Essays in Mechanism Design

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Contents

Chapter 1

Introduction

This thesis is comprised of three essays in mechanism design. The first essay/chapter investigates the stability property of trading mechanisms of an internet platform that conducts trades between buyers and sellers. The second chapter considers the allocation of a single object among a set of agents whose valuations are interdependent. The third chapter is about the allocation of multiple units of a good among a set of agents who have private valuations for one unit of good.

We provide a brief description of each chapter below.

1.1 Stability and double auction design

This chapter is about an internet platform which is trying to conduct trades between buyers and sellers. Each seller is in possession of a single unit of homogenous good and there are several buyers each of whom desires a single unit of the good. The utility functions of both buyers and sellers are quasi-linear with the valuations of buyers and sellers being private information.

Our contribution is to introduce a novel consideration in the mechanism design problem for the platform in this model in addition to the standard ones of (interim) incentive-compatibility and (interim) individual-rationality. Coalitions of buyers and sellers are free to set up their own trading mechanisms and divide the surplus among themselves. This threat by the buyers and sellers will impact the optimal expected revenue of the platform. If the platform tries to extract "too much revenue", a coalition of buyers and sellers can *block* the mechanism offered by the platform and trade with a mechanism of their own choice.

Specifically, we analyze the structure of revenue optimal mechanisms for the platform when single buyer-seller coalitions can threaten to secede from the platform's mechanism.

We look for revenue-optimal mechanisms in the class of *stable* mechanisms. The notion of stability in private information settings has received a great deal of attention in the literature beginning with Wilson (1978). Wilson defines an *interim* core in an incomplete information setting. Dutta and Vohra (2005) refine the notion of interim core and define the credible core. They prove the non-emptiness of incentive compatible credible core in an auction model. Apart from the interim core, there are two other types of cores: the ex-post core and the ex-ante core, depending on the stage (ex-post or ex-ante) in which the agents form coalitions.

The notion of stability in private information settings that we use is called single-buyer-single-seller (SBSS) ex-ante stability. It is based on the notion of ex-ante incentive compatible core given by Forges et al. (2002a) for an exchange economy setting. It was also used recently by Bikhchandani (2017) who studies ex-ante stability of matching mechanisms in a model with one-sided incomplete information and nontransferable utilities. In our model, the agents form coalitions before their private information is revealed. If a buyer-seller pair receive a strictly higher expected payoff (before the realization of their privately observed valuations) via a bilateral trading mechanism, they will block the mechanism proposed by the platform. A platform's mechanism is SBSS ex-ante stable if no buyer-seller pair can block it in this sense.

Our first set of results concerns the SBSS ex-ante stability of several well-known double auctions. We consider the trade reduction mechanism, its special case the McAfee double auction and the positive-spread posted-price mechanism. In all cases, we assume that the values of buyers and sellers are independently and identically distributed with a uniform distribution over a unit interval. We show by the means of explicit computation that these mechanisms are not stable for a particular value of number of buyers and sellers. In all cases, the buyer and seller can block using the posted-price mechanism given by Hagerty and Rogerson (1985). It is clear from these examples that stability imposes constraints on mechanism chosen by the platform.

The main result of the chapter concerns the expected revenue-maximizing mechanism that satisfies interim incentive-compatibility, interim individual-rationality and SBSS ex-ante stability. Using ideas in Myerson (1981) and Myerson and Satterthwaite (1983) we identify a revenue optimal mechanism without stability constraints, i.e. a mechanism that maximizes expected revenue subject to interim incentive-compatibility and interim individual rationality. We show that this mechanism is not SBSS ex-ante stable. There is therefore, genuine tension between stability and revenue maximization. Finally, we provide a revenue-maximizing mechanism that satisfies the stability constraints.

1.2 Budget-balanced mechanisms for single-object allocation problems with interdependent values

In this chapter we consider the problem of allocating a single object among a set of agents in an interdependent value setting. Each agent receives a signal about the valuation of the object. Her valuation depends on the signals received by all other agents. The mechanism must satisfy the properties of ex-post incentive compatibility, ex-post individual-rationality, budget-balance and ex-post efficiency.

In our model, the Green-Laffont impossibility continues to hold.* We consider two types of mechanisms. The first are *signal-ranking mechanisms* (or s-ranking mechanisms). Agents report their signals and are ranked according to these reports. The s-ranking allocation rule assigns a probability for receiving the object to each agent. Transfers for agents are determined accordingly. The *valuation-ranking mechanisms* or v-ranking mechanisms on the other hand, assign probabilities for receiving the object based on the ranking of agents' valuations.

We show that a ranking allocation rule that is strategy-proof and can be implemented by budget-balanced transfers in the private-value case is also an ex-post incentive compatible (EPIC) and ex-post individually rational (EPIR) s-ranking allocation rule that can be implemented with budget-balanced (BB) transfers provided the valuation functions satisfy an additive separability condition. An immediate consequence of this result is that the s-ranking mechanism where the agents with the highest and second-highest ranking signals receive the object with probabilities $1 - \frac{1}{n}$ and $\frac{1}{n}$ respectively (i.e. the Green-Laffont allocation vector) is EPIC, EPIR and implementable by budget-balanced transfers if the valuation functions satisfy additive separability condition, single-crossing and symmetry. We show by means of an example that the result does not hold. We also show that the allocation rule of the mechanism that maximizes worst-case efficiency ratio given by Long et al. (2017) is the s-ranking allocation rule which maximizes worst-case efficiency ratio among all EPIC, EPIR and BB s-ranking mechanisms when valuation functions are of a specific form that satisfies SAS condition, single-crossing, symmetry. We then provide an example to show that this mechanism is no longer optimal when the valuation functions are not symmetric.

*See Nath et al. (2015) for further details.

For v-ranking mechanisms, first we show that it is necessary for valuation functions to satisfy single-crossing for the mechanism to be EPIC. Then we show that a ranking allocation rule that is strategy-proof and can be implemented by budget-balanced transfers in the private-value case is also an EPIC and EPIR v-ranking allocation rule that can be implemented with budget-balanced transfers provided the valuation functions satisfy the additive separability condition and single-crossing. Under an additional condition of symmetry of valuation functions, the allocation functions for s-ranking mechanisms and v-ranking mechanisms are allocation equivalent. Moreover, the agents have the same payment functions and get the same utility from allocation equivalent, EPIC, EPIR and BB s-ranking and v-ranking mechanisms.

Another approach to the impossibility result is to allocate the object only to the agent with the highest signal but with probability less than one. The object is thrown away or retained by the seller with the remaining probability. The agent who is allocated the object makes a payment which is redistributed among all the agents ensuring budget balancedness. Such mechanisms were called *probability-burning mechanisms* by Mishra and Sharma (2018) and studied in private valuation models. We explore the feasibility of such mechanisms in the interdependent valuation case. For a semi-separable class of valuation functions, we show that a particular probability-burning mechanism is EPIC, EPIR and BB. For additively separable and symmetric class of valuation functions, we design another probability-burning mechanism and show that it is welfare-maximizing in the class of EPIC, EPIR, BB mechanisms that allocate only to the agents with topmost signal and satisfy an additional property called *equal treatment at equal signals*.

1.3 Probability-burning mechanisms in multiple-good allocation problems

The final chapter considers the problem of allocating m units of a good among n agents. Each agent demands a single unit of good the valuation of which is his private information. The mechanism must have the usual properties viz. incentive-compatibility, individual-rationality, budget-balance and efficiency. This contrasts with the approach of Dastidar (2017) who focuses on goals of efficiency and revenue generation for mechanisms that allocate a scarce good to a set of agents. In our model, due to the Green-Laffont impossibility result (Green and Laffont (1979)), no mechanism can simultaneously satisfy efficiency, incentive-compatibility and budget-balance. So, one of the properties must be relaxed in order to find a mechanism which satisfies two properties and a weakened version of the third. In a multi-unit allocation problem, Guo and Conitzer (2014) weaken the budget-balance condition and study linear redistribution mechanisms. Gujar and Narahari (2008) extend the analysis to the case of multiple heterogenous goods. In this chapter, we relax the property of efficiency and looks within the class of incentive-compatible and budget-balanced mechanisms. We follow the approach given by Mishra and Sharma (2018) called *probability-burning mechanisms*.

The chapter has two objectives. The first is to extend the mechanism of Mishra and Sharma (2018) to the multi-good allocation problem. We propose the *equal-probability-burning mechanism* which allocates a single unit of good to each of the top m highest-valued agents with equal probability. The probability is auctioned through a multi-unit Vickrey auction and the revenue collected is redistributed back to the agents which ensures budget-balance. Some of the allocation probability is burnt at some valuation profiles for each unit of good. We then compare the welfare properties of this mechanism with some other mechanisms that are budget-balanced, dominant strategy incentive-compatible and individually-rational. These mechanisms are the multi-unit extension of Green-Laffont mechanism and the single-unit burning mechanism given by Guo and Conitzer (2014).

We find that the worst-case efficiency ratio of multi-unit Green-Laffont mechanism is higher than that of equal-probability-burning mechanism. If the number of agents is greater than the threshold level $m + \frac{m^2}{2} + \sqrt{m(m^2-1) + \frac{m^4}{4}}$, the worst-case efficiency ratio of the equal-probability-burning mechanism is greater than that of the single-unit burning mechanism. The expected total welfare of equal-probability-burning mechanism is less than that of the multi-unit Green-Laffont mechanism but converges to it as n increases.

The second objective is to design probability-burning mechanism with reserve prices. Goods are allocated only if the valuations of at least m agents are above the reserve price. In this case each of the m agents with the highest ranked valuations is given a good with equal probability. The allocation probability depends on the relationship between the reserve price and the valuations of $(m+1)^{th}$ and $(m+2)^{th}$ ranked agents. We show that the mechanism is budget-balanced, individually-rational and dominant strategy incentive-compatible.

Our main goal is to demonstrate that introducing reserve prices may increase the expected welfare of agents. For this purpose we assume that valuations are uniformly distributed. In the restricted setting of $n = 4$ and $m = 2$, we show that the optimal reserve price is non-zero. For a single-good model we explicitly compute the optimal reserve price and show that the expected total welfare with the reserve price is greater than the expected total welfare in the mechanism of Mishra and Sharma (2018).

Chapter 2

Stability and double auction design

2.1 Introduction

In recent years, there has been a phenomenal rise in number and size of internet platforms. Sharing economy companies like Uber and Airbnb are platforms that facilitate the connection between millions of buyers and sellers everyday. The combined size of these platforms was 1.5 per cent of US gross domestic product. i.e. USD 273 billion in 2015*.

A platform generates revenue by matching buyers and sellers. On one side of the market are sellers who hold various goods, and on the other side are buyers who wish to purchase them from the sellers. Both buyers and sellers have valuations for the goods. Since the platform is unaware of the valuations of buyers and sellers, it must use a "mechanism" in which agents bid for the various goods. The mechanism then specifies the set of buyers and sellers who trade with each other and the prices they pay and receive. The platform typically charges a margin between buyer's payment and seller's receipt. Mechanisms are typically required to satisfy incentive compatibility (truthful elicitation of valuations) and individual rationality (ensuring voluntary participation). An early example of an analysis of such issues is Myerson and Satterthwaite (1983). They considered the problem of a platform/broker in a bilateral trading environment (a single buyer and a seller) which attempts to maximize its expected revenue.

We consider the design of mechanisms from the perspective of platform in the double auction setting. There is a single homogenous good, several sellers each in possession of a single unit of good and several buyers each of whom desires a single unit of the good. The

*See Measuring the digital economy, IMF, 2018 utility functions of both buyers and sellers are quasi-linear with the valuations of buyers and sellers being private information.

Our contribution is to introduce a novel consideration in the mechanism design problem for the platform in this model in addition to the standard ones of (interim) incentive-compatibility and (interim) individual-rationality. Coalitions of buyers and sellers are free to set up their own trading mechanisms and divide the surplus among themselves. It is clear that this will impact the optimal expected revenue of the platform. If the platform tries to extract "too much revenue", a coalition of buyers and sellers can *block* the mechanism offered by the platform and trade with a mechanism of their own choice. We restrict attention to the case where the blocking coalition consists of a *single* buyer-seller pair. This assumption considerably simplifies the analysis. It is also realistic - forming larger blocking coalitions will involve greater coordination between agents. We analyze the structure of revenue optimal mechanisms for the platform when single buyer-seller coalitions can threaten to secede from the platform's mechanism.

It is clear from the foregoing discussion that we are looking for revenue optimal mechanisms in the class of *stable* mechanisms. The notion of stability in private information settings has received a great deal of attention in the literature beginning with Wilson (1978). Wilson defines an *interim* core in an incomplete information setting. Apart from the interim core, there are two other types of cores, the ex-post core and the ex-ante core, depending on the stage (ex-post or ex-ante) in which the agents form coalitions.

Ex-post stability is the most demanding of the three notions. In this case, agents form coalitions after they have communicated their private information and the outcomes specified by the mechanism have been implemented. A weaker notion is the interim core where agents form coalitions after their private information is revealed to them.[†] The weakest notion of stability is ex-ante stability. Here agents form coalitions before their private information is revealed. If a buyer-seller pair receive a strictly higher expected payoff (before the realization of their privately observed valuations) via a bilateral trading mechanism, they will block the mechanism proposed by the platform. A platform's mechanism is single-buyer-single-seller (SBSS) ex-ante stable if no buyer-seller pair can block in this sense. Forges et al. (2002a) refer to this approach as the one that occurs "behind the veil of ignorance".

Our first set of results concerns the SBSS ex-ante stability of several well-known double auctions. We consider the trade reduction mechanism, its special case the McAfee double auction and the positive-spread posted-price mechanism. In all cases, we assume that buyer and seller are independently and identically distributed with a uniform distribution over a

[†]A detailed discussion of interim stability and ex-post stability can be found in Forges et al. (2002b).

unit interval. We show by the means of explicit computation that these mechanisms are not stable for a particular value of number of buyers and sellers. In all cases, the buyer and seller can block using the posted-price mechanism given by Hagerty and Rogerson (1985). It is clear from these examples that stability imposes constraints on mechanism chosen by the platform.

The main result of the chapter concerns the expected revenue-maximizing mechanism that satisfies interim incentive-compatibility, interim individual-rationality and SBSS ex-ante stability. Using ideas in Myerson (1981) and Myerson and Satterthwaite (1983) we identify a revenue optimal mechanism without stability constraints, i.e. a mechanism that maximizes expected revenue subject to interim incentive-compatibility and interim individual rationality. We show that this mechanism is not SBSS ex-ante stable. There is therefore, genuine tension between stability and revenue maximization. Finally, we provide a revenue-maximizing mechanism that satisfies the stability constraints.

The unconstrained revenue-maximizing mechanism allocates the goods to the buyers with highest virtual valuation and sellers with lowest virtual costs till a buyer's virtual valuation becomes lower than a seller's virtual cost. The constrained revenue-maximizing mechanism, on the other hand allocates on the basis of adjusted virtual valuation. The adjusted virtual valuation of buyer is higher than its virtual valuation and the adjusted virtual cost of a seller is lower than its virtual cost. There will be more trades taking place as a result, resulting in higher ex-ante expected utility of buyers and sellers but the platform will earn less revenue.

The chapter proceeds as follows. In Section 2.2 we give a brief overview of recent work on stability of mechanisms. We describe the model and define some special mechanisms that we use throughout the chapter in Section 2.3. In Section 2.4 we discuss the SBSS ex-ante stability property of all the mechanisms. Section 2.5 starts with generalization of Myerson-Satterthwaite model of revenue-maximization to a general market. We prove that it is not generally SBSS ex-ante stable. We then find a mechanism that maximizes the revenue of platform in class of SBSS ex-ante stable mechanisms. Section 2.6 concludes.

2.2 Related Literature

Like in our chapter, Forges (2004) considers the notion of the ex-ante incentive-compatible core in the context of a two-sided assignment game. The paper demonstrates the non-emptiness of the core when players have common values. Bikhchandani (2017) applies the notion of ex-ante stability to a matching model with one-sided incomplete information but where the utilities are nontransferable. Chen and Hu (2017) do the same in a two-sided incomplete information matching model.

In recent work Bochet and Ilkilic (2017) use the notion of bilateral stability (introduced by Kranton and Minehart (2001)) to show that there generally does not exist a bilaterally stable mechanism that satisfies strategyproofness, individual rationality and efficiency in a bipartite network setting. If the network does not have cycles, the Vickrey auction is the unique mechanism that satisfies all these properties. The notion used in the paper is related to the ex-post core - the agents form a blocking coalition after the mechanism has already allocated the goods and the transfers have taken place.

A few papers have considered the interim core. Peivandi and Vohra (2021) use a notion of interim core called B-blocking to show that there does not exist a stable and interim incentive-compatible mechanism for a centralized market. The threat of a mechanism to which a coalition can deviate induces inefficiency in the trading mechanism of the market. Dutta and Vohra (2005) use a refined notion of interim core called the credible core. They prove the non-emptiness of incentive compatible credible core in an auction model. The credible core is a refinement of the concept of interim incentive compatible core i.e. a coalition can credibly identify an informational event such that the types of agents involved in the event prefer the new mechanism to the existing one.

2.3 The model and basic definitions

There are N buyers $B = \{b_1, b_2, \ldots, b_N\}$ and N sellers $S = \{s_1, s_2, \ldots, s_N\}$ in a market[‡]. Let K be the grand coalition of the buyers and sellers i.e. $K = B \cup S$. A typical buyer and seller will be denoted by b_i and s_j respectively. Each seller has a unit of an indivisible good and all the goods are identical to each other. Each buyer demands at most one unit of the good. Buyer b_i has valuation (type) v_i for the good, which is his private information. Similarly, each seller s_j has cost (type) c_j of supplying the good, which is the private information of the seller. We assume that v_i and c_j are independent random variables with support $[0, 1]$. The distribution function for v_i, $i \in \{1, 2, \ldots, N\}$ is F, with an associated density function f. The distribution function for c_j, $j \in \{1, 2, \ldots, N\}$ is G, with an associated density function g.

A complete type profile denoted by $(v, c) = (v_1, \ldots, v_N, c_1, \ldots, c_N)$ is a $2N$ tuple of valuations of buyers and sellers. The space of type profiles is $T = [0, 1]^{2N}$. For every buyer $b_i \in B$, let $T \setminus \{b_i\} \equiv \{(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_N, c_1, \ldots, c_N)\}$ i.e. it denotes the set of $2N - 1$

[‡]The number of buyers and sellers being equal makes the analysis tractable.

types excluding that of buyer b_i. Clearly, $T \setminus \{b_i\} = [0,1]^{2N-1}$. Similarly, for every seller $s_j \in S$, $T \setminus \{s_j\} \equiv \{(v_1, \ldots, v_N, c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_N)\}$ i.e. it denotes the set of $2N-1$ types excluding that of buyer s_j.

It will be assumed throughout the chapter that the distributions F and G satisfy the *regularity condition.*

Definition **2.1** *Let $\phi(v_i) = v_i - \frac{1-F(v_i)}{f(v_i)}$ and $\psi(c_j) = c_j + \frac{G(c_j)}{g(c_j)}$ for each i and $j \in \{1, 2, \ldots, N\}$. The distributions F and G satisfy regularity if the functions $\phi(v_i)$ and $\psi(c_j)$ are non-decreasing in v_i and c_j, respectively. The functions $\phi(v_i)$ and $\psi(c_j)$ shall be referred to as the virtual valuation functions of b_i and s_j at type v_i and c_j, respectively.*

The regularity conditions are standard conditions in the literature. They considerably simplify the analysis of the optimal mechanisms. They are weak conditions satisfied by, for instance, the uniform distribution.

Let coalition A be a non-empty subset of K. A *mechanism* for A elicits type of each agent in A and specifies the allocations and transfers of these agents. Formally, a mechanism $\mathcal{M}^A = (q^A, t^A)$ is a collection of functions $(q^A_{b_i \in A}, q^A_{s_j \in A}, t^A_{b_i \in A}, q^A_{s_j \in A})$.

$$q^A_{b_i}, q^A_{s_j} : [0,1]^{|A|} \to [0,1]$$

$$t^A_{b_i}, t^A_{s_j} : [0,1]^{|A|} \to \mathbb{R}$$

For every type profile (v, c), $q^A_{b_i}(v, c)$ is the probability of buyer b_i receiving the good and $q^A_{s_j}(v, c)$ is the probability of seller s_j selling his good. Furthermore, $t^A_{b_i}(v, c)$ is the payment made by the buyer b_i, while $t^A_{s_j}(v, c)$ is the amount received by seller s_j.

A mechanism is required to be *feasible* i.e. the sum of allocation probabilities of buyers is equal to the sum of allocation probabilities of sellers. Formally, we require $\sum\limits_{b_i \in A} q^A_{b_i}(v, c) = \sum\limits_{s_j \in A} q^A_{s_j}(v, c)$ for every $(v, c) \in [0,1]^{|A|}$.

Two mechanisms will be of particular interest in the chapter. The first is for the grand coalition K and is denoted by $\mathcal{M} = (q, t)$ (we suppress the superscript). The second is for a coalition $A = \{b_i, s_j\}$. It is called a bilateral mechanism and denoted by $\mathcal{M}^A$.

Definition **2.2** *The mechanism $\mathcal{M}^A$ satisfies the no-budget deficit (NBD) condition if*

$$\int\limits_{(v,c) \in [0,1]^{|A|}} \Big(\sum_{b_i \in A} t^A_{b_i}(v, c) - \sum_{s_j \in A} t^A_{s_j}(v, c) \Big) f(v) g(c) dv dc \geq 0$$

If a mechanism satisfies the NBD condition for the grand coalition K, then the platform can never make negative ex-ante revenue.

The ex-post utilities of a buyer b_i and a seller c_j in mechanism $\mathcal{M}^A$ are:

$$U^A_{b_i}(v,c) = q^A_{b_i}(v,c)v_i - t^A_{b_i}(v,c)$$

$$U^A_{s_j}(v,c) = t^A_{s_j}(v,c) - q^A_{s_j}(v,c)c_j$$

The interim utilities of a buyer b_i and a seller s_j are:

$$U_{b_i}(v_i) = \int\limits_{(v_{-i},c)\in T\setminus b_i} (q_{b_i}(v,c)v_i - t_{b_i}(v,c))f(v_{-i})g(c)dv_{-i}dc$$

$$U_{s_j}(c_j) = \int\limits_{(v,c_{-j})\in T\setminus s_j} (t_{s_j}(v,c) - q_{s_j}(v,c)c_j)f(v)g(c_{-j})dvdc_{-j}$$

Also, the ex-ante expected utilities of a buyer b_i and a seller s_j are:

$$U_{b_i} = \int\limits_{(v,c)\in T} (q_{b_i}(v,c)v_i - t_{b_i}(v,c))f(v)g(c)dvdc$$

$$U_{s_j} = \int\limits_{(v,c)\in T} (t_{s_j}(v,c) - q_{s_j}(v,c)c_j)f(v)g(c)dvdc$$

If a buyer b_i's valuation is v_i, $\overline{q}_{b_i}(v_i)$ is the expected probability of receiving a good and $\overline{t}_{b_i}(v_i)$ is the expected transfer defined as,

$$\overline{q}_{b_i}(v_i) = \int\limits_{(v_{-i},c)\in T\setminus b_i} q_{b_i}(v,c)f(v_{-i})g(c)dv_{-i}dc$$

$$\overline{t}_{b_i}(v_i) = \int\limits_{(v_{-i},c)\in T\setminus b_i} t_{b_i}(v,c)f(v_{-i})g(c)dv_{-i}dc$$

Similarly, $\overline{q}_{s_j}(c_j)$ and $\overline{t}_{s_j}(c_j)$ are defined for the seller c_j. If its type is c_j, then,

$$\overline{q}_{s_j}(c_j) = \int\limits_{(v,c_{-j})\in T\setminus s_j} q_{c_j}(v,c)f(v)g(c_{-j})dvdc_{-j}$$

$$\overline{t}_{s_j}(c_j) = \int\limits_{(v,c_{-j})\in T\setminus s_j} t_{c_j}(v,c)f(v)g(c_{-j})dvdc_{-j}$$

Fix a mechanism $\mathcal{M}$. We define the following properties:

DEFINITION **2.3** *A mechanism* $\mathcal{M} = (q^e, t^e)$ *is* ***ex-post efficient*** *if for each* $(v, c) \in T$, $q^e_{b_i}(v, c)$ *and* $q^e_{s_j}(v, c)$ *for all* i *and* $j \in \{1, 2, \ldots, N\}$, *it solves the following problem,*

$$\max_{(q_{b_i}, q_{s_j} \forall i,j=1,2,\ldots,N)} \Big(\sum_{i=1}^{N} q_{b_i} v_i - \sum_{j=1}^{N} q_{s_j} c_j \Big)$$

$$s.t. \quad \sum_{i=1}^{N} q_{b_i} = \sum_{j=1}^{N} q_{s_j} \leq N$$

$$and \ 0 \leq q_{b_i}, q_{s_j} \leq 1, \forall \ i, j \in \{1, 2, \ldots, N\}$$

It is straightforward to characterize the ex-post efficient allocation rules at a type profile (v, c). Arrange all the buyers in the descending order of their valuations i.e. $v_{(i)}$ denotes the i^{th} highest valuation among all buyers. Similarly, arrange the sellers in ascending order of their costs i.e. $c_{(j)}$ denotes the seller with j^{th} lowest cost. Let k be the highest index such that $v_{(k)} \geq c_{(k)}$ and $v_{(k+1)} < c_{(k+1)}$. In an ex-post efficient allocation rule, the k top buyers trade with the k sellers with lowest costs.

Now we define a few notions of incentive compatibility and individual rationality.

DEFINITION **2.4** *A mechanism* $\mathcal{M} = (q, t)$ *is* ***ex-post incentive-compatible*** *if for each* $(v, c) \in T$, $b_i \in B$ *and* $s_j \in S$, *the following holds:*

$$q_{b_i}(v, c)v_i - t_{b_i}(v, c) \geq q_{b_i}(v'_i, v_{-i}, c)v_i - t_{b_i}(v'_i, v_{-i}, c) \ , \forall v'_i \in [0, 1]$$

$$t_{s_j}(v, c) - q_{s_j}(v, c)c_j \geq t_{s_j}(v, c'_j, c_{-j}) - q_{s_j}(v, c'_j, c_{-j})c_j \ , \forall c'_j \in [0, 1]$$

According to the definition, no agent is better-off by misrepresenting their own type no matter what types of other players have been realized. This is clearly the same as dominant strategy incentive-compatibility. A weaker notion of incentive compatibility requires the agents to be better off in the interim sense.

DEFINITION **2.5** *A mechanism* $\mathcal{M} = (q, t)$ *is* ***interim incentive-compatible*** *if for each* $b_i \in B$ *and* $s_j \in S$ *the following holds:*

$$\overline{q}_{b_i}(v_i)v_i - \overline{t}_{b_i}(v_i) \geq \overline{q}_{b_i}(v'_i)v_i - \overline{t}_{b_i}(v'_i) \ , \forall v_i, v'_i \in [0, 1]$$

$$\overline{t}_{s_j}(c_j) - \overline{q}_{s_j}(c_j)c_j \geq \overline{t}_{s_j}(c'_j) - \overline{q}_{s_j}(c'_j)c_j \ , \forall c_j, c'_j \in [0, 1]$$

Here, revealing one's true type yields an interim expected utility that is at least as great as the one obtained by misrepresentation assuming that all other agents are telling the truth.

Definition **2.6** *A mechanism* $\mathcal{M} = (q, t)$ *is* ***ex-post individually-rational*** *if for each* $(v, c) \in T$ *and* $b_i \in B$ *and* $s_j \in S$*, the following holds:*

$$q_{b_i}(v, c)v_i - t_{b_i}(v, c) \geq 0$$

$$t_{s_j}(v, c) - q_{s_j}(v, c)c_j \geq 0$$

Individual rationality ensures that agents participate voluntarily in the mechanism. In the case of ex-post individual rationality, each agent obtains non-negative ex-post utility irrespective of the type profile. We also define a weaker notion of individual rationality where each agent is better-off in expectations.

Definition **2.7** *A mechanism* $\mathcal{M} = (q, t)$ *is* ***interim individually-rational*** *if for each* $(v, c) \in T$ *and* i *and* $j \in \{1, 2, \ldots, N\}$ *the following holds:*

$$\overline{q}_{b_i}(v_i)v_i - \overline{t}_{b_i}(v_i) \geq 0 \ , \ \forall v_i \in [0, 1]$$

$$\overline{t}_{s_j}(c_j) - \overline{q}_{s_j}(c_j)c_j \geq 0 \ , \ \forall c_j \in [0, 1]$$

2.3.1 Some special mechanisms

We shall now describe some well-known mechanisms from the literature.

We begin with the *posted price mechanism*. The mechanism specifies a fixed price p. For the type vector (v, c), let k be the highest index[§] such that $v_{(k)} \geq p \geq c_{(k)}$ and $v_{(k+1)} < p$ or $c_{(k+1)} > p$. In this case, k goods are transferred from the sellers to the buyers. In particular, buyers with k highest valuations receive a good each from sellers with k lowest costs. Each buyer who receives a good pays p and each seller who transfers a good receives p.

A generalization of the posted price mechanism is the *the positive spread posted price mechanism*. The mechanism specifies two prices p_1 and p_2 with $p_1 > p_2$. Let k be the highest index such that $v_{(k)} \geq p_1$ and $v_{(k+1)} < p_1$. Also, $c_{(k)} \leq p_2$ and $c_{(k+1)} > p_2$. Once again the k-highest valuation buyers (i.e. those above $v_{(k)}$) receive an good while k-lowest cost sellers i.e. with valuations less than $c_{(k)}$ sell their goods. Each trading buyer pays p_1 while each trading seller receives p_2. Clearly, the positive spread posted price mechanism reduces to the posted price mechanism when $p_1 = p_2$.

[§]Recall that in the type vector (v, c), the buyers are ranked in descending order of their types and the sellers are ranked in ascending order of their types. So, $v_{(1)}$ is the highest valuation among all the buyers. Also, $c_{(1)}$ is the lowest cost among all the sellers.

The *trade reduction mechanism* was introduced in McAfee (1992). Let k be the highest index such that $v_{(k)} \geq c_{(k)}$ and $v_{(k+1)} < c_{(k+1)}$. In this case, $k-1$ units of goods are traded. Buyers with the highest $k-1$ valuations i.e. buyers with valuation greater than or equal to $v_{(k-1)}$ receive goods from sellers with lowest $k-1$ costs i.e. sellers with costs lower than or equal to $c_{(k-1)}$. The k^{th} ranked buyer and seller do not trade. Each buyer who trades pays $v_{(k)}$ i.e. an amount equal to the valuation of k^{th}-highest buyer, while each seller who trades receives $c_{(k)}$ i.e. an amount equal to the cost of k^{th}-lowest seller. The trade reduction mechanism is similar to the positive spread posted price mechanism in which the prices are endogenously determined.

The *McAfee double auction* (McAfee (1992)) computes k which is the highest index such that $v_{(k)} \geq c_{(k)}$ and $v_{(k+1)} < c_{(k+1)}$. The mechanism specifies price p, where $p = \frac{v_{(k+1)}+c_{(k+1)}}{2}$. If $p \in [c_{(k)}, v_{(k)}]$, then k trades take place where trading buyers pay p and trading sellers receive p. Otherwise, buyers with the top $k-1$ valuations trade with sellers with the $k-1$ lowest costs. The buyers that receive a good pay $v_{(k)}$ and the sellers that sell their good get $c_{(k)}$.

It is well known that all the four mechanisms are ex-post incentive compatible and ex-post individually rational. However, none of them are ex-post efficient.

We also describe a well-known bilateral trading mechanism (for one buyer and one seller) that we repeatedly use in our analysis. We will refer to this mechanism as the MS mechanism following Myerson and Satterthwaite (1983). The mechanism is described below.

For all $\alpha \geq 0$ define functions ϕ_α and ψ_α as follows:

$$\phi_\alpha(v) = v - \alpha\Big(\frac{1-F(v)}{f(v)}\Big) \text{ and} \tag{2.1}$$

$$\psi_\alpha(c) = c + \alpha\frac{G(c)}{g(c)} \tag{2.2}$$

For type profile (v, c) of agents, allocations for buyer b and seller s are given by $(q_b^\alpha(v,c), q_s^\alpha(v,c))$ where,

$$q_b^\alpha(v,c) = \begin{cases} 1, & \text{if } \phi_\alpha(v) \geq \psi_\alpha(c) \\ 0, & \text{if } \phi_\alpha(v) < \psi_\alpha(c) \end{cases} \tag{2.3}$$

$$q_s^\alpha(v,c) = \begin{cases} 1, & \text{if } \phi_\alpha(v) \geq \psi_\alpha(c) \\ 0, & \text{if } \phi_\alpha(v) < \psi_\alpha(c) \end{cases}$$

The value of α^* is obtained by solving equation given below.

$$\int_0^1\int_0^1 \left(q_b^\alpha(v,c)\Big(v-\frac{1-F(v)}{f(v)}\Big)-q_s^\alpha(v,c)\Big(c+\frac{G(c)}{g(c)}\Big)\right)f(v)g(c)dvdc=0 \tag{2.4}$$

The final allocation probabilities are $(q_b^{\alpha^*}(v,c), q_s^{\alpha^*}(v,c))$¶. The transfer functions $(t_b^{\alpha^*}(v,c), t_s^{\alpha^*}(v,c))$ can be chosen such that the mechanism is interim incentive-compatible and interim individually-rational.

Myerson and Satterthwaite (1983) show the existence of such a mechanism. They also show that it maximises the sum of ex-ante expected utilities of the buyer and the seller in the class of interim incentive-compatible and interim individually rational mechanisms. An example of ex-ante efficient mechanism is the Chatterjee-Samuelson mechanism given in Myerson and Satterthwaite (1983). When both F and G are uniform, the mechanism at a particular type profile (v,c) of agents is as follows,

$$q_b(v,c)=q_s(v,c)=\begin{cases}1, & \text{if } v-c\geq\frac{1}{4}\\ 0, & \text{otherwise}\end{cases}$$

$$t_b(v,c)=t_s(v,c)=\begin{cases}\frac{1}{3}(v+c+\frac{1}{2}), & \text{if } v-c\geq\frac{1}{4}\\ 0, & \text{otherwise}\end{cases}$$

2.4 Stability of Mechanisms

Consider the case where there are 2 buyers and 2 sellers. Assume for simplicity that F and G are uniformly distributed in $[0,1]$. Suppose the platform proposes the trade reduction mechanism. Agents decide before the realisation of their types whether to trade through the mechanism proposed by the platform or to trade among themselves. If they trade among themselves, they do so using the posted price mechanism. The ex-ante expected utility‖ of a buyer in the trade reduction mechanism is $U_{b_1}=0.0166$ and that of a seller is $U_{s_1}=0.0166$. On the other hand, the ex-ante expected utilities of the buyer and the seller in the posted price mechanism** are $U_b^P=0.0625$ and $U_s^P=0.0625$, respectively. Clearly, both buyer and

¶Gresik and Satterthwaite (1989) generalized the result and found the ex-ante efficient mechanism when there are multiple buyers and sellers in the market. In section 2.5, we comment on the difference between the α^* corresponding to this mechanism and to the one obtained by solving the constrained optimization problem within the $\mathcal{M}^\alpha$ class of mechanisms.

‖The calculations of ex-ante expected utilities of agents are included in the Appendix.

**The calculations can be found once again in the Appendix.

seller are better-off ex-ante (i.e. before the realisation of types) by trading through the posted price mechanism. They will, therefore, *block* the mechanism proposed by the platform. This motivates the consideration of the requirement of *stability* for mechanisms.

We follow Holmstrom and Myerson (1983) and Vohra (1999) in defining various notions of stability. These depend on when the agents decide to form the coalition i.e. before the realisation of their types (ex-ante), after the realisation of types but agents only being able to observe their own types (interim), and when agents can observe everyone's types (ex-post).

Assumption - We assume that all the deviating coalitions consist of only one buyer and one seller. Unless explicitly mentioned, we will refer to A as the bilateral coalition consisting of a single buyer b_i and a single seller s_j.

Let $A \subset K$ be a coalition. Consider interim incentive compatible mechanisms, $\mathcal{M} = (q, t)$ and $\mathcal{M}^A = (q^A, t^A)$ for K and A, respectively. Let $U_k^{\mathcal{M}^A}$ and $U_k^{\mathcal{M}}$ be the ex-ante expected utility of agent $k \in A$ in mechanism $\mathcal{M}^A$ and $\mathcal{M}$, respectively. The mechanism $\mathcal{M}^A$ *ex-ante dominates* mechanism $\mathcal{M}$ if

$$U_k^{\mathcal{M}^A} > U_k^{\mathcal{M}} \ , \forall k \in A$$

Definition **2.8** *A mechanism $\mathcal{M}$ is* ***Single-buyer-single-seller (SBSS) ex-ante stable*** *if it is not ex-ante dominated by any interim incentive-compatible mechanism $\mathcal{M}^A$ for any coalition A.*

If the interim incentive-compatible mechanism $\mathcal{M}$ for the grand coalition K is such that no coalition A of a buyer and a seller can increase the ex-ante expected utility of each agent by trading among themselves through another interim incentive-compatible mechanism $\mathcal{M}^A$, then the mechanism $\mathcal{M}$ is SBSS ex-ante stable. The agents form a coalition ex-ante i.e. before they receive their private information.

The coalition A has an *interim objection* to $\mathcal{M}$ if there exists an interim incentive-compatible mechanism $\mathcal{M}^A$ such that

$$\begin{gathered} U_{b_i}^{\mathcal{M}^A}(v_i) > U_{b_i}^{\mathcal{M}}(v_i) \ \forall v_i \in [0,1], b_i \in A \text{ and} \\ U_{s_j}^{\mathcal{M}^A}(c_j) > U_{s_j}^{\mathcal{M}}(c_j) \ \forall c_j \in [0,1], s_j \in A \end{gathered}$$

Definition **2.9** *A mechanism $\mathcal{M}$ is* ***Single-buyer-single-seller (SBSS) interim stable*** *if no coalition A has an interim objection to it.*

In the interim stage, each agent observes only his own type but not that of others. A coalition can form at this stage and decide to adopt an interim incentive-compatible mechanism $\mathcal{M}^A$. If each agent of each type in A does strictly better using this mechanism than in the mechanism for the grand coalition under consideration, then A will have an objection to the grand mechanism. The latter is SBSS interim stable if no agent has an objection.

The coalition A has an *ex-post objection* to $\mathcal{M}$ at type vector (v_A^*, c_A^*) if there exists an interim incentive compatible mechanism $\mathcal{M}^A$ and a profile of valuations $(v_A^*, v_{K\setminus A}^*, c_A^*, c_{K\setminus A}^*) \in T$ such that

$$U_k^{\mathcal{M}^A}(v_A^*, c_A^*) > U_k^{\mathcal{M}}(v_A^*, v_{K\setminus A}^*, c_A^*, c_{K\setminus A}^*) \; \forall k \in A$$

Definition **2.10** *A mechanism $\mathcal{M}$ is* ***Single-buyer-single-seller (SBSS) ex-post stable*** *if no coalition A has an ex-post objection to it.*

In the SBSS ex-post stable case, agents form a coalition after the types of all agents can be commonly observed. If there does not exists a coalition A and a type profile for each agent in A which is strictly better-off by trading via an interim incentive compatible mechanism $\mathcal{M}^A$, then the interim incentive compatible mechanism $\mathcal{M}$ is SBSS ex-post stable.

We note that SBSS interim stability is stronger than SBSS ex-ante stability.

Proposition **2.1** *If an interim incentive-compatible mechanism is SBSS interim stable, then it is SBSS ex-ante stable.*

Proof: Consider a mechanism $\mathcal{M}$ that is SBSS interim stable. Therefore, for any coalition A, and for any interim incentive-compatible mechanism $\mathcal{M}^A$ we have,

$$\begin{aligned} U_{b_i}^{\mathcal{M}}(v_i) &\geq U_{b_i}^{\mathcal{M}^A}(v_i) \text{ for every } v_i \in [0,1] \\ U_{s_j}^{\mathcal{M}}(c_j) &\geq U_{s_j}^{\mathcal{M}^A}(c_j) \text{ for every } c_j \in [0,1] \end{aligned}$$

For the buyer b_i, this implies,

$$\int_0^1 U_{b_i}^{\mathcal{M}}(v_i) f(v_i) dv_i \geq \int_0^1 U_{b_i}^{\mathcal{M}^A}(v_i) f(v_i) dv_i$$

Rewriting, we get $U_{b_i}^{\mathcal{M}} \geq U_{b_i}^{\mathcal{M}^A}$. Similarly, $U_{s_j}^{\mathcal{M}} \geq U_{s_j}^{\mathcal{M}^A}$. The mechanism $\mathcal{M}$ gives at least as much ex-ante expected utilities to agents that they get from the mechanism $\mathcal{M}^A$. Hence, the mechanism $\mathcal{M}$ is not ex-ante dominated by any interim incentive-compatible mechanism $\mathcal{M}^A$ for any coalition A. Thus, the mechanism $\mathcal{M}$ is SBSS ex-ante stable. ■

We check for the stability of mechanisms that we described earlier. We first show that the trade reduction mechanism and McAfee double auction are not SBSS ex-post stable.

Proposition **2.2** *The trade reduction mechanism and the McAfee double auction mechanism are not SBSS ex-post stable.*

Proof: Let $(v, c) = (v_1, \ldots, v_N, c_1, \ldots, c_N)$ be a type profile such that $v_1 > v_2 > \ldots > v_k > v_{k+1} > \ldots > v_N$, $c_1 < c_2 < \ldots < c_k < c_{k+1} < \ldots < c_N$, $v_k > c_k$ and $v_{k+1} < c_{k+1}$. In the trade reduction mechanism buyer k and seller k do not trade. However this pair can trade at any price in the open interval (c_k, v_k) and be strictly better-off. Therefore the trade reduction mechanism is not SBSS ex-post stable.

Let (v, c) be a type profile as in the previous paragraph satisfying the additional restriction $\frac{v_{k+1}+c_{k+1}}{2} \notin [c_k, v_k]$. Once again buyer k and seller k do not trade in the McAfee double auction. However, this pair can trade at any price in the open interval (c_k, v_k) and be strictly better-off. Thus, this pair of buyer and seller have an ex-post objection to the McAfee double auction mechanism. Clearly, the McAfee double auction is not SBSS ex-post stable. ■

Next we consider SBSS ex-ante stability. A bilateral coalition can block by using any interim incentive-compatible mechanism. A considerable simplification is obtained by applying the next proposition where it is shown a mechanism is stable if and only if no bilateral coalition gets lower aggregate utility than the one guaranteed by the MS mechanism.

Proposition **2.3** *A mechanism $\mathcal{M}$ is SBSS ex-ante stable if and only if, for all bilateral coalitions $A = \{b_i, s_j\}$, $U_{b_i}^{\mathcal{M}} + U_{s_j}^{\mathcal{M}} \geq U_{b_i}^{\mathcal{MS}} + U_{s_j}^{\mathcal{MS}}$.(Recall that $U_{b_i}^{\mathcal{MS}}$ and $U_{s_j}^{\mathcal{MS}}$ are the ex-ante expected utilities of buyer and seller in MS mechanism.)*

Proof: Sufficiency. Suppose

$$U_{b_i}^{\mathcal{M}} + U_{s_j}^{\mathcal{M}} \geq U_{b_i}^{\mathcal{MS}} + U_{s_j}^{\mathcal{MS}} \tag{2.5}$$

holds all coalitions for all coalitions $A = \{b_i, s_j\}$. We claim that $\mathcal{M}$ is stable. Suppose this is false. Then there exists some coalition $A = \{b_i, s_j\}$ and an interim incentive-compatible mechanism $\mathcal{M}'$ for A such that A blocks $\mathcal{M}$ with $\mathcal{M}'$. Therefore $U_{b_i}^{\mathcal{M}'} > U_{b_i}^{\mathcal{M}}$ and $U_{s_j}^{\mathcal{M}'} > U_{s_j}^{\mathcal{M}}$. Hence,

$$U_{b_i}^{\mathcal{M}'} + U_{s_j}^{\mathcal{M}'} > U_{b_i}^{\mathcal{M}} + U_{s_j}^{\mathcal{M}} \tag{2.6}$$

From (2.5) and (2.6),

$$U_{b_i}^{\mathcal{M}'} + U_{s_j}^{\mathcal{M}'} > U_{b_i}^{\mathcal{MS}} + U_{s_j}^{\mathcal{MS}}$$

But this contradicts the property of MS mechanism that it maximizes the sum of ex-ante expected utilities of agents.

Necessity. Suppose $\mathcal{M}$ is SBSS ex-ante stable. We claim that the inequality $U_{b_i}^{\mathcal{M}} + U_{s_j}^{\mathcal{M}} \geq U_{b_i}^{\mathcal{MS}} + U_{s_j}^{\mathcal{MS}}$ holds for all coalitions A. Suppose this is false for a coalition $A = \{b_i, s_j\}$ i.e. $U_{b_i}^{\mathcal{M}} + U_{s_j}^{\mathcal{M}} < U_{b_i}^{\mathcal{MS}} + U_{s_j}^{\mathcal{MS}}$.

If $U_{b_i}^{\mathcal{M}} < U_{b_i}^{\mathcal{MS}}$ and $U_{s_j}^{\mathcal{M}} < U_{s_j}^{\mathcal{MS}}$, then the mechanism MS blocks the mechanism $\mathcal{M}$. This contradicts the assumption that $\mathcal{M}$ is SBSS ex-ante stable.

Assume without loss of generality therefore that, $U_{b_i}^{\mathcal{M}} \geq U_{b_i}^{\mathcal{MS}}$ and $U_{s_j}^{\mathcal{MS}} > U_{s_j}^{\mathcal{M}}$. Let $U_{b_i}^{\mathcal{MS}} + U_{s_j}^{\mathcal{MS}} - (U_{b_i}^{\mathcal{M}} + U_{s_j}^{\mathcal{M}}) = \kappa > 0$. Consider another mechanism $\mathcal{M}' = (q^{\mathcal{MS}}, t')$ for the coalition A. The allocation function is same as that of MS mechanism but the transfer function is constructed as follows. Let $\epsilon > 0$ be such that $\epsilon < \min\{\kappa, U_{s_j}^{\mathcal{MS}} - U_{s_j}^{\mathcal{M}}\}$. Also, let $\Delta = U_{s_j}^{\mathcal{MS}} - U_{s_j}^{\mathcal{M}} - \epsilon$. The transfer functions are:

$$t'_{b_i}(v, c) = t_{b_i}^{\mathcal{MS}}(v, c) + \Delta$$

$$t'_{s_j}(v, c) = t_{s_j}^{\mathcal{MS}}(v, c) - \Delta$$

Since the difference in the transfers of the agents in the MS and $\mathcal{M}'$ mechanisms are independent of type profile (v, c), and MS mechanism is interim incentive-compatible, $\mathcal{M}'$ is also interim incentive-compatible.

The ex-ante expected utilities of agents in mechanism $\mathcal{M}'$ are, $U_{b_i}^{\mathcal{M}'} = U_{b_i}^{\mathcal{MS}} + \Delta$ and $U_{s_j}^{\mathcal{M}'}(c_j) = U_{s_j}^{\mathcal{MS}}(c_j) - \Delta$. By construction of Δ it follows that $U_{s_j}^{\mathcal{M}} < U_{s_j}^{\mathcal{M}'}$ since $U_{s_j}^{\mathcal{MS}} > U_{s_j}^{\mathcal{M}} + \epsilon$, and also $U_{b_i}^{\mathcal{M}} < U_{b_i}^{\mathcal{M}'}$ since this implies that $U_{b_i}^{\mathcal{M}} < U_{b_i}^{\mathcal{MS}} + \Delta = U_{b_i}^{\mathcal{MS}} + U_{s_j}^{\mathcal{MS}} - U_{s_j}^{\mathcal{M}} - \epsilon$ which simplifies to $\epsilon < \kappa$. Hence, mechanism $\mathcal{M}'$ ex-ante dominates $\mathcal{M}$, i.e. $\mathcal{M}$ is not SBSS ex-ante stable. This is a contradiction. ■

We examine the SBSS ex-ante stability property of the trade reduction mechanism and the McAfee double auction when both F and G distributions are uniform on $[0, 1]$. Figure 2.1 below shows the ex-ante expected utility of an agent in the trade reduction mechanism and the McAfee double auction for different market sizes. The blue, red and green curves correspond to McAfee double auction, trade reduction mechanism and MS mechanism, respectively. Note that the model is symmetric, so the ex-ante expected utilities of buyers and sellers are equal in each of the three mechanisms.

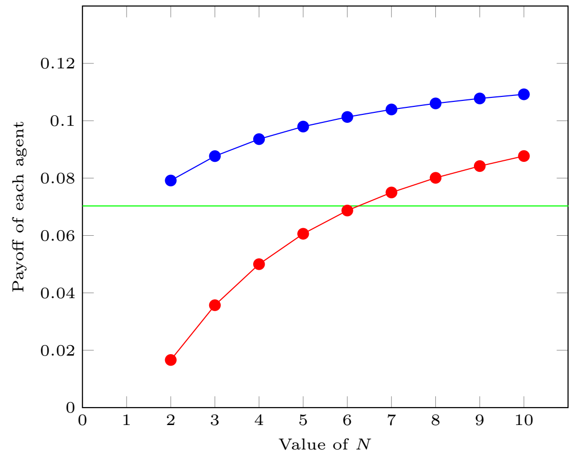


Figure 2.1: Payoff of agents in McAfee double auction and Trade reduction mechanism

It is evident that the McAfee double auction is SBSS ex-ante stable for all $N \geq 2$. For $N \leq 6$, the trade reduction mechanism is not SBSS ex-ante stable[††]. However, for $N \geq 7$, it does turn out to be SBSS ex-ante stable.

We now look at the positive spread posted price mechanism and then find the revenue-maximizing mechanism for the platform.

2.5 Expected revenue-maximizing mechanism for a platform

Suppose a platform is earning revenue by matching large numbers of buyers and sellers. If it attempts to extract "too much surplus" from the market, buyers and sellers will presumably be tempted to reject the service of the platform and set up a trading mechanism by themselves. For instance, suppose the platform uses a positive spread posted price mechanism charging prices p_1 and p_2 to buyers and sellers respectively. If the distributions F and G are uniform on $[0, 1]$, and $N = 2$, the expected revenue of the platform is:

$$4(p_1 - p_2)\Big(\frac{1}{2}p_2^2(1-p_1)^2 + p_1(1-p_1)(p_2 - \frac{p_2^2}{2}) + \frac{1}{2}p_2(1-p_2)(1-p_1)^2\Big)$$

Revenue maximization yields a unique maximum at $p_1 = 0.6847$ and $p_2 = 0.3152$. Computations reveal $U_{b_i} = U_{s_j} = 0.023$. Clearly, $U_{b_i} + U_{s_j} = 0.046 < U_b^{\mathcal{MS}} + U_s^{\mathcal{MS}} = 0.1406$. Applying Proposition 2.3, the revenue-maximizing positive spread posted price mechanism is not SBSS ex-ante stable, i.e it is infeasible for the platform.

[††] The ex-ante expected utility of an agent in trade reduction mechanism when $N = 6$ is 0.0687, whereas in the MS mechanism the agent gets utility of 0.0703.

There is clearly a tension between stability of a mechanism and the expected revenue that the platform earns from it. Our goal is to design revenue-maximizing mechanisms, which also ensure that the agents do not break away from the mechanism offered by the platform.

Before analyzing revenue-maximizing mechanisms with stability constraints, we consider the benchmark case of revenue-maximizing mechanisms without stability constraints. This analysis closely follows Myerson and Satterthwaite (1983) who solve the problem for the case of a single buyer and seller.

2.5.1 The benchmark case: expected revenue-maximization without stability constraints

We look for a mechanism that maximizes the expected revenue for the platform in the class of feasible, interim incentive-compatible and interim individually-rational mechanism. Myerson and Satterthwaite (1983) finds such a mechanism when there is only one buyer and one seller in the market. We extend their result to the case when there are N buyers and N sellers.

The next proposition characterizes the class of interim incentive-compatible mechanisms. These results are standard in the literature. However a proof is provided in the Appendix for convenience.

Proposition **2.4** *A mechanism (q, t) is interim incentive-compatible if and only if 1, 2 and 3 hold:*

1. *$q_{b_i}(v,c)$ is increasing in v_i for all $i \in \{1, 2, \dots, N\}$.*
2. *$q_{s_j}(v, c)$ is decreasing in c_j for all $j \in \{1, 2, \dots, N\}$.*
3. $\bar{t}_{b_i}(v_i) = \bar{t}_{b_i}(0) + v_i\bar{q}_{b_i}(v_i) - \int\limits_0^{v_i} \bar{q}_{b_i}(x)dx$ *,for all* $i \in \{1, 2, \dots, N\}$

   $\bar{t}_{s_j}(c_j) = \bar{t}_{s_j}(1) + c_j\bar{q}_{s_j}(c_j) + \int\limits_{c_j}^{1} \bar{q}_{s_j}(x)dx$ *,for all* $j \in \{1, 2, \dots, N\}$

This proposition is used to prove another proposition which characterizes expected revenue π_0 of the platform.

Recall that the virtual valuation functions for b_i and s_j are $\phi(v_i)$ and $\psi(c_j)$, respectively.

Proposition 2.5 *For any interim incentive-compatible mechanism (q, t), the expected revenue of the platform is given by,*

$$\pi_0 = \int_0^1 \dots \int_0^1 \Big(\sum_{i=1}^{N} q_{b_i}(v,c)\phi(v_i) - \sum_{j=1}^{N} q_{s_j}(v,c)\psi(c_j) \Big) f(v)g(c)dvdc$$
$$- \sum_{i=1}^{N} U_{b_i}(0) - \sum_{j=1}^{N} U_{s_j}(1) \quad (2.7)$$

Proof: The revenue of the platform is given by:

$$\pi_0 = \mathbb{E}_{b,s}\Big[\sum_{i=1}^{N} t_{b_i}(v,c) - \sum_{j=1}^{N} t_{s_j}(v,c) \Big]$$

$$= \sum_{i=1}^{N} \mathbb{E}_{b,s}[v_i q_{b_i}(v,c) - U_{b_i}(v,c)] - \sum_{j=1}^{N} \mathbb{E}_{b,s}[c_j q_{s_j}(v,c) + U_{s_j}(v,c)]$$

Substituting the expression for utility functions of agents from proof of Proposition 2.4,

$$= \int_0^1 \dots \int_0^1 \Big(\sum_{i=1}^{N} v_i q_{b_i}(v,c) - \sum_{j=1}^{N} c_j q_{s_j}(v,c) \Big) f(v)g(c)dvdc - \sum_{i=1}^{N} U_{b_i}(0)$$

$$+ \sum_{i=1}^{N} \int_0^1 \dots \int_0^1 q_{b_i}(v,c) \frac{1 - F(v_i)}{f(v_i)} f(v)g(c)dvdc - \sum_{j=1}^{N} U_{s_j}(1)$$

$$+ \sum_{j=1}^{N} \int_0^1 \dots \int_0^1 q_{s_j}(v,c) \frac{F(c_j)}{f(c_j)} f(v)g(c)dvdc$$

Rearranging these terms gives us the expression for the revenue of the platform. Hence, proved. ■

We identify a mechanism, and then show that it is feasible, interim incentive-compatible, interim individually-rational and maximizes the revenue of the platform in the class of interim incentive-compatible mechanisms.

Define a mechanism $\mathcal{M}^* = (q^*, t^*)$ as follows. Arrange the virtual reservation values of the $2N$ agents $(\phi(v_1), \phi(v_2), \dots, \phi(v_N),\ \psi(c_1), \psi(c_2), \dots, \psi(c_N))$ in descending order. Let $\mathcal{R}_{b_i}(v,c)$ and $\mathcal{R}_{s_j}(v,c)$ be the ranks of the virtual valuation of b_i and s_j in the profile (v,c)

respectively[‡‡]. Allocation functions for buyers and sellers are described below.

$$q^*_{b_i}(v,c) = \begin{cases} 1 & \text{if } \mathcal{R}_{b_i}(v,c) \le N \\ 0 & \text{if } \mathcal{R}_{b_i}(v,c) > N \end{cases} \quad \forall i \in \{1,2,\dots,N\} \tag{2.8}$$

$$q^*_{s_j}(v,c) = \begin{cases} 1 & \text{if } \mathcal{R}_{s_j}(v,c) \ge N \\ 0 & \text{if } \mathcal{R}_{s_j}(v,c) < N \end{cases} \quad \forall j \in \{1,2,\dots,N\}$$

Let $B(v,c)$ and $S(v,c)$ denote the set of buyers and sellers respectively who trade i.e. $B(v,c) = \{i \mid \mathcal{R}_{b_i}(v,c) \le N\}$ and $S(v,c) = \{j \mid \mathcal{R}_{s_j}(v,c) \ge N\}$. Also let i^* denote the buyer with lowest ranked virtual valuation among the buyers who are allocated goods and let j^* denote the seller with highest ranked virtual valuation among sellers who sell the good.

Payments for buyers and sellers are now described.

$$t^*_{b_i}(v,c) = \begin{cases} \min\{\tilde{v} | \tilde{v} > 0 \text{ and } \phi(\tilde{v}) \ge \psi(c_{j^*})\} & \text{if } i \in B(v,c) \\ 0 & \text{if } i \notin B(v,c) \end{cases}$$

$$t^*_{s_j}(v,c) = \begin{cases} \max\{\tilde{c} | \tilde{c} < 1 \text{ and } \psi(\tilde{c}) \le \phi(v_{i^*})\} & \text{if } j \in S(v,c) \\ 0 & \text{if } j \notin S(v,c) \end{cases}$$

Observe that $\mathcal{M}^*$ is feasible as the number of units sold by the seller is equal to the number of units bought by the buyers.

Theorem **2.1** *Assume F and G satisfy regularity, i.e. the virtual valuation functions ϕ and ψ are increasing. The mechanism $\mathcal{M}^*$ maximizes the expected revenue of platform in the class of interim incentive-compatible and interim individually-rational mechanisms.*

Proof: The platform solves the following optimization problem:

$$\max_{(q_{b_i}, q_{s_j} \forall i,j=1,2,\dots,N)} \pi_0$$

$$\text{s.t., } \mathbb{E}_{-b_i}[q_{b_i}(v,c)v_i - t_{b_i}(v,c)] \ge 0 \tag{2.9}$$

$$\mathbb{E}_{-s_j}[t_{s_j}(v,c) - q_{s_j}(v,c)c_j] \ge 0 \tag{2.10}$$

[‡‡]Let $N = 3$. Let the type profile (v,c) be such that $\phi(v_1) = 0.6$, $\phi(v_2) = 0.1$, $\phi(v_3) = 0.9$) and $\psi(c_1) = 0.7$, $\psi(c_2) = 0.2$, $\psi(c_3) = 0.8$. Ranking them in descending order we get $(\phi(v_3), \psi(c_3), \psi(c_1), \phi(v_1), \psi(c_2), \phi(v_2))$. Here, $\mathcal{R}_{b_1}(v,c) = 4$, $\mathcal{R}_{b_2}(v,c) = 6$, $\mathcal{R}_{b_3}(v,c) = 1$, $\mathcal{R}_{s_1}(v,c) = 3$, $\mathcal{R}_{s_2}(v,c) = 5$, $\mathcal{R}_{s_3}(v,c) = 2$.

$$\forall i, j \in \{1, 2, \ldots, N\} \text{ and } \forall v_i, c_j \in [0, 1]$$

Let $\mathcal{M} = (q, t)$ be an interim incentive-compatible mechanism. According to Proposition 2.5, the expected revenue of the platform is given by

$$\pi_0 = \int_0^1 \ldots \int_0^1 H(v, c) f(v) g(c) dv dc - \sum_{i=1}^{N} U_{b_i}(0) - \sum_{j=1}^{N} U_{s_j}(1) \tag{2.11}$$

where $H(v, c) = \sum_{i=1}^{N} q_{b_i}(v, c)\phi(v_i) - \sum_{j=1}^{N} q_{s_j}(v, c)\psi(c_j)$ and the allocation functions $q_{b_i}(v, c)$ and $q_{s_j}(v, c)$ are increasing in v_i and c_j, respectively. Clearly, π_0 will be maximized if:

1. $H(v, c)$ is maximized at each value of the type profile (v, c)
2. $U_{b_i}(0) = U_{s_j}(1) = 0$ for all i and $j \in \{1, 2, \ldots, N\}$
3. inequalities (2.9) and (2.10) are satisfied.

We will show that 1, 2 and 3 are achieved by the mechanism $\mathcal{M}^*$.

Fix an arbitrary type profile (v, c). Assume without loss of generality, $v_1 \geq v_2 \geq \ldots \geq v_N$ and $c_1 \leq c_2 \leq \ldots \leq c_N$. Since ϕ and ψ are increasing, $\phi(v_1) \geq \phi(v_2) \geq \ldots \geq \phi(v_N)$ and $\psi(c_1) \leq \psi(c_2) \leq \ldots \leq \psi(c_N)$. It follows from inspection that $H(v, c)$ is maximized by the allocation rule $q^*(v, c)$ of mechanism $\mathcal{M}^*$. For instance, if k is the highest index such that $\phi(v_k) \geq \psi(c_k))$ and $\phi(v_{k+1}) \geq \psi(c_{k+1}))$, then $H(v, c)$ decreases if buyer $k+1$ and seller $k+1$ trade.

The allocation functions $q_{b_i}(v, c)$ and $q_{s_j}(v, c)$ are clearly non-decreasing in v_i and c_j respectively.

The mechanism $\mathcal{M}^*$ is interim incentive-compatible. To see this, pick any buyer b_i. If at any type profile (v, c), $\phi(v_i) < \phi(v_{i^*})$, no good is allocated to b_i. If the buyer misreports v_i' such that $\phi(v_i') > \phi(v_{i'^*})$, b_i is allocated an good. The buyer gets a utility of $v_i - \phi^{-1}(\psi(c_{j'^*})) < 0$. This is because b_i was not allocated the good when he reported the truth as his virtual valuation was less than $\psi(c_{j^*})$ and $\psi(c_{j'^*}) \geq \psi(c_{j^*})$. So, b_i has no incentive to misreport. Also, b_i has no incentive to misreport $v_i' < v_i$. Similarly, it can be shown that no seller has any incentive to misreport. So, the mechanism is ex-post incentive compatible, and hence it is interim incentive-compatible.

To see how 2 is achieved, notice that if there is a buyer b_i with valuation $v_i = 0$ his virtual reservation is $\phi(0) = -\frac{1}{f(0)}$ which is clearly less than zero. According to $\mathcal{M}^*$, b_i will be given a good only if there exists a seller with a lower virtual valuation. However the lowest possible virtual valuation of a seller zero. Hence, b_i is never allocated a good. Also $t_{b_i}(0, v_{-i}, c) = 0$ for all c. Hence, $U_{b_i}(0) = 0$.

Similarly, a seller s_j with valuation $c_j = 1$ has virtual reservation $\psi(1) = 1 + \frac{1}{g(1)}$ which is clearly greater than one. According to $\mathcal{M}^*$, s_j sells the good only if there exists a buyer with a higher virtual valuation. However the highest possible virtual valuation of a buyer is always less than 1. Hence, s_j never sells the good. The transfer for s_j is $t_{s_j}(v, 1, c_{-j}) = 0$. Hence, $U_{s_j}(1) = 0$.

The arguments in the previous two paragraphs show that requirement 2 is satisfied by $\mathcal{M}^*$.

By construction, $H(v, c) \geq 0$ for all type profiles (v, c). Since requirement 2 holds, $\mathcal{M}^*$, satisfies the NBD condition.

Finally we verify interim individual-rationality. We know from Proposition 2.4 that the interim expected utility of buyer b_i with valuation v_i is

$$U_{b_i}(v_i) = U_{b_i}(0) + \int_0^{v_i} \overline{q}^*_{b_i}(x)dx$$

Similarly, interim expected utility of seller s_j with valuation c_j is,

$$U_{s_j}(c_j) = U_{s_j}(1) + \int_{c_j}^{1} \overline{q}^*_{s_j}(y)dy$$

Since $U_{b_i}(0) = U_{s_j}(1) = 0$ and $\overline{q}^*_{b_i}(v_i)$ and $\overline{q}^*_{s_j}(c_j)$ are non-negative for all v_i and c_j respectively, $U_{b_i}(v_i) \geq 0$ and $U_{s_j}(c_j) \geq 0$. Hence, mechanism $\mathcal{M}^*$ satisfies interim individual-rationality. This proves the result. ■

2.5.2 Stability of the revenue-maximizing mechanism $\mathcal{M}^*$

The ex-ante expected utility of a buyer b_i and seller s_j in $\mathcal{M}^*$ are given by the expressions below:

$$U_{b_i} = \sum_{r=0}^{N-1} \frac{N!(N-1)!}{(r!)^2((N-r-1)!)^2} \int_{\phi^{-1}(0)}^{1} \int_{0}^{\psi^{-1}(\phi(v_i))} (v_i - \phi^{-1}(\psi(c_{r+1})))(F(v_i))^{N-r-1}(1-F(v_i))^r$$
$$(F(c_{r+1}))^r(1-F(c_{r+1}))^{N-r-1} f(v_i) f(c_{r+1}) dc_{r+1} dv_i \quad (2.12)$$

$$U_{s_j} = \sum_{r=0}^{N-1} \frac{N!(N-1)!}{(r!)^2((N-r-1)!)^2} \int_{\phi^{-1}(0)}^{1} \int_{0}^{\psi^{-1}(\phi(v_i))} (\psi^{-1}(\phi(v_{r+1})) - c_j)(F(v_i))^{N-r-1}(1-F(v_i))^r$$
$$(F(c_j))^r(1-F(c_j))^{N-r-1} f(v_{r+1}) f(c_j) dc_j dv_{r+1} \quad (2.13)$$

The sum $U_{b_i} + U_{s_j}$ is computed for small values of N numerically in the case when F and G are uniform (both on $[0,1]$) and in the case where F is uniform and $G(c) = c^2$ (both on $[0,1]$).

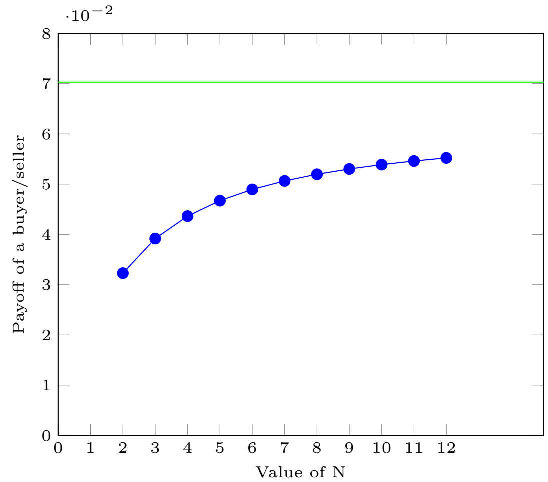
It is apparent from the graphs in Figure 2.2 that $\mathcal{M}^*$ is not SBSS ex-ante stable for the values of N shown. Thus an explicit stability constraint may be binding in the revenue-maximization problem for the platform.

Table 2.1 shows the payoff that an agent (buyer or seller) obtains in the three main mechanisms introduced in the previous sections. The unconstrained revenue-maximizing mechanism gives less payoff compared to both the trade reduction mechanism and the McAfee mechanism. The payoff of the trade reduction mechanism rises rapidly whereas the payoff in the unconstrained revenue-maximizing mechanism converges to 0.03125 and hence the gains from trade of agents is far lesser. Clearly, as the platform squeezes the agents more, they get lesser utility. So, they will try to block the mechanism by trading among themselves as they are guaranteed higher ex-ante payoffs.

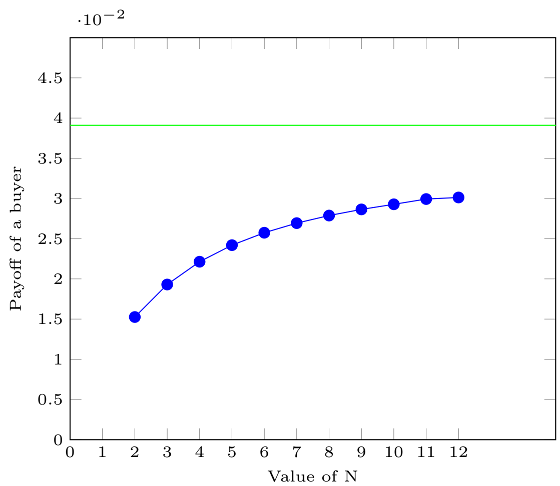
Value of N	Trade reduction	McAfee	Unconstrained
2	0.0166	0.0792	0.023
3	0.0357	0.0876	0.0265
4	0.0500	0.0936	0.0244
5	0.0606	0.0979	0.0249
6	0.0687	0.1013	0.0264
7	0.0750	0.1039	0.0279
8	0.0801	0.1060	0.0291
9	0.0842	0.1077	0.0301
10	0.0877	0.1092	0.0308

Table 2.1: Utility of an agent when both F and G are uniform

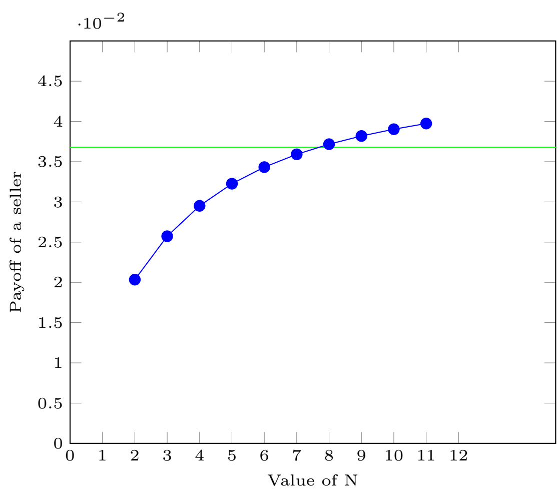
We turn to analysis of revenue-optimal mechanism when there are stability constraints in the next sub-section.



F and G are uniform



F is uniform and $G(c) = c^2$



F is uniform and $G(c) = c^2$

Figure 2.2: Payoff of agents in unconstrained revenue-maximizing mechanism for different distributions F and G

2.5.3 Revenue-maximizing mechanism with stability constraints

The revenue-optimization for the platform in the presence of stability constraints is as follows:

$$\max_{(q_{b_i}, q_{s_j} \forall i,j=1,2,\ldots,N)} \pi_0$$

$$\text{s.t., } \mathbb{E}_{-b_i}[q_{b_i}(v,c)v_i - t_{b_i}(v,c)] \geq 0 \tag{2.14}$$

$$\mathbb{E}_{-s_j}[t_{s_j}(v,c) - q_{s_j}(v,c)c_j] \geq 0 \tag{2.15}$$

$$\forall\ i,j \in \{1,2,\ldots,N\} \text{ and } \forall\ v_i, c_j \in [0,1]$$

$$U_{b_i} + U_{s_j} \geq U_{b_i}^{\mathcal{MS}} + U_{s_j}^{\mathcal{MS}},\ \forall\ i,j \in \{1,2,\ldots,N\} \tag{2.16}$$

In addition to constraints (2.14) and (2.15), we now have additional constraints specified by inequality (2.16). These are extra N^2 constraints, one for each possible buyer-seller coalition. The constraints ensure that an arbitrary buyer b_i, seller s_j pair do not reject the platform ex-ante and set-up their own trading mechanism.

There are significant difficulties involved in solving the problem above in full generality. In particular, it is not apparent which of the constraints in (2.16) are binding. In order to make progress we restrict attention to a class of sub-mechanisms and follow the technique developed in Gresik and Satterthwaite (1989).

Let $\sigma_B : B \to B$ be a bijection. For every valuation profile $v = (v_1, v_2, \ldots, v_N)$, let $v_{\sigma_B^{-1}} = (v_{\sigma_B^{-1}(1)}, v_{\sigma_B^{-1}(2)}, \ldots, v_{\sigma_B^{-1}(N)})$. In other words, i^{th} buyer's valuation is now the valuation of buyer $\sigma_B^{-1}(i)$ in the profile v. Similarly, Let $\sigma_S : S \to S$ be a bijection. For every cost profile $c = (c_1, c_2, \ldots, c_N)$, let $c_{\sigma_S^{-1}} = (c_{\sigma_S^{-1}(1)}, c_{\sigma_S^{-1}(2)}, \ldots, c_{\sigma_S^{-1}(N)})$. In other words, j^{th} seller's valuation is now the valuation of seller $\sigma_S^{-1}(j)$ in the profile c.

Let v_σ be the valuation profile which is obtained by interchanging the valuations v_i and $v_{\sigma(i)}$ in (v,c) with rest of the valuations and costs remaining same.

Definition **2.11** *An allocation function is symmetric if for all (v,c) and for all permutations σ_B and σ_S*

(i) $q_{b_i}(v,c) = q_{b_{\sigma(i)}}(v_{\sigma_B^{-1}}, c)$, $\forall i \in B$

(ii) $q_{s_j}(v,c) = q_{s_{\sigma(j)}}(v, c_{\sigma_S^{-1}})$, $\forall j \in S$

The definition of symmetric allocation function is with respect to either buyers or sellers. While comparing the allocation probability of a buyer like in part (i) of the definition, only the buyers are permuted. Similarly, in part (ii) of the definition, only the sellers are permuted upon. This is because, the allocation probability of an agent (whether buyer or seller) does not depend on the other agent (seller or buyer) to which it is matched i.e. a buyer can receive a good from any seller and a seller can sell the good to any buyer.

EXAMPLE **2.1** Consider the case where $N = 3$ i.e. there are 3 buyers and 3 sellers in the market. Also, let $\sigma_B(1) = 3$, $\sigma_B(2) = 1$, $\sigma_B(3) = 2$, $\sigma_S(1) = 2$, $\sigma_S(2) = 3$ and $\sigma_S(3) = 1$. If the mechanism is symmetric, this means, for buyer b_1 and buyer b_3, and for any type profile (v_1, v_2, v_3, c), the allocation function

$$q_{b_1}(v_1, v_2, v_3, c) = q_{b_3}(v_2, v_3, v_1, c)$$

Also, notice that if $v_1 = v_2 \in [0, 1]$, $\overline{q}_{b_1}(v_1) = \overline{q}_{b_2}(v_2)$ i.e. for the same valuation, the interim allocation probabilities of buyers are equal.

Now, we define a symmetric mechanism.

DEFINITION **2.12** *A mechanism $\mathcal{M}$ is symmetric if its allocation function is symmetric.*

For a symmetric mechanism, the utility of all the buyers is equal and that of all the sellers is also equal. Hence, all the stability constraints become identical. By suppressing the indices of buyer and seller, the single constraint can be written as

$$U_b + U_s \geq U_b^{\mathcal{MS}} + U_s^{\mathcal{MS}} \tag{2.17}$$

The platform now solves the following revenue-optimization problem:

$$\max_{(q_b, q_s)} \int_0^1 \cdots \int_0^1 N\Big(q_b(v,c)\Big(v - \frac{(1-F(v))}{f(v)}\Big) - q_s(v,c)\Big(c + \frac{G(c)}{g(c)}\Big)\Big) f(v)g(c)dvdc - NU_b(0) - NU_s(1)$$

$$\text{s.t., } \mathbb{E}_{-b}[q_b(v,c)v - t_b(v,c)] \geq 0 \tag{2.18}$$

$$\mathbb{E}_{-s}[t_s(v,c) - q_s(v,c)c] \geq 0 \tag{2.19}$$

$$\forall v, c \in [0, 1]$$

$$U_b + U_s \geq U_b^{\mathcal{MS}} + U_s^{\mathcal{MS}}, \tag{2.20}$$

In order to solve the optimization problem, we rely on techniques developed by Myerson and Satterthwaite (1983) even though their problem is different from ours - they are interested in finding a mechanism that maximizes the sum of ex-ante expected utilities of a buyer and seller while we need to find a mechanism that maximizes the revenue of the platform subject to the stability constraint.

Pick $\alpha \in [0,1]$. We describe below a class of symmetric mechanisms $\mathcal{M}^\alpha = (q^\alpha, t^\alpha)$.

The α-virtual valuations for the buyers and sellers are the same as those described for the MS mechanism (equations (2.1) and (2.2)) i.e.

$$\phi_\alpha = v - \alpha \frac{1 - F(v)}{f(v)}$$

$$\psi_\alpha = c + \alpha \frac{G(c)}{g(c)}$$

Fix a valuation profile (v, c). Arrange the α-virtual values of the $2N$ agents $(\phi_\alpha(v_1), \phi_\alpha(v_2), \dots, \phi_\alpha(v_N),\ \psi_\alpha(c_1), \psi_\alpha(c_2), \dots, \psi_\alpha(c_N))$ in descending order. Let $\mathcal{R}_{b_i}(v,c)$ and $\mathcal{R}_{s_j}(v,c)$ be the ranks of the α-virtual valuations of b_i and s_j in the profile (v,c) respectively. The allocation functions $q^\alpha_{b_i}(v,c)$ for buyer b_i and $q^\alpha_{s_j}(v,c)$ for seller s_j are as follows:

$$q^\alpha_{b_i}(v,c) = \begin{cases} 1 & \text{if } \mathcal{R}_{b_i}(v,c) \leq N \\ 0 & \text{if } \mathcal{R}_{b_i}(v,c) > N \end{cases} \quad \forall i \in \{1, 2, \dots, N\} \tag{2.21}$$

$$q^\alpha_{s_j}(v,c) = \begin{cases} 1 & \text{if } \mathcal{R}_{s_j}(v,c) \geq N \\ 0 & \text{if } \mathcal{R}_{s_j}(v,c) < N \end{cases} \quad \forall j \in \{1, 2, \dots, N\}$$

Let $B_\alpha(v,c)$ and $S_\alpha(v,c)$ denote the set of buyers and sellers respectively who trade. Also let i^* denote the buyer with lowest ranked virtual valuation among the buyers who are allocated goods and let j^* denote the seller with highest ranked virtual valuation among sellers who sell the good.

Payments for buyers and sellers are now described.

$$t^\alpha_{b_i}(v,c) = \begin{cases} \min\{\tilde{v} | \tilde{v} > 0 \text{ and } \phi_\alpha(\tilde{v}) \geq \psi_\alpha(c_{j^*})\} & \text{if } i \in B_\alpha(v,c) \\ 0 & \text{if } i \notin B_\alpha(v,c) \end{cases}$$

$$t^\alpha_{s_j}(v,c) = \begin{cases} \max\{\tilde{c} | \tilde{c} < 1 \text{ and } \psi_\alpha(\tilde{c}) \leq \phi_\alpha(v_{i^*})\} & \text{if } j \in S_\alpha(v,c) \\ 0 & \text{if } j \notin S_\alpha(v,c) \end{cases}$$

The mechanism $\mathcal{M}^{\alpha}$ is a straightforward generalization of mechanism $\mathcal{M}^{*}$ described in the previous section which maximizes the expected revenue of the platform. The only difference is that the virtual valuations of buyers and sellers are adjusted by a factor of α. For $\alpha = 1$, the allocation functions $(q^{\alpha}_{b_i}(v,c), q^{\alpha}_{s_j}(v,c))$ correspond to the allocation functions of the unconstrained revenue-maximizing mechanism $\mathcal{M}^{*}$ i.e. $(q^{*}_{b_i}(v,c), q^{*}_{s_j}(v,c))$. For $\alpha = 0$, they correspond to the allocation functions of the ex-post efficient mechanism $(q^{e}_{b_i}(v,c), q^{e}_{s_j}(v,c))$[§§].

The α-virtual valuation of buyer b_i is decreasing in α and the α-virtual valuation of seller s_j is increasing in α. When α increases, the buyers and sellers lose opportunities for trading.

For the mechanism $\mathcal{M}^{\alpha}$, the stability constraint in (2.17) can be written as,

$$\int_0^1 \cdots \int_0^1 \Big(q^{\alpha}_{b}(v,c)\frac{1-F(v)}{f(v)} + q^{\alpha}_{s}(v,c)\frac{F(c)}{f(c)}\Big) f(v)g(c)dvdc + U_b(0) + U_s(1) \geq U_b^{\mathcal{MS}} + U_s^{\mathcal{MS}} \quad (2.22)$$

The left-hand side of inequality (2.22) is the sum of the ex-ante expected utilities of a buyer and a seller in $\mathcal{M}^{\alpha}$ mechanism. Notice that $U_b(0) = U_s(1) = 0$[¶¶]. Hence the relevant constraint will reduce to

$$\int_0^1 \cdots \int_0^1 \Big(q^{\alpha}_{b}(v,c)\frac{1-F(v)}{f(v)} + q^{\alpha}_{s}(v,c)\frac{F(c)}{f(c)}\Big) f(v)g(c)dvdc \geq \gamma \quad (2.23)$$

where $\gamma = U_b^{\mathcal{MS}} + U_s^{\mathcal{MS}}$.

Fix the mechanism $\mathcal{M}^{\alpha}$ and consider the following function $C(\alpha)$:

$$C(\alpha) = \int_0^1 \cdots \int_0^1 \Big(q^{\alpha}_{b}(v,c)\frac{1-F(v)}{f(v)} + q^{\alpha}_{s}(v,c)\frac{F(c)}{f(c)}\Big) f(v)g(c)dvdc - \gamma \quad (2.24)$$

It is clear from inspection of (2.23) that $\mathcal{M}^{\alpha}$ is stable if and only if $C(\alpha) \geq 0$.

If $C(1) \geq 0$ the unconstrained revenue-maximizing mechanism $\mathcal{M}^{*}$ satisfies the stability constraint and the revenue-maximizing mechanism is also the optimal mechanism in the presence of stability constraints. On the other hand, if $C(1) < 0$, the unconstrained revenue-maximizing mechanism is not stable. In this case, we want an α less than one such that the stability constraint is satisfied with equality. According to the next Proposition, such an α can be found.

[§§]Recall the definition in Section 2.3.

[¶¶]The proof for this is similar to the one given in the proof of Theorem 2.1 for mechanism $\mathcal{M}^{*}$.

Proposition **2.6** *If $C(1) < 0$, there exists an $\alpha^* \in (0,1)$ such that $C(\alpha^*) = 0$.*

Proof: Suppose $C(1) < 0$. If $C(0) > 0$ and the function C is decreasing and continuous in α, the result follows by an application of the Intermediate Value Theorem.

According to Myerson and Satterthwaite (1983), the ex-post efficient mechanism (in the class of interim incentive-compatible, interim individually-rational mechanisms) offered by the platform to the buyers and sellers requires that the trade be subsidized by the platform. The amount of subsidy is:

$$\pi_0 + \sum_{i=1}^{N} U_{b_i} + \sum_{j=1}^{N} U_{s_j} = \int_0^1 \dots \int_0^1 \Big(\sum_{i=1}^{N} q^0_{b_i}(v,c)(v_i - \frac{1-F(v_i)}{f(v_i)}) - \sum_{j=1}^{N} q^0_{s_j}(v,c)(c_j + \frac{F(c_j)}{f(c_j)})\Big) f(v)g(c)dvdc < 0 \quad (2.25)$$

Rewriting the expression on the left-hand side of (2.25), we get,

$$\int_0^1 \dots \int_0^1 \Big(q^0_{b_i}(v,c)\frac{1-F(v_i)}{f(v_i)} + q^0_{s_j}(v,c)\frac{F(c_j)}{f(c_j)}\Big) f(v)g(c)dvdc > \frac{1}{N}\int_0^1 \dots \int_0^1 \Big(\sum_{i=1}^{N} q^0_{b_i}(v,c)v_i - \sum_{j=1}^{N} q^0_{s_j}(v,c)c_j\Big) f(v)g(c)dvdc \quad (2.26)$$

The expression on the right-hand side of inequality (2.26) is $\frac{1}{N}^{th}$ of the expected potential gains from trade for N buyers and N sellers i.e. the expected gains from trade of a pair of one buyer and one seller in ex-post efficient mechanism (there are N such pairs). This value is increasing in N. Specifically, for any $N \geq 2$, this expression will be greater than its value at $N = 1$.

$$\frac{1}{N}\int_0^1 \dots \int_0^1 \Big(\sum_{i=1}^{N} q^0_{b_i}(v,c)v_i - \sum_{j=1}^{N} q^0_{s_j}(v,c)c_j\Big) f(v)g(c)dvdc > \int_0^1 \int_0^v \Big(v - c\Big) f(v)g(c)dvdc \quad (2.27)$$

The expression on the right-hand side of inequality (2.27) is the expected gains from trade of one buyer and one seller in ex-post efficient mechanism. We also know that in the one buyer and one seller case, an ex-post efficient and interim incentive-compatible mechanism gives a higher sum of ex-ante expected utilities to the agents than the MS mechanism. Hence,

$$\int_0^1\int_0^v \Big(v-c\Big)f(v)g(c)dvdc > \gamma \tag{2.28}$$

From (2.26), (2.27) and (2.28), $C(0) > 0$. It remains to show that $C(\alpha)$ is decreasing in α and is continuous. The arguments to establish these properties closely follow those in Gresik and Satterthwaite (1989).

It is easy to see that $\phi_\alpha(v_i)$ is decreasing in α for each buyer b_i. Similarly, $\psi_\alpha(c_j)$ is increasing in α for each seller s_j. Observe that for every profile (v,c) and $\alpha' > \alpha$, $q_{b_i}^{\alpha'}(v,c) \leq q_{b_i}^{\alpha}(v,c)$ and $q_{s_j}^{\alpha'}(v,c) \leq q_{s_j}^{\alpha}(v,c)$. It follows immediately that $C(\alpha)$ is decreasing in α. Informally, an increase in α shrinks the set of profiles where trade takes place, lowering the expected utilities of both buyers and sellers.

We now demonstrate the continuity of the function $C(\cdot)$. Observe that $C(\alpha)$ can be rewritten in the manner shown below:

$$C(\alpha) = \int_0^1 \overline{q}_{b_i}^{\alpha}(v_i)\frac{1-F(v_i)}{f(v_i)}f(v_i)dv_i + \int_0^1 \overline{q}_{s_j}^{\alpha}(c_j)\frac{G(c_j)}{g(c_j)}g(c_j)dc_j - \gamma$$

Now, $C(\alpha)$ is continuous if $\overline{q}_{b_i}^{\alpha}(v_i)$ and $\overline{q}_{s_j}^{\alpha}(c_j)$ are continuous. Consider the smallest value of v_i for which buyer i's α-virtual reservation value ranks in the top N,

$$\tau_{b_i}(v_{-i}, c, \alpha) = \min\,\{v_i|\mathcal{R}_{b_i}(v,c,\alpha) \geq N\}$$

For seller j, $\tau_{s_j}(v, c_{-j}, \alpha)$ is defined analogously. The functions τ_{b_i} and τ_{s_j} are continuous as the virtual reservation functions are all continuous. Now, $\overline{q}_{b_i}^{\alpha}(v_i)$ can be written as,

$$\overline{q}_{b_i}^{\alpha}(v_i) = \int_0^1 \dots \int_0^1 q_{b_i}^{\alpha}(v,c)f(v_{-i})g(c)dv_{-i}dc$$

$$= \int_0^1 \dots \int_0^1 \Big(\int_0^{\tau_{s_N}} g(c_N)dc_N\Big)f(v_{-i})g(c_{-N})dv_{-i}dc_{-N}$$

This is continuous as τ_{s_N} and all the density functions are continuous. Similarly, $\overline{q}_{s_j}^{\alpha}(c_j)$ is also continuous. Hence, $C(\alpha)$ is continuous.

The Intermediate Value Theorem can now be applied to conclude that there exists $\alpha^* \in (0,1)$ such that $C(\alpha^*) = 0$. This completes the proof of the proposition. ■

We claim that α^*-adjusted mechanism $\mathcal{M}^{\alpha^*}$ is indeed the one which maximizes the revenue of the platform in the class of interim incentive-compatible, interim individually-rational and SBSS ex-ante stable mechanisms.

THEOREM **2.2** *$\mathcal{M}^{\alpha^*}$ maximizes the revenue of the platform in the class of interim incentive-compatible, interim individually-rational and SBSS ex-ante stable mechanisms.*

Proof: The Lagrangian for the optimization problem is given below:

$$L(q_b, q_s, \lambda) \equiv \int_0^1 \dots \int_0^1 N\Big(q_b(v,c)\Big(v - \frac{(1-F(v))}{f(v)}\Big) - q_s(v,c)\Big(c + \frac{G(c)}{g(c)}\Big)\Big)$$

$$f(v)g(c)dvdc - NU_b(0) - NU_s(1) + \lambda\Big(U_b + U_s - \gamma\Big)$$

$$= \int_0^1 \dots \int_0^1 N\Big(q_b(v,c)\Big(v - (1-\frac{\lambda}{N})\frac{(1-F(v))}{f(v)}\Big) - q_s(v,c)\Big(c + (1-\frac{\lambda}{N})\frac{G(c)}{g(c)}\Big)\Big)$$

$$f(v)g(c)dvdc - (N-\lambda)U_b(0) - (N-\lambda)U_s(1) - \lambda\gamma$$

Let $H(v,c) = q_b(v,c)\Big(v - (1-\frac{\lambda}{N})\frac{(1-F(v))}{f(v)}\Big) - q_s(v,c)\Big(c + (1-\frac{\lambda}{N})\frac{G(c)}{g(c)}\Big)$. The Lagrangian can be rewritten as:

$$L(q_b, q_s, \lambda) \equiv \int_0^1 \dots \int_0^1 H(v,c)f(v)g(c)dvdc - (N-\lambda)U_b(0) - (N-\lambda)U_s(1) - \lambda\gamma \quad (2.29)$$

The Lagrangian will be maximized if the following conditions hold for some $\lambda \geq 0$:

1. $H(v,c)$ is maximized at each value of the type profile (v,c)
2. $U_{b_i}(0) = U_{s_j}(1) = 0$ for all $i,\, j \in \{1, 2, \dots, N\}$

We will show that 1 and 2 are achieved by allocation rule $q^{\alpha^*}(v,c)$ of the mechanism $\mathcal{M}^{\alpha^*}$ where $\alpha^* = 1 - \frac{\lambda}{N}$. Thus,

$$H(v,c) = q_b(v,c)\Big(v - \alpha^*\frac{(1-F(v))}{f(v)}\Big) - q_s(v,c)\Big(c + \alpha^*\frac{G(c)}{g(c)}\Big)$$

$$= q_b(v,c)\phi_{\alpha^*}(v) - q_s(v,c)\psi_{\alpha^*}(c)$$

As part of our maintained hypothesis, we assume that $\alpha^* \in [0,1]$. This ensures that $\phi_{\alpha^*}(v)$ and $\psi_{\alpha^*}(c)$ are increasing in v and c, respectively.

We now derive the allocation rule $q_b(v,c)$ and $q_s(v,c)$ that maximize the Lagrangian in (2.29). Fix an arbitrary type profile (v,c). Assume without loss of generality, $v_1 \geq v_2 \geq \ldots \geq v_N$ and $c_1 \leq c_2 \leq \ldots \leq c_N$. Since ϕ_{α^*} and ψ_{α^*} are increasing, $\phi_{\alpha^*}(v_1) \geq \phi_{\alpha^*}(v_2) \geq \ldots \geq \phi_{\alpha^*}(v_N)$ and $\psi_{\alpha^*}(c_1) \leq \psi_{\alpha^*}(c_2) \leq \ldots \leq \psi_{\alpha^*}(c_N)$. It follows from inspection that $H(v,c)$ is maximized by the allocation rule $q^{\alpha^*}(v,c)$ of mechanism $\mathcal{M}^{\alpha^*}$. For instance, if k is the highest index such that $\phi_{\alpha^*}(v_k) \geq \psi_{\alpha^*}(c_k))$ and $\phi_{\alpha^*}(v_{k+1}) \geq \psi_{\alpha^*}(c_{k+1}))$, then $H(v,c)$ decreases if buyer $k+1$ and seller $k+1$ trade.

As each mechanism in $\mathcal{M}^{\alpha}$ class of mechanisms satisfies the conditions $U_{b_i}(0) = U_{s_j}(1) = 0$ for all $i,j \in \{1,2,\ldots,N\}$, the mechanism $\mathcal{M}^{\alpha^*}$ also satisfies these conditions.

These arguments establish that the functions $q_b^{\alpha^*}(v,c)$ and $q_s^{\alpha^*}(v,c)$ will maximize the Lagrangian in (2.29). Note that the functions are exactly the same as the allocation functions of mechanism $\mathcal{M}^{\alpha^*}$.

Finally, Proposition 2.6 establishes that such an $\alpha^* \in (0,1)$ exists. Clearly, λ shall always be positive. Thus, theorem is proved. ■

Before proceeding forward, we briefly highlight how our mechanism $\mathcal{M}^{\alpha^*}$ is different from the α^*-mechanism given by Gresik and Satterthwaite (1989). Although they look within the $\mathcal{M}^{\alpha}$ class of interim incentive-compatible and interim individually-rational mechanisms just like in our chapter, their goal is different as there is no trading platform or broker in their model. They find the ex-ante efficient mechanism for the agents in this class of mechanisms. Their optimal mechanism is strongly budget-balanced. In our model, the platform accrues surplus and the optimal mechanism satisfies the NBD condition.

We illustrate the revenue-maximizing mechanism $\mathcal{M}^{\alpha^*}$ through an example below:

Example **2.2** Let both F and G be uniform distributions.

$$\phi(v_i,\lambda) = v_i - (1 - \frac{\lambda}{N})(1 - v_i) \text{ ,and}$$

$$\psi(c_j,\lambda) = c_j + (1 - \frac{\lambda}{N})c_j$$

The trade takes place when,

$$v_i - c_j \geq \frac{1 - N\lambda}{2 - N\lambda} = \mu$$

The deviating mechanism gives the payoff of:

$$\overline{U}_b = \int_{1/4}^{1} \int_{0}^{v-1/4} (1-v)dcdv = 0.0703$$

Similarly, $\overline{U}_s = 0.0703$. So, $\overline{U}_b + \overline{U}_s = 0.1406$. Let $N = 2$. The constraint is:

$$\int_0^1 \dots \int_0^1 \Big(q^{\alpha}_{b_i}(v,c)(1 - v_i) + q^{\alpha}_{s_j}(v,c)c_j \Big) f(v)g(c)dvdc = 0.1406$$

Solving, we get $\mu = 0.279$. Table 2.2 lists different values of μ as N changes. As N increases beyond 3, the value of μ keeps decreasing. This μ is the wedge between the buyer's valuation and the seller's cost which is the result of both the agents having private information. In the unconstrained revenue-maximizing mechanism, the size of this wedge was 0.5. The stability constraints, which guarantee ex-ante utility for the agents, decrease the size of the wedge. This suggests that more trades are happening. Also, the size of the wedge goes down as the market size increases making the market more efficient.

Value of N	Value of μ
2	0.279
3	0.281
4	0.2789
5	0.2758
6	0.2731
7	0.2707
8	0.2688
9	0.2671
10	0.2660

Table 2.2: Value of μ when both F and G are uniform

Figure 2.3 shows the revenue of the platform in both the revenue-maximizing mechanisms. As the market size N increases, the revenue of the platform in the constrained revenue-maximizing mechanism diverges from the revenue in the unconstrained revenue-maximizing mechanism. The agents are able to exert a cost on the platform which keeps increasing as the number of buyers and sellers increase. Figure 2.4 shows the revenue of the platform per buyer-seller pair for both the unconstrained and the constrained optimization mechanisms.

2.6 Conclusion

This chapter is an attempt to study the stability properties of a trading mechanism offered by an internet platform to a market with multiple buyers and sellers. The unconstrained

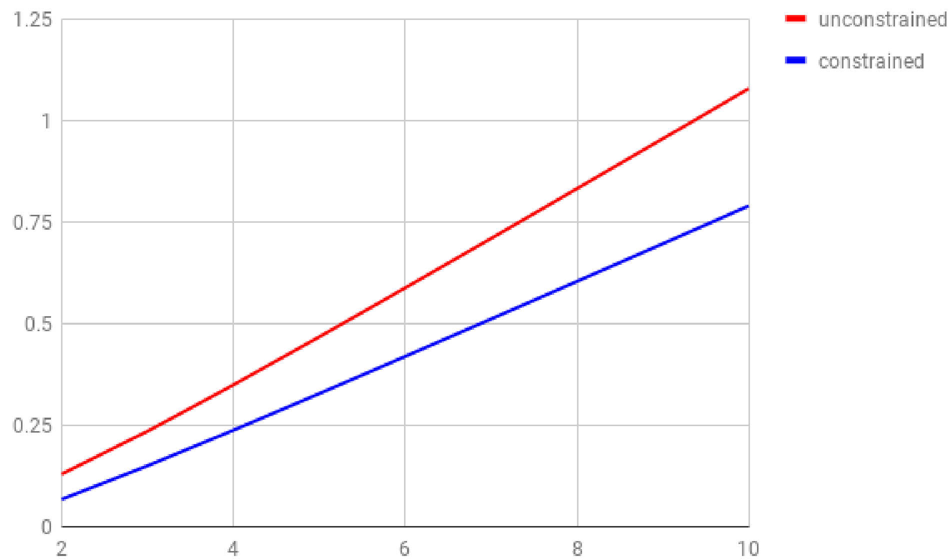


Figure 2.3: Revenue of platform for two optimization mechanisms

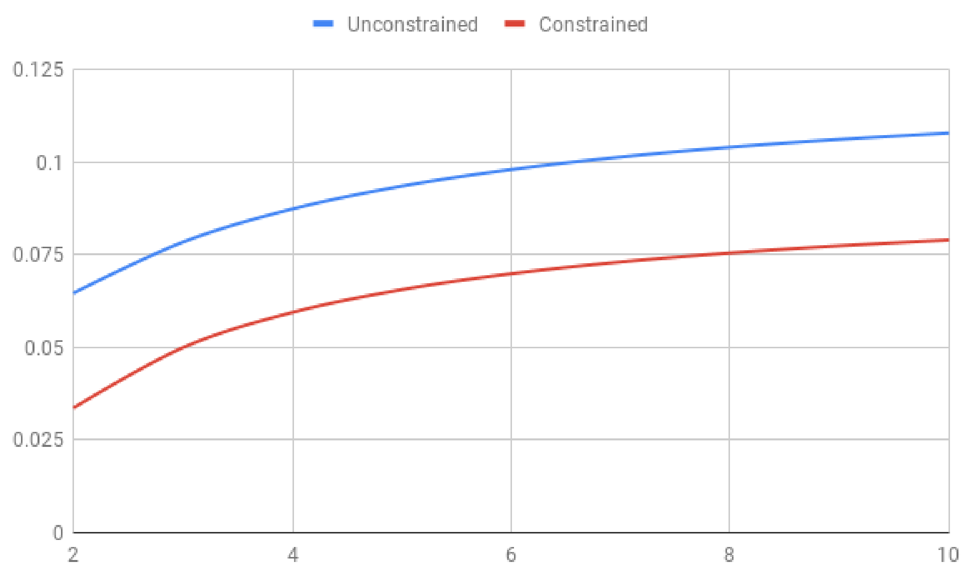


Figure 2.4: Revenue of platform per buyer-seller pair for two optimization mechanisms

revenue-maximizing mechanism for the platform is not SBSS ex-ante stable. We then find the revenue-optimizing mechanism for the platform which is SBSS ex-ante stable. In future, we would like to study different notions of interim incentive-compatible core in our environment.

2.7 Appendix

We calculate ex-ante expected utilities of buyers and sellers in various mechanisms. We assume the distribution F and G are uniform. There are 2 buyers and 2 sellers. The set of players is $\{b_1, b_2, s_1, s_2\}$. Let the type profile be (v_1, v_2, c_1, c_2).

Trade reduction mechanism

Consider buyer b_1. He is allocated a good only when:

1. $v_1 > v_2$, $c_1 < c_2$, $v_1 > c_1$, $v_2 > c_2$
2. $v_1 > v_2$, $c_2 < c_1$, $v_1 > c_2$, $v_2 > c_1$

In both the cases buyer b_1 has the higher of the two valuations. In the first case, seller s_1 has the lower of the two costs and in the second case, seller s_2 has the lower of the two costs. If the valuation of buyer b_1 is higher than the valuation of buyer b_2 and the costs of the sellers are less than the valuation of buyer b_2, then the good is allocated to buyer b_1. Only one trade takes place in which b_1 is allocated a good and the seller having the lowest cost sells a good. The ex-ante expected utility of the buyer b_1 is therefore,

$$U_{b_1} = \int_0^1 \int_0^{v_2} \int_{v_2}^1 \int_0^{c_2} (v_1 - v_2) dc_1 dv_1 dc_2 dv_2 + \int_0^1 \int_0^{v_2} \int_{v_2}^1 \int_0^{c_1} (v_1 - v_2) dc_2 dv_1 dc_1 dv_2$$

$$= \int_0^1 \int_0^{v_2} \frac{1}{2} c_2 (1 - v_2)^2 dc_2 dv_2 + \int_0^1 \int_0^{v_2} \frac{1}{2} c_1 (1 - v_2)^2 dc_1 dv_2 = \frac{1}{60}$$

McAfee double auction

Cases	Allocation probability			
	q_{b_1}	q_{b_2}	q_{s_1}	q_{s_2}
$v_1 > v_2,\ c_1 < c_2,\ v_1 > c_1,\ v_2 > c_2,\ p = 0.5 \in [c_2, v_2]$	1	1	1	1
$v_1 > v_2,\ c_1 < c_2,\ v_1 > c_1,\ v_2 > c_2,\ p = 0.5 \notin [c_2, v_2]$	1	0	1	0
$v_1 > v_2,\ c_1 < c_2,\ v_1 > c_1,\ v_2 < c_2,\ p = \frac{v_2+c_2}{2} \in [c_1, v_1]$	1	0	1	0
$v_2 > v_1,\ c_1 < c_2,\ v_2 > c_1,\ v_1 > c_2,\ p = 0.5 \in [c_2, v_1]$	1	1	1	1

Table 2.3: Allocation of goods in McAfee double auction

The cases when b_1 is allocated a good are listed in Table 2.3. In the first three cases, b_1 has the higher of the two valuations i.e. $v_1 > v_2$ and s_1 has the lower of the two costs i.e. $c_1 < c_2$. Buyer b_1 is allocated the good if price $p = 0.5 \in [c_2, v_2]$ and $v_1 > c_1,\ v_2 > c_2$. In the second case when $p = 0.5 \notin [c_2, v_2]$, b_1 still gets the good but at price of v_2. In the third case, $v_1 > c_1,\ v_2 < c_2$ and $p = \frac{v_2+c_2}{2} \in [c_1, v_1]$. So, b_1 still gets the good at this price. In the fourth case, b_1 is the lower of two valuations and $v_1 > c_2,\ v_2 > c_1,\ p = 0.5 \in [c_2, v_1]$. He gets the good at price $p = 0.5$. As the sellers are ex-ante identical, interchanging the sellers will yield four more cases equivalent to the ones listed in Table 2.3. The ex-ante expected utility of b_1 is therefore,

$$U_{b_1} = 2\int_{1/2}^{1}\int_{0}^{1/2}\int_{v_2}^{1}\int_{0}^{c_2} (v_1 + v_2 - 1)dc_1dv_1dc_2dv_2 + 2\int_{1/2}^{1}\int_{1/2}^{v_2}\int_{v_2}^{1}\int_{0}^{c_2} (v_1 - v_2)dc_1dv_1dc_2dv_2 +$$

$$2\int_{0}^{1/2}\int_{0}^{v_2}\int_{v_2}^{1}\int_{0}^{c_2} (v_1 - v_2)dc_1dv_1dc_2dv_2 + 2\int_{0}^{1}\int_{0}^{c_2}\int_{(v_2+c_2)/2}^{1}\int_{0}^{(v_2+c_2)/2} \left(v_1 - \frac{v_2 + c_2}{2}\right)dc_1dv_1dv_2dc_2 = 0.0792$$

Proof of Proposition 2.4:

Sufficiency. Suppose q_{b_i} and q_{s_j} are increasing and decreasing, respectively for each i and j. Consider the following transfer rule

$$\bar{t}_{b_i}(v_i) = v_i\bar{q}_{b_i}(v_i) - \int_{0}^{v_i} \bar{q}_{b_i}(x)dx$$

$$\bar{t}_{s_j}(c_j) = c_j\bar{q}_{s_j}(c_j) + \int_{c_j}^{1} \bar{q}_{s_j}(x)dx$$

If the buyer b_i reports $v_i' > v_i$ his interim utility is $U_{b_i}(v_i') = v_i\overline{q}_{b_i}(v_i') - v_i'\overline{q}_{b_i}(v_i') + \int_0^{v_i'} \overline{q}_{b_i}(x)dx$. The difference in interim utility of b_i is $U_{b_i}(v_i) - U_{b_i}(v_i') = \int_0^{v_i} \overline{q}_{b_i}(x)dx - \overline{q}_{b_i}(v_i')(v_i - v_i') - \int_0^{v_i'} \overline{q}_{b_i}(x)dx = -\int_{v_i}^{v_i'} \overline{q}_{b_i}(x)dx + \overline{q}_{b_i}(v_i')(v_i' - v_i)$. As $\overline{q}_{b_i}(v_i)$ is increasing in v_i, $\overline{q}_{b_i}(v_i') \geq \overline{q}_{b_i}(v_i)$. This means $U_{b_i}(v_i) - U_{b_i}(v_i') \geq 0$. Hence, b_i has no incentive to misreport. Similarly, it can be shown that the buyer has no incentive to report any $v_i' < v_i$. Also, through similar arguments it can be shown that no seller has any incentive to misreport. The mechanism is interim incentive-compatible.

Necessity. Suppose (q, t) is interim incentive-compatible and interim individually-rational. Let v_i and v_i' be two possible valuations for buyer b_i. Then by interim incentive-compatibility,

$$U_{b_i}(v_i) = \overline{q}_{b_i}(v_i)v_i - \overline{t}_{b_i}(v_i) \geq \overline{q}_{b_i}(v_i')v_i - \overline{t}_{b_i}(v_i') \tag{2.30}$$

$$U_{b_i}(v_i') = \overline{q}_{b_i}(v_i')v_i' - \overline{t}_{b_i}(v_i') \geq \overline{q}_{b_i}(v_i)v_i' - \overline{t}_{b_i}(v_i) \tag{2.31}$$

Adding, we have

$$(v_i' - v_i)\overline{q}_{b_i}(v_i') \geq U_{b_i}(v_i') - U_{b_i}(v_i) \geq (v_i' - v_i)\overline{q}_{b_i}(v_i) \tag{2.32}$$

If $v_i' > v_i$, then $\overline{q}_{b_i}(v_i') \geq \overline{q}_{b_i}(v_i)$ follows immediately. This establishes that $q_{b_i}(v, c)$ is increasing in v_i. By a similar argument, $\overline{q}_{s_j}(c_j)$ is decreasing in c_j for all $j = 1, 2, \ldots, N$.

Also, dividing all the sides of expression in (2.32) by $v_i' - v_i$, taking limits and noticing that the derivative of $\overline{q}_{b_i}(v_i)$ exists almost everywhere, by the Sandwich Theorem,

$$\frac{dU_{b_i}(v_i)}{dv_i} = \overline{q}_{b_i}(v_i) \text{ almost everywhere.}$$

Integrating, we get,

$$U_{b_i}(v_i) = U_{b_i}(0) + \int_0^{v_i} \overline{q}_{b_i}(x)dx$$

Similarly, for every seller j, $\overline{q}_{s_j}(c_j)$ is decreasing, and,

$$U_{s_j}(c_j) = U_{s_j}(1) + \int_{c_j}^{1} q_{s_j}(y)dy$$

The ex-ante expected utility of a buyer i can be written as,

$$U_{b_i} = \int_0^1 U_{b_i}(v_i) f(v_i) dv_i$$

$$= U_{b_i}(0) + \int_0^1 \Big(\int_0^{v_i} \overline{q}_{b_i}(x) dx \Big) f(v_i) dv_i$$

Integrating by parts,

$$U_{b_i} = U_{b_i}(0) + F(1) \int_0^1 \overline{q}_{b_i}(x) dx - 0 - \int_0^{v_i} \overline{q}_{b_i}(v_i) F(v_i) dv_i$$

$$= U_{b_i}(0) + \int_0^1 \overline{q}_{b_i}(v_i)(1 - F(v_i)) dv_i$$

Or,

$$U_{b_i} = U_{b_i}(0) + \int_0^1 \dots \int_0^1 q_{b_i}(v,c) \frac{(1 - F(v_i))}{f(v_i)} f(v) g(c) dv dc$$

Similarly, for every seller j,

$$U_{s_j} = U_{s_j}(1) + \int_0^1 \dots \int_0^1 q_{s_j}(v,c) \frac{G(c_j)}{g(c_j)} f(v) g(c) dv dc$$

Hence, proved. ■

Chapter 3

Budget-balanced mechanisms for single-object allocation problems with interdependent values

3.1 Introduction

Consider a family in which a bequest such as a house (which we shall henceforth call an object) has to be divided among potential heirs or agents. Depending on the will, each agent can potentially be allocated the whole bequest or a part of it. Each agent has a valuation for the object which is private information. Agents can also be compensated by transfers i.e. they can transfer money among each other. There is no outside agency which can provide subsidies to the participants nor can any surplus accrue which is not redistributed. It is also desirable to award the bequest to the agent who has the highest valuation for it. This is clearly a problem in mechanism design. The goal is to identify mechanisms that provide incentives to reveal their private information truthfully, is budget-balanced and efficient.

Recent literature on this question has studied this question in the private value setting. According to the well-known Green-Laffont impossibility result (see Green and Laffont (1979) for details) it is impossible to design a mechanism that meets all three objectives i.e. is strategy-proof, efficient and budget-balanced. The impossibility result necessitates a second-best approach. One such approach was formulated by Green and Laffont (1979). An agent, say i, is selected with uniform probability. The remaining agents participate in a Vickrey auction where the agent with highest bid wins the bequest and pays an amount equal to the second-highest bid. This amount is transferred to agent i. This mechanism is obviously budget-balanced. It is also strategy-proof since truth-telling is a dominant strategy for agents participating in the Vickrey auction while agent i's private information plays no role. It is clearly not efficient since agent i could be the agent with highest valuation among all the agents.

The Green-Laffont idea has been significantly generalized by Long et al. (2017). They introduce *ranking mechanisms* where the allocation is decided on the basis of the ranks of announced valuations. The Green-Laffont mechanism is a particular ranking mechanism in which the agent with the highest valuation receives the object with probability $1 - \frac{1}{n}$ and the agent with second-highest valuation gets the object with probability $\frac{1}{n}$. They characterize the class of strategy-proof and budget-balanced mechanisms and also find the ranking mechanism that maximizes the worst-case efficiency within this class.

We consider the same problem in an interdependent value setting. Each agent receives a signal about the valuation of the object. Her valuation depends on the signals received by all other agents. This is a familiar model in mechanism design and auction theory (see, for example, Milgrom and Weber (1982) and Dasgupta and Maskin (2000)). The appropriate truth-telling notion in a model with interdependent valuations is ex-post incentive compatibility. According to the requirement, each agent has the incentive to reveal his signal truthfully assuming that other agents are reporting their signals truthfully. Efficiency in this context is ex-post efficiency according to which the agent with highest valuation receives the object. In our model, the Green-Laffont impossibility continues to hold* - see Nath et al. (2015) for example.

We consider two types of mechanisms. The first are *signal-ranking mechanisms* (or s-ranking mechanisms). Agents report their signals and are ranked according to these reports. The s-ranking allocation rule assigns a probability for receiving the object to each agent. Transfers for agents are determined accordingly. Valuation-ranking mechanisms or v-ranking mechanisms on the other hand, assign probabilities for receiving the object based on the ranking of agents' valuations.

Consider the class of s-ranking mechanisms. Our first observation is that the naive Green-Laffont idea no longer works. An agent can no longer be "excluded" since her signal determines the valuations of all agents. The signal from excluded agent must be elicited since the valuations of the agents participating in the Vickrey auction depend on it. If the surplus from the Vickrey auction is transferred to the excluded agent then he will typically have an incentive to misreport his signal. Details can be found in Section 3.6.1.

*In some special cases like the sequencing problem it might be possible to reconcile the three properties e.g. Hain and Mitra (2004).

Nevertheless, we show that a ranking allocation rule that is strategy-proof and can be implemented by a budget-balanced transfers in the private-value case is also an ex-post incentive compatible (EPIC) and ex-post individually rational (EPIR) s-ranking allocation rule that can be implemented with budget-balanced (BB) transfers provided the valuation functions satisfy an additive separability condition. An immediate consequence of this result is that the s-ranking mechanism where the agents with the highest and second-highest ranking signals receive the object with probabilities $1 - \frac{1}{n}$ and $\frac{1}{n}$ respectively (i.e. the Green-Laffont allocation vector) is EPIC, EPIR and implementable by budget-balanced transfers if the valuation functions satisfy additive separability condition, single-crossing and symmetry. We show by means of an example that the result does not hold. We also show that the allocation rule of the mechanism that maximizes worst-case efficiency ratio given by Long et al. (2017) is the s-ranking allocation rule which maximizes worst-case efficiency ratio among all EPIC, EPIR and BB s-ranking mechanisms when valuation functions are of a specific form that satisfies SAS condition, single-crossing and symmetry. We then provide an example to show that this mechanism is no longer optimal when the valuation functions are not symmetric.

For v-ranking mechanisms, first we show that it is necessary for valuation functions to satisfy single-crossing for the mechanism to be EPIC. Then we show that a ranking allocation rule that is strategy-proof and can be implemented by a budget-balanced transfers in the private-value case is also an EPIC and EPIR v-ranking allocation rule that can be implemented with budget-balanced transfers provided the valuation functions satisfy the additive separability condition and single-crossing. Under an additional condition of symmetry of valuation functions, the allocation functions for s-ranking mechanisms and v-ranking mechanisms are allocation equivalent. Moreover, the agents have the same payment functions and get the same utility from allocation equivalent, EPIC, EPIR and BB s-ranking and v-ranking mechanisms.

Another approach to the impossibility result is to allocate the object only to the agent with the highest signal but with probability less than one. The object is thrown away or retained by the seller with the remaining probability. The agent who is allocated the object makes a payment which is redistributed among all the agents ensuring budget balancedness. Such mechanisms were called *probability-burning mechanisms* by Mishra and Sharma (2018) and studied in private valuation models. We explore the feasibility of such mechanisms in the interdependent valuation case. For a semi-separable class of valuation functions, we show that a particular probability-burning mechanism is EPIC, EPIR and budget balanced (BB). For additively separable and symmetric collection of valuation functions, we design another probability-burning mechanism and show that it is welfare-maximizing in the class of EPIC, EPIR, BB mechanisms that allocate only to the agents with topmost signal and satisfy an additional property called *equal treatment at equal signals*.

The literature of mechanism design in interdependent value setting emphasizes the importance of single-crossing for efficient mechanisms to be EPIC (see d'Aspremont and Gerard-Varet (1982) and Dasgupta and Maskin (2000) for example). In this chapter, by relaxing efficiency we identify two mechanisms viz. s-ranking mechanisms and probability-burning mechanisms that are EPIC, EPIR and BB for valuation functions that satisfy additively separable or semi-separable condition but not single-crossing.

This chapter proceeds as follows. Section 3.2 provides a literature survey. The model and basic definitions are introduced in Section 3.3. Section 3.4 discusses the signal-ranking mechanisms and Section 3.5 discusses the valuation-ranking mechanisms. Section 3.6 extends the discussion to welfare properties of these mechanisms and the failure of Green-Laffont mechanism. Section 3.7 discusses probability-burning mechanisms. Section 3.8 is the conclusion.

3.2 Literature review

Many papers have explored the single-object allocation problem with budget-balance in the private value setting. Long et al. (2017) explores the class of dominant strategy incentive compatible and BB ranking mechanisms. Their worst-case efficient ranking mechanism coincides with the Green-Laffont mechanism for $n \leq 8$ and allocates to more than two agents when $n > 8$ where n is the number of agents. Long (2019) extends this concept to a multi-object model. Mishra and Sharma (2018) introduce the probability-burning mechanisms. They find the Pareto optimal probability-burning mechanism in the class of top-only, BB and dominant strategy incentive compatible mechanisms.

There is an extensive literature on the property of ex-post incentive compatibility in interdependent-value models. For an efficient mechanism to be EPIC, the valuation functions must satisfy the single-crossing condition (for further detailed discussion see d'Aspremont and Gerard-Varet (1982), Perry and Reny (1999), Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001) and Bergemann and Morris (2005)). But there are many settings in which the valuation functions do not satisfy the single-crossing condition - see Eden et al. (2018) for a discussion.

In the private value setting, Long et al. (2017) give the necessary and sufficient condition for a mechanism to be budget-balanced. They prove that a monotone allocation rule can be implemented by a dominant strategy incentive compatible and budget-balanced mechanism if and only if it satisfies residual balancedness. A similar result was obtained by Yenmez

(2015) in an interdependent-value matching model.

3.3 The model

An object has to be allocated among the set of agents $N = \{1, 2, \ldots, n\}$. Each agent $i \in N$ receives a signal s_i which is his private information. The signals are independently and identically distributed in the unit interval $S = [0, 1]$. A signal profile $s \in S^n$ is an n-tuple $s = (s_1, s_2, \ldots, s_n)$. Also $s_{-i} \in S^{n-1}$ is the signal profile $s_{-i} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ containing all signals in s except that of agent i. Since we are investigating an interdependent values model, we assume that each agent $i \in N$ has a valuation function $v_i : S^n \to \mathbb{R}_+$. Thus i's valuation of the object depends on the signals received by *all* agents. In contrast, an agent's valuation in a private values model depends only on his own signal and can without loss of generality be assumed to be the signal itself.

An allocation rule is a map $f : S^n \to [0, 1]^n$ where $f_i(s)$ denotes the probability of allocation of the object to agent i when the signal profile is s. The allocation probabilities are assumed to satisfy the feasibility condition $\sum\limits_{i \in N} f_i(s) \leq 1$ for every $s \in S^n$. The payment rule of agent i is $p_i : S^n \to \mathbb{R}$. A mechanism M is pair $(f, p) \equiv (f_1, f_2, \ldots, f_n, p_1, p_2, \ldots, p_n)$ and gives utility of $v_i(s)f_i(s) - p_i(s)$ to agent i for all $i = 1, 2, \ldots, n$ and $s \in S^n$.

The following properties of a mechanism will be relevant for this chapter.

- The mechanism $M \equiv (f, p)$ is ex-post incentive compatible (EPIC) if for every $i \in N$, every $s_{-i} \in S^{n-1}$, and every $s_i, s_i' \in S$, we have

  $$v_i(s)f_i(s_i, s_{-i}) - p_i(s_i, s_{-i}) \geq v_i(s)f_i(s_i', s_{-i}) - p_i(s_i', s_{-i})$$

  Ex-post incentive compatibility ensures that each agent prefers to report his own signal truthfully if other agents are truthful.

- The mechanism $M \equiv (f, p)$ is ex-post individually rational (EPIR) if for every $i \in N$ and every $s \in S^n$, we have

  $$v_i(s)f_i(s) - p_i(s) \geq 0$$

  This property guarantees that every agent gets a non-negative utility at every signal profile if all agents report truthfully.

- The mechanism $M \equiv (f, p)$ is budget-balanced (BB) if for every $s \in S^n$, we have

  $$\sum_{i \in N} p_i(s) = 0$$

The mechanism neither generates a surplus nor runs a deficit at any signal profile.

3.4 Signal-ranking mechanisms

We analyze signal-ranking mechanisms in this section. These mechanisms are adaptations of the ranking mechanisms in Long et al. (2017).

Fix an arbitrary signal profile s. We partition the set of agents into equivalence classes depending on the ranks of their signals. In particular $s[1]$ is the set of agents with the highest signals, $s[2]$ is the set of agents with the second-highest signal and so on. Formally,

$$s[1] = \{i \in N | s_i \geq s_j \ \forall j \in N\}$$

and

$$s[k] = \{i \in N \setminus (\bigcup_{k'=1}^{k-1} s[k']) : s_i \geq s_j \ \forall j \in N \setminus (\bigcup_{k'=1}^{k-1} s[k'])\}$$

Let L be the greatest integer such that $s[L] \neq \emptyset$.[†] Clearly $1 \leq L \leq n$. Let $|s[k]|$, $k \in \{1, \ldots, L\}$ denote the cardinality of the set $s[k]$, i.e. it is the number of agents whose signals are ranked k. Note $\sum_{k=1}^{L} |s[k]| = n$. Let $|[s]| = (|s[1]|, |s[2]|, \ldots, |s[L]|)$.

Suppose $N = \{1, 2, 3, 4, 5, 6\}$ and $s = (0.4, 0.9, 0.9, 0.3, 0.9, 0.3)$. Then $s[1] = \{2, 3, 5\}$, $s[2] = \{1\}$ and $s[3] = \{4, 6\}$. Also $L = 3$ and $|[s]| = (3, 1, 2)$.

Let $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ be an n-tuple of real numbers such that $1 \geq \pi_1 \geq \pi_2 \geq \ldots \geq \pi_n \geq 0$. Let $\alpha = (\alpha_1, \ldots, \alpha_R)$ be a vector of R strictly positive integers such that $\sum_{r=1}^{R} \alpha_r = n$. For all $r = 1, 2, \ldots, R$, let A_r denote the partial sum $\alpha_1 + \alpha_2 + \ldots + \alpha_r$. Finally, let $\langle \pi(\alpha) \rangle$ be a R-dimensional vector whose r^{th} component is $\langle \pi(\alpha) \rangle_r = \sum\limits_{k=A_{r-1}+1}^{A_r} \pi_k$ for $r \geq 2$ and $\langle \pi(\alpha) \rangle_{r=1} = \sum\limits_{k=1}^{A_1} \pi_k$. In other words, the first component of $\langle \pi(\alpha) \rangle$ is the sum of the first α_1 terms of π, the second component is the sum of the next α_2 terms of π and so on.

Let $\pi = (0.45, 0.25, 0.15, 0.1, 0.05, 0)$ and $\alpha = (2, 2, 1, 1)$. Then the vector of partial sums $A = (2, 4, 5, 6)$ and $\langle \pi(\alpha) \rangle = (0.70, 0.25, 0.05, 0)$. Thus the first component of $\langle \pi(\alpha) \rangle$ is the sum of the first two terms of π, the second component is the sum of the next two terms of π, the third component is the fifth term of π and the fourth component is the sixth term of π.

We are now ready to define s-ranking allocation rules.

[†]We suppress the dependence of L on s for notational convenience.

DEFINITION **3.1** *An allocation rule f is a signal-ranking (s-ranking) allocation rule if there exists $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ such that $1 \geq \pi_1 \geq \pi_2 \geq \ldots \geq \pi_n \geq 0$ and $\sum\limits_{i \in N} \pi_i = 1$ such that for all s and $k = 1, \ldots, L$ we have:*

(i) $\sum\limits_{i \in s[k]} f_i(s) = \langle \pi(|[s]|) \rangle_k$*, and*

(ii) $f_i(s) = f_j(s)$ *whenever* $i, j \in s[k]$.

A mechanism (f, p) is a s-ranking mechanism if f is a s-ranking allocation rule.

We illustrate the idea of an s-ranking mechanism by combining the earlier examples, i.e. $N = \{1, 2, 3, 4, 5, 6\}$, $s = (0.4, 0.9, 0.9, 0.3, 0.9, 0.3)$ and $\pi = (0.45, 0.25, 0.15, 0.1, 0.05, 0)$. Then $|[s]| = (3, 1, 2)$ and $\langle \pi(|[s]|) \rangle = (0.85, 0.1, 0.05)$. The agents with the highest ranking signals, $s[1] = \{2, 3, 5\}$ together "share" the allocation probability 0.85, i.e. each of these agents receives the object with probability $\frac{0.85}{3}$. The agent with the second-highest signal, $s[2] = \{1\}$ gets the object with probability 0.1 while the agents with the third-highest signal $s[3] = \{4, 6\}$ "share" the allocation probability 0.05, i.e. each receives the object with probability 0.025.

The key to our result on s-ranking mechanisms is the particular structure we impose on valuation functions. We describe this below.

DEFINITION **3.2** *The valuation functions $v_i : S^n \to \mathbb{R}_+$, $i = 1, \ldots, n$ satisfy the Strong Additive Separability (SAS) condition if there exists $n+1$ increasing functions $g_i : S \to \mathbb{R}_+$, $i \in \{1, \ldots, n\}$ and $h : S \to \mathbb{R}_+$ with $g_i(0) = h(0) = 0$ for all $i \in \{1, \ldots, n\}$ such that, for all $s \in S^n$ and $i \in \{1, \ldots, n\}$, we have $v_i(s) = g_i(s_i) + \sum_{j \neq i} h(s_j)$.*

It is important to note that the h function used to "aggregate the signals of other agents" is common to all agents. Two examples of valuation functions that are not members of the SAS class will be provided later in the section.

Our main result for this section is the following:

THEOREM **3.1** *Assume agents' valuation functions satisfy the SAS condition. Assume further that the s-ranking allocation rule $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ satisfies the equation*

$$\sum_{j \in N} (-1)^j \binom{n-1}{j-1} \pi_j = 0. \tag{3.1}$$

Then there exist payment functions $p = (p_1, p_2, \ldots, p_n)$ such that the s-ranking mechanism (π, p) is EPIC, BB and EPIR.

The proof of the Theorem can be found in the Appendix. Here we outline the main features of the argument. Our first step is to show that there exist payment functions p such that (π, p)-allocation is EPIC whenever the valuation functions are increasing in own signals. This is satisfied by the valuation functions in the SAS class since the g_i functions are increasing. Payments for the allocation rule can then be obtained by applying the Revenue Equivalence formula for an interdependent value model (see Roughgarden and Talgam-Cohen (2016))

$$p_i(s) = v_i(s)f_i(s) - v_i(0, s_{-i})f_i(0, s_{-i}) - \int_0^{s_i} f_i(x, s_{-i})\frac{\partial v_i(x, s_{-i})}{\partial s_i}dx \ , \forall i \in N \tag{3.2}$$

By assuming $p_i(0, s_{-i}) = 0$ for all $i \in N$, individual rationality is ensured. A condition ensuring budget-balance can be obtained from the more general condition of Yenmez (2015). We introduce some notation before stating the result. Denote by $R(s)$ the revenue collected from the mechanism, i.e.

$$R(s) = \sum_{i \in N} p_i(s)$$

For any signal profile s, let $N_s^0 = \{i \in N | s_i = 0\}$. For any signal profile s and any $T \subseteq N$, $(0_T, s_{-T})$ denotes the signal profile where the signals of all agents in T is 0 and each agent $i \notin T$ has signal s_i, the i^{th} component of s.

Theorem **3.2 (Yenmez (2015))** *An EPIC mechanism $M \equiv (f, p)$ is budget-balanced if and only if*

$$\sum_{T \subseteq N} (-1)^{|T|} R(0_T, s_{-T}) = 0 \tag{3.3}$$

for all signal profiles s. Here $R(s)$ is computed from the payment functions $p_i(s)$ given in Condition 3.2.

Condition 3.3 is called the *residual balancedness* condition. We show that Condition 3.1 implies that the s-ranking allocation rule is residually balanced, thus implying budget-balance. Condition 3.1 is similar to the condition derived by Long et al. (2017) for allocation rules that are strategy-proof, BB and EPIR in the private values case. Also, they give the unique payment functions for mechanisms that are strategy-proof, budget-balanced and satisfy the condition of symmetry. We adapt their payment functions for a mechanism that is EPIC and BB:

$$p_i(s) = \begin{cases} -\dfrac{1}{|N_s^0|} \sum\limits_{T \subseteq N: N_s^0 \subseteq T} \dfrac{(-1)^{|T \setminus N_s^0|}}{C(|T|, |N_s^0|)} R(0_T, s_{-T}) & , \text{ if } i \in N_s^0 \\ v_i(s) f_i(s) - v_i(0, s_{-i}) f_i(0, s_{-i}) - \int\limits_0^{s_i} f_i(x, s_{-i}) \frac{\partial v_i(x, s_{-i})}{\partial s_i} dx & \\ -\dfrac{1}{|N_s^0| + 1} \sum\limits_{T \subseteq N: (N_s^0 \cup \{i\}) \subseteq T} \dfrac{(-1)^{|T \setminus N_s^0| - 1}}{C(|T|, |N_s^0| + 1)} R(0_T, s_{-T}) & , \text{ if } i \notin N_s^0 \end{cases}$$

The next two examples illustrate two s-ranking mechanisms that satisfy all the required properties.

Example **3.1** Let $N = \{1, 2, 3\}$. Assume that the valuation functions are of the form $v_i(s) = g_i(s_i) + \sum_{j \neq i} h(s_j)$ where g_i and h are increasing. Consider the s-ranking mechanism $\pi = \left(\frac{2}{3}, \frac{1}{3}, 0\right)$. It clearly satisfies Condition 3.1.

Pick an arbitrary signal profile s. Let $s_{(1)} > s_{(2)} > s_{(3)}$, i.e. (i) is the agent with the i^{th} ranked signal. Payments at s are given by:

$$p_{(i)}(s) = \begin{cases} \frac{1}{3} g_{(1)}(s_{(2)}) + \frac{1}{3} g_{(1)}(s_{(3)}) - \frac{1}{3} g_{(2)}(s_{(3)}) + \frac{1}{2} h(s_{(2)}) + \frac{1}{6} h(s_{(3)}) & , \text{ if } i = 1 \\ \frac{1}{3} g_{(2)}(s_{(3)}) - \frac{1}{3} g_{(1)}(s_{(3)}) + \frac{1}{6} h(s_{(1)}) - \frac{1}{6} h(s_{(3)}) & , \text{ if } i = 2 \\ -\frac{1}{3} g_{(1)}(s_{(2)}) - \frac{1}{6} h(s_{(1)}) - \frac{1}{2} h(s_{(2)}) & , \text{ if } i = 3 \end{cases}$$

It is easy to verify that $\sum\limits_{i \in N} p_{(i)}(s) = 0$. Note that the budget-balance property depends crucially on the fact that the function h is the same for all agents. In order to verify ex-post incentive-compatibility, it has to be verified that no agent can gain by misrepresenting her signal in a manner that changes the rank of her signal. We only confirm a special case for agent 1 where $s_2 > s_3 > s_1$. Truth-telling by agent 1 gives her the object with zero probability while she receives a payment of $\frac{1}{3} g_2(s_3) + \frac{1}{6} h(s_2) + \frac{1}{2} h(s_3)$. Suppose 1 announces s_1' such that $s_1' > s_2 > s_3$. She now receives the object with probability $\frac{2}{3}$ and has to pay $\frac{1}{3} g_1(s_2) + \frac{1}{3} g_1(s_3) - \frac{1}{3} g_2(s_3) + \frac{1}{2} h(s_2) + \frac{1}{6} h(s_3)$. The net change in payoff from misrepresentation is $\frac{2}{3} g_1(s_1) - \frac{1}{3}(g_1(s_2) + g_1(s_3)) = \frac{1}{3}(g_1(s_1) - (g_1(s_2)) + \frac{1}{3}(g_1(s_1) - g_1(s_3)) < 0$ where the last inequality follows from the fact that g_1 is an increasing function. It can be verified that no misrepresentation is profitable for any agent 1.

Example **3.2** Let $N = \{1, 2, 3, 4\}$ agents. The valuation functions are given by $v_i(s) = s_i + \beta(\sum\limits_{j \neq i} s_j)$ where $\beta > 0$. Consider the s-ranking mechanism $\pi = \left(\frac{4}{9}, \frac{5}{18}, \frac{1}{6}, \frac{1}{9}\right)$ which satisfies Condition 3.1.

Pick an arbitrary signal profile s and let $s_{(1)} \geq s_{(2)} \geq s_{(3)} \geq s_{(4)}$. Payments at s are given by:

$$p_{(i)}(s) = \begin{cases} \left(\frac{1}{6} + \frac{7}{27}\beta\right) s_{(2)} + \left(\frac{1}{36} + \frac{19}{108}\beta\right) s_{(3)} + \frac{4}{27}\beta s_{(4)} & \text{, if } i = 1 \\ \frac{5}{54}\beta s_{(1)} + \left(\frac{1}{36} + \frac{1}{108}\beta\right) s_{(3)} - \frac{1}{54}\beta s_{(4)} & \text{, if } i = 2 \\ -\frac{2}{27}\beta s_{(1)} - \left(\frac{1}{12} + \frac{11}{108}\beta\right) s_{(2)} - \frac{7}{54}\beta s_{(4)} & \text{, if } i = 3 \\ -\frac{1}{54}\beta s_{(1)} - \left(\frac{1}{12} + \frac{17}{108}\beta\right) s_{(2)} - \left(\frac{1}{18} + \frac{5}{27}\beta\right) s_{(3)} & \text{, if } i = 4 \end{cases}$$

It can again be verified that sum of agents' payments is zero and that no agent can gain by misrepresentation.

We have seen that the s-ranking mechanism $(\frac{2}{3}, \frac{1}{3}, 0)$ is EPIC, BB and EPIR provided that the valuation functions belong to the SAS class. The next two examples show that this result may not hold if valuation functions do not satisfy the SAS assumption.

Example **3.3** Let $N = \{1, 2, 3\}$. The valuation functions are $v_1(s) = s_1 + 0.5(s_2 + s_3)$, $v_2(s) = s_2 + 0.4(s_1 + s_3)$ and $v_3(s) = s_3 + 0.5(s_1 + s_2)$. The valuation functions violate the SAS assumption since the function used to aggregate the signals of other agents is different for each agent. Consider the s-ranking mechanism $\pi = (\frac{2}{3}, \frac{1}{3}, 0)$.

Pick an arbitrary signal profile s where $s_1 > s_2 > s_3 > 0$. Table 3.1 below computes the value of $R(0_T, s_{-T})$ for various values of $T \subseteq N$.

$R(s_1, s_2, s_3)$	$\frac{1}{3}v_1(s_2, s_2, s_3) + \frac{1}{3}v_1(s_3, s_2, s_3) + \frac{1}{3}v_2(s_1, s_3, s_3)$
$R(0, s_2, s_3)$	$\frac{1}{3}v_2(0, s_3, s_3) + \frac{1}{3}v_2(0, 0, s_3) + \frac{1}{3}v_3(0, s_2, 0)$
$R(s_1, 0, s_3)$	$\frac{1}{3}v_1(s_3, 0, s_3) + \frac{1}{3}v_1(0, 0, s_3) + \frac{1}{3}v_3(s_1, 0, 0)$
$R(s_1, s_2, 0)$	$\frac{1}{3}v_1(s_2, s_2, 0) + \frac{1}{3}v_1(0, s_2, 0) + \frac{1}{3}v_2(s_1, 0, 0)$
$R(s_1, 0, 0)$	$\frac{1}{6}v_2(s_1, 0, 0) + \frac{1}{6}v_3(s_1, 0, 0)$
$R(0, s_2, 0)$	$\frac{1}{6}v_1(0, s_2, 0) + \frac{1}{6}v_3(0, s_2, 0)$
$R(0, 0, s_3)$	$\frac{1}{6}v_1(0, 0, s_3) + \frac{1}{6}v_2(0, 0, s_3)$
$R(0, 0, 0)$	0

Table 3.1: Value of $R(0_T, s_{-T})$ for various $T \subseteq N$.

Therefore,

$$\sum_{T \subseteq N} (-1)^{|T|} R(0_T, s_{-T}) = R(s_1, s_2, s_3) - R(0_1, s_{-1}) - R(0_2, s_{-2}) - R(0_3, s_{-3}) + R(0_{12}, s_{-12})$$

$$+ R(0_{13}, s_{-13}) + R(0_{23}, s_{-23}) - R(0_{123}, s_{-123})$$

$$\begin{aligned}
&= R(s_1,s_2,s_3) - R(0,s_2,s_3) - R(s_1,0,s_3) - R(s_1,s_2,0) + R(0,0,s_3) \\
&\quad + R(0,s_2,0) + R(s_1,0,0) - R(0,0,0) \\
&= \frac{1}{3}v_1(s_2,s_2,s_3) + \frac{1}{3}v_1(s_3,s_2,s_3) + \frac{1}{3}v_2(s_1,s_3,s_3) + \frac{1}{6}v_2(s_1,0,0) \\
&\quad + \frac{1}{6}v_3(s_1,0,0) + \frac{1}{6}v_1(0,s_2,0) + \frac{1}{6}v_3(0,s_2,0) + \frac{1}{6}v_1(0,0,s_3) \\
&\quad + \frac{1}{6}v_2(0,0,s_3) - \frac{1}{3}v_2(0,s_3,s_3) - \frac{1}{3}v_2(0,0,s_3) - \frac{1}{3}v_3(0,s_2,0) \\
&\quad - \frac{1}{3}v_1(s_3,0,s_3) - \frac{1}{3}v_1(0,0,s_3) - \frac{1}{3}v_3(s_1,0,0) - \frac{1}{3}v_1(s_2,s_2,0) \\
&\quad - \frac{1}{3}v_1(0,s_2,0) - \frac{1}{3}v_2(s_1,0,0) \\
&= \frac{1}{3}(s_2 + 0.5(s_2+s_3)) + \frac{1}{3}(s_3 + 0.5(s_2+s_3)) + \frac{1}{3}(s_3 + 0.4(s_1+s_3)) \\
&\quad + \frac{1}{6}(0.4s_1) + \frac{1}{6}(0.5s_1) + \frac{1}{6}(0.5s_2) + \frac{1}{6}(0.5s_2) + \frac{1}{6}(0.5s_3) + \frac{1}{6}(0.4s_3) \\
&\quad - \frac{1}{3}(s_3 + 0.4s_3) - \frac{1}{3}(0.4s_3) - \frac{1}{3}(0.5s_2) - \frac{1}{3}(s_3 + 0.5s_3) - \frac{1}{3}(0.5s_3) \\
&\quad - \frac{1}{3}(0.5s_1) - \frac{1}{3}(s_2 + 0.5s_2) - \frac{1}{3}(0.5s_2) - \frac{1}{3}(0.4s_1) \\
&= -\frac{1}{6}(0.1s_1 + 0.1s_3) \neq 0
\end{aligned}$$

It follows from Theorem 3.2 that π is not BB.

Example **3.4** Let $N = \{1,2,3\}$. The valuation functions are $v_1(s) = s_1 + s_2s_3$, $v_2(s) = s_2 + s_1s_3$ and $v_3(s) = s_3 + s_1s_2$. Once again the SAS condition is violated. Consider the following allocation probabilities $\pi = (\frac{2}{3}, \frac{1}{3}, 0)$. Pick an arbitrary signal profile s such that $s_1 > s_2 > s_3 > 0$. Using Table 3.1 above we compute $R(0_T, s_{-T})$ for various values of $T \subseteq N$. Using the particular form of the valuation functions, we obtain:

$$\begin{aligned}
\sum_{T\subseteq N}(-1)^{|T|}R(0_T,s_{-T}) &= R(s_1,s_2,s_3) - R(0_1,s_{-1}) - R(0_2,s_{-2}) - R(0_3,s_{-3}) + R(0_{12},s_{-12}) \\
&\quad + R(0_{13},s_{-13}) + R(0_{23},s_{-23}) - R(0_{123},s_{-123}) \\
&= R(s_1,s_2,s_3) - R(0,s_2,s_3) - R(s_1,0,s_3) - R(s_1,s_2,0) + R(0,0,s_3) \\
&\quad + R(0,s_2,0) + R(s_1,0,0) - R(0,0,0) \\
&= \frac{1}{3}(s_2 + s_2s_3) + \frac{1}{3}(s_3 + s_2s_3) + \frac{1}{3}(s_3 + s_1s_3) \\
&\quad - \frac{1}{3}(s_3) - \frac{1}{3}(s_3) - \frac{1}{3}(s_2) = \frac{1}{3}(2s_2s_3 + s_1s_3) \neq 0
\end{aligned}$$

Once again, the mechanism is not residually balanced. So, Theorem 3.2 implies that π is not budget-balanced.

3.5 Valuation ranking mechanisms

In a valuation ranking or v-ranking mechanism, the allocation probabilities are determined by the ranks of the valuations of the agents. In other words, agents announce their signals which are transformed by the designer using the valuation functions. These computed valuations are then ranked and allocation probabilities determined according to these ranks. We define v-ranking mechanisms below - the definition is identical to that of s-ranking mechanisms except that signals are replaced by valuations.

Let $v = (v_1, v_2, \ldots v_n)$ be an n-tuple of valuations. Let $v[1]$ be the set of agents with the highest valuations, $v[2]$ the set of agents with the second-highest valuation and so on. Formally,

$$v[1] = \{i \in N | v_i(s) \geq v_j(s) \ \forall j \in N\}$$

and

$$v[k] = \{i \in N \setminus (\bigcup_{k'=1}^{k-1} v[k']) : v_i(s) \geq v_j(s) \ \forall j \in N \setminus (\bigcup_{k'=1}^{k-1} v[k'])\}$$

Let L be the greatest integer such that $v[L] \neq \emptyset$. Clearly $1 \leq L \leq n$. Let $|v[k]|$, $k \in \{1, \ldots, L\}$ denote the cardinality of the set $v[k]$, i.e. it is the number of agents whose signals are ranked k. Note $\sum_{k=1}^{L} |v[k]| = n$. Let $|[v]| = (|v[1]|, |v[2]|, \ldots, |v[L]|)$.

Let $\rho = (\rho_1, \rho_2, \ldots, \rho_n)$ be an n-tuple of real numbers such that $1 \geq \rho_1 \geq \rho_2 \geq \ldots \geq \rho_n \geq 0$. Let $\alpha = (\alpha_1, \ldots, \alpha_R)$ be a vector of R strictly positive integers such that $\sum_{r=1}^{R} \alpha_r = n$. For all $r = 1, \ldots, R$, let A_r denote the partial sum $\alpha_1 + \alpha_2 + \ldots + \alpha_r$. Finally, let $\langle \rho(\alpha) \rangle$ be a R-dimensional vector whose r^{th} component is $\langle \rho(\alpha) \rangle_r = \sum\limits_{k=A_{r-1}+1}^{A_r} \rho_k$ for $r \geq 2$ and $\langle \rho(\alpha) \rangle_{r=1} = \sum\limits_{k=1}^{A_1} \rho_k$. In other words, the first component of $\langle \rho(\alpha) \rangle$ is the sum of the first α_1 terms of ρ, the second component is the sum of the next α_2 terms of ρ and so on.

Definition **3.3** *An allocation rule f is a valuation-ranking (v-ranking) allocation rule if there exists $\rho = (\rho_1, \rho_2, \ldots, \rho_n)$ such that $1 \geq \rho_1 \geq \rho_2 \geq \ldots \geq \rho_n \geq 0$ and $\sum\limits_{i \in N} \rho_i = 1$ such that for all s and $k \in \{1, \ldots, L\}$ we have:*

(i) $\sum\limits_{i \in v[k]} f_i(s) = \langle \rho(|[v]|) \rangle_k$, *and*

(ii) $f_i(s) = f_j(s)$ *whenever* $i, j \in v[k]$.

A mechanism (f, p) is a v-ranking mechanism if f is a v-ranking allocation rule.

Like an s-ranking mechanism, a valuation ranking mechanism is a n-tuple of non-negative real numbers adding to one. The efficient allocation rule is an example of v-ranking allocation rule where the allocation probabilities are $\rho = (1, 0, 0, \ldots, 0)$. We shall say that a v-mechanism is *trivial* if $\rho = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$. A v-mechanism is non-trivial if it is not trivial, i.e. there exists $i \in \{1, \ldots, n-1\}$ such that $\rho_i > \rho_{i+1}$.

In order to address the question of v-ranking mechanisms that are EPIC , BB and EPIR, we introduce the following property of valuation functions.

DEFINITION **3.4** *The valuation functions $v_i : [0,1]^n \to \mathbb{R}_+$, $i \in N$ satisfy single-crossing if for every $i, j \in N$, every $s_{-i} \in S_{-i}$ and every $s_i > s_i'$,*

$$v_i(s_i, s_{-i}) - v_i(s_i', s_{-i}) > v_j(s_i, s_{-i}) - v_j(s_i', s_{-i})$$

The single-crossing condition is a familiar condition in auction theory with interdependent values (see Perry and Reny (1999), Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001) and Bergemann and Morris (2005)). It requires increases in the signal value of agent i to affect the valuation of i more than that of any other agent j.

Our main result in this section is the following:

THEOREM **3.3** *(i) If a non-trivial v-ranking mechanism is EPIC, the valuation functions must satisfy single-crossing.*

(ii) Suppose the valuation functions satisfy the SAS condition and single-crossing. Assume further that the v-ranking allocation rule $\rho = (\rho_1, \rho_2, \ldots, \rho_n)$ satisfies the equation

$$\sum_{j \in N} (-1)^j \binom{n-1}{j-1} \rho_j = 0. \tag{3.4}$$

there exist payment functions $p = (p_1, p_2, \ldots, p_n)$ such that the v-ranking mechanism (ρ, p) is EPIC, BB and EPIR.

The proof of the Theorem is contained in the Appendix. Part (i) is a consequence of the fact that signal monotonicity is a sufficient condition for monotonicity of the allocation function (see the proof of Theorem 3.1). If single-crossing is violated, an increase in the signal of agent i may lead to a greater change in the valuation of agent j than of i, causing a reversal in the valuation rankings of i and j. This would lead to a failure of EPIC. Part (ii) follows from a suitable application of Theorem 3.2. We note that SAS valuation functions satisfy single-crossing if they satisfy the additional requirement that $g_i(x') - g_i(x) > h(x') - h(x)$ for all $x' > x$ and $x', x \in S$, $i \in \{1, 2, \ldots, n\}$.

We illustrate the v-ranking mechanism through an example. Consider the setting described in Example 3.1 of previous section. Let $\rho = (\frac{2}{3}, \frac{1}{3}, 0)$. Pick an arbitrary signal profile s. At this signal profile let $v_{[1]}(s) > v_{[2]}(s) > v_{[3]}(s)$, i.e. $[i]$ is the agent with the i^{th} ranked valuation. The corresponding signals are $(s_{[1]}, s_{[2]}, s_{[3]})$. Payments at this profile s are given by:

$$p_{[i]}(s) = \begin{cases} \frac{1}{3}g_{[1]}(\kappa_{12}(s_{[2]})) + \frac{1}{3}g_{[1]}(\kappa_{13}(s_{[3]})) - \frac{1}{3}g_{[2]}(\kappa_{23}(s_{[3]})) + \frac{1}{2}h(s_{[2]}) + \frac{1}{6}h(s_{[3]}) & \text{, if } i = 1 \\ \frac{1}{3}g_{[2]}(\kappa_{23}(s_{[3]})) - \frac{1}{3}g_{[1]}(\kappa_{13}(s_{[3]})) + \frac{1}{6}h(s_{[1]}) - \frac{1}{6}h(s_{[3]}) & \text{, if } i = 2 \\ -\frac{1}{3}g_{[1]}(\kappa_{12}(s_{[2]})) - \frac{1}{6}h(s_{[1]}) - \frac{1}{2}h(s_{[2]}) & \text{, if } i = 3 \end{cases}$$

where,

$$\kappa_{ij}(s_{[j]}) = \inf\{s_{[i]} \in S \mid g_{[i]}(s_{[i]}) - h(s_{[i]}) \geq g_{[j]}(s_{[j]}) - h(s_{[j]})\}$$

The function $\kappa_{ij}(\cdot)$ (where $i < j$) is the minimum value of $s_{[i]}$ at which the valuation of agent ranked $[i]$ is equal to the valuation of agent ranked $[j]$. For valuation functions that satisfy SAS condition, there exists $\hat{s}_{[i]} \in S$ such that

$$v_{[i]}(\hat{s}_{[i]}, s_{-[i]}) = v_{[j]}(\hat{s}_{[i]}, s_{-[i]}) \Longrightarrow g_{[i]}(\hat{s}_{[i]}) - h(\hat{s}_{[i]}) = g_{[j]}(s_{[j]}) - h(s_{[j]})$$

Since $g_{[i]}(0) = g_{[j]}(0) = h(0) = 0$ and the functions $g_{[i]}$ and h are continuous and monotonic, the Intermediate Value Theorem guarantees the existence of $\hat{s}_{[i]} \in (0, s_{[i]})$ which satisfies this equation.

It can be easily verified that $\sum\limits_{i \in N} p_{[i]}(s) = 0$. As in the s-ranking mechanism case, the budget-balance property depends on the function h being the same for all agents. To verify EPIC, consider agent 1. Pick a signal profile s such that $v_2(s) > v_3(s) > v_1(s)$. Truth-telling by agent 1 gives her the object with zero probability while she receives a payment of $\frac{1}{3}g_2(\kappa_{23}(s_3)) + \frac{1}{6}h(s_2) + \frac{1}{2}h(s_3)$. Suppose 1 announces s_1' such that $s_1' > s_1$ and let $v_1(s_1', s_{-1}) > v_2(s_1', s_{-1}) > v_3(s_1', s_{-1})$ at this new signal profile (s_1', s_2, s_3). She now receives the object with probability $\frac{2}{3}$ and has to pay $\frac{1}{3}g_1(\kappa_{12}(s_2)) + \frac{1}{3}g_1(\kappa_{13}(s_3)) - \frac{1}{3}g_2(\kappa_{23}(s_3)) + \frac{1}{2}h(s_2) + \frac{1}{6}h(s_3)$. The net change in payoff from misrepresentation is $\frac{2}{3}g_1(s_1) - \frac{1}{3}(g_1(\kappa_{12}(s_2)) + g_1(\kappa_{13}(s_3))) = \frac{1}{3}(g_1(s_1) - (g_1(\kappa_{12}(s_2))) + \frac{1}{3}(g_1(s_1) - g_1(\kappa_{13}(s_3))) < 0$ where the last inequality follows from the fact that g_1 is an increasing function and $\kappa_{12}(s_2) > s_1$ and $\kappa_{13}(s_3) > s_1$. If the valuations are ranked as $v_2(s_1', s_{-1}) > v_1(s_1', s_{-1}) > v_3(s_1', s_{-1})$ at this new signal profile (s_1', s_2, s_3), then agent 1 is allocated the object with probability $\frac{1}{3}$ and pays $\frac{1}{3}g_1(\kappa_{13}(s_3)) - \frac{1}{3}g_2(\kappa_{23}(s_3)) + \frac{1}{6}h(s_2) - \frac{1}{6}h(s_3)$. The net change in payoff from misrepresentation is $\frac{1}{3}(g_1(s_1) - g_1(\kappa_{13}(s_3))) < 0$

where the inequality follows because $\kappa_{13}(s_3) > s_1$. It can similarly be verified that no misrepresentation is profitable for any other agent.

Under certain assumptions on valuation functions an s-ranking allocation rule and a "corresponding" v-ranking allocation rule are "equivalent".

DEFINITION **3.5** *The valuation functions $v_i : [0,1]^n \to \mathbb{R}_+$, $i \in N$ satisfy symmetry if for every $i \in N$, for any permutation $\sigma : N \to N$ and all signal profiles $s = (s_1, s_2, \ldots, s_n)$, we have*

$$v_i(s_1, s_2, \ldots, s_n) = v_{\sigma(i)}(s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(n)})$$

If the valuation functions are symmetric, the valuation of an agent i at signal profile $s = (s_1, s_2, \ldots, s_n)$ is equal to the valuation of agent $\sigma(i)$ at signal profile $(s_{\sigma(1)}, s_{\sigma(2)}, \ldots, s_{\sigma(n)})$ for any permutation of the agents. For example if the permutation interchanges two agents, say i and j leaving others unchanged, then symmetry implies $v_i(s_1, \ldots, s_i, \ldots, s_j, \ldots, s_n) = v_j(s_1, \ldots, s_j, \ldots, s_i, \ldots, s_n)$ and $v_k(s_1, \ldots, s_i, \ldots, s_j, \ldots, s_n) = v_k(s_1, \ldots, s_j, \ldots, s_i, \ldots, s_n)$ for all $k \neq i, j$.

The examples below show that the two conditions viz. single-crossing and symmetry are independent. In each case $N = \{1, 2, 3\}$.

EXAMPLE **3.5** The valuation functions are $v_1(s) = s_1 + 0.2(s_2 + s_3)$, $v_2(s) = s_2 + 0.2(s_1 + s_3)$ and $v_3(s) = s_3 + 0.2(s_1 + s_2)$. Both single-crossing and symmetry are satisfied.

EXAMPLE **3.6** The valuation functions are $v_1(s) = s_1 + s_2 s_3$, $v_2(s) = s_2 + 0.5 s_1$ and $v_3(s) = s_3$. They satisfy single-crossing condition but not satisfy symmetry.

EXAMPLE **3.7** The valuation functions are $v_1(s) = s_1 + 2(s_2 + s_3)$, $v_2(s) = s_2 + 2(s_1 + s_3)$ and $v_3(s) = s_3 + 2(s_1 + s_2)$. Symmetry is satisfied but not single-crossing.

EXAMPLE **3.8** The valuation functions are $v_1(s) = s_1 + 2s_2 + 3s_3$, $v_2(s) = s_2 + 3s_1 + 5s_3$ and $v_3(s) = s_3 + 7s_1 + 4s_2$. Neither single-crossing condition nor symmetry are satisfied.

We say that two allocation rules are *allocation equivalent* if they result in the same allocation at every signal profile. Note that both s-ranking and v-ranking allocation rules are n-tuples of decreasing real numbers adding up to one. Hence an s-ranking allocation rule can be interpreted as a v-ranking allocation rule and vice-versa.

Proposition 3.1 *Assume that the valuation functions satisfy single-crossing and symmetry. The s-ranking allocation rule π and the v-ranking allocation rule ρ are allocation equivalent.*

Proposition 3.1 in conjunction with our earlier results leads to the following result.

Proposition 3.2 *Assume that the valuation functions satisfy the SAS condition, single-crossing and symmetry. Assume further that the s-ranking allocation rule π satisfies Condition 3.1. Let p be a payment function such that the s-ranking mechanism (π, p) is EPIC, BB and EPIR (Theorem 3.1). Then the v-ranking mechanism (π, p) is EPIC, BB and EPIR. Moreover the two mechanisms give all agents the same utility at every signal profile.*

The proofs of both Propositions are in the Appendix.

3.6 Further remarks on ranking mechanisms

In this section, we show that a natural adaptation of the Green-Laffont mechanism to the interdependent value setting does not satisfy ex-post incentive-compatibility. We also analyze s-ranking mechanisms from the perspective of efficiency in a special case.

3.6.1 The Green-Laffont mechanism

Recall that in the private values model, the Green-Laffont mechanism picks an agent uniformly at random from the set of agents. The object is allocated using the Vickrey auction among rest of the agents. The revenue that is generated is given to the agent that was removed. The mechanism allocates the object to the agents with the highest and second-highest valuations with probability $1 - \frac{1}{n}$ and $\frac{1}{n}$ respectively, at every valuation profile.

Consider the interdependent values model where valuation functions satisfy the SAS condition, single-crossing and symmetry, i.e. $v_i(s) = g(s_i) + \sum\limits_{j \neq i} h(s_j)$ where g and h are increasing functions and $g'(x) > h'(x)$ for all x. In addition, suppose the Vickrey auction on the private values case is replaced by the generalized Vickrey auction (see Krishna (2009) for a detailed discussion on this mechanism) whose allocation rules and payment functions (f, p) are as follows:

$$f_i(s) = \begin{cases} 1 & \text{, if } v_i(s) \geq \max\limits_{j \neq i} v_j(s) \\ 0 & \text{, otherwise} \end{cases}$$

and

$$p_i(s) = \begin{cases} v_i(\kappa(s_{-i}), s_{-i}) & , \text{ if } f_i(s) = 1 \\ 0 & , \text{ if } f_i(s) = 0 \end{cases}$$

where $\kappa(s_{-i}) = \inf \{s' \in S | v_i(s', s_{-i}) \geq \max_{j \neq i} v_j(s', s_{-i})\}$.

It is well-known that the assumptions on valuation functions ensure that the generalized Vickrey auction among the "remaining" $n-1$ agents, is EPIC[‡]. However, the mechanism as a whole is not EPIC. In order to see this consider the case where $N = \{1, 2, 3\}$ and the signal profile s satisfies $s_1 > s_2 > s_3$. In view of our assumption on valuation functions, we have $v_1(s) > v_2(s) > v_3(s)$. If agent 1 is removed, the relative ranking of the other agents doesn't change. The object is allocated to 2 who pays $v_2(s_1, s_3, s_3)$ to agent 1. Similarly, if agent 2 is removed, agent 1 is allocated the object and pays $v_1(s_2, s_2, s_3)$ to agent 2. If agent 3 is removed, agent 1 receives the object and pays $v_1(s_3, s_2, s_3)$ to agent 3. The final payments are as follows:

$$\begin{aligned} p_1(s) &= \frac{1}{3}(v_1(s_2, s_2, s_3) + v_1(s_3, s_2, s_3) - v_2(s_1, s_3, s_3)) \\ p_2(s) &= \frac{1}{3}(v_2(s_1, s_3, s_3) - v_1(s_3, s_2, s_3)) \\ p_3(s) &= -\frac{1}{3}v_1(s_2, s_2, s_3) \end{aligned}$$

If agent 3 reports s_3' where $s_1 > s_2 > s_3' > s_3$, he receives a payoff of $\frac{1}{3}v_1(s_2, s_2, s_3')$. This is higher than the payoff of $\frac{1}{3}v_1(s_2, s_2, s_3)$ received when he reports his signal truthfully since $h(\cdot)$ is increasing. Clearly the mechanism is not EPIC. Note however that the s-ranking mechanism $\pi = (1 - \frac{1}{n}, \frac{1}{n}, 0, \ldots, 0)$ satisfies Condition 3.1, and can therefore be implemented by a mechanism that is EPIC, budget balanced and EPIR. Moreover the mechanism is equivalent to the v-ranking mechanism $\rho = (1 - \frac{1}{n}, \frac{1}{n}, 0, \ldots, 0)$.

3.6.2 Efficiency comparisons

In this section, we focus on the welfare properties of s-ranking mechanisms. Pick an arbitrary signal profile s with $s_{(1)} \geq s_{(2)} \geq \ldots \geq s_{(n)}$ i.e. $s_{(1)}$ is the highest-ranked signal and $s_{(n)}$ is the lowest-ranked signal. Corresponding to this ranking of signals the valuations of agents are $(v_{(1)}(s), v_{(2)}(s), \ldots, v_{(n)}(s))$. The actual ranking of valuations at signal profile s satisfies $v_{[1]}(s) \geq v_{[2]}(s) \geq \ldots \geq v_{[n]}(s)$ i.e. $v_{[1]}(s)$ is the highest-ranked valuation and $v_{[n]}(s)$ is

[‡]See Krishna (2009) where it is shown that single-crossing condition is sufficient for ex-post incentive compatibility of generalized Vickrey auction

the lowest-ranked valuation. In general, $v_{[i]}(s) \neq v_{(i)}(s)$; however if the valuation functions satisfy single-crossing condition and symmetry $v_{[i]}(s) = v_{(i)}(s)$ for all $i \in \{1, 2, \ldots, n\}$ and all signal profiles s.

The maximum possible welfare at any signal profile s is generated when the object is allocated with probability one to the agent with highest valuation at s i.e. the maximum welfare is $v_{[1]}(s)$. The *efficiency ratio* of a s-ranking mechanism at signal profile s is

$$\frac{\sum_{i \in N} f_{(i)}(s) v_{(i)}(s)}{v_{[1]}(s)} = \frac{\pi_1 v_{(1)}(s) + \pi_2 v_{(2)}(s) + \ldots + \pi_n v_{(n)}(s)}{v_{[1]}(s)}$$

This is the ratio of the welfare achieved by the mechanism achieved by s-ranking mechanism and the maximum possible welfare at the signal profile s. Notice that we are assuming that mechanisms satisfy the BB property so that there is no loss of welfare because money has to be "thrown away". The *worst-case efficiency ratio* μ for the s-ranking mechanism is the lowest possible value of efficiency ratio across all signal profiles, i.e.

$$\mu = \inf_{s \in S^n} \frac{\sum_{i \in N} f_{(i)}(s) v_{(i)}(s)}{v_{[1]}(s)}$$

Our objective is to find a mechanism that maximizes the value of μ in the class of EPIC, EPIR and BB s-ranking mechanisms. This problem does not appear to be easy when valuation functions belong to the general SAS class. We show that if we restrict attention to a symmetric subclass of SAS valuation functions, the optimal s-ranking mechanism coincides with the *optimal ranking mechanism* of Long et al. (2017). The latter mechanism maximizes μ in the class of dominant strategy incentive-compatible and BB ranking mechanisms in the private-value setting. We also show through examples that the optimal mechanism may be different when the valuation functions are not symmetric.

The allocation rule of the optimal ranking mechanism of Long et al. (2017), $\pi^* = (\pi_1, \ldots, \pi_m, \ldots, \pi_n)$ is defined as follows:

$$\pi_i^* = \begin{cases} 1 - \frac{m-1}{C(n-2,m-1)+m} & \text{, if } i = 1 \\ \frac{1}{C(n-2,m-1)+m} & \text{, if } i \in \{2, 3, \ldots, m\} \\ 0 & \text{, otherwise} \end{cases}$$

where,

$$m \in \arg \min_{2 \leq i \leq (n-1), i \ \mathit{even}} \frac{i-1}{C(n-2, i-1) + i}$$

The object is allocated only to the agents with the m highest signals. For $n \leq 8$, $m = 2$. This corresponds to the Green-Laffont allocation rule in which the object is allocated only to the agents with the highest and second-highest valuations.

We now state the main result.

Theorem **3.4** *Suppose the valuation functions satisfy SAS condition. In addition, assume $g_i(x) = \gamma h(x)$ for all $i \in \{1, 2, \ldots, n\}$ for some $\gamma > 1$, Then the s-ranking mechanism with allocation rule π^* maximizes the worst-case efficiency ratio among all s-ranking mechanisms that satisfy EPIC, EPIR and BB.*

The proof is in the Appendix. Proposition 3.2 applies since valuation functions satisfy symmetry and single-crossing. As a result, the problem of finding an optimal s-ranking mechanism reduces to the problem solved by Long et al. (2017) in the private-value setting.

The next example demonstrates that the s-ranking mechanism π^* may not be optimal if the valuation functions do not satisfy symmetry.

Example **3.9** Let $N = \{1, 2, 3\}$. The valuation functions are $v_1(s) = 100s_1 + s_2 + s_3$, $v_2(s) = 2s_2 + s_1 + s_3$ and $v_3(s) = 2s_3 + s_1 + s_2$. The valuation functions satisfy SAS condition and single-crossing but violate symmetry. Consider an arbitrary s-ranking allocation $\pi = (\pi_1, \pi_2, \pi_3)$. Suppose $s_2 \geq s_1 \geq s_3$. So the ranking of valuation can either be $v_1(s) \geq v_2(s) \geq v_3(s)$ if $99s_1 \geq s_2 \geq s_3$ or $v_2(s) \geq v_1(s) \geq v_3(s)$ if $s_2 \geq 99s_1 \geq s_3$. The efficiency ratios are respectively,

$$\frac{\pi_1 v_2(s) + \pi_2 v_1(s) + \pi_3 v_3(s)}{v_1(s)} \text{ and } \frac{\pi_1 v_2(s) + \pi_2 v_1(s) + \pi_3 v_3(s)}{v_2(s)}$$

Combining all the cases, the worst-case efficiency of the mechanism is:

$$\begin{aligned}
\mu &= \inf_{(s_1,s_2,s_3)\in S^3} \frac{\sum\limits_{i\in N} f_i(s) v_{[i]}(s)}{\max\limits_{i\in N} \{v_i(s)\}} \\
&= \inf \Bigg\{ \Bigg(\frac{\pi_1 v_3(s) + \pi_2 v_2(s) + \pi_3 v_1(s)}{v_3(s)} \Bigg| s_3 \geq s_2 \geq s_1, s_3 \geq s_2 \geq 99s_1 \Bigg) \\
&\cup \Bigg(\frac{\pi_1 v_3(s) + \pi_2 v_2(s) + \pi_3 v_1(s)}{v_1(s)} \Bigg| s_3 \geq s_2 \geq s_1, 99s_1 \geq s_3 \geq s_2 \Bigg) \\
&\cup \Bigg(\frac{\pi_1 v_3(s) + \pi_2 v_2(s) + \pi_3 v_1(s)}{v_3(s)} \Bigg| s_3 \geq s_2 \geq s_1, s_3 \geq 99s_1 \geq s_2 \Bigg) \\
&\cup \Bigg(\frac{\pi_1 v_3(s) + \pi_2 v_1(s) + \pi_3 v_2(s)}{v_3(s)} \Bigg| s_3 \geq s_1 \geq s_2, s_3 \geq 99s_1 \geq s_2 \Bigg)
\end{aligned}$$

$$
\begin{aligned}
&\cup \Big(\frac{\pi_1 v_3(s)+\pi_2 v_1(s)+\pi_3 v_2(s)}{v_1(s)}\Big| s_3 \geq s_1 \geq s_2, 99s_1 \geq s_3 \geq s_2\Big)\\
&\cup \Big(\frac{\pi_1 v_1(s)+\pi_2 v_2(s)+\pi_3 v_3(s)}{v_1(s)}\Big| s_1 \geq s_2 \geq s_3, 99s_1 \geq s_2 \geq s_3\Big)\\
&\cup \Big(\frac{\pi_1 v_1(s)+\pi_2 v_3(s)+\pi_3 v_2(s)}{v_1(s)}\Big| s_1 \geq s_3 \geq s_2, 99s_1 \geq s_3 \geq s_2\Big)\\
&\cup \Big(\frac{\pi_1 v_2(s)+\pi_2 v_3(s)+\pi_3 v_1(s)}{v_2(s)}\Big| s_2 \geq s_3 \geq s_1, s_2 \geq s_3 \geq 99s_1\Big)\\
&\cup \Big(\frac{\pi_1 v_2(s)+\pi_2 v_3(s)+\pi_3 v_1(s)}{v_1(s)}\Big| s_2 \geq s_3 \geq s_1, 99s_1 \geq s_2 \geq s_3\Big)\\
&\cup \Big(\frac{\pi_1 v_2(s)+\pi_2 v_3(s)+\pi_3 v_1(s)}{v_2(s)}\Big| s_2 \geq s_3 \geq s_1, s_2 \geq 99s_1 \geq s_3\Big)\\
&\cup \Big(\frac{\pi_1 v_2(s)+\pi_2 v_1(s)+\pi_3 v_3(s)}{v_2(s)}\Big| s_2 \geq s_1 \geq s_3, s_2 \geq 99s_1 \geq s_3\Big)\\
&\cup \Big(\frac{\pi_1 v_2(s)+\pi_2 v_1(s)+\pi_3 v_3(s)}{v_1(s)}\Big| s_2 \geq s_1 \geq s_3, 99s_1 \geq s_2 \geq s_3\Big)\Big\}\\
&= \inf\Big\{\frac{1}{2}+\frac{1}{2}\pi_1, \frac{1}{100}+\frac{99}{100}\pi_1, \frac{105}{303}+\frac{1}{101}\pi_1, \frac{106}{153}-\frac{49}{51}\pi_1\Big\} \qquad (3.5)
\end{aligned}
$$

The last equality is obtained by finding the minimum value of each of the terms in the parentheses according to the respective constraints on s_1, s_2 and s_3. We also know that from Condition (3.1), $-\pi_1 + 2\pi_2 - \pi_3 = 0$. So, $\pi_2 = \frac{1}{3}$ and $\pi_3 = \frac{2}{3} - \pi_1$. By substituting these, we get the expression in (3.5). Notice that this expression is a function of π_1 where $\pi_1 \in [\frac{1}{3}, \frac{2}{3}]$. The maximum value of μ is the maxima of this function which occurs at $\pi_1 = \frac{5351}{15000} \approx 0.36$ and it is $\mu = 0.35$. Also, $\mu_{\pi^*} = 0.05$. The mechanism $\pi = (0.36, 0.33, 0.31)$ has a higher μ than the mechanism π^*. Hence, the mechanism of Long et al. (2017) does not maximize the worst-case efficiency ratio in the interdependent-value model.

3.7 Probability-burning mechanisms

In the s-ranking and v-ranking mechanisms, the object can be allocated to agents which do not have the highest-ranking signal. In this section, we consider the case when the object is allocated only to the agent with the highest signal with certain probability and if the allocation does not take place then either rest of the probability is burnt or the object is destroyed. Such a mechanism is called a probability-burning mechanism. Mishra and Sharma (2018) introduced these mechanisms in the private-value setting and studied the allocation of a single object. We now study these mechanisms in our model of agents having interdependent valuations.

We begin this section by defining the probability-burning mechanism. When agents' valuations satisfy a special condition, we describe a probability-burning mechanism and show that it is EPIC, EPIR and BB. In a restricted setting with 3 agents we find the probability-burning mechanism that is welfare-maximizing in the class of EPIC, EPIR and BB mechanisms which allocate the object only to the agent with the highest signal.

A *probability-burning allocation function* f satisfies the following properties: for all signal profiles s,

(i) $f_i(s) = 0$ for all $i \in s[k]$, $k \in \{2, 3, \dots, L\}$

(ii) $\sum\limits_{i \in N} f_i(s) \leq 1$.

A probability-burning mechanism is a pair (f, p) where f is a probability-burning allocation function. Note that a probability-burning allocation function assigns object with positive probability only to the agent who has the highest signal. However, the object may not be allocated with probability one. It allows for the possibility that at some signal profile s, $\sum\limits_{i \in N} f_i(s) < 1$, i.e. object is wasted or probability is "burnt". This is a violation of efficiency and will occur when probability-burning mechanisms are required to additionally satisfy incentive-compatibility and budget-balance.

The mechanism we will now describe depends on valuation functions which follow a special structure.

Definition **3.6** *The valuation functions $v_i : S^n \to \mathbb{R}_+$, $i = \{1, \dots, n\}$ satisfy the Additive Semi-separability (AS) condition if there exists increasing functions $g : S^{n-1} \to \mathbb{R}_+$ and $h : S^{n-1} \to \mathbb{R}_+$ which are weakly increasing functions with $g(x_1, x_2, \dots, x_{n-1}) \geq h(x_1, x_2, \dots, x_{n-1})$ $\forall (x_1, x_2, \dots, x_{n-1}) \in S^{n-1}$ such that, for all $s \in S^n$ and $i \in \{1, \dots, n\}$, we have $v_i(s) = g(s_{-i}) + \sum\limits_{j \neq i} h(s_{-j})$.*

For each valuation function, the g function leaves out the agent's own signal in its $n-1$ arguments and each of the h functions leaves out one of the $n-1$ signals of other agents. Both the g and the h functions are common to all agents. The valuation function v_i is the sum of g function and $(n-1)$ number of h functions (s_i is an argument in each of the h functions). Hence, each valuation function v_i is a function of its own signal s_i in general, but it may happen that for some specific functions g and h the valuation functions of some agents may not depend on their own signals. The following examples elucidate the AS condition.

Example **3.10** Let $N = \{1, 2, 3\}$. Let $g(x, y) = \max\{x, y\}$ and $h(x, y) = x$. The valuation functions are

$$\begin{aligned} v_1(s_1, s_2, s_3) &= g(s_2, s_3) + h(s_1, s_2) + h(s_1, s_3) \\ &= \max\{s_2, s_3\} + s_1 + s_1 = \max\{s_2, s_3\} + 2s_1 \\ v_2(s_1, s_2, s_3) &= g(s_1, s_3) + h(s_1, s_2) + h(s_2, s_3) = \max\{s_1, s_3\} + s_1 + s_2 \\ v_3(s_1, s_2, s_3) &= g(s_1, s_2) + h(s_1, s_3) + h(s_2, s_3) = \max\{s_1, s_2\} + s_1 + s_2 \end{aligned}$$

Example **3.11** Let $N = \{1, 2, 3\}$. Let $g(x, y) = x$ and $h(x, y) = xy$. The valuation functions are $v_1(s) = s_2 + s_2 s_1 + s_1 s_3$, $v_2(s) = s_1 + s_1 s_2 + s_2 s_3$ and $v_3(s) = s_1 + s_1 s_3 + s_2 s_3$.

Example **3.12** Let $N = \{1, 2, 3, 4\}$. Let $g(x, y, z) = xy + z$ and $h(x, y, z) = z$. The valuation functions are

$$\begin{aligned} v_1(s_1, s_2, s_3, s_4) &= g(s_2, s_3, s_4) + h(s_1, s_2, s_3) + h(s_1, s_2, s_4) + h(s_1, s_3, s_4) \\ &= s_2 s_3 + s_4 + s_3 + s_4 + s_4 = s_2 s_3 + s_3 + 3s_4 \\ v_2(s_1, s_2, s_3, s_4) &= g(s_1, s_3, s_4) + h(s_1, s_2, s_3) + h(s_1, s_2, s_4) + h(s_2, s_3, s_4) \\ &= s_1 s_3 + s_4 + s_3 + s_4 + s_4 = s_1 s_3 + s_3 + 3s_4 \\ v_3(s_1, s_2, s_3, s_4) &= g(s_1, s_2, s_4) + h(s_1, s_2, s_3) + h(s_1, s_3, s_4) + h(s_2, s_3, s_4) \\ &= s_1 s_2 + s_4 + s_3 + s_4 + s_4 = s_1 s_2 + s_3 + 3s_4 \\ v_4(s_1, s_2, s_3, s_4) &= g(s_1, s_2, s_3) + h(s_1, s_2, s_4) + h(s_2, s_3, s_4) + h(s_1, s_3, s_4) \\ &= s_1 s_2 + s_3 + s_4 + s_4 + s_4 = s_1 s_2 + s_3 + 3s_4 \end{aligned}$$

We now describe the mechanism M^{pb}. Let there be a signal profile s such that $s_1 \geq s_2 \geq \ldots \geq s_n$. The allocation functions of agents are

$$f_i(s) = \begin{cases} \frac{1}{|s[1]|} \left(\frac{(n-1)h(s_n, s_n, \ldots, s_n) + h(s_{n-1}, s_{n-1}, \ldots, s_{n-1})}{g(s_2, s_3, \ldots, s_n) + h(s_2, s_3, \ldots, s_n) + \sum\limits_{j \neq 1,2} h(s_2, s_2, s_3, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n)} \right) & \text{, if } i \in s[1] \\ 0 & \text{, otherwise} \end{cases}$$

The payment function of each agent is given by the Revenue equivalence formula

$$p_i(s) = p_i(0, s_{-i}) + v_i(s) f_i(s) - \int_0^{s_i} f_i(x, s_{-i}) \frac{\partial v_i(x, s_{-i})}{\partial s_i} dx \tag{3.6}$$

where,

$$p_i(0, s_{-i}) = \begin{cases} -h(s_n, s_n, \ldots, s_n) & \text{, if } i \in \{1, 2, 3, \ldots, n-1\} \\ -h(s_{n-1}, s_{n-1}, \ldots, s_{n-1}) & \text{, if } i = n \end{cases}$$

Here, $h(s_n, s_n, \ldots, s_n)$ is the value of function $h(\cdot)$ at signal profile $(s_n, s_n, \ldots, s_n)$ and $h(s_{n-1}, s_{n-1}, \ldots, s_{n-1})$ is the value at the signal profile $(s_{n-1}, s_{n-1}, \ldots, s_{n-1})$. For the signal profile s such that $s_1 \geq s_2 \geq \ldots \geq s_n$, the denominator in the expression of $f_i(s)$ is the valuation of agent 1 computed at the signal profile $(s_2, s_2, s_3, s_4, \ldots, s_n)$ where $s_1 = s_2$ i.e. the minimum value of s_1 at which the object is allocated to agent 1. As in the probability-burning mechanism of Mishra and Sharma (2018), the redistribution amounts of all agents except the agent with lowest-ranked signal depend only on the value of lowest-ranked signal. The redistribution amount of agent with lowest-ranked signal depends only on the value of signal of second-lowest ranked agent.

We illustrate the mechanism M^{pb} by taking specific valuation functions. The allocation probabilities and payments of agents are computed at different signal profiles.

Example **3.13** Let $N = \{1, 2, 3\}$. Let signal profile be such that $s_3 > s_1 > s_2$. The mechanism M^{pb} is

$$f_3(s) = \frac{2h(s_2, s_2) + h(s_1, s_1)}{g(s_1, s_2) + h(s_1, s_1) + h(s_2, s_1)}$$

and $f_1(s) = f_2(s) = 0$. From the definition of AS condition, $g(s_1, s_2) > h(s_2, s_2)$ as g is increasing in all its arguments and $g(x, y) \geq h(x, y)$ for all $(x, y) \in S^2$. Also, $h(s_1, s_2) > h(s_2, s_2)$. So, $2h(s_2, s_2) + h(s_1, s_1) < g(s_1, s_2) + h(s_1, s_1) + h(s_1, s_2)$. The allocation probability $f_3(s)$ is feasible. Agent 3 pays $f_3(s)v_3(s_1, s_2, s_1)$ which is

$$\frac{2h(s_2, s_2) + h(s_1, s_1)}{g(s_1, s_2) + h(s_1, s_1) + h(s_2, s_1)}(g(s_1, s_2) + h(s_1, s_1) + h(s_2, s_1)) = 2h(s_2, s_2) + h(s_1, s_1)$$

The payments are $p_3(s) = h(s_2, s_2) + h(s_1, s_1)$, $p_1(s) = -h(s_2, s_2)$ and $p_2(s) = -h(s_1, s_1)$. Agent 3 pays $2h(s_2, s_2) + h(s_1, s_1)$ and receives a redistribution amount of $h(s_2, s_2)$. Agent 1 receives amount of $h(s_2, s_2)$ and agent 2 receives $h(s_1, s_1)$.

Example **3.14** Let $N = \{1, 2, 3\}$. Let $g(x, y) = \max\{x, y\}$ and $h(x, y) = x$. The valuation functions are $v_1(s) = \max\{s_2, s_3\} + 2s_1$, $v_2(s) = \max\{s_1, s_3\} + s_1 + s_2$ and $v_3(s) = \max\{s_1, s_2\} + s_1 + s_2$. Let signal profile be such that $s_2 > s_3 > s_1$. The allocation probabilities are

$$f_2(s) = \frac{2s_1 + s_3}{3s_3}$$

and $f_1(s) = f_3(s) = 0$. The payments are $p_2(s) = s_1 + s_3$, $p_3(s) = -s_1$ and $p_1(s) = -s_3$. Agent 2 pays $2s_1 + s_3$ and receives a redistribution transfer of s_1. Agents 3 and 1 receive redistribution amounts of s_1 and s_3, respectively.

EXAMPLE **3.15** Let $N = \{1, 2, 3\}$. Let $g(x, y) = x$ and $h(x, y) = xy$. The valuation functions are $v_1(s) = s_2 + s_2s_1 + s_1s_3$, $v_2(s) = s_1 + s_1s_2 + s_2s_3$ and $v_3(s) = s_1 + s_1s_3 + s_2s_3$. Let the signal profile be such that $s_1 \geq s_2 \geq s_3$. Following 4 cases are possible after resolving the tie-breaking between agents:

- $s_1 > s_2 > s_3$
  The allocation probabilities are
  $$f_1(s) = \frac{2s_3^2 + s_2^2}{s_2 + s_2^2 + s_2s_3}$$
  and $f_2(s) = f_3(s) = 0$. The payments are $p_1(s) = s_3^2+s_2^2$, $p_2(s) = -s_3^2$ and $p_3(s) = -s_2^2$. Agent 1 pays $2s_3^2+s_2^2$ and receives a redistribution amount of s_3^2. Agents 2 and 3 receive redistribution amount of s_3^2 and s_2^2.

- $s_1 = s_2 > s_3$
  The allocation probabilities are
  $$f_1(s) = f_2(s) = \frac{1}{2}\Big(\frac{2s_3^2 + s_2^2}{s_2 + s_2^2 + s_2s_3}\Big)$$
  and $f_3(s) = 0$. Agent 1 and 2 each pay $\frac{1}{2}(2s_3^2 + s_2^2)$. Agent 1 and 2 each receive s_3^2 and agent 3 receives s_2^2.

- $s_1 > s_2 = s_3$
  The allocation probabilities are
  $$f_1(s) = \frac{2s_3^2 + s_2^2}{s_2 + s_2^2 + s_2s_3} = \frac{3s_2}{2s_2 + 1}$$
  and $f_2(s) = f_3(s) = 0$. Agent 1 pays $3s_2^2$. Each agent receives an amount of s_2^2.

- $s_1 = s_2 = s_3$
  The allocation probabilities are
  $$f_1(s) = f_2(s) = f_3(s) = \frac{1}{3}\Big(\frac{2s_3^2 + s_2^2}{s_2 + s_2^2 + s_2s_3}\Big) = \frac{1}{3}\Big(\frac{3s_2}{2s_2 + 1}\Big)$$
  Agents 1, 2 and 3 each pay s_2^2. Also, each agent receives an amount of s_2^2. So, $p_1(s) = p_2(s) = p_3(s) = 0$.

In each of these 4 cases, some probability is burnt at all possible signal profiles. Also, the semi-separability condition on valuation functions plays an important role in maintaining budget-balance in all the examples.

Every agent gets non-negative utility in each of these examples, hence the mechanism is EPIR. It is also EPIC as we illustrate through one of the examples. Consider the valuation functions as in Example 3.13. Let $s_1 > s_2 > s_3$. The allocation probabilities are

$$f_1(s) = \frac{2h(s_3, s_3) + h(s_2, s_2)}{g(s_2, s_3) + h(s_2, s_2) + h(s_2, s_3)}$$

and $f_2(s) = f_3(s) = 0$. The payments are $p_1(s) = h(s_2, s_2) + h(s_3, s_3)$, $p_2(s) = -h(s_3, s_3)$ and $p_3(s) = -h(s_2, s_2)$. Suppose agent with signal s_3 misreports to s_3' such that $s_3' > s_1 > s_2$. His allocation probability is

$$f_3(s) = \frac{2h(s_2, s_2) + h(s_1, s_1)}{g(s_1, s_2) + h(s_1, s_1) + h(s_1, s_2)}$$

and $f_1(s) = f_2(s) = 0$. The payments are $p_3(s) = h(s_2, s_2) + h(s_1, s_1)$, $p_1(s) = -h(s_2, s_2)$ and $p_2(s) = -h(s_1, s_1)$. The utility of agent with signal s_3' is

$$\frac{2h(s_2, s_2) + h(s_1, s_1)}{g(s_1, s_2) + h(s_1, s_1) + h(s_1, s_2)}(g(s_1, s_2) + h(s_2, s_3) + h(s_1, s_3)) - (h(s_2, s_2) + h(s_1, s_1))$$

Net gain in utility of the agent is

$$\begin{aligned}
&\frac{2h(s_2, s_2) + h(s_1, s_1)}{g(s_1, s_2) + h(s_1, s_1) + h(s_1, s_2)}(g(s_1, s_2) + h(s_2, s_3) + h(s_1, s_3)) - (h(s_2, s_2) + h(s_1, s_1)) - h(s_2, s_2)\\
&= (2h(s_2, s_2) + h(s_1, s_1))\frac{g(s_1, s_2) + h(s_2, s_3) + h(s_1, s_3) - (g(s_1, s_2) + h(s_1, s_1) + h(s_1, s_2))}{g(s_1, s_2) + h(s_1, s_1) + h(s_1, s_2)}\\
&= (2h(s_2, s_2) + h(s_1, s_1))\frac{(h(s_2, s_3) - h(s_1, s_1)) + (h(s_1, s_3) - h(s_1, s_2))}{g(s_1, s_2) + h(s_1, s_1) + h(s_1, s_2)}
\end{aligned}$$

As h is increasing in both the arguments, $h(s_2, s_3) < h(s_1, s_1)$ and $h(s_1, s_3) < h(s_1, s_2)$. The expression in the numerator is negative. This means

$$(2h(s_2, s_2) + h(s_1, s_1))\frac{(h(s_2, s_3) - h(s_1, s_1)) + (h(s_1, s_3) - h(s_1, s_2))}{g(s_1, s_2) + h(s_1, s_1) + h(s_1, s_2)} < 0$$

So, the agent has no incentive to misreport to s_3'. Similarly, it can be shown that no other agent has any incentive to misreport.

The next result shows that the general mechanism M^{pb} also satisfies EPIC, EPIR and BB.

Proposition **3.3** *Mechanism M^{pb} is EPIC, EPIR and BB.*

The proof of the proposition is in the Appendix.

3.7.1 A welfare-maximizing probability-burning mechanism

In this subsection, we describe another probability-burning mechanism and study its welfare properties. We consider a model of 3 agents i.e. $N = \{1, 2, 3\}$, and the valuation functions that satisfy SAS condition and symmetry. The valuation functions are $v_i(s) = g(s_i) + \sum_{j \neq i} h(s_j)$ for all $i \in \{1, 2, \ldots, n\}$. Consider the following mechanism M^{pbo}.

$$f_i(s) = \begin{cases} \dfrac{1}{|s[1]|}\left(\dfrac{2(g(s_3) + 2h(s_3)) + g(s_2) + 2h(s_2)}{3(g(s_2) + h(s_2) + h(s_3))}\right) & \text{, if } i \in s[1] \\ 0 & \text{, otherwise} \end{cases}$$

And the payment function of each agent is given by the Revenue Equivalence formula as in (3.6) where,

$$p_i(0, s_{-i}) = \begin{cases} -\frac{1}{3}(g(s_3) + 2h(s_3)) & \text{, if } i = 1, 2 \\ -\frac{1}{3}(g(s_2) + 2h(s_2)) & \text{, if } i = 3 \end{cases}$$

This mechanism is similar to the one described by Mishra and Sharma (2018) and extends their mechanism to interdependent-value model. The object is auctioned using the generalized Vickrey auction. The agent with highest ranked signal at a signal profile s receives the object with probability $f_1(s)$ as his valuation is the highest among all the agents. The agent pays $f_1(s)((\beta + 1)s_2 + \beta s_3)$. This amount is redistributed to all the agents. The mechanism is EPIC, BB and EPIR. The following example illustrates this mechanism for a specific valuation function.

Example **3.16** Let $g(x) = x$ and $h(y) = \beta y$. The valuation functions are $v_i(s) = s_i + \beta(\sum_{j \neq i} s_j)$ where $\beta > 0$. Let there be a signal profile such that $s_2 \geq s_3 \geq s_1$. Following 4 cases are possible after resolving the tie-breaking between agents:

- $s_2 > s_3 > s_1$

  The allocation probabilities are

$$f_2(s) = \left(\frac{1}{3}s_3 + \frac{2}{3}s_1\right)\left(\frac{1 + 2\beta}{(\beta + 1)s_3 + \beta s_1}\right)$$

  and $f_1(s) = f_3(s) = 0$. The payments are $p_2(s) = \frac{1}{3}(1+2\beta)(s_1+s_3)$, $p_1(s) = -(1+2\beta)\frac{s_3}{3}$ and $p_3(s) = -(1+2\beta)\frac{s_1}{3}$. Agent 2 pays $\frac{1}{3}(1+2\beta)(2s_1+s_3)$ and receives a redistribution amount of $(1+2\beta)\frac{s_1}{3}$. Agents 1 and 3 receive redistribution amount of $(1+2\beta)\frac{s_3}{3}$ and $(1+2\beta)\frac{s_1}{3}$ respectively.

- $s_2 = s_1 > s_3$
  The allocation probabilities are
  $$f_2(s) = f_1(s) = \frac{1}{2}\left(\frac{1}{3}s_3 + \frac{2}{3}s_1\right)\left(\frac{1+2\beta}{(\beta+1)s_3 + \beta s_1}\right)$$
  and $f_3(s) = 0$. Agent 2 and 1 each pay $\frac{1}{6}(1+2\beta)(2s_1+s_3)$. Agent 2 and 1 each receive $(1+2\beta)\frac{s_3}{3}$ and agent 3 receives $(1+2\beta)\frac{s_1}{3}$.

- $s_2 > s_1 = s_3$
  The allocation probabilities are
  $$f_2(s) = \left(\frac{1}{3}s_3 + \frac{2}{3}s_3\right)\left(\frac{1+2\beta}{(\beta+1)s_3 + \beta s_3}\right) = 1$$
  and $f_1(s) = f_3(s) = 0$. Agent 2 pays $(1+2\beta)s_1$. Each agent receives an amount of $(1+2\beta)\frac{s_1}{3}$.

- $s_2 = s_1 = s_3$
  The allocation probabilities are
  $$f_2(s) = f_1(s) = f_3(s) = \frac{1}{3}$$
  Agent 1, 2 and 3 each pay $(1+2\beta)\frac{s_3}{3}$. Also, each agent receives an amount of $(1+2\beta)\frac{s_3}{3}$. So, $p_1(s) = p_2(s) = p_3(s) = 0$.

We now turn to the welfare properties of the mechanism. The welfare that a budget-balanced probability-burning mechanism M generates at a signal profile s is

$$W_M(s) = \sum_{i \in N} f_i(s)v_i(s) = f_1(s)v_1(s)$$

We compare the total welfare generated by M^{pb} with that of M^{pbo} through an example. Let $N = \{1, 2, 3\}$ and let the valuation functions of agents be $v_1(s) = \frac{2}{3}s_1 + s_2 + s_3$, $v_2(s) = \frac{2}{3}s_2 + s_1 + s_3$ and $v_3(s) = \frac{2}{3}s_3 + s_1 + s_2$. They satisfy the SAS condition with symmetry. They also satisfy the AS condition[§]. Consider a signal profile such that $s_1 > s_2 > s_3$. The allocation probabilities are

$$f_1^{pb}(s) = \frac{4s_3 + 2s_2}{5s_2 + 3s_3} \text{ and } f_1^{pbo}(s) = \frac{16s_3 + 8s_2}{15s_2 + 9s_3}$$

[§]This can be seen by rewriting valuation functions as $v_1(s) = \frac{2}{3}(s_2+s_3) + \frac{1}{3}(s_1+s_2) + \frac{1}{3}(s_1+s_3)$ and similarly for $v_2(\cdot)$ and $v_3(\cdot)$ where $g(x,y) = \frac{2}{3}(x+y)$ and $h(x,y) = \frac{1}{3}(x+y)$.

Taking their difference we get,

$$f_1^{pbo}(s) - f_1^{pb}(s) = \frac{16s_3 + 8s_2}{15s_2 + 9s_3} - \frac{4s_3 + 2s_2}{5s_2 + 3s_3} = \frac{(12s_3^2 + 10s_2^2 + 26s_2s_3)}{(5s_2 + 3s_3)(15s_2 + 9s_3)} > 0$$

So, $f_1^{pbo}(s) > f_1^{pb}(s)$ for all $s \in S^n$. This implies $W^{pbo}(s) > W^{pb}(s)$ for all $s \in S^n$. The mechanism M^{pbo} *welfare-dominates* the mechanism M^{pb}. In fact, we will now show that M^{pbo} is welfare-undominated in the class of EPIR, BB, EPIC mechanisms that allocate the object to agents with topmost signal only and that satisfy an additional property called equal treatment at equal signals. It can be seen that mechanism M^{pbo} satisfies ETES as all the agents who have highest signal are allocated the object with same allocation probability.

We first define this additional property which is based on a similar property given by Mishra and Sharma (2018) in private-value setting.

Definition **3.7** *A mechanism M satisfies equal treatment at equal signals (ETES) if for every signal profile s in which $s_i = s_j$ for any $i, j \in N$ the following is true*

$$f_i(s) = f_j(s) \text{ and } p_i(s) = p_j(s)$$

We now define our notion of welfare-maximization. Let $\mathcal{M}$ be the class of EPIC, EPIR, BB, ETES mechanisms that allocate the object to the agents with the highest signal.

Definition **3.8** *A mechanism M is welfare-maximizing in the class $\mathcal{M}$ of mechanisms if*

$$W_{M'}(s) \geq W_M(s) \quad \forall s \in S^n$$

for all $M' \in \mathcal{M}$.

A mechanism M is welfare-maximizing if it welfare-dominates all other mechanisms in the class $\mathcal{M}$. By adapting the arguments in Mishra and Sharma (2018) in their private-value model to our interdependent-value setting we can prove the following:

Theorem **3.5** *Assume agents' valuation functions satisfy the SAS condition and symmetry. The mechanism M^{pbo} is welfare-maximizing in the class of EPIC, EPIR, BB and ETES mechanisms that allocate the object only to the agents with the highest signal.*

Proof: The valuation functions of agents are $v_i(s) = g(s_i) + \sum_{j \neq i} h(s_j)$. Let $M \equiv (f, p)$ be a probability-burning mechanism that is EPIC, EPIR, BB and ETES.

If the signal profile is $s = (a, a, a)$, ETES ensures that the object is allocated to each agent with probability $\frac{1}{3}$. Hence, by (3.6), the payment of each agent is

$$p_i(s) = p_i(0, s_{-i}) + \frac{1}{3}v_i(a, a, a), \ \forall i \in \{1, 2, 3\}$$

As $v_1(a, a, a) = v_2(a, a, a) = v_3(a, a, a)$, and the mechanism is BB i.e. $\sum_{i \in N} p_i(s) = 0$, we have

$$\sum_{i \in N} p_i(0, s_{-i}) = -v_1(a, a, a)$$

By ETES property, $p_1(0, s_{-1}) = p_2(0, s_{-2}) = p_3(0, s_{-3}) = -\frac{1}{3}v_1(a, a, a)$.

Now, change the signals of second and third agent. Let the signal profile be (a, b, b) where $a > b$. The payment of each agent is

$$\begin{aligned}
p_1(s) &= p_1(0, s_{-1}) + v_1(a, b, b) - (v_1(a, b, b) - v_1(b, b, b)) \\
&\quad = p_1(0, b, b) + v_1(b, b, b) = -\frac{1}{3}v_1(b, b, b) + v_1(b, b, b) = \frac{2}{3}v_1(b, b, b) \\
p_2(s) &= p_2(0, s_{-2}) = p_2(a, 0, b) \\
p_3(s) &= p_3(0, s_{-3}) = p_3(a, b, 0)
\end{aligned}$$

Following the budget-balance condition, and the fact that $p_2(a, 0, b) = p_3(a, b, 0)$ (this follows from ETES as agent 2 and 3 have same signal b), we get

$$p_2(a, 0, b) = p_3(a, b, 0) = -\frac{1}{3}v_1(b, b, b)$$

Now consider the signal profile (a, a, b) where $a > b$. As M satisfies ETES, the object is allocated to both agent 1 and 2 with equal probability. So $f_1(s) = f_2(s)$. The payments of each agent is

$$\begin{aligned}
p_1(s) &= p_1(0, s_{-1}) + f_1(s)v_1(a, a, b) = p_1(0, a, b) + f_1(s)v_1(a, a, b) \\
p_2(s) &= p_2(0, s_{-2}) + f_1(s)v_2(a, a, b) = p_2(a, 0, b) + f_2(s)v_2(a, a, b) \\
p_3(s) &= p_3(0, s_{-3}) = p_3(a, a, 0)
\end{aligned}$$

As $v_1(a, a, b) = v_2(a, a, b)$, we have $p_1(0, a, b) = p_2(a, 0, b) = -\frac{1}{3}v_1(b, b, b)$.

Hence, for any EPIC, EPIR, BB and ETES probability-burning mechanism, at the signal profile (a, b, c) such that $a > b > c$, the following is true:

$$\sum_{i \in N} p_i(0, s_{-i}) = -\frac{1}{3}(2v_1(c, c, c) + v_1(b, b, b)) \tag{3.7}$$

Now we prove that the mechanism $\mathcal{M}^{pbo}$ maximizes welfare of agents. At signal profiles (a, a, a) and (a, b, b), the mechanism allocates the object without burning any probability. So, no mechanism can do better at these signal profiles. Now consider the signal profile (a, a, b). Let there be a mechanism $\mathcal{M}' \equiv (f', p')$ in the class $\mathcal{M}$ of mechanisms which generates higher welfare at this signal profile. So,

$$\begin{aligned}
p_1'(s) &= p_1'(0, s_{-1}) + f_1'(s)v_1(a, a, b) = p_1'(0, a, b) + f_1'(s)v_1(a, a, b) \\
p_2'(s) &= p_2'(0, s_{-2}) + f_1'(s)v_2(a, a, b) = p_2'(a, 0, b) + f_2'(s)v_2(a, a, b) \\
p_3'(s) &= p_3'(0, s_{-3}) = p_3'(a, a, 0)
\end{aligned}$$

From (3.7), $\sum\limits_{i \in N} p_i'(0, s_{-i}) = -\frac{1}{3}(2v_1(b, b, b) + v_1(a, a, a))$. Using budget-balance condition and substituting the expression of valuation functions, we get,

$$f_1'(s) = f_2'(s) = \frac{1}{2}\left(\frac{2(g(b) + 2h(b)) + g(a) + 2h(a)}{3(g(a) + h(a) + h(b))}\right)$$

This is same allocation probability as that of mechanism $\mathcal{M}^{pbo}$. Now consider the signal profile (a, b, c) where $a > b > c$. Let $f_1'(a, b, c) > f_1^{pbo}(a, b, c)$. We have

$$\begin{aligned}
p_1'(s) &= p_1'(0, s_{-1}) + f_1'(s)v_1(s) - \int\limits_b^a f_1'(x, s_{-1})\frac{\partial v_1(x, s_{-1})}{\partial s_1}dx \\
p_2'(s) &= p_2'(0, s_{-2}) \\
p_3'(s) &= p_3'(0, s_{-3})
\end{aligned}$$

Adding these, the expression on right-hand side is,

$$\begin{aligned}
\sum_{i \in N} p_i'(s) = \sum_{i \in N} p_i'(0, s_{-i}) + f_1'(s)v_1(s) - \int\limits_b^a f_1'(x, s_{-1})\frac{\partial v_1(x, s_{-1})}{\partial s_1}dx \qquad (3.8) \\
\geq \sum_{i \in N} p_i'(0, s_{-i}) + f_1'(s)v_1(s) - f_1'(s)(v_1(a, b, c) - v_1(b, b, c)) \\
= \sum_{i \in N} p_i'(0, s_{-i}) + f_1'(s)v_1(b, b, c) \\
> \sum_{i \in N} p_i^{pbo}(0, s_{-i}) + f_1^{pbo}(s)v_1(b, b, c) = 0
\end{aligned}$$

The first inequality follows from the the fact that the function $f_1(s)$ is increasing in s_1. The last inequality is due to $\sum\limits_{i \in N} p_i'(0, s_{-i}) = \sum\limits_{i \in N} p_i^{pbo}(0, s_{-i}) = -\frac{1}{3}(2v_1(c, c, c) + v_1(b, b, b))$.

By budget-balance the expression in (3.8) must be equal to zero. Hence, we get a contradiction. So, $f_1^{pbo}(s) \geq f_1'(s)$ at signal profile (a, b, c). ■

A more general result for an arbitrary number of agents remains an open question.

3.8 Conclusion

This chapter introduces a new and simple approach of designing EPIC, EPIR and BB mechanisms for interdependent valuation model. The s-ranking mechanisms relax ex-post efficiency condition to obtain ex-post incentive compatibility without requiring the valuations to satisfy single-crossing condition. Also, when the valuations are additively separable, the s-ranking allocation rule can be implemented with BB transfers. In future, we would like to characterize the complete set of valuation functions for which a BB mechanism can exist.

We also study probability-burning mechanisms by restricting the class of valuation functions to semi-separable form. If the restrictions are relaxed, the existence of probability-burning mechanisms needs to be explored. Also, finding whether there exists a welfare-maximizing probability-burning mechanism for an arbitrary number of agents remains an open problem.

3.9 Appendix

Before proving Theorem 3.1, we first prove the following lemma which provides the sufficient condition for a s-ranking allocation rule to be ex-post implemented by a payment rule.

Lemma **3.1** *If the valuation functions $(v_1(s), v_2(s), \ldots, v_n(s))$ are increasing in their own signal, then there exists a payment rule p such that the s-ranking mechanism (π, p) is EPIC.*

Proof: We show that the signal-ranking mechanism is EPIC. Consider an arbitrary signal profile s and without loss of generality let $s_1 > s_2 > \ldots > s_n$. Let the payment of agent ranked i is:

$$\begin{aligned} p_i(s) = v_i(s)\pi_i - &\int_0^{s_i} f_i(x, s_{-i}) \frac{\partial v_i(x, s_{-i})}{\partial s_i} dx \\ = &\sum_{j=1}^{n-i} v_i(s_{i+j}, s_{-i})(\pi_{i+j-1} - \pi_{i+j}) \end{aligned} \tag{3.9}$$

If the agent i reports $s_{i'} > s_i$ such that his rank is $i' < i$, the payment of the agent is

$$p_{i'}(s_{i'}, s_{-i'}) = v_{i'}(s_{i'}, s_{-i'})\pi_{i'} - \int_0^{s_{i'}} f_{i'}(x, s_{-i'}) \frac{\partial v_{i'}(x, s_{-i'})}{\partial s_{i'}} dx$$

$$= \sum_{j=1}^{i-i'-1} v_{i'}(s_{i'+j}, s_{-i'})(\pi_{i'+j-1} - \pi_{i'+j}) + v_{i'}(s_{i'-1}, s_{-i'})(\pi_{i-1} - \pi_i)$$
$$+ \sum_{j=i+1}^{n-1} v_{i'}(s_j, s_{-i'})(\pi_j - \pi_{j+1})$$

Let $\Delta u(s_{i'}, s_i) = v_i(s)\pi_i - p_i(s) - \big(v_{i'}(s)\pi_{i'} - p_{i'}(s_{i'}, s_{-i'})\big)$ be the difference between utilities obtained by agent ranked i with signal type s_i when he reports the true signal and when he falsely reports $s_{i'}$. Hence,

$$\Delta u(s_{i'}, s_i) = v_i(s)\pi_i - v_{i'}(s)\pi_{i'} - \sum_{j=1}^{n-i} v_i(s_{i+j}, s_{-i})(\pi_{i+j-1} - \pi_{i+j}) + \sum_{j=1}^{i-i'-1} v_{i'}(s_{i'+j}, s_{-i'})(\pi_{i'+j-1} - \pi_{i'+j})$$
$$+ v_{i'}(s_{i'-1}, s_{-i'})(\pi_{i-1} - \pi_i) + \sum_{j=i+1}^{n-1} v_{i'}(s_j, s_{-i'})(\pi_j - \pi_{j+1})$$
$$= -v_i(s)(\pi_{i'} - \pi_i) + \sum_{j=1}^{i-i'} v_{i'}(s_{i'+j-1}, s_{-i'})(\pi_{i'+j-1} - \pi_{i'+j})$$
$$= \sum_{j=1}^{i-i'} \big(v_{i'}(s_{i'+j-1}, s_{-i'}) - v_i(s_i, s_{-i})\big)(\pi_{i'+j-1} - \pi_{i'+j}) \tag{3.10}$$

As the function $v_i(s_i, s_{-i})$ is increasing in s_i, the expression in (3.10) is positive. Similarly if $s_{i'} < s_i$, then also $\Delta u(s_{i'}, s_i) > 0$. Hence, the mechanism is EPIC. ■

Proof of Theorem 3.1:

Let the valuation functions satisfy SAS. Consider a s-ranking allocation π. From Lemma 3.1, there exists a EPIC payment rule p. We first show that when π satisfies the Condition 3.1 then it is residually balanced. Consider a signal profile s such that $s_1 > s_2 > \ldots > s_n > 0$. So, using (3.2) and putting $p_i(0, s_{-i}) = 0$ for all the agents, the revenue that is generated is

$$R(s) = \sum_{i \in N} p_i(s) = \sum_{i \in N} \Big(v_i(s)\pi_i - \int_0^{s_i} f_i(x, s_{-i}) \frac{\partial v_i(x, s_{-i})}{\partial s_i} dx \Big)$$
$$= \sum_{i \in N} \Big(\big(g_i(s_i) + \sum_{j \neq i} h(s_j)\big)\pi_i - \int_0^{s_i} f_i(x, s_{-i}) \frac{\partial g_i(x, s_{-i})}{\partial s_i} dx \Big)$$
$$= \sum_{k=1}^{n} \Big(\sum_{l=1, l \neq k}^{n} \pi_l \Big) h(s_k) + \sum_{j=1}^{n-1} (\pi_j - \pi_{j+1})\big(g_1(s_{j+1}) + g_2(s_{j+1}) + \ldots + g_j(s_{j+1})\big) \tag{3.11}$$

Let $T \subseteq N$ be such that the last ranked agent n does not belong to the set of agents T and $|T| = n - m$. In the profile $(0_T, s_{-T})$, rank of agent n is m. The set $T \cup n$ contains $n - m + 1$ agents.

Now we can rewrite the condition of residual balancedness as,

$$\sum_{T \subseteq N} (-1)^{|T|} R(0_T, s_{-T}) = \sum_{T \subseteq N : n \in T} (-1)^{|T|} R(0_T, s_{-T}) + \sum_{T \subseteq N : n \notin T} (-1)^{|T|} R(0_T, s_{-T})$$
$$= \sum_{T \subseteq N : n \notin T} (-1)^{|T|} \Big(R(0_T, s_{-T}) - R(0_{T \cup \{n\}}, s_{-(T \cup \{n\})}) \Big)$$

From (3.11), we can compute

$$R(0_T, s_{-T}) - R(0_{T \cup \{n\}}, s_{-(T \cup \{n\})}) = \Big(\sum_{k=1, k \neq |N \setminus T|}^{n} \pi_k \Big) h(s_n) + (\pi_{|N \setminus T| - 1} - \pi_{|N \setminus T|}) \sum_{j \in N \setminus (T \cup \{n\})} g_j(s_n)$$

The residual balancedness condition now becomes

$$\sum_{T \subseteq N} (-1)^{|T|} R(0_T, s_{-T}) = \sum_{T \subseteq N : n \notin T} (-1)^{|T|} \Big(\Big(\sum_{k=1, k \neq |N \setminus T|}^{n} \pi_k \Big) h(s_n) + (\pi_{|N \setminus T| - 1} - \pi_{|N \setminus T|}) \sum_{j \in N \setminus (T \cup \{n\})} g_j(s_n) \Big)$$
$$= \Big(-\pi_1 + \binom{n-1}{1} \pi_2 - \binom{n-1}{2} \pi_3 + \ldots + (-1)^n \binom{n-1}{n-1} \pi_n \Big) h(s_n)$$
$$+ \Big(-\pi_1 + \binom{n-1}{1} \pi_2 - \binom{n-1}{2} \pi_3 + \ldots + (-1)^n \binom{n-1}{n-1} \pi_n \Big) \Big(\sum_{j=1}^{n-1} g_j(s_n) \Big)$$

If π satisfies Condition 3.1, the above expression is equal to zero. Hence, the s-ranking allocation rule satisfies residual balancedness condition. Following Theorem 3.2, there exist transfers such that s-ranking mechanism is BB and EPIC. ■

Proof of Theorem 3.3:

Proof of part (i): Let M^v be a v-ranking mechanism with allocation probabilities $(\rho_1, \rho_2, \ldots, \rho_n)$ and payment rules $(p_1, p_2, \ldots, p_n)$ are given by the Revenue Equivalence principle as in (3.2).

Consider a signal profile s and let $v_1(s) \geq v_2(s) \geq \ldots \geq v_n(s)$. The payment of agent ranked i is:

$$p_i(s) = v_i(s)\rho_i - v_i(0, s_{-i}) f_i(0, s_{-i}) - \int_0^{s_i} f_i(x, s_{-i}) \frac{\partial v_i(x, s_{-i})}{\partial s_i} dx$$
$$= \sum_{j=1}^{n-i} v_i(\kappa_{i,i+j}(s_{-i}), s_{-i})(\rho_{i+j-1} - \rho_{i+j}) - v_i(0, s_{-i}) f_i(0, s_{-i}) + \rho_n v_i(\max\{0, \kappa_{i,0}(s_{-i})\}, s_{-i})$$

where,

$$\kappa_{i,i+j}(s_{-i}) = \inf \{y \in S \mid v_i(y, s_{-i}) \geq v_{i+j}(y, s_{-i})\} \tag{3.12}$$

and,

$$\kappa_{i,0}(s_{-i}) = \inf \{y \in S \mid v_i(y, s_{-i}) \geq 0\}$$

If the agent i reports $s'_i > s_i$ such that his rank is $k < i$, the payment of the agent is

$$p_k(s'_i, s_{-i}) = v_k(s'_i, s_{-i})\rho_k - v_k(0, s_{-i})f_k(0, s_{-i}) - \int_0^{s'_i} f_k(x, s_{-i})\frac{\partial v_k(x, s_{-i})}{\partial s'_i}dx$$

$$= \sum_{j=1}^{i-i'-1} v_{i'}(\kappa_{i',i'+j}(s_{-i'}), s_{-i'})(\rho_{i'+j-1} - \rho_{i'+j}) + v_{i'}(\kappa_{i',i'-1}(s_{-i'}), s_{-i'})(\rho_{i-1} - \rho_i) +$$

$$\sum_{j=i+1}^{n-1} v_{i'}(\kappa_{i',j}(s_{-i'}), s_{-i'})(\rho_j - \rho_{j+1}) - v_{i'}(0, s_{-i'})f_i(0, s_{-i'}) + \rho_n v_{i'}(\max\{0, \kappa_{i,0}(s_{-i})\}, s_{-i'})$$

Let $\Delta u(s_{i'}, s_i) = v_i(s)\rho_i - p_i(s) - \big(v_{i'}(s)\rho_{i'} - p_{i'}(s_{i'}, s_{-i'})\big)$ be the difference between utilities obtained by agent ranked i with signal type s_i when he reports the true signal and when he falsely reports $s_{i'}$. Hence,

$$\begin{aligned}
\Delta u(s_{i'}, s_i) = v_i(s)\rho_i - v_{i'}(s)\rho_{i'} - &\sum_{j=1}^{n-i} v_i(\kappa_{i,i+j}(s_{-i}), s_{-i})(\rho_{i+j-1} - \rho_{i+j}) \\
&+ \sum_{j=1}^{i-i'-1} v_{i'}(\kappa_{i',i'+j}(s_{-i'}), s_{-i'})(\rho_{i'+j-1} - \rho_{i'+j}) \\
&+ v_{i'}(\kappa_{i',i'-1}(s_{-i'}), s_{-i'})(\rho_{i-1} - \rho_i) + \sum_{j=i+1}^{n-1} v_{i'}(\kappa_{i',j}(s_{-i'})(\rho_j - \rho_{j+1}) \\
= -v_i(s)(\rho_{i'} - \rho_i) + &\sum_{j=1}^{i-i'} v_{i'}(\kappa_{i',i'+j-1}(s_{-i'}), s_{-i'})(\rho_{i'+j-1} - \rho_{i'+j}) \\
= &\sum_{j=1}^{i-i'} \big(v_{i'}(\kappa_{i',i'+j-1}(s_{-i'}), s_{-i'}) - v_i(s_i, s_{-i})\big)(\rho_{i'+j-1} - \rho_{i'+j})
\end{aligned} \tag{3.13}$$

Notice the expression within the first parenthesis of (3.13). Consider any $j \in \{1, 2, \ldots, i - i'\}$. We prove that $\kappa_{i',i'+j-1}(s_{-i'}) \geq s_i$ for all j. Suppose if this is not true. Then $s_i > \kappa_{i',i'+j-1}(s_{-i'})$. By the single-crossing condition,

$$v_i(s_i, s_{-i}) - v_i(\kappa_{i',i'+j-1}(s_{-i'}, s_{-i}) > v_{i'+j-1}(s_i, s_{-i}) - v_{i'+j-1}(\kappa_{i',i'+j-1}(s_{-i'}, s_{-i})$$

Rearranging, we get

$$v_i(s_i, s_{-i}) - v_{i'+j-1}(s_i, s_{-i}) > v_i(\kappa_{i',i'+j-1}(s_{-i'}, s_{-i}) - v_{i'+j-1}(\kappa_{i',i'+j-1}(s_{-i'}, s_{-i})$$

The expression on right-hand side is zero. Hence,

$$v_i(s_i, s_{-i}) - v_{i'+j-1}(s_i, s_{-i}) > 0$$

But this is a contradiction because agent ranked i is ranked lower than agent ranked $i'+j-1$ at the type profile s. Hence, $\kappa_{i',i'+j-1}(s_{-i'}) \geq s_i$ for all j. Hence, the expression in (3.13),

$$\sum_{j=1}^{i-i'} \big(v_{i'}(\kappa_{i',i'+j-1}(s_{-i'}), s_{-i'}) - v_i(s_i, s_{-i})\big)(\rho_{i'+j-1} - \rho_{i'+j}) \geq 0$$

Similarly if $s_{i'} < s_i$, then also $\Delta u(s_{i'}, s_i) > 0$. Hence, the mechanism is EPIC.

Proof of part (ii):

Let the valuation functions satisfy SAS and single-crossing. Consider a v-ranking allocation ρ. From part (i) of this theorem, there exists a EPIC payment rule p. We first show that when an v-ranking allocation rule satisfies the Condition 3.4 then it is residually balanced. Consider a signal profile s such that $v_1(s) > v_2(s) > \ldots > v_n(s)$. So, using (3.2) and putting $p_i(0, s_{-i}) = 0$ for all the agents, the revenue that is generated is

$$\begin{aligned} R(s) = \sum_{i \in N} p_i(s) &= \sum_{i \in N}\Big(v_i(s)\rho_i - \int_0^{s_i} f_i(x, s_{-i})\frac{\partial v_i(x, s_{-i})}{\partial s_i}dx\Big) \\ &= \sum_{i \in N}\Big(\big(g_i(s_i) + \sum_{j \neq i} h(s_j)\big)\rho_i - \int_0^{s_i} f_i(x, s_{-i})\frac{\partial g_i(x, s_{-i})}{\partial s_i}dx\Big) \\ &= \sum_{k=1}^{n}\big(\sum_{l=1, l\neq k}^{n} \rho_l\big)h(s_k) + \sum_{j=1}^{n-1}(\rho_j - \rho_{j+1})\big(g_1(\kappa_{1,j+1}(s_{j+1})) + g_2(\kappa_{2,j+1}(s_{j+1})) \\ &\quad + \ldots + g_j(\kappa_{j,j+1}(s_{j+1}))\big) \end{aligned} \tag{3.14}$$

Let $T \subseteq N$ be such that the last ranked agent n does not belong to the set of agents T and $|T| = n - m$. In the profile $(0_T, s_{-T})$, rank of agent n is m. The set $T \cup n$ contains $n - m + 1$ agents.

The condition of residual balancedness is,

$$\sum_{T \subseteq N}(-1)^{|T|}R(0_T, s_{-T}) = \sum_{T \subseteq N: n \notin T}(-1)^{|T|}\Big(R(0_T, s_{-T}) - R(0_{T\cup\{n\}}, s_{-(T\cup\{n\})})\Big)$$

From (3.14), we can compute

$$R(0_T, s_{-T}) - R(0_{T\cup\{n\}}, s_{-(T\cup\{n\})}) = \big(\sum_{k=1, k\neq|N\setminus T|}^{n} \rho_k\big)h(s_n) + (\rho_{|N\setminus T|-1} - \rho_{|N\setminus T|})\sum_{j \in N\setminus(T\cup\{n\})} g_j(\kappa_{j,n}(s_n))$$

The residual balancedness condition now becomes

$$\begin{aligned}\sum_{T\subseteq N}(-1)^{|T|}R(0_T,s_{-T}) &= \sum_{T\subseteq N: n\notin T}(-1)^{|T|}\Big(\big(\sum_{k=1,k\neq |N\setminus T|}^{n}\rho_k\big)h(s_n)+(\rho_{|N\setminus T|-1}-\rho_{|N\setminus T|})\sum_{j\in N\setminus(T\cup\{n\})}g_j(\kappa_{j,n}(s_n))\Big)\\ &= \Big(-\rho_1+\binom{n-1}{1}\rho_2-\binom{n-1}{2}\rho_3+\ldots+(-1)^n\binom{n-1}{n-1}\rho_n\Big)h(s_n)\\ &\quad+\Big(-\rho_1+\binom{n-1}{1}\rho_2-\binom{n-1}{2}\rho_3\\ &\quad+\ldots+(-1)^n\binom{n-1}{n-1}\rho_n\Big)\big(\sum_{j=1}^{n-1}g_j(\kappa_{j,n}(s_n))\big)\end{aligned}$$

If ρ satisfies Condition 3.4, the above expression is equal to zero. Hence, the v-ranking allocation rule satisfies residual balancedness condition. Following Theorem 3.2, there exist transfers such that v-ranking mechanism is BB and EPIC. ■

Before proving Propositions 3.1 and 3.2 we prove the following lemma.

LEMMA **3.2** *If the valuation functions satisfy single-crossing and symmetry, the following is true for any two agents $i, j \in N$, and at every signal profile s:*

(i) $s_i > s_j \Leftrightarrow v_i(s) > v_j(s)$, *and*

(ii) $s_i = s_j \Leftrightarrow v_i(s) = v_j(s)$.

Proof: Let σ_{ij} be the permutation such that $\sigma_{ij}(k) = k$ for all $k \neq i, j$, $\sigma_{ij}(i) = j$ and $\sigma_{ij}(j) = i$. (ii) follows directly from definition of symmetry. For (i), let there be a signal profile s. Pick any two agents i and j. As v_i and v_j satisfy single-crossing condition, this implies that for any $s_i > s'_i$,

$$v_i(s_i, s_{-i}) - v_i(s'_i, s_{-i}) > v_j(s_i, s_{-i}) - v_j(s'_i, s_{-i}) \tag{3.15}$$

If $s'_i = s_j$, and let $s_i = \theta_1$ and $s_j = \theta_2$, then (3.15) can be written as,

$$\begin{aligned}&v_i(s_1, s_2, \ldots, \theta_1, \ldots,\theta_2, \ldots, s_n) - v_i(s_1, s_2, \ldots, \theta_2, \ldots, \theta_2, \ldots, s_n) > \\ &\qquad v_j(s_1,s_2, \ldots, \theta_1, \ldots, \theta_2, \ldots, s_n) - v_j(s_1, s_2, \ldots, \theta_2, \ldots, \theta_2, \ldots, s_n)\end{aligned} \tag{3.16}$$

As the valuation functions satisfy symmetry, the permutation σ_{ij} implies that the second term on both sides of (3.16) are identical

$$v_i(s_1, s_2, \ldots, \theta_2, \ldots, \theta_2, \ldots, s_n) = v_j(s_1, s_2, \ldots, \theta_2, \ldots, \theta_2, \ldots, s_n)$$

Hence, from (3.16),

$$v_i(s_1, s_2, \ldots, \theta_1, \ldots, \theta_2, \ldots, s_n) > v_j(s_1, s_2, \ldots, \theta_1, \ldots, \theta_2, \ldots, s_n)$$

This proves (i). ■

Proof of Proposition 3.1:

This follows directly from Lemma 3.2. As the ranking of signals and valuations coincides, the s-ranking allocation rule π and v-ranking allocation rule ρ are the same. The s-ranking mechanism and v-ranking mechanism are allocation equivalent. ■

Proof of Proposition 3.2:

Consider the allocation equivalent s-ranking allocation rule π and v-ranking allocation rule ρ. Their payments are also equivalent which we now prove. From Theorem 3.3, there exists a payment rule such that the v-ranking allocation ρ is EPIC. Let the v-ranking mechanism be (ρ, p). Let signal profile be such that $s_1 > s_2 > \ldots > s_n$. Agent ranked i makes payment of

$$\begin{aligned} p_i(s) &= v_i(s)f_i(s) - v_i(0, s_{-i})f_i(0, s_{-i}) - \int_0^{s_i} f_i(x, s_{-i})\frac{\partial v_i(x, s_{-i})}{\partial s_i}dx \\ &= -v_i(0, s_{-i})f_i(0, s_{-i}) + \sum_{j=1}^{n-i} v_i(\kappa_{i,i+j}(s_{-i}), s_{-i})(\rho_{i+j-1} - \rho_{i+j}) + \rho_n v_i(\max\{0, \kappa_{i,0}(s_{-i})\}, s_{-i}) \end{aligned}$$

Here, as in (3.12) we have

$$\kappa_{i,i+j}(s_{-i}) = \inf\{y \in S \mid v_i(y, s_{-i}) \geq v_{i+j}(y, s_{-i})\} \tag{3.17}$$

From Lemma 3.2, $v_i(y, s_{-i}) = v_{i+j}(y, s_{-i})$ only if $s_i = s_{i+j}$. Hence,

$$\begin{aligned} p_i(s) &= -v_i(0, s_{-i})f_i(0, s_{-i}) + \sum_{j=1}^{n-i} v_i(s_{i+j}, s_{-i})(\rho_{i+j-1} - \rho_{i+j}) + \rho_n v_i(0, s_{-i}) \\ &= \sum_{j=1}^{n-i} v_i(s_{i+j}, s_{-i})(\rho_{i+j-1} - \rho_{i+j}) \end{aligned} \tag{3.18}$$

Comparing (3.18) with (3.9), the allocation equivalent EPIC s-ranking mechanism with the allocation rule π has the same payment as the payment of v-ranking mechanism. If the allocation rule satisfies Condition 3.1, then from Theorem 3.1, the payment rule can also be BB. Hence, v-ranking mechanism is also EPIC and BB. ■

Proof of Theorem 3.4:

Let there be an arbitrary s-ranking mechanism $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$. The efficiency ratio of the mechanism at arbitrary signal profile s is:

$$\frac{\pi_1 v_{[1]}(s) + \pi_2 v_{[2]}(s) + \ldots + \pi_n v_{[n]}(s)}{v_{[1]}(s)} = \pi_1 + \pi_2 \frac{v_{[2]}(s)}{v_{[1]}(s)} + \ldots + \pi_n \frac{v_{[n]}(s)}{v_{[1]}(s)}$$

The worst-case efficiency ratio is

$$\mu = \inf_{s \in S^n} \left(\pi_1 + \pi_2 \frac{v_{[2]}(s)}{v_{[1]}(s)} + \ldots + \pi_n \frac{v_{[n]}(s)}{v_{[1]}(s)} \right)$$

If the signal profile is such that $s_1 \geq s_2 \geq \ldots \geq s_n$, then

$$\begin{aligned}
\mu &= \pi_1 + \pi_2 \frac{v_2(s)}{v_1(s)} + \ldots + \pi_n \frac{v_n(s)}{v_1(s)} \\
&= \pi_1 + \pi_2 \left(\frac{\gamma h(s_2) + \sum\limits_{j \neq 2} h(s_j)}{\gamma h(s_1) + \sum\limits_{j \neq 1} h(s_j)} \right) + \ldots + \pi_n \left(\frac{\gamma h(s_n) + \sum\limits_{j \neq n} h(s_j)}{\gamma h(s_1) + \sum\limits_{j \neq 1} h(s_j)} \right)
\end{aligned}$$

The minimum value of each of the $n-1$ ratios is $\frac{1}{\gamma}$ and the minima occurs at $(s_1, s_2, \ldots, s_n) = (1, 0, 0, \ldots, 0)$. Hence,

$$\mu = \pi_1 + \frac{1}{\gamma}(\pi_2 + \pi_3 + \ldots + \pi_n) = \left(1 - \frac{1}{\gamma}\right)\pi_1 + \frac{1}{\gamma}$$

The optimization problem is:

$$\begin{aligned}
\max_{(\pi_1, \pi_2, \ldots, \pi_n)} \quad & \left(1 - \frac{1}{\gamma}\right)\pi_1 + \frac{1}{\gamma} \\
\text{s.t.} \qquad\qquad & \pi_i \geq 0 \qquad \forall i \in \{1, 2, \ldots, n\} \\
\pi_1 + \pi_2 + \ldots + \pi_n &= 1 \\
\sum_{j \in N} (-1)^j \binom{n-1}{j-1} \pi_j &= 0 \\
\pi_{i+1} - \pi_i &\leq 0 \qquad \forall i \in \{1, 2, \ldots, n-1\}
\end{aligned}$$

This optimization problem is equivalent to the optimization problem solved by Long et al. (2017) to find the optimal worst-case efficient ranking mechanism in the class of dominant strategy incentive compatible and BB mechanisms. This is because all the constraints in both the problems are identical and the objective function above is a monotonic transformation of objective function of their optimization problem. Hence, π^* also solves our optimization problem and maximizes the worst-case efficiency ratio. ■

Proof of Proposition 3.3:

We first prove that the probability-burning mechanism is BB and EPIR.

Payments of agents are:

$$\begin{aligned}
p_1(s) &= -h(s_n, s_n, \ldots, s_n) \\
&\quad + \Big(\frac{(n-1)h(s_n, s_n, \ldots, s_n) + h(s_{n-1}, s_{n-1}, \ldots, s_{n-1})}{g(s_{-1}) + h(s_2, s_3, \ldots, s_n) + \sum\limits_{j\neq 1,2} h(s_2, s_2, s_3, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n)}\Big)\big(g(s_{-1}) \\
&\quad + \sum_{j\neq 1} h(s_{-j})\big) - \Big(\frac{(n-1)h(s_n, s_n, \ldots, s_n) + h(s_{n-1}, s_{n-1}, \ldots, s_{n-1})}{g(s_{-1}) + h(s_2, s_3, \ldots, s_n) + \sum\limits_{j\neq 1,2} h(s_2, s_2, s_3, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n)}\Big) \\
&\quad \big(\sum_{j\neq 1} h(s_{-j}) - h(s_2, s_3, \ldots, s_n) - \sum_{j\neq 1,2} h(s_2, s_2, s_3, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n)\big) \\
&= -h(s_n, s_n, \ldots, s_n) + \\
&\quad + \Big(\frac{(n-1)h(s_n, s_n, \ldots, s_n) + h(s_{n-1}, s_{n-1}, \ldots, s_{n-1})}{g(s_{-1}) + h(s_2, s_3, \ldots, s_n) + \sum\limits_{j\neq 1,2} h(s_2, s_2, s_3, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n)}\Big)\big(g(s_{-1}) + \\
&\quad h(s_2, s_3, \ldots, s_n) + \sum_{j\neq 1,2} h(s_2, s_2, s_3, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n)\big) \\
&= -h(s_n, s_n, \ldots, s_n) + (n-1)h(s_n, s_n, \ldots, s_n) + h(s_{n-1}, s_{n-1}, \ldots, s_{n-1}) \\
&= (n-2)h(s_n, s_n, \ldots, s_n) + h(s_{n-1}, s_{n-1}, \ldots, s_{n-1}) \\
p_i(s) &= -h(s_n, s_n, \ldots, s_n) \ \forall i \in \{2, 3, \ldots, n-1\} \\
p_n(s) &= -h(s_{n-1}, s_{n-1}, \ldots, s_{n-1})
\end{aligned}$$

Clearly, $\sum\limits_{i\in N} p_i(s) = 0$. The mechanism is BB.

The utility of agents are:

$$\begin{aligned}
u_1(s) &= v_1(s)f_1(s) - p_1(s) \\
&= \Big(\frac{(n-1)h(s_n, s_n, \ldots, s_n) + h(s_{n-1}, s_{n-1}, \ldots, s_{n-1})}{g(s_{-1}) + h(s_2, s_3, \ldots, s_n) + \sum\limits_{j\neq 1,2} h(s_2, s_2, s_3, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n)}\Big)\big(g(s_{-1}) \\
&\quad + \sum_{j\neq 1} h(s_{-j})\big) - \big((n-2)h(s_n, s_n, \ldots, s_n) + h(s_{n-1}, s_{n-1}, \ldots, s_{n-1})\big)
\end{aligned}$$

$$= \Big(\frac{(n-1)h(s_n, s_n, \ldots, s_n) + h(s_{n-1}, s_{n-1}, \ldots, s_{n-1})}{g(s_{-1}) + h(s_2, s_3, \ldots, s_n) + \sum\limits_{j\neq 1,2} h(s_2, s_2, s_3, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n)}\Big)\Big(\sum_{j\neq 1} h(s_{-j})$$

$$- h(s_2, s_3, \ldots, s_n) - \sum_{j\neq 1,2} h(s_2, s_2, s_3, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n)\Big) + h(s_n, s_n, \ldots, s_n)$$

$$u_i(s) = h(s_n, s_n, \ldots, s_n)\ \forall i \in \{2, 3, \ldots, n-1\}$$

$$u_n(s) = h(s_{n-1}, s_{n-1}, \ldots, s_{n-1})$$

As each valuation function is weakly increasing in its own signal,

$$\sum_{j\neq 1} h(s_{-j}) - h(s_2, s_3, \ldots, s_n) - \sum_{j\neq 1,2} h(s_2, s_2, s_3, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n) \geq 0$$

Hence, $u_1(s) \geq 0$. If $h(\cdot)$ function is non-negative then $u_i(s) \geq 0$ for all $i \neq 1$. Hence, the mechanism is EPIR.

Suppose, agent ranked n misreports to $s'_n > s_1$. His utility if he reports truthfully is $u_n(s) = h(s_{n-1}, s_{n-1}, \ldots, s_{n-1})$. The agent's new utility is

$$u_n(s'_n, s_{-n}) = \Big(\frac{(n-1)h(s_{n-1}, s_{n-1}, \ldots, s_{n-1}) + h(s_{n-2}, s_{n-2}, \ldots, s_{n-2})}{g(s_{-n}) + h(s_1, s_2, \ldots, s_{n-1}) + \sum\limits_{j\neq 1,n} h(s_1, s_2, s_3, \ldots, s_{j-1}, s_{j+1}, \ldots, s_1)}\Big)\Big(g(s_{-n})$$

$$+ \sum_{j\neq n} h(s_{-j})\Big) - \Big((n-2)h(s_{n-1}, s_{n-1}, \ldots, s_{n-1}) + h(s_{n-2}, s_{n-2}, \ldots, s_{n-2})\Big)$$

Let $\Delta u(s_{i'}, s_i) = v_i(s)f_i(s) - p_i(s) - \big(v_i(s)f_i(s'_i, s_{-i}) - p_i(s'_i, s_{-i})\big)$ be the difference between utilities obtained by agent ranked i with signal type s_i when he reports the true signal and when he falsely reports $s_{i'}$. Hence,

$$\Delta u(s_{i'}, s_i) = -\Big(\frac{(n-1)h(s_n, s_n, \ldots, s_n) + h(s_{n-1}, s_{n-1}, \ldots, s_{n-1})}{g(s_{-1}) + h(s_2, s_3, \ldots, s_n) + \sum\limits_{j\neq 1,2} h(s_2, s_2, s_3, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n)}\Big)\Big(\sum_{j\neq n} h(s_{-j})$$

$$- h(s_1, s_2, \ldots, s_{n-1}) - \sum_{j\neq 1,n} h(s_1, s_2, s_3, \ldots, s_{j-1}, s_{j+1}, \ldots, s_1)\Big)$$

$$= \Big(\frac{(n-1)h(s_n, s_n, \ldots, s_n) + h(s_{n-1}, s_{n-1}, \ldots, s_{n-1})}{g(s_{-1}) + h(s_2, s_3, \ldots, s_n) + \sum\limits_{j\neq 1,2} h(s_2, s_2, s_3, \ldots, s_{j-1}, s_{j+1}, \ldots, s_n)}\Big)$$

$$\times \Big(h(s_1, s_2, \ldots, s_{n-1}) + \sum_{j\neq 1,n} h(s_1, s_2, s_3, \ldots, s_{j-1}, s_{j+1}, \ldots, s_1) - \Big(\sum_{j\neq n} h(s_{-j})\Big)$$

As each valuation function is weakly increasing in its own signal, the expression in the bigger parenthesis on the right side is non-negative. Hence, $\Delta u(s_{i'}, s_i) \geq 0$. Similarly we can show that any agent ranked higher than agent n has no incentive to misreport. Hence, the mechanism is EPIC. ■

Chapter 4

Probability-burning mechanisms in multiple-good allocation problems

4.1 Introduction

We consider the problem of allocating m units of a good among n agents. Each agent demands a single unit of good the valuation of which is his private information. Agents can give or receive payments but aggregate payments must be zero. There are several instances in real-life where such a problem arises, for instance, in allocating bequests among various claimants (see Coombs (2013) for a real-life example).

Other properties that mechanisms are required to satisfy are incentive-compatibility, individual-rationality and efficiency*. A standard result in mechanism design theory is the Green-Laffont impossibility result (Green and Laffont (1979)). According to it, no mechanism can simultaneously satisfy efficiency, incentive-compatibility and budget-balance. So, one of the properties must be relaxed in order to find a mechanism which satisfies two properties and a weakened version of the third.

In this chapter, we relax the property of efficiency and look within the class of incentive-compatible and budget-balanced mechanisms. We follow the approach of Mishra and Sharma (2018). They consider a single-good allocation problem and their mechanism allocates the good only to the agent with the highest valuation. Some of the allocation probability is burnt at some valuation profiles. Such a mechanism is called a *probability-burning mechanism*.

*This contrasts with the approach of Dastidar (2017) who focuses on goals of efficiency and revenue generation for mechanisms that allocate a scarce object to a set of agents.

The chapter has two objectives. The first is to extend the mechanism of Mishra and Sharma (2018) to the multi-good allocation problem. We propose the *equal-probability-burning mechanism* which allocates a single unit of good to each of the top m highest-valued agents with equal probability. The probability is auctioned through a multi-unit Vickrey auction and the revenue collected is redistributed back to the agents which ensures budget-balance. We then compare the welfare properties of this mechanism with some other mechanisms that are budget-balanced (BB), dominant strategy incentive-compatible (DSIC) and individually-rational (IR). These mechanisms are the multi-unit extension of Green-Laffont mechanism and the single-unit burning mechanism given by Guo and Conitzer (2014).

We find that the worst-case efficiency ratio of multi-unit Green-Laffont mechanism is higher than that of equal-probability-burning mechanism. If the number of agents is greater than $m + \frac{m^2}{2} + \sqrt{m(m^2-1) + \frac{m^4}{4}}$ the worst-case efficiency ratio of the equal-probability-burning mechanism is greater than that of single-unit burning mechanism. The expected total welfare of equal-probability-burning mechanism is less than that of the multi-unit Green-Laffont mechanism but converges to it as n increases.

The second objective is to design probability-burning mechanism with reserve prices. Goods are allocated only if the valuations of at least m agents are above the reserve price. In this case each of the m agents with the highest ranked valuations is given a good with equal probability. The allocation probability depends on the relationship between the reserve price and the valuations of $(m+1)^{th}$ and $(m+2)^{th}$ ranked agents. We show that the mechanism is BB, IR and DSIC.

Our main goal is to demonstrate that introducing reserve prices may increase the expected welfare of agents. For this purpose we assume that valuations are uniformly distributed. In the restricted setting of $n = 4$ and $m = 2$, we show that the optimal reserve price is non-zero. For a single-good model we explicitly compute the optimal reserve price and show that the expected total welfare with the reserve price is greater than the expected total welfare in the mechanism of Mishra and Sharma (2018) (henceforth called the MS mechanism).

This chapter proceeds as follows. Section 4.2 discusses the literature survey. The model and basic definitions are introduced in Section 4.3. Section 4.4 describes the equal-probability-burning mechanism for multiple units of good and discusses the welfare properties. Section 4.5 describes the mechanism when there is a reserve price. Section 4.6 is the conclusion.

4.2 Related Literature

There are many papers that explore the relaxation of efficiency in the Green-Laffont impossibility result[†]. The simplest mechanism is the Green-Laffont mechanism which allocates the good to agent with highest valuation with a probability of $1 - \frac{1}{n}$ and to the second-highest agent with probability $\frac{1}{n}$. Long et al. (2017) define a class of ranking mechanisms in which the good is allocated to agents not having the highest valuation with positive probability. They find the least inefficient mechanism in the class of BB and DSIC mechanisms which allocates the good to $\frac{n}{2}$ agents. Long (2019) extends the result to multi-unit case. Long (2018) finds the least inefficient mechanism in a class of envy-free, BB and DSIC ranking mechanisms.

Mishra and Sharma (2018) find the Pareto optimal probability-burning mechanism for a single good in the class of BB, IR and DSIC mechanisms. As number of agents grow, their mechanism converges to efficiency and the ex-ante expected welfare converges to that of Green-Laffont mechanism. Some of the probability is necessarily burnt in their mechanism. This technique of destroying the good has also been explored by de Clippel et al. (2014). They propose a deterministic mechanism in which the burning of units of good depends on the valuation of agents but they optimize in the class of weakly BB, IR and DSIC mechanisms. Vikram (2021) studies similar problem in an interdependent value setting. He identifies three types of mechanisms - signal-ranking mechanisms, valuation-ranking mechanisms and probability-burning mechanisms and gives conditions on valuation functions under which these mechanisms satisfy incentive-compatibility, individual-rationality and budget-balance.

For multi-unit case, Guo and Conitzer (2014) study two types of linear redistribution mechanisms for allocating multiple units of good while maintaining strict budget-balance. One way is to partition the agents and goods into two sets each and allocate the sets of goods arbitrarily to the sets of agents through separate VCG mechanisms. The revenue generated by one set of agents is redistributed equally to all the agents in the other set. These are called partition mechanisms and it turns out that Green-Laffont mechanism is the least inefficient mechanism in this class. The other type of mechanism is the single-unit burning mechanism. A unit of good is burnt with some probability and rest of the units of good are allocated to highest valued agents. Gujar and Narahari (2008) extend their result to the case of multiple heterogenous goods.

[†]Refer to Mishra and Sharma (2018) for a detailed literature survey on different ways in which the Green-Laffont impossibility result can be relaxed.

4.3 The model and basic definitions

There are m identical units of a good which are to be allocated among n agents. We assume throughout that $n \geq m + 2$[‡]. Let the set of agents be $N = \{1, 2, \ldots, n\}$. Each agent demands only one good and has a valuation v_i for the good which is his private information. The valuations are independently and identically distributed in the unit interval $V = [0, 1]$ according to the distribution function G and corresponding density function g. A valuation profile is $v = (v_1, v_2, \ldots, v_n)$. Denote the agent with i^{th} highest valuation in any valuation profile by $v_{(i)}$. The agents are ranked as $v_{(1)} \geq v_{(2)} \geq \ldots \geq v_{(n)}$ where $v_{(1)}$ is the highest valuation and $v_{(n)}$ is the lowest valuation.

An allocation rule is a map $f : V^n \to [0, 1]^n$ where $f_i(v)$ denotes the probability of allocation of a unit of good to agent i when the valuation profile is $v = (v_1, v_2, \ldots, v_n)$. The allocation probabilities are assumed to satisfy the feasibility condition $\sum\limits_{i \in N} f_i(v) \leq m$ for every $v \in V^n$. The payment rule of agent i is $p_i : V^n \to \mathbb{R}$. A mechanism M is pair $(f, p) \equiv (f_1, f_2, \ldots, f_n, p_1, p_2, \ldots, p_n)$ and gives utility of $v_i f(v) - p_i(v)$ to agent i for all $i = 1, 2, \ldots, n$ and $v \in V^n$. A mechanism must satisfy the following properties:

- A mechanism $M \equiv (f, p)$ is dominant strategy incentive-compatible (DSIC) if for every $i \in N$, and every $v_{-i} \in V^{n-1}$, and for every $v_i, v_i' \in V$

$$v_i f_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) \geq v_i f_i(v_i', v_{-i}) - p_i(v_i', v_{-i})$$

- A mechanism $M \equiv (f, p)$ is individually-rational (IR) if for every $i \in N$, and every $v \in V^n$,

$$v_i f_i(v) - p_i(v) \geq 0$$

- A mechanism $M \equiv (f, p)$ is budget-balanced (BB) if for every $v \in V^n$,

$$\sum_{i \in N} p_i(v) = 0$$

We define some welfare measures for mechanisms. The total welfare of a mechanism at any valuation profile $v \in V^n$ is

$$W^M(v) = \sum_{i \in N} (v_i f_i(v) - p_i(v))$$

‡If $n < m$, the problem is trivial as each agent is always allocated a unit of good.

For a budget-balanced mechanism, the total welfare becomes

$$W^M(v) = \sum_{i \in N} v_i f_i(v)$$

The best possible welfare that a budget-balanced mechanism can achieve is

$$W^*(v) = v_{(1)} + v_{(2)} + v_{(3)} + \ldots + v_{(m)}$$

We define the worst-case efficiency ratio for a budget-balanced mechanism.

DEFINITION **4.1** *For a budget-balanced mechanism $M \equiv (f, p)$, the worst-case efficiency ratio is given by*

$$\alpha^M = \min_{v \in V^n} \frac{W^M(v)}{W^*(v)}$$

This is the minimum ratio of total welfare generated by a mechanism and the best possible welfare among all valuation profiles.

Another measure of welfare is the expected total welfare.

DEFINITION **4.2** *Given a distribution G, for a budget-balanced mechanism $M \equiv (f, p)$, the expected total welfare is given by*

$$\mathbb{E}[W^M] = \int_0^1 \int_0^1 \ldots \int_0^1 \Big(\sum_{i \in N} v_i f_i(v) \Big) g(v_n) g(v_{n-1}) \ldots g(v_1) dv_n dv_{n-1} \ldots dv_1$$

Denote by $v[k]$ the set of agents who have the k^{th} highest valuation at v. Formally,

$$v[1] = \{i \in N | v_i \geq v_j \ \forall j \in N\}$$

At any valuation profile v, the set of agents can be partitioned into disjoint sets $v[1], v[2], \ldots, v[n_0(v)]$ such that $\bigcup_{k=1}^{n_0(v)} v[k] = N$. Here, $v[k]$ is given by,

$$v[k] = \{i \in N \setminus (\bigcup_{k'=1}^{k-1} v[k']) : v_i \geq v_j \ \forall j \in N \setminus (\bigcup_{k'=1}^{k-1} v[k'])\}$$

Also, $n_0(v)$ is the index corresponding to set of agents with lowest valuation at a valuation profile v. If m units of a good are to be allocated, let the set of goods be partitioned into $m_1, m_2, \ldots, m_{n_0(v)}$ such that $\sum_{k=1}^{n_0(v)} m_k = m$. The set of agents $v[k]$ is allocated $m_k \leq |v[k]|$ units of good for all $k = 1, 2, \ldots, n_0(v)$.

Now consider the special case in which the allocation of all units of good is done as follows. Pick a valuation profile $v \in V^n$. First, for $k = 1$ the agents in $v[1]$ are allocated $m_1 = |v[1]|$ units of good. Then for $k = 2$ agents in $v[2]$ are allocated $m_2 = |v[2]|$ and so on for each k till $k = \overline{m}(v) - 1$. The index $\overline{m}(v)$ corresponds to the set of agents of lowest valuation i.e. $v[\overline{m}(v)]$ to which the remaining units of good $m_{\overline{m}(v)}$ are allocated such that $m_1 + m_2 + \ldots + m_{\overline{m}(v)} = m$ and $\overline{m}(v) \leq |v[\overline{m}(v)]|$. The agents in set $v[\overline{m}(v)+1], \ldots, v[n_0(v)]$ are never allocated any unit of the good at any valuation profile.

Definition **4.3** *An allocation rule f is efficient if at every valuation profile $v \in V^n$*

(i) the agents in $v[k]$ for each $k \in \{1, 2, \ldots, \overline{m}(v) - 1\}$ are allocated $m_k = |v[k]|$ units of good

(ii) the agents in $v[\overline{m}(v)]$ are allocated the remaining units of good

(iii) the agents in $v[k]$ for each $k \in \{\overline{m}(v) + 1, \ldots, n_0\}$ are never allocated any unit of good,

and the allocation probabilities are such that

$$\sum_{i \in \cup_{k=1}^{\overline{m}(v)} v[k]} f_i(v) = m$$

A mechanism $M \equiv (f, p)$ is efficient if allocation rule f is efficient.

Efficiency requires all units to be allocated to the top m highest valued agents at any valuation profile with probability one.

A *probability-burning allocation function f* satisfies the following properties: for all valuation profiles v,

(i) $f_i(v) = 0$ for all $i \in v[k]$, $k \in \{\bar{m}(v) + 1, \ldots, n_0(v)\}$

(ii) $\sum_{i \in N} f_i(v) \leq m$.

A probability-burning mechanism is a pair (f, p) where f is a probability-burning allocation function. Note that a probability-burning allocation function assigns the units of good with positive probability only to agents who have the m highest valuations. However, all units of the goods may not be allocated with probability one. It allows for the possibility that at some valuation profile v, $\sum_{i \in N} f_i(v) < m$, i.e. units of the good are wasted or probability is "burnt". This is a violation of efficiency and will occur when probability-burning mechanisms are required to additionally satisfy incentive-compatibility and budget-balance.

4.4 The Equal-probability-burning mechanism

Our goal in this section is to introduce the equal-probability-burning mechanism and compare its welfare properties with two other BB, IR and DSIC mechanisms. We begin with an informal description.

In the equal-probability-burning (EP) mechanism, all units of good are allocated through a multi-unit Vickrey auction. The probability auctioned for each unit of good is the same and it depends on the valuations of $(m+1)^{th}$ and $(m+2)^{th}$ ranked agents i.e. the agents immediately after top m valuations. All agents who receive a unit of good pay the same amount each which is a convex combination of valuations of $(m+1)^{th}$ and $(m+2)^{th}$ agents. As m agents receive one unit each, the total revenue generated is $\frac{m}{n}\big((n-m-1)v_{(m+2)}+(m+1)v_{(m+1)}\big)$. All the revenue is distributed back to the agents and the amount that each agent receives does not depend on the agent's own valuation. The structure of EP mechanism is similar to that of MS mechanism and is its straight-forward extension to the multi-unit case.

When there are ties in valuations of agents, the units of good are allocated in a sequential manner with the agents having highest valuation being allocated first, the agents having second-highest valuation being allocated next and so on until all the units have been assigned. We now provide a detailed description of the mechanism by including the tie-breaking rules:

1. The agents report their valuations $v_1, v_2, \ldots, v_n$. The set of agents is partitioned into the sets $v[1], v[2], \ldots, v[n_0]$ and $\overline{m}(v)$ is computed.
2. The allocation is done as follows:
   (a) Allocate $m_1 = |v[1]|$ units of good to agents in set $v[1]$ with each agent receiving one unit of good with probability $\big(1-\frac{(m+1)}{n}\big)+\big(\frac{(m+1)}{n}\big)\frac{v_{(m+2)}}{v_{(m+1)}}$.
   (b) Repeat this for each $k \in \{2,3,\ldots,\overline{m}(v)-1\}$ such that each set of agents $v[k]$ receives $m_k = |v[k]|$ units of good.
   (c) Allocate $m_{\overline{m}(v)}$ units of good to agents in the set $v[\overline{m}(v)]$. Each agent is allocated a unit of good with probability $\frac{m_{\overline{m}(v)}}{|v[\overline{m}(v)]|}\big(\frac{n-m-1}{n}+\frac{(m+1)}{n}\frac{v_{(m+2)}}{v_{(m+1)}}\big)$.
   (d) Agents in the sets $v[\overline{m}(v)+1], \ldots, v[n_0(v)]$ are not allocated anything.
3. Agents who receive a unit of good pay $\big(1-\frac{(m+1)}{n}\big)v_{(m+1)}+\big(\frac{m+1}{n}\big)v_{(m+2)}$ each.
4. The surplus is redistributed as follows:
   (a) Agents 1 to $m+1$ receive $\frac{m}{n}v_{(m+2)}$ each.

(b) Agents $m+2$ to n receive $\frac{m}{n}v_{(m+1)}$ each.

The allocation functions are:

$$f_{(i)}^{EP}(v) = \begin{cases} \frac{m_k}{|v[k]|}\left(\left(1 - \frac{(m+1)}{n}\right) + \frac{(m+1)}{n}\frac{v_{(m+2)}}{v_{(m+1)}}\right) & \text{, if } i \in v[k] \text{ for } k \in \{1, 2, \dots, \overline{m}(v)\} \\ 0 & \text{, otherwise} \end{cases}$$

The payment functions are given by:

$$p_{(i)}^{EP}(v) = p_{(i)}^{EP}(0, v_{-(i)}) + v_{(i)} f_{(i)}^{EP}(v) - \int_0^{v_{(i)}} f_{(i)}^{EP}(x, v_{-(i)})dx$$

Here,

$$p_{(i)}^{EP}(0, v_{-(i)}) = \begin{cases} -\frac{m}{n}v_{(m+2)} & \text{, if } i \in \{1, 2, \dots, m+1\} \\ -\frac{m}{n}v_{(m+1)} & \text{, if } i \in \{m+2, \dots, n\} \end{cases}$$

The following examples illustrate the mechanism. In each case, let $N = \{1, 2, 3, 4, 5\}$ and $m = 2$.

Example **4.1** Let there be a valuation profile such that $v_2 = v_3 = v_4 > v_1 > v_5$. Here, $v[1] = \{2, 3, 4\}$, $v[2] = \{1\}$ and $v[3] = \{5\}$ and $\overline{m}(v) = 1$. The allocation probabilities are $f_2(v) = f_3(v) = f_4(v) = \frac{2}{3}\left(\frac{2}{5} + \frac{3}{5}\frac{v_1}{v_4}\right)$ and $f_1(v) = f_5(v) = 0$. Agents 2, 3 and 4 pay $\frac{2}{3}\left(\frac{2}{5} + \frac{3}{5}\frac{v_1}{v_4}\right)v_4$ each. Agents $2, 3$ and 4 receive $\frac{2}{5}v_1$ each and agents 1 and 5 receive $\frac{2}{5}v_4$ each. So, $p_2(v) = p_3(v) = p_4(v) = \frac{4}{15}v_4$ and $p_1(v) = p_5(v) = -\frac{2}{5}v_4$.

Example **4.2** Let there be a valuation profile such that $v_2 > v_5 = v_1 = v_4 = v_3$. Here, $v[1] = \{2\}$, $v[2] = \{1, 3, 4, 5\}$. So, $m_1 = 1$ and $\overline{m}(v) = 2$. The allocation probabilities are $f_2(v) = \frac{2}{5} + \frac{3}{5}\frac{v_4}{v_1} = \frac{2}{5} + \frac{3}{5} = 1$ and $f_5(v) = f_1(v) = f_4(v) = f_3(v) = \frac{1}{4}\left(\frac{2}{5} + \frac{3}{5}\frac{v_4}{v_1}\right) = \frac{1}{4}$. Agent 2 pays v_1 and agents $1, 3, 4$ and 5 pay $\frac{v_1}{4}$. All agents receive $\frac{2}{5}v_1$ each. So, $p_2(v) = \frac{3}{5}v_1$ and $p_1(v) = p_3(v) = p_4(v) = p_5(v) = -\frac{3}{20}v_1$.

Example **4.3** Let there be a valuation profile such that $v_5 > v_1 > v_3 > v_4 > v_2$. Here, $v[1] = \{5\}$, $v[2] = \{1\}$, $v[3] = \{3\}$, $v[4] = \{4\}$, $v[5] = \{2\}$. So, $m_1 = 1$ and $\overline{m}(v) = 2$. The allocation probabilities are $f_5(v) = f_1(v) = \frac{2}{5} + \frac{3}{5}\frac{v_4}{v_3}$ and $f_3(v) = f_4(v) = f_2(v) = 0$. Agents 5 and 1 pay $\left(\frac{2}{5} + \frac{3}{5}\frac{v_4}{v_3}\right)v_3$ each. Agents $5, 1$ and 3 receive $\frac{2}{5}v_4$ each and agents 4 and 2 receive $\frac{2}{5}v_3$ each. So, $p_5(v) = p_1(v) = \frac{2}{5}v_3 + \frac{1}{5}v_4$, $p_3(v) = -\frac{2}{5}v_4$ and $p_4(v) = p_2(v) = -\frac{2}{5}v_3$.

As is evident from the examples, the mechanism is BB. The mechanism is also IR as each agent gets non-negative utility. We illustrate that it is DSIC. In the Example 4.3, let agent 4 report v'_4 such that $v'_4 > v_5 > v_1 > v_3 > v_2$. Agent 4 is allocated the good with probability $\frac{2}{5} + \frac{3}{5}\frac{v_3}{v_1}$ and has to pay $\frac{2}{5}v_1 + \frac{1}{5}v_3$. The change in his utility is

$$\Big(\frac{2}{5} + \frac{3}{5}\frac{v_3}{v_1}\Big)v_4 - \Big(\frac{2}{5}v_1 + \frac{1}{5}v_3\Big) - \frac{2}{5}v_3 = \Big(\frac{2}{5} + \frac{3}{5}\frac{v_3}{v_1}\Big)(v_4 - v_1) < 0$$

Agent 4 has no incentive to report v'_4. Similarly, it can be shown that no agent has any incentive to misreport, i.e. the mechanism is DSIC.

The next proposition generalizes this.

Proposition **4.1** *The EP mechanism is BB, IR and DSIC.*

The proof is in the Appendix.

In the next subsection, we compare EP mechanism with different BB, IR and DSIC mechanisms found in the literature.

4.4.1 Comparison of welfare properties of mechanisms

We first give a brief description of two mechanisms that are found in the literature of allocation of goods among agents. The two mechanisms are the multi-unit version of the Green-Laffont (GL) mechanism and the single-unit burning (SU) mechanism as given by Guo and Conitzer (2014). Then we compare the welfare properties and worst-case efficiency properties of equal-probability-burning mechanism with these mechanisms.

Multi-unit Green-Laffont mechanism: There are m units of a good which are to be allocated to n agents. An agent is picked at random and a multi-unit Vickrey auction is conducted among rest of the agents. The revenue that is generated is given to the agent that was picked out. For instance, suppose there are 4 agents and 2 units of a good. Let there be a valuation profile such that $v_3 > v_1 > v_4 > v_2$. If agent 1 is excluded then agents 3 are 4 are allocated the units of good and they pay v_2 each which is given to agent 1. Formally, the allocation and transfer functions are:

$$f^{GL}_{(i)}(v) = \begin{cases} 1 - \frac{1}{n} & \text{, if } i \in \{1, 2, \dots, m\} \\ \frac{m}{n} & \text{, if } i = m+1 \\ 0 & \text{, otherwise} \end{cases}$$

The payment functions are given by:

$$p^{GL}_{(i)}(v) = \begin{cases} \left(1 - \frac{m+1}{n}\right)v_{(m+1)} & \text{, if } i \in \{1, 2, \ldots, m\} \\ 0 & \text{, if } i = m+1 \\ -\frac{m}{n}v_{(m+1)} & \text{, if } i \in \{m+2, \ldots, n\} \end{cases}$$

EXAMPLE **4.4** Let $N = \{1, 2, 3, 4\}$ and $m = 2$. Let there be a valuation profile such that $v_3 > v_1 > v_4 > v_2$. Agents 3 and 1 get one unit of good each with allocation probability $f_3(v) = f_1(v) = \frac{3}{4}$. Agent 4 gets a unit of good with probability $f_4(v) = \frac{1}{2}$. Agent 3 and 1 pay $\frac{v_4}{4}$ each and agent 2 receives $\frac{v_4}{2}$.

By construction, the mechanism is BB. Also, as the multi-unit Vickrey auction is DSIC and IR, the GL mechanism is DSIC and IR.

Single-unit burning mechanism: There are m units of a good which are to be allocated to n agents. Out of these m units, $m-1$ units of good are allocated by the standard multi-unit Vickrey auction. With a probability of $\frac{m-1}{n-1}$, the good that is left is allocated and with probability of $\frac{n-m}{n-1}$ the unit of good is burnt. If the m^{th} good is allocated, all the agents pay $v_{(m+1)}$ each, and if it is burnt then they pay v_m each. Each agent receives a redistribution amount of $\frac{m-1}{n-1}$ times the valuation of m^{th} highest agent from among the rest of the $n-1$ agents. For instance, let $n = 5$ and $m = 3$. Let valuation profile be $v_4 > v_1 > v_5 > v_3 > v_2$. Agents 4 and 1 are allocated one unit of good each. The third unit of good is allocated with probability $\frac{1}{2}$ to agent 5 and with rest of the probability it is burnt. If the third unit of good is allocated, then agents 4, 1 and 5 pay v_3 each. If the unit of good is destroyed, then agents 4 and 1 pay v_5 each. Agent 4, 1 and 5 receive $\frac{v_3}{2}$ and agents 3 and 2 receive $\frac{v_5}{2}$. Formally, suppose the valuations of agents are $v_1 > v_2 > \ldots > v_n$. The allocation and transfer functions are:

$$f^{SU}_{(i)}(v) = \begin{cases} 1 & \text{, if } i \in \{1, 2, \ldots, m-1\} \\ \frac{m-1}{n-1} & \text{, if } i = m \\ 0 & \text{, otherwise} \end{cases}$$

$$p^{SU}_{(i)}(v) = p^{SU}_{(i)}(0, v_{-(i)}) + v_{(i)} f^{SU}_{(i)}(v) - \int_0^{v_{(i)}} f^{SU}_{(i)}(x, v_{-(i)})dx$$

Here,

$$p_{(i)}(0, v_{-(i)}) = \begin{cases} -\left(\frac{m-1}{n-1}\right)v_{(m+1)} & \text{, if } i \in \{1, 2, \ldots, m\} \\ -\left(\frac{m-1}{n-1}\right)v_{(m)} & \text{, if } i \in \{m+1, \ldots, n\} \end{cases}$$

Example **4.5** Let $N = \{1,2,3,4,5\}$ and $m = 3$. Let there be a valuation profile such that $v_4 > v_1 > v_5 > v_3 > v_2$. Agents 4 and 1 get one unit of good each with allocation probability $f_4(v) = f_1(v) = 1$. Agent 5 gets a unit of good with probability $f_3(v) = \frac{1}{2}$. The payments are $p_4(v) = p_1(v) = \frac{v_5}{2}$, $p_5(v) = 0$, $p_3(v) = p_2(v) = -\frac{v_5}{2}$. Suppose agent 3 reports v_3' such that $v_3' > v_4 > v_1 > v_5 > v_2$. He gets a unit of good with probability 1 and has to pay $\frac{v_1}{2}$. His utility is $v_3 - \frac{v_1}{2}$. The change in utility is $v_3 - \frac{v_1}{2} - \frac{v_5}{2} = \frac{1}{2}(v_3 - v_1 + v_3 - v_5) < 0$. The agent has no incentive to misreport as v_3'. Similarly, none of the other agents has any incentive to misreport. The mechanism is DSIC. Also, each agent gets non-negative utility. So the mechanism is IR. The sum of payments is zero and hence the mechanism is BB.

The Table 4.1 gives the payoff of each agent in the 3 mechanisms when $N = \{1,2,3,4,5\}$ and $m = 2$. Notice that only the GL mechanism allocates the good to agents below top m valuations. The SU mechanism allocates the units of good unequally. It allocates all the units of good except one with probability of 1 and with positive probability destroys the last unit of good. The lowest ranked agent to be allocated the good in GL and SU mechanisms does not pay or receive any amount.

$n=5, m=2$	Agents				
Mechanisms	(1)	(2)	(3)	(4)	(5)
EP	$(\frac{2}{5} + \frac{3}{5}\frac{v_{(4)}}{v_{(3)}}, \frac{1}{5}v_{(4)} + \frac{2}{5}v_{(3)})$	$(\frac{2}{5} + \frac{3}{5}\frac{v_{(4)}}{v_{(3)}}, \frac{1}{5}v_{(4)} + \frac{2}{5}v_{(3)})$	$(0, -\frac{2}{5}v_{(4)})$	$(0, -\frac{2}{5}v_{(3)})$	$(0, -\frac{2}{5}v_{(3)})$
GL	$(\frac{4}{5}, \frac{2}{5}v_{(3)})$	$(\frac{4}{5}, \frac{2}{5}v_{(3)})$	$(\frac{2}{5}, 0)$	$(0, -\frac{2}{5}v_{(3)})$	$(0, -\frac{2}{5}v_{(3)})$
SU	$(1, \frac{3}{4}v_{(2)})$	$(\frac{1}{4}, 0)$	$(0, -\frac{1}{4}v_{(2)})$	$(0, -\frac{1}{4}v_{(2)})$	$(0, -\frac{1}{4}v_{(2)})$

Table 4.1: Allocation probabilities and payments of agents in different mechanisms

Given a valuation profile v, the welfare generated by these 3 mechanisms are:

$$\begin{aligned}
W^{EP}(v) &= \left(\left(1 - \frac{m+1}{n}\right) + \left(\frac{m+1}{n}\right)\frac{v_{(m+2)}}{v_{(m+1)}}\right)(v_{(1)} + v_{(2)} + \ldots + v_{(m)}) \\
W^{GL}(v) &= \left(1 - \frac{1}{n}\right)v_{(1)} + \ldots + \left(1 - \frac{1}{n}\right)v_{(m)} + \frac{m}{n}v_{(m+1)} \\
&= \left(1 - \frac{1}{n}\right)(v_{(1)} + v_{(2)} + \ldots + v_{(m)}) + \frac{m}{n}v_{(m+1)} \\
W^{SU}(v) &= (v_{(1)} + v_{(2)} + \ldots + v_{(m-1)}) + \left(\frac{m-1}{n-1}\right)v_{(m)}
\end{aligned}$$

Suppose $n = 4$ and $m = 2$. The welfares of these three mechanisms are

$$W^{GL}(v) = \frac{3}{4}(v_{(1)} + v_{(2)}) + \frac{1}{2}v_{(3)}$$

$$W^{EP}(v) = (\frac{1}{4} + \frac{3}{4}\frac{v_4}{v_3})(v_{(1)} + v_{(2)})$$
$$W^{SU}(v) = v_{(1)} + \frac{1}{3}v_{(2)}$$

If $v = (0.9, 0.8, 0.7, 0.1)$, then $W^{GL} > W^{EP}$ and if $v = (0.9, 0.8, 0.7, 0.7)$ then $W^{GL} < W^{EP}$. So, there exists a set of valuation profiles with a positive Lebesgue measure where EP mechanism generates more welfare than the GL mechanism.

We state the main result of this section.

THEOREM **4.1** *1.* $\alpha^{GL} > \alpha^{EP}$

2. $\alpha^{EP} > \alpha^{SU}$ *if* $n > m + \frac{m^2}{2} + \sqrt{m(m^2-1) + \frac{m^4}{4}}$.

3. For uniform distribution G, $\mathbb{E}[W^{GL}] - \mathbb{E}[W^{EP}] = O\big(\frac{1}{n^2}\big)$

Proof: The worst-case efficiency ratios for the three mechanisms are

$$\alpha^{EP} = 1 - \frac{m+1}{n}$$
$$\alpha^{GL} = 1 - \frac{1}{n}$$
$$\alpha^{SU} = \frac{n(m-1)}{m(n-1)}$$

Clearly, the worst-case efficiency ratio of GL mechanism is higher than that of EP mechanism for all values of m. The worst-case efficiency ratio of EP mechanism is higher than that of the SU mechanism if

$$1 - \frac{m+1}{n} > \frac{n(m-1)}{m(n-1)}$$
$$\text{or } n > m + \frac{m^2}{2} + \sqrt{m(m^2-1) + \frac{m^4}{4}}$$

The expected total welfare of GL mechanism is:

$$\begin{aligned}
\mathbb{E}[W^{GL}] &= \mathbb{E}\Big[\Big(1 - \frac{1}{n}\Big)v_{(1)} + \ldots + \Big(1 - \frac{1}{n}\Big)v_{(m)} + \frac{m}{n}v_{(m+1)}\Big] \\
&= \Big(1 - \frac{1}{n}\Big)(E[v_{(1)}] + E[v_{(2)}] + \ldots + E[v_{(m)}]) + \frac{m}{n}E[v_{(m+1)}] \\
&= \Big(1 - \frac{1}{n}\Big)\Big(\frac{n}{n+1} + \frac{n-1}{n+1} + \ldots + \frac{n-m+1}{n}\Big) + \frac{m}{n}\Big(\frac{n-m}{n}\Big) \\
&= \frac{m}{2n(n+1)}\Big((2n-m+1)(n-1) + 2(n-m)\Big)
\end{aligned}$$

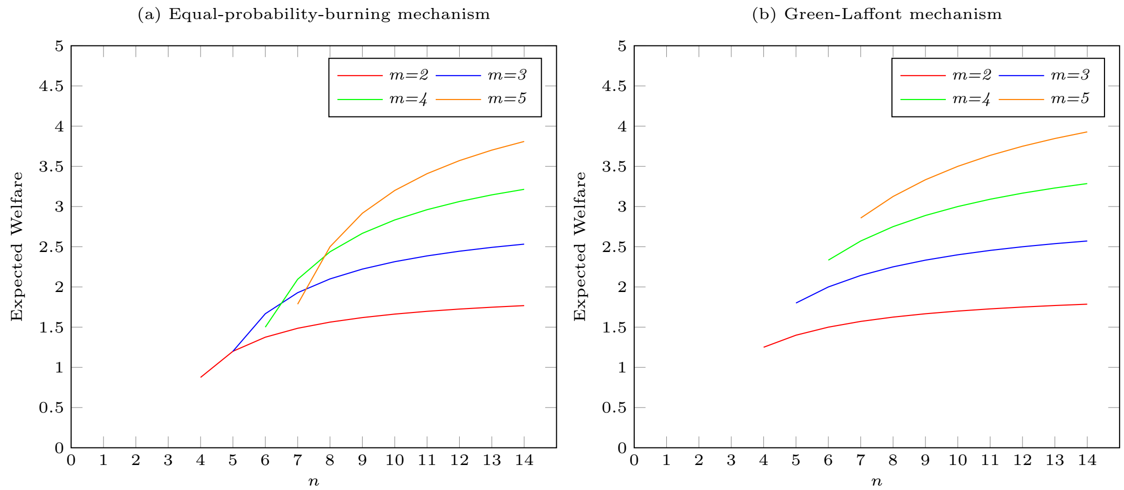


Figure 4.1: Expected welfare of mechanims for uniform distribution

The expected total welfare of EP mechanism is:

$$\begin{aligned}
\mathbb{E}[W^{EP}] &= \mathbb{E}\Big[\Big(\Big(1 - \frac{(m+1)}{n}\Big) + \big(\frac{m+1}{n}\big)\frac{v_{(m+2)}}{v_{(m+1)}}\Big)(v_{(1)} + v_{(2)} + \ldots + v_{(m)})\Big] \\
&= \Big(1 - \frac{m+1}{n}\Big)(\mathbb{E}[v_{(1)}] + \mathbb{E}[v_{(2)}] + \ldots + \mathbb{E}[v_{(m)}]) \\
&\quad + \frac{(m+1)}{n}\Big(\mathbb{E}\Big[\frac{v_{(m+2)}}{v_{(m+1)}}v_{(1)}\Big] + \mathbb{E}\Big[\frac{v_{(m+2)}}{v_{(m+1)}}v_{(2)}\Big] + \ldots + \mathbb{E}\Big[\frac{v_{(m+2)}}{v_{(m+1)}}v_{(m)}\Big]\Big) \\
&= \frac{m(2n-m+1)}{2(n+1)}\Bigg(\frac{n-m-1}{n} + \Big(\frac{(m+1)}{n}\Big)\Big(\frac{n-m-1}{n-m}\Big)\Bigg) \\
&= \frac{m(n-m-1)(2n-m+1)}{2n(n-m)}
\end{aligned}$$

The expected total welfare of SU mechanism is:

$$\begin{aligned}
\mathbb{E}[W^{SU}] &= \mathbb{E}\Big[(v_{(1)} + v_{(2)} + \ldots + v_{(m-1)} + \frac{m-1}{n-1}v_{(m)})\Big] \\
&= \mathbb{E}[v_{(1)}] + \mathbb{E}[v_{(2)}] + \ldots + \mathbb{E}[v_{(m-1)}] + \frac{m-1}{n-1}\mathbb{E}[v_{(m)}] \\
&= \frac{(2n-m)(m-1)}{2(n-1)}
\end{aligned}$$

Now,

$$\begin{aligned}
\mathbb{E}[W^{GL}] - \mathbb{E}[W^{EP}] = \frac{m}{2n(n+1)}\Big((2n-m+1)(n-1) + 2(n-m)\Big) \\
- \frac{m(n-m-1)(2n-m+1)}{2n(n-m)}
\end{aligned}$$

$$= \frac{m(m+1)}{2n(n-m)} \tag{4.1}$$

Thus, $\mathbb{E}[W^{GL}] > \mathbb{E}[W^{EP}]$ for all values of n and m. As n increases the difference between the two values decreases. The expression in (4.1) converges to zero at the rate $\frac{1}{n^2}$. ■

In Figure 4.1, notice that for small values of n, the gap between the ex-ante expected welfares of the two mechanisms is large irrespective of values of m. Irrespective of the number of units of goods, the difference between expected total welfare of GL mechanism and the EP mechanism approaches zero as the number of agents increases. Thus, for large values of n, both the mechanisms give approximately the same expected total welfare. As illustrated by Mishra and Sharma (2018) in the single-good case, the comparison between the expected total welfare of the two mechanisms is difficult for a general distribution of valuations.

4.5 Probability-burning mechanisms with a reserve price

In this section we show that introducing a reserve price in the equal probability burning mechanism may improve the expected welfare generated by probability-burning mechanisms.

Consider the following variant of the probability-burning mechanism. Fix a reserve price r. The allocation function allocates a unit of good to the top m agents only when the valuation of *each* of them is above r. Even if the valuation of m^{th} agent falls below the reserve price, no good is allocated to any agent. The allocation probabilities depend on the valuations of $(m+1)^{th}$ and $(m+2)^{th}$ agents. The mechanism differs from the equal-probability-burning mechanism when the valuation of $(m+2)^{th}$ agent drops below the reserve price. When the valuation profile is such that $v_{(m+1)} \geq r > v_{(m+2)}$, the allocation probabilities depend on the reserve price and $v_{(m+1)}$, and if $v_{(m+1)} < r$, the allocation probabilities are constant values.

We give a formal description of the mechanism which we call the BR mechanism. Recall $\overline{m}(v)$ from the previous section. Let $m_r(v)$ be the index corresponding to agents with least valuation such that $v_i \geq r$ for $i \in v[m_r(v)]$ and $v_i < r$ for all $i \in v[k]$ where $k \in \{m_r(v)+1, \ldots, n_0(v)\}$.

1. The agents report their valuations $v_1, v_2, \ldots, v_n$. The set of agents is partitioned into the sets $v[1], v[2], \ldots, v[n_0]$, and $\overline{m}(v)$ and $m_r(v)$ are computed.

2. If $\overline{m}(v) \leq m_r(v)$, then all the m units are allocated through the top-only allocation rule.

The allocation probabilities are calculated depending on whether $v_{(m+1)}$ and $v_{(m+2)}$ are higher than r or not and are as follows:

(a) If $\overline{m}(v) \leq m_r(v)$ and $v_{(m+2)} \in v[k]$ for some $k \leq m_r(v)$, then for each unit of good the probability $\frac{m_k}{|v[k]|}\big((1-\frac{(m+1)}{n})+\frac{(m+1)}{n}\frac{v_{(m+2)}}{v_{(m+1)}}\big)$ where $k \in \{1,2,\ldots,\overline{m}(v)\}$ is allocated through the multi-unit Vickrey auction with reserve price.

(b) If $\overline{m}(v) \leq m_r(v)$, $v_{(m+1)} \in v[j]$ for some $j \leq m_r(v)$, and $v_{(m+2)} \in v[m_r(v)+1]$, then for each unit of good the probability $\frac{m_k}{|v[k]|}\big((1-\frac{(m+1)}{n})+\frac{(m+1)}{n}\frac{r}{v_{(m+1)}}\big)$ where $k \in \{1,2,\ldots,\overline{m}(v)\}$ is allocated through the multi-unit Vickrey auction with reserve price.

(c) If $\overline{m}(v) = m_r(v)$ and $v_{(m+1)} \in v[\overline{m}(v)+1]$, then for each unit of good the probability $1-\frac{m}{n}$ is allocated through the multi-unit Vickrey auction with reserve price.

3. Agents make the following payment:

(a) If $\overline{m}(v) \leq m_r(v)$ and $v_{(m+2)} \in v[k]$ for some $k \leq m_r(v)$, then agents in sets $i \in v[k]$ where $k \in \{1,2,\ldots,\overline{m}(v)\}$ pay an amount of $\frac{m_k}{|v[k]|}\Big((1-\frac{(m+1)}{n})+\frac{(m+1)}{n}\frac{v_{(m+2)}}{v_{(m+1)}}\Big)v_{(m+1)}$ each.

(b) If $\overline{m}(v) \leq m_r(v)$, $v_{(m+1)} \in v[j]$ for some $j \leq m_r(v)$, and $v_{(m+2)} \in v[m_r(v)+1]$, then agents in sets $i \in v[k]$ where $k \in \{1,2,\ldots,\overline{m}(v)\}$ pay an amount of $\frac{m_k}{|v[k]|}\Big((1-\frac{(m+1)}{n})+\frac{(m+1)}{n}\frac{r}{v_{(m+1)}}\Big)v_{(m+1)}$ each.

(c) If $\overline{m}(v) = m_r(v)$ and $v_{(m+1)} \in v[\overline{m}(v)+1]$, then agents in sets $i \in v[k]$ where $k \in \{1,2,\ldots,\overline{m}(v)\}$ pay an amount of $\big(1-\frac{m}{n}\big)r$ each.

4. The generated revenue is reallocated to the agents as follows:

(a) If $\overline{m}(v) \leq m_r(v)$ and $v_{(m+2)} \in v[k]$ for some $k \leq m_r(v)$, then each of the top $m+1$ agents receives an amount of $\frac{m}{n}v_{(m+2)}$ and rest $n-m-1$ agents receive an amount of $\frac{m}{n}v_{(m+1)}$ each.

(b) If $\overline{m}(v) \leq m_r(v)$, $v_{(m+1)} \in v[j]$ for some $j \leq m_r(v)$, and $v_{(m+2)} \in v[m_r(v)+1]$, then each of the top $m+1$ agents receives an amount of $\frac{m}{n}r$ and rest of the $n-m-1$ agents receive an amount of $\frac{m}{n}v_{(m+1)}$ each.

(c) If $\overline{m}(v) = m_r(v)$ and $v_{(m+1)} \in v[\overline{m}(v)+1]$, then the lowest $n-m$ agents receive an amount $\frac{m}{n}r$ each.

5. If $\overline{m}(v) > m_r(v)$, no unit of good is allocated to any agent and there is no payment or reallocation to any agent.

The allocation functions of BR mechanism are:

$$f_{(i)}^{BR}(v) = \begin{cases} \frac{m_k}{|v[k]|}\left(\left(1-\frac{m+1}{n}\right)+\frac{m+1}{n}\frac{v_{(m+2)}}{v_{(m+1)}}\right) & \text{, if } i \in v[k], k \in \{1,2,\ldots,\overline{m}(v)\}, v_{(m+1)}, v_{(m+2)} \geq r \\ \frac{m_k}{|v[k]|}\left(\left(1-\frac{m+1}{n}\right)+\frac{m+1}{n}\frac{r}{v_{(m+1)}}\right) & \text{, if } i \in v[k], k \in \{1,2,\ldots,\overline{m}(v)\}, v_{(m+1)} \geq r, v_{(m+2)} < r \\ \frac{m_k}{|v[k]|}\left(1-\frac{m}{n}\right) & \text{, if } i \in v[k], k \in \{1,2,\ldots,\overline{m}(v)\}, v_{(m)} \geq r, v_{(m+1)}, v_{(m+2)} < r \\ 0 & \text{, otherwise} \end{cases}$$

The payment functions are given by:

$$p_{(i)}^{BR}(v) = p_{(i)}^{BR}(0, v_{-(i)}) + v_{(i)} f_{(i)}^{BR}(v) - \int_0^{v_{(i)}} f_{(i)}^{BR}(x, v_{-(i)})dx$$

Here,

$$p_{(i)}^{BR}(0, v_{-(i)}) = \begin{cases} -\frac{m}{n}v_{(m+2)} & \text{, if } i \in \{1,2,\ldots,m+1\} \text{ and } v_{(m+1)}, v_{(m+2)} \geq r \\ -\frac{m}{n}v_{(m+1)} & \text{, if } i \in \{m+2,\ldots,n\} \text{ and } v_{(m+1)} \geq r \\ -\frac{m}{n}r & \text{, if } i \in \{1,2,\ldots,m+1\}, v_{(m)}, v_{(m+1)} \geq r, v_{(m+2)} < r \\ -\frac{m}{n}r & \text{, if } i \in \{m+2,\ldots,n\}, v_{(m)} \geq r, v_{(m+1)}, v_{(m+2)} < r \end{cases}$$

The following examples illustrate the mechanism. In each case, let $N = \{1, 2, 3, 4, 5\}$ and $m = 2$.

Example **4.6** Consider the valuation profile v where $v_3 > v_2 > v_4 > r > v_1 > v_5$. The allocation probabilities are $f_3(v) = f_2(v) = \frac{2}{5} + \frac{3}{5}\frac{r}{v_4}$ and $f_4(v) = f_1(v) = f_5(v) = 0$. Agents 3 and 2 pay $\left(\frac{2}{5} + \frac{3}{5}\frac{r}{v_4}\right)v_4$ each. Agents $3, 2$ and 4 receive $\frac{2}{5}r$ each and agents 1 and 5 receive $\frac{2}{5}v_4$ each. So, $p_3(v) = p_2(v) = \frac{2}{5}v_4 + \frac{1}{5}r$, $p_4(v) = -\frac{2}{5}r$ and $p_1(v) = p_5(v) = -\frac{2}{5}v_4$.

Example **4.7** Consider the valuation profile v where $v_2 > v_5 > r > v_1 > v_4 > v_3$. The allocation probabilities are $f_2(v) = f_5(v) = \frac{3}{5}$ and $f_1(v) = f_4(v) = f_3(v) = 0$. Agents 2 and 5 pay $\frac{3}{5}r$ each. Agents $1, 4$ and 3 receive $\frac{2}{5}r$ each. So, $p_2(v) = p_5(v) = \frac{3}{5}r$, and $p_1(v) = p_4(v) = p_3(v) = -\frac{2}{5}r$.

Example **4.8** Consider the valuation profile v where $v_5 > v_1 > v_3 > v_4 > r > v_2$. The allocation probabilities are $f_5(v) = f_1(v) = \frac{2}{5} + \frac{3}{5}\frac{v_4}{v_3}$ and $f_3(v) = f_4(v) = f_2(v) = 0$. Agents 5 and 1 pay $\left(\frac{2}{5} + \frac{3}{5}\frac{v_4}{v_3}\right)v_3$ each. Agents $5, 1$ and 3 receive $\frac{2}{5}v_4$ each and agents 4 and 2 receive $\frac{2}{5}v_3$ each. So, $p_5(v) = p_1(v) = \frac{2}{5}v_3 + \frac{1}{5}v_4$, $p_3(v) = -\frac{2}{5}v_4$ and $p_4(v) = p_2(v) = -\frac{2}{5}v_3$.

Clearly, the mechanism is BB and also IR as each agent gets non-negative utility. We illustrate that it is DSIC as well. In the Example 4.7, let agent 3 report v_3' such that $v_2 > v_3' > v_5 > r > v_1 > v_4$. Agent 3 is allocated the good with probability $\frac{2}{5} + \frac{3}{5}\frac{r}{v_5}$ and has to pay $\frac{2}{5}v_5 + \frac{1}{5}r$. The change in his utility is

$$\Big(\frac{2}{5} + \frac{3}{5}\frac{r}{v_5}\Big)v_3 - \Big(\frac{2}{5}v_5 + \frac{1}{5}r\Big) - \frac{2}{5}r = \Big(\frac{2}{5} + \frac{3}{5}\frac{r}{v_5}\Big)(v_3 - v_5) < 0$$

Agent 3 has no incentive to report v_3'. Similarly, it can be shown that no agent has any incentive to misreport. The mechanism is DSIC. The next proposition generalizes this.

Proposition 4.2 *The BR mechanism is BB, IR and DSIC.*

The proof is in the Appendix. The next subsection discusses the welfare properties of this mechanism.

4.5.1 Welfare properties

In auction theory, the seller's revenue can be improved by setting a reserve price (see Krishna (2009) for details). In our setting where budgets are balanced, we show nevertheless that reserve prices are useful in improving the expected welfare generated by the mechanism.

We first give the formal description of BR mechanism when there is a single good. Substituting $m = 1$, the allocation functions are as follows:

$$f^{BR}_{(i)}(v) = \begin{cases} \frac{1}{|v[1]|}\Big(\big(1 - \frac{2}{n}\big) + \frac{2}{n}\frac{v_{(3)}}{v_{(2)}}\Big) & \text{, if } i \in v[1], v_{(2)}, v_{(3)} \geq r \\ \frac{1}{|v[1]|}\Big(\big(1 - \frac{2}{n}\big) + \frac{2}{n}\frac{r}{v_{(2)}}\Big) & \text{, if } i \in v[1], v_{(2)} \geq r, v_{(3)} < r \\ 1 - \frac{1}{n} & \text{, if } i \in v[1], v_{(1)} \geq r, v_{(2)}, v_{(3)} < r \\ 0 & \text{, otherwise} \end{cases}$$

The payment functions are given by:

$$p^{BR}_{(i)}(v) = p^{BR}_{(i)}(0, v_{-(i)}) + v_{(i)} f^{BR}_{(i)}(v) - \int_0^{v_{(i)}} f^{BR}_{(i)}(x, v_{-(i)})dx$$

Here,

$$p^{BR}_{(i)}(0, v_{-(i)}) = \begin{cases} -\frac{v_{(3)}}{n} & \text{, if } i \in \{1, 2\} \text{ and } v_{(2)}, v_{(3)} \geq r \\ -\frac{v_{(2)}}{n} & \text{, if } i \in \{3, \ldots, n\} \text{ and } v_{(2)} \geq r \\ -\frac{r}{n} & \text{, if } i \in \{1, 2\}, v_{(1)}, v_{(2)} \geq r, v_{(3)} < r \text{ or if } i \in \{2, 3, \ldots, n\}, v_{(1)} \geq r, v_{(2)} < r \\ 0 & \text{, if } i = 1, \text{ and } v_{(1)} \geq r, v_{(2)} < r \end{cases}$$

The total welfare generated by this mechanism is

$$W^{BR}(v) = \begin{cases} \left(\left(1-\frac{2}{n}\right)+\frac{2}{n}\frac{v_{(3)}}{v_{(2)}}\right)v_{(1)} & \text{, if } v_{(1)}, v_{(2)}, v_{(3)} \geq r \\ \left(\left(1-\frac{2}{n}\right)+\frac{2}{n}\frac{r}{v_{(2)}}\right)v_{(1)} & \text{, if } v_{(1)}, v_{(2)} \geq r \text{ and } v_{(3)}, v_{(4)}, \ldots, v_{(n)} < r \\ \left(\frac{n-1}{n}\right)v_{(1)} & \text{, if } v_{(1)} \geq r \text{ and } v_{(2)}, v_{(3)}, \ldots, v_{(n)} < r \\ 0 & \text{, otherwise} \end{cases}$$

Before we compare the welfare generated by this mechanism with that of the probability-burning mechanism of Mishra and Sharma (2018), we describe the MS mechanism in detail. The allocation rule is as follows:

$$f^{MS}_{(i)}(v) = \begin{cases} \frac{1}{|v[1]|}\left(\left(1-\frac{2}{n}\right)+\frac{2}{n}\frac{v_{(3)}}{v_{(2)}}\right) & \text{, if } i \in v[1] \\ 0 & \text{, otherwise} \end{cases}$$

The payment functions are given by:

$$p^{MS}_{(i)}(v) = p^{MS}_{(i)}(0, v_{-(i)}) + v_{(i)} f^{MS}_{(i)}(v) - \int_0^{v_{(i)}} f^{MS}_{(i)}(x, v_{-(i)})dx$$

Here,

$$p^{MS}_{(i)}(0, v_{-(i)}) = \begin{cases} -\frac{v_{(3)}}{n} & \text{, if } i \in \{1,2\} \\ -\frac{v_{(2)}}{n} & \text{, if } i \in \{3,\ldots,n\} \end{cases}$$

For a valuation profile v such that $v_{(1)} \geq r$ and $v_{(2)} < r$, the welfare generated by BR mechanism is greater than that of MS mechanism if

$$1-\frac{1}{n} > \left(1-\frac{2}{n}\right)+\frac{2}{n}\frac{v_{(3)}}{v_{(2)}} \tag{4.2}$$

$$\text{or, } v_{(2)} > 2v_{(3)}$$

So, setting a reserve price does improve the total welfare of the agents at some valuation profiles as the allocation probability is greater in the BR mechanism. Also, note that the worst-case efficiency ratio of this mechanism is zero i.e. $\alpha^{BR} = 0$.

We state the main result of this section.

Theorem 4.2 *Assume G to be uniformly distributed. Then,*

1. *for $m = 2$, $n = 4$, $r = 0$ does not maximize the expected total welfare of BR mechanism*

2. *for $m = 1$, $n \geq 3$ the reserve price that maximizes the expected total welfare of BR mechanism is $r^* = \frac{1}{e}$, and*

3. *for $m = 1$ and any n, $\mathbb{E}[W^{BR}] > \mathbb{E}[W^{MS}]$.*

Proof: The expected total welfare when $m = 2$, $n = 4$ is:

$$\mathbb{E}[W^{BR}] = \mathbb{E}\Big[\Big(\frac{1}{4} + \frac{3}{4}\frac{v_{(4)}}{v_{(3)}}\Big)(v_{(1)} + v_{(2)})|v_{(1)}, v_{(2)}, v_{(3)}, v_{(4)} \geq r\Big] Pr(v_{(1)}, v_{(2)}, v_{(3)}, v_{(4)} \geq r)$$
$$+ \mathbb{E}\Big[\Big(\frac{1}{4} + \frac{3}{4}\frac{r}{v_{(3)}}\Big)(v_{(1)} + v_{(2)})|v_{(1)}, v_{(2)}, v_{(3)} \geq r \cap v_{(4)} < r\Big] Pr(v_{(1)}, v_{(2)}, v_{(3)} \geq r$$
$$\cap\, v_{(4)} < r)$$
$$+ \mathbb{E}\Big[\Big(\frac{1}{2}\Big)(v_{(1)} + v_{(2)})|v_{(1)}, v_{(2)} \geq r \cap v_{(3)}, v_{(4)} < r\Big] Pr(v_{(1)}, v_{(2)} \geq r$$
$$\cap\, v_{(3)}, v_{(4)} < r)$$

As G is the uniform distribution, substituting in the above expression and taking derivative with respect to r, we get

$$\frac{d\mathbb{E}[W^{BR}]}{dr} = \frac{3}{2}r(3r^2 + 8r - 6\log(r) - 11)$$

For $r \in (0, 0.219)$, $\frac{d\mathbb{E}[W^{BR}]}{dr} > 0$. The expected total welfare increases as the reserve price increases from zero. Hence, when there is a reserve price the expected total welfare of agents is higher than when there is no reserve price.

The expected total welfare of BR mechanism when $m = 1$ is

$$\mathbb{E}[W^{BR}] = \mathbb{E}\Big[\Big((1 - \frac{2}{n}) + \frac{2}{n}\frac{v_{(3)}}{v_{(2)}}\Big)v_{(1)}|v_{(1)}, v_{(2)}, v_{(3)} \geq r\Big] Pr(v_{(1)}, v_{(2)}, v_{(3)} \geq r)$$
$$+ \mathbb{E}\Big[\Big((1 - \frac{2}{n}) + \frac{2}{n}\frac{r}{v_{(2)}}\Big)v_{(1)}|v_{(1)}, v_{(2)} \geq r \text{ and } v_{(3)} < r\Big] Pr(v_{(1)}, v_{(2)} \geq r \text{ and } v_{(3)} < r)$$
$$+ \mathbb{E}\Big[\Big(1 - \frac{1}{n}\Big)v_{(1)}|v_{(1)} \geq r, v_{(2)} < r\Big] Pr(v_{(1)} \geq r, v_{(2)} < r)$$
$$= \int_r^1\int_r^{v_1}\int_r^{v_2}\int_0^{v_3} \cdots \int_0^{v_{n-1}} n!\Big((1 - \frac{2}{n}) + \frac{2}{n}\frac{v_3}{v_2}\Big)v_1 g(v_n)g(v_{n-1})\dots g(v_1)dv_n \dots dv_1$$
$$+ \int_r^1\int_r^{v_1}\int_0^{r}\int_0^{v_3} \cdots \int_0^{v_{n-1}} n!\Big((1 - \frac{2}{n}) + \frac{2}{n}\frac{r}{v_2}\Big)v_1 g(v_n)g(v_{n-1})\dots g(v_1)dv_n \dots dv_1$$
$$+ \frac{(n-1)}{n}\int_r^1\int_0^{r}\int_0^{v_2} \cdots \int_0^{v_{n-1}} (n!)v_1 g(v_n)g(v_{n-1})\dots g(v_1)dv_n \dots dv_1$$

$$=\int_r^1\int_r^{v_1}\int_r^{v_2} \frac{n!}{(n-3)!}\Big(\big(1-\frac{2}{n}\big)+\frac{2}{n}\frac{v_3}{v_2}\Big)v_1 G^{n-3}(v_3)g(v_3)g(v_2)g(v_1)dv_3dv_2dv_1$$

$$+\int_r^1\int_r^{v_1} \frac{n!}{(n-2)!}\Big(\big(1-\frac{2}{n}\big)+\frac{2}{n}\frac{r}{v_2}\Big)v_1 G^{n-2}(r)g(v_2)g(v_1)dv_2dv_1$$

$$+\frac{(n-1)}{n}\int_r^1 \frac{n!}{(n-1)!}v_1 G^{n-1}(r)g(v_1)dv_1$$

$$=\frac{n!}{(n-2)!}\int_r^1\int_r^{v_1}\Big(G^{n-2}(v_2)+\int_r^{v_2}\frac{2}{nv_2}G^{n-3}(v_3)dv_3\Big)v_1 g(v_2)g(v_1)dv_2dv_1$$

$$+\frac{(n-1)}{n}\int_r^1 \frac{n!}{(n-1)!}v_1 G^{n-1}(r)g(v_1)dv_1$$

Taking the derivative with respect to r, we get

$$\frac{d\mathbb{E}[W^{BR}]}{dr}=-(n-1)rg(r)G^{n-1}(r)+(n-1)^2g(r)G^{n-2}(r)\int_r^1 v_1g(v_1)dv_1$$

$$-n(n-1)G^{n-2}(r)g(r)\int_r^1 v_1g(v_1)dv_1-(n-1)G^{n-3}(r)\int_r^1\int_r^{v_1}\frac{2v_1}{v_2}g(v_2)g(v_1)dv_2dv_1 \quad (4.3)$$

As G is the uniform distribution, equating (4.3) with zero at $r=r^*$, we have

$$-\frac{n}{2}(n-1)(r^*)^{n-2}(1-(r^*)^2)+\frac{1}{2}(n-1)(r^*)^{n-2}((r^*)^2-2\log(r^*)-1)$$
$$+\frac{n-1}{2}((n-1)(r^*)^{n-2}-(n+1)(r^*)^n)=0$$

Simplifying, we get $r^*=\frac{1}{e}=0.367$. The maximum indeed occurs at this point as at $r=r^*$,

$$\frac{d^2\mathbb{E}[W^{BR}]}{dr^2}=-(n-1)(n-2)r^{n-3}(1+\log(r))-(n-1)r^{n-3}=-\frac{(n-1)}{e^{n-3}}<0$$

The expected total welfare at this reserve price is

$$\mathbb{E}[W^{BR}]=\frac{n-2}{n-1}+\frac{1}{(n-1)e^{n-1}}$$

For $n=3$, Figure 4.2 plots the expected total welfare of the agents with respect to the reserve price. From Mishra and Sharma (2018), the MS mechanism has expected total welfare of

$$\mathbb{E}[W^{MS}]=\frac{n-2}{n-1}$$

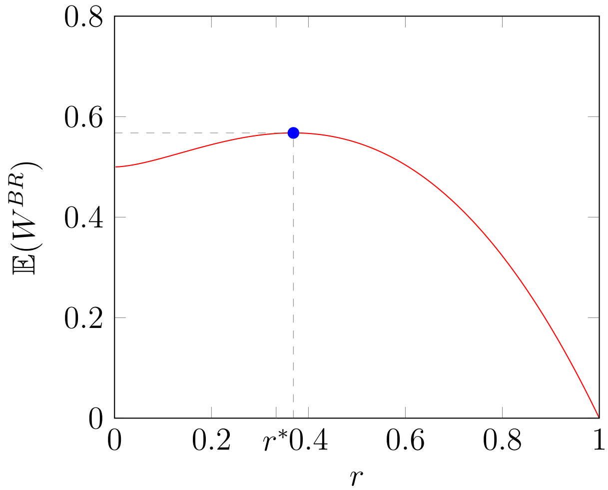


Figure 4.2: Expected total welfare of agents for uniform distribution when n=3

Clearly, $\mathbb{E}[W^{BR}] > \mathbb{E}[W^{MS}]$. ■

For the specific case of $n = 4$ and $m = 2$ we show that the expected total welfare of BR mechanism is strictly greater than when there is no reserve price (at $r = 0$ the BR mechanism is same as the EP mechanism). Also, the BR mechanism generates higher expected total welfare than the MS mechanism and as the value of n increases, the two values converge. Setting a reserve price $r > 0$ is beneficial for the mechanism designer. There is no allocation if there are not at least m agents whose valuations are above the reserve price. This leads to loss of welfare for the agents. But at some valuation profiles like the example in (4.2), the BR mechanism allocates the units of good with higher probability as compared to the MS mechanism. Thus, the BR mechanism generates higher welfare as compared to the MS mechanism which also offsets the welfare losses incurred by not allocating when the agents have low valuations.

4.6 Conclusion

In this chapter, we study probability-burning mechanisms for allocation of multiple units of good. We propose the equal-burning-mechanism and study some of its welfare properties. It remains to be seen whether the EP mechanism is welfare-undominated in the class of BB, DSIC and IR mechanisms that allocate only to the topmost agents. Also, whether it is possible to design a mechanism that allocates unequally to the agents and also generates higher total welfare compared to the EP mechanism is also an open question.

Then we propose a probability-burning mechanism when there is a reserve price. We find that the expected total welfare of such a mechanism for single good is higher than the expected total welfare of MS mechanism when the valuations are drawn from the uniform distribution. It remains to be seen whether this result still holds when the valuations are drawn from a general distribution. It will also be interesting to study how the reserve price that maximizes the expected total welfare of agents changes as m and n increase.

4.7 Appendix

Proof of Proposition 4.1:

Consider a valuation profile $v \in V^n$ such that $v_{(1)} > v_{(2)} > \ldots > v_{(n)}$. The payments made by top m agents are $\left(1 - \frac{m+1}{n} + \frac{m+1}{n}\frac{v_{(m+2)}}{v_{(m+1)}}\right)v_{(m+1)}$ each. So, $p_{(1)}(v) = \ldots = p_{(m)}(v) = \left(1 - \frac{m+1}{n}\right)v_{(m+1)} + \left(\frac{1}{n}\right)v_{(m+2)}$ and $p_{(m+1)}(v) = -\frac{m}{n}v_{(m+2)}$ and $p_{(m+2)}(v) = \ldots = p_{(n)}(v) = -\frac{m}{n}v_{(m+1)}$. Clearly, the payments are balanced i.e. $\sum\limits_{i \in N} p_{(i)}(v) = 0$. Each agent gets a non-negative utility. So the mechanism is IR.

Pick any agent i from among the lowest $n-m-1$ ranked agents. The agent's utility is $\frac{m}{n}v_{m+1}$. Let agent i misreport as $v'_{i'}$ such that his rank in new valuation profile is i' and is among the top m agents in new valuation profile. The agent's utility from misreporting is

$$\left(1 - \frac{m+1}{n} + \frac{m+1}{n}\frac{v'_{(m+2)}}{v'_{(m+1)}}\right)v_{(i)} - \left(1 - \frac{(m+1)}{n} + \frac{(m+1)}{n}\frac{v'_{(m+2)}}{v'_{(m+1)}}\right)v_{(m)} + \frac{m}{n}v_{(m+1)}$$

The net change in utility is

$$\left(1 - \frac{m+1}{n} + \frac{m+1}{n}\frac{v'_{(m+2)}}{v'_{(m+1)}}\right)v_{(i)} - \left(1 - \frac{(m+1)}{n} + \frac{(m+1)}{n}\frac{v'_{(m+2)}}{v'_{(m+1)}}\right)v_{(m)} + \frac{m}{n}v_{(m+1)} - \frac{m}{n}v_{(m+1)}$$
$$= \left(1 - \frac{m+1}{n} + \frac{m+1}{n}\frac{v'_{(m+2)}}{v'_{(m+1)}}\right)(v_{(i)} - v_{(m)}) < 0$$

So, the agent i has no incentive to misreport.

Suppose the $(m+1)^{th}$ ranked agent is picked. Agent's utility when he reports truthfully is $\frac{m}{n}v_{(m+2)}$. Let him misreport as $v'_{i'}$ such that his rank is among the top m agents in new valuation profile. The agent's utility from misreporting is

$$\left(1 - \frac{m+1}{n} + \frac{m+1}{n}\frac{v'_{(m+2)}}{v'_{(m+1)}}\right)v_{(m+1)} - \left(1 - \frac{(m+1)}{n} + \frac{(m+1)}{n}\frac{v'_{(m+2)}}{v'_{(m+1)}}\right)v_{(m)} + \frac{m}{n}v_{(m+2)}$$

The net change in utility is

$$\Big(1-\frac{m+1}{n}+\frac{m+1}{n}\frac{v^{'}_{(m+2)}}{v^{'}_{(m+1)}}\Big)v_{(m+1)}-\Big(1-\frac{(m+1)}{n}+\frac{(m+1)}{n}\frac{v^{'}_{(m+2)}}{v^{'}_{(m+1)}}\Big)v_{(m)}+\frac{m}{n}v_{(m+2)}-\frac{m}{n}v_{(m+2)}$$
$$=\Big(1-\frac{m+1}{n}+\frac{m+1}{n}\frac{v^{'}_{(m+2)}}{v^{'}_{(m+1)}}\Big)(v_{(m+1)}-v_{(m)})<0$$

So, the agent ranked $m+1$ has no incentive to misreport.

Similarly, no other agent has any incentive to misreport. Hence, the EP mechanism is DSIC. ■

Proof of Proposition 4.2:

If $v_{(m+2)} \geq r$, the mechanism is same as EP mechanism and we have proved the properties in Proposition 4.1.

Consider a valuation profile $v \in V^n$ such that $v_{(1)} > v_{(2)} > \ldots > v_{(m+1)} > r > v_{(m+2)} > \ldots > v_{(n)}$. The payments made by top m agents are $\big(1-\frac{m+1}{n}+\frac{m+1}{n}\frac{r}{v_{(m+1)}}\big)v_{(m+1)}$ each. So, $p_{(1)}(v) = \ldots = p_{(m)}(v) = \big(1-\frac{m+1}{n}\big)v_{(m+1)} + \big(\frac{1}{n}\big)r$ and $p_{(m+1)}(v) = -\frac{m}{n}r$ and $p_{(m+2)}(v) = \ldots = p_{(n)}(v) = -\frac{m}{n}v_{(m+1)}$. Clearly, the payments are balanced i.e. $\sum\limits_{i\in N} p_{(i)}(v) = 0$.

When the valuation profile is such that $v_{(1)} > v_{(2)} > \ldots > v_{(m)} > r > v_{(m+1)} \ldots > v_{(n)}$, the payments made by top m agents are $\big(1-\frac{m}{n}\big)r$ each. So, $p_{(1)}(v) = \ldots = p_{(m)}(v) = \big(\frac{n-m}{n}\big)r$ and $p_{(m+1)}(v) = p_{(m+2)}(v) = \ldots = p_{(n)}(v) = -\frac{m}{n}r$. Clearly, the payments are balanced i.e. $\sum\limits_{i\in N} p_{(i)}(v) = 0$.

In both these cases, each agent gets a non-negative utility. So the mechanism is IR.

Consider a valuation profile $v \in V^n$. Let $v_1 \geq v_2 \geq \ldots \geq v_n$. The valuation profile is partitioned into the sets $v[1], v[2], \ldots, v[n_0(v)]$. The top-only allocation is such that the units of good are allocated only to agents in $v[1], v[2], \ldots, v[\overline{m}(v)]$ if $\overline{m}(v) \leq m_r(v)$. There are 3 possible cases

1. If $\overline{m}(v) \leq m_r(v)$ and $v_{(m+2)} \in v[k]$ for some $k \leq m_r(v)$
   If any agent i misreports to any value $v_i' \geq r$, the mechanism is same as equal-probability-burning mechanism which is DSIC.

   If any agent $i \in v[k]$ where $k \in \{1, 2, \ldots, \overline{m}(v)\}$ misreports as $v_i' < r$, he gets a utility of $\frac{v_{(m+2)}}{n}$ and

$$\frac{m_k}{|v[k]|}\Big(1-\frac{2}{n}+\frac{2}{n}\frac{v_3}{v_2}\Big)v_1-\frac{m_k}{|v[k]|}\Big(1-\frac{2}{n}+\frac{2}{n}\frac{v_3}{v_2}\Big)v_2+\frac{v_{(m+2)}}{n}-\frac{v_{(m+2)}}{n}<0$$

If any agent $i \in v[k]$ where $k \in \{\overline{m}(v)+1,\ldots,m_r(v)\}$ misreports as $v_i' < r$, he gets a utility of $\frac{v_{(m+1)}}{n}$ which is same as he gets at v_i.

2. If $\overline{m}(v) \leq m_r(v)$, $v_{(m+1)} \in v[j]$ where $j \leq m_r(v)$, and $v_{(m+2)} \in v[m_r(v)+1]$
   If any agent $i \in v[k]$ where $k \in \{1,2,\ldots,\overline{m}(v)\}$ misreports to any value $v_i' < r$, he gets a utility of $\frac{m}{n}r$ and

$$\frac{m_k}{|v[k]|}\Big(1-\frac{(m+1)}{n}+\frac{(m+1)}{n}\frac{r}{v_{(m+1)}}\Big)v_i-\frac{m_k}{|v[k]|}\Big(1-\frac{(m+1)}{n}+\frac{(m+1)}{n}\frac{r}{v_{(m+1)}}\Big)v_{(m+1)}+\frac{m}{n}r-\frac{m}{n}r<0$$

   Suppose $v_{(m+1)} \in v[\overline{m}(v)+1]$. If any agent $i \in v[k]$ where $k \in \{1,2,\ldots,\overline{m}(v)\}$ misreports to any value $v_{(m+1)} \geq v_i'$, he gets a utility of $\frac{m}{n}r$ and

$$\frac{m_k}{|v[k]|}\Big(1-\frac{(m+1)}{n}+\frac{(m+1)}{n}\frac{r}{v_{(m+1)}}\Big)v_i-\frac{m_k}{|v[k]|}\Big(1-\frac{(m+1)}{n}+\frac{(m+1)}{n}\frac{r}{v_{(m+1)}}\Big)v_{(m+1)}+\frac{m}{n}r-\frac{m}{n}r<0$$

   If any agent $i \in v[k]$ where $k \in \{m_r(v)+1,\ldots,n_0(v)\}$ misreports to any value $v_i' \geq v_m$, he gets a utility of $\frac{m_k'}{|v'[k]|}\Big(1-\frac{(m+1)}{n}+\frac{(m+1)}{n}\frac{v_{(m+1)}}{v_m}\Big)v_i-\frac{m_k'}{|v'[k]|}\Big(1-\frac{(m+1)}{n}+\frac{(m+1)}{n}\frac{v_{(m+1)}}{v_m}\Big)v_m+\frac{m}{n}v_m$ and

$$\frac{m_k'}{|v'[k]|}\Big(1-\frac{(m+1)}{n}+\frac{(m+1)}{n}\frac{v_{(m+1)}}{v_m}\Big)v_i-\frac{m_k'}{|v'[k]|}\Big(1-\frac{(m+1)}{n}+\frac{(m+1)}{n}\frac{v_{(m+1)}}{v_m}\Big)v_m +\frac{m}{n}v_{(m+1)}-\frac{m}{n}v_{(m+1)}<0$$

   If any agent $i \in v[k]$ where $k \in \{m_r(v)+1,\ldots,n_0(v)\}$ misreports to any value $v_{(m+1)} > v_i' \geq r$ then he gets a utility of $\frac{m}{n}v_{(m+1)}$ and this is same as he gets at v_i.

3. If $\overline{m}(v) = m_r(v)$ and $v_{(m+1)} \in v[\overline{m}(v)+1]$

   If any agent $i \in v[k]$ where $k \in \{\overline{m}(v)+1,\ldots,n_0(v)\}$ misreports to any value $v_i' \geq v_m$, he gets a utility of $\frac{m_k'}{|v'[k]|}\Big(1-\frac{(m+1)}{n}+\frac{(m+1)}{n}\frac{r}{v_m}\Big)v_i-\frac{m_k'}{|v'[k]|}\Big(1-\frac{(m+1)}{n}+\frac{(m+1)}{n}\frac{r}{v_m}\Big)v_m+\frac{m}{n}r$ and

$$\frac{m_k'}{|v'[k]|}\Big(1-\frac{(m+1)}{n}+\frac{(m+1)}{n}\frac{r}{v_m}\Big)v_i-\frac{m_k'}{|v'[k]|}\Big(1-\frac{(m+1)}{n}+\frac{(m+1)}{n}\frac{r}{v_m}\Big)v_m+\frac{m}{n}r-\frac{m}{n}r<0$$

   If any agent $i \in v[k]$ where $k \in \{\overline{m}(v)+1,\ldots,n_0(v)\}$ misreports to any value $v_m > v_i' \geq r$, he gets a utility of $\frac{m}{n}r$ and this is same as the utility he gets at v_i.

Hence, the BR mechanism is DSIC. ■

References

Bergemann, D. and S. Morris (2005): "Robust mechanism design," *Econometrica*, 73(6), 1771–1813.

Bikhchandani, S. (2017): "Stability with one-sided incomplete information," *Journal of Economic Theory*, 168, 372–399.

Bochet, O. and R. Ilkilic (2017): "Incentive-Compatible and Stable Trade on Networks," *Journal of Economic Theory*, 169, 218–233.

Chen, Y. and G. Hu (2017): "A theory of stability in matching with incomplete information," *Working paper.*

Coombs, A. (2013): "How to avoid estate fight among your heirs?" *Wall Street Journal*, Online; accessed 4-August-2021, https://www.wsj.com/articles/SB10001424052702303932504579252593886768168.

Dasgupta, P. and E. Maskin (2000): "Efficient auctions," *Quarterly Journal of Economics*, 115(2), 341–388.

d'Aspremont, C. and L. Gerard-Varet (1982): "Bayesian incentive compatible beliefs," *Journal of Mathematical Economics*, 10(1), 83–103.

Dastidar, K. (2017): "Oligopoly, Auctions and Market Quality," *Springer: Tokyo, Japan.*

de Clippel, G., V. Naroditskiy, M. Polukarov, A. Greenwald, and N. Jennings (2014): "Destroy to save," *Games and Economic Behavior*, 86, 392–404.

Dutta, B. and R. Vohra (2005): "Incomplete information, credibility and the core," *Mathematical Social Sciences*, 50, 148–165.

Eden, A., M. Feldman, A. Fiat, and K. Goldner (2018): "Interdependent values without single-crossing," *Proceedings of the 2018 ACM Conference on Economics and Computation (EC '18). ACM, New York, NY, USA*, 369–369.

Forges, F. (2004): "The ex-ante incentive compatible core of the assignment game," *Mathematical Social Sciences*, 47, 135–151.

Forges, F., J.-F. Mertens, and R. Vohra (2002a): "The ex ante incentive compatible core in the absence of wealth effects," *Econometrica*, 70.

Forges, F., E. Minelli, and R. Vohra (2002b): "Incentives and the core of an exchange economy: a survey," *Journal of Mathematical Economics*, 38.

Green, J. and J. Laffont (1979): "Incentives in Public Decision-making," *Elsevier, Amsterdam.*

Gresik, T. and M. Satterthwaite (1989): "The Rate at which a Simple Market Converges to Efficiency as the Number of Traders Increases: An Asymptotic Result for Optimal Trading Mechanisms," *Journal of Economic Theory*, 48, 304–332.

Gujar, S. and Y. Narahari (2008): "Redistribution of VCG payments in assignment of heterogeneous objects," *Proceedings of the Fourth Workshop on Internet and Network Economics (WINE)*, 438–445.

Guo, M. and V. Conitzer (2014): "Better redistribution with inefficient allocation in multi-unit auctions," *Artificial Intelligence*, 216, 287–308.

Hagerty, K. and W. Rogerson (1985): "Robust trading mechanisms," *Journal of Economic Theory*, 42, 94–107.

Hain, D. and M. Mitra (2004): "Simple sequencing problems with interdependent costs," *Games and Economic Behavior*, 48, 271–291.

Holmstrom, B. and R. Myerson (1983): "Efficient and durable decision rules with incomplete information," *Econometrica*, 51, 1799–1819.

Jehiel, P. and B. Moldovanu (2001): "Efficient design with interdependent valuations," *Econometrica*, 69(5), 1237–1259.

Kranton, R. and D. Minehart (2001): "A Theory of Buyer-Seller Networks," *American Economic Review*, 91, 485–508.

Krishna, V. (2009): "Auction Theory," *Academic Press.*

Long, Y. (2018): "Envy free and budget balanced assignment of identical objects," *Social Choice and Welfare*, 50(4), 705–719.

——— (2019): "Optimal budget-balanced ranking mechanisms to assign identical objects," *Economic Theory*, 70, 467–502.

Long, Y., D. Mishra, and T. Sharma (2017): "Balanced ranking mechanisms," *Games and Economic Behavior*, 105, 9–39.

McAfee, R. (1992): "A Dominant Strategy Double Auction," *Journal of Economic Theory*, 56, 434–450.

Milgrom, P. and R. Weber (1982): "A theory of auctions and competitive bidding," *Econometrica*, 1089–1122.

Mishra, D. and T. Sharma (2018): "A simple budget balanced mechanism," *Social Choice and Welfare*, 169, 218–233.

Myerson, R. (1981): "Optimal auction design," *Mathematics of Operations Research*, 6(1), 58–73.

Myerson, R. and M. Satterthwaite (1983): "Efficient mechanisms for bilateral trading," *Journal of Economic Theory*, 23, 265–281.

Nath, S., O. Zoeter, Y. Narahari, and C. Dance (2015): "Dynamic mechanism design with interdependent valuations," *Review of Economic Design*, 19(3), 211–228.

Peivandi, A. and R. Vohra (2021): "Instability of centralized markets," *Econometrica*, 89, 163–179.

Perry, M. and P. Reny (1999): "An ex-post efficient auction," *Working paper, Hebrew University and University of Chicago.*

Roughgarden, T. and I. Talgam-Cohen (2016): "Optimal and robust mechanism design with interdependent values," *ACM Transactions on Economics and Computation*, 4(3), 18:1–18:34.

Vikram, A. (2021): "Budget balanced mechanisms for single-good allocation problems with interdependent values," *Working paper.*

Vohra, R. (1999): "Incomplete information, incentive compatibility and the core," *Journal of Economic Theory*, 86, 123–147.

Wilson, R. (1978): "Information, efficiency and the core of an economy," *Econometrica*, 14, 807–816.

Yenmez, M. (2015): "Incentive compatible market design with applications," *International Journal of Game Theory*, 44, 543–569.