer and cost of the seller.

distribution of v and c are F and G respectively:

$$\widehat{H}(v,1) = F(v) \qquad \forall v$$
$$\widehat{H}(1,c) = G(c) \qquad \forall c$$

Let $\mathcal{H}$ denote the set of all joint distributions consistent with (F,G). Note that the true distribution $H \in \mathcal{H}$. Since F and G are continuous functions, we can find well defined joint probability density function denoted by h that generates joint distribution consistent with (F,G). [2]

Definition 11 *A mechanism* (q,t_b,t_s) *is* **Bayesian incentive compatible (BIC)** *with respect to a prior* $\widehat{H} \in \mathcal{H}$ *if*

$$\mathbb{E}_{c,\widehat{H}}\big[vq(v,c) - t_b(v,c)\big] \geq \mathbb{E}_{c,\widehat{H}}\big[vq(v',c) - t_b(v',c)\big] \qquad \forall\ v,v' \in [0,1]$$
$$\mathbb{E}_{v,\widehat{H}}\big[t_s(v,c) - cq(v,c)\big] \geq \mathbb{E}_{v,\widehat{H}}\big[t_s(v,c') - cq(v,c')\big] \qquad \forall\ c,c' \in [0,1],$$

where $\mathbb{E}_{c,\widehat{H}}$ *denotes the conditional expectation of* c *given valuation* v *using joint distribution* $\widehat{H}$ *and* $\mathbb{E}_{v,\widehat{H}}$ *denotes the conditional expectation of* v *given cost* c *using joint distribution* $\widehat{H}$ *A mechanism* (q,t_b,t_s) *is* **marginal-consistent Bayesian incentive compatible (M-BIC)** *if it is BIC with respect to all priors* $\widehat{H} \in \mathcal{H}$.

Clearly, a DSIC mechanism is BIC with respect to all priors. Hence, it is M-BIC.

3.2.2 Other desiderata

It is natural to impose two additional constraints on mechanisms in the bilateral trading problem: (a) participation constraint (b) budget-balance constraint.

Definition 12 *A mechanism* (q,t_b,t_s) *is* **ex-post individually rational (EIR)** *if for every*

[2]For example, $h(v,c) = f(v)g(c)$ where

$$f(v) = \begin{cases} \dfrac{dF(v)}{dv} & \text{if } F \text{ is differentiable at } v \\ 1 & \text{otherwise.} \end{cases} \qquad g(c) = \begin{cases} \dfrac{dG(c)}{dc} & \text{if } G \text{ is differentiable at } c \\ 1 & \text{otherwise.} \end{cases}$$

By Lebesgue's Theorem for the differentiability of monotone functions, marginal distributions F and G are differentiable almost everywhere. This makes h consistent with (F,G).

(v, c)

$$vq(v,c) - t_b(v,c) \geq 0$$
$$t_s(v,c) - cq(v,c) \geq 0$$

A weaker version of individual rationality is the following.

Definition 13 *A mechanism* (q, t_b, t_s) *is* **interim individually rational (IIR)** *with respect to a prior* $\widehat{H} \in \mathcal{H}$ *if*

$$\mathbb{E}_{c,\widehat{H}}\big[vq(v,c) - t_b(v,c)\big] \geq 0 \qquad \forall\ v \in [0,1]$$
$$\mathbb{E}_{v,\widehat{H}}\big[t_s(v,c) - cq(v,c)\big] \geq 0 \qquad \forall\ c \in [0,1]$$

A mechanism (q, t_b, t_s) *is* **marginal-consistent interim individually rational (M-IIR)** *if it is IIR with respect to all priors* $\widehat{H} \in \mathcal{H}$*.*

The M-IIR participation constraint is the analogue of M-BIC incentive constraint we had introduced earlier. Since the designer is uncertain about the true prior, she wants to design mechanisms which satisfy these stronger notions of IC and IR constraints. Note that these are still weaker than DSIC and EIR constraints.

We also introduce two notions of budget-balance constraints.

Definition 14 *A mechanism* (q, t_b, t_s) *is*

- **budget-balanced (BB)** *if for all* (v, c)*,* $t_b(v,c) = t_s(v,c)$
- ϵ**-budget-balanced (**ϵ**-BB)** *given* $\epsilon > 0$*, if for all* (v, c)*,* $|t_b(v,c) - t_s(v,c)| \leq \epsilon$*.*

The proofs construct a DSIC and EIR mechanism, but a particular type of DSIC and EIR mechanism. We call it the *block mechanism*. For this, we divide the type space $[0,1]^2$ into n^2 squares for any positive integer n. We do so in the usual way: for each $k \in \{0, 1, \ldots, n\}$, let $v_k = c_k = \frac{k}{n}$. Then, a *block* (square) is defined as:

$$B_{k,\ell} := [v_{k-1}, v_k) \times (c_{\ell-1}, c_\ell] \qquad \forall\ k, \ell \in \{1, \ldots, n\}$$

with the usual convention that $v_0 = c_0 = 0$. Clearly, $B_{k,\ell}$ is different for different values of n. But we supress the dependence of $B_{k,\ell}$ on the value of n for notational simplicity, unless it is necessary to be explicit.

Definition 15 *A mechanism* (q^n, t^n_b, t^n_s) *is an* n-**block mechanism** *if for each block* $B_{k\ell}$ *and for every* $(v, c), (v', c') \in B_{k\ell}$*, we have*

$$\begin{aligned} q^n(v, c) &= q^n(v', c') \\ t^n_i(v, c) &= t^n_i(v', c') \qquad \forall\, i \in \{b, s\} \end{aligned}$$

We will impose these notions of budget-balancedness, individual rationality and incentive compatibility to define three classes of mechanisms.

3.2.3 Three classes of mechanisms

We consider three classes of mechanisms. These mechanisms use different notions of IC, IR, and BB constraints.

$$\begin{aligned} \mathcal{M}_B &= \{(q, t_b, t_s) : \ (q, t_b, t_s) \text{ is M-BIC, M-IIR, and BB}\} \\ \mathcal{M}_D &= \{(q, t_b, t_s) : \ (q, t_b, t_s) \text{ is DSIC, EIR, BB and } n\text{-block mechanism}\} \end{aligned}$$

Then, for a given $\epsilon > 0$, define

$$\mathcal{M}_\epsilon = \{(q, t_b, t_s) : \ (q, t_b, t_s) \text{ is DSIC, EIR, and } \epsilon\text{-BB}\}$$

We establish equivalence results for these classes of mechanisms in terms of robust efficiency gains.

Also, we define $\mathcal{M}_{\widehat{D}} = \{(q, t_b, t_s) : \ (q, t_b, t_s) \text{ is DSIC, EIR, and BB}\}$.

As a designer, we are interested in evaluating the *worst efficiency* of a mechanism in these classes. Formally, given a mechanism (q, t_b, t_s), its **robust efficiency gain** is

$$\textsc{Eff}(q, t_b, t_s) = \inf_{\widehat{H} \in \mathcal{H}} \mathbb{E}_{(v,c),\widehat{H}}\big[(v - c)q(v, c)\big]$$

In each classes of these mechanism, we can then define the *optimal robust mechanism*. Given an $\epsilon > 0$, for any $\mathcal{M} \in \{\mathcal{M}_\epsilon, \mathcal{M}_B, \mathcal{M}_D\}$,

$$\textsc{Eff}(\mathcal{M}) = \sup_{(q,t_b,t_s)\in\mathcal{M}} \textsc{Eff}(q, t_b, t_s)$$

Any mechanism in $\mathcal{M}$ which attains worst case efficiency gain equal to $\textsc{Eff}(\mathcal{M})$ will be an *optimal robust mechanism* in $\mathcal{M}$.

3.3 Main results

There are two main results of the paper. The first theorem says that robust efficiency gain in the class of mechanisms $\mathcal{M}_B$ can be made arbitrarily close to robust efficiency gain in class of prior-free mechanisms. The precise statement is the following.

Theorem 5 *For every $\epsilon > 0$, there exists $\eta(\epsilon) > 0$ and $\lim_{\epsilon\downarrow 0} \eta(\epsilon) = 0$ such that*

$$\textsc{Eff}(\mathcal{M}_B) - \textsc{Eff}(\mathcal{M}_{\eta(\epsilon)}) \leq \epsilon.$$

The second main theorem compares robust efficiency gains in $\mathcal{M}_B$ with the class of mechanisms in $\mathcal{M}_D$.

Theorem 6 $\textsc{Eff}(\mathcal{M}_B) = \textsc{Eff}(\mathcal{M}_D)$.

The proof of these theorems reveal that the equivalence results are stronger than what the statements suggest.

Since $\mathcal{M}_B \supset \mathcal{M}_{\widehat{D}} \supset \mathcal{M}_D$, Theorem 6 implies that $\textsc{Eff}(\mathcal{M}_B) = \textsc{Eff}(\mathcal{M}_{\widehat{D}})$. This establishes equivalence between the robust BIC and DSIC mechanisms with additional constraints in budget balancedness and individual rationality.

3.4 Proofs of main results

We construct n- block mechanisms satisfying DSIC and EIR mechanisms to prove the equivalence theorems.

3.4.1 Proof of Theorem 5

For proof of Theorem 5, we construct a sequence of n-block mechanisms starting from a mechanism in $\mathcal{M}_B$. We show that each of these block mechanisms is DSIC and EIR. The constructed block mechanism need not be BB. But, for sufficiently high value of n, it is arbitrarily close to budget balancedness. For sufficiently high value of n, can come arbitrarily close to the robust efficiency gain of the original mechanism.

Given a mechanism $(q, t_b, t_s) \in \mathcal{M}_B$ and positive integer $n > 1$, define a new mechanism (q^n, t_b^n, t_s^n) as follows. First, we define q^n: for each block $B_{k,\ell}$ and for each $(v, c) \in B_{k,\ell}$

$$q^n(v,c) = \begin{cases} 0 & \text{if } k = 0 \text{ or } \ell = n \text{ or } \int\limits_{B_{k-1,\ell}} q(x,y)dxdy = 0 \text{ or } \int\limits_{B_{k,\ell+1}} q(x,y)dxdy = 0 \\ n^2 \int\limits_{B_{k,\ell}} q(x,y)dxdy & \text{otherwise} \end{cases} \tag{3.1}$$

It is clear that q^n is feasible: $n^2 \int\limits_{B_{k,\ell}} q(x,y)dxdy \leq n^2 \int\limits_{B_{k,\ell}} dxdy = 1$ for each k, ℓ. We show that q^n is *ex-post monotone*: for each v, v', c, c' with $v' > v$ and $c' < c$, $q^n(v,c) \leq q^n(v',c)$ and $q^n(v,c) \leq q^n(v,c')$.

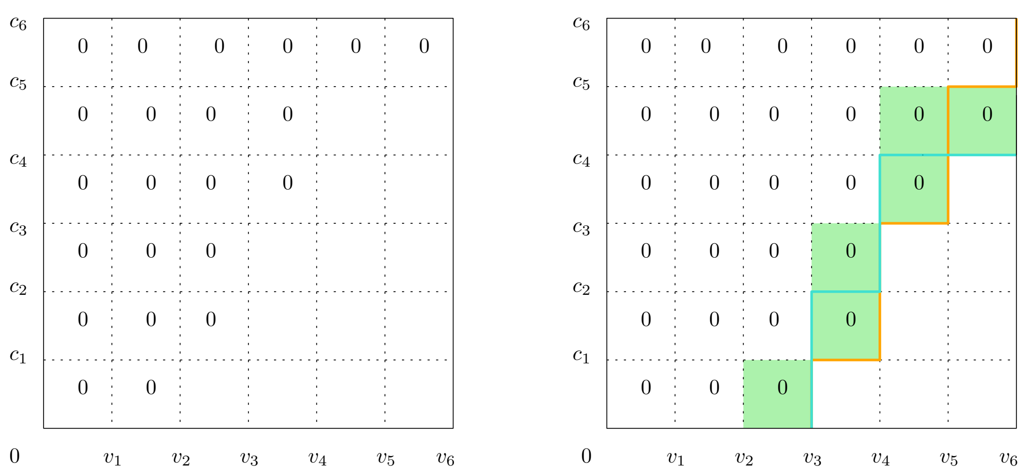


Figure 3.1: Relating allocations: From q to q_6'

The figure 3.1 illustrates how the allocation probabilities under the two mechanisms are different. The entry zero in block implies that the average allocation probability over the block is zero. The figure on the left has blocks with the same average probability as q. From this block mechanism, we can derive block mechanism, q^6. The average probabilities match for mechanism q and q^6 on all but few blocks that coloured in green. The green coloured blocks have positive allocation probability for mechanism q but zero allocation probability under mechanism q^6.

The allocation function q^6 weakly decreases the allocation probability for each block and increases the number of blocks with zero probability of allocation, but this happens for only at max $2n$ blocks out of n^2 blocks, for a general allocation function, q_n. As n increases, the share of such blocks would be insignificant and would have the same efficiency gains as the one from just averaging out the allocation probabilities.

Lemma 4 *Allocation rule q^n is ex-post monotone.*

Proof: The proof goes in many steps.

STEP 1. In this step, we show that q satisfies a monotonicity property and the mechanism (q, t_b, t_s) satisfies a version of the *payoff equivalence* formula. Fix a v and $(c_{\ell-1}, c_\ell]$. Define a joint density $\hat{h}^v$ follows:

$$\hat{h}^v(x, y) = \begin{cases} n & \text{if } x = v, y \in (c_{\ell-1}, c_\ell] \\ 0 & \text{if } x = v, y \notin (c_{\ell-1}, c_\ell] \\ h(x, y) & \text{otherwise} \end{cases} \tag{3.2}$$

Hence, $\hat{h}^v$ coincides with h everywhere except for $h(v, y)$ for all v. Hence, the marginals of $\hat{h}^v$ and h coincide everywhere, implying $\hat{h}^v \in \mathcal{H}$.

Now, consider a pair of M-BIC constraints for buyer: $v, v' \in [0, 1]$ with $v > v'$. Consider the M-BIC prior constraints of mechanism (q, t_b, t_s) for prior $\hat{h}^v \in \mathcal{H}$.

$$\int\limits_y vq(v,y)\hat{h}^v(v,y)dy - \int\limits_y t_b(v,y)\hat{h}^v(v,y)dy \geq \int\limits_y vq(v',y)\hat{h}^v(v,y)dy - \int\limits_y t_b(v',y)\hat{h}^v(v,y)dy$$

$$\iff \int\limits_{c_{\ell-1}}^{c_\ell} vq(v,y)dy - \int\limits_{c_{\ell-1}}^{c_\ell} t_b(v,y)dy \geq \int\limits_{c_{\ell-1}}^{c_\ell} vq(v',y)dy - \int\limits_{c_{\ell-1}}^{c_\ell} t_b(v',y)dy \tag{3.3}$$

Analogously, the M-BIC constraint from v' to v with prior $\hat{h}^{v'}$ gives

$$\int\limits_{c_{\ell-1}}^{c_\ell} v'q(v',y)dy - \int\limits_{c_{\ell-1}}^{c_\ell} t_b(v',y)dy \geq \int\limits_{c_{\ell-1}}^{c_\ell} v'q(v,y)dy - \int\limits_{c_{\ell-1}}^{c_\ell} t_b(v,y)dy \tag{3.4}$$

Adding (3.3) and (3.4) gives

$$(v - v') \int\limits_{c_{\ell-1}}^{c_\ell} \big[q(v,y) - q(v',y)\big]dy \geq 0$$

Since $v > v'$, we get for all $\ell \in \{1, \ldots, n\}$,

$$\int\limits_{c_{\ell-1}}^{c_\ell} q(v,y)dy \geq \int\limits_{c_{\ell-1}}^{c_\ell} q(v',y)dy \tag{3.5}$$

Analogously, for each $c < c'$, we get for all $k \in \{1, \ldots, n\}$,

$$\int\limits_{v_{k-1}}^{v_k} q(x,c)dx \geq \int\limits_{v_{k-1}}^{v_k} q(x,c')dx \tag{3.6}$$

Next, using (3.3) gives us for every v and every ℓ,

$$u_b(v,\ell) \geq \int\limits_{c_{\ell-1}}^{c_\ell} vq(v',y)dy - \int\limits_{c_{\ell-1}}^{c_\ell} t_b(v',y)dy,$$

where $u_b(v,\ell) := \int\limits_{c_{\ell-1}}^{c_\ell} vq(v,y)dy - \int\limits_{c_{\ell-1}}^{c_\ell} t_b(v,y)dy$. Rewriting

$$u_b(v,\ell) \geq u_b(v',\ell) + (v-v') \int\limits_{c_{\ell-1}}^{c_\ell} q(v',y)dy$$

Hence, $u_b(\cdot,\ell)$ is convex in the first argument for every ℓ and $\int\limits_{c_{\ell-1}}^{c_\ell} q(v',y)dy$ is the subgradient of $u_b(\cdot,\ell)$ at v'. By the fundamental theorem of calculus, we can thus write for every v and every ℓ

$$u_b(v,\ell) = u_b(0,\ell) + \int\limits_0^v \Big(\int\limits_{c_{\ell-1}}^{c_\ell} q(x,y)dy \Big) dx \tag{3.7}$$

We use this to prove ex-post monotonicity of q^n in the next step.

Step 2. Fix $v \in [v_{k-1}, v_k)$ for some k. If $q^n(v,c) > 0$ and $q^n(v,c') > 0$ for some c, c' with $c' > c$, we argue that $q^n(v,c) \geq q^n(v,c')$. Suppose $(v,c) \in B_{k,\ell}$ and $(v,c') \in B_{k,\ell'}$. If $\ell = \ell'$, we are done since $q^n(v,c') = q^n(v,c)$. So, $c' > c$ implies $\ell' > \ell$. Then,

$$\frac{1}{n^2} q^n(v,c') = \int\limits_{c_{\ell'-1}}^{c_{\ell'}} \int\limits_{v_{k-1}}^{v_k} q(x,y)dxdy \leq \int\limits_{c_{\ell-1}}^{c_\ell} \int\limits_{v_{k-1}}^{v_k} q(x,y)dxdy = \frac{1}{n^2} q^n(v,c),$$

where we used (3.6) for the inequality.

Now, suppose $q^n(v,c) = 0$ for some c and we will show that $q^n(v,c') = 0$ for all $c' > c$. Suppose $(v,c) \in B_{k,\ell}$ and $(v,c') \in B_{k,\ell'}$. If $\ell = \ell'$, we are done since $q^n(v,c') = q^n(v,c)$. Else, $\ell' > \ell$. Hence, $\ell \neq n$. Since $q^n(v,c) = 0$, this means $q^n(v,c'') = 0$ for all $c'' \in (c_\ell, c_{\ell+1}]$. If $\ell' = \ell + 1$, we are done. Else, we repeatedly apply this procedure to get $q^n(v,c') = 0$.

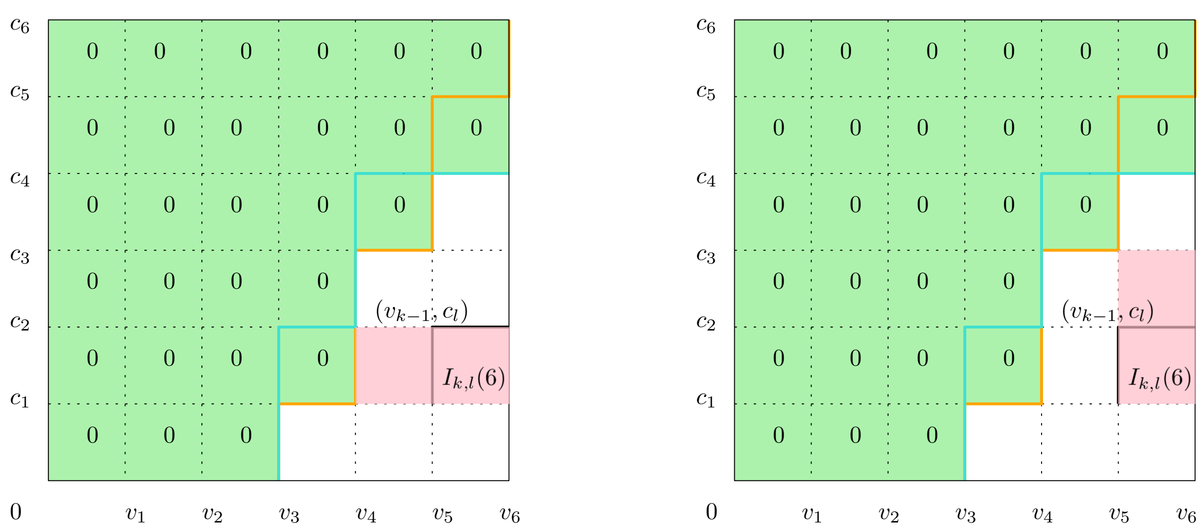
This shows that $q^n(v,c) \geq q^n(v,c')$ for all $c < c'$ and all v. An analogous proof using (3.5) can be done to show that $q^n(v,c) \leq q^n(v',c)$ for all $v' > v$ and for all c. ■

By Lemma 4, since q^n is monotone, we can define an EIR and DSIC mechanism using standard revenue equivalence techniques: payments at the lowest types are set to zero and local incentive constraints bind to give payments at all types. In particular, for every $(v, c) \in B_{k,\ell}$,

$$t_b^n(v,c) = \begin{cases} 0 & \text{if } k = 0 \\ t_b^n(v_{k-2}, c_\ell) + v_{k-1}\big[q^n(v_{k-1}, c_\ell) - q^n(v_{k-2}, c_\ell)\big] & \text{otherwise.} \end{cases} \tag{3.8}$$

Similarly, for every $(v, c) \in B_{k,\ell}$,

$$t_s^n(v,c) = \begin{cases} 0 & \text{if } \ell = n \\ t_b^n(v_k, c_{\ell+1}) + c_\ell\big[q^n(v_k, c_\ell) - q^n(v_k, c_\ell + 1)\big] & \text{otherwise.} \end{cases} \tag{3.9}$$



(a) Payment of buyer

(b) Payment of seller

Figure 3.2: Relating payment rules: (t_b, t_s) and (t_b^6, t_s^6)

The payment of buyer and seller is such that the agents with valuation at boundary of squares in the block are indifferent between reporting their true valuation and misreporting valuation in next block as shown in the figure 3.2. The corresponding blocks are shown in pink colour. Thus, we have a DSIC and EIR mechanism (q^n, t_b^n, t_s^n) and gives the next lemma.

Lemma 5 *(q^n, t_b^n, t_s^n) is DSIC and EIR mechanism.*

Lemma 6 *As $n \to \infty$, we have* $\lim_{n\to\infty} |t_b^n(v,c) - t_s^n(v,c)| = 0$.

Proof: We start by defining the following functions:

1. $$\tilde{v}(c) = \inf\left[\left\{ v : \lim_{r\to\infty} r^2 \int\limits_{c-\frac{1}{r}}^{c} \int\limits_{v}^{v+\frac{1}{r}} q(x,y)\ dx\ dy > 0 \right\} \cup \{1\}\right]$$

2. $$\tilde{c}(v) = \sup\left[\left\{ c : \lim_{r\to\infty} r^2 \int\limits_{c-\frac{1}{r}}^{c} \int\limits_{v}^{v+\frac{1}{r}} q(x,y)\ dx\ dy > 0 \right\} \cup \{0\}\right]$$

Consider arbitrary $(v,c) \in \Theta$. We define $\tilde{\Theta} = \{(v,c) : v > \tilde{v}(c), (v,c) \in \Theta\}$. By definition of $\tilde{v}(c)$, $\tilde{c}(v)$ and (q^n, t_b^n, t_s^n), it follows that if $(v,c) \in \Theta - \tilde{\Theta}$, we have $q^n(v,c) = 0$ and $t_b^n(v,c) = t_s(v,c) = 0$.[3] Thus, we have budget balancedness for such values.

Now we just need to show η-budget balancedness for $(v,c) \in \tilde{\Theta}$. Formally, we show that as $n \to \infty$, we have $|t_b^n(v,c) - t_s^n(v,c)| \to |t_b(v,c) - t_s(v,c)|$. We start by finding relation between t_b and t_b^n.

By equation (3.7), we get

$$\int\limits_{c-\frac{1}{n}}^{c} t_b(x,y)\ dy = \int\limits_{c-\frac{1}{n}}^{c} t_b(0,y)\ dy + \int\limits_{c-\frac{1}{n}}^{c} x\ q(x,y)\ dy - \int\limits_{c-\frac{1}{n}}^{c} \left[\int_0^x q(r,y)\ dr\right]\ dy, \quad \forall x \tag{3.10}$$

Integrating over x from v to $v + \frac{1}{n}$, we have

$$\begin{aligned}\int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} t_b(x,y)\ dx\ dy = \frac{1}{n} \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} t_b(0,y)\ dy + \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} x\ q(x,y)\ dx\ dy \\ - \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} \left[\int_0^x q(r,y)\ dr\right]\ dx\ dy\end{aligned} \tag{3.11}$$

[3]Note that $v \le \tilde{v}(c)$ iff $c \ge \tilde{c}(v)$.

Integration by parts gives

$$\int\limits_{v}^{v+\frac{1}{n}} x\ q(x,y)\ dx\ dy = \left[x\ \int_0^x q(r,y)\ dr\ dx\right]_v^{v+\frac{1}{n}} - \int\limits_{v}^{v+\frac{1}{n}} \left[\int_0^x q(r,y)\ dr\right]\ dx$$

$$= \left[\left(v+\frac{1}{n}\right)\ \int_0^{v+\frac{1}{n}} q(r,y)\ dr\ dx - v\ \int_0^v q(r,y)\ dr\ dx\right] - \int\limits_{v}^{v+\frac{1}{n}} \left[\int_0^x q(r,y)\ dr\right]\ dx$$

$$= v\ \int_v^{v+\frac{1}{n}} q(r,y)\ dr\ dx + \frac{1}{n}\int_0^{v+\frac{1}{n}} q(r,y)\ dr\ dx - \int\limits_{v}^{v+\frac{1}{n}} \left[\int_0^x q(r,y)\ dr\right]\ dx \tag{3.12}$$

Substituting equation (3.12) in equation (3.11) and multiplying the resultant equation by n^2, we get

$$n^2 \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} t_b(x,y)\ dx\ dy$$

$$= n \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} t_b(0,y)\ dy + v\left[n^2 \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} q(x,y)\ dx\ dy\right] + n\left[\int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} q(x,y)\ dx\ dy\right]$$

$$- 2n^2 \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} \left[\int_0^x q(r,y)\ dr\right]\ dx\ dy$$

$$= n \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} t_b(0,y)\ dy + t_b^n(v,c) - t_b^n\left(v-\frac{1}{n},c\right) + vq^n\left(v-\frac{1}{n},c\right)$$

$$+ n \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} q(x,y)\ dx\ dy - 2n^2 \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} \left[\int_0^x q(r,y)\ dr\right]\ dx\ dy$$

By using relation between (q^n, t_b^n) of consecutive blocks from equation (3.8), we get

$$\begin{aligned} n^2 \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} t_b(x,y)\ dx\ dy = n \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} t_b(0,y)\ dy + t_b^n(v,c) + \sum_{i=1}^{k-1} \frac{1}{n} q^n\left(v - \frac{i}{n}, c\right) \\ - t_b^n\left(v - \frac{k}{n}, c\right) + \left(v - \frac{k-1}{n}\right) q^n\left(v - \frac{k}{n}, c\right) \\ + n \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} q(x,y)\ dx\ dy - 2n^2 \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} \left[\int_0^x q(r,y)\ dr\right]\ dx\ dy \end{aligned}$$

where $k(n) = \min\left\{k : q^n\left(v - \frac{k}{n}, c\right) = 0, k \in \mathbb{N}\right\}$. [4]

By definition of k and individual rationality, $t_b^n\left(v - \frac{k}{n}, c\right) = q^n\left(v - \frac{k}{n}, c\right) = 0$.

Also, since $\sum_{i=1}^{k-1} \frac{1}{n} q^n\left(v - \frac{i}{n}, c\right) = n \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v-\frac{k-1}{n}}^{v} q(x,y)\ dx\ dy$, we get

$$\begin{aligned} & n^2 \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} t_b(x,y)\ dx\ dy \\ &= n \int\limits_{c-\frac{1}{n}}^{c} \int_{v}^{v+\frac{1}{n}} t_b(0,y) dy + t_b^n(v,c) + n \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v-\frac{k-1}{n}}^{v} q(x,y)\ dx\ dy + n \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} q(x,y)\ dx\ dy \\ &\quad - 2n^2 \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} \left[\int\limits_{\tilde{v}(c)}^{x} q(r,y)\ dr\right]\ dx\ dy - 2n^2 \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} \left[\int_0^{\tilde{v}(c)} q(r,y)\ dr\right]\ dx\ dy \end{aligned} \tag{3.13}$$

[4] It is well defined function as $v > \tilde{v}$ and by definition of q^n, whenever (v,c) lie on boundary of grid, $q^n(v,c) > 0$.

Thus, we have

$$n^2 \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} t_b(x,y)\ dx\ dy = t_b^n(v,c) + A_b(n,v,c) + B_b(n,v,c) + C_b(n,v,c)$$

where

$$\begin{aligned}
A_b(n,v,c) &= n \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v-\frac{k-1}{n}}^{v} q(x,y)\ dx\ dy - n \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{\tilde{v}(c)}^{v} q(x,y)\ dx\ dy \\
B_b(n,v,c) &= n \int\limits_{c-\frac{1}{n}}^{c} \int_{\tilde{v}(c)}^{v+\frac{1}{n}} q(x,y)\ dx\ dy - 2n^2 \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} \left[\int\limits_{\tilde{v}(c)}^{x} q(r,y)\ dr \right] dx\ dy \\
C_b(n,v,c) &= -\ 2n^2 \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} \left[\int_{0}^{\tilde{v}(c)} q(r,y)\ dr \right] dx\ dy + n \int\limits_{c-\frac{1}{n}}^{c} \int_{v}^{c+\frac{1}{n}} t_b(0,y)\ dy
\end{aligned}$$

Thus, for $v > \tilde{v}(c)$, we have

$$\lim_{n\to\infty} n^2 \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} t_b(x,y)\ dx\ dy = \lim_{n\to\infty} t_b^n(v,c) \tag{3.14}$$

Analogously, for each (v,c) such that $c < \tilde{c}(v)$, we have

$$\lim_{n\to\infty} n^2 \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} t_s(x,y)\ dx\ dy = \lim_{n\to\infty} t_s^n(v,c) \tag{3.15}$$

Combining the last two inequalities, in the case where $v > \tilde{v}(c)$ and $c < \tilde{c}(v)$, we get

$$\lim_{n\to\infty} t_b^n(v,c) - t_s^n(v,c) = \lim_{n\to\infty} n^2 \int\limits_{c-\frac{1}{n}}^{c} \int\limits_{v}^{v+\frac{1}{n}} [t_b(x,y) - t_s(x,y)] \; dx \; dy = 0 \tag{3.16}$$

The last equality follows from budget balancedness of (q, t_b, t_s).

Hence, we have shown that for $(v,c) \in \tilde{\Theta}$, the η-budget balancedness holds for large enough n.

From budget balancedness for $(v,c) \in \Theta - \tilde{\Theta}$ and η- budget balancedness for $(v,c) \in \tilde{\Theta}$, we get $\forall (v,c) \in \Theta$,

$$\lim_{n\to\infty} |t_b^n(v,c) - t_s^n(v,c)| = 0.$$

■

Lemma 7 *The efficiency gains generated by sequence of mechanism (q^n, t_b^n, t_s^n) converges to efficiency gains from mechanism (q, t_b, t_s), $\forall \widehat{H} \in \mathcal{H}$.*

Proof:

The efficiency gains of mechanism (q^n, t_b^n, t_s^n) is given by

$$\begin{aligned} &\int\limits_{\Theta} (v-c) \; q^n(v,c) \; d\widehat{H}(v,c) \\ =&\sum_{k,\ell} \int\limits_{B_{k,\ell}(n)} (v-c) \; q^n(v,c) \; d\widehat{H}(v,c) \end{aligned}$$

We would use the following lemma directly at this point. It will be proved in Section 3.6.

Lemma 8 *For any square $B_{k,\ell}(w)$, we have*

$$\lim_{m\to\infty} \int\limits_{B_{k,\ell}(w)} (v-c)\ q^{wm}(v,c)\ d\widehat{H}(v,c) = \int\limits_{B_{k,\ell}(w)} (v-c) q(v,c)\ d\widehat{H}(v,c) + \frac{2}{w} \int\limits_{B_{k,\ell}(w)} q(v,c)\ d\widehat{H}(v,c) \tag{3.17}$$

Summing over (k,ℓ) in equation (3.17), we get

$$\begin{aligned}\sum_{k,\ell} \Bigg| \int\limits_{B_{k,\ell}(w)} (x-y)\ q^{wm}(x,y)\ d\widehat{H}(x,y) - \int\limits_{B_{k,\ell}(w)} (x-y)\ q(x,y)\ d\widehat{H}(x,y) \Bigg| \\ \leq \frac{2}{w} \sum_{k,\ell} \left[\int\limits_{B_{k,\ell}(w)} q(x,y)\ d\widehat{H}(x,y) \right] \\ \leq \frac{2}{w}\end{aligned}$$

The last inequality follows from the fact that $\forall (x,y), q(x,y) \leq 1$ and $\widehat{H}$ is a probability measure.

Thus, as $w \to \infty$, $m \to \infty$, we have

$$\sum_{k,\ell} \int\limits_{B_{k,\ell}(w)} (x-y)\ q^{wm}(x,y)\ d\widehat{H}(x,y) \to \sum_{k,\ell} \int\limits_{B_{k,\ell}(w)} (x-y)\ q(x,y)\ d\widehat{H}(x,y)$$

Note that we had partitioned the entire space of valuation into $(wm)^2$ squares. Choosing $w = m = \sqrt{n}$, we can see that for all $\widehat{H} \in \mathcal{H}$, as $n \to \infty$, efficiency gains of (q^n, t_b^n, t_s^n) converges to efficiency gains of mechanism (q, t_b, t_s). ■

By combining Lemma 5, Lemma 6 and Lemma 7, for all $\widehat{H} \in \mathcal{H}$ and for all $(q, t_s, t_b) \in$

$\mathcal{M}_B$,

$$\mathbb{E}_{(v,c),\widehat{H}}\big[(v-c)q(v,c)\big] = \lim_{n\to\infty} \mathbb{E}_{(v,c),\widehat{H}}\big[(v-c)q^n(v,c)\big] \text{ and } \lim_{n\to\infty} |t_b^n(v,c) - t_s^n(v,c)| = 0$$

For every $\epsilon > 0$, for all $\widehat{H} \in \mathcal{H}$, $(q, t_s, t_b) \in \mathcal{M}_B$, there exists $\eta(\epsilon)$ such that $\lim_{\epsilon\downarrow 0} \eta(\epsilon) = 0$, $(q^n, t_b^n, t_s^n) \in \mathcal{M}_{\eta(\epsilon)}$ for large n and

$$\mathbb{E}_{(v,c),\widehat{H}}\big[(v-c)q(v,c)\big] \le \mathbb{E}_{(v,c),\widehat{H}}\big[(v-c)q^n(v,c)\big] + \epsilon.$$

Taking infimum over $\widehat{H} \in \mathcal{H}$, we get that for every $\epsilon > 0$ and for all $(q, t_s, t_b) \in \mathcal{M}_B$, there exists $\eta(\epsilon)$ such that $\lim_{\epsilon\downarrow 0} \eta(\epsilon) = 0$ and for large n, we have $(q^n, t_b^n, t_s^n) \in \mathcal{M}_{\eta(\epsilon)}$ and

$$\textsc{Eff}(q, t_b, t_s) \le \textsc{Eff}(q^n, t_b^n, t_s^n) + \epsilon.$$

Taking supremum over all $(q, t_b, t_s) \in \mathcal{M}_B$ and using the fact that $(q^n, t_b^n, t_s^n) \in \mathcal{M}_{\eta(\epsilon)}$, we get that for every $\epsilon > 0$, there exists $\eta(\epsilon)$ such that $\lim_{\epsilon\downarrow 0} \eta(\epsilon) = 0$ and

$$\textsc{Eff}(\mathcal{M}_B) \le \textsc{Eff}(\mathcal{M}_{\eta(\epsilon)}) + \epsilon.$$

3.4.2 Proof of Theorem 6

We define class of mechanisms, $\mathcal{M}_\eta^B := \{(q^n, t_b^n, t_s^n) : (q, t_b, t_s) \in \mathcal{M}_B, (q^n, t_b^n, t_s^n) \in \mathcal{M}_\eta\}$. Here, (q^n, t_b^n, t_s^n) is constructed using (3.1), (3.8) and (3.9) from mechanism (q, t_b, t_s).

Lemma 9 *As $\eta \downarrow 0$, we have $\textsc{Eff}(\mathcal{M}_\eta^B) \to \textsc{Eff}(\mathcal{M}_D)$.*

Proof: We start by mentioning two results for n-block mechanisms implementable in dominant strategy. The proof of the following observations provided in Section 3.6.

Observation 1 *For a n-block mechanism (q, t_b, t_s) implementable in dominant strategy, for $k \in \{1, 2, \ldots, n-1\}$ and $\ell \in \{1, 2, \ldots, n-1\}$, we have*

$$t_b(v_k, c_\ell) - t_s(v_k, c_\ell) = (v_k - c_\ell)q(v_k, c_\ell) - \frac{1}{n}\left[\sum_{i=0}^{k-1} q(v_i, c_\ell) + \sum_{i=l+1}^{n} q(v_k, c_i)\right] + t_b(v_0, c_\ell) - t_s(v_k, c_n)$$

Observation 2 *The constructed n-block mechanism, (q^n, t_b^n, t_s^n) has the property that for all n, for any $k < \ell$, we have $q^n(v_k, c_\ell) = 0$.*

Hagerty and Rogerson (1987) shows that a block mechanism q_0 which is DSIC, BB and EIR is implementable by posted price mechanism. In particular, such a mechanism with n blocks can be represented by a vector $\mathbf{u}_{(n-1)\times 1}$ where i^{th} element, $u_i = q_0(v_i, c_i)$ where $\sum_{i=1}^{n-1} u_i \leq 1$.[5]

Consider a constructed n-block mechanism $M = (q^n, t_b^n, t_s^n) \in \mathcal{M}_\eta^B$. We construct from M a block mechanism (q_0, t_0, t_0) which is DSIC, BB and EIR. Let vector $\mathbf{u}$ be the corresponding vector for the block mechanism, (q_0, t_0, t_0) and be defined as

$$u_i = q^n(v_i, c_i) + \Delta_n, \quad i \in \{1, \ldots, n-1\} \tag{3.18}$$

Here, Δ_n is a carefully chosen constant. We will define it later.

We show that allocation function q_0 is well defined and $\lim_{\eta\to 0}[q^n(v,c) - q_0(v,c)] = 0$ in three steps.

Step 1. The allocation function, q_0 defined above is well defined iff $\sum_{i=1}^{n-1} u_i \leq 1$.

From observation 1, for $k > \ell$, we have [6]

$$q^n(v_k, c_\ell) = \frac{1}{(v_k - c_\ell)}\left(t_b(v_k, c_\ell) - t_s(v_k, c_\ell) + \frac{1}{n}\left[\sum_{i=0}^{k-1} q^n(v_i, c_\ell) + \sum_{i=l+1}^{n} q^n(v_k, c_i)\right]\right)$$

Iteratively replacing expression of allocation rules on the right hand side of equation and using the observation 2, we get

$$\begin{aligned} q^n(v_k, c_\ell) &= r_n(k,\ell) + \sum_{i=l}^{k-1} q^n(v_i, c_i) \quad \text{ where} \\ r_n(k,\ell) &= \frac{n}{k-\ell}\Bigg(t_b^n(v_k, c_\ell) - t_s^n(v_k, c_\ell) \\ &\quad + \frac{1}{k-1}\left[t_b^n(v_{k-1}, c_\ell) - t_s^n(v_{k-1}, c_\ell) + t_b^n(v_k, c_{\ell-1}) - t_s^n(v_k, c_{\ell-1}) + \frac{1}{k-\ell-2}(\ldots)\right]\Bigg) \end{aligned}$$

[5] See Corollary 3 in Hagerty and Rogerson (1987) for details.
[6] For the concerned block mechanisms, $t_b^n(v_0, c_\ell) = t_0(v_0, c_\ell) = t_s^n(v_k, c_n) = t_0(v_k, c_n) = 0$.

We choose $\Delta_n = \dfrac{r_n(n,1)}{n-1}$. Note that $\displaystyle\sum_{i=1}^{n-1} u_i = \sum_{i=1}^{n-1} q^n(v_i, c_i) + r_n(n,1) = q^n(v_n, c_1)$. Since $q^n(v_{n-1}, c_1) \le 1$, it must be true that $\sum_{i=1}^{n-1} u_i \le 1$.

Step 2. We find the expression for $q^n(v,c) - q_0(v,c)$ for all $(v,c) \in \Theta$. Fix n and a utility profile $(v,c) \in B_{k,\ell}(n)$. Note that k, ℓ corresponding to v, c depends on n.

Case 1. For $(v,c) \in B_{k,l}(n)$ where $k < \ell$ for all n : From observation 2, it follows that $q^n(v,c) = 0$. In Hagerty and Rogerson (1987), the construction from vector $\mathbf{u}$ is such that $q_0(v,c) = 0$. Thus, $q^n(v,c) - q_0(v,c) = 0$.

Case 2. For $(v,c) \in B_{k,k}(n)$: By construction of q_0, we have $q^n(v,c) - q_0(v,c) = \Delta_n = \dfrac{r_n(n,1)}{n-1}$.

Case 3. For $(v,c) \in B_{k,l}(n)$ where $k > \ell$: As $k > \ell$, we have $v > c$.

$$\begin{aligned} q^n(v,c) - q_0(v,c) &= r_n(k,\ell) - (k-\ell)\Delta_n \\ &= r_n(k,\ell) - \frac{k-\ell}{n-1} r_n(n,1) \end{aligned}$$

Step 3. We show that $\lim_{\eta \downarrow 0}[q^n(v,c) - q_0(v,c)] = 0$.

As $\eta \to 0$, we have $n \to \infty$. For a utility profile $(v,c) \in B_{k,\ell}(n)$, as $n \to \infty$, $k, \ell \to \infty$ since $\lim_{n\to\infty} \dfrac{k}{n} = v$ and $\lim_{n\to\infty} \dfrac{\ell}{n} = c$.

As $M \in \mathcal{M}_\eta^B$, we have

$$|r_n(k,\ell)| \le \frac{n}{k-\ell}\left(\eta + \frac{1}{k-1}\left[2\eta + \frac{1}{k-\ell-2}(\dots)\right]\right)$$

It follows

$$\lim_{\eta\to 0} |r_n(k,\ell)| \leq \lim_{n\to\infty} \frac{\eta}{v-c} = 0$$
$$\lim_{\eta\to 0} |r_n(n,1)| \leq \lim_{n\to\infty} \eta = 0.$$

Putting these values in expression for $q^n(v,c) - q_0(v,c)$, we get $\lim_{\eta\to 0} q^n(v,c) - q_0(v,c) = 0$, for all $(v,c) \in \Theta$.

This further implies that for all $\widehat{H} \in \mathcal{H}$, for all (q^n, t_b^n, t_s^n), as $\eta \to 0$, we have

$$\mathbb{E}_{(v,c),\widehat{H}}\big[(v-c)q^n(v,c)\big] \to \mathbb{E}_{(v,c),\widehat{H}}\big[(v-c)q_0(v,c)\big].$$

Thus, for all (q^n, t_b^n, t_s^n), as $\eta \to 0$, we have

$$\textsc{Eff}(q^n, t_b^n, t_s^n) \leq \textsc{Eff}(q_0, t_0, t_0).$$
$$\lim_{\eta\to 0} \textsc{Eff}(\mathcal{M}_\eta^B) \leq \textsc{Eff}(\mathcal{M}_D)$$

■

From proof of Theorem 5 and Lemma 9, we get

$$\textsc{Eff}(\mathcal{M}_B) \leq \textsc{Eff}(\mathcal{M}_D)$$

But $\textsc{Eff}(\mathcal{M}_B) \geq \textsc{Eff}(\mathcal{M}_D)$ since $\mathcal{M}_B \subseteq \mathcal{M}_D$. This implies $\textsc{Eff}(\mathcal{M}_B) = \textsc{Eff}(\mathcal{M}_D)$.

3.5 Conclusion

Combining Lemma 5, Lemma 6 and Lemma 7 and Lemma 9, we get the following result: for every $\epsilon > 0, (q, t_b, t_s) \in \mathcal{M}_B, \widehat{H} \in \mathcal{H}$, there exist $(q_0, t_0, t_0) \in \mathcal{M}_D$ such that $\textsc{Eff}(q, t_b, t_s) - \textsc{Eff}(q^n, t_b^n, t_s^n) \leq \epsilon$. This result is much stronger than the theorems stated in Section 3.3.

Theorem 6 implies that focus on robust mechanism in the class of block mechanism implementable in dominant strategy, with ex-post individual rationality and budget balancedness. This simplifies the mechanism designer's problem significantly since Hagerty and Rogerson (1987) showed that mechanisms in $\mathcal{M}_D$ are implementable by random posted price mechanisms, which is a much simpler class of mechanisms.

3.6 Appendix

3.6.1 Proof of Lemma 8

Step 1. We define $\hat{q}^r(v,c) := r^2 \int\limits_{B_{k',\ell'}(r)} q(x,y)\ dx\ dy$ where $(v,c) \in B_{k',\ell'}(r)$.

Recall the following definitions:

1. $\tilde{v}(c) = \inf \left[\left\{ v : \lim\limits_{r\to\infty} r^2 \int\limits_{c-\frac{1}{r}}^{c} \int\limits_{v}^{v+\frac{1}{r}} q(x,y)\ dx\ dy > 0 \right\} \cup \{1\} \right]$

2. $\tilde{c}(v) = \sup \left[\left\{ c : \lim\limits_{r\to\infty} r^2 \int\limits_{c-\frac{1}{r}}^{c} \int\limits_{v}^{v+\frac{1}{r}} q(x,y)\ dx\ dy > 0 \right\} \cup \{0\} \right]$

We would show the convergence between probability of allocation for the given mechanism, $q^n(v,c)$ and $\hat{q}^n(v,c)$.

Let $C = \{(v,c) : v = \tilde{v}(c) \text{ or } c = \tilde{c}(v)\}$ and $\mathcal{D}(wm) = \{B_{k',\ell'}(wm) : (v,c) \in B_{k',\ell'}(wm) \cap C\}$.

Notice that for a given r, $q^r(v,c) \neq r^2 \int\limits_{B_{k,\ell}(r)} q(x,y)\ dx\ dy = \hat{q}^r(v,c)$ for squares, $B_{k',\ell'}(r)$ containing a point in C. In particular, restricted to $B_{k,\ell}(w)$, the values differ over $\mathcal{D}(wm)$ only. It follows directly from the definition of $q^{wm}(v,c)$. We show this below.

Suppose for contradiction that there exists $(v,c) \in B_{k',\ell'}(wm) \notin \mathcal{D}(wm)$ and $\hat{q}^{wm}(v,c) \neq q^{wm}(v,c)$. Note that $\hat{q}^{wm}(v,c) \neq q^{wm}(v,c)$ implies that $\hat{q}^{wm}(v,c) > 0 = q^{wm}(v,c)$. Since $\hat{q}^{wm}(v,c) > 0$, we have $x > \tilde{v}(y)$ and $y < \tilde{c}(x)$ for all $(x,y) \in B_{k',\ell'}(wm)$.[7] Also, by definition of q^{wm}, $q^{wm}(v,c) = 0$ implies that $\hat{q}^{wm}(v - \frac{1}{wm}, c) = 0$ or $\hat{q}^{wm}(v, c + \frac{1}{wm}) = 0$. Combining with the fact $\hat{q}^{wm} > 0$, we get $\tilde{v}(y) \leq x$ or $\tilde{c}(x) \geq y$, for some $(x,y) \in B_{k',\ell'}(wm)$. A contradiction.

[7] The inequalities are strict as $B_{k',\ell'}(wm) \notin \mathcal{D}(wm)$.

Using the above fact, for all $B_{k,\ell(w)}$ we have

$$\begin{aligned}
\int\limits_{B_{k,\ell}(w)} \hat{q}^{wm}(v,c)\, d\widehat{H}(v,c) - \int\limits_{B_{k,\ell}(w)} q^{wm}(v,c)\, d\widehat{H}(v,c)& \\
&= \sum_{B_{i,j}(wn)\subseteq B_{k,\ell}(w)} \left[\int\limits_{B_{i,j}(wn)} \hat{q}^{wm}(v,c)\, d\widehat{H}(v,c) - \int\limits_{B_{i,j}(wn)} q^{wm}(v,c)\, d\widehat{H}(v,c) \right] \\
&\leq \sum_{D\in\mathcal{D}(wm)} \int\limits_{D} \hat{q}^{wm}(v,c)\, d\widehat{H}(v,c) \\
&\leq \sum_{D\in\mathcal{D}(wm)} \widehat{H}(D)
\end{aligned}$$

The first inequality follows from the fact that $\forall(v,c)$, $\hat{q}^{wm}(v,c) \geq q^{wm}(v,c)$ and non-negativity of q^{wm}. The last inequality follows from the fact that $\hat{q}^{wm}(v,c) \leq 1$ by definition.

Thus, for all $B_{k,\ell(w)}$, as $m \to \infty$, we have

$$\begin{aligned}
&\lim_{m\to\infty} \left[\int\limits_{B_{k,\ell}(w)} \hat{q}^{wm}(v,c)\, d\widehat{H}(v,c) - \int\limits_{B_{k,\ell}(w)} q^{wm}(v,c)\, d\widehat{H}(v,c) \right] \\
&\leq \lim_{m\to\infty} \sum_{D\in\mathcal{D}(wm)} \widehat{H}(D) \\
&= \widehat{H}(B) = 0
\end{aligned}$$

Altenatively, for all $B_{k,\ell(w)}$, $m \to \infty$, we have

$$\int\limits_{B_{k,\ell}(w)} q^{wm}(v,c)\, d\widehat{H}(v,c) \to \int\limits_{B_{k,\ell}(w)} \hat{q}^{wm}(v,c)\, d\widehat{H}(v,c)$$

Step 2. We would show the convergence between probability of allocation for the given mechanism, $q(v,c)$ and $\hat{q}^n(v,c)$, which would later be used to prove the lemma 7.

To simplify notations, we consider two measures on the borel sigma algebra: Lebesgue measure, λ and given probability measure, $\widehat{H}$. We use the concept of simple functions to show convergence in probabilities.

Consider a rectangle, I. We can divide it into α^2 equal sized blocks by cutting each side into α equal parts. We define $\mathcal{S}(I, \alpha)$ as the collection of these α^2 equal sized blocks.

Let $\psi : B_{k,\ell}(w) \to \mathbb{R}$ be a simple function defined as $\psi(v, c) = \sum_i^r a_i \mathcal{X}_{A_i}(x)$ where $\mathcal{X}_{A_i}$ is indicator function, A_i are disjoint measurable sets and $\psi(v, c) \geq q(v, c),\ \forall (v, c)$. Also, we have $\cup_{i=1}^r A_i = B_{k,\ell}(w)$.

Fix collection of measurable sets, $\mathcal{A} = \{A_1, A_2, \ldots, A_r\}$. Since A_i is a measurable set, for every $\epsilon > 0$, there exists subsequence of squares (for large enough m_i), $\{S_j^i : j \in \mathbb{N}\} \subset \mathcal{S}(B_{k,\ell}(w),\ m_i)$ such that

(i) $\cup_j S_j^i \supseteq A_i$

(ii) $\lambda(\cup_j S_j^i) < \lambda(A_i) + \epsilon$ and

(iii) $\widehat{H}(\cup_j S_j^i) < \widehat{H}(A_i) + \epsilon$

Choose $m = \max_i m_i$. This ensures that above conditions hold simultaneously for all A_i.

Note that

$$\begin{aligned}\lambda\left(\left(\cup_j S_j^i\right) \cap \left(\cup_{t\neq i} A_t\right)\right) &= \lambda\left(\cup_j S_j^i\right) + \lambda\left(\cup_{t\neq i} A_t\right) - \lambda\left(\left(\cup S_j^i\right) \cup \left(\cup_{t\neq i} A_t\right)\right) \\ &< \lambda(A_i) + \epsilon + \sum_{t\neq i} \lambda(A_t) - \lambda(B_{k,\ell}(w)) = \epsilon\end{aligned}$$

The first equality follows from additivity of measure, λ. The strict inequality follows from definition of S_j^i and additivity of measure λ.

As a result, for any set $I \in \mathcal{S}(B_{k,\ell}(w),\ m)$ such that $\lambda(I \cap A_i) \neq \emptyset$ and $\lambda\left(I \cap \left(\cup_{t\neq i} A_t\right)\right) \neq \emptyset$, we have $\lambda\left(I \cap \left(\cup_{t\neq i} A_t\right)\right) < \epsilon$. Analogously, we also have $\widehat{H}\left(I \cap \left(\cup_{t\neq i} A_t\right)\right) < \epsilon$.

Now, we show that $\int\limits_{B_{k,\ell}(w)} \hat{q}_{wm}(x,y) d\widehat{H}(x,y) \leq \int\limits_{B_{k,\ell}(w)} \psi(x,y) d\widehat{H}(x,y)$ as $m \to \infty$.

Consider arbitrary $I \in \mathcal{S}(B_{k,\ell}(w),\ m)$. For $(v,c) \in I$, we have

$$\begin{aligned}
\hat{q}^{wm}(v,c) &= \frac{\int_I q(x,y)\ d\lambda(x,y)}{\lambda(I)} \\
&\leq \sum_t a_t \frac{\lambda(A_t \cap I)}{\lambda(I)} \\
&\leq a_i \frac{\lambda(A_i \cap I)}{\lambda(I)} + \max_t a_t \sum_{t \neq i} \frac{\lambda(A_t \cap I)}{\lambda(I)}
\end{aligned}$$

The equality follows from the definition of $\hat{q}_n(v,c)$. The first inequality follows from definition of function of ψ and the last inequality follows from definition of max.

Thus,

$$\int\limits_{A_i} \hat{q}^{wm}(x,y)\ d\widehat{H}(x,y) \leq \sum_{I \in \mathcal{S}(B_{k,\ell}(w),\ m)} \left[a_i \frac{\lambda(A_i \cap I)}{\lambda(I)} + \max_t a_t \sum_{t \neq i} \frac{\lambda(A_t \cap I)}{\lambda(I)} \right] \widehat{H}(A_i \cap I)$$

We partition the set $\mathcal{S}(B_{k,\ell}(w),\ m)$ on the basis of whether I intersects measurable sets other than A_i or not, with positive measure. Consider $\mathcal{R}(A_i) = \{I : \lambda(I \cap \cup_{t \neq i} A_t) = 0\}$.

For $(v,c) \in I \in \mathcal{R}(A_i)$, $\hat{q}^{wm}(v,c) \leq a_i$. This implies

$$\sum_{I \in \mathcal{R}(A_i)} \hat{q}^{wm}(x,y)\widehat{H}(A_i \cap I) \leq a_i \sum_{I \in \mathcal{R}(A_i)} \widehat{H}(A_i \cap I) \tag{3.19}$$

For $(v,c) \in I \notin \mathcal{R}(A_i)$, we have

$$\begin{aligned}
\sum_{I \notin \mathcal{R}(A_i)} \hat{q}^{wm}(x,y)\widehat{H}(A_i \cap I) &< \sum_{I \notin \mathcal{R}(A_i)} \left[a_i \frac{\lambda(A_i \cap I)}{\lambda(I)} + \max_t a_t \sum_{t \neq i} \frac{\lambda(A_t \cap I)}{\lambda(I)} \right] \widehat{H}(A_i \cap I) \\
&\leq \sum_{I \notin \mathcal{R}(A_i)} \left[a_i + \max_t (a_t - a_i) \right] \widehat{H}(A_i \cap I) \\
&= a_i \sum_{I \notin \mathcal{R}(A_i)} \widehat{H}(A_i \cap I) + \max_t (a_t - a_i) \sum_{I \notin \mathcal{R}(A_i)} \widehat{H}(A_i \cap I) \\
&< a_i \sum_{I \notin \mathcal{R}(A_i)} \widehat{H}(A_i \cap I) + \max_t (a_t - a_i)\epsilon
\end{aligned} \tag{3.20}$$

Adding (3.19) and (3.20), we get that $\forall i \in \{1, \ldots, r\}$

$$\int_{A_i} \hat{q}^{wm}(x,y) \; d\widehat{H}(x,y) < a_i \widehat{H}(A_i) + \max_t (a_t - a_i)\epsilon$$

Summing over i, we get

$$\int_{B_{k,\ell}(w)} \hat{q}^{wm}(x,y) \; d\widehat{H}(x,y) < \int_{B_{k,\ell}(w)} \psi(x,y) \; d\widehat{H}(x,y) + \sum_i \max_t (a_t - a_i)\epsilon$$

As the above equation holds for all $\epsilon > 0$, we get

$$\int_{B_{k,\ell}(w)} \hat{q}^{wm}(x,y) \; d\widehat{H}(x,y) \leq \int_{B_{k,\ell}(w)} \psi(x,y) \; d\widehat{H}(x,y) \tag{3.21}$$

Now consider $\phi : B_{k,\ell}(w) \to \mathbb{R}$ be a simple function defined as $\phi(v,c) = \sum_i^r a_i \mathcal{X}_{A_i}(x)$ where

$\mathcal{X}_{A_i}$ is indicator function, A_i are disjoint measurable sets and $\phi(v,c) \leq q(v,c),\ \forall (v,c)$. Also, we have $\cup_{i=1}^{r} A_i = B_{k,\ell}(w)$.

Analogously, we can show that $\int\limits_{B_{k,\ell}(w)} \hat{q}_{wm}(x,y)\ d\widehat{H}(x,y) \geq \int\limits_{B_{k,\ell}(w)} \phi(x,y)\ d\widehat{H}(x,y)$ as $m \to \infty$.

Since $q(v,c)$ is integrable w.r.t. to H, we have

$$\inf_{\psi} \int\limits_{B_{k,\ell}(w)} \psi(x,y)\ d\widehat{H}(x,y) = \sup_{\phi} \int\limits_{B_{k,\ell}(w)} \phi(x,y)\ d\widehat{H}(x,y)$$

Using the above fact with lower and upper bound for $\int\limits_{B_{k,\ell}(w)} q^{wm}(x,y)\ d\widehat{H}(x,y)$, we get that as $m \to \infty$,

$$\int\limits_{B_{k,\ell}(w)} \hat{q}^{wm}(x,y)\ d\widehat{H}(x,y) \to \int\limits_{B_{k,\ell}(w)} q(x,y)\ d\widehat{H}(x,y)$$

Step 3. We use the observations in Step 1 and Step 2 to prove lemma.

Combining the convergences in Step 1 and Step 2, as $m \to \infty$,

$$\int\limits_{B_{k,\ell}(w)} q^{wm}(x,y)\ d\widehat{H}(x,y) \to \int\limits_{B_{k,\ell}(w)} q(x,y)\ d\widehat{H}(x,y) \tag{3.22}$$

Note that

$$\left| \int\limits_{B_{k,\ell}(w)} (x-y)\ q^{wm}(x,y)\ d\widehat{H}(x,y) - \int\limits_{B_{k,\ell}(w)} (x-y)\ q(x,y)\ d\widehat{H}(x,y) \right|$$
$$\leq \left(\min_x B_{k,\ell}(w) - \inf_y B_{k,\ell}(w)\right) \left| \int\limits_{B_{k,\ell}(w)} q^{wm}(x,y) d\widehat{H}(x,y) - \int\limits_{B_{k,\ell}(w)} q(x,y) d\widehat{H}(x,y) \right|$$
$$+ \frac{1}{w}\left[\int\limits_{B_{k,\ell}(w)} \left(q^{wm}(x,y) + q(x,y)\right) d\widehat{H}(x,y)\right] \tag{3.23}$$

Using equation (3.22) and (3.23), we get that as $m \to \infty$, we have

$$\int\limits_{B_{k,\ell}(w)} (x-y)\ q^{wm}(x,y)\ d\widehat{H}(x,y) \to \int\limits_{B_{k,\ell}(w)} (x-y)\ q(x,y)\ d\widehat{H}(x,y) + \frac{2}{w}\left[\int\limits_{B_{k,\ell}(w)} q(x,y)\ d\widehat{H}(x,y)\right]$$

3.6.2 Proof of observations in proof of Theorem 6

Proof of Observation 1

Consider arbitrary (v_k, c_ℓ). By equation (3.7), we have

$$t_b(v_k, c_\ell) = v_k q(v_k, c_\ell) - \frac{1}{n}\sum_{i=1}^{k-1} q(v_i, c_\ell) + t_b(v_0, c_\ell)$$
$$t_s(v_k, c_\ell) = c_\ell q(v_k, c_\ell) + \frac{1}{n}\sum_{i=l+1}^{n} q(v_k, c_i) + t_s(v_k, c_n)$$

Thus, subtracting the above two equations, we get

$$t_b(v_k, c_\ell) - t_s(v_k, c_\ell) = (v_k - c_\ell) q(v_k, c_\ell) - \frac{1}{n}\left[\sum_{i=0}^{k-1} q(v_i, c_\ell) + \sum_{i=l+1}^{n} q(v_k, c_i)\right] + t_b(v_0, c_\ell) - t_s(v_k, c_n)$$

Proof of Observation 2

Fix a v and $(c_{\ell-1}, c_\ell]$. Define a joint density $\hat{h}^v$ as follows:

$$\hat{h}^v(x,y) = \begin{cases} n & \text{if } x = v, y \in (c_{\ell-1}, c_\ell] \\ 0 & \text{if } x = v, y \notin (c_{\ell-1}, c_\ell] \\ h(x,y) & \text{otherwise} \end{cases}$$

By M-IIR of $(q, t_b, t_s) \in \mathcal{M}_B$,

$$\int_{c_{\ell-1}}^{c_\ell} (t_b^n(v,y) - t_s^n(v,y)) \; dy \le \int_{c_{\ell-1}}^{c_\ell} (v-y)q(v,y) \; dy$$

The inequality holds for all v. By integrating over v from v_k to v_{k+1}, we get

$$\int_{v_k}^{v_{k+1}} \int_{c_{\ell-1}}^{c_\ell} (t_b^n(x,y) - t_s^n(x,y)) \; dy \; dx \le \int_{v_k}^{v_{k+1}} \int_{c_{\ell-1}}^{c_\ell} (x-y)q(x,y) \; dy \; dx \tag{3.24}$$

Analogously, by BB of $(q, t_b, t_s) \in \mathcal{M}_B$, we get

$$\int_{v_k}^{v_{k+1}} \int_{c_{\ell-1}}^{c_\ell} (t_b^n(x,y) - t_s^n(x,y)) \; dy \; dx = 0 \tag{3.25}$$

Combining the inequalities (3.24) and (5.3), we get

$$\int_{v_k}^{v_{k+1}} \int_{c_{\ell-1}}^{c_\ell} (v-c)q(x,y) \; dy \; dx = 0 \tag{3.26}$$

For $k < \ell - 1$, we have $(x-y) < 0$ for all $(x,y) \in [v_k, v_{k+1}) \times (c_{\ell-1}, c_\ell]$. To satisfy the above inequality, we must have $q(v,c) = 0$ a.e.

Thus, for $k < \ell$, we have $q^n(v_k, c_\ell) = 0$ by definition of q^n.

Chapter 4

Optimal robust mechanism in bilateral trading

4.1 Introduction

We consider an environment of bilateral trading in the private values model. The value of the buyer and the cost of the seller are jointly distributed but this distribution is not known to the designer. The designer only knows the marginal distributions of the buyer's type and the seller's type. Our objective is to design a mechanism that is robust to this uncertainty of the designer. In particular, we want to design a mechanism that maximizes *expected welfare guarantee*. The expected welfare guarantee of a mechanism is the *worst or minimum* expected gains from trade where the minimum is taken over all joint distributions consistent with the known marginals. We characterize the worst joint distribution giving the expected welfare guarantee of a dominant strategy incentive compatible, ex-post individually rational, and budget-balanced mechanism. We use this to show that a *deterministic* posted price mechanism is an optimal robust mechanism.

Our result contributes to the growing literature on robust mechanism design pioneered by Bergemann and Morris. This literature tries to answer the Wilson critic (Wilson, 1987) of mechanisms that rely on the informational assumptions of the designer. Bergemann and Morris (2005) finds environments in which the ex-post implementation is equivalent to interim implementation for all types; the equivalence holds for separable environments, for example, implementation of social choice function, a quasi-linear environment with no restriction on transfers.

In information setting similar to ours, where only marginal distributions of bidders are known, He and Li (2020) finds that in an auction environment, second price auction with no reserve price is asymptotically optimal. The optimality of mechanism is the minimum revenue guarantee for the possible joint distributions, similar to our approach of minimum efficiency guarantee. Similar approach was adopted by Carroll (2017) where the principal wants to screen an agent with multi-dimensional type with correlation. The principal just knows the marginal distribution of each component of agent's type and the mechanism is evaluated by the worst case expected profit over all the possible joint distributions.

Our choice of DSIC as a solution concept is without loss of generality, Alternatively, we can look at BIC and IIR mechanisms which are robust to a set of joint distributions consistent with the given marginals. In Chapter 3, we explore the consequences of such *marginal robust BIC and IIR* and show that every such mechanism satisfying BB has the same expected welfare guarantee that a DSIC, EIR, and BB mechanism.

Our paper adds to the long literature on the bilateral trading problem, which is inspired by the impossibility result in Myerson and Satterthwaite (1983). It assumes independent private values and derives the expected welfare-maximizing mechanism – see also Chatterjee and Samuelson (1983) for a description of equilibria of a particular bilateral trading mechanism. While these papers assume common knowledge of priors, our paper follows the models in robust mechanism design literature and relaxes this assumption.

Another strand of literature looks at robustness with respect to information structure (Bergemann et al., 2017; Brooks and Du, 2020; Carroll, 2018). In those environments, there is "fixed" prior distribution over valuations that results from distribution over state spaces and the associated joint distribution over valuations. Then, there is an information structure that determines how the signals would be generated. The information structure affects the strategy of players as it affects the posterior beliefs about valuations. Our approach is different in the sense that that only the information about priors is common knowledge, not the joint distribution. Secondly, they consider general information structures whereas we consider a particular information structure where the signal of each player reveals the true valuation of that player.

As part of the proof, we prove min-max theorem over deterministic posted price mechanisms that extends to then the class of bayesian implementable mechanisms. It is similar to Brooks and Du (2020) that proves a strong min-max theorem but for informationally robust mechanism.

In section 4.2, we introduce the model and designer's problem. In section 4.3, we characterise the worst distribution given any deterministic posted price mechanism and use it to find the optimal robust mechanism restricted to the class of deterministic posted price mechanism. mechanism. In section 4.4, we have the main result of the paper that in the class of DSIC mechanisms, a deterministic posted price is an optimal robust mechanism. We also show equivalence between the optimal robust efficiency gains and minimum guaranteed efficiency gains had the designer known the true distribution of valuation of the buyer and cost of the seller.

4.2 Model

We consider the private values model of bilateral trading.[1] There is a single object for trade, which the seller can produce and the buyer is willing to buy. The valuation of the buyer for the object and the cost of the seller for producing the object are jointly distributed according to a distribution H. The marginal distribution of valuation of the buyer is denoted by F and the marginal distribution of cost of the seller is denoted by G. Though the joint distribution H is common knowledge among agents (the buyer and the seller), the designer does not know H. However, she knows the marginal distributions F and G. We assume that the valuations of buyers lie in $V = [0, \overline{v}]$ and costs lie in $C = [0, \overline{c}]$. We define $\Theta = V \times C$.

We assume that marginal distribution of valuations, F and G are continuous i.e. there are no atoms.[2] There are infinitely many possible joint distributions consistent with the given marginal distributions of agents. The joint distribution does not affect the set from which mechanism can be chosen since all the properties are prior free, but it affects the

[1] The model in same as the model in Chapter 3.

[2] We relax this assumption about the continuity of F and G in the appendix. The results are qualitatively similar to the continuous case.

efficiency gains from a mechanism.

We use marginal density of valuation throughout as it is easy to work with. From the given marginal distribution functions, we can determine the density functions as follows[3]:

$$f(v) = \begin{cases} \dfrac{dF(v)}{dv} & \text{if } F \text{ is differentiable at } v \\ 1 & \text{otherwise.} \end{cases}$$

$$g(c) = \begin{cases} \dfrac{dG(c)}{dc} & \text{if } G \text{ is differentiable at } c \\ 1 & \text{otherwise.} \end{cases}$$

A joint probability density $\hat{h}$ of (v, c) is *consistent* with (f, g) if the marginal density of v and c are f and g respectively:

$$\int_c \hat{h}(v,c)\ dc = f(v) \qquad \forall v \in V$$
$$\int_v \hat{h}(v,c)\ dv = g(c) \qquad \forall c \in C$$

Let $\mathcal{H}^d$ denote the set of all joint densities consistent with (f, g).[4]

Definition 16 *A mechanism* (q, t_b, t_s) *is* **dominant strategy incentive compatible (DSIC)** *if for every* $(v, c) \in \Theta$

$$vq(v,c) - t_b(v,c) \geq vq(v',c) - t_b(v',c) \qquad \forall\ v' \in V$$
$$t_s(v,c) - cq(v,c) \geq t_s(v,c') - cq(v,c') \qquad \forall\ c' \in C$$

Definition 17 *A mechanism* (q, t_b, t_s) *is* **ex-post individually rational (EIR)** *if for every* (v, c)

$$vq(v,c) - t_b(v,c) \geq 0$$
$$t_s(v,c) - cq(v,c) \geq 0$$

Definition 18 *A mechanism* (q, t_b, t_s) *is* **ex-post budget balanced (BB)** *if for every*

[3]By Lebesgue's Theorem for the differentiability of monotone functions, marginal distributions F and G are differentiable almost everywhere. As a result, the efficiency gains value would be unaffected by the choice of f and g at the points where the marginal distributions are not differentiable.

[4]All the notations associated with distributions in previous chapter are analogously defined for densities.

(v, c)

$$t_b(v,c) = t_s(v,c).$$

A budget balanced mechanism can be represented by a pair, (q, t).

Objective of designer: Consider an arbitrary budget balanced mechanism, $(q,t) \in \mathcal{M}$. The trade probability and the true joint probability density, h will determine the efficiency of mechanism and is given by $\mathbb{E}_{(v,c),h}\big[(v-c)q(v,c)\big]$.

For any mechanism, efficiency depends on joint probability density of valuations which is unknown to designer. A natural measure of efficiency of a mechanism would be the worst efficiency, which is refereed to as *robust efficiency gains* of a mechanism (q, t) and is given as

$$\textsc{Eff}(q,t) = \inf_{\hat{h} \in \mathcal{H}^d} \mathbb{E}_{(v,c),\hat{h}}\big[(v-c)q(v,c)\big]$$

The designer would find a mechanism that maximises robust efficiency gains in a class of mechanisms.

Let $\mathcal{M}_{\widehat{D}}$ be the class of mechanisms satisfying dominant strategy incentive compatibility, budget balancedness and ex-post individual rationality. Hagerty and Rogerson (1987) shows that mechanism in class of block mechanisms, any mechanism in $\mathcal{M}_{\widehat{D}}$ can be implemented by posted price mechanisms- randomisation of the deterministic posted price mechanisms. This combined with Theorem 6 of Chapter 3 implies that the designer can focus without loss of generality on the class of posted price mechanisms.

Let $\mathcal{M}_D$ be the collection of all posted price mechanisms. The objective of the designer is to find optimal mechanism (q^*, t^*), where

$$\textsc{Eff}(\mathcal{M}_D) := \sup_{(q,t) \in \mathcal{M}_D} \textsc{Eff}(q,t) = \textsc{Eff}(q^*, t^*)$$

4.3 Worst distributions

Now we restrict our attention to deterministic posted price mechanisms. In such mechanisms, there is a posted price p. If the valuation of buyer, v is greater than the posted price, p and valuation of seller, c is less than posted price, p, then the trade occurs with certainty and price, p will be charged as payment. If the valuation of buyer, v is less than the posted price, p or valuation of seller, c is greater than posted price, p, then trade does not occur with certainty and no payment is made.

A deterministic posted price mechanism, $M^p = (q^p, t^p)$ is defined as follows

$$q^p(v,c) = \begin{cases} 1 & \text{if } v > p \text{ and } c < p \\ 0 & \text{if } v < p \text{ or } c > p \end{cases}$$

$$t^p(v,c) = \begin{cases} p & \text{if } v > p \text{ and } c < p \\ 0 & \text{if } v < p \text{ or } c > p \end{cases}$$

When $v = p$ or $c = p$, the tie between trading and not trading can be broken in anyway we want.

The efficiency gains of M^p for true joint probability density h is given by

$$\begin{aligned} &\int_{v>p}\int_{c<p} (v-c)\ h(v,c)\ dc\ dv \\ &\quad= \int_{v>p}\int_{c<p} v \cdot h(v,c)\ dc\ dv - \int_{v>p}\int_{c<p} c \cdot h(v,c)\ dc\ dv \\ &\quad= \int_{v>p} v \left[\int_{c<p} h(v,c)\ dc\right]\ dv - \int_{c<p} c \left[\int_{v>p} h(v,c)\ dc\right]\ dv \end{aligned} \tag{4.1}$$

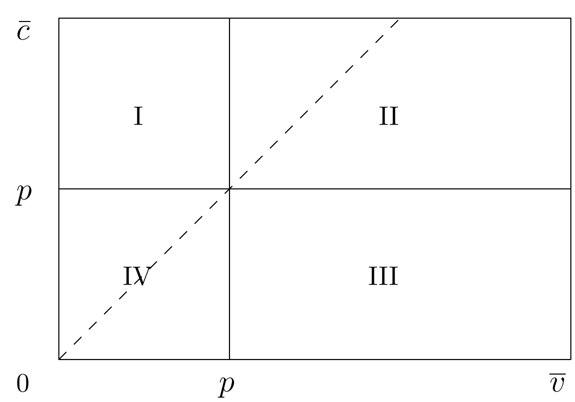


Figure 4.1: Trade region

Consider Figure 4.1. The value of the buyer and the cost of the seller is represented horizontally and vertically, respectively. Given a posted price mechanism M^p, the trade occurs only in region III of Figure 4.1 and will be referred as trade region.

For a given deterministic posted price mechanism, we characterise the set of joint distributions satisfying the marginal distribution of valuations that minimises the efficiency gains. To do that, we would use the concept of redistributing the mass which is explained in the next section.

4.3.1 Redistribution of mass

Consider a joint probability density, $\hat{h} \in \mathcal{H}^d$ such that rectangles A and D have same mass, say m in the corresponding regions. We introduce the idea of redistribution of mass from A and D to B and C. The redistribution would reduce the mass in regions A and D to zero whereas the mass in regions B and D will increase by mass, m.

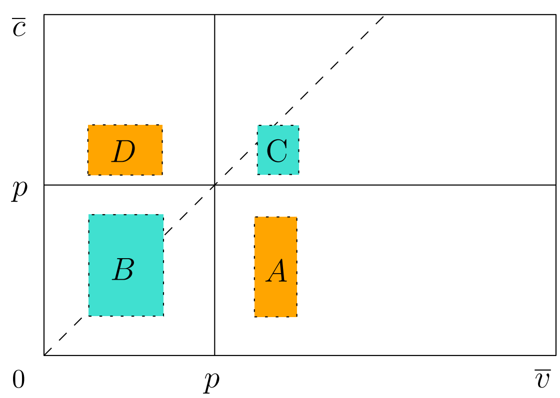


Figure 4.2: Redistributing Mass

For a given $\hat{h}$, consider a new joint probability density h'

$$h'(v,c) = \begin{cases} 0 & \text{if } (v,c) \in A \cup D \\ \\ \dfrac{\int_{y:(v,y)\in B\cup D} \hat{h}(v,y)\ dy \int_{x:(x,c)\in B\cup A} \hat{h}(x,c)\ dx}{\int_{(x,y)\in B\cup A} \hat{h}(x,y)\ dx\ dy} & \text{if } (v,c) \in B \\ \\ \dfrac{\int_{y:(v,y)\in C\cup A} \hat{h}(v,y)\ dy \int_{x:(x,c)\in C\cup D} \hat{h}(x,c)\ dx}{\int_{(x,y)\in C\cup D} \hat{h}(x,y)\ dx\ dy} & \text{if } (v,c) \in C \\ \\ \hat{h}(v,c) & \text{otherwise.} \end{cases} \tag{4.2}$$

The above joint probability density is constructed such that marginal density of a valuation restricted to just the rectangles $A,\ B,\ C$ and D remains unchanged. Since $\hat{h} \in \mathcal{H}^d$, we get $h' \in \mathcal{H}^d$. To be more specific, the marginal density of c over rectangle A and marginal density of v over rectangle D is shifted to rectangle B. The marginal density of c over rectangle D and marginal density of v over rectangle A is shifted to rectangle C. This gives us the marginal densities over each of the rectangles.

Given the marginal densities over a rectangle, we can construct a joint probability density function where the random variables are independent. Suppose $\alpha(x)$ and $\gamma(y)$ are the marginal densities over a rectangular region, $R \equiv X \times Y$. A possible joint probability density, where the random variables are independent in rectangular region is given by $\dfrac{\alpha(x)\ \gamma(y)}{\int_X \alpha(x)dx} \equiv \dfrac{\alpha(x)\ \gamma(y)}{\int_Y \gamma(y)dy}$. Using this fact, we get $h'(v,c)$.

Notice that such a redistribution will decrease the mass in trade region and reduce the efficiency gains of mechanism M^p. We will use the redistributions of the above form to characterise the worst distribution for a given posted price mechanism.

4.3.2 Characteristics of worst distribution for a given posted price mechanism

Fix mechanism M^p. We define $\ell_p := \int_p^{\overline{v}} f(v)\ dv - \int_p^{\overline{c}} g(c)\ dc$. It is the minimum mass from a joint distribution that must lie in the trade region in order to meet the requirement of mass imposed by the marginal distributions of valuations. Note that ℓ_p is a parameter and depends on just the given marginal distributions.

If $\ell_p \leq 0$, there is a possibility where the entire mass can be distributed such that mass in region corresponding to efficiency gains is zero. This possibility is shown in Figure 4.3.

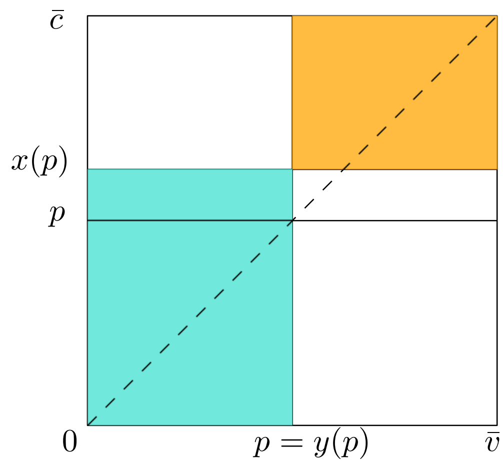


Figure 4.3: Worst distribution given posted price mechanism with price, p

Since efficiency gains is a non-negative number, for the given posted price mechanism, the robust efficiency gain must be 0.

Now we characterise the worst distribution when $\ell_p > 0$. Note that if this is the case, a distribution depicted in Figure 4.3 is not feasible. Thus, any distribution consistent with given marginal distribution of valuations must have positive mass in trade region.

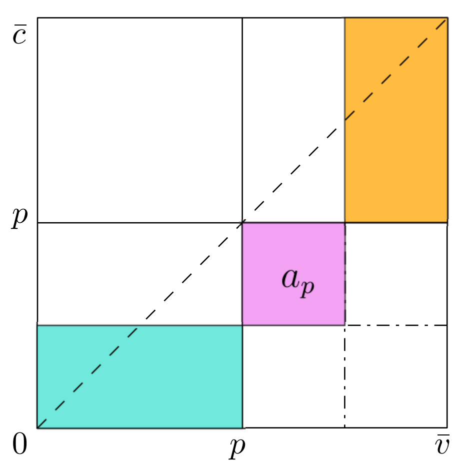


Figure 4.4: Mass in trade region

Let $\hat{h}^p$ be the worst distribution for mechanism M^p. We define $a_p := \int\limits_p^{\bar{v}} \int\limits_0^p \hat{h}^p(x,y)\ dy\ dx$

For $\ell_p > 0$, we must have $a_p > 0$. Such a situation is depicted in Figure 4.4.

We prove few useful lemmas that characterise the worst case distribution, given a deterministic posted price mechanism, M^p.

Lemma 10 *If $a_p > 0$, then $z := \int\limits_0^p \int\limits_p^{\bar{c}} \hat{h}^p(x,y)\ dy\ dx = 0$.*

Proof: Suppose not for contradiction. If $a_p,\ z > 0$, then we can find rectangle A and rectangle D with mass $m > 0$ as shown in Figure 4.5.

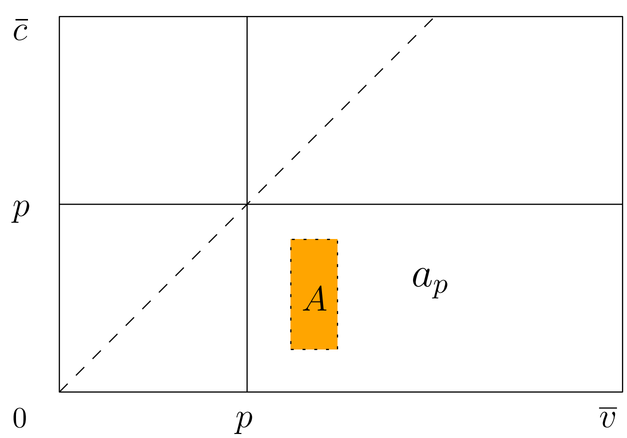


Figure 4.5: Revenue decreasing redistribution

Redistribute mass from A and D to B and C in the manner described by equation (4.2). As a result, efficiency gains will decrease, contradicting the assumption that we started with worst distribution. ■

To further characterise the worst case distribution, we define the set G as the smallest open rectangle with vertex (p,p) and diagonally opposite vertex, (v,c) where $v > p$, and $c < p$, such that $\int_{(x,y)\in G} \hat{h}^p(x,y)\ dy\ dx = a_p$.

Formally, let $R_{k,\ell} \equiv (p,k) \times (\ell,p)$. We have $G \equiv \text{int}\left\{\cap_{(k,\ell)\in T} R_{k,\ell}\right\}$ where $T = \{(k,l) :$

$\int_p^k \int_\ell^p \hat{h}^p(x,y)\ dx\ dy = a_p\}$

We define two rectangles,

$$L_G = \{(v,c) : \exists(x,c) \text{ such that } (x,c) \in G,\ v \in [0,p)\} \text{ and}$$
$$U_G = \{(v,c) : \exists(v,y) \text{ such that } (v,y) \in G,\ c \in (p,\overline{c})\}$$

L_G is the rectangle to the left of G and U_G is a rectangle upward G.

In Figure 4.6, the dotted red rectangle shows the boundary of set G. In the next lemma, we show that mass of area to the left, L_G and above, U_G is zero.

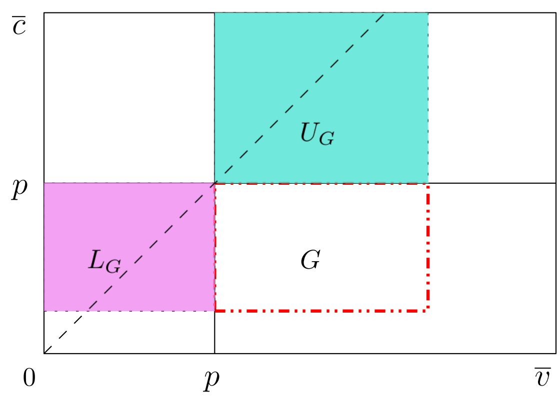


Figure 4.6: Zero mass in left and upper region to G

Lemma 11 *For worst distribution $\hat{h}^p$, mass in region L_G and U_G is zero.*

$$\int_{L_G} \hat{h}^p(x,y)\ dx\ dy = \int_{U_G} \hat{h}^p(x,y)\ dx\ dy = 0$$

Proof: We start with the proof for L_G region.

Suppose for contradiction that there exists a rectangle, with non-zero mass, m in region L_G.. Let D' be the rectangle in L_G with mass, m.[5] Now, consider rectangle A' defined as follows:

$$A' = \{(v,c) : c \in \{x : (x,y) \in D'\}\} \cap G$$

[5]Without loss of generality, we could have considered $D' = L_G$.

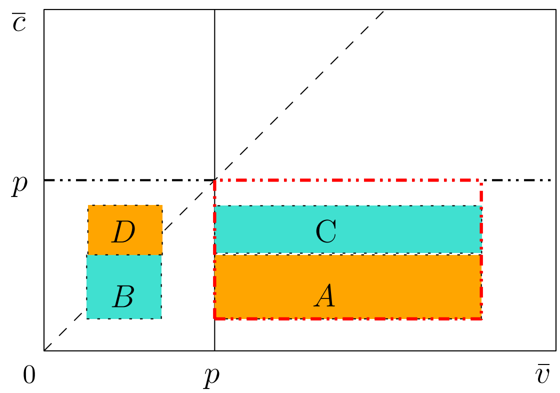


Figure 4.7: Revenue decreasing redistribution for L_G region

Since, D' has positive mass, $A' \neq \emptyset$. Note that we can find $D \subseteq D'$ and $A \subseteq A'$ such that D and A have same mass and $\{c : (v, c) \in A\} \cap \{c : (v, c \in D)\} = \emptyset$ and $\inf D' \geq \sup A'$. To find these A and D, one can find a horizontal line cutting regions A' and D'. As the horizontal line moves upward, the mass below the line in rectangle A' will increase and mass above the line in rectangle D' will decrease.[6] As a result, one can choose horizontal line for region close to the bottom boundary of set G and keep it moving upward until the mass equalises in region above the horizontal line in rectangle D' and region below horizontal line in rectangle A'.

Now consider redistribution from A and D to B and C as per equation (4.2). Notice that as a result, the efficiency gains will decrease, contradicting the assumption that we started with worst distribution.

Analogously, we could argue for upper rectangle, U_G. The redistribution corresponding to this case is shown in Figure 4.8:

■

When $\ell_p > 0$, for consistency with the marginals and the lemmas 10 and 11, we will have regions with mass b and d as shown in Figure 4.9.

[6]For any line passing through G, the mass of region A is strictly positive; it follows from definition of G.

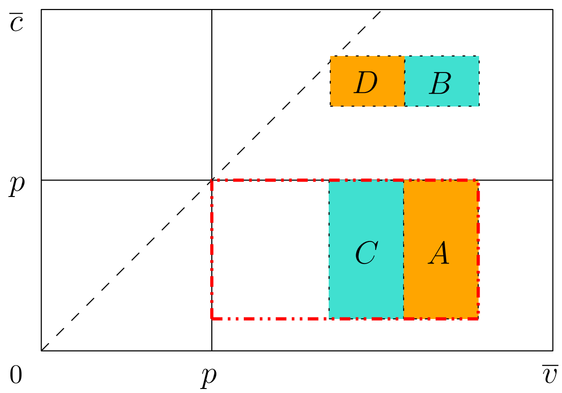


Figure 4.8: Revenue decreasing redistribution for U_G region

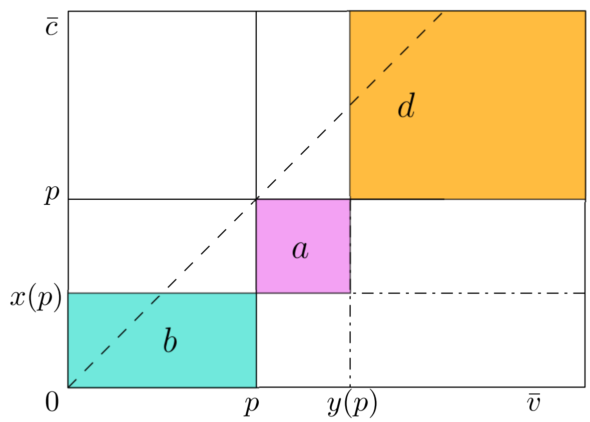


Figure 4.9: Worst distribution when $\ell_p > 0$

Here,

$$
\begin{aligned}
b &= \int_0^{x(p)} g(c)\ dc = \int_0^p f(v)\ dv \\
d &= \int_p^{\bar{c}} g(c)\ dc = \int_{y(p)}^{\overline{v}} f(v)\ dv \\
a &= \int_0^p g(c)\ dc - \int_0^p f(v)\ dv
\end{aligned}
\tag{4.3}
$$

Note that here, $x(p) < p$. [7] For $p = 0$, by definition, $x(p) = 0$ and the efficiency gains is zero.

From the analysis above, depending upon ℓ_p, we will have either of the two distributions as worst distribution. When $\ell_p \leq 0$, the efficiency gains is zero.

[7] For $\ell_p < 0$, $x(p) > p$.

Whenever the equations (4.3) are satisfied and $\ell_p > 0$, irrespective of how the mass is distributed within three rectangles, the efficiency gains are same and given by

$$\int_p^{y(p)} vf(v)\ dv - \int_{x(p)}^{p} cg(c)\ dc$$

where

$$\int_0^{x(p)} g(c)\ dc = \int_0^{p} f(v)\ dv$$
$$\int_{y(p)}^{\overline{v}} f(v)\ dv = \int_p^{\overline{c}} g(c)\ dc$$
$$x(p) \leq p$$

This gives us the first proposition of our paper.

Proposition 2 *The optimal robust mechanism within the class of deterministic posted price mechanism is mechanism with posted price p^*, where p^* is solution to*

$$\arg\max_{p\in\mathbb{R}_+} \quad \int_p^{y(p)} vf(v)\ dv - \int_{x(p)}^{p} cg(c)\ dc$$

$$\textit{where } G(x(p)) = F(p), F(y(p)) = G(p) \textit{ and } x(p) \leq p$$

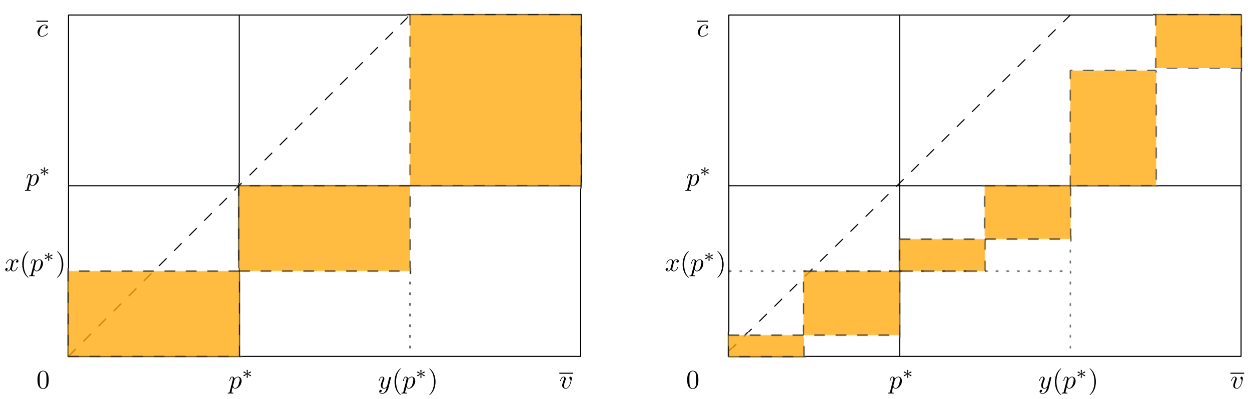
Let $A = \max_p \int_p^{y(p)} vf(v)\ dv - \int_{x(p)}^{p} cg(c)\ dc$. This represents the robust efficiency gains corresponding to optimal robust mechanism.

Proposition 2 characterized that the optimal robust mechanism in the class of deterministic mechanisms. Our main result will show that this mechanism is also the optimal robust mechanism if we allowed for random mechanisms.

4.4 Main result

In this section, we show that the mechanism M^{p^*} discussed in Proposition 2 is in fact the optimal robust mechanism. We use the observation that for a collection of consistent joint distributions, $\mathcal{F}(p^*)$, mechanism M^{p^*} is the optimal mechanism and as a result, M^{p^*} is optimal robust mechanism. [8]

For posted price mechanism M^{p^*}, there is a worst distribution of the form that we found in previous section. In that form, we had three rectangles with positive mass. Consider the collection of consistent joint distributions, $\mathcal{F}(p^*)$ having finer mass distribution than the obtained worst distribution. For illustration, consider Figure 4.10. Three segments of buyer's valuation further divided into two equal parts. Given the six segments for buyer's valuation, we can six segments of valuation of seller ensuring consistency in marginal distribution. Now, we will have six smaller rectangle with mass.



(a) Coarse distribution

(b) Fine distribution

Figure 4.10: Joint distributions in collection $\mathcal{F}(p)$

Such a finer mass distribution ensures that efficiency gains by mechanism, M^{p^*} for these joint distributions is same as the guaranteed efficiency gains associated with mechanism as these finer distributions are of the form of worst distribution associated with M^{p^*}.

Consider any posted price mechanism, M with distribution function, $G(p)$ over prices. The efficiency gains from M for a finer distribution can possibly be greater than A. But as we make the distributions finer and finer, the efficiency gains converges to the robust efficiency gains from M. As a result, the efficiency gains from the posted price mechanism is convex combination of robust efficiency gains of mechanisms, M^p. Since M^{p^*} is the optimal robust mechanism in the class of deterministic posted price mechanisms, p^*, the robust

[8] For all ϵ, the efficiency gains of mechanism, M from the associated worst distribution cannot exceed the one corresponding to M^{p^*} by an amount more than ϵ.

efficiency gains corresponding to M^{p^*} will be greater than those corresponding to posted price mechanism. It follows that M^{p^*} is optimal robust mechanism in $\mathcal{M}_D$.

Theorem 7 *The posted price mechanism M^{p^*} is an optimal robust mechanism.*

The theorem implies that if we are interested in optimal robust mechanisms in bilateral trading setting, we can restrict ourselves to a very simple class of mechanism- deterministic posted price mechanism.

We have considered the max-min problem till now, where designer wants to design an optimal robust mechanism and the efficiency gains associated with optimal robust mechanism gives the lower bound on the efficiency gains that can be realised if the designer optimally chooses the mechanism. The alternate lower bound for efficiency gains will be the one in which there is uncertainty about the true joint probability density but after the realised value of joint probability density, the designer can optimally choose mechanism. We will expect that the efficiency gains in such a situation will be higher as the designer gets to choose mechanism after realisation of true joint probability density in comparison to choosing the mechanism before realisation in case of max-min problem. Though we get that these lower bounds actually coincide and we have the following theorem:

Theorem 8 *The value of max-min and min-max of efficiency gains are equal for the class of DSIC mechanisms with BB and IR.*[9]

$$\inf_{\hat{h}\in\mathcal{H}^d} \sup_{(q,t)\in\mathcal{M}_D} \int_v \int_c (v-c)\hat{h}(v,c)\ q(v,c)\ dc\ dv = \sup_{(q,t)\in\mathcal{M}_D} \inf_{\hat{h}\in\mathcal{H}^d} \int_v \int_c (v-c)\hat{h}(v,c)\ q(v,c)\ dc\ dv$$

The above theorem holds because restricted to collection $\mathcal{F}(p^*)$, p^* maximises efficiency gains. Restricted to these joint densities, the guaranteed efficiency gains had we chosen the mechanism optimally for true joint probability density is $A = \textsc{Eff}(M^{p^*})$. Combined with the fact that min-max is not less than max-min in any problem, we get the value of max-min and min-max of efficiency gains equal.

[9] The equality would hold even for more general setting of BIC mechanisms, BB and interim individually rational.

4.5 Appendix

4.5.1 Proof of Theorem 7

For a given p, we define a collection of consistent joint densities as follows:
$\mathcal{F}(p) = \{\hat{h} : \exists\, n \in \mathbb{N}\ \&\ \{c_i\}_{i\in\mathbb{N}} \text{ s.t. } \int_0^{\frac{k}{n}t+s} f(v)\, dv = \int_0^{c_k} g(c)\, dc = \int_0^{\frac{k}{n}t+s} \int_0^{c_k} \hat{h}(v,c)\, dv\, dc, \forall k \in \{1,\dots,n\}, \forall (t,s) \in \{(0,p),(p,y(p)-p),(y(p),\overline{v}-y(p))\}\}$.[10]

The collection $\mathcal{F}(p)$ is collection of consistent joint distributions that are finer than the worst distribution constructed in section 4.3 and generates efficiency gains of A. This can be viewed in Figure 4.11.

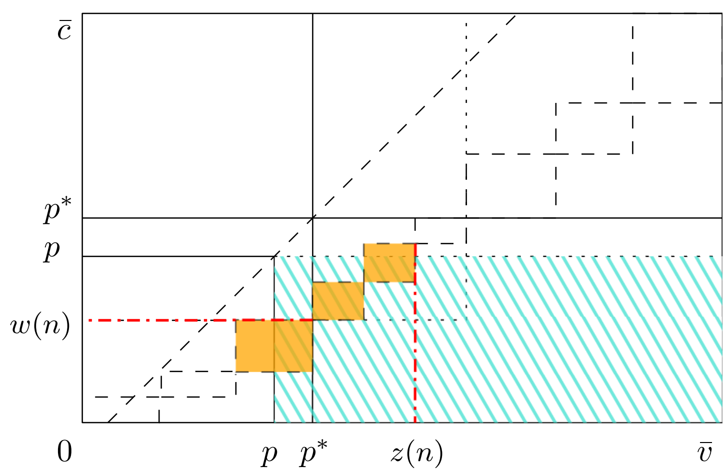


Figure 4.11: Sequence of joint distributions

Notice that there are sets $\{0, v_1, \dots, v_n\}$ and $\{0, c_1, \dots, c_n\}$ such that there is mass only in rectangles of form $[v_{k-1}, v_k] \times [c_{k-1}, c_k]$ where $k \in \{1, \dots, n\}$.

Consider an arbitrary joint probability density, $\hat{h}_n(v,c) \in \mathcal{F}(p^*)$. From analysis of max-min exercise, we know that just the marginal density in the region of trade matters for

[10] We will sometimes use $\hat{h}_n \in \mathcal{F}(p^*)$ to denote the joint probability density with n partitions in each of three rectangles in three regions.

calculation of efficiency gains. Thus, the efficiency gains for posted price mechanism-M^p is upper bounded by [11]

$$
\begin{aligned}
&\int_p^{z(n)} vf(v)dv - \int_{w(n)}^p cg(c)dc \text{ where}\\
&w(n) = \inf\left\{c_k : \int_0^p f(v)dv \leq \int_0^{c_k} g(c)\ dc, k \in \{1,\ldots,n\}\right\} \text{ and}\\
&z(n) = \inf\left\{v_k : \int_0^p g(c)dc \leq \int_0^{v_k} f(v)dv, k \in \{1,\ldots,n\}\right\}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\int_0^{p+\delta_1(n)} f(v)\ dv &= \int_0^{w(n)} g(c)\ dc\\
\int_0^{z(n)} f(v)\ dv &= \int_0^{p+\delta_2(n)} g(c)\ dc
\end{aligned}
$$

Since δ_1 and δ_2 are strictly decreasing in n,

$$
\begin{gathered}
w(n) \to x(p),\ z(n) \to y(p)\ \&\\
\int_p^{z(n)} vf(v)dv - \int_{w(n)}^p cg(c)dc \to \int_p^{y(p)} vf(v)dv - \int_{x(p)}^p cg(c)dc \leq A
\end{gathered}
$$

The efficiency gains of posted price mechanism, (q,t) is given by

$$
\lim_{n\to\infty} \mathbb{E}_{M,\hat{h}_n\in\mathcal{F}(p^*)} \left(\int_p^{z(n)} vf(v)dv - \int_{w(n)}^p cg(c)dc\right) \leq A
$$

Since the above relation holds for arbitrary posted price mechanism, we get

$$
\sup_{(q,t)\in\mathcal{M}_D} \inf_{\hat{h}\in\mathcal{F}(p^*)} \int_v \int_c (v-c)\hat{h}(v,c)\ q(v,c)\ dc\ dv \leq A
$$

As $\mathcal{F}(p^*) \subset \mathcal{H}^d$, we get

$$
\text{EFF}(\mathcal{M}_D) = \sup_{(q,t)\in\mathcal{M}_D} \inf_{\hat{h}\in\mathcal{H}^d} \int_v \int_c (v-c)\hat{h}(v,c)\ q(v,c)\ dc\ dv \leq A
$$

[11] The diagram shows for $p < p^*$. Argument is same for $p > p^*$.

But

$$\textsc{Eff}(\mathcal{M}_D) \geq \textsc{Eff}(M^{p^*}) = A$$

Thus, we get $\textsc{Eff}(\mathcal{M}_D) = A$ and M^{p^*} is an optimal robust mechanism in $\mathcal{M}_D$.

4.5.2 Proof of Theorem 8

By definition of inf and the fact that $\hat{h}_n(v,c) \in \mathcal{F}(p^*)$

$$\inf_{\hat{h}\in\mathcal{F}(p^*)} \sup_{(q,t)\in\mathcal{M}_D} \int_v \int_c (v-c)\hat{h}(v,c)\ q(v,c)\ dc\ dv \leq \lim_{n\to\infty} \sup_{(q,t)\in\mathcal{M}_D} \int_v \int_c (v-c)\hat{h}_n(v,c)\ q(v,c)\ dc\ dv \tag{4.4}$$

By optimality of M^{p^*} for very fine joint distributions,

$$\begin{aligned} \lim_{n\to\infty} \sup_{(q,t)\in\mathcal{M}_D} \int_v \int_c (v-c)\hat{h}_n(v,c)\ q(v,c)\ dc\ dv &= \lim_{n\to\infty} \int_{v>p^*} \int_{c<p^*} (v-c)\ \hat{h}_n(v,c)\ dc\ dv \\ &= A \end{aligned} \tag{4.5}$$

The last equality follows from the fact that for any joint probability density, $\hat{h}_n(v,c) \in \mathcal{F}(p^*)$, the efficiency gains is A as it is of the form of worst distribution for mechanism M^{p^*}. Combining (4.4) and (4.5) and the fact $\mathcal{H}^d \supset \mathcal{F}(p^*)$ that we get

$$\inf_{\hat{h}\in\mathcal{H}^d} \sup_{(q,t)\in\mathcal{M}_D} \int_v \int_c (v-c)\hat{h}(v,c)\ q(v,c)\ dc\ dv \leq A \tag{4.6}$$

But,

$$\begin{aligned} \inf_{\hat{h}\in\mathcal{H}^d} \sup_{(q,t)\in\mathcal{M}_D} \int_v \int_c (v-c)\hat{h}(v,c)\ q(v,c)\ dc\ dv &\geq \sup_{(q,t)\in\mathcal{M}_D} \inf_{\hat{h}\in\mathcal{H}^d} \int_v \int_c (v-c)\hat{h}(v,c)\ q(v,c)\ dc\ dv \\ &\equiv A \end{aligned} \tag{4.7}$$

Equations (4.6) and (4.7) establishes equivalence between min-max and max-min exercise.

Chapter 5

Asymmetric auctions with entry

5.1 Introduction

A common assumption in auction theory and mechanism design is that the set of bidders is determined exogenously. This may not be a realistic assumption in many settings. An important setting where entry is endogenous is the government procurement auctions. The suppliers need to incur significant fixed costs before entering the bidding process. These auctions with entry induce different incentives from suppliers than in the standard auctions. We study such auctions with entry from an asymmetric set of suppliers and characterize the *optimal* auction.

Our results extend the result of McAfee and McMillan (1987), who showed that a second-price auction with an anonymous participation fee is an optimal auction when suppliers are *symmetric*. This is in contrast to the standard auction models, where the optimal auction involves a *reserve price* (Myerson, 1981). In our model, suppliers are asymmetric in terms of fixed cost and distribution of per unit cost, but they are ordered in a stochastic sense: the cost distribution of suppliers can be ordered in terms of first order stochastic dominance.

We show that the second price auction with *non-anonymous participation fee* is an optimal auction, i.e., an auction which minimizes the expected cost of procurement. This optimal auction is also ex-ante efficient: maximizes the social welfare (expected total cost of procurement). Our optimal auction involves supplier specific participation fee and the procurer extracts the full ex-ante surplus from the suppliers. The results do not require any assumption on the joint distribution of costs of suppliers except the stochastic ordering of the distributions across suppliers.

We apply our main result to a two period model where the auctioneer needs to procure a good repeatedly in two periods. The suppliers get cost advantage in the second period by supplying in the first period. This cost advantage is modeled as follows: a supplier who supplies in the first period gets to draw her variable cost from a stochastically better distribution. We identify simple conditions on the primitives of the model such that split awards (i.e., procuring from multiple suppliers) dominate single sourcing (i.e., procuring from a single supplier) in the first period of our two period model. The simple intuition for this result is that split awards give the benefit of cost reduction to more firms in the second period. But splitting the award in the first period also increases cost of procurement in the first period. We show that if the "marginal" benefit of achieving this cost reduction in the second period is higher for lower levels of production in the first period than at higher levels, split awards will reduce expected cost of procurement.

Note that in a single period model, single sourcing is optimal, where the lowest cost supplier supplies the entire unit of the good. However, in many instances of repeated procurement auctions, we observe that multiple suppliers (split award) supply the good. For example, vaccine procurement is done repeatedly among the same set of potential suppliers and many countries practice split awards to procure vaccines. Our application justifies the split awards in a simple model involving two periods, where suppliers who are awarded the contract in the first period get a (variable) cost advantage in the second period.

The literature on split awards versus single sourcing is divided on which auction performs better. An exact statement depends on the auction environment. Perry and Sákovics (2003), Anton et al. (2010), Gilbert and Klemperer (2000), Gong et al. (2012) identify settings where split awards are better than single sourcing. Our result contributes to this literature. The detailed literature review is presented in section 5.6.

5.2 An example

The procurer must procure a divisible object from set of potential suppliers.[1] Consider three potential suppliers, $i \in \{1, 2, 3\}$. The fixed cost are $f_1 = 0.1, f_2 = 0.2$ and $f_3 = 0.3$. The per unit costs are drawn from $[0, 1]$ using distribution $G_i = x^i$. The fixed cost and cost distributions are common knowledge. Notice that firm 1 has lowest fixed cost and the cost distribution is stochastically dominated, making it most efficient among the potential supplier, followed by firm 2.

The auctioneer announces a mechanism. Each firm decides to enter into auction without realising their per unit cost. Then auction takes place according to pre-announced auction format.

We want to find the optimal mechanism and its characteristics- Should the mechanism give subsidy or charge participation fee? How many suppliers enter into auction in equilibrium? Which subset of suppliers participate in auction? Should the mechanism have single sourcing or split award? How does optimal auction change with introduction of learning by doing?

We present the asymmetric model formally in the next section.

5.3 The asymmetric model

There is a universe of suppliers denoted by $\mathcal{S}$. A buyer needs to procure one unit of a divisible good. Entry of each supplier $i \in \mathcal{S}$ involves a fixed cost f_i. Once suppliers enter, they have capacity to produce the entire unit. Further, once a supplier i enters, it realizes a per unit cost c_i. Each c_i is drawn from $[0, K]$ using distribution G_i with positive density function g_i. We will write $g_{-i}(c_{-i}) \equiv \times_{j \neq i} g_j(c_j)$. If supplier i supplies $q_i \in [0, 1]$ units of the good and is paid a transfer t, her utility is $t - c_i q_i$, where q_i may either be interpreted as the fraction of total good being supplied.

[1] Such a situation is captured when outside option for procurer imposes high cost.

Ex-ante ordered suppliers. We will assume that suppliers are ex-ante ordered in the following sense:

$$\text{Fixed cost ordering:} \qquad f_1 \leq f_2 \leq \ldots \leq f_{|\mathcal{S}|}$$
$$\text{Ex-ante ordering:} \qquad G_1(x) \geq G_2(x) \geq \ldots \geq G_{|\mathcal{S}|}(x) \qquad \forall\, x \in [0, K].$$

Since supplier 1 has the lowest fixed cost and for every x, its probability of having per unit cost lower than x is the highest among all suppliers, supplier 1 is the most ex-ante efficient supplier. The next ex-ante efficient supplier is supplier 2, and so on.

Mechanism. The buyer needs to announce a mechanism without knowing which suppliers will participate in a mechanism (such a choice of mechanism determines the entry decision of suppliers). Using the revelation principle, we only focus on direct mechanisms, where suppliers directly report their costs and the designer computes allocations and transfers. Consider a mechanism when a set of suppliers $N \subseteq \mathcal{S}$ (with $|N| = n$) enter. A mechanism is a tuple $(Q, T) \equiv \{Q_i, T_i\}_{i \in N}$, where $Q_i : [0, K]^n \to [0, 1]$ is the allocation rule and $T_i : [0, K]^n \to \mathbb{R}$ is the transfer rule for supplier i.[2] Allocations have to be feasible: for every profile of costs $c \equiv (c_1, \ldots, c_n)$,

$$\sum_{i \in N} Q_i(c) = 1.$$

So, we assume that the buyer has to procure the entire unit. Also, transfers can be potentially negative: a *participation* fee in the auction. Transfers can be positive when no quantity is allocated: a *subsidy*.

Incentive constraints. We will impose Bayesian incentive constraints on the mechanism. The following notation will be useful to define it. Let

$$q_i(c_i) := \int_{[0,K]^{n-1}} Q_i(c_i, c_{-i}) g_{-i}(c_{-i}) dc_{-i}$$

[2] Since the buyer does not know which set of suppliers will participate, a mechanism must announce an allocation rule and transfer rule for every possible set of participating suppliers N. We suppress this notation from the definition of mechanism for making notation simpler.

be the interim quantity allocated to *type* c_i (who reports c_i to the mechanism). Let

$$t_i(c_i) := \int_{[0,K]^{n-1}} T_i(c_i, c_{-i}) g_{-i}(c_{-i}) dc_{-i}$$

be the interim transfer amount of type c_i. The interim utility of type c_i from truthful reporting is denoted by $u_i(c_i) := t_i(c_i) - c_i q_i(c_i)$.

Definition 19 *A mechanism (Q, T) is Bayesian incentive compatible (IC) for a set of suppliers N if for every $i \in N$ and for every $c_i, c_i' \in [0, K]$*

$$u_i(c_i) \geq t_i(c_i') - c_i q_i(c_i')$$

Note that a mechanism (Q, T) may be IC for a set of suppliers $N \subseteq \mathcal{S}$ but may not be IC for another set of suppliers $N' \subseteq \mathcal{S}$. The interim terms u_i, t_i, q_i all depend on the distribution of cost of suppliers and will be different for different set of suppliers.

Simple algebraic manipulation of RHS of IC constraints give $u_i(c_i) \geq u_i(c_i') + (c_i' - c_i) q_i(c_i')$. Using standard Myersonian techniques, we can show (Börgers and Krahmer (2015)) that u_i is convex (and hence, absolutely continuous and differentiable almost everywhere), and its derivative (almost everywhere) is $-q_i$: $\frac{du_i(c_i)}{dc_i} = -q_i(c_i)$. Since u_i is convex, q_i is decreasing: lower cost gives you higher interim quantities. These facts in fact characterize the IC constraints: u_i convex and $-q_i$ being a subgradient of u_i. Hence, we can write for any IC mechanism (Q, T), the interim payoff of supplier of type c_i is

$$u_i(c_i) = u_i(K) - \int_K^{c_i} q_i(x) dx = u_i(K) + \int_{c_i}^{K} q_i(x) dx, \tag{5.1}$$

where this equation follows from the fundamental theorem of calculus and the fact that $-q_i(c_i)$ is the subgradient of u_i at c_i. Conversely, any mechanism (Q, T) which satisfies monotonicity, i.e. q_i is decreasing with increasing c_i for each c_i, and satisfies Equation (5.1) is IC. These are standard facts and can be derived using well-known techniques (Börgers and Krahmer (2015)).

So, the expected payment of the buyer from mechanism (Q, T), when a set of N suppliers enter is

$$\textsc{Pay}(Q, T; N) = \sum_{i \in N} \int_0^K t_i(c_i) g_i(c_i) dc_i$$

The expected payment (and all notations in the mechanism like q_i, t_i, u_i) depend on the set of suppliers who participate in the mechanism.

Note that a blind minimization of payment does not make sense because this expression can be minimized by setting $t_i(c_i) = 0$ and $q_i(c_i) = 0$ for all i and for all c_i (i.e. not procuring anything). So, the minimization has to be under feasibility constraints: $\sum_{i \in N} Q_i(c) = 1$.

Individual rationality. It is usual to impose interim individual rationality constraints on the mechanisms. This will mean $u_i(c_i) \geq 0$ for all c_i, which is equivalent to requiring $u_i(K) \geq 0$ (due to Equation 5.1). But imposing such a constraint may not make sense if the entry decision is ex-ante. So, the right participation constraint is ex-ante payoff in the mechanism is at least the fixed cost:

$$\int_0^K u_i(c_i) g_i(c_i) dc_i \geq f_i \tag{5.2}$$

Note that u_i depends on the set of suppliers who have entered. So, ex-ante payoff of a supplier depends on the set of suppliers who decide to enter (along with i).

Using Equation (5.1), the above constraint simplifies to

$$\begin{aligned} u_i(K) + \int_0^K \int_{c_i}^K \big[q_i(x) dx\big] g_i(c_i) dc_i \geq f_i \\ \iff u_i(K) + \int_0^K G_i(c_i) q_i(c_i) dc_i \quad \geq f_i \qquad \text{(changing order of integration)} \end{aligned} \tag{5.3}$$

So, if the set of suppliers N enter, an expected cost minimizing buyer must solve the following optimization problem:

$$\min \sum_{i \in N} \int_0^K t_i(c_i) g_i(c_i) dc_i$$

$$\text{s.t.} \qquad \text{IC constraints hold}$$

$$u_i(K) + \int_0^K G_i(c_i) q_i(c_i) dc_i \geq f_i \qquad \forall\ i \in N$$

$$\sum_{i \in N} Q_i(c) = 1 \qquad \forall\ c.$$

A mechanism which solves the above optimization program (for a fixed N) is called an N-*optimal* mechanism. Let the value of the optimal solution be denoted as $\text{PAY}(Q, T; N)$. Note that $\text{PAY}(Q, T; N)$ is the minimum expected payment of the buyer over all IC mechanisms that induce the set of suppliers N to enter.

Definition 20 *A mechanism* (Q, T) **induces N-entry** *if*

$$u_i(K) + \int_0^K G_i(c_i) q_i(c_i) dc_i \geq f_i \qquad \forall\ i \in N \tag{5.4}$$

and it is IC for the set of suppliers N: for every $i \in N$ and for every $c_i \in [0, K]$,

$$u_i(c_i) \geq t_i(c_i') + c_i q_i(c_i') \qquad \forall\ c_i' \in [0, K], \tag{5.5}$$

where u_i, t_i, q_i are computed with respect to the set of suppliers N.

If a mechanism induces N-entry, then the ex-ante IR constraint must hold for the set of suppliers in N and must not hold for the set of suppliers in $\mathcal{S} \setminus N$. Since a mechanism always allows to charge participation fee, one way to ensure suppliers in $\mathcal{S} \setminus N$ to not enter is to charge a high participation fee. Hence, we do not need to explicitly mention this non-participation constraint.

Definition 21 *A mechanism* (Q, T) *is* **optimal** *if there exists $N \subseteq \mathcal{S}$ such that it induces*

N-entry and for every $N' \subseteq \mathcal{S}$ and every (Q', T') that induces N'-entry, we have

$$\text{PAY}(Q, T; N) \leq \text{PAY}(Q', T'; N')$$

5.4 The result: efficiency and cost minimization

Theorem 9 *A second-price auction with a supplier-specific non-negative participation fee is an optimal mechanism. Further, this optimal mechanism satisfies the following properties:*

1. *There exists an integer n^* such that the optimal mechanism induces $[n^*]$-entry, where $[n^*] = \{1, \ldots, n^*\}$.*
2. *The participation fee of any supplier $i \in [n^*]$ is such that the ex-ante payoff of i is zero.*
3. *The participation fee of any supplier $i > n^*$ equals the participation fee of supplier n^*.*

Remark. Note that suppliers are ordered: $f_1 \leq \ldots \leq f_{|\mathcal{S}|}$ and $G_1(x) \geq \ldots \geq G_{|\mathcal{S}|}(x)$ for all $x \in [0, K]$. As will be clear in the proof, n^* will be such that the mechanism will be *ex-ante efficient*, i.e., n^* will be chosen such that ex-ante cost of procurement is minimized.

Proof: Let (Q, T) be an N-optimal mechanism. Define a new mechanism (Q^*, T^*) for N-agents as follows.[3] First, allocation rule Q^* allocates the entire unit to the lowest cost supplier at all profiles. Then, for all $i \in N$, define

$$u_i^*(K) := f_i - \int_0^K \prod_{j \neq i : j \in N} (1 - G_j(x)) G_i(x) dx. \tag{5.6}$$

Using (5.1) and the fact that Q^* is monotone, T^* can be defined from Q^* and $u_i^*(K)$ for each i such that IC constraints hold. Since Q^* selects the lowest cost supplier, for each i and for each x

$$q_i^*(x) = \prod_{j \neq i : j \in N} (1 - G_j(x))$$

[3]Notation. If we write a mechanism (Q^*, T^*), we denote its interim variables as q^*, t^*, u^*. Similarly, if we write a mechanism as $(\hat{Q}, \hat{T})$, we denote its interim variables as $\hat{q}, \hat{t}, \hat{u}$, and so on.

Hence, by (5.6) and (5.3), ex-ante IR constraints hold with equality for each supplier in N in mechanism(Q^*, T^*). For suppliers who do not belong to N, we can always set a high participation fee and violate their ex-ante IR constraints.

So, (Q^*, T^*) is an IC and ex-ante IR mechanism which induces N-entry. Also, note that since (Q, T) is an ex-ante IR mechanism, for every i

$$u_i^*(K) + \int_0^K q_i^*(x)G_i(x)dx = f_i \leq u_i(K) + \int_0^K q_i(x)G_i(x)dx \tag{5.7}$$

Now, note that

$$\begin{aligned}
\text{PAY}(Q,T;N) &= \sum_i \Big[\int_0^K t_i(c_i)g_i(c_i)dc_i\Big] \\
&= \sum_i \Big[\int_0^K \big[u_i(c_i) + c_i q_i(c_i)\big] g_i(c_i)dc_i\Big] \\
&= \sum_i \Big[u_i(K) + \int_0^K \big[\int_{c_i}^K q_i(x)dx + c_i q_i(c_i)\big] g_i(c_i)dc_i\Big] \\
&= \sum_i \Big[u_i(K) + \int_0^K G_i(c_i)q_i(c_i)dc_i + \int_0^K c_i q_i(c_i)g_i(c_i)dc_i\Big]
\end{aligned}$$

A similar equation holds for the expected payment in (Q^*, T^*). If (Q^*, T^*) is not N-optimal then, we must have

$$\begin{gathered}
\sum_i \Big[u_i(K) + \int_0^K G_i(c_i)q_i(c_i)dc_i + \int_0^K c_i q_i(c_i)g_i(c_i)dc_i\Big] \\
< \sum_i \Big[u_i^*(K) + \int_0^K G_i(c_i)q_i^*(c_i)dc_i + \int_0^K c_i q_i^*(c_i)g_i(c_i)dc_i\Big]
\end{gathered}$$

Using (5.7), we get (we write $g(c) \equiv \times_{i \in N} g_i(c_i)$)

$$\begin{aligned}
\sum_i \Big[\int_0^K c_i q_i(c_i) g_i(c_i) dc_i \Big] &< \sum_i \Big[\int_0^K c_i q_i^*(c_i) g_i(c_i) dc_i \Big] \\
\Rightarrow \sum_{i \in N} \Big[\int_{[0,K]^n} c_i Q_i(c) g(c) dc \Big] &< \sum_{i \in N} \Big[\int_{[0,K]^n} c_i Q_i^*(c) g(c) dc \Big] \\
\Rightarrow \int_{[0,K]^n} \Big[\sum_{i \in N} Q_i(c) c_i \Big] g(c) dc &< \int_{[0,K]^n} \sum_{i \in N} \Big[Q_i^*(c) c_i \Big] g(c) dc = \int_{[0,K]^n} c_{[1]} g(c) dc,
\end{aligned}$$

where $c_{[1]}$ is the cost of the lowest cost supplier at cost profile c. This means there is some generic cost profile c such that

$$\sum_{i \in N} c_i Q_i(c) < c_{[1]},$$

which is a contradiction to the definition of $c_{[1]}$.

Hence, for any N, the expected payment from the N-optimal mechanism (Q^N, T^N) is

$$\begin{aligned}
\textsc{Pay}(N) &= \sum_{i \in N} \Big[u_i^N(K) + \int_0^K G_i(c_i) q_i^N(c_i) dc_i + \int_0^K c_i q_i^N(c_i) g_i(c_i) dc_i \Big] \\
&= \sum_{i \in N} f_i + \sum_{i \in N} \int_0^K c_i q_i^N(c_i) g_i(c_i) dc_i, \qquad (5.8)
\end{aligned}$$

where the second equality follows from (5.7).

Now, for every i and every $x \in [0, K]$,

$$q_i^N(x) = \prod_{j \neq i : j \in N} (1 - G_j(x)). \qquad (5.9)$$

Fix $k \in N$ and write

$$\textsc{Pay}(N) = \sum_{i \in N} f_i + \int_0^K x q_k^N(x) g_k(x) dx + \sum_{i \in N \setminus \{k\}} \int_0^K x q_i^N(x) g_i(x) dx$$

Let (Q^{N-k}, T^{N-k}) be an $N \setminus \{k\}$-optimal mechanism. Then, note that for any $j \in N \setminus \{k\}$,

$q_j^{N-k}(x) = \prod_{i \in N \setminus \{k,j\}} (1 - G_i(x))$. Hence, $q_j^N(x) = q_j^{N-k}(x)(1 - G_k(x))$. Hence, we can write

$$
\begin{aligned}
\textsc{Pay}(N) &= \sum_{i \in N} f_i + \int_0^K x q_k^N(x) g_k(x) dx + \sum_{j \in N \setminus \{k\}} \int_0^K x q_j^N(x) g_j(x) dx \\
&= \sum_{i \in N} f_i + \int_0^K x q_k^N(x) g_k(x) dx + \sum_{j \in N \setminus \{k\}} \int_0^K x q_j^{N-k}(x) g_j(x)(1 - G_k(x)) dx
\end{aligned}
$$

Further, for every x

$$
\frac{dq_k^N(x)}{dx} = -\sum_{j \neq k} g_j(x) \prod_{i \in N \setminus \{k,j\}} (1 - G_i(x)) = -\sum_{j \neq k} g_j(x) q_j^{N-k}(x) \tag{5.10}
$$

Hence, we can rewrite for every i,

$$
\begin{aligned}
\int_0^K x q_k^N(x) g_k(x) dx &= \Big[[x q_k^N(x) G_k(x)]_0^K - \int_0^K \frac{d[x q_k^N(x)]}{dx} G_k(x) dx \Big] \\
&= -\Big[\int_0^K \big[q_k^N(x) - x \sum_{j \in N \setminus \{k\}} g_j(x) q_j^{N-k}(x) \big] G_k(x) dx \Big]
\end{aligned} \tag{5.11}
$$

Using this, we rewrite

$$
\begin{aligned}
\textsc{Pay}(N) &= \sum_{i \in N} f_i + \int_0^K x q_k^N(x) g_k(x) dx + \sum_{j \in N \setminus \{k\}} \int_0^K x q_j^{N-k}(x) g_j(x)(1 - G_j(x)) dx \\
&= \sum_{i \in N} f_i - \int_0^K q_k^N(x) G_k(x) dx + \sum_{j \in N \setminus \{k\}} \int_0^K x g_j(x) q_j^{N-k}(x) G_k(x) dx \\
&+ \sum_{j \in N \setminus \{k\}} \int_0^K x q_j^{N-k}(x) g_j(x)(1 - G_k(x)) dx \\
&= \sum_{i \in N} f_i - \int_0^K q_k^N(x) G_k(x) dx + \sum_{j \in N \setminus \{k\}} \int_0^K x q_j^{N-k}(x) g_j(x) dx
\end{aligned} \tag{5.12}
$$

Equation (5.11) shows a recursive relation between optimal payment with N suppliers and optimal payment with $N \setminus \{k\}$ suppliers. Denoting the optimal revenue from any subset of suppliers S as $\textsc{Pay}(S)$, we note that

$$\textsc{Pay}(N \setminus \{k\}) = \sum_{j \in N \setminus \{k\}} f_j + \sum_{j \in N \setminus \{k\}} \int_0^K x q_j^{N-k}(x) g_j(x) dx$$

Using (5.11) with this, we get

$$\textsc{Pay}(N) = \textsc{Pay}(N \setminus \{k\}) + f_k - \int_0^K q_k^N(x) G_k(x) dx \tag{5.13}$$

Let (Q^*, T^*) be an optimal mechanism and suppose it induces N^*-entry. We argue that if $i \in N^*$, then for all $k < i$ we have $k \in N^*$. Suppose not. Then, $i \in N^*$ and $k < i$ such that $k \notin N^*$. Then, let $N^{**} = N^* \setminus \{i\} \cup \{k\}$. We will denote the optimal N^{**} mechanism as (Q^{**}, T^{**}). Hence,

$$q_k^{**}(x) = \prod_{j \neq k : j \in N^{**}} (1 - G_j(x)) = \prod_{j \neq i : j \in N^*} (1 - G_j(x)) = q_i^*(x). \tag{5.14}$$

Using Equation (5.12), we get

$$\begin{aligned}
\textsc{Pay}(N^*) - \textsc{Pay}(N^{**}) &= (f_i - f_k) - \int_0^K x q_i^*(x) G_i(x) dx + \int_0^K x q_k^{**}(x) G_k(x) dx \\
&= (f_i - f_k) + \int_0^K x q_k^{**}(x) \big(G_k(x) - G_i(x)\big) dx \qquad \text{(using (5.14))} \\
&\geq 0,
\end{aligned}$$

where the last inequality follows since $f_i \geq f_k$ and $G_i(x) \leq G_k(x)$. Hence, (Q^{**}, T^{**}) with N^{**} set of suppliers is also optimal. This implies that there is some n^* such that the optimal mechanism induces entry of set of suppliers $[n^*] = \{1, \ldots, n^*\}$. Using Equation (5.8), we should choose n^* to minimize

$$\textsc{Pay}([n^*]) = \sum_{i=1}^{n^*} f_i + \sum_{i=1}^{n^*} \int_0^K c_i q_i^{[n^*]}(c_i) g_i(c_i) dc_i \tag{5.15}$$

Payment expression (5.15) is also the expression of ex-ante cost of procurement from $[n^*]$ set of suppliers. Hence, minimizing this also leads to an ex-ante efficient mechanism.

If the optimal mechanism $(Q^{[n^*]}, T^{[n^*]})$ induces entry of $[n^*]$ suppliers, for any $i \in [n^*]$, $u_i^{[n^*]}(K)$ is the utility of supplier i with the highest possible type (per unit cost). Since any supplier with cost K has zero (interim) probability of winning, a negative value of $u_i^{[n^*]}(K)$ indicates a (positive) participation fee of supplier i. We show below that for all $i \in [n^*]$, the participation fee of i is positive, i.e., $u_i^{[n^*]}(K) \leq 0$.

Consider any $i < k$ and $i, k \in [n^*]$. Note that

$$q_i^{[n^*]}(x)(1 - G_i(x)) = q_k^{[n^*]}(x)(1 - G_k(x)). \tag{5.16}$$

Hence,

$$\begin{aligned}
u_i^{[n^*]}(K) &= f_i - \int_0^K q_i^{[n^*]}(x) G_i(x) dx \\
&= f_i - \int_0^K q_k^{[n^*]}(x)(1 - G_k(x)) \frac{G_i(x)}{1 - G_i(x)} dx \qquad \text{(using (5.16))} \\
&= f_i - \int_0^K q_k^{[n^*]}(x) G_k(x) \frac{1 - G_k(x)}{G_k(x)} \frac{G_i(x)}{1 - G_i(x)} dx \\
&\leq f_k - \int_0^K q_k^{[n^*]}(x) G_k(x) dx \qquad \text{(since } f_k \geq f_i \text{ and } G_i(x) \geq G_k(x) \text{ for all } x\text{)} \\
&= u_k^{[n^*]}(K).
\end{aligned}$$

Consider the optimal mechanism for $[n^* - 1] \equiv N^* \setminus \{n^*\}$ suppliers. By optimality of $[n^*]$

and using (5.13), we can now argue that

$$0 \geq \textsc{Pay}([n^*]) - \textsc{Pay}([n^*-1]) = f_{n^*} - \int_0^K q_{n^*}^{[n^*]}(x) G_{n^*}(x) dx = u_{n^*}^{[n^*]}(K),$$

where the last inequality follows from (5.7). Hence, we see that $u_i^{[n^*]}(K) \leq u_{n^*}^{[n^*]}(K) \leq 0$ for all $i \in [n^*]$. This shows that the participation fee $(-u_i^{[n^*]}(K))$ is non-negative for all the suppliers.

Finally, for any $i > n^*$, if supplier i with per unit cost x enters, her interim allocation probability will be at most $q_{n^*}^{[n^*]}(x)$. So, if we set a participation fee of $u_{n^*}^{[n^*]}(K)$, her ex-ante payoff is less than or equal to

$$u_{n^*}^{[n^*]}(K) + \int_0^K G(x) q_{n^*}^{[n^*]}(x) dx = f_{n^*} \leq f_i,$$

where the first equality follows by (5.7) and the second inequality follows since $i > n^*$. Thus, if we set a participation fee of $u_{n^*}^{[n^*]}(K)$ for all the suppliers $i > n^*$, they will not enter. ■

The proof also reveals that if the suppliers are ex-ante identical (equal fixed cost and identical distribution), then the participation fee of all the suppliers is the same. This is the result in McAfee and McMillan (1987).

In one-period model, allowing supplier specific participation fee/subsidy before realisation of marginal cost allows the buyer to completely extract the surplus from every supplier. This effectively reduces the expected procurement cost to expected social cost, i.e., the expected cost of production and fixed cost of all the entrant firms. Given any number of suppliers, it is best for the buyer to procure from the supplier with lowest cost of production (single-sourcing). This can be implemented using second price auction with supplier-specific fees.

5.5 A two period model of repeated entry

We now consider a two period where suppliers choose to enter in each period. We can imagine two different products being procured by a buyer in the two periods, but the products are related so that procuring the first product gives some cost advantage in producing the second product. This may be the case in vaccine procurement, where having produced a vaccine for a certain disease may give relative advantage to a supplier if it decides to produce a vaccine of a related disease. The precise model is formalized below.

There are two periods. In each period, an object needs to be procured by a buyer from a universe of suppliers $\mathcal{S}$. There is a fixed cost f in producing each of the object. This fixed cost is assumed to be the same for all the suppliers and for both the periods. We assume that the buyer cannot commit to period 2 mechanism in period 1.

In period 1, all the suppliers are symmetric in terms of per unit cost: their unit costs are drawn from $[0, K]$ using distribution G. However, suppliers have asymmetric unit costs in the second period. In particular, if supplier i supplied $q_i \in [0, 1]$ units in period 1, then her unit cost distribution in period 2 is "stochastically dominant" in the following sense. Let H_i denote the unit cost distribution of supplier i in period 2 with density h_i. Then, we assume that for every $q_i \in [0, 1]$

$$\frac{h_i(x)}{1 - H_i(x)} = \phi(q_i)\frac{g(x)}{1 - G(x)} \qquad \forall\, x \in [0, K] \tag{5.17}$$

where $\phi : [0, 1] \to \mathbb{R}_+$ is an increasing, continuously differentiable function with $\phi(0) = 1$. Note that $\frac{h_i(x)}{1-H_i(x)}$ is the *hazard rate* of distribution H_i. Hence, the new distribution H_i *hazard rate* dominates the original distribution G. This particular form of hazard rate dominance is for analytical tractability.

We can write (5.17) equivalently as for every $q_i \in [0, 1]$

$$H_i(x) = 1 - (1 - G(x))^{\phi(q_i)} \qquad \forall\, x \in [0, K] \tag{5.18}$$

Since hazard rate dominance implies first-order stochastic dominance, the suppliers can be ordered as in Theorem 9. Hence, Theorem 9 applies in period 2.

Period 2 problem. Since the period 2 is the last period, we know from Theorem 9 that the buyer must conduct a second-price auction with asymmetric participation fees, and extract all the surplus. In particular, the ex-ante payoff of each supplier is zero in period 2. Given this, we want to investigate if procuring from a single supplier is optimal in the first period.

Period 1 problem. Since the ex-ante payoff of each supplier is zero in period 2, incentive and individual rationality constraints in period 1 does not affect payoffs in period 2. Note that Theorem 9 applies to period 1 (with symmetric suppliers as in McAfee and McMillan (1987)). Suppose we consider an alternate mechanism where $s \in [0, 1]$ fraction of the object is given to lowest cost supplier and $(1 - s)$ fraction of the object is given to the second lowest cost supplier. Since this allocation rule is monotone, we can use revenue equivalence to construct a mechanism where IR constraints are binding (just as in Theorem 9). Suppose n_1 suppliers enter in period 1 (since suppliers are symmetric in period 1, the identity of the suppliers are not important) due to this mechanism. Further, suppose n_2 suppliers enter in the optimal mechanism for period 2. Let the set of suppliers who enter in period 2 be N_2.

Total cost of procurement in both the periods can be computed using Equation (5.8). For period 1, note that a type x supplier gets the object with probability

$$s(1 - G(x))^{n_1-1} + (1 - s)(n_1 - 1)G(x)(1 - G(x))^{n_1-2}$$

Hence, the total expected cost in period 1 is

$$\textsc{Pay}(n_1) = n_1 f + n_1 \int_0^K xg(x)\Big[s(1 - G(x))^{n_1-1} + (1 - s)(n_1 - 1)G(x)(1 - G(x))^{n_1-2}\Big]dx$$

To compute the total expected cost in period 2, we note that there are three types of suppliers in period 1:

(a) lowest cost supplier in period 1 who gets s fraction of the object and its cost distribution in period 2 changes to $1 - (1 - G(x))^{\phi(s)}$ for all x;

(b) second lowest cost supplier in period 1 who gets $(1 - s)$ fraction of the object and its cost distribution changes to $1 - (1 - G(x))^{\phi(1-s)}$ for all x;

(c) suppliers who do not supply, and their cost distribution remains G.

The losers from period 1 would participate in period 2 for sufficiently low value of fixed cost f.

So, expected cost from the losers of period 1 in period 2 is (again, using (5.8))

$$\textsc{Pay}_\ell(N_2) = (n_2 - 2)f + (n_2 - 2)\int_0^K x(1 - G(x))^{n_2 - 3 + \phi(s) + \phi(1-s)} g(x)dx$$

Expected cost from the lowest cost supplier of period 1 in period 2 is (noting that its density is $\phi(s)(1 - G(x))^{\phi(s)-1}g(x)$)

$$\textsc{Pay}_1(N_2) = f + \phi(s)\int_0^K x(1 - G(x))^{n_2 - 3 + \phi(1-s) + \phi(s)} g(x)dx$$

Finally, the expected cost from the second lowest cost supplier of period 1 in period 2 is (noting that its density is $\phi(1-s)(1 - G(x))^{\phi(1-s)-1}g(x)$)

$$\textsc{Pay}_1(N_2) = f + \phi(1-s)\int_0^K x(1 - G(x))^{n_2 - 3 + \phi(s) + \phi(1-s)} g(x)dx$$

Hence, total cost of procurement in period 2 is

$$\textsc{Pay}(n_2) = n_2 f + (n_2 + \phi(s) + \phi(1-s) - 2)\int_0^K x(1 - G(x))^{n_2 - 3 + \phi(s) + \phi(1-s)} g(x)dx$$

Let $\psi(s) = (n_2 + \phi(s) + \phi(1-s) - 2)$. Then, note that

$$\begin{aligned}
\textsc{Pay}(n_2) &= n_2 f + \psi(s)\int_0^K x(1 - G(x))^{\psi(s)-1} g(x)dx \\
&= n_2 f - \int_0^K x\frac{d(1 - G(x))^{\psi(s)}}{dx}dx \\
&= n_2 f + \int_0^K (1 - G(x))^{\psi(s)}dx
\end{aligned}$$

We also collect the term dependent on s in $\textsc{Pay}(n_1)$, which we denote as $\textsc{Pay}(n_1; s)$,

$$\textsc{Pay}(n_1; s) = n_1 \int_0^K xg(x)\Big[s(1-G(x))^{n_1-1} + (1-s)(n_1-1)G(x)(1-G(x))^{n_1-2}\Big]dx$$

The effect of change in s on $\textsc{Pay}(n_1; s)$ is given by

$$\begin{aligned}\frac{\partial \textsc{Pay}(n_1; s)}{\partial s} &= n_1 \int_0^K xg(x)\Big[s(1-G(x))^{n_1-1} - s(n_1-1)G(x)(1-G(x))^{n_1-2}\Big]dx \\ &= c_{[1]} - c_{[2]},\end{aligned}$$

where $c_{[i]}$ is the expected cost of the i-th lowest supplier in period 1.

Analogously, we collect the term dependent on s in $\textsc{Pay}(n_2)$, which we denote as $\textsc{Pay}(n_2; s)$,

$$\textsc{Pay}(n_2; s) = \int_0^K (1-G(x))^{\psi(s)}dx$$

The effect of change in s on $\textsc{Pay}(n_2; s)$ is given by

$$\begin{aligned}\frac{\partial \textsc{Pay}(n_2; s)}{\partial s} &= \frac{d\psi(s)}{ds}\int_0^K (1-G(x))^{\psi(s)-1}\log(1-G(x))dx \\ &= \left(\frac{d\phi(s)}{ds} - \frac{d\phi(1-s)}{ds}\right)\int_0^K (1-G(x))^{\psi(s)-1}\log(1-G(x))dx\end{aligned}$$

We introduce some notations for simplification.

We define $\beta(a) = \int_0^K (1-G(x))^{a-1}\log(1-G(x))dx$. Note that $\beta(a) < 0$ for all a and β is bounded above by $\beta(|\mathcal{S}| + 2\phi(1) - 3) < 0$ in our problem as $\beta$and ϕ are increasing functions and $n_2 \leq |\mathcal{S}|$.

We define a measure of concavity of function ϕ as

$$\textsc{Conc}(\phi) = -\left(\frac{d\phi(s)}{ds} - \frac{d\phi(1-s)}{ds}\right)\Bigg|_{s=1}.$$

Note that $\textsc{Conc}(\phi) > 0$ for a strictly concave ϕ since $\dfrac{d\phi(s)}{ds} - \dfrac{d\phi(1-s)}{ds} < 0$ for $s > \frac{1}{2}$.

Combined with the fact that β is bounded above by negative number, we get that $\dfrac{\textsc{Pay}(n_2; s)}{\partial s}\Bigg|_{s=1}$ tends to infinity as $\textsc{Conc}(\phi)$ tends to infinity. As a result, for sufficiently low f and ϕ with large enough $\textsc{Conc}(\phi)$, we have

$$\frac{\partial \textsc{Pay}(n_2; s)}{\partial s}\Bigg|_{s=1} > K \geq c_{[2]} - c_{[1]} = -\frac{\partial \textsc{Pay}(n_1; s)}{\partial s}.$$

For such f and ϕ, decreasing the share slightly from $s = 1$, decreases the total cost of procurement. We summarize this finding in a theorem below.

Theorem 10 *For sufficiently low f and ϕ with sufficiently high $\textsc{Conc}(\phi)$, split award dominates single sourcing in period 1.*

5.6 Related literature

The optimal mechanism with entry depends heavily on the auction environment. McAfee and McMillan (1987) considers a symmetric environment with common knowledge about entry cost. They show that setting the reserve price equal to seller's cost is optimal for society as well as seller. Further, they derive the revenue maximizing mechanism for the seller in this symmetric environment. Levin and Smith (1994) has a model similar to McAfee and McMillan (1987) but with stochastic entry and extends the result on reserve price to this environment. They show that limiting the entry can can be beneficial to the auctioneer. Ye et al. (2004) shows optimality of second price sealed auction in symmetric environment with informative signals to bidders, potentially changing the posterior about cost of entrants.

Samuelson (1985) also considers symmetric environment but assumes that the the bidders/suppliers realise their cost before entry. The tradeoff between maximising social surplus and minimising procurement cost emerges. For cost minimisation, the procurer should set the ceiling price lower than its cost of procurement from third party whereas for social surplus maximisation, ceiling price should be set equal to buyer's cost. The paper also shows that policies limiting the number of potential bidders may be welfare improving. Li and Zheng (2009) analyses first price auction in three different symmetric auction environments: stochastic and deterministic entry before realisation of cost; and entry after realisation of cost. They show that the relationship between number of potential bidders and cost of procurement is ambiguous for all the environments. Moreno and Wooders (2011) considers private heterogenous entry costs of potential bidders and show that revenue maximising reserve is above seller's value and an entry cap combined with admission generates higher revenue than admission fee along with reserve price. Chen and Kominers (2021) show thats that charging admission fee can sometimes dominate the benefit of additional bidders in environment with uncertainty over entry costs.

The literature on single sourcing vs split awards is mixed and depends on the auction environment. The desirability of split awards is supported by Anton et al. (2010), Gilbert and Klemperer (2000), Perry and Sákovics (2003). The splitting awards incentivise the potential bidders to enter to invest, which either through reduction in cost of the bidder or through increased entry reduces the cost of procurement for the auctioneer relative to single sourcing. Gilbert and Klemperer (2000) shows in situations where there is high investment cost for entry, rationing would incentivise players to make sunk investment to enter the market. The modelling is different from ours; the decision to undertake investment by potential bidders affects their valuation/cost in their model, unlike our model. In a similar model but with deterministic modelling of marginal cost as a function of investment, Gong et al. (2012) characterises the conditions for dominance of single sourcing and split awards.

Perry and Sákovics (2003) shows that for procurer, split award can dominate single sourcing in situations by increasing the number of players that enter into auction and lowering the cost of procurement. Anton et al. (2010) shows that in some setting where the uncertainty regarding scale economies is large, split award outcomes are efficient. Klotz and Chatterjee (1995b) shows that even in one-shot procurement split awards dominate sole sourcing for sufficiently risk averse bidders. Klotz and Chatterjee (1995a) shows dual sourcing in an environment of production learning; unlike our paper, they consider that learning is deterministic.

5.7 Concluding remarks

We find that in asymmetric environment single sourcing is optimal. The result is driven by two features of the model: realisation of per-unit cost after entry and the possibility of charging participation fee. The procurer can extract the full surplus and minimisation of cost of procurement maximises the social surplus. This equalisation between social surplus maximisation and procurement cost minimisation makes single sourcing optimal. In two period model with learning, we find that for environments where small experience makes huge difference in learning (highly concave ϕ), split award dominates single sourcing. Even in this model, the procurement cost minimisation maximises social surplus. The learning in the model creates possibility of increasing welfare by splitting award.

The asymmetric participation fee is difficult to implement in real world. Firstly, it depends heavily on the information about cost distributions. Secondly, it is difficult to justify discrimination of bidders in real world. It would be interesting to find the optimal mechanism in asymmetric model of entry without participation fee.

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