PROJECT REPORT

on

RAINBOW VERTEX COLORING

A Thesis to be Submitted

in Partial Fulfilment of the Requirements

for the Degree of

Master of Technology

by

DIPTIMAN GHOSH

Roll No : CS1924

under the supervision of

Dr. Sourav Chakraborty

Advanced Computing and Microelectronics Unit

Indian Statistical Institute

Kolkata-700108, India

Certificate

This is to certify that the dissertation entitled **'Rainbow Vertex Coloring'** submitted by Diptiman Ghosh to Indian Statistical Institute, Kolkata, in partial fulfillment for the award of the degree of Master of Technology in Computer Science is a bonafied record of work carried out by him under my supervision and guidance. The dissertation has fulfilled all the requirements as per the regulations of this institute and in my opinion, has recorded the standard needed for submission.

9/7/21

Date

Sourav Chakraborty

Acknowledgement

I would like to acknowledge my advisor, Dr. Sourav Chakraborty, Associate Professor, Advanced Computing and Microelectronics Unit, Indian Statistical Institute, Kolkata, for his guidance and support. I am highly obliged to him for giving me opportunity to complete my M.Tech Dissertation under his supervision. I would like to acknowledge Sudipta Ghosh (M.Tech CS, ISIK) for his effort in solving this problem. And I am very much thankful to my family for their constant support.

Diptiman Ghosh

09.07.21

Date

Diptiman Ghosh
M.Tech. in Computer Science
ISI Kolkata

Abstract

The concept of rainbow connection was introduced by Chartrand et al.. It has become a new and active subject in graph theory. On this topic a book was written by Li and Sun and there is a survey paper also by Li, Shi and Sun. From then many researches on rainbow edge coloring is going on. Krivelevich and Yuster have defined vertex variant on rainbow connection. On rainbow vertex connection also research has been started from then. Rainbow vertex coloring on powers of trees have been solved in [9]. They gave a linear time algorithm to color vertices such that the graph will be rainbow vertex connected. In our knowledge rainbow edge coloring on powers of trees has not been solved yet. In this work we will use similar type of idea of rainbow vertex coloring to find rainbow edge coloring on powers of trees. We will give a linear time algorithm to color edges such that the graph will be rainbow edge connected. Sudipta Ghosh worked on squares of trees in his M.Tech Dissertation and in this work I will extend his work for higher power of trees.

Contents

1 Introduction

Graph connectivity and coloring is a well researched topic. Rainbow Coloring is a combination of coloring and connectivity problem in graphs. Chartrand et al.[5] first mentioned about rainbow edge coloring. One recent such variant rainbow vertex coloring problem was defined by Krivelevich and Yuster and has received significant attention [7].

1.1 Motivation:

Between any two agencies, there will be some intermediate agencies in which the information passes through one or more secure paths with large number of passwords required. Passwords all should be distinct in the information path between any two agencies. So we need minimum no of passwords such that on any information path between any two agencies passwords should not be repeated. And this can be solved by graph theory and here rainbow coloring concept comes.

1.2 Rainbow Coloring:

A edge colored graph is said to be rainbow edge connected if between every pair of vertices in the graph, there exist a path connecting the pair where color of every edge in that path is distinct. Such type of path is called rainbow path. The minimum number of colors required to make a graph rainbow connected, is known as rainbow connection number ($rc(G)$). Caro et al. [1] conjectured that computing rainbow connection number of a graph is a NP-Hard problem. This conjecture is proved by Chakrobarty et al. [3]. A vertex colored graph is said to be rainbow vertex-connected if between any pair of its vertices, there is a path whose internal vertices are colored with distinct colors. This vertex coloring may not be a proper graph coloring, as an example, a complete graph is rainbow vertex-connected under the coloring that assigns the same color to every vertex. The Rainbow Vertex Coloring (RVC) problem takes as input a graph G and an integer k and asks whether G has a coloring with k colors under which it is rainbow vertex-connected. The rainbow vertex connection number of a graph G is the smallest number of colors needed in one such coloring and is denoted $rvc(G)$. More recently, Li et al. defined a stronger variant of this problem by requiring that the rainbow paths connecting the pairs of vertices are also shortest paths between those pairs. In this case we say the graph is strong rainbow vertex-connected. The analogous computational problem is called Strong Rainbow Vertex Coloring (SRVC) and the corresponding parameter is denoted by $srvc(G)$.

1.3 Previous work

In the section Literature Survey (section 4) we have mentioned the results on rainbow vertex coloring which we have studied. Whole updated survey on rainbow coloring can be found in [8].

Sudipta Ghosh in his M.Tech Dissertation solve the rainbow connection problem for squares of trees. In this work we extend his work for higher powers of trees. In this work we have proved this following theorem.

Theorem 1.1. *If G be power of trees (T^k where T is a tree and $k \geq 3$), then $rc(G) \in \{diam(G), diam(G)+1\}$,and the corresponding optimal rainbow coloring can be found in the time that is linear in the size of G.*

1.4 Thesis Outline

Throughout the thesis we proceed in the following way.
In section 2 we have defined rainbow coloring for edge and vertex variant both. And after that we have defined some basic terminology in graph theory which are needed to understand our work. We have studied some research papers based on rainbow vertex coloring. We have also mentioned some graph classes on which rainbow vertex coloring is solved on those papers.
In section 3 we have mentioned a small survey on rainbow vertex coloring. First we have mentioned hardness of rainbow vertex coloring. Also we have mentioned some graph classes on rainbow vertex coloring is hard. And after that we have mentioned some graph classes on rainbow vertex coloring can be solved in polynomial time. And then we have mentioned some interesting result on rainbow edge coloring on random graphs.

In section 4 we have discussed our work on powers of trees and proved the previously mentioned theorem.

2 Preliminaries and Definition

Graph connectivity and coloring is a well researched topic. Rainbow Coloring is a combination of coloring and connectivity problem in graphs. Chartrand et al. first mentioned about rainbow edge coloring and after that rainbow vertex coloring also was defined.

Definition 2.1. *A path in a vertex-colored graph G is a rainbow vertex path if all its internal vertices have distinct colors. G is rainbow vertex-connected if there is a rainbow vertex path between every pair of its vertices.*

Definition 2.2. *Rainbow Vertex Coloring (rvc) is the decision problem in which we are given a connected (uncolored) graph G and an integer k, and the task is to decide whether the vertices of G can be colored with at most k colors such that G is rainbow vertex-connected. The rainbow vertex connection number of G, denoted by $rvc(G)$, is the minimum k such that G has a rainbow vertex coloring with k colors.*

Definition 2.3. *A stronger variant of rainbow vertex coloring was introduced by Li et al. A vertex colored graph G is strongly rainbow vertex connected if between every pair of vertices of G, there is a shortest path that is also a rainbow vertex path. The Strong Rainbow Vertex Coloring (srvc) problem takes as input a connected (uncolored) graph G and an integer k, and the task is to decide whether the vertices of H can be colored such that G is strongly rainbow vertex-connected. This definition is the vertex variant of the Strong Rainbow Coloring problem.*

Definition 2.4. *Let $G = (V, E)$ be a graph and $c : E \rightarrow \{1, 2, 3, ..., r\}$,$r \in \mathbb{N}$, where adjacent edge can be colored same. For any two arbitrary vertices u and v, if $\exists$ a path between u and v such that every edge in that path is of different color, then that path is called rainbow path and u and v is called rainbow connected. If for every pair of vertices in a graph is rainbow connected, then that graph is called rainbow connected graph. The minimum number of colors needed to make a graph rainbow connected is rainbow connection number of that graph denoted as $rc(G)$.*

We will firstly define some basic definitions in graph theory and also some graph classes on which rainbow vertex coloring has been studied.

2.1 Basic definitions and notations

We will find rainbow edge coloring on powers of trees in our work later. Therefore first we will define what is power of a graph.

Definition 2.5. *The* k-th Power *of a graph, denoted by G^k where $k \geq 1$, is defined as follows: $V(G^k) = V(G)$. Two vertices u and v are adjacent in $V(G^k)$ if and only if the distance between vertices u and v in G, i.e., $dist_G(u, v) \leq k$.*

Definition 2.6. *The* eccentricity *of a vertex v is $ecc(v) := \max_{x \in V(G)} d(v, x)$. The* radius *of G is $rad(G) := \min_{x \in V(G)} ecc(x)$. The* diameter *of G is $diam(G) := \max_{x \in V(G)} ecc(x)$.*

Definition 2.7. *A center of a graph G is a vertex c for which eccentricity(c) in minimum and equal to radius of G.*

Later to find rainbow edge coloring on powers of trees we will use diameter and centre again and again in lemmas.

Definition 2.8. *A dominating set of G is a set $D \subseteq V$ such that every vertex in V - D is adjacent to at least one vertex in D. If G[D] is connected, then D is a connected dominating set. The minimum size of a connected dominating set in G, denoted by $\gamma_c(G)$, is known as the connected domination number of G. [***o***]*

This parameter provides an upper bound on the rainbow. vertex connection number of a connected graph, since G becomes rainbow vertex-connected by simply coloring all vertices of the connected dominating set distinctly, and the remaining vertices with any of the already used colors.[6]

2.2 Definition of some graph classes

Some graph class definitions are mentioned here. A detailed background on these graph classes can be found, for example, in the book by Brandstädt, Le, and Spinrad.[6]. On these graph classes rainbow vertex coloring has been studied. We will mention the results on these graph classes in the next section.

Definition 2.9. *A graph is an apex graph if it contains a vertex (called an apex) whose removal results in a planar graph.[6]*

Definition 2.10. *A graph is chordal if all of its induced simple cycles are of length 3. Some well-known sub classes of chordal graphs are interval graphs, split graphs, and block graphs.[6]*

Definition 2.11. *A graph is an interval graph if it is chordal and it contains no triple of non-adjacent vertices, such that there is a path between every two of them that does not contain a neighbor of the third. Another way to interpret interval graph is an interval graph is an undirected graph where each vertex represents an interval in real line and two vertex is connected by an edge if the corresponding intervals has non-empty intersection.[6]*

Definition 2.12. *A graph is a split graph if its vertex set can be partitioned into an independent set and a clique.*

Definition 2.13. *A graph is a block graph if every bi connected component (block) of G is a complete graph.*

Definition 2.14. *Let σ be a permutation of the integers between 1 and n. We can make a graph G_σ on vertex set [n] in the following way. Vertices i and j are adjacent in G_σ if and only if they appear in σ in the opposite order of their natural order. A graph on n vertices is a permutation graph if it is isomorphic to G_σ for some permutation σ of the integers between 1 and n. A graph is a bipartite permutation graph if it is both a bipartite graph and a permutation graph.*

Definition 2.15. *An independent triple of vertices x, y, z in a graph G is an asteroidal triple (AT), if between every pair of vertices in the triple, there is a path that does not contain any neighbour of the third. A graph without asteroidal triples is called an AT-free graph.*

On these graph classes rainbow vertex coloring has been studied. In Sudipta Ghosh's M.Tech Dissertation rainbow edge coloring has been surveyed.

3 Literature Review

First we will mention about hardness of rainbow vertex coloring. In the next subsection we will also mention rainbow vertex coloring result on some graph classes in which classes rainbow vertex coloring can be solved in polynomial time. Hardness of rainbow edge coloring and some rainbow edge coloring result on some graph classes in which classes rainbow edge coloring can be solved in polynomial time has been surveyed in Sudipta Ghosh's M.Tech Dissertation work. In the next subsection we will also mention work on rainbow edge coloring on random graphs.

3.1 Hardness of Rainbow Coloring

Theorem 3.1. *$rvc(G)$ is NP-complete for every $k \geq 2$. It is also NP-hard to approximate $rvc(G)$ within a factor of $2 - \epsilon$ unless $P \neq NP$, for any $\epsilon > 0$.[6]*

Theorem 3.2. *$srvc(G)$ is NP-complete for every $k \geq 2$. It is NP-hard to approximate $srvc(G)$ within a factor of $n^{\frac{1}{2}-\epsilon}$ unless $P \neq NP$, for any $\epsilon > 0$.[6]*

rvc and *srvc* is NP complete on the following graph classes.

Theorem 3.3. *For bipartite graph of diameter 4, to decide whether rvc and srvc is $\leq k$ is NP-complete for every $k \geq 3$. Moreover,it is NP-hard to approximate both $rvc(G)$ and $srvc(G)$ within a factor of $n^{\frac{1}{3}-\epsilon}$, for every $\epsilon > 0$.(Heggerness et al.)[6]*

This theorem can be proved using this idea: Let H be a hypergraph on n vertices. Then in polynomial time we can construct a bipartite graph G of diameter 4 and with $O(n^3)$ vertices such that for any $k \in [n]$, H has a proper k-coloring if and only if G has a $(k+1)$-coloring under which G is (strongly) rainbow vertex-connected. Moreover, if H is a planar graph, then G is an apex graph.

Theorem 3.4. *For bipartite apex graph of diameter 4, to decide whether rvc and srvc is $\leq k$ is NP-complete. Moreover,it is NP-hard to approximate both $rvc(G)$ and $srvc(G)$ within a factor of $n^{\frac{5}{4}-\epsilon}$, for every $\epsilon > 0$.(Heggerness et al.)[6]*

This result is particularly interesting since no hardness result was known on a sparse graph class (like apex graphs) for any of the variants of rainbow coloring.

Theorem 3.5. *For split graph of diameter* 3*, to decide whether rvc and srvc is* $\leq k$ *is NP-complete for every* $k \geq 3$*. Moreover,it is* NP*-hard to approximate both* $rvc(G)$ *and* $srvc(G)$ *within a factor of* $n^{\frac{1}{3}-\epsilon}$*, for every* $\epsilon > 0$*.(Heggerness et al.)*

3.2 Result on some graph classes

Theorem 3.6. *For a block graph, or a unit interval graph, rvc and srvc can be solved in linear time. For interval graph, rvc can be solved in linear time (Heggerness et al.).[6]*

For interval graph $rvc = diam(G) - 1$ and for block graph $srvc =$ no of cut vertices.

Theorem 3.7. *rvc is linear-time solvable on planar graphs for every fixed* k*.*

Conjecture: A diametral path of a graph G is a shortest path whose length is equal to $diam(G)$. A graph is a diametral path if every connected induced subgraph has a dominating diametral path. Let G be a diametral path graph. Then $rvc(G) = diam(G) - 1$(Heggerness et al).[6]

Theorem 3.8. *If* G *is a permutation graph on* n *vertices, then* $rvc(G) = diam(G) - 1$ *and the corresponding rainbow vertex coloring can be found in* $O(n^2)$ *time(Heggerness et al.).[6]*

Theorem 3.9. *If* G *is a split strongly chordal graph with* l *cut vertices, then* $rvc(G) = srvc(G) = max(diam(G) - 1, l)$*(Heggerness et al.).[6]*

Conjecture and Open Problem: Complexity of finding rainbow color on AT free graphs i.e graphs do not contain asteroidal triple (ex: interval graphs, permutation graphs) and strongly chordal graphs(ex: power of trees, split strongly chordal graphs) (Hggerness et al.)[6]

3.3 Result of Rainbow edge coloring on Random Graphs

Let $G = G(n, p)$ denote the binomial random graph on n vertices with edge probability p. Some work on rainbow edge coloring has been done on random graphs and some interesting result has been found.

Theorem 3.10. *Caro et al. proved that* $p = \frac{logn}{n}$ *is the sharp threshold for the property* $rc(G(n,p)) \leq 2$.*[2]*

He and Liang studied further the rainbow connectivity of random graphs. They obtain the sharp threshold for the property $rc(G) \leq d$ where d is constant.

Li and Sun worked on the rainbow connectivity of the binomial graph at the connectivity threshold p $= \frac{logn+\omega}{n}$ where $\omega = o(logn)$.

We know $diam(G)$ is the lower bound of rainbow edge coloring. In the following theorem a pretty interesting result has been found. For random graphs rainbow edge coloring is asymptotically equal to the diameter with high probability.

Theorem 3.11. *Let* $G = G(n,p)$, $p = \frac{logn+\omega}{n}$, $\omega \to \infty$, $\omega = o(logn)$, *Also, let* Z_1 *be the number of vertices of degree 1 in G. Then, with high probability(whp)* $rc(G) \sim max\ (Z_1, L)$. *It is known that whp the diameter of G(n,p) is asymptotic to L for p as in the above range. Here* $L = \frac{logn}{loglogn}$.*[4]*

Theorem 3.12. *Let* $G = G(n,r)$ *be a random r-regular graph where* $r \geq 3$ *is a fixed integer. Then, whp* $rc(G) = O(log^4 n)$ *when* $r = 3$ *and* $O(log^{2\theta_r} n)$ *when* $r \geq 4$, *where* $\theta_r = \frac{log(r-1)}{log(r-2)}$.*[4]*

4 Our Work: Rainbow edge coloring for powers of trees

In this section we will discuss on rainbow edge coloring of T^k (T^k is k th power of tree T). Sudipta Ghosh has discussed rainbow edge coloring on square of trees in his M.Tech Dissertation. So in this section I will extend his work for $k \geq 3$. Though rainbow vertex coloring of powers of trees is discussed in previous research, but as of our knowledge rainbow edge coloring on powers of trees is not discussed till now.

We know $diam(G)$ is the lower bound of the rainbow connection number of a graph G. For power of Trees we have showed the following result.

Theorem 4.1. ***For powers of tree T^k, rainbow connection number*** $\in \{diam(T^k), diam(T^k) + 1\}$

Therefore like squares of trees same type of results hold for higher powers of trees, but here to prove the above theorem we have to consider more cases than cases in square of trees. The diameter of T is always even if the centre of T is a single vertex. So always diam(T) $= 0 \pmod 2$ holds if the centre of T is a single vertex. but for higher power of trees if centre of T is single vertex we have to consider whether $diam(T) = 0 \pmod 2$ or not. We will discuss different such cases through different lemmas.

To prove the above theorem some definitions are required to be known.

Definition 4.1. ***Branch:*** *If one endpoint of an edge is centre, then if the edge is removed the tree fall apart in two parts. A branch is the part that doesn't contain the centre. If the centre contains single vertex, then the no of branches is nothing but equal to the degree of the centre. A subbranch of B will be denoted by B'. Subbranch is a branch which uses some vertices of main branch.*

Definition 4.2. ***Layer:*** *We define layer i as the set of all vertices with distance $\lfloor \frac{diam(T)}{2} \rfloor - i$ to the center of T. layer of a vertex v will be denoted by $l(v)$.*

Now we will prove the theorem through different cases and lemmas. Rainbow connection number of a graph G will be denoted by $rc(G)$ and shortest rainbow connection number of a graph G will be denoted by $src(G)$.

Lemma 1. ***Suppose*** T ***is a tree and it has single vertex in centre,*** $diam(T) \geq 3k$ ***and*** $diam(T) = 0(mod\ k)$***, and there are at least three branches from the center with maximum length. Then*** $src(T^k) \geq rc(T^k) \geq diam(T^k) + 1$***.***

Proof.

Suppose B_1, B_2, B_3 are three branches from the centre with maximum length (as our assumption).

Divide the layers 1, 2, . . . , $\frac{diam(T)-1}{2}$ in blocks of size k. If a block has size k then we can say it a complete block. Then the topmost block may be or may not be a complete block. Let n be the number of complete blocks in B_1. Suppose a_1 , a_2 , . . . are vertices in B_1 in layer $0(mod\ k)$. That means a_1 , a_2 , . . . be the topmost vertices in the complete blocks in B_1. Similarly, let b_1 , b_2 , . . . be the topmost vertices in the complete blocks in B_2. That means b_1 , b_2 , . . . are all vertices in B_2 in layer $0(mod\ k)$. Suppose $d(x, y)$ is distance between two vertices x and y.

Suppose v_1, v_2, v_3 are layer 0 vertices in those maximum branches B_1, B_2, B_3 respectively. Then shortest distance between each pairwise v_i and v_j will be $diam(T^k)$ in the graph T^k(as among all pairs of vertices for these three pairs shortest path distance will be maximum) and those shortest paths are unique. As $diam(T) = 0(mod\ k)$ so $d(a_1, b_1)$ will be either 0 or k. Now we want to find the shortest path between v_1 and v_2. From v_1 follow $0(mod\ k)$ layers in B_1 to reach a_1 and from v_2 follow $0(mod\ k)$ layers in B_2 to reach b_1. If $d(a_1, b_1) = 0$ then a_1 and b_1 are nothing but centre. If $d(a_1, b_1) = k$ then we have to use an edge from a_1 to b_1. So this shortest path is unique. For other v_i and v_j pairs shortest path can be found similarly.

Suppose on a contrary assume $rc(T^k) = diam(T^k)$. So, if we use $diam(T^k)$ colors then certainly we have to follow those shortest paths to get rainbow path between each pair v_i and v_j.

Claim: It will not be possible to make rainbow colored path for all pairs v_i and v_j with $diam(T^k)$ colors.

Proof: Shortest path between v_1 and v_2 and shortest path between v_1 and v_3 will share some edges of B_1. The non shared portion in those two shortest paths should consist of edges with same color. That means the portion of shortest path between v_1 and v_2 which is in B_2 and portion of shortest path between v_1 and v_3 which is in B_3 should consist of edges with same color if we assume the contrary assumption. But if we consider v_2 to v_3 shortest path, it is nothing but combination of edges of those non shared portions in B_2 and B_3 and also it is unique shortest path between v_2 and v_3. But it will not be rainbow colored path because those non shared portions in B_2 and B_3 consists of same colored edges as we have mentioned before. So, it will not be possible to make rainbow colored path between each pair of vertices using $diam(T^k)$ colors. ■

So $src(T^k) \geq rc(T^k) \geq diam(T^k) + 1$.

□

We have shown in lemma 1 $diam(T^k)$ no of colors is not sufficient for this case, but in lemma 2 we will show $diam(T^k) + 1$ no of colors is sufficient.

Lemma 2. ***If*** T ***is a tree and it has single vertex in the centre and*** $diam(T) \geq 3k$ ***and*** $diam(T) = 0$***(****mod k****) and at least three branches of maximum length from the centre, then*** $rc(T^k) = diam(T^k) + 1$***.***

Proof.

let $l(v_i)$ denotes layer of vertex v_i.
$c(v_iv_j)$ denotes color of edge v_iv_j. Let $l(v_i) < l(v_j)$.

Coloring Procedure:

$$c(v_iv_j) = \begin{cases} c_1 & \text{if } v_j \text{ is center and } l(v_i) \not\equiv 0 \pmod k \\ c_2 & \text{if } v_j \text{ is center and } l(v_i) \equiv 0 \pmod k \\ l(v_j) & \text{if } l(v_j) \equiv 0, -1 \pmod k \\ c_1 & \text{otherwise} \end{cases}$$

Claim: There exist rainbow colored path between each pair of vertices.

Proof: Suppose u is a vertex in B_1 and v is a vertex in B_2. We want to find the rainbow colored path between u and v. In B_1 from u follow the path using $0(mod\ k)$ layered vertices. So we are using $0(mod\ k)$ colored edges. And now use c_2 colored edge to reach the centre from top $0(mod\ k)$ layered vertex in B_1 . From the centre use color c_1 edge to reach nearest $-1(mod\ k)$ layered vertex to the centre in B_2 (if v is in topmost block instead we have to use c_1 colored edge from centre to reach v). Now follow the path using $-1(\text{mod } k)$ layered vertices using $-1(mod\ k)$ colored edges to reach v. So, it will be a rainbow colored path. Edges of other branches will be colored similarly depending on the conditions of coloring procedure. If u is in B_i and v is in B'_i (Recall the definition of subbranch), then we can assume B'_i as some other branch B_j (may be B_2), follow the path similarly as we have reached from B_1 to B_2. So in this process we can find rainbow colored path between any pair of vertices. ■

No of colors:

Now we will show the number of colors has been used is actually $diam(T^k) + 1$. Suppose $l = \frac{diam(T)}{2} - 1$. We have divided those layers in blocks of size k. Notice that two colors are used in every complete block. There are $\lfloor \frac{l}{k} \rfloor$ complete blocks, so $2\lfloor \frac{l}{k} \rfloor$ colors for those blocks. Recall a_1 and b_1 mentioned in lemma 1. d(a_1,b_1) is either 0 or k as per our assumption. So, for the second case $diam(T^k)$ will be $2\lfloor \frac{l}{k} \rfloor + 1$. And we have used two extra colors c_1 and c_2 except $2\lfloor \frac{l}{k} \rfloor$ colors. So we are using $diam(T^k) + 1$ colors. For the first case diam(T^k) will be $2\lfloor \frac{l}{k} \rfloor$. Basically we are using then one extra color apart from $2\lfloor \frac{l}{k} \rfloor$ colors. So in this case also we are using $diam(T^k) + 1$ colors.

□

Lemma 3. ***If*** T ***is a tree and it has single vertex in centre and*** $diam(T) \geq 2k+1$ ***and*** $diam(T) \neq 0 \pmod k$***, then*** $rc(T^k) = diam(T^k)$***.***

Proof.

Suppose $l(v)$ denotes layer of vertex v. Suppose v_iv_j is an edge and $l(v_j) > l(v_i)$. c(v_iv_j) denotes color of that edge. We want to color the edges such that between each pair of vertices there exist a rainbow path.

Coloring Procedure:

$$c(v_iv_j) = \begin{cases} l(v_j) & \text{if } l(v_j) \equiv 0, -1 \pmod{k} \end{cases}$$

We can have some observations seeing the conditions mentioned in lemma. Let z be the center vertex. There will be two longest branches from z. Let B_1 and B_2 be two longest branches from z. If one longest branch exists, then there will be two centre and that is contrary to our assumption. Recall the definition of complete block in lemma 1. The topmost block won't be complete, if complete then $diam(T) = 0 \pmod{k}$, contrary to our assumption.

No of colors used so far:

Suppose $l = \frac{diam(T)}{2} - 1$. We have divided those layers in blocks of size k. Notice that two colors are used in every complete block. There are $\lfloor \frac{l}{k} \rfloor$ complete blocks, so $2\lfloor \frac{l}{k} \rfloor$ colors have been used in those blocks.

Recall a_1 and b_1 mentioned in lemma 1. Now we will consider three cases and complete the coloring to find rainbow path between each pair of vertices.

Case 1:

Suppose that $d(a_1, b_1) > k$. We know $d(a_1, z) < k$ and $d(b_1, z) < k$. We claim that $diam(T^k) = 2\lfloor \frac{l}{k} \rfloor + 2$. Let $u \in B_1$, $v \in B_2$ be vertices in layer 0. A u, v-path contains a vertex in every complete block in B_1, a vertex in every complete block in B_2 and a vertex in a topmost incomplete block or z. All in all, these are $2\lfloor \frac{l}{k} \rfloor + 1$ internal vertices. So $2\lfloor \frac{l}{k} \rfloor + 2$ edges are used and it is nothing but $diam(T^k)$ length path. So we can use two more colors in the coloring. Suppose these colors are c_1 and c_2. No of blocks in one branch is n. Suppose nearest $-1 \pmod{k}$ layered vertex to centre is denoted by p_1

Coloring Procedure:

$$c(v_iv_j) = \begin{cases} c_1 & \text{if } l(v_i) = a_1 \\ c_2 & \text{if } l(v_i) = p_1 \\ c_2 & \text{color of edge with one endpoint in } B_1 \text{ and other endpoint in } B_2 \\ c_1 & \text{color of other edges in topmost block} \end{cases}$$

Then, for any two vertices u and v, to find the rainbow path between u and v use the $0 \pmod{k}$ layers to go from u to z and then use $-1 \pmod{k}$ layers to go from z to v. First we are using $0 \pmod{k}$ colored edges and then c_1 colored edge to reach z from a_1 and from z then use c_2 colored edge to reach $-1 \pmod{k}$ layered vertex (instead if v is in topmost block with layer higher than layer of a_1 we can reach v direct from z using c_2 colored edge) and then use $-1 \pmod{k}$ colored edges to reach v. So that will be rainbow colored path. If v is in topmost $0 \pmod{k}$ layer, u to a_1 path will be same and after that from a_1 use c_2 colored edge to reach a vertex in B_2 next to the centre and use c_1 colored edge to reach v from that vertex.

Case 2:

Now suppose that $d(a_1, b_1) = k$. It follows that $k \mid diam(T)$, a contradiction with the assumptions of the lemma.

Case 3:

Now suppose that $d(a_1, b_1) \leq k - 1$. Here we can use one more color. Color all uncolored edges with one color (say c_1). Let u and v be two vertices.

Use the $0 \pmod{k}$ layers to go from u to a_1 and then from a_1 reach $-1 \pmod{k}$ layered vertex using c_1 colored edge (from a_1 we can use c_1 colored edge to reach v which is in B_2 or B_1 whatever if v's layer greater or equal to layer of a_1) and then follow $-1 \pmod{k}$ vertices using $-1 \pmod{k}$ colored edges to reach v.

□

Lemma 4. ***If*** T ***be a tree and it has single vertex in the centre;*** $diam(T) \geq 3k$***,*** $diam(T) = 0 \pmod{k}$ ***and*** T ***has two branches of maximum length, then*** $rc(T^k) = diam(T^k)$

Proof.

Let B_1 and B_2 be the branches of maximum length and B_3 represents all other branches. In the time of coloring we represent an edge by ab where $l(a) < l(b)$ and in the time of rainbow path finding we want to find rainbow path between two vertices u and v. Recall block partition in lemma 1. If $d(a_1, b_1) = 0$ then diam(T^k) will be even. In this case a_1 and b_1 both will be centre. If $d(a_1, b_1) = k$ then diam(T^k) will be odd.

We first consider the case when $diam(T^k)$ is odd.

Here we will mention path and with that also will mention the color of the required edges. Other edges can be colored arbitrarily. So, we are trying to find rainbow path between u and v.

Case 1. u is in B_1 and v is in B_2 :

Subcase (i): u is in any layer except $-1 \pmod k$ and v is in any layer except $-1 \pmod k$):
Coloring Procedure:

1. In B_1 for the edge ab if $l(a)$ is in 0 th layer or $l(b)$ is in k th layer or $l(a)$ is in 0 th layer and $l(b)$ is in k th layer both then the edge color will be c_1 (for now don't consider edge with one endpoint in $k-1$ th layer)
2. In B_1 for the edge ab if $l(a)$ is in $0 \pmod k$ th layer or $l(b)$ is in $0 \pmod k$ th layer or $l(a)$ is in $0 \pmod k$ th layer and $l(b)$ is in $0 \pmod k$ th layer both then the edge color will be $l(a) - 1$ (for now don't consider edge with one endpoint in $-1 \pmod k$)
3. In B_2 for the edge ab If $l(a)$ is in 0 th layer or $l(b)$ is in k th layer or $l(a)$ is in 0 th layer and $l(b)$ is in k th layer both then the edge color will be c_0 (for now don't consider edge with one endpoint in $k-1$))
4. In B_2 for the edge ab if $l(a)$ is in $0 \pmod k$ th layer or $l(b)$ is in $0 \pmod k$ th layer or $l(a)$ is in $0 \pmod k$ th layer and $l(b)$ is in $0 \pmod k$ th layer both then the edge color will be $l(a)$ (for now don't consider edge with one endpoint in $-1 \pmod k$)
5. Edge from $0 \pmod k$ layered vertex in B_1 to $0 \pmod k$ layered vertex in B_2 will be colored c .

Path:

From u go to higher nearest $0 \pmod k$ layer by edge colored with $-1 \pmod k$ or c_1 (for the case when layer of $u < k$) and then follow c colored edges to reach $0 \pmod k$ vertex in B_2 and then follow $0 \pmod k$ layers in B_2 using $0 \pmod k$ colored edges to reach nearest

$0 \pmod k$ layer to v and then follow $0 \pmod k$ colored edge or c_0 colored edge (for the case when layer of $v < k$) to reach v .

Subcase (ii): If u is in any layer except $-1 \pmod k$ and v is in $-1 \pmod k$ layer:

Coloring Procedure:

1. In B_2 and B_1 for the edge ab If $l(a) = 0 \pmod k$ and $l(b) = -1 \pmod k$ then edge color will be c.
2. In B_2 and B_1 for the edge ab If $l(b) = k - 1$ color will be c.
3. In B_2 If $l(a)$ is in $-1 \pmod k$ th layer or $l(b)$ is in $-1 \pmod k$ th layer or $l(a)$ is in $-1 \pmod k$ th layer and $l(b)$ is in $-1 \pmod k$ th layer both then the edge color will be $l(a)$ (for now don't consider edge with one endpoint in $-1 \pmod k$)
4. In B_2 From topmost $-1 \pmod k$ layer to centre edge color will be c_1.
5. In B_1 If $l(a)$ is in $-1 \pmod k$ th layer or $l(b)$ is in $-1 \pmod k$ th layer or $l(a)$ is in $-1 \pmod k$ th layer and $l(b)$ is in $-1 \pmod k$ th layer both then the edge color will be $l(a) + 1$ (for now don't consider edge with one endpoint in $-1 \pmod k$))
6. In B_1 From topmost $-1 \pmod k$ layer to centre edge color will be c_0.

Path:

From u go to higher nearest $-1 \pmod k$ layer by edge with color $0 \pmod k$ or c and then follow $-1 \pmod k$ layers using $0 \pmod k$ colored edges to reach top $-1 \pmod k$ layered vertex in B_1 and then use c_0 colored edge to reach centre and then use c_1 colored edge to reach top $-1 \pmod k$ layered vertex in B_2 and then follow $-1 \pmod k$layer in B_2 using $-1 \pmod k$ colored edges to reach v .

Subcase (iii): If u is in $-1 \pmod k$ layer and v is in any layer except $0 \pmod k$:

Coloring Procedure:

1. Edge between $-1 \pmod k$ layered vertex next to centre in B_1 and $-1 \pmod k$ layered vertex next to centre in B_2 will be colored c_0.

Path:

From u follow $-1 \pmod k$ layers using $0 \pmod k$ colored edges to reach top $-1 \pmod k$ layered vertex in B_1 and then use c_0 colored edge to reach centre and then use c_1 colored edge to reach top $-1 \pmod k$ layered vertex in B_2 and then follow $-1 \pmod k$ layer in

B_2 using $-1 \pmod k$ colored edges to reach nearest $-1 \pmod k$ layered vertex to v and then use $-1 \pmod k$ or c colored edge to reach v .

Subcase (iv): If u is in $-1 \pmod k$ layer and If v is in $0 \pmod k$ layer.

Coloring Procedure:

1. In B_1 and B_2 from topmost $0 \pmod k$ layer to centre edge color will be c.

Path:

From u follow $-1 \pmod k$ layers using $0 \pmod k$ colored edges to reach top $-1 \pmod k$ layered vertex in B_1 and then use c_0 colored edge to reach centre (then we have to use c colored edge to reach v if necessary and stop) and then use c_1 colored edge to reach top $-1 \pmod k$ layered vertex in B_2 (then we have to use c colored edge to reach v if necessary and stop) and then follow $-1 \pmod k$ layer in B_2 using $-1 \pmod k$ colored edges to reach higher nearest $-1 \pmod k$ layered vertex to v and then use c colored edge to reach v .

Case 2. u is in B_1 and v is in B_1' :

Subcase (i): If u is in any layer and v is in any layer except $-1 (\text{mod } k)$:

Coloring Procedure:

1. If $l(b)$ = top $-1 \pmod k$ layered vertex and $l(a)$ = top $-1 \pmod k$ layered vertex, then edge color will be $l(a) - 1$ in B_1 and in B_2 this edge color will be $l(a)$.

Path:

From u go to higher nearest $-1 \pmod k$ layer by edge with color $0 \pmod k$ or c and then follow $-1 \pmod k$ layers using $0 \pmod k$ colored edges to reach top $-1 \pmod k$ layered vertex in B_1 and then use c_0 colored edge to reach $0 \pmod k$ layered vertex in B_1' and then follow $0 \pmod k$ layer in B_1' using $-1 \pmod k$ colored edges to reach v .

Subcase (ii): If u is in any layer and v is in $-1 \pmod k$ layer:

Coloring Procedure:

In B_1 if the two endpoints of edge are $-1 \pmod k$ layered vertex and $\geq 0 \pmod k$ layered vertex, color will be c_0 (if one endpoint in B_1 and another endpoint in B_1' with this condition then it is also true).

Path:

Use c colored edge (to reach lower $0 \pmod k$ layered vertex from $-1 \pmod k$ layered vertex) or $-1 \pmod k$ colored edge (to reach higher $0 \pmod k$ layered vertex from other u) and follow $0 \pmod k$ layer vertices using $-1 \pmod k$ colored edges and use c_0 colored edge to reach $-1 \pmod k$ layered vertex in B_1' and then use $-1 \pmod k$ colored edges to reach v.

Using the same path we can consider the case $B_2 - B_2'$.

Case 3. u is in B_1 and v is in B_3:

Subcase (i): If u is in any layer except $-1 \pmod k$ and v is in layer $0 \pmod k$):

Coloring Procedure:

1. In B_3 If $l(a)$ is in $0 \pmod k$ th layer or $l(b)$ is in $0 \pmod k$ th layer or $l(a)$ is in $0 \pmod k$ th layer and $l(b)$ is in $0 \pmod k$ th layer both then the edge color will be $l(a)$ (for now don't consider edge with one endpoint in $1 \pmod k$ and don't consider $l(b) = k$).
2. In B_3 edge between centre and top $0 \pmod k$ vertex will be c_0.
3. If $l(a)$ is in $1 \pmod k$ th layer or $l(b)$ is in $1 \pmod k$ th layer or $l(a)$ is in $1 \pmod k$ th layer and $l(b)$ is in $1 \pmod k$ th layer both then the edge color will be $l(b) - 2$ (for now don't consider edge with one endpoint in $0 \pmod k$ and don't consider $l(b) =$ centre).

Path:

From u use c_1 colored edge or $-1 \pmod k$ colored edge to reach higher nearest $0 \pmod k$ layered vertex and then use c colored edge to reach centre and then follow $0 \pmod k$ layered vertices using $0 \pmod k$ colored edges to reach v.

Subcase (ii): If u is layer $-1 \pmod k$ and v is layer $0 \pmod k$):

Coloring Procedure:

1. In B_3 If $l(a) = 1 \pmod k$ and $l(b) = 0 \pmod k$ then edge color will be c.
2. Edge between B_1 and B_3 will be c_0.

Path:

From u follow $-1 \pmod{k}$ layered vertices using $0 \pmod{k}$ colored edges and then use c_0 colored edge to reach $1 \pmod{k}$ layered vertex in B_3 and then follow $1 \pmod{k}$ layered vertices using $-1 \pmod{k}$ colored edges and then use c colored edge to reach v.

Subcase (iii): If u is in any layer and v is in any layer except $0 \pmod{k}$):

Coloring Procedure:

No new coloring will be required for this case.

Path:

From u go to $-1 \pmod{k}$ layered vertex using $0 \pmod{k}$ colored edge or c colored edge and then follow $-1 \pmod{k}$ layered vertices using $0 \pmod{k}$ colored edges and then use c_0 colored edge to reach $1 \pmod{k}$ layered vertex in B_3 and then follow $1 \pmod{k}$ layered vertices using $-1 \pmod{k}$ colored edges to reach v.

Case 4. u is in B_2 and v is in B_3:

Subcase (i): If u is in any layer and v is in layer $0 \pmod{k}$:

Coloring Procedure: No new coloring will be required for this case.

Path:

From u use c colored edge or $-1 \pmod{k}$ colored edge to reach nearest $-1 \pmod{k}$ layered vertex and then follow $-1 \pmod{k}$ layered vertices using $-1 \pmod{k}$ colored edges and then use c_1 colored edge to reach centre and then follow $0 \pmod{k}$ layered vertices using $0 \pmod{k}$ colored edges or c_0 colored edge(to reach top $0 \pmod{k}$ vertex) to reach v.

Subcase (ii): If u is in $-1 \pmod{k}$ and v is in layer except $0 \pmod{k}$:

Coloring Procedure:

No new coloring is required for this case.

Path:

From u follow $-1 \pmod{k}$ layered vertices using $-1 \pmod{k}$ colored edges and then use c_1 colored edge to reach centre and then follow $0 \pmod{k}$ layered vertices using $0 \pmod{k}$ colored edges and then use c colored edge to reach v.

Subcase (iii): If u is in any layer except $-1 \pmod k$ and v is in any layer except $0 \pmod k$):

Coloring Procedure:

Edge between top $1 \pmod k$ layer vertex in B_2 and top $1 \pmod k$ layer vertex in B_3 will be colored c.

Path:

From u go to $0 \pmod k$ layered vertex using $0 \pmod k$ colored edge or c_0 colored edge and then follow $0 \pmod k$ layered vertices using $0 \pmod k$ colored edges and then use c colored edge to reach $1 \pmod k$ layered vertex in B_3 and then follow $1 \pmod k$ layered vertices using $-1 \pmod k$ colored edges to reach v.

Case 5. u is in B_3 and v is in B_3'

Subcase (i): If u is in any layer and v is in layer except $0 \pmod k$)):

Coloring Procedure:

$0 \pmod k$ layered vertex in B_3 to B_3' color will be c_0.

Path:

From u use c or $0 \pmod k$ colored edge to reach $0 \pmod k$ layered vertex and then follow $0 \pmod k$ layered vertices using $0 \pmod k$ colored edges and then use c_0 edge to reach top $1 (\mathrm{mod}\ k)$ vertex in B_3' and then follow $1 \pmod k$ layered vertices using $-1 \pmod k$ colored edges to reach v.

Subcase (ii): If u is in $0 \pmod k$ layer and v is in layer $0 \pmod k$):

Coloring Procedure:

No new coloring will be required for this case.

Path:

From u follow $0 \pmod k$ layered vertices using $0 \pmod k$ colored edges and then use c_0 edge to reach top $1 \pmod k$ vertex in B_3' and then follow $1 \pmod k$ layered vertices using $-1 \pmod k$ colored edges to reach lower nearest $1 \pmod k$ vertex and then use c colored edge to reach v .

Subcase (iii): If u is in layer except $0 \pmod k$ layer and v is in layer $0 \pmod k$):

Coloring Procedure:

No new coloring will be required for this case.

Path:

From u follow $1 \pmod k$ layered vertices using $-1 \pmod k$ colored edges and then use c_0 edge to reach in B_3' top $0 \pmod k$ vertex and then follow $0 \pmod k$ layered vertices using $0 \pmod k$ colored edges to reach v .

Now suppose diam(T^k) is even.

Case 1. u is in B_1 and v is in B_2 :

Subcase (i): u is in any layer except $-1 \pmod k$ and v is in any layer except $-1 \pmod k$):

Coloring Procedure:

1. In B_1 if $l(a)$ is in 0 th layer or $l(b)$ is in k th layer or $l(a)$ is in 0 th layer and $l(b)$ is in k th layer both then the edge color will be c_1 (for now don't consider edge with one endpoint in $k-1$).
2. In B_1 if $l(a)$ is in $0 \pmod k$ th layer or $l(b)$ is in $0 \pmod k$ th layer or $l(a)$ is in $0 \pmod k$ th layer and $l(b)$ is in $0 \pmod k$ th layer both then the edge color will be $l(a)-1$ (for now don't consider edge with one endpoint in $-1 \pmod k$))).
3. In B_2 if $l(a)$ is in 0 th layer or $l(b)$ is in k th layer or $l(a)$ is in 0 th layer and $l(b)$ is in k th layer both then the edge color will be c_0 (for now don't consider edge with one endpoint in $k-1$).
4. In B_2 if $l(a)$ is in $0 \pmod k$ th layer or $l(b)$ is in $0 \pmod k$ th layer or $l(a)$ is in $0 \pmod k$ th layer and $l(b)$ is in $0 \pmod k$ th layer both then the edge color will be $l(a)$ (for now don't consider edge with one endpoint in $-1 \pmod k$)).

Path:

From u go to nearest $0 \pmod k$ layer by edge with color $-1 \pmod k$ or c_1(for the case when layer of $u < k$) and then follow $0 \pmod k$ layers using $-1 \pmod k$ colored edges to reach centre and then follow $0 \pmod k$ layers in B_2 using $0 \pmod k$ colored edges to reach nearest $0 \pmod k$ layer to v and then follow $0 \pmod k$ colored edge or c_0 colored edge(for the case when layer of $v < k$) to reach v .

Subcase (ii): If u is in any layer except $-1 \pmod k$ and v is in $-1 \pmod k$ layer:

Coloring Procedure:

1. In B_2 if $l(b) = 0 \pmod k$ and $l(a) = -1 \pmod k$ then edge color will be c_0.
2. In B_1 if $l(b) = 0 \pmod k$ and $l(a) = -1 \pmod k$ then edge color will be c_1.

Path:

From u go to nearest $0 \pmod k$ layer by edge with color $0 \pmod k$ or c_1(for the case when layer of $u < k$) and then follow $0 \pmod k$ layers using $-1 \pmod k$ colored edges to reach centre and then go to nearest $0 \pmod k$ layer in B_2 using $0 \pmod k$ colored edges and then use c_0 colored edge to reach v .

Subcase (iii): If u is in $-1 \pmod k$ layer and v is in any layer except $0 \pmod k$:

Coloring Procedure:

1. Edge between $-1 \pmod k$ layered vertex next to centre in B_1 and $-1 \pmod k$ layered vertex next to centre in B_2 will be colored c_0.

Path:

From u use $-1 \pmod k$ layered vertices using $0 \pmod k$ colored edges and then use c_0 colored edge to reach $-1 \pmod k$ layered vertex in B_2 and then use $-1 \pmod k$ layered vertices using $-1 \pmod k$ colored edges to reach nearest $-1 \pmod k$ layered vertex to v with high level and then use $-1 \pmod k$ layered edge to reach v. From u to reach centre use color c_1 edge instead of edge between B_1 and B_2.

Subcase (iv): If u is in $-1 \pmod k$ layer and v is in layer $0 \pmod k$:

Coloring Procedure:

No new coloring is required for this case.

Path:

From u use c_1 colored edge to reach $0 \pmod k$ layered vertex and then start using 0

(mod k) layered vertices using $-1 \pmod k$ colored edges to reach centre and then use $0 \pmod k$ layered vertices to reach v using $0 \pmod k$ colored edges.

Case 2. u is in B_1 and v is in B_1' :

Subcase (i): If u is in any layer and v is in any layer except $-1(\text{mod } k)$:

Coloring Procedure:

In B_1 if $l(b)$ = top $-1 \pmod k$ layered vertex layer and $l(a)$ = top $0 \pmod k$ layered vertex layer, then edge color will be $l(a) - 1$ and in B_2 this edge color will be $l(a)$.

Path:

From u use c_0 colored edge or $0 \pmod k$ colored edge to reach nearest $-1 \pmod k$ layered vertex and then follow $-1 \pmod k$ layered vertices using $0 \pmod k$ colored edges to reach the vertex next to the centre and then start using $0 \pmod k$ layered vertices using $-1 \pmod k$ colored edges to reach nearest $0 \pmod k$ layered vertex to v with high level and then use $-1 \pmod k$ or c_1 colored edge to reach v .

Subcase (ii): If u is in any layer and v is in $-1 \pmod k$ layer:

Coloring Procedure:

No new coloring is required for this case.

Path:

Use c_1 colored edge or $-1 \pmod k$ colored edge to reach nearest $0 \pmod k$ layered vertex and then follow $0 \pmod k$ layered vertices using $-1 \pmod k$ colored edges to reach top $-1 \pmod k$ layered vertex and then use $0 \pmod k$ colored edges to reach v.

Using the same path we can consider the case $B_2 - B_2'$.

Case 3. u is in B_1 and v is in B_3:

Subcase (i): If u is in any layer and v is in layer $0 \pmod k$:

Coloring Procedure:

1. In B_3 if $l(a)$ is in $0 \pmod k$ th layer or $l(b)$ is in $0 \pmod k$ th layer or $l(a)$ is in 0

(mod k) th layer and $l(b)$ is in $0 \pmod k$ th layer both then the edge color will be $l(a)$ (for now don't consider edge with one endpoint in $1 \pmod k$ and don't consider $l(b) = k$).

Path:

From u use c_1 colored edge or $-1 \pmod k$ colored edge to reach nearest $0 \pmod k$ layered vertex and then follow $0 \pmod k$ layered vertices using $-1 \pmod k$ colored edges and after reaching centre follow $0 \pmod k$ vertices using $0 \pmod k$ colored edges in B_3 to reach v .

Subcase (ii): If u is in any layer and v is in layer except $0 \pmod k$:

Coloring Procedure:

1. In B_3 if $l(a)$ is in $1 \pmod k$ th layer or $l(b)$ is in $1 \pmod k$ th layer or $l(a)$ is in $1 \pmod k$ th layer and $l(b)$ is in $1 \pmod k$ th layer both then the edge color will be $l(b) - 2$ (for now don't consider edge with one endpoint in $0 \pmod k$ and don't consider $l(b) =$ centre).
2. Edges between B_1 and B_3 will be c_1.

Path:

From u use c_0 colored edge or $0 \pmod k$ colored edge to reach nearest $-1 \pmod k$ layered vertex and then follow $-1 \pmod k$ layered vertices using $0 \pmod k$ colored edges and then use c_1 colored edge to reach $1 \pmod k$ layered vertex in B_3 and then follow $1 \pmod k$ vertices using $-1 \pmod k$ colored edges in B_3 to reach v .

Case 4. u is in B_2 and v is in B_3:

Subcase (i): If u is in any layer and v is in layer $0 \pmod k$:

Coloring Procedure:

No new coloring is required for this case:

Path:

From u use c_1 colored edge or $-1 \pmod k$ colored edge to reach nearest $-1 \pmod k$ layered vertex and then follow $-1 \pmod k$ layered vertices using $-1 \pmod k$ colored edges and by using c_0 colored edge after reaching centre follow $0 \pmod k$ vertices using $0 \pmod k$ colored edges in B_3 to reach v .

Subcase (ii): If u is in any layer and v is in layer except $0 \pmod k$):

Coloring Procedure:

1. In B_3 if $l(b)$ = centre layer and $l(a) \geq$ (layer of centre - k + 1) then the color of the edge will be c_1.

Path:

From u use c_0 colored edge or $0 \pmod k$ colored edge to reach nearest $0 \pmod k$ layered vertex and then follow $0 \pmod k$ layered vertices using $0 \pmod k$ colored edges and after reaching centre then use c_1 colored edge to reach $1 \pmod k$ layered vertex in B_3 and then follow $1 \pmod k$ vertices using $-1 \pmod k$ colored edges in B_3 to reach v .

Case 5. u is in B_3 and v is in B'_3

Subcase (i): If u is in any layer and v is in layer except $0 \pmod k$)):

Coloring Procedure:

1. In B_3 if $l(a) < k$ and $l(b) = 0$ then the edge color will be c_0.

Path:

From u use c_0 or $0 \pmod k$ colored edge to reach $0 \pmod k$ layered vertex and then follow $0 \pmod k$ layered vertices using $0 \pmod k$ colored edges and after reaching centre use c_1 colored edge to reach first $1 \pmod k$ vertex and then follow $1 \pmod k$ layered vertices using $-1 \pmod k$ colored edges to reach v .

Subcase (ii): If u is in layer $0 \pmod k$ layer and v is in layer $0 \pmod k$):

Coloring Procedure:

1. In B_3 if $l(a) = 0 \pmod k$ and $l(b) = 1 \pmod k$ then the edge color will be c_0.

Path:

From u follow $0 \pmod k$ layered vertices using $0 \pmod k$ colored edges and after reaching centre use c_1 colored edge to reach top $1 \pmod k$ vertex and follow $1 \pmod k$ layered vertices using $-1 \pmod k$ colored edges and use c_0 edge to reach v .

Subcase (iii): If u is in layer except $0 \pmod k$ layer and v is in layer $0 \pmod k$:

Coloring Procedure:

No new coloring will be required for this case.

Path:

From u follow $1 \pmod k$ layered vertices using $-1 \pmod k$ colored edges and by using c_1 colored edge after reaching centre follow $0 \pmod k$ layered vertices using $0 \pmod k$ colored edges to reach v.

Now we want to find the no of colors in both cases. In first case when diam(T^k) is odd then the topmost block in B_1 or B_2 will be incomplete. Length of shortest path between two layer 0 vertices of B_1 and B_2 will be diam(T^k). As $d(a_!, b_1) = k$, so we have to use an edge from B_1 to B_2. If $l = \frac{diam(T)}{2} - 1$, then there will be $\lfloor \frac{l}{k} \rfloor$ complete blocks in each branch and each block we are using two colors. And also we are using extra color c, so basically we are using diam(T^k) colors.

For the second case $d(a_!, b_1) = 0$, so we have to cover each complete blocks (all blocks are complete) to go through shortest path between two layer 0 vertices of B_1 and B_2 and in each block we are using two colors. So, in this case also we are using diam(T^k) colors.

□

Lemma 5. ***If*** T ***be a tree and it has single vertex in the centre and*** $diam(T) < 2k$***, then*** $rc(T^k) = diam(T^k)$ ***and if*** $diam(T) = 2k$ ***then*** $rc(T^k) = diam(T^k) + 1$

Proof.

We want to find rainbow color path between two vertices u and v. First consider the case when $diam(T) = 2k$:

Coloring Procedure:

Edge whose one endpoint is centre and other endpoint is next vertex to centre will be colored 2. Color of edges with two endpoints in different branches or subbranches will be 2. Other edges whose one endpoint centre will be colored 1. Other edges will be colored 3.

Claim: Each pair of vertices has a rainbow path.

Proof: If u and v in different branch, then from u reach centre by color 1 edge, then use color 2 edge to reach vertex next to the centre on the branch in which v is and then from there use color 3 edge to reach v. If u is next to the centre, instead of reaching the centre by color 2 edge we can directly go to next vertex to centre on the branch in which v is.

If v is on subbranch of branch in which u is, in that case from v use color 2 edge to reach a vertex on branch in which u is. Then use color 3 edge to reach u from that vertex. This path is from v to u.

■

Now consider the case when $diam(T) < 2k$:

Coloring Procedure:

Edges with two endpoints in different branches or subbranches will be colored 2. Other edges will be colored 1.

Claim: Each pair of vertices has a rainbow path.

Proof: Suppose length of branch in which v is is less than length of branch in which u is (one branch is shorter as $diam(T) < 2k$). From v use color 2 edge to reach a vertex on branch in which u is. This edge exists as $diam(T) < 2k$. Then use color 1 edge to reach u from that vertex.

■

In both cases $diam(T^k)$ is 2. For the first case we are using 3 colors (i.e $diam(T^k) + 1$). For the second case we are using 2 colors (i.e $diam(T^k)$).

□

Lemma 6. ***If*** T ***be a tree and it has two vertices in centre, then*** $rc(T^k) = diam(T^k)$

Proof.

We want coloring of edges such that for any pair of two vertices there exists a rainbow colored path between them. We will proceed through two cases.

Case 1. First consider the case when $diam(T) \geq 2k + 1$.

Let z_1 and z_2 be two centre vertices in T and let B_i be the branches of z_i including z_i. They are the maximum length branches, if one is shorter then there will be a single centre and that will be a contradiction to our assumption. Again we will proceed through two cases. Recall a_1 and b_1 mentioned in lemma 1.
First consider the case when $d(a_1, b_1) \leq k$. The other case will be discussed later.
Suppose B_i' is a subbranch of B_i. B_i is a branch including z_i. we may assume B_i's are the maximal branches. B_i is the representative of all branches from z_i. Color of particular edges which are required to get rainbow colored path between any two vertices are mentioned here. Other edges can be colored arbitrarily.

Coloring Procedure:

1. Color of edges in B_i' will be similar to same type edges in B_i (same type edge means layer of two endpoints of edge are same).

2. Color the edges with one endpoint in B_1 and another endpoint in B_2 with a new color suppose c_1.

3. Color the edges with one endpoint in B_i and another endpoint in B_i' with c_1.

4. Color of edges with anyone endpoint in incomplete blocks of B_1 or B_2 are c_1.

5. Color the edges between $0 \pmod{k}$ th layer vertex and $-1 \pmod{k}$ th layered vertex with c_1.

6. Edges with upper end (upper end means which endpoint of edge is in higher layer) kl and $kl - 1$ th layer vertex should be colored kl and $kl - 1$ respectively in B_1. Edges with lower end kl and $kl - 1$ th layer vertex should be colored $k(l + 1)$ and $k(l + 1) - 1$ respectively in B_1.

7. In B_2 the coloring will be reversed just. That means edges with upper end kl and $kl - 1$ th layer vertex should be colored $kl - 1$ and kl respectively in B_2. Edges with lower end kl and $kl - 1$ th layer vertex should be colored $k(l + 1) - 1$ and $k(l + 1)$ respectively in B_2.

8. Other edges can be colored arbitrarily.

Claim: There exist rainbow colored path between each pair of vertices.

Proof:

Choose two vertices u and v. We want to find rainbow colored path between them. We will proceed through several cases.

Subcase (i): u **is in** B_1 **and** v **is in** B_2

Use layers $0 \pmod k$ to reach a_1 from u. So all edges which are used till now are colored with $0 \pmod k$. From a_1 then use the edge to reach b_1 in B_2. This edge exists as $d(a_1, b_1) \leq k$ as we have supposed. The color of this edge is c_1 as this edge has one endpoint in B_1 and another endpoint in B_2. Then take layers $0 \pmod k$ to reach v. So in that case we are using edges with color $-1 \pmod k$. If we assume u is the layer 0 vertex in B_1 and v is the layer 0 vertex in B_2, then this is the shortest path and it is the $diam(T^k)$ length path. So we are using $diam(T^k)$ no of colors.

Subcase (ii): u **is in** B_1 **and** v **is in** B_1'

Suppose the common ancestor of u and v is in layer i. Suppose i is in complete block. Now assume $k(p-1) \leq i \leq kp$. From u use layers $0 \pmod k$ to reach vertex which is in layer kp. So we are using edges colored with $0 \pmod k$. If $i = kp-1$ or kp then take the edge to reach the vertex which is in $kp-1$ th layer in B_i'. If i is except $kp-1$ or kp then take the edge to reach the vertex which is in $kp-1$ th layer in B_i . Color of this edge is c_1. Then use layers $-1 \pmod k$ to reach the vertex v. Now we are using edges with color $-1 \pmod k$. And if i is in incomplete block then after reaching kp th vertex similarly we have to go $kp-1$ th vertex in the other branch using c_1 colored edge. This edge exists as $d(a_1, z_1) < k-1$.

Subcase (iii): u **is in incomplete block of** B_1 **and** v **is in complete block of** B_1**.**

Take c_1 colored edge to reach a_1 from u. Suppose v is in layer i. Now assume $k(p-1) \leq i \leq kp$ and $i \neq -1 \pmod k$. Now go to vertex in layer kp using layers $0 \pmod k$ (So we are using edges with color $0 \pmod{k}$) and then take edge with color kp to go v. But if v is in layer $-1 \pmod{k}$, then after using c_1 colored edge to reach $-1 \pmod{k}$, follow the layers $-1 \pmod{k}$ instead of using layers $0 \pmod{k}$(So we are using edges colored with $-1 \pmod{k}$).

Subcase (iv): **u is in complete block of B_1 and v is in complete block of B_1**
From u use layers $0 \pmod{k}$ to reach v. So we are using edges colored with $0 \pmod{k}$. If u and v are in incomplete block then they will have an edge.

Subcase (v): **u is in B_1 and v is in some other branch from z_1**
Use $0 \pmod{k}$ layers in B_1 and after that use c_1 colored edge to reach $-1 \pmod{k}$ layered vertex in other branch and then use $-1 \pmod{k}$ layered vertices to reach v.

Subcase (vi): **case 2, case 3, case 4, case 5** can be solved similarly for the branch B_2.

Now we will proceed through second case when $d(a_1, b_1) > k$.

Coloring Procedure:

Now if $d(a_1, b_1) > k$ then there is a minor change of coloring procedure. Edges from vertices of B_1 to z_2 will be colored c_1. Edges from vertices of B_2 to z_1 will be colored c_2. Color of the edge with one endpoint in incomplete block of B_1 and other endpoint in layer $0 \pmod{k}$ of B_1 will be c_1. Color of the edge with one endpoint in incomplete block of B_1 and other endpoint in layer $-1 \pmod{k}$ of B_2 will be c_2. In B_2 color change will be done reversely (c_2 in place of c_1 and c_1 in place of c_2). Other edges can be colored similarly as we have mentioned before.

Rainbow Path:

To find the rainbow path there will be a minor difference. To find rainbow path between a vertex in B_1 and a vertex in B_2 we have used a edge from B_1 to B_2 with color c_1 (in case 1). But in this case we won't find that certain edge. So we have to follow color c_1 edge from that certain vertex in B_1 to reach z_2 and then we have to use c_2 colored edge to reach that certain vertex in B_2 from z_2. And for case 2 if the common ancestor is in incomplete block then first follow $0 \pmod k$ layers with $0 \pmod k$ colored edges and c_1 colored edge from u to reach the common ancestor and after that follow c_2 colored edge and then $-1 \pmod k$ colored edges to reach v using $-1 \pmod k$ layers.

No of colors:

Now we will show the number of colors has been used is actually $diam(T^k)$. Suppose $l = \frac{diam(T)}{2} - 1$. We have divided those layers in blocks of size k. Notice that two colors are used in every complete block. There are $\lfloor \frac{l}{k} \rfloor$ complete blocks, so $2\lfloor \frac{l}{k} \rfloor$ colors for those blocks. If $d(a_1, b_1) < k$ then $diam(T^k)$ will be $2\lfloor \frac{l}{k} \rfloor + 1$. And we have used one extra colors c_1 except $2\lfloor \frac{l}{k} \rfloor$ colors. If $d(a_1, b_1) > k$ $diam(T^k)$ will be $2\lfloor \frac{l}{k} \rfloor + 2$. Basically we are using then two extra colors c_1 and c_2 apart from $2\lfloor \frac{l}{k} \rfloor$ colors. So in both cases we are using $diam(T^k)$ colors.

■

Case 2. Now we consider the case when $diam(T) \leq 2k$.

Coloring Procedure:

There is an edge between z_1 and z_2. So suppose u is in branch of z_1. Each edge with one endpoint z_i and other endpoint in branch of z_i will be colored 1. Each edge with one endpoint z_i and other endpoint in branch of z_j where $i \neq j$ will be colored 2. Edges with two endpoint in different branches and subbranches will be colored 2. Other edges will be colored 1.

<u>Claim:</u> Each pair of vertices has rainbow color path.

<u>Proof:</u>

Subcase (i): **If u and v are in branch of different z_i:**

Suppose length of branch of u is greater than length of branch of v and if u is in z_1 branch and v is in z_2 branch, use color 1 edge to reach z_1 from u and then use color 2 edge to reach v from z_1. This edge exists as $diam(T) \leq 2k$.

Subcase (ii): **If u and v are in branch of same z_i:**

If common ancestor of u and v is centre, from v use color 2 edge to reach lower next vertex to z_i on the branch of u and then use color 1 edge to reach u from that vertex. First edge exists as $diam(T) \leq 2k$. And if common ancestor is not centre then from u use color 2 edge to reach lower next vertex to common ancestor on branch of v and from there use color 1 edge to reach v. First edge exists as $diam(T) \leq 2k$.

So only 2 colors are needed and also $diam(T^k) = 2$. So in this case also we are using $diam(T^k)$ colors. ■

□

Lemma 7. ***For powers of tree T^k, rainbow connection number $\in$ {$diam(T^k)$, $diam(T^k) + 1$}***

Proof. It can be proved using previous mentioned lemmas. □

5 References

References

[1] Yair Caro et al. "On Rainbow Connection". In: *The Electronic Journal of Combinatorics* 15.R57 (2008). URL: https://doi.org/10.37236/781.

[2] Yair Caro et al. "On rainbow connection". In: *The Electronic Journal of Combinatorics* 15.R57 (2008).

[3] Sourav Chakraborty et al. "HARDNESS AND ALGORITHMS FOR RAINBOW CONNECTIVITY". In: *COMBIN. OPTIM* 21 (2009), pp. 243–254.

[4] Charalampos and E.Tsourakakis. "Mathematical and Algorithmic Analysis of Network and Biological Data". PhD thesis. Carnegie Mellon University, 2013.

[5] Gary Chartrand, G.L. Johns, and Kathleen A. McKeon. "RAINBOW CONNECTION IN GRAPHS." In: *MATHEMATICA BOHEMICA* 133 (2008), pp. 85–98.

[6] Pinar Heggernes et al. "Rainbow Vertex Coloring Bipartite Graphs and Chordal Graphs". In: *43rd International Symposium on Mathematical Foundations of Computer Science (MFCS 2018)* 83 (2018), . 83:1–83:13.

[7] Michael Krivelevich and Raphael Yuster. "The Rainbow Connection of a Graph Is (at Most) Reciprocal to Its Minimum Degree". In: *Journal of Graph Theory* 63 (Jan. 2009), pp. 185–191. DOI: 10.1002/jgt.20418.

[8] Xueliang Li and Yuefang Sun. "An Updated Survey on Rainbow Connections of Graphs- A Dynamic Survey". In: *Theory and Applications of Graphs* 00 (Jan. 2017), pp. 1–65. DOI: 10.20429/tag.2017.000103.

[9] Paloma T. Lima, Erik Jan van Leeuwen, and Marieke van der Wegen. "Algorithms for the Rainbow Vertex Coloring Problem on Graph Classes". In: *45th International Symposium on Mathematical Foundations of Computer Science (MFCS 2020)*. Vol. 170.

Leibniz International Proceedings in Informatics (LIPIcs). 2020, 63:1–63:13. ISBN: 978-3-95977-159-7.