Horizontal, Contact and Partially Horizontal Immersions in Fat Distributions

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Contents

List of Notations

$\langle S \rangle$	Linear subspace generated by a subset S in a vector space
$C^\infty(\Sigma, M)$	Space of smooth maps $\Sigma \to M$
$\hom(E, F)$	Bundle morphisms between vector bundles E and F
$\mathrm{Op}(A)$	An arbitrary, unspecified open set containing A
$\mathcal{D}$	A distribution on a manifold M
Ω	Curvature 2-form associated to $\mathcal{D}$
Ω_K	Curvature 2-form associated to the distribution K
ξ	A contact structure on a manifold
Ξ	A holomorphic contact structure on a complex manifold
$\mathfrak{D}$	A differential operator
$\mathfrak{L}_u$	Linearization of $\mathfrak{D}$, at a function u
ΓX	Space of local sections of a fiber bundle X
$X^{(r)}$	The r^{th} jet bundle associated to a fiber bundle X
$\mathcal{R}$	A differential relation in the jet bundle $X^{(r)}$ for some $r \geq 0$
$\Gamma\mathcal{R}$	Space of sections of $X^{(r)}$ taking values in $\mathcal{R}$
$\mathrm{Sol}\,\mathcal{R}$	Space of sections of X, whose r^{th}-jet prolongation belongs to $\Gamma\mathcal{R}$
$\tilde{\mathcal{R}}$	An extension relation associated to $\mathcal{R}$
$\mathfrak{D}^{\mathsf{Hor}}$	An operator $C^\infty(\Sigma, M) \to \Omega^1(T\Sigma, TM/\mathcal{D})$
$\mathfrak{D}^{\mathsf{Cont}}$	An operator $C^\infty(\Sigma, M) \to \Omega^1(K, TM/\mathcal{D})$, where $K \subset T\Sigma$

For notations used in Chapter 5, we refer to Table 5.1 on Page 118.

Chapter 1

Introduction

A *distribution* $\mathcal{D}$ on a manifold M is a smooth subbundle of the tangent bundle TM. The rank of the vector bundle is defined as the rank of the distribution. A distribution is generally viewed as a smooth assignment of vector subspaces $\mathcal{D}_x \subset T_xM$ to the points $x \in M$.

For a given distribution $\mathcal{D}$ on a manifold M, we can consider the vector fields on M, which are sections of $\mathcal{D}$. We say $\mathcal{D}$ is *involutive* if the Lie bracket of any two local sections of $\mathcal{D}$ is again a section of the same kind. In other words the vector fields in $\mathcal{D}$ are closed under the Lie bracket operation. By the Frobenius Theorem, a distribution is involutive precisely when it is *integrable*, that is, through each point of the manifold M, there is an (immersed) submanifold $\mathcal{L}$, such that the tangent space $T_x\mathcal{L}$ equals $\mathcal{D}_x$ at each point $x \in \mathcal{L}$. Thus, the dimension of the integral submanifolds is the same as the rank of $\mathcal{D}$.

The *non-integrable* distributions are not only plentiful but they also exhibit rich structures. Here we are particularly interested in *bracket generating* distributions which lie at the polar opposite end to involutive distributions. Explicitly, a distribution $\mathcal{D}$ is said to be *bracket generating* if the successive Lie bracket operations on vector fields in $\mathcal{D}$ generate the whole tangent bundle. These distributions have received a wide attention because of their close connection to physical questions related to constrained motion ([Car10, BCG$^+$91, Mon02]).

Chow ([Cho39]) proved that if $\mathcal{D}$ is bracket generating then any two points of M can be joined by a smooth curve which is tangent to $\mathcal{D}$ at each point, in contrast with the involutive distributions where two points can be joined by such a curve if and only if they lie on the same integral submanifold. Furthermore, Ge ([Ge93]) established that any continuous curve joining two points can actually be C^0-approximated by a curve which is everywhere tangent to $\mathcal{D}$.

An immediate question that arises next is what could be the maximum possible dimension k of a submanifold (immersed or embedded) through each point of M, which is everywhere tangent to $\mathcal{D}$? Any such submanifold is called *horizontal* to the given distribution $\mathcal{D}$. More generally, we may have the ambitious goal of classifying $\mathcal{D}$-horizontal immersions and embeddings up to homotopy. This question has been well-studied in many instances and the answer to this is usually given in the language of h-principle.

Among all the bracket-generating distributions, the *contact structures* have been studied most extensively ([Gei08]). These are corank 1 distributions on odd-dimensional manifolds, which are maximally non-integrable. In other words, a contact structure ξ is locally given by a 1-form α such that, $\alpha \wedge (d\alpha)^n$ is non-vanishing, where the dimension of the manifold is $2n+1$. It can be easily seen that the maximal dimension of a horizontal submanifold of ξ as above is n. These are called *Legendrians*. Locally, there are plenty of n-dimensional horizontal (Legendrian) submanifolds. Globally, Legendrian immersions and (loose) Legendrian embeddings are completely understood in terms of h-Principle ([Gro86, Duc84, Mur12]). Beyond the corank 1 situation, very few cases are completely known. *Engel structures*, which are certain rank 2 distribution on 4-dimensional manifolds ([Eng89]), have been studied in depth in the recent years, and the question of existence and classification of horizontal loops in a given Engel structure has been solved ([Ada10, CdP18]). Horizontal immersions on product of contact manifolds have also been studied in [D'A94].

The contact and Engel distributions mentioned above have several interesting properties. The simplest invariant for distribution germs is given by a pair of integers (n, r) where $n = \dim M$ and $r =$ rank $\mathcal{D}$. The germs of contact and Engel structures are *generic* in their respective classes and they also happen to be *stable*. These distributions admits local framing which generates finite dimensional nilpotent lie algebras. The only other generic class of distributions generating finite dimensional Lie algebras are the even contact structures and the 1-dimensional distributions. All of them lie in the range $r(n-r) \leq n$ ([Mon93]). But in the range $r(n-r) > n$, the study of a generic distribution becomes difficult due to the presence of function moduli.

The contact distributions are the simplest kind of *strongly* bracket generating distribution. A distribution $\mathcal{D}$ is called strongly bracket generating if every non-vanishing vector field along $\mathcal{D}$, about a point $x \in M$, Lie bracket generates the tangent space T_xM. Strongly bracket generating distributions are also referred to as *fat distributions* in the literature. In fact, in corank 1, fat distributions are the same as the contact ones. The germs of fat distributions in higher corank, are far from being generic ([Ray68]). However, they are interesting in their own right and have been well-studied ([Ge93, Mon02]).

The notion of contact structures can be extended verbatim to complex manifolds. These are complex, corank 1-subbundles of the holomorphic tangent bundle $T^{(1,0)}M$ of a complex manifold

M, with $\dim_{\mathbb{C}} M = 2n+1$, given locally by holomorphic 1-forms α satisfying $\alpha \wedge (d\alpha)^n \neq 0$. The h-principle for holomorphic Legendrian embeddings of an open Riemann surface into certain holomorphic contact manifolds has been studied in [FL18b, FL18a]. If one forgets the complex structure of a given holomorphic contact distribution, one gets a corank 2-distribution on a manifold of real dimension $4n+2$, which enjoys the fatness property ([Mon02]). There is also a quaternionic analogue of contact structures but defined from a different point of view.

In this thesis we focus on fat distributions of corank > 1. We look at some specific classes of fat distributions in corank > 1 and study horizontal immersions and other classes of maps into them. Let us briefly mention the contents of the chapters and the main theorems proved therein.

Chapter 2

In this chapter we discuss the preliminaries of distributions and introduce the notion of its curvature form. Given any distribution $\mathcal{D}$, we can consider the quotient map $\lambda : TM \to TM/\mathcal{D}$ as a $TM/\mathcal{D}$-valued 1-form on M. It then induces a $TM/\mathcal{D}$-valued 2-form $\Omega : \Lambda^2\mathcal{D} \to TM/\mathcal{D}$ defined as follows:

$$\Omega(X,Y) = -\lambda([X,Y]), \quad \text{for all } X, Y \in \Gamma\mathcal{D}$$

Ω is called the *curvature form* of the distribution $\mathcal{D}$; it plays a crucial role in the classification of horizontal immersions and other classes of maps into $\mathcal{D}$.

We also give a brief review of the sheaf techniques in the theory of h-principle, followed by the Nash-Gromov Implicit Function Theorem for smooth differential operators and its implications in sheaf theory. Differential equations or inequalities governing space of sections of a fibre-bundle can be realized by a subset $\mathcal{R}$ in an appropriate jet space. A section of the jet bundle having its image in $\mathcal{R}$ is called a *formal solution* of the differential relation. h-principle means that a formal solution can be homotoped to a solution of the given relation, in other words, presence of h-principle reduces a differential problem to an algebraic one.

Chapter 3

In this chapter, we revisit homotopy classification of K-contact immersions ([Gro86]) and, in particular, that of horizontal immersions in a general distribution ([Gro86, Gro96]) following the h-principle theory. Given a distribution $\mathcal{D}$ on M and a distribution K on Σ, an immersion $u : \Sigma \to K$ is called *K-contact* if the derivative map $du : T\Sigma \to TM$ maps K into $\mathcal{D}$. In particular, for $K = T\Sigma$, the K-contact immersions are nothing but the $\mathcal{D}$-horizontal immersions. It is easy to see that a K-contact immersion must necessarily pull-back the curvature form Ω

of $\mathcal{D}$ onto the curvature form Ω_K of K in an appropriate sense. If $K = T\Sigma$ then $\Omega_K = 0$ and the curvature condition reduces to the isotropy condition.

K-contact maps can be seen as solutions to a first order partial differential equations associated to a differential operator $\mathfrak{D}$ defined on the space of smooth maps $C^\infty(\Sigma, M)$ and taking values in $TM/\mathcal{D}$-valued 1-forms on Σ. The operator is known to be infinitesimally invertible on an open subset of $C^\infty(\Sigma, M)$ consisting of Ω-*regular* immersions, which are defined by an open condition on the 1-jet space. This brings us to the Nash-Gromov implicit function theorem discussed in Chapter 2. The role of Ω in regularity is not always explicit; indeed, if the distribution $\mathcal{D}$ is contact or Quaternionic contact then every immersion is Ω-regular.

Complete h-principle can be obtained for Ω-regular K-contact immersions provided Σ is an open manifold. Explicitly, every formal Ω-regular K-contact immersion satisfying the curvature condition can be homotoped to a genuine Ω-regular K-contact immersion. However, to obtain h-principle on a general Σ one requires certain extensibility criteria to be satisfied. This is referred to as *overregularity condition* in [Gro96] by Gromov. Our goal here is to give a detailed exposition on the homotopy classification of K-contact overregular immersions based on the Nash-Moser Implicit Function Theorem and the general theory of h-principle discussed in Chapter 2.

Chapter 4

The K-contact immersions in a contact manifold $(M, \mathcal{D})$ are automatically Ω-regular. They are known to satisfy the C^0-dense h-principle ([Gro86]). Moreover, if K is also contact then we have the following existence result due to Gromov.

Theorem ([Gro86])**.** *If ξ is the standard contact structure on $\mathbb{R}^{2n+1}$ and K is a cotrivializable contact structure on a manifold Σ, then an arbitrary map $\Sigma \to M$ can be C^0-approximated by a isocontact immersion $(\Sigma, K) \to (\mathbb{R}^{2n+1}, \xi)$, provided $2n + 1 \geq 3 \dim \Sigma$.*

The special case of horizontal immersions into contact structures was also studied by Duchamp in [Duc84].

In this chapter we consider certain fat distributions of corank $p > 1$, and obtain some new results as a consequence of the h-principle proved in the previous chapter. The detailed proof rests upon the internal structure of the distribution $\mathcal{D}$.

Though fat distributions are not generic we have a good hold on them for the following reason. For every 1-form α annihilating $\mathcal{D}$, the restriction of the 2-form $d\alpha$ to the distribution is nondegenerate. Therefore, we can represent the curvature form locally by a p-tuple of nondegenerate 2-forms at each point. This gives an equivalent characterization of Ω-regularity, which is easily tractable. We introduce a new invariant on the class of fat corank 2 distributions called 'degree' and look at the degree 2 distributions. This is the real analogue of holomorphic contact structures.

We obtain the following h-principle for horizontal immersions into degree 2 fat distributions.

Theorem ([BD20],Theorem 4.2.1, Theorem 4.2.4)**.** *Suppose that $\mathcal{D}$ is a degree 2 fat distribution on a manifold M. Then $\mathcal{D}$-horizontal Ω-regular immersions $\Sigma \to (M, \mathcal{D})$ satisfy the h-principle, provided $\operatorname{rk} \mathcal{D} \geq 4 \dim \Sigma + 4$. Consequently, there exists a regular horizontal immersion $\Sigma \to (M, \mathcal{D})$, provided $\operatorname{rk} \mathcal{D} \geq \max\{4 \dim \Sigma + 4,\ 5 \dim \Sigma + 2\}$.*

Similar results have been proved for horizontal immersions in Quaternionic contact manifolds as well.

Theorem (Theorem 4.2.14, Theorem 4.2.17)**.** *Suppose $\mathcal{D}$ is a quaternionic contact structure on M. Then $\mathcal{D}$-horizontal immersions $\Sigma \to (M, \mathcal{D})$ satisfy the h-principle, provided $\operatorname{rk} \mathcal{D} \geq 4 \dim \Sigma + 4$. Consequently, there exists a $\mathcal{D}$-horizontal immersion $\Sigma \to (M, \mathcal{D})$, provided $\operatorname{rk} \mathcal{D} \geq \max\{4 \dim \Sigma + 4,\ 5 \dim \Sigma - 3\}$.*

We also prove the h-principle and existence of isocontact immersions in degree 2 fat distributions.

Theorem (Theorem 4.2.23, Theorem 4.2.26)**.** *Suppose $\mathcal{D}$ is a degree 2 fat distribution on M and K is a contact structure on Σ. Then K-isocontact immersions $(\Sigma, K) \to (M, \mathcal{D})$ satisfy the h-principle, provided $\operatorname{rk} \mathcal{D} \geq 2 \operatorname{rk} K + 4$. Furthermore, there exists a K-isocontact immersion $(\Sigma, K) \to (M, \mathcal{D})$, provided the following conditions holds :*

- $\operatorname{rk} \mathcal{D} \geq \max\{2 \operatorname{rk} K + 4,\ 3 \operatorname{rk} K - 2\}$, *and*
- *one of the following two conditions holds true,*
  - *both K and $\mathcal{D}$ are cotrivial*
  - $H^2(\Sigma) = 0$

Chapter 5

In this chapter, we study *partially horizontal* immersions, introduced by Gromov in [Gro96]. An immersion $u : \Sigma \to (M, \mathcal{D})$ is called m*-horizontal* if the inverse image of $\mathcal{D}$ under the derivative map $du : T\Sigma \to TM$ is a rank m distribution on Σ. In particular, if $m = \dim \Sigma$, then the m-horizontal immersions are precisely the $\mathcal{D}$-horizontal ones. On the other hand, when

$m = \dim\Sigma - \operatorname{cork}\mathcal{D}$, the induced distribution on Σ has the same corank as that of $\mathcal{D}$ and m-horizontal immersions are simply the immersions that are transverse to $\mathcal{D}$.

We study m-horizontal immersions in generality following the ideas outlined in [Gro96] and obtain the following results.

Theorem (Theorem 5.2.3)**.** *Let $\mathcal{D}$ be a corank p distribution on M. Then, transverse immersions $\Sigma \to (M, \mathcal{D})$ satisfy the h-principle, provided $\dim M > \dim\Sigma$.*

Theorem (Theorem 5.2.5, Corollary 5.2.7)**.** *Suppose $\mathcal{D}$ is a corank p fat distribution on M. For $m = \dim\Sigma-(p-1)$, the m-horizontal immersions $\Sigma \to (M, \mathcal{D})$ satisfy the h-principle, provided $\operatorname{rk}\mathcal{D} > 2m$. Furthermore, if $\mathcal{D}$ is cotrivial and Σ admits a cotrivial subbundle of corank $(p-1)$, then there exists an m-horizontal immersion $\Sigma \to (M, \mathcal{D})$, provided $\dim M \geq 3\dim\Sigma - p + 1$.*

Theorem (Theorem 5.2.8)**.** *Let $\mathcal{D}$ be a quaternionic contact structure on M. For $m = \dim\Sigma - 1$, the m-horizontal immersions $\Sigma \to (M, \mathcal{D})$ satisfy the h-principle, provided $\operatorname{rk}\mathcal{D} \geq 4m + 4$.*

Chapter 6

In this chapter we return to the horizontal immersion problem for a corank 2 fat distribution $\mathcal{D}$ on a 6 dimensional manifold. Any such distribution is automatically a degree 2 fat distribution. But the results of Chapter 4 do not give h-principle for immersed horizontal loops in $\mathcal{D}$ due to dimension restriction.

However, this does not rule out the possibility of getting h-principle for horizontal immersions when $\dim D \geq 4\dim\Sigma$, since the system is underdetermined. The result of [AFL17] also supports this claim for holomorphic contact structures.

Theorem ([AFL17])**.** *Let Ξ be the standard holomorphic contact structure on $\mathbb{C}^{2n+1}$ and Σ be a connected, open Riemann surface. Then, the space of holomorphic Legendrian embeddings $\Sigma \to \mathbb{C}^{2n+1}$ is weak homotopy equivalent to the space of continuous maps $\Sigma \to S^{4n-1}$*

The real distribution $\mathcal{D}$ underlying a holomorphic contact distribution is fat. Moreover, there are 1-forms λ^1, λ^2 defining $\mathcal{D}$ such that $\ker d\lambda_1 = \ker d\lambda_2$ is generated by a pair of vector fields Z_1, Z_2, which further satisfy $\lambda^i(Z_j) = \delta_{ij}$ and $[Z_1, Z_2] = 0$. We shall refer to such vector fields as *Reeb-like* vector fields.

Using the implicit function theorem due to Hamilton ([Ham82]) we prove the following.

Theorem ([Bho20],Corollary 6.4.4)**.** *Suppose $\mathcal{D}$ is a fat distribution on a $\mathbb{R}^6$, which admits (local) Reeb-like vector fields. Then horizontal immersions $\Sigma \to (M, \mathcal{D})$ satisfy the local h-principle.*

By solving the algebraic problem we get the main result of this chapter.

Theorem ([Bho20],Theorem 6.4.5)**.** *A distribution $\mathcal{D}$ as in the above theorem, admits germs of horizontal submanifolds of dimension 2.*

The existence of 2-dimensional horizontal germs suggests that we may get h-principle for horizontal loops in $\mathcal{D}$, possibly with a new regularity condition.

Chapter 2

Preliminaries : Distributions and h-Principles

In the first half of this chapter we shall recall the preliminaries of distributions on a manifold and recall some results that are pertinent to our work in this thesis. The second half of this chapter is devoted to the sheaf-theoretic and the analytic theory of h-principle introduced by Gromov in [Gro86].

2.1 Distributions

All manifolds and maps, unless mentioned otherwise, are considered to be smooth. The background material for this section can be found in [Mon02, BCG+91, Gei08].

Definition 2.1.1. A *distribution* $\mathcal{D}$ on a manifold M is a smooth sub-bundle of the tangent bundle TM. The *rank* of the distribution is defined as the rank of $\mathcal{D}$ as a subbundle and *corank* of $\mathcal{D}$ is the integer $\dim M - \operatorname{rk} \mathcal{D}$.

A vector field on M will be referred to as a *vector field in* $\mathcal{D}$ if it is a section of $\mathcal{D}$. In short, we shall write $X \in \mathcal{D}$ to mean that X is a vector field taking values in $\mathcal{D}$; that is, $X(x) \in \mathcal{D}_x$ for every $x \in M$. The space of smooth sections of $\mathcal{D}$ will be denoted by $\Gamma(\mathcal{D})$.

For any two local sections $X, Y \in \mathcal{D}$ we have the local field given by their Lie bracket $[X, Y]$. We can define a sheaf $[\mathcal{D}, \mathcal{D}]$ by prescribing its stalk as follows :

$$[\mathcal{D}, \mathcal{D}]_x = \left\{ [X, Y]_x \;\middle|\; X, Y \in \mathcal{D} \text{ are local sections about } x \right\}$$

Though $[\mathcal{D}, \mathcal{D}]_x \subset T_x M$ is a linear subspace, $\dim[\mathcal{D}, \mathcal{D}]_x$ need not be constant in x and hence the $[\mathcal{D}, \mathcal{D}]$ is not, in general, a distribution. Observe that, given any two sheaves $\mathcal{E}, \mathcal{F}$ of vector fields, we may similarly define the sheaf $[\mathcal{E}, \mathcal{F}]$ by taking Lie brackets of local sections of $\mathcal{E}$ and $\mathcal{F}$ respectively. Hence, we may recursively define the sheaves $\mathcal{D}^i$ for all $i \geq 1$:

$$\mathcal{D}^{i+1} = \mathcal{D}^i + [\mathcal{D}, \mathcal{D}^i], \quad \mathcal{D}^1 = \mathcal{D}$$

In this thesis, we will only consider the distributions $\mathcal{D} \subset TM$ for which each $\mathcal{D}^i$ is again a distribution. As we shall see, many interesting examples are of this type.

Definition 2.1.2. A distribution $\mathcal{D} \subset TM$ is called *involutive* if we have $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$.

This leads to the notion of integrability.

Integrable Distribution

Let us begin with the definition.

Definition 2.1.3. A distribution $\mathcal{D} \subset TM$ is called *integrable* if through each point $x_0 \in M$, there is an immersed submanifold $N \subset M$ such that, $T_xN = \mathcal{D}_x$ for all $x \in N$.

Clearly, any rank 1 distribution is involutive and through each point of M there exists an integral curve to the 1-dimensional distribution. In fact, these two concepts are equivalent by a famous theorem due to Frobenius.

Theorem 2.1.4. *A distribution $\mathcal{D} \subset TM$ is integrable if and only if $\mathcal{D}$ is involutive.*

An involutive (or integrable) distribution defines a *foliation* $\mathcal{F}$ on M, by partitioning the manifold into integral submanifolds which are referred to as the *leaves* of the foliation. Indeed, an integrable distribution is precisely the tangent distributions $T\mathcal{F}$ of some foliation $\mathcal{F}$ on M.

Non-Integrable Distribution

A distribution $\mathcal{D} \subset TM$ is *non-integrable* (or *nonholonomic*) if $\mathcal{D}$ is not involutive, that is if we have $[\mathcal{D}, \mathcal{D}] \not\subset \mathcal{D}$. It turns out that a generic distribution is not only non-integrable, but furthermore they are *bracket-generating*.

Definition 2.1.5. A distribution $\mathcal{D} \subset TM$ is called *bracket-generating* if successive Lie brackets of (local) sections of $\mathcal{D}$ span the tangent bundle TM.

Thus, if $\mathcal{D} \subset TM$ is bracket generating, then for each $x \in M$, there exists a positive integer $r(x)$, depending on x, such that

$$\mathcal{D}^{r(x)}|_x = T_xM$$

If $r(x) = r$ for all x, then $TM = \mathcal{D}^r$ and we say that $\mathcal{D}$ is $(r-1)$-step bracket generating. We shall discuss below some important classes of bracket generating distributions.

Definition 2.1.6. A *contact structure* $\xi \subset TM$ is a corank 1 distribution on a manifold of dimension $2n+1$, such that ξ is locally given as $\xi \underset{loc.}{=} \ker\alpha$ for some (local) 1-form $\alpha \in \Omega^1(M)$, satisfying the nondegeneracy condition,

$$\alpha \wedge (d\alpha)^n \neq 0,$$

which is equivalent to saying that the 2-form $d\alpha|_\xi$ is nondegenerate. The 1-form α is called a *contact form* and the pair (M, ξ) is called a *contact manifold*.

Every odd-dimensional Euclidean manifold $\mathbb{R}^{2n+1}$ has a canonical contact structure defined by the 1-form

$$\alpha = dz - \sum_{i=1}^{n} y_i dx^i,$$

where $\{z, x^i, y_i, 1 \leq i \leq n\}$ is any global coordinates system on $\mathbb{R}^{2n+1}$. It follows from the Darboux theorem that any contact structure locally looks like this.

Theorem 2.1.7. *Given any contact structure $\xi \subset TM$ on a manifold M of dimension $2n+1$, we have that around each $x \in M$ there exists some coordinate neighborhood (U, z, y_i, x^i), such that*

$$\xi|_U = \ker\Big(dz - \sum_i y_i dx^i\Big)$$

Any such choice of neighborhood as above is known as a *Darboux neighborhood*. Using this we can then get a local framing for the contact structure as,

$$\xi|_U = \Big\langle \partial_{y_i},\ \partial_{x^i} - y_i dz; \quad 1 \leq i \leq n \Big\rangle$$

Observe that the local frame $\{\partial_{y_1}, \ldots, \partial_{y_n}\}$ is involutive.

Another interesting class of bracket generating distribution is given by the Engel structures.

Definition 2.1.8. An *Engel distribution* is a rank 2 distribution $\mathcal{D}$ on a 4-dimensional manifold M such that, $\mathcal{D}^2 = \mathcal{D} + [\mathcal{D}, \mathcal{D}]$ is rank 3 distribution and $\mathcal{D}^3 = \mathcal{D}^2 + [\mathcal{D}, \mathcal{D}^2]$ is all of TM.

In particular, an Engel distribution is 2-step bracket-generating. The standard Engel structure on $\mathbb{R}^4$ is given as the common kernel of two 1-forms,

$$\alpha = dz - ydx, \quad \text{and } \beta = dw - zdx,$$

where $\{x, y, z, w\}$ are canonical coordinates on $\mathbb{R}^4$. We see that, $\mathcal{D} = \ker\alpha \cap \ker\beta$ admits a global frame,

$$\mathcal{D} = \left\langle \partial_y,\ \partial_x + y\partial_z + z\partial_w \right\rangle$$

Similar to the contact structures, the Engel distributions also have canonical representations.

Theorem 2.1.9. *Given any Engel structure $\mathcal{D} \subset TM$ on a 4-dimensional manifold M, we have that around each $x \in M$ there exists some coordinate neighborhood (U, x, y, z, w), such that*

$$\mathcal{D}|_U = \ker\left(dz - ydx\right) \cap \ker\left(dw - zdx\right)$$

Contact and Engel structures have many similar properties. In this connection, let us mention a striking result by Montgomery.

Theorem 2.1.10 ([Mon93])**.** *A generic distribution of rank r on a manifold of dimension n, satisfying*

$$r(n - r) > n$$

does not admit any local frame, which Lie bracket generates a finite dimensional Lie algebra.

To understand how this relates to contact and Engel structures, we observe that there are only three possible solutions of $r(n - r) \leq n$.

- $r = 1$: We have the line fields. Since these are clearly involutive, any Lie algebra generated by a local frame is 1-dimensional.
- $r = n-1$: When n is an odd number, a generic distribution germ is a contact distribution. From the discussion above, we have certain local frame fields, which Lie bracket generates an n-dimensional Lie algebra, known as the *(real) Heisenberg algebra*.

  When n is an even number, we have an analogous distribution, known as the *even contact structure*, which exhibits very similar properties.
- $r = 2$ and $n = 4$: Any generic distribution germ of this type is an Engel structure. Again looking back at the local frame given above, we see that this frame Lie bracket generates a 4-dimensional Lie algebra, known as the *Engel algebra*.

In the range $r(n-r) > n$, there are infinitely many *non-isomorphic* distribution germs of rank r on n-dimensional manifolds. This makes it considerably difficult to study a generic distribution of higher corank.

Horizontal Curves and Loops

Given a distribution $\mathcal{D} \subset TM$, we ask whether any two points $a, b \in M$ with $a \neq b$, can be joined by a path γ which is everywhere tangential to $\mathcal{D}$. This question can be rephrased as a boundary value problem for smooth functions $\gamma : [0,1] \to M$:

$$\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}, \quad \forall t \in [0,1], \text{ such that } \gamma(0) = a, \quad \gamma(1) = b.$$

Any curve satisfying the above differential condition is called a *$\mathcal{D}$-horizontal curve*.

If $\mathcal{D} \subset TM$ is integrable, then the answer to the question is in the negative. Indeed, any horizontal path in an integrable distribution is restricted to some leaf of the underlying foliation. Thus for a pair of points residing in different leaves, there is no horizontal path joining them.

On the other hand, for a bracket-generating distribution we essentially get that any tangent direction on M can be obtained as a successive Lie brackets of vectors in $\mathcal{D}$. Consequently, we get a positive answer for bracket-generating distributions in the form of Chow's theorem.

Theorem 2.1.11 ([Cho39]). *Suppose $\mathcal{D} \subset TM$ is a bracket generating distribution on a connected manifold M. Then, for any two points $a, b \in M$ there is a smooth $\mathcal{D}$-horizontal path joining them.*

Chow's theorem has many interesting implications. Let a, b be two fixed points on M. Given a distribution $\mathcal{D} \subset TM$, let $\Omega_{a,b}(\mathcal{D})$ denote the space of all smooth $\mathcal{D}$-horizontal paths in M joining a and b. That is,

$$\Omega_{a,b}(\mathcal{D}) = \left\{ \gamma : [0,1] \to M \;\middle|\; \gamma(0) = a,\ \gamma(1) = b,\ \dot{\gamma}(t) \in \mathcal{D}|_{\gamma(t)} \right\}$$

Chow's theorem says that $\Omega_{a,b}(\mathcal{D}) \neq \emptyset$ if $\mathcal{D}$ is bracket-generating. The following result of Ge shows that $\Omega_{a,b}(\mathcal{D})$, in fact, contains plenty of curves.

Theorem 2.1.12 ([Ge93]). *If $\mathcal{D}$ is bracket-generating, then $\Omega_{a,b}(\mathcal{D})$ is weakly homotopy equivalent to the space $\Omega_{a,b}$ of all smooth paths in M joining a to b. In fact the inclusion map,*

$$\Omega_{a,b}(\mathcal{D}) \hookrightarrow \Omega_{a,b}$$

induces isomorphism in each of the homotopy groups. In particular, any smooth path joining a, b *is path-homotopic to a* $\mathcal{D}$*-horizontal path.*

The paths in the above theorem are not immersed. So now we may modify the question as follows : does there exists a smooth *immersion* (or an *embedding*), joining two points, or more generally, whether there are *closed* horizontal immersed curves? We do have positive answer for contact and Engel structures.

Theorem 2.1.13 ([EM02])**.** *Given a contact structure* $\xi \subset TM$*, the* ξ*-horizontal immersions* $S^1 \to M$ *satisfy the complete* h*-principle.*

Here h-principle means that the problem of finding an immersion $S^1 \to M$ can be reduced to an algebraic problem. We recall the formal definition of h-principle in the next section of this chapter.

In fact any embedded closed loop in $\mathbb{R}^3$, also known as *knots*, can be C^0-approximated by an embedded horizontal loop, where we consider the standard contact structure on $\mathbb{R}^3$ ([Gei08, pg. 101]). However the *embedded* Legendrian loops do not abide by the h-principle even in the simplest possible case of $S^1 \to \mathbb{R}^3$, with the standard contact structure. Indeed, there are infinitely many topologically trivial embeddings of loops, i.e unknots, which are *not* homotopic in the space of horizontal embeddings ([EF09]).

Now let us look at our other prominent example, that is an Engel structure $\mathcal{D} \subset TM$. In case of Engel structures, complete h-principle does not hold due to the presence of some *rigid* curves. An Engel structure contains a line field $\mathcal{W}$ given by

$$\mathcal{W} = \left\{ W \in \mathcal{D}^2 \;\middle|\; [W, \mathcal{D}^2] \subset \mathcal{D}^2 \right\}$$

Locally, with respect to a choice of standard Darboux chart (Theorem 2.1.9), this line field is given by ∂_w. The rigid curves are integral curves of this line field. Up to reparametrization, there is a unique $\mathcal{D}$-horizontal curve γ, joining two points in the same leaf of $\mathcal{W}$ and satisfying $\partial_w \gamma \neq 0$ ([BH93]). This exhibits a certain (local) rigidity of the $\mathcal{D}$-horizontal curves. In fact rigid curves are singular points for the horizontality operator.

The existence of rigid curves in the Engel structure, impairs the possibility of getting any h-principle for $\mathcal{D}$-horizontal loops. But we can restrict ourselves to a class of *regular* curves. These are $\mathcal{D}$-horizontal immersions $\gamma : S^1 \to M$ which are *not everywhere tangential* to $\mathcal{W}$; in other words γ is transverse to $\mathcal{W}$ at some point. Then we have the following theorem.

Theorem 2.1.14 ([CdP18]). *The h-principle holds for $\mathcal{D}$-horizontal embeddings $S^1 \to M$ which are not everywhere tangent to $\mathcal{W}$.*

Horizontal Maps

Definition 2.1.15. Given a distribution $\mathcal{D} \subset TM$ on M and a manifold Σ, any smooth immersion $u : \Sigma \to M$ is called *$\mathcal{D}$-horizontal* if the derivative map du maps into $\mathcal{D}$, i.e, if

$$\operatorname{Im} du_\sigma \subset \mathcal{D}_{u(\sigma)}, \quad \text{for each } \sigma \in \Sigma$$

Question. Given a distribution $\mathcal{D}$, what is the maximum dimension of a $\mathcal{D}$-horizontal submanifold passing through a point in M?

The question of existence of horizontal immersions of higher dimensional manifolds is intimately related to the curvature form of a distribution.

Curvature of a Distribution

Given a distribution $\mathcal{D} \subset TM$ we have the natural quotient map,

$$\lambda : TM \to TM/\mathcal{D},$$

so that $\mathcal{D} = \ker \lambda$. We can treat λ as a $TM/\mathcal{D}$-valued 1-form on the manifold M. We shall denote the space of $TM/\mathcal{D}$ valued 1-forms on M by

$$\Omega^1(TM, TM/\mathcal{D}) = \Gamma \hom(TM, TM/\mathcal{D})$$

Choosing a *local* trivialization of the bundle $TM/\mathcal{D}$ over some open set $U \subset M$, we can write $TM/\mathcal{D}|_U = \langle e_1, \ldots, e_p \rangle$, where $p = \operatorname{rk} TM/\mathcal{D} = \operatorname{cork} \mathcal{D}$ and $\{e_i\}$ are some sections of the bundle $TM/\mathcal{D}$ over U. Then we may write, $\lambda = \sum_{s=1}^p \lambda^s \otimes e_s$, where λ^s are local 1-forms defined on U. We have, $\mathcal{D}|_U = \bigcap_{s=1}^p \ker \lambda^s$. Clearly, choosing a different trivialization, we will end up with a different set of 1-forms defining $\mathcal{D}$. Unless necessary, we will denote $\lambda \underset{loc.}{=} (\lambda^s)$ without referring to the trivialization.

Let $\omega^s = d\lambda^s|_\mathcal{D}$, $s = 1, \ldots, p$. Then, for any pair of local vector fields $X, Y \in \mathcal{D}$,

$$\omega^s(X, Y) = d\lambda^s(X, Y) = X\lambda^s(Y) - Y\lambda^s(X) - \lambda^s([X, Y]) = -\lambda^s([X, Y])$$

Definition 2.1.16. Given a distribution $\mathcal{D}$ on M, the *curvature form* Ω of $\mathcal{D}$ is defined as follows:

$$\Omega(X,Y) = -[X,Y] \mod \mathcal{D} = -\lambda([X,Y])$$

where X,Y are local sections of $\mathcal{D}$.

It is clear that $\mathcal{D}$ is involutive if and only if $\Omega = 0$. Thus the curvature measures the defect of $\mathcal{D}$ from being integrable.

From the discussion above, we see that for any choice of trivialization $\lambda \underset{loc.}{=} (\lambda^s)$ we have, $\Omega \underset{loc.}{=} (\omega^s)$ where $\omega^s = d\lambda^s|_{\mathcal{D}}$. Let us first make the following observation.

Proposition 2.1.17. *The curvature form Ω is $C^\infty(M)$-linear.*

Proof. Suppose $X,Y \in \mathcal{D}$ are some local sections. For any $f,g \in C^\infty(M)$ we have,

$$[fX,gY] = fX(g)Y + g[fX,Y] = fX(g)Y - gY(f)X + fg[X,Y]$$

which implies

$$\Omega(fX,gY) = -[fX,gY] \mod \mathcal{D} = -fg[X,Y] \mod \mathcal{D} = fg\Omega(X,Y)$$

This proves the claim. □

Hence, the curvature form Ω of $\mathcal{D}$ can be equivalently defined as a $TM/\mathcal{D}$ valued 2-form on $\mathcal{D}$,

$$\Omega : \Lambda^2\mathcal{D} \to TM/\mathcal{D},$$

given by,

$$\Omega(X,Y) = -\lambda([\tilde{X},\tilde{Y}]_x), \quad X,Y \in \mathcal{D}_x,\ x \in M,$$

where $\tilde{X},\tilde{Y}$ are arbitrary local sections of $\mathcal{D}$ extending X,Y respectively.

Let us also discuss how to get the curvature 2-form Ω from the quotient map $\lambda : TM \to TM/\mathcal{D}$ directly.

Proposition 2.1.18. *If $\mathcal{D} = \ker\lambda$ then the curvature form of $\mathcal{D}$ is given as, $\Omega = d_\nabla\lambda|_{\mathcal{D}}$, for any choice of connection ∇ on $TM/\mathcal{D}$.*

Proof. Fix some connection ∇ on the bundle $TM/\mathcal{D}$. Then we have the $TM/\mathcal{D}$-valued 2-form $d_\nabla\lambda$ defined as,

$$d_\nabla\lambda(X,Y) = \nabla_X\lambda(Y) - \nabla_Y\lambda(X) - \lambda([X,Y])$$

for (local) vector fields X, Y. Now if we restrict to $\mathcal{D}$, i.e, if we have local sections $X, Y \in \mathcal{D} = \ker\lambda$, we see,

$$d_\nabla\lambda(X,Y) = -\lambda([X,Y]) = -[X,Y] \mod \mathcal{D} = \Omega(X,Y)$$

Hence we have, $\Omega = d_\nabla\lambda|_{\mathcal{D}}$. $\square$

Dual Curvature

Given a distribution $\mathcal{D} \subset TM$, we can define a subbundle $\mathrm{Ann}(\mathcal{D})$ of the cotangent bundle T^*M, called the *annihilator bundle* of $\mathcal{D}$, as follows :

$$\mathrm{Ann}(\mathcal{D})_x = \Big\{\alpha \in T_x^*M \;\Big|\; \alpha \text{ vanishes over } \mathcal{D}_x\}, \quad \text{for } x \in M$$

There is a canonical bundle isomorphism,

$$\mathrm{Ann}(\mathcal{D}) \cong (TM/\mathcal{D})^*,$$

induced by the nondegenerate pairing $\mathrm{Ann}(\mathcal{D}) \times TM/\mathcal{D} \to \mathbb{R}$, defined by, $(\alpha, X \mod \mathcal{D}) \mapsto \alpha(X)$. Using this identification, we can dualize the bundle map $\Omega : \Lambda^2\mathcal{D} \to TM/\mathcal{D}$ and get the *dual curvature map*,

$$\omega : \mathrm{Ann}(\mathcal{D}) \to \Lambda^2\mathcal{D}^*.$$

Explicitly, we have,

$$\omega(\alpha) = d\alpha|_{\mathcal{D}}, \quad \text{for any local secion } \alpha \in \mathrm{Ann}(\mathcal{D}).$$

Hence, any choice of (local) frames $\{\lambda^1, \ldots, \lambda^p\}$ of $\mathrm{Ann}(\mathcal{D})$, defines a representation of Ω as $\Omega \underset{loc.}{=} \big(\omega(\lambda^s)\big)$.

Observation. If $\mathcal{D}$ is 1-step bracket generating, then Ω is an epimorphism and hence ω is an injective bundle map. Therefore, the components of the curvature form Ω are linearly independent. And conversely.

Suppose that $\mathcal{D} = \bigcap_{s=1}^p \ker\lambda^s$ for some global 1-forms $\lambda^s \in \Omega^1(M)$. If $u : \Sigma \to M$ is a $\mathcal{D}$-horizontal immersion, then du_x maps into $\mathcal{D}$ and hence $\lambda^s \circ du_x = 0$ for all s. Therefore, u satisfies the following system of equations :

$$u^*\lambda^s = 0, \quad s = 1, \ldots, p$$

Taking exterior derivative on both sides, we get

$$u^*\omega^s = 0, \quad s = 1, \dots, p$$

where $\omega^s = d\lambda^s|_{\mathcal{D}}$. Hence, every horizontal immersion u satisfies

$$u^*\Omega = 0,$$

where Ω is the curvature form of $\mathcal{D}$. We shall refer to this as the *isotropy condition* or *curvature condition*.

This relation imposes certain obstruction to the existence of smooth horizontal immersions. For example, if $\xi = \ker \alpha$ is a contact distribution on M, then $d\alpha$ restricts to a symplectic form on ξ. Since any horizontal immersion $\Sigma \to M$ is $d\alpha$-isotropic, we must have $\dim \Sigma \leq \frac{1}{2}$ rank ξ.

Legendrian Immersions and h-Principle

Definition 2.1.19. Given a contact structure $\xi \subset TM$ on a manifold M, a ξ-horizontal immersion $\Sigma \subset M$ is called a *Legendrian immersion* if $\dim \Sigma = \frac{1}{2} \operatorname{rk} \xi$.

Example 2.1.20. Given any manifold M, there is a standard contact structure ξ on the first jet space $J^1(M, \mathbb{R}) = T^*M \times \mathbb{R}$ given as, $\xi = \ker \left(dz - \pi^*\lambda\right)$, where λ is the tautological 1-form on the cotangent bundle, z is the coordinate along $\mathbb{R}$ and $\pi : J^1(M, \mathbb{R}) \to T^*M$ is the projection map. Now, for any smooth map $f : M \to \mathbb{R}$, the 1-jet prolongation $j^1_f : M \to J^1(M, \mathbb{R})$ is a Legendrian embedding. Indeed,

$$(j^1_f)^*\left(dz - \pi^*\lambda\right) = d(z \circ j^1_f) - (\pi \circ j^1_f)^*\lambda = df - (df)^*\lambda = df - df = 0$$

We have a generalization of Theorem 2.1.13.

Theorem 2.1.21 ([Duc84])**.** *Legendrian immersions satisfy the complete h-principle. In particular, any formal Legendrian immersion can be homotoped to a genuine Legendrian immersion.*

A *formal* Legendrian immersion is by definition a bundle map $F : T\Sigma \to TM$, satisfying the following algebraic conditions :

- F is a bundle monomorphism, with $\operatorname{Im} F \subset \xi$, and
- $F^*\Omega = 0$, where Ω is the curvature form of ξ.

The above h-principle does not extend to Legendrian embeddings. However a special class of Legendrian embeddings, called "loose Legendrian embeddings", are amenable to homotopy classification.

Theorem 2.1.22 ([Mur12]). *Loose Legendrian embeddings in a contact manifold of dimension ≥ 5 satisfy the complete h-principle.*

Isocontact Immersions

Definition 2.1.23. Given contact structures $\xi \subset TM$ and $K \subset T\Sigma$ on the manifolds M and Σ respectively, an immersion $u : \Sigma \to M$ is called *isocontact* if $K = du^{-1}(\xi)$.

We have the following theorem by Gromov.

Theorem 2.1.24. [Gro86, pg. 339] *Given the standard contact structure ξ on $\mathbb{R}^{2n+1}$ and a cotrivializable contact structure K on Σ, with $\dim \Sigma = 2m+1$, an arbitrary map $\Sigma \to M$ admits a fine C^0-approximation by isocontact immersions $(\Sigma, K) \to (\mathbb{R}^{2m+1}, \xi)$, provided $n \geq 3m+1$ holds.*

Isocontact immersions and more generally embeddings in arbitrary contact structures also abide by the h-principle ([Dat97, EM02]).

In this thesis we consider certain class of bracket generating distributions of corank > 1 and one of our goal is to study the existence of horizontal and isocontact immersions for such distributions. This leads us to the general theory of h-principle.

2.2 h-Principle

We shall first recall the basic terminology and then briefly review the sheaf theoretic and analytic techniques of the theory of h-principle following [Gro86].

2.2.1 Sheaf Theoretic Techniques in h-Principle

The goal of h-principle is to solve a differential system by homotoping a *formal* solution to a genuine solution. Now, any differential system can be understood as a certain system of algebraic equations or inequalities defined on the jet bundles of sections of some fibration. Let us formalize these notions.

Throughout this section $p : X \to V$ will denote a smooth fiber bundle and the r-jet bundle of sections of X will be denoted by $p^{(r)} : X^{(r)} \to V$.

Definition 2.2.1. An r^{th}-order *partial differential relation* (or simply a *relation*) for sections of $p : X \to V$ is a subset $\mathcal{R} \subset X^{(r)}$ in the r-jet space $X^{(r)}$. An open subset of the jet space $X^{(r)}$ will be referred to as an *open relation*.

We shall denote the space of sections of $p : X \to V$ and $p^{(r)} : X^{(r)} \to V$ by $\Gamma(X)$ and $\Gamma(X^{(r)})$ respectively. The space $\Gamma(X)$ will be endowed with the C^∞-compact open topology while the space $\Gamma(X^{(r)})$ will have the C^0-compact-open topology. There is a canonical r-jet map,

$$j^{(r)} : \Gamma(X) \to \Gamma(X^{(r)}),$$

which takes a section f to its r-jet prolongation j^r_f. A section of the r-jet bundle is said to be a *holonomic* section if it lies in the image of j^r.

For any relation $\mathcal{R}$, we shall now introduce some subspaces of $\Gamma(X)$ and $\Gamma(X^{(r)})$.

Definition 2.2.2. A smooth section of X is said to be a *solution* of $\mathcal{R}$ if its r-jet prolongation has its image in $\mathcal{R}$. A continuous section of $X^{(r)}$ whose image is contained in $\mathcal{R}$ is called a *formal solution* of $\mathcal{R}$.

We shall denote,

- $\operatorname{Sol}\mathcal{R}$ as the space of solutions of $\mathcal{R}$.
- $\Gamma\mathcal{R}$ as the space of formal solutions of $\mathcal{R}$.

The r-jet map, $j^r : \operatorname{Sol}\mathcal{R} \to \Gamma\mathcal{R}$ identifies the solution space $\operatorname{Sol}\mathcal{R}$ with the holonomic sections of $\mathcal{R}$.

Definition 2.2.3. A relation $\mathcal{R} \subset X^{(r)}$ is said to satisfy the *ordinary h-principle* if any formal solution of $\mathcal{R}$ can be homotoped to a holonomic section while keeping the homotopy completely within $\Gamma\mathcal{R}$; in other words if,

$$\pi_0(j^r) : \pi_0\big(\operatorname{Sol}\mathcal{R}\big) \to \pi_0\big(\Gamma\mathcal{R}\big)$$

is surjective.

Hence, h-principle reduces an analytical problem to an algebraic problem. We say $\mathcal{R} \subset X^{(r)}$ satisfies the *parametric* or *complete* h-principle if the map j^r is a weak homotopy equivalence, i.e, if the map,

$$\pi_i(j^r) : \pi_i\big(\operatorname{Sol}\mathcal{R}\big) \to \pi_i\big(\Gamma\mathcal{R}\big),$$

is an isomorphism for each $i \geq 0$. Therefore, the parametric h-principle completely classifies the solution space of $\mathcal{R}$.

Definition 2.2.4. A relation $\mathcal{R} \subset X^{(r)}$ is said to satisfy the *C^0-dense h-principle* if,

- the usual h-principle holds for $\mathcal{R}$, and
- for any $F_0 \in \Gamma\mathcal{R}$ with base map $f_0 = \operatorname{bs} F_0$ and for any arbitrary neighborhood U of $\operatorname{Im} f_0$ in X, we can choose the homotopy $F_t \in \Gamma\mathcal{R}$ joining F_0 to a holonomic $F_1 = j^r_{f_1}$ in such a way, that the base map $f_t = \operatorname{bs} F_t$ satisfies $\operatorname{Im} f_t \subset U$ for all $t \in [0,1]$.

Given any relation $\mathcal{R}$ we now define two topological sheaves: The sheaf of solutions of $\mathcal{R}$ and the sheaf of sections of $\mathcal{R}$, which will be often referred to as $\operatorname{Sol}\mathcal{R}$ and $\Gamma\mathcal{R}$, respectively.

Before going any further, we recall some general theory of topological sheaves and define some concepts which will be used later. Recall that a *topological sheaf* Φ over a smooth manifold V, assigns

- to each open subset $U \subset V$ a topological space $\Phi(U)$, and
- to each pair of open sets (U, U'), with $U' \subset U \subset V$, a continuous map (known as the *restriction map*) $\Phi(U) \to \Phi(U')$.

For an arbitrary subset C of V, we define $\Phi(C)$ as the direct limit,

$$\Phi(C) = \lim_{\substack{C \subset U \\ U \subset V \text{ is open}}} \Phi(U)$$

Thus, an element of $\Phi(C)$ can be represented by an element of $\Phi(U)$, where U is some open set containing C. Keeping up with the definition of direct limits, this open set is not kept fixed and for notational convenience we denote this by $\operatorname{Op}(C)$. In a similar fashion, we define $\operatorname{Op}(v)$ for any $v \in V$ as some arbitrarily small (and not fixed) open neighborhood of v. We shall not consider the direct limit topology on $\Phi(C)$, rather we shall work with the weaker notion of quasi-topological structures ([Gro86, pg. 36]) on them, which is sufficient for our purpose. In particular, by a 'continuous' map $f : Q \to \Phi(C)$, we shall mean that there is an open subset U containing C such that $f_q \in \Phi(U)$, for all $q \in Q$.

Definition 2.2.5. A sheaf homomorphism $\alpha : \Phi \to \Psi$ is called a *local weak homotopy equivalence*, if the induced map $\alpha_v : \Phi_v \to \Psi_v$ at the stalk level is a weak homotopy equivalence for each $v \in V$. The map α is a *weak homotopy equivalence* if $\alpha_U : \Phi(U) \to \Psi(U)$ is a weak homotopy equivalence for any open $U \subset V$.

For our purpose it is enough to consider sheaves that arise in connection with space of sections of a fibration in which case the morphisms assigned to a pair of open sets are the obvious restriction maps. More specifically, we shall be interested in the sheaves $\Phi = \mathrm{Sol}\,\mathcal{R}$ and $\Psi = \Gamma\mathcal{R}$, associated to some relation $\mathcal{R}$.

Flexible and Micro-flexible Sheaves

A (topological) sheaf Φ on V is said to be *flexible* if for every pair of compact sets (B, A), with $A \subset B$, the restriction map $\Phi(B) \to \Phi(A)$ is a Serre fibration. Flexibility is an important property of a sheaf as apparent from the theorem below.

Theorem 2.2.6 (Homomorphism Theorem)**.** [Gro86, pg. 77] *Let $\alpha : \Phi \to \Psi$ be a sheaf homomorphism between two flexible sheaves. Then α is a weak homotopy equivalence if α is a local weak homotopy equivalence.*

Now let us look at the sheaves $\Phi = \mathrm{Sol}\,\mathcal{R}$ and $\Psi = \Gamma\mathcal{R}$, for some relation $\mathcal{R} \subset X^{(r)}$. It turns out that Ψ is *always* flexible. Moreover, for many relations, the r-jet map $j^r : \mathrm{Sol}\,\mathcal{R} \to \Gamma\mathcal{R}$ is easily seen to be a local weak homotopy equivalence. Hence, the h-principle for $\mathcal{R}$ would follow if we can prove that the solution sheaf is flexible. However, the solution sheaf $\Phi = \mathrm{Sol}\,\mathcal{R}$ fails to be flexible in general and it is not easy to get around it. A property called *micro*flexibility, which is weaker than flexibility, comes to the rescue.

A continuous map $p : X \to Y$ is called a *micro-fibration*, if for an arbitrary polyhedron P and for any commutative diagram,

$$\begin{tikzcd} P \times \{0\} \arrow[r, "\tilde{f}_0"] \arrow[d, hook] & X \arrow[d, "p"] \\ P \times [0,1] \arrow[r, "f"'] & Y \end{tikzcd}$$

there exists an $\varepsilon > 0$ and a continuous map $F : P \times [0, \varepsilon] \to Y$ such that

$$p \circ F = f|_{P\times[0,\varepsilon]} \quad \text{and} \quad F|_{P\times 0} = \tilde{f}_0.$$

Note that if ε can be chosen to be 1 then p is a fibration.

Definition 2.2.7. A topological sheaf Φ is called *microflexible* if the restriction map $\Phi(B) \to \Phi(A)$ is a microfibration for each pair of compact sets (B, A), with $A \subset B \subset V$.

Example 2.2.8. It is easy to see that every *open* relation is *locally integrable* (i.e, every jet in $\mathcal{R}$ extends to a local holonomic section of $\mathcal{R}$) and hence the solution sheaf is microflexible.

Solution sheaves of many non-open relations are also microflexible, e.g., we shall see later in this section that the sheaf of 'regular' solutions of a partial differential equation is microflexible if the associated differential operator is 'infinitesimally invertible'.

The passage from microflexibility to flexibility depends on the local symmetry of the relation under consideration.

Action of the Pseudogroup $\mathrm{Diff}(V)$

Let $\mathrm{Diff}(V)$ denote the pseudogroup of local diffeomorphisms of a manifold V and Γ be the sheaf of sections of a fiber bundle over V. A pair of elements $(\varphi, f) \in \mathrm{Diff}(V) \times \Gamma$ is said to be compatible if f is defined on the image of φ. By an action of $\mathrm{Diff}(V)$ on Γ we mean a partial map $\psi : \mathrm{Diff}(V) \times \Gamma \to \Gamma$, defined only on compatible pairs and having all the properties of an ordinary action. In other words, if $\varphi : U \to U'$ is a diffeomorphism between two open subsets of V and $f \in \Gamma(U')$ then the element $\psi(\varphi, f)$, denoted by $\varphi.f$, belongs to $\Gamma(U)$. We can similarly define an action of a sub-pseudogroups of $\mathrm{Diff}(V)$on Γ.

The simplest example of a $\mathrm{Diff}(V)$-action is seen on the space of C^∞-maps between two manifolds, $C^\infty(V, W)$, which is given by $(\varphi, f) \mapsto f\circ\varphi$, where $f \in C^\infty(V, W)$ and $\varphi \in \mathrm{Diff}(V)$.

A subsheaf Φ of Γ is said to be $\mathrm{Diff}(V)$-*invariant* if $\varphi : U \to U'$ maps $\Phi(U')$ into $\Phi(U)$, under an action described as above. If Φ is the solution sheaf of some relation $\mathcal{R}$ then an action on Φ induces an action on $\Gamma\mathcal{R}$.

Consider the product manifold $V \times \mathbb{R}$ and let $\pi : V \times \mathbb{R} \to V$ be the canonical projection map. Define $\mathrm{Diff}(V \times \mathbb{R}, \pi)$ to be the space of all fiber-preserving (local) diffeomorphisms of $V \times \mathbb{R}$; in other words, $f \in \mathrm{Diff}(V \times \mathbb{R}, \pi)$ if $\pi \circ f = \pi$.

Theorem 2.2.9 (Flexibility Theorem)**.** [Gro86, pg. 78] *Suppose Φ is a microflexible sheaf over the manifold $V \times \mathbb{R}$ and Φ is $\mathrm{Diff}(V \times \mathbb{R}, \pi)$-invariant. Then the restriction sheaf $\Phi|_{V\times 0}$ is a flexible sheaf over $V = V \times 0$.*

Note that $V \times 0 \subset V \times \mathbb{R}$ is a closed subset. Thus a section of $\Phi|_V$ over some open set $U \subset V$ is understood as a section defined over an arbitrary open neighborhood of $U \times 0$ in $V \times \mathbb{R}$. An immediate consequence of the flexibility theorem is the following h-principle 'near' V.

Theorem 2.2.10. *Let $\mathcal{R}$ be a relation defined for sections over $V \times \mathbb{R}$. Suppose, the solution sheaf Φ of $\mathcal{R}$ is microflexible and $\mathrm{Diff}(V \times \mathbb{R}, \pi)$-invariant. If $\mathcal{R}$ satisfies the local parametric h-principle then it satisfies the parametric h-principle near $V \times 0$.*

This leads us into Gromov's famous h-principle theorem on open manifolds.

Theorem 2.2.11. [Gro86, pg. 79] *Any $\mathrm{Diff}(V)$-invariant, open relation $\mathcal{R}$, over an open manifold V, satisfies the parametric h-principle.*

Remark 2.2.12. It is easy to note that the openness condition on $\mathcal{R}$ in the above theorem can be relaxed by the following two conditions: (a) the solution sheaf of $\mathcal{R}$ is microflexible, and (b) $\mathcal{R}$ satisfies the local parametric h-principle.

On the other hand, the openness condition on V in Theorem 2.2.11 may not be relaxed and there are easy counter-examples where the h-principle fails. However, this theorem is a crucial step towards getting an analogous result for certain relations on *closed* manifolds.

h-Principle For Closed Manifolds

In order to deal with relations $\mathcal{R}$ over closed manifolds V, the idea is to embed the manifold in question in the *open* manifold $\tilde{V} = V \times \mathbb{R}$ and transform the problem in hand to an h-principle problem of an auxiliary relation $\tilde{\mathcal{R}}$ on $\tilde{V}$. This is where the flexibility theorem plays a crucial role.

Let us first formally introduce the notion of an *extension* of a relation. For simplicity we shall assume that the fiber bundles $X \to V$ are natural bundles ([Gro86, pg. 145]), so that there is a natural action of $\mathrm{Diff}(V)$ on the space of sections $\Gamma(X)$.

Definition 2.2.13. Let $\mathcal{R} \subset X^{(r)}$ be a relation. By an *extension* we mean a bundle $\tilde{X} \to \tilde{V}$ over the manifold $\tilde{V} = V \times \mathbb{R}$, along with a relation $\tilde{\mathcal{R}} \subset \tilde{X}^{(r)}$, such that the following conditions are satisfied.

1. There is a fiber-preserving morphism,

$$ev : \Gamma\tilde{X}|_{V\times 0} \to \Gamma X,$$

   such that the induced map,

$$\begin{aligned} ev_* : \tilde{X}^{(r)}|_{V\times 0} &\to X^{(r)} \\ j^r_x &\mapsto j^r_{ev(x)} \end{aligned}$$

   maps $\mathcal{R}|_{V\times 0}$ into $\mathcal{R}$.

2. There is an open cover $\mathcal{O}$ of Σ by contractible coordinate charts, closed under finite (nonempty) intersections, such that,

$$ev|_O : \Gamma\tilde{\mathcal{R}}|_{O\times 0} \to \Gamma\mathcal{R}|_O$$

is surjective for each $O \in \mathcal{O}$. This will be referred to as the 'local surjectivity' of the extension.

Notation : If $\mathcal{R}$ has an extension $\tilde{\mathcal{R}}$ then we adopt the following notations for the sheaves.

$$\tilde{\Phi} = \operatorname{Sol}\tilde{\mathcal{R}} \quad \text{and} \quad \tilde{\Psi} = \Gamma\tilde{\mathcal{R}}.$$

To keep the notation light, we shall denote the induced sheaf morphisms $\tilde{\Phi}|_{V\times 0} \to \Phi$ and $\tilde{\Psi}|_{V\times 0} \to \Psi$ by ev as well. The map ev and the covering $\mathcal{O}$ will not be mentioned explicitly, unless necessary.

Remark 2.2.14. We should point out that this notion of extension is in the similar vein of [dP76], where the author considered open relations only.

As an application of the flexibility theorem we now get the following.

Theorem 2.2.15. *Let $\mathcal{R}$ be an r^{th}-order relation over V, which admits an extension $\tilde{\mathcal{R}}$ over $\tilde{V} = V \times \mathbb{R}$, such that,*

1. *$\tilde{\Phi}|_{V\times 0}$ is flexible, and*

2. *$\tilde{\mathcal{R}}$ satisfies the local h-principle.*

Suppose further that,

3. *for each compact set $C \subset O \in \mathcal{O}$ and for each solution $u \in \operatorname{Sol}\mathcal{R}|_{\operatorname{Op}C}$, the jet map*

$$j^r : ev^{-1}(u) \to ev^{-1}(F = j^r_u)$$

in the following diagram,

$$\begin{tikzcd} ev^{-1}(u) \arrow[r, hook] \arrow[d, dashed, "j^r"'] & \tilde{\Phi}|_{\operatorname{Op}C\times 0} \arrow[r, "ev"] \arrow[d] & \Phi|_{\operatorname{Op}C} \arrow[d] & u \arrow[d, mapsto] \\ ev^{-1}(F) \arrow[r, hook] & \Gamma\tilde{\mathcal{R}}|_{\operatorname{Op}C\times 0} \arrow[r, "ev"'] & \Gamma\mathcal{R}|_{\operatorname{Op}C} & F = j^r_u \end{tikzcd}$$

induces surjective map on the set of path components.

Then the relation $\mathcal{R}$ satisfies the C^0-dense h-principle.

Let us give a proof of this h-principle, which is essentially done via a cell-wise induction. We fix the extension $\tilde{\mathcal{R}}$ of the relation $\mathcal{R}$, along with the open cover $\mathcal{O}$ of Σ, so that the map $ev : \tilde{\Psi}|_{O\times 0} \to \Psi|_O$ is surjective on sections for each $O \in \mathcal{O}$.

Next we fix a triangulation $\{\Delta^\alpha\}$ on V, so that each simplex Δ^α satisfies,

$$\Delta^\alpha \subset O_\alpha \text{ for some } O_\alpha \in \mathcal{O}.$$

For any simplex $\Delta = \Delta^\alpha$ we denote,

$$O_\Delta = \bigcap_{\Delta \subset \Delta^\beta} O_\beta$$

Since any Δ is contained in at most finitely many simplices, the intersection is finite and thus $O_\Delta \in \mathcal{O}$. Let us also fix some convention about arbitrarily small open sets.

Convention about $\mathrm{Op}(\cdot)$ and $\tilde{\mathrm{Op}}(\cdot)$ For any subset $A \subset V$, $\mathrm{Op}(A)$ will denote an unspecified open neighborhood of A, which may change in the course of the proof of Theorem 2.2.15. Similarly, $\tilde{\mathrm{Op}} A$ will denote an arbitrary open neighborhood of A in $\mathrm{Op}(A) \times \mathbb{R}$. Furthermore, we will also assume that $\mathrm{Op}\,\Delta \subset O_\Delta$ and $\mathrm{Op}\,\partial\Delta \subset O_\Delta$, for any cell Δ of the triangulation.

The following lemma is the base case for the induction involved in the proof.

Lemma 2.2.16. *Fix some 0-simplex $v \in \Sigma$. Given any $F \in \Psi$, there exists a C^0-small homotopy $F_t \in \Psi$ and open sets V_1, V_2 satisfying,*

$$v \subset V_1 \subset \bar{V}_1 \subset V_2 \subset \bar{V}_2 \subset O_v,$$

where $\bar{V}_i$ is the closure of V_i, such that,

- $F_0 = F$
- *F_1 is holonomic on V_1*
- *F_t is constant and equals F on $\{v\} \cup (V \setminus V_2)$.*

Proof. Since $\tilde{\mathcal{R}}$ is an extension of $\mathcal{R}$, we get some arbitrary lift $\tilde{F} \in \tilde{\Psi}|_v$ of $F \in \Psi|_v$, along the sheaf map $ev : \tilde{\Psi}|_{V\times 0} \to \Psi$. Then from the hypothesis (3) applied to the compact set $C = \{v\}$, we get a path $\tilde{G}_t \in \tilde{\Psi}|_v$ joining $\tilde{F}$ to a holonomic section $\tilde{G}_1$, such that $\tilde{G}_t|_v = \tilde{F}|_v$ is fixed. Set, $F^v_t = ev(\tilde{G}_t)$. Then $F^v_t|_v = F|_v$ and F^v_1 is holonomic on $\mathrm{Op}(v)$.

We now need to extend this homotopy to all over Σ. Fix open sets V_1, V_2, with $v \subset V_1 \subset \bar{V}_1 \subset V_2 \subset \bar{V}_2 \subset O_v$. Next get a cutoff function $\rho : V \to [0,1]$, which is identically 1 on $\bar{V}_1$ and

$\operatorname{supp} \rho \subset V_2$. Define,

$$F_t(\sigma) = \begin{cases} F^v_{\rho(\sigma)t}(\sigma), \text{ if } \sigma \in \bar{V}_2 \\ F(\sigma), \text{ if } \sigma \in V \setminus V_2 \end{cases}$$

It is then easy to see that F_t is the required homotopy. The homotopy can be made arbitrarily C^0-small by choosing the open set O_v sufficiently small. $\square$

The next lemma is the crux of the proof, i.e, the induction step. Note that we are considering top-dimensional simplices of V as well.

Lemma 2.2.17. *Suppose $\Delta \subset V$ is some $i+1$-cell, for $i \geq 0$. Given $F \in \Psi$ such that $F|_{\mathrm{Op}\,\partial\Delta}$ is holonomic. Then there exists a C^0-small homotopy $F_t \in \Psi$ and open sets V_1, V_2, W_1 satisfying,*

$$\Delta \subset V_1 \subset \bar{V}_1 \subset V_2 \subset \bar{V}_2 \subset O_\Delta \quad \textit{and} \quad \partial\Delta \subset W_1 \subset \bar{W}_1 \subset V_1 \cap \mathrm{Op}\,\partial\Delta \subset O_\Delta,$$

such that,

- $F_0 = F$
- F_1 *is holonomic on* V_1
- F_t *is constant and equals* F *on* $W_1 \cup \big(V \setminus V_2\big)$.

Proof. Since $\tilde{\mathcal{R}}$ is an extension, we first obtain some arbitrary lift $\tilde{F} \in \tilde{\Psi}|_\Delta$ of $F|_{\mathrm{Op}\,\Delta} \in \Psi|_\Delta$, along the map ev. This is possible since the simplex Δ is contained in some $O \in \mathcal{O}$. Now, as we are given that $F|_{\mathrm{Op}\,\partial\Delta}$ is holonomic, using the hypothesis (3) for the compact set $C = \partial\Delta$, we obtain a homotopy

$$\tilde{G}^{\partial\Delta}_t \in \tilde{\Psi}|_{\partial\Delta}$$

joining $\tilde{F}|_{\mathrm{Op}\,\partial\Delta}$ to a holonomic section $\tilde{G}^{\partial\Delta}_1 \in \tilde{\Psi}|_{\partial\Delta}$. Let us denote,

$$\tilde{G}^{\partial\Delta}_1 = j^r_{\tilde{u}^{\partial\Delta}},$$

for some regular solution $\tilde{u}^{\partial\Delta} : \tilde{\mathrm{Op}}\partial\Delta \to M$. Furthermore, under the map $ev : \tilde{\Psi}|_{\partial\Delta} \to \Psi|_{\partial\Delta}$ we have that $ev(G_t) = F|_{\mathrm{Op}\,\partial\Delta}$ is constant.

Next, recall that the sheaf $\tilde{\Psi}|_\Delta$ is flexible. Consider the diagram,

$$\begin{tikzcd} 0 \arrow[r, "\tilde{F}"] \arrow[d, hook] & \tilde{\Psi}|_\Delta \arrow[d] \\ I \arrow[r, "G^{\partial\Delta}_t"'] & \tilde{\Psi}|_{\partial\Delta} \end{tikzcd}$$

We then have a homotopy lift $\tilde{G}^\Delta_t : [0,1] \to \tilde{\Psi}|_\Delta$, which is fixed on $\tilde{\mathrm{Op}}\partial\Delta$. In particular we have,

$$\tilde{G}^\Delta_1|_{\tilde{\mathrm{Op}}\partial\Delta} = \tilde{G}^{\partial\Delta}_1 = j^r_{\tilde{u}^{\partial\Delta}}.$$

Now we consider the map of *fibrations* as follows.

$$\begin{tikzcd}
\eta^{-1}(\tilde{u}^{\partial\Delta}) \arrow[r, hook] \arrow[d, "J"'] & \tilde{\Phi}|_\Delta \arrow[r, "\eta"] \arrow[d, "J"] & \tilde{\Phi}|_{\partial\Delta} \arrow[d, "J"] & \tilde{u}^{\partial\Delta} \arrow[d, mapsto] \\
\chi^{-1}(\tilde{G}^\Delta_1|_{\tilde{\mathrm{Op}}\partial\Delta}) \arrow[r, hook] & \tilde{\Psi}|_\Delta \arrow[r, "\chi"'] & \tilde{\Psi}|_{\partial\Delta} & j^r_{\tilde{u}^{\partial\Delta}} = \tilde{G}^\Delta_1|_{\tilde{\mathrm{Op}}\partial\Delta}
\end{tikzcd}$$

Here η is indeed a fibration, as $\tilde{\Phi}|_V$ is assumed to be flexible. Now the rightmost and the middle $J = j^r$ are *local* weak homotopy equivalences by the hypothesis (2). Hence they are in fact weak homotopy equivalences by an application of the homomorphism theorem (Theorem 2.2.6). By the 5-lemma argument, we then have,

$$J : \eta^{-1}(\tilde{u}) \to \chi^{-1}\big(j^r_{\tilde{u}^{\partial\Delta}}\big)$$

is a weak homotopy equivalence. Now,

$$\tilde{G}^\Delta_1 \in \chi^{-1}(j^r_{\tilde{u}^{\partial\Delta}})$$

Hence we have a path

$$\tilde{H}_t \in \chi^{-1}(j^r_{\tilde{u}^{\partial\Delta}})$$

joining $\tilde{G}^\Delta_1$ to some holonomic section $\tilde{H}_1 = j^r_{\hat{u}^\Delta}$, where $\hat{u}^\Delta : \tilde{\mathrm{Op}}\Delta \to M$ is a regular solution. In particular, this homotopy is fixed on $\tilde{\mathrm{Op}}\partial\Delta$. We have the concatenated homotopy,

$$\tilde{F}_t : \tilde{F} \sim_{\tilde{G}^\Delta_t} \tilde{G}^\Delta_1 \sim_{\tilde{H}_t} j^r_{\hat{f}^\Delta}.$$

Set $F^\Delta_t = ev(\tilde{F}_t)$. Then, $F^\Delta_0 = F|_\Delta$ and F^Δ_1 is holonomic on $\mathrm{Op}\,\Delta$. Furthermore, as observed, F^Δ_t is fixed on $\mathrm{Op}\,\partial\Delta$.

Lastly, we need to extend F^Δ_t to all of V, keeping it F outside $\mathrm{Op}\,\Delta$. Fix open sets, $\Delta \subset V_1 \subset \bar{V}_1 \subset V_2 \subset \bar{V}_2 \subset \mathrm{Op}\,\Delta$ and $\partial\Delta \subset W_1 \subset \bar{W}_1 \subset V_1 \cap \mathrm{Op}\,\partial\Delta$. Next get a cutoff function $\rho : V \to [0,1]$ which is identically 1 on $\bar{V}_1$ and $\mathrm{supp}\,\rho \subset V_2$. Define,

$$F_t(\sigma) = \begin{cases} F^\Delta_{\rho(\sigma)t}(\sigma), & \text{if } \sigma \in \bar{V}_2 \\ F(\sigma), & \text{if } \sigma \in V \setminus V_2 \end{cases}$$

It is easy to see that F_t is the required homotopy. The homotopy can be made arbitrarily C^0-small by choosing the open set O_Δ sufficiently small. $\square$

We may now proceed to prove the h-principle.

Proof of Theorem 2.2.15 . We show that for a given $F \in \Psi$, there is a homotopy $F_t \in \Psi$ such that $F_0 = F$ and F_1 is a holonomic section. The proof is done by a cell-wise induction.

Step 0 **:** For each 0-simplex $v \in V$, using Lemma 2.2.16, we get a homotopy $F_t^v \in \Psi$, which is holonomic on $\mathrm{Op}(v)$ and is identically F on $V \setminus \mathrm{Op}(v)$. But then all these homotopies patch together nicely and we have a homotopy $F_t^0 \in \Psi$ such that $F_0^0 = F$ and F_1^0 is holonomic on $\mathrm{Op}\, V^{(0)}$, neighborhood of the 0-skeleton $V^{(0)}$. Clearly, $F_t^0 = F$ on $V \setminus \mathrm{Op}\, V^{(0)}$.

Step 1 **:** For each 1-simplex Δ of V, using Lemma 2.2.17, we get a homotopy $F_t^\Delta \in \Psi|_\Delta$ such that F_1^Δ is holonomic on $\mathrm{Op}(\Delta)$. Also, $F_t^\Delta = F_1^0$ on $\mathrm{Op}\, \partial\Delta \cup \big(\Sigma \setminus \mathrm{Op}\, \Delta\big)$. Hence all these homotopies patch together nicely and we get, $F_t^1 \in \Psi$ such that $F_0^1 = F_1^0$ and F_1^1 is holonomic on $\mathrm{Op}\, V^{(1)}$, neighborhood of the 1-skeleton $V^{(1)}$. Clearly, $F_t^1 = F_1^1$ on $\Sigma \setminus \mathrm{Op}\, V^{(1)}$

Step $i+1$ **:** Suppose we have $F_1^i \in \Psi$ which is holonomic on $\mathrm{Op}\, V^{(i)}$. For each $i+1$-simplex Δ, using Lemma 2.2.17, we get a homotopy $F_t^\Delta \in \Psi$ such that F_1^Δ is holonomic on $\mathrm{Op}\, \Delta$. Also, $F_t^\Delta = F_1^i$ on $\mathrm{Op}\, \partial\Delta \cup \big(\Sigma \setminus \mathrm{Op}\, \Delta\big)$. Hence all these homotopies patch together nicely and we get, $F_t^{i+1} \in \Psi$ such that $F_t^{i+1} = F_1^i$ and F_1^{i+1} is holonomic on $\mathrm{Op}\, V^{(i+1)}$, neighborhood of the $i+1$-skeleton $V^{(i+1)}$. Clearly, $F_t^{i+1} = F_1^i$ on $V \setminus \mathrm{Op}\, V^{(i+1)}$.

The induction stops once we have performed step k where $k = \dim V$. We end up with a sequence of homotopies in Ψ. Concatenating all of them we have the homotopy,

$$F_t : F \sim_{F_t^0} F_1^0 \sim_{F_t^1} F_1^1 \sim \cdots \sim_{F_t^{k-1}} F_1^{k-1} \sim_{F_t^k} F_1^k.$$

Clearly $F_t \in \Psi$ is the desired homotopy joining F to a *holonomic* section $F_1 = F_1^k \in \Psi$. Since at each stage the homotopy can be chosen to be arbitrarily C^0-small and since there are finitely many stages, we see that F_t can be made arbitrary C^0-small as well. This concludes the proof. $\square$

Remark 2.2.18. Note that the setup of Theorem 2.2.15 requires the extension relation $\tilde{\mathcal{R}}$ to be defined over the manifold $\tilde{V} = V \times \mathbb{R}$, even though in the course of the proof we are only using the fact that we have *local* lifts along $ev : \Gamma\tilde{\mathcal{R}}|_O \to \Gamma\mathcal{R}$ over some contractible open set

$O \subset V$. In fact, given a section $F \in \Psi$ we require a good open cover of V, depending on F, say, $\mathcal{O} = \{O_i\}_{i \in \Lambda}$ for some index set Λ, which has the following property : For each $i \in \Lambda$ there exists,

$$\text{a natural bundle } \tilde{X}_i \to \tilde{O}_i = O_i \times \mathbb{R} \text{ and a relation } \tilde{\mathcal{R}}_i \subset \tilde{X}^{(r)}$$

so that, (a) $\tilde{\mathcal{R}}_i$ is an extension of $\mathcal{R}|_{O_i}$ and (b) $\tilde{\mathcal{R}}_i$ satisfies the hypothesis of Theorem 2.2.15 for the manifold O_i. Then the proof of Theorem 2.2.15 goes on without any change. Consequently, $\mathcal{R}$ satisfies the C^0-dense h-principle. We will apply the theorem in this general setup in Chapter 5.

2.2.2 Differential Operators

In this section we shall see that solution sheaves of a large class of *non-open* relations, which appear in connection with differential equations, are microflexible and furthermore those relations satisfy the local h-principle.

Definition 2.2.19. Let $p : X \to V$ be a smooth fibration and $G \to V$ be a smooth vector bundle. A *differential operator* of order r is a map $\mathfrak{D} : \Gamma X \to \Gamma G$ given by a smooth bundle morphism $\Delta : X^{(r)} \to G$ which satisfies, $\mathfrak{D}(x) = \Delta \circ j^r_x$ for any (local) section $x \in \Gamma X$. Therefore, we have the commuting diagram,

$$\begin{tikzcd}
X^{(r)} \arrow[rr, "\Delta"] \arrow[dr] & & G \arrow[dl] \\
& V \arrow[ul, bend left, "j^r_x"] \arrow[ur, bend right, "\mathfrak{D}(x)"'] &
\end{tikzcd}$$

The bundle map Δ is known as the *symbol* of the operator $\mathfrak{D}$. For $\alpha \geq 0$, we define a bundle map $\Delta^{(\alpha)} : X^{(r+\alpha)} \to G^{(\alpha)}$ given by,

$$\Delta^{(\alpha)}(j^{r+\alpha}_x(v)) = j^r_{\mathfrak{D}(x)}(v).$$

Unless mentioned otherwise, $X \to V$ will denote a smooth fibration, $G \to V$ will denote a smooth vector bundle and $\mathfrak{D} : \Gamma X \to \Gamma G$ a smooth differential operator.

Linearization of a Differential Operator

Given a differential operator $\mathfrak{D} : \Gamma X \to \Gamma G$, the derivative at a point should quantify the infinitesimal change in the value of the operator for small perturbation of the source. First we need to define the tangent space of ΓX at some "point" $x \in \Gamma X$.

Tangent space of $\Gamma(X)$: We first consider the case $X = M \times N \to M$, so that we have the identification $\Gamma X = C^\infty(M, N)$. A curve in $C^\infty(M, N)$ is a continuous map $\gamma : [0, 1] \to C^\infty(M, N)$. By a smooth curve we mean that the homotopy $f : M \times [0, 1] \to N$ defined by $f(x, t) = \gamma_t(x)$ is smooth. Differentiating f at $t = 0$, we get a vector field ξ along f_0, given by $\xi = \frac{d}{dt}\big|_{t=0} f_t$. In fact, we can identify the tangent space to $C^\infty(M, N)$ by the section space of the vector bundle f^*TM.

We carry out the same idea for the general case of a fiber bundle $\pi : X \to V$ as well. For some $x \in \Gamma X$, consider a smooth family of sections $\{x_t\}_{t\in(-\epsilon,\epsilon)}$ so that $x_0 = x$. Now for any fixed point $v \in V$, $t \mapsto x_t(v)$ is a smooth curve in the fiber $X_v = \pi^{-1}(v)$. Hence, taking derivative at $t = 0$, we have that $\xi(v) = \frac{d}{dt}\big|_{t=0} x_t(v)$, which is a tangent vector to the fiber $\pi^{-1}(v)$ of X, at $v \in V$. The tangent vectors to X which are tangential to the fibers are referred to as vertical tangent vectors. The space of all vertical tangent vectors form a subbundle of TX which will be denoted by $T^{vert}X$. It is easy to note that $T^{vert}X = \ker d\pi$. Clearly, ξ is then a section to $T^{vert}X$, defined along $x : V \to X$. Thus we have the following identification of the tangent space of $\Gamma(X)$ at $x \in \Gamma X$ as,

$$T_x\Gamma X = \Gamma x^*(T^{vert}X)$$

Now the linearization of $\mathfrak{D} : \Gamma X \to \Gamma G$ at some section $x \in \Gamma X$ is a linear map between their respective tangent spaces,

$$T_x\mathfrak{D} : T_x\Gamma X \to T_{\mathfrak{D}(x)}\Gamma G.$$

We have identified, $T_x\Gamma X = \Gamma x^* T^{vert}X$. Since G is a vector bundle, ΓG is a vector bundle as well and we may canonically identify $T_{\mathfrak{D}(x)}\Gamma G = \Gamma G$. Now, for any $\xi \in \Gamma x^* T^{vert}X$, choose some representative family of sections $\{x_t\}_{t\in(-\epsilon,\epsilon)}$, such that $x_0 = x$ and $\xi(v) = \frac{d}{dt}|_{t=0} x_t(v)$ for $v \in V$. Then, the differential $T_x\mathfrak{D} : \Gamma x^* T^{vert}X \to \Gamma G$ is given as,

$$T_x\mathfrak{D}(\xi) = \lim_{t\to 0} \frac{1}{t}\Big(\mathfrak{D}(x_t) - \mathfrak{D}(x)\Big),$$

which is a *linear* differential operator of the same order as $\mathfrak{D}$. We will call $T_x\mathfrak{D}$ the *linearization* of the operator $\mathfrak{D}$ at x. The linearity here means $\mathbb{R}$-linearity and not to be confused with $C^\infty(V)$-linearity.

Remark 2.2.20. By choosing a suitable topology, we can make sure that the differential defined above makes sense in an infinite dimensional manifold setup. For a rigorous treatment, one may look into the beautiful monograph by Palais ([Pal66]). In this thesis, we may safely work under the assumption that most of the natural operations from finite dimensional differential topology can be performed in the infinite dimensional setup as well.

Universal Inversion of a Linear Differential Operator

A differential operator $L : \Gamma X \to \Gamma G$, where both $X \to V$ and $G \to V$ are vector bundles, is called a *linear* differential operator if $L(x + y) = L(x) + L(y)$ and $L(\lambda x) = \lambda L(x)$ for all $x, y \in \Gamma(X)$ and $\lambda \in \mathbb{R}$. Note that in this case both $\Gamma(X)$ and $\Gamma(G)$ are infinite-dimensional vector spaces.

A linear differential operator L is *underdetermined* if we have $\operatorname{rk} X > \operatorname{rk} G$. Similarly, we say L is *overdetermined* if $\operatorname{rk} X < \operatorname{rk} G$ and is *determined* if $\operatorname{rk} X = \operatorname{rk} G$.

Observe that given two vector bundles X, G over V, any linear differential operator $\Gamma X \to \Gamma G$ of order r, is determined by the symbol map $X^{(r)} \to G$. In fact, the space of linear differential operators of order r can be identified with the space of sections ΓH, where $H = \hom(X^{(r)}, G)$ is a bundle over V.

Let $L : \Gamma X \to \Gamma G$ be an under-determined linear differential operator of order r. A linear differential operator $M : \Gamma G \to \Gamma X$ of some order s, is called a *right inverse* of L if $L \circ M = \mathrm{Id}$.

Let $\mathcal{A} \subset H^{(s)}$ be an open subset of the jet bundle and $A = \operatorname{Sol} \mathcal{A}$. Hence, A is an open subspace in the space of all r-th order linear differential operators $\Gamma(X) \to \Gamma(G)$. By a *universal* right inversion of the operators A, we mean a differential operator $\mathfrak{M} : A \times \Gamma G \to \Gamma X$ such that for any $L \in A$ we have, $L \circ \mathfrak{M}(L, \_) = \mathrm{Id}$, i.e,

$$L\big(\mathfrak{M}(L, g)\big) = g, \text{ for any } g \in \Gamma G$$

Similarly we can define a (universal) left inversion as well.

Infinitesimal Inversion of a Differential Operator

Definition 2.2.21. A differential operator $\mathfrak{D} : \Gamma X \to \Gamma G$ of order r, is said to be *infinitesimally invertible* over a set $A \subset \Gamma X$ if we have a family of *differential* operators $M_x : \Gamma G \to \Gamma x^* T^{vert} X$ of order s, such that the following holds.

- There exists an *open* set $\mathcal{A} \subset X^{(d)}$ such that A consists of precisely the C^∞-solutions of $\mathcal{A}$, i.e., $A = \operatorname{Sol}(\mathcal{A})$.

- The global operator $M(x,g) = M_x(g)$ for $x \in A$ and $g \in \Gamma G$ is a differential operator which has order d in $x \in S$ and order s in $g \in \Gamma G$. This is defined by a smooth map $\mathcal{A} \oplus G^{(s)} \to T^{vert}X$.

- The operator M_x is a right inverse to $L_x = T_x\mathfrak{D}$, i.e, we have

$$L_x\big(M_x(g)\big) = g, \quad \text{for } x \in A \text{ and } g \in \Gamma G$$

The number d is called the *defect* of the inversion and the class of maps, over which inversion exists, will be often referred to as $\mathcal{A}$-*regular maps*. Note that A is an open subset in the fine C^∞-topology. We have that M is an universal right inversion of order s for the family $\{L_x \mid x \in A\}$.

Observation 2.2.22. Let us make a few observation about the definition.

- In contrast with the classical implicit function theorem in the finite dimensional case, where we only ask for the surjectivity at one point, here we demand that $T_x\mathfrak{D}$ be (right) invertible for all x which belongs to the *open* set of maps A. Indeed, it is crucial that the set of maps is the solution space for an *open* relation $\mathcal{A} \subset X^{(d)}$.

- The requirement that $M(x,g) = M_x(g)$ be a differential operator takes into account that the family of right inverses $\{M_x\}$ is smooth in x in a certain sense.

Now we can state the main theorem concerning infinitesimal inversion.

Theorem 2.2.23. [Gro86, pg. 117] *Given that $\mathfrak{D} : \Gamma X \to \Gamma G$ is a smooth differential operator of order r. Suppose $\mathfrak{D}$ is infinitesimally invertible, with defect d and order s, over the set $A \subset \Gamma X$. Then, for each $x \in A$, there exists a family of open sets $\mathcal{B}_x \subset \Gamma G$ and operators $\mathfrak{D}_x^{-1} : \mathcal{B}_x \to A$ such that the following holds.*

- **Neighborhood Property :** *Each $\mathcal{B}_x$ contains some open subset of the 0-section in ΓG. Furthermore, the union $\mathcal{B} = \bigcup_{x\in A}\{x\} \times \mathcal{B}_x \subset A \times \Gamma G$ is an open subset.*

- **Normalization Property :** $\mathfrak{D}_x^{-1}(\mathbf{0}) = \mathbf{x}$ *for each $x \in A$, where $\mathbf{0}$ represents the zero-section.*

- **Inversion Property :** *For each $x \in A$ and $g \in \mathcal{B}_x$ we have, i.e, for $(x,g) \in \mathcal{B}$, we have*

$$\mathfrak{D}\big(\mathfrak{D}_x^{-1}(g)\big) = \mathfrak{D}(x) + g.$$

- **Continuity Property :** *For any smooth $g \in \mathcal{B}_x$, the section $\mathcal{D}_x^{-1}(g)$ is smooth. Furthermore the operator $\mathfrak{D}^{-1} : \mathcal{B} \to A$ defined as $\mathfrak{D}^{-1}(x, g) = \mathfrak{D}_x^{-1}(g)$ is jointly continuous*

- **Locality Property :** *There exists some auxiliary metric on the manifold V, such that the value of the section $\mathfrak{D}_x^{-1}(g) : V \to X$ at any $v \in V$ depends only on the value of the sections x, g on the unit ball $B_v(1)$ of radius 1 around v. In other words,*

$$(x, g)|_{B_v(1)} = (x', g')|_{B_v(1)} \Rightarrow \mathfrak{D}_x^{-1}(g)|_v = \mathfrak{D}_{x'}^{-1}(g')|_v.$$

As a consequence, we get the following implicit function theorem.

Theorem 2.2.24 (Nash-Gromov Implicit Function Theorem)**.** [Gro86, pg. 118] *Suppose $\mathfrak{D} : \Gamma X \to \Gamma G$ is a differential operator of order r, which is infinitesimally invertible over $A = \mathrm{Sol}(\mathcal{A})$, where the inversion has order s and defect d. Set, $\bar{s} = \max\{d, 2r + s\}$. Then for any any $x_0 \in A$, there exists a $C^{s+\bar{s}+1}$-open neighborhood $\mathcal{B}_0 \subset \Gamma G$ of the 0-section, such that for any smooth $g \in \mathcal{B}_0$, there exists a smooth solution $x \in A$ for the equation,*

$$\mathfrak{D}(x) = \mathfrak{D}(x_0) + g.$$

In particular $\mathfrak{D}$ is an open map when restricted to $\mathcal{A}$-regular maps.

Remark 2.2.25. Note the importance of the fine topology in the implicit function theorem. We get that the open neighborhood $\mathcal{B}_0$ in the above is C^σ-small, where

$$\sigma = s + \bar{s} + 1 = s + 1 + \max\{d, 2r + s\}.$$

In other words, the implicit function theorem states that for any $x_0 \in A$, we are able to solve the equation $\mathfrak{D}(x) = g$, whenever g is C^σ-close to $\mathfrak{D}(x_0)$.

The above theory of Differential operators was developed by Gromov based on the seminal work of J. Nash on smooth isometric embedding ([Nas56]). Nash's method of inversion was really elegant, and it was expounded upon by J. Moser in [Mos61, Mos66]. In simple terms, the inversion is obtained through an iterative process like Newton's method of finding a solution of an (nonlinear) equation, which incorporates certain smoothing operators so that the sequence of approximate solutions do converge to a genuine smooth solution in the limit.

Existence of Local Solutions

Let $\mathcal{R} \subset X^{(r)}$ be an arbitrary relation. By a *local solution* of $\mathcal{R}$ around some $v \in V$, we mean a local section $x \in \Gamma X$ defined over some $\mathrm{Op}(v)$, so that $j^r_x : \mathrm{Op}(v) \to \mathcal{R}$. A local section $x : \mathrm{Op}(v) \to X$ is called an *infinitesimal solution* to $\mathcal{R}$ at $v \in V$ if $j^r_x(v) \in \mathcal{R}$. Note that a local solution is clearly an infinitesimal solution at each point of its domain of definition. On the other hand, if $\mathcal{R}$ is an *open* relation, then shrinking $\mathrm{Op}(v)$ as necessary, we can make sure that an infinitesimal solution of $\mathcal{R}$ at v is in fact a local solution. A relation $\mathcal{R}$ is *locally integrable* if every jet in $\mathcal{R}$ extends to a local solution of $\mathcal{R}$.

Let us now define these notions in the context of an r-th order differential operator $\mathfrak{D} : \Gamma X \to \Gamma G$. Fix some $g \in \Gamma G$.

Definition 2.2.26. A section $x \in \Gamma X$ defined on $\mathrm{Op}(v)$, is called an *infinitesimal solution* to the equation $\mathfrak{D}(x) = g$, of order α, at the point $v \in V$ if

$$j^\alpha_{\mathfrak{D}(x)-g}(v) = 0.$$

The section is a called a *local solution*, if $\mathfrak{D}(x) = g$ holds on $\mathrm{Op}(v)$.

Note that the equation is defined on the $(r+\alpha)$-jet space of section of X. Now for every $\alpha \geq 0$, we define a relation $\mathcal{R}^\alpha = \mathcal{R}^\alpha(\mathfrak{D}, g)$ as,

$$\mathcal{R}^\alpha = \left\{ j^{r+\alpha}_x(v) \in X^{(r+\alpha)} \;\middle|\; x \text{ is an infinitesimal solution to } \mathfrak{D}(x) = g \text{ of order } \alpha, \text{ at } v \in V \right\}$$

Observe that the C^∞-solutions of $\mathcal{R}^\alpha$ are precisely the smooth solutions of the equation $\mathfrak{D}(x) = g$. In particular, the relations $\mathcal{R}^\alpha$ have the same set of C^∞-solutions for all $\alpha \geq 0$.

Now assume that $\mathfrak{D}$ is infinitesimally invertible over an open relation $\mathcal{A} \subset X^{(d)}$. Then we consider the relations $\mathcal{R}_\alpha$, for $\alpha \geq d - r$, as follows :

$$\mathcal{R}_\alpha = \mathcal{R}_\alpha(\mathfrak{D}, g, \mathcal{A}) = \mathcal{R}^\alpha(\mathfrak{D}, g) \cap \left(p^{r+\alpha}_d\right)^{-1}(\mathcal{A}) \subset X^{r+\alpha},$$

where $p^{r+\alpha}_d : X^{(r+\alpha)} \to X^{(d)}$ is the jet projection map.

The relations $\mathcal{R}_\alpha$ are of primary interest to us. We note that C^∞-solutions of $\mathcal{R}_\alpha$ are precisely the solutions of the equation $\mathfrak{D}(x) = g$ which are $\mathcal{A}$-regular as well. As before, the solution spaces $\mathrm{Sol}\,\mathcal{R}_\alpha$ are all the same for $\alpha \geq d - r$. On the other hand, the space of sections $\Gamma\mathcal{R}_\alpha$ are distinct, as the relations $\mathcal{R}_\alpha$ are sitting inside different jet spaces. Let us now denote,

$$\Phi = \mathrm{Sol}\,\mathcal{R}_\alpha \quad \text{and} \quad \Psi_\alpha = \Gamma\mathcal{R}_\alpha, \text{ for } \alpha \geq d - r$$

We also have the jet map $j^{r+\alpha} : \Phi \to \Psi_\alpha$ for each $\alpha \geq d - r$. We then have the following.

Theorem 2.2.27. [Gro86, pg. 120] *The sheaf $\Phi = \mathrm{Sol}(\mathcal{R}_\alpha)$ of $\mathcal{A}$-regular solutions to the equation $\mathfrak{D}(x) = g$ is microflexible for $\alpha \geq d - r$.*

Theorem 2.2.28. [Gro86, pg. 119] *$j^{r+\alpha} : \Phi \to \Psi_\alpha$ is a* local *weak homotopy equivalence whenever we have,*

$$\alpha \geq s + \max\{d, 2r + s\}$$

The above discussion culminates in the following h-principle over open manifolds.

Theorem 2.2.29. *Let V be an open manifold. Suppose $\mathfrak{D} : \Gamma X \to \Gamma G$ is a differential operator of order r, which admits infinitesimal inversion over some $A = \mathrm{Sol}\,\mathcal{A}$, so that the inversion is of order s and defect d. For any fixed $g \in \Gamma G$, denote by $\mathcal{R}_\alpha = \mathcal{R}_\alpha(\mathfrak{D}, g, \mathcal{A})$ the relation of $\mathcal{A}$-regular, α-infinitesimal solutions to the equation $\mathfrak{D}(x) = g$. Suppose the solution sheaf Φ of $\mathcal{R}_\alpha$ is $\mathrm{Diff}(V)$-invariant. Then for $\alpha \geq s + \max\{d, 2r + s\}$, the relation $\mathcal{R}_\alpha$ abides by the parametric h-principle. That is, the jet map $j^{r+\alpha} : \mathrm{Sol}(\mathcal{R}_\alpha) \to \Gamma\mathcal{R}_\alpha$ is a weak homotopy equivalence.*

The proof is immediate from Remark 2.2.12.

A Stronger Version of the Implicit Function Theorem

Let us look back at the jet prolongation map again. Specifically, observe that in Theorem 2.2.28, as we deform a given infinitesimal solution $j_x^{r+\alpha}(v) \in \mathcal{R}_\alpha|_v$ at some $v \in V$ to a local solution x_1 over $\mathrm{Op}(v)$, we have no control over the value of the section $x_t(v)$ at the point v. Can we get a local homotopy which is stationary at v?

More generally, we may also ask whether we can we get a solution to the differential equation in a neighborhood of a submanifold V_0, provided there is a C^∞ map which solves $\mathcal{R}_\alpha$ at all points of V_0. This is not totally unreasonable to expect, as this can be viewed as a Cauchy initial value problem. Indeed, we have the following stronger version of Theorem 2.2.24.

Theorem 2.2.30. [Gro86, pg. 143] *Let $\mathfrak{D} : \Gamma X \to \Gamma G$ be a differential operator of order r, which is infinitesimally invertible over $A = \mathrm{Sol}(\mathcal{A})$, with inversion of defect d and order s. Suppose $x_0 \in A$, $g_0 = \mathfrak{D}(x_0)$, and $V_0 \subset V$ is a closed submanifold V of positive codimension, without boundary. If $g \in \Gamma(G)$ is such that $j_{g_0}^\alpha = j_g^\alpha$ on points of V_0 for*

$$\alpha \geq 2r + 3s + \bar{s} = 2r + 3s + \max\{d, 2r + s\},$$

then there exists an $x \in A$ satisfying $\mathfrak{D}(x) = g$ on $\mathrm{Op}(V_0)$ and

$$j^{2r+s-1}_{x_0} = j^{2r+s-1}_{x}, \text{ on points of } V_0.$$

Taking V_0 to be a single point we get the following corollary.

Corollary 2.2.31. *Every infinitesimal solution x_0 of $\mathcal{R}_\alpha$ at $v \in V$ is homotopic to a local solution x_1 of $\mathcal{R}_\alpha$, given that*

$$\alpha \geq 2r + 3s + \max\{d, 2r + s\}.$$

Furthermore, the homotopy x_t satisfies the condition $j^{2r+s-1}_{x_t}(v) = j^{2r+s-1}_{x}(v)$ for all $t \in [0, 1]$.

In the next chapter, we shall see how the results of this section come into play in proving the h-principle for horizontal and (iso)contact immersions.

Chapter 3

Revisiting h-Principle for K-Contact Immersions

Unless mentioned otherwise, M will denote a smooth manifold, with a fixed corank p-distribution $\mathcal{D}$, having curvature form Ω. The goal of this chapter is to discuss the h-principle for horizontal immersions and other classes of maps into $(M, \mathcal{D})$. The results proven in this chapter are not new and an outline of these results and their proofs can be found in [Gro86, EM02]. We shall present here a detailed proof using the general theory of h-principle that have been discussed in Chapter 2. The main result of this chapter is stated in Theorem 3.2.7.

3.1 K-contact Immersions

Recall that a smooth immersion $u : \Sigma \to M$ is $\mathcal{D}$-horizontal if the differential du maps $T\Sigma$ into $\mathcal{D}$. In other words, a horizontal map u satisfies the equation $du^{-1}(\mathcal{D}) = T\Sigma$. This viewpoint gives rise to a natural generalization of horizontal maps where we fix a distribution $K \subset T\Sigma$ and ask for maps $u : \Sigma \to M$ such that du maps K into $\mathcal{D}$. We are thus led into defining K-contact maps.

Definition 3.1.1. [Gro86, pg. 338] Given a distribution $K \subset T\Sigma$, we say a map $u : \Sigma \to (M, \mathcal{D})$ is K-*contact*, if we have that

$$du(K_\sigma) \subset T_{u(\sigma)}\mathcal{D}, \quad \text{for each } \sigma \in \Sigma$$

In other words, u is K-contact if $K \subset du^{-1}(\mathcal{D})$. If $K = T\Sigma$, then K-contact maps $u : \Sigma \to (M, \mathcal{D})$ are precisely the $\mathcal{D}$-horizontal maps.

In what follows below, Σ will denote an arbitrary manifold and K will denote an arbitrary distribution on it, unless mentioned otherwise.

Definition 3.1.2. A K-contact map $u : (\Sigma, K) \to (M, \mathcal{D})$ is called K-*isocontact* (or, simply *isocontact*) if we have $K = du^{-1}(\mathcal{D})$.

Now observe that for any contact map $u : (\Sigma, K) \to (M, \mathcal{D})$, we have an induced bundle map,

$$\begin{aligned} \tilde{du} : T\Sigma/K &\longrightarrow u^*TM/\mathcal{D} \\ X \mod K &\longmapsto du(X) \mod \mathcal{D} \end{aligned}$$

which is well-defined since $du(K) \subset \mathcal{D}$. In fact, we have the following commutative diagram,

$$\begin{tikzcd} T\Sigma \arrow[r, "du"] \arrow[d, "\mu"'] & TM \arrow[d, "\lambda"] \\ T\Sigma/K \arrow[r, "\tilde{du}"'] & TM/\mathcal{D} \end{tikzcd}$$

where λ and μ are the quotient maps, defining $\mathcal{D}$ and K respectively. We now observe a simple characterization of K-isocontact *immersions*, which follows from easy dimension counting argument.

Observation 3.1.3. A contact immersion $u : (\Sigma, K) \to (M, \mathcal{D})$ is isocontact if and only if the bundle map $\tilde{du}$ is injective.

Hence, for an isocontact immersion $(\Sigma, K) \to (M, \mathcal{D})$ to exist, the following numerical constraints must necessarily be satisfied,

$$\operatorname{rk} K \le \operatorname{rk} \mathcal{D} \quad \text{and} \quad \operatorname{cork} K \le \operatorname{cork} \mathcal{D}.$$

The Curvature Condition for (Iso)contact Maps

K-contactness automatically imposes a differential condition involving the curvatures of the distributions.

Proposition 3.1.4. *Given a K-contact map $u : (\Sigma, K) \to (M, \mathcal{D})$ we have the following commutative diagram,*

$$\begin{tikzcd} \Lambda^2 K \arrow[r, "du"] \arrow[d, "\Omega_K"'] & \Lambda^2 \mathcal{D} \arrow[d, "\Omega_{\mathcal{D}}"] \\ T\Sigma/K \arrow[r, "\tilde{du}"'] & TM/\mathcal{D} \end{tikzcd}$$

where $\Omega_K, \Omega_{\mathcal{D}}$ are the curvature forms of K and $\mathcal{D}$ respectively. In other words we have,

$$u^*\Omega_{\mathcal{D}}|_K = \tilde{du} \circ \Omega_K$$

Proof. Let $X, Y \in K|_\sigma$. Choose some local extensions $\tilde{X}, \tilde{Y} \in K$ of X, Y respectively, around $\sigma \in \Sigma$. Let U be a trivializing neighborhood for the subbundle $\mathcal{D}$ around $u(\sigma) \in M$, so that we may write, $\mathcal{D}|_U = \bigcap_{s=1}^p \ker \lambda^s$ for some local 1-forms $\lambda^s \in \Omega^1(U)$. Then, $\Omega_{\mathcal{D}}|_U \underset{loc.}{=} (\omega^s)$, where $\omega^s = d\lambda^s|_{\mathcal{D}}$. Since u is K-contact, in particular, we have $u^*\lambda^s|_K = 0$. Consequently,

$$u^*d\lambda^s(X, Y) = d(u^*\lambda^s)(X, Y) = -u^*\lambda^s\big([\tilde{X}, \tilde{Y}]_\sigma\big)$$

On the other hand, from Definition 2.1.16 we have,

$$\Omega_K(X, Y) = -[\tilde{X}, \tilde{Y}]_\sigma \mod K_\sigma$$

Hence for $X, Y \in K_\sigma$ as above,

$$\begin{aligned}
\tilde{du} \circ \Omega_K(X, Y) &= \tilde{du}\big(-[\tilde{X}, \tilde{Y}]_\sigma \mod K_\sigma\big) = -u_*[\tilde{X}, \tilde{Y}] \mod \mathcal{D}_{u(\sigma)} \\
&= \Big(\lambda^s\Big(-u_*[\tilde{X}, \tilde{Y}]\Big)\Big) = \Big(u^*\lambda^s\big(-[\tilde{X}, \tilde{Y}]_\sigma\big)\Big) = \Big(u^*d\lambda^s(X, Y)\Big) \\
&= \Big(u^*\omega^s(X, Y)\Big) = u^*\Omega_{\mathcal{D}}(X, Y)
\end{aligned}$$

Since $X, Y \in K_\sigma$ is arbitrary, we have proved the claim. $\square$

If $K = T\Sigma$, then $\Omega_K = \Omega_{T\Sigma} = 0$ and hence for a horizontal immersion $u : \Sigma \to M$ we get back the *isotropy* condition, namely,

$$u^*\Omega_{\mathcal{D}} = \tilde{du} \circ \Omega_{T\Sigma} = 0.$$

Contact Immersion Operator

We shall now see that the (iso)contact immersions $(\Sigma, K) \to (M, \mathcal{D})$ appear as the solutions of certain first order partial differential equation. For simplicity, let us assume that $\mathcal{D} = \bigcap_{s=1}^p \ker \lambda^s$ for global 1-forms $\lambda^s \in \Omega^1(M)$. It is easy to note that for any map $u : \Sigma \to M$,

$$du(K) \subset \mathcal{D} \Leftrightarrow u^*\lambda^s|_K = 0 \text{ for each } 1 \le s \le p$$

Now, consider the operator,

$$\begin{aligned}\mathfrak{D}^{\mathsf{Cont}} : C^\infty(\Sigma, M) &\to \Gamma \hom(K, \mathbb{R}^p)\\ u &\mapsto \left(u^*\lambda^s|_K\right)\end{aligned}$$

so that u is K-contact if and only if $\mathfrak{D}^{\mathsf{Cont}}(u) = 0$.

Fixing some coordinates $\{x^i\}$ any $\{y^\mu\}$ respectively on Σ and M, we may write, $\lambda^s \underset{loc.}{=} \lambda^s_\mu dy^\mu$ and then we have,

$$u^*\lambda^s = u^*\left(\lambda^s_\mu dy^\mu\right) = \left(\lambda^s_\mu \circ u\right)\partial_i u^\mu dx^i$$

We see that $\mathfrak{D}^{\mathsf{Cont}}$ is indeed a first order differential operator in the sense of Definition 2.2.19 and it is determined by the bundle map $\Delta : J^1(\Sigma, M) \to \hom(K, \mathbb{R}^p)$ given by,

$$\Delta\left(j^1_u(x)\right) = \left(X \mapsto \left((\lambda^s_\mu \circ u)(x)\partial_i u^\mu dx^i(X)\right)^p_{s=1}\right), \text{ where } X \in K_x.$$

In other words, if $(x, y, F : T_x\Sigma \to T_yM) \in J^1(\Sigma, M)|_{(x,y)}$, then we have

$$\Delta(x, y, F) = \left(x, (F^*\lambda^s|_{K_x})\right).$$

Linearization of $\mathfrak{D}^{\mathsf{Cont}}$

We shall denote the linearization operator of $\mathfrak{D}^{\mathsf{Cont}}$ at a map $u : \Sigma \to M$ by $\mathfrak{L}^{\mathsf{Cont}}_u$. Since $T_uC^\infty(\Sigma, M) = \Gamma u^*TM$, we have that

$$\mathfrak{L}^{\mathsf{Cont}}_u = T_u\mathfrak{D}^{\mathsf{Cont}} : \Gamma u^*TM \to \Gamma \hom(K, \mathbb{R}^p).$$

Suppose $\xi \in \Gamma u^*TM$. Let $u_t : \mathrm{Op}(\sigma) \to M$ be a smooth family of maps such that $u_0 = u$ on $\mathrm{Op}(\sigma)$ and $\xi(\sigma) = \frac{d}{dt}|_{t=0} u_t(\sigma)$. Then the linearization operator is given as,

$$\mathfrak{L}^{\mathsf{Cont}}_u(\xi)(X) = \frac{d}{dt}\Big|_{t=0} \mathfrak{D}^{\mathsf{Cont}}(u_t)(X) = \frac{d}{dt}\Big|_{t=0} u_t^*\lambda^s(X), \quad \text{for local section } X \text{ of } K.$$

By the Cartan formula,

$$\frac{d}{dt}\Big|_{t=0} u_t^*\lambda^s(X) = u_0^*\Big(d\lambda^s(\xi, u_*X) + d\big(\lambda^s(\xi)\big)(X)\Big).$$

We shall write down the operator $\mathfrak{L}_u^{\mathsf{Cont}}$ succinctly as,

$$\mathfrak{L}_u^{\mathsf{Cont}}(\xi) = \Big(\iota_\xi d\lambda^s + d\big(\iota_\xi \lambda^s\big)\Big)\Big|_K.$$

Note that since $\xi \in \Gamma u^*TM$ is a vector field *along* the map $u : \Sigma \to M$, the contraction $\iota_\xi d\lambda^s$ is interpreted as a 1-form on $T\Sigma$, defined by the formula,

$$\big(\iota_\xi d\lambda^s\big)_\sigma(X) = (d\lambda^s)_{u(\sigma)}(\xi_\sigma, du_\sigma(X)), \quad \text{for } X \in T_\sigma\Sigma.$$

Similarly we interpret, $\iota_\xi \lambda^s|_\sigma = \lambda^s|_{u(\sigma)}(\xi_\sigma)$ for $\sigma \in \Sigma$.

Infinitesimal Inversion of $\mathfrak{D}^{\mathsf{Cont}}$

Having identified the linearization operator $\mathfrak{L}_u^{\mathsf{Cont}} : \Gamma u^*TM \to \Gamma \hom(K, \mathbb{R}^p)$, we restrict it to the subspace $\Gamma u^*\mathcal{D}$. Note that for any $\xi \in \Gamma u^*\mathcal{D}$ we have, $\iota_\xi \lambda^s = 0$ and thus the restricted linearization operator has the following simple description :

$$\begin{aligned} \mathcal{L}_u^{\mathsf{Cont}} : \Gamma u^*\mathcal{D} &\to \Gamma \hom(K, \mathbb{R}^p) \\ \xi &\mapsto \Big(\iota_\xi d\lambda^s\Big)\Big|_K = \Big(X \mapsto d\lambda^s(\xi, u_* X)\Big) \end{aligned}$$

Observe that $\mathcal{L}_u^{\mathsf{Cont}}$ is, in fact, $C^\infty(\Sigma)$-linear and hence is given by a bundle map, $u^*\mathcal{D} \to \hom(K, \mathbb{R}^p)$. If this bundle map is an epimorphism, then it has a right inverse, also given by a bundle map; in other words, we have a 0^{th}-order inversion for the differential operator $\mathcal{L}_u^{\mathsf{Cont}}$. In fact, by using a Riemannian metric on M, we can get a *continuous* family of right inverses, over the set of maps,

$$A = \Big\{ u : \Sigma \to M \;\Big|\; u \text{ is an immersion and } \mathcal{L}_u^{\mathsf{Cont}} \text{ is a bundle epimorphism} \Big\}.$$

Observe that A is precisely the solution space of the relation $\mathcal{A} \subset J^1(\Sigma, M)$ consisting of tuples $(\sigma, y, F : T_\sigma\Sigma \to T_y M)$, such that,

- F is injective, and
- the linear map,

$$\begin{aligned} \mathcal{D}_y &\to \hom(K, \mathbb{R}^p) \\ \xi &\mapsto \Big(X \mapsto d\lambda^s(\xi, FX)\Big) \end{aligned}$$

is surjective.

If we wish to study K-*iso*contact immersions, then in the light of Observation 3.1.3, F must also satisfy,

- $\operatorname{rk}(\lambda^s \circ F) \geq \operatorname{cork} K$

Clearly, $\mathcal{A}$ is an *open* relation, and as we have already noted, the operator $\mathfrak{D}^{\mathsf{Cont}}$ is infinitesimally invertible over $A = \operatorname{Sol}(\mathcal{A})$.

A smooth solution u of $\mathcal{A}$ (that is, $u \in A$) will be referred to as a $(d\lambda^s)$-*regular* immersion. In general, $(d\lambda^s)$-regularity depends on our choice of defining 1-forms λ^s for $\mathcal{D}$. But it turns out that the space of $(d\lambda^s)$-regular, K-contact immersions $(\Sigma, K) \to (M, \mathcal{D})$, is independent of any such choice. Suppose $u : \Sigma \to M$ is a K-contact immersion. In particular, we have that, $du(K) \subset \mathcal{D}$. Then from Definition 2.1.16 it is clear that

$$\mathcal{L}_u^{\mathsf{Cont}}(\xi) = u^*\Big(\iota_\xi \Omega\Big)\big|_K,$$

where Ω is the curvature 2-form of $\mathcal{D}$.

Definition 3.1.5. A K-contact immersion $u : (\Sigma, K) \to (M, \mathcal{D})$ is called Ω-*regular* if the bundle map,

$$\mathcal{L}_u^{\mathsf{Cont}} : u^*\mathcal{D} \to \hom(K, u^*TM/\mathcal{D})$$
$$\xi \mapsto \iota_\xi \Omega|_K$$

is an epimorphism.

Remark 3.1.6. In simple terms, Ω-regularity of a K-contact immersion $u : \Sigma \to M$ is equivalent to the solvability of the following *algebraic* system in local vector fields $\xi \in \Gamma\mathcal{D}$:

$$\Omega(\xi, u_* X_i) = G_i, \quad 1 \leq i \leq \operatorname{rk} K,$$

where G_i are arbitrary smooth functions on Σ. Here (X_i) is some choice of local frame of K. In particular, if $K = T\Sigma$, then for every $\sigma \in \Sigma$, the subspace $\operatorname{Im} du_\sigma$ is Ω-isotropic in $\mathcal{D}_{u(\sigma)}$. Therefore, in order to solve the algebraic system for arbitrary G_i, we must have ([Gro96, pg. 251]),

$$\operatorname{rk}\mathcal{D} - \dim\Sigma \geq \operatorname{cork}\mathcal{D} \times \dim\Sigma.$$

When $\mathcal{D} \subset TM$ is not cotrivial

For a general distribution $\mathcal{D} \subset TM$, we may write $\mathcal{D} = \ker \lambda$, where the quotient map $\lambda : TM \to TM/\mathcal{D}$ is treated as a $TM/\mathcal{D}$-valued 1-form. Then for any map $u : \Sigma \to M$, we have that $u^*\lambda$ is a $u^*TM/\mathcal{D}$-valued 1-form on Σ. We wish to study the operator $u \mapsto u^*\lambda|_K$, whose zeroes are precisely the K-contact maps $(\Sigma, K) \to (M, \mathcal{D})$.

Consider the space of maps, $\mathcal{B} = C^\infty(\Sigma, M)$. For each $u \in \mathcal{B}$, we have the infinite dimensional vector space,

$$\mathcal{E}_u = \Gamma \hom(K, u^*TM/\mathcal{D}).$$

Let $\mathcal{E} \to \mathcal{B}$ be an infinite-dimensional vector bundle over $\mathcal{B}$, having $\mathcal{E}_u$ as the fibre over $u \in \mathcal{B}$. The operator $u \mapsto u^*\lambda|_K$ can then be viewed as a section of this bundle. Next, we fix a connection ∇ on $TM/\mathcal{D}$. This enables us to get a parallel transport on $\mathcal{E} \to \mathcal{B}$. Recall that, $T_u\mathcal{B} = \Gamma u^*TM$, and the vertical tangent space at $u^*\lambda|_K$ is isomorphic to $\mathcal{E}_u$. We can then define the *linearization* operator at $u \in \mathcal{B}$ as,

$$\begin{aligned} \mathfrak{L}_u^{\mathsf{Cont}} : \Gamma u^*TM &\to \Gamma \hom(K, u^*TM/\mathcal{D}) \\ \xi &\mapsto \big(\iota_\xi d_\nabla \lambda + d\iota_\xi \lambda\big)|_K \end{aligned}$$

As before, we restrict $\mathfrak{L}_u^{\mathsf{Cont}}$ to the subspace $\Gamma u^*\mathcal{D}$ to get the operator

$$\begin{aligned} \mathcal{L}_u^{\mathsf{Cont}} : \Gamma u^*\mathcal{D} &\to \Gamma \hom(K, u^*TM/\mathcal{D}) \\ \xi &\mapsto \iota_\xi d_\nabla \lambda|_K \end{aligned}$$

which is $C^\infty(\Sigma)$-linear, and hence is given by a bundle map, $u^*\mathcal{D} \to \hom(K, u^*TM/\mathcal{D})$. We say an immersion $u \in \mathcal{B}$ is $d_\nabla\lambda$-regular if the bundle map defined by the operator $\mathcal{L}_u^{\mathsf{Cont}}$ is surjective. We do not distinguish between notations for the operator and the bundle map.

The notion of $d_\nabla\lambda$-regularity depends very much on the choice of the connection ∇ on $TM/\mathcal{D}$. But if $u : (\Sigma, K) \to (M, \mathcal{D})$ is K-contact, then as a consequence of Proposition 2.1.18, the $d_\nabla\lambda$-regularity is equivalent to Ω-regularity, and hence the notion is independent of the choice of connection.

The relation $\mathcal{R}^{\mathsf{Cont}}$

We now define a first order relation in $J^1(\Sigma, M)$.

Definition 3.1.7. Given subbundles $K \subset T\Sigma$ and $\mathcal{D} \subset TM$, we define $\mathcal{R}^{\mathsf{Cont}} \subset J^1(\Sigma, M)$ as the first order relation consisting of 1-jets $(x, y, F : T_x\Sigma \to T_yM)$ satisfying the following :

1. $F(K_x) \subset \mathcal{D}_y$
2. F is injective and Ω-regular
3. F abides by the curvature condition,

$$F^*\Omega|_{K_x} = \tilde{F} \circ \Omega_{K_x},$$

   where and Ω_K is the curvature form of K.

We also have a subrelation $\mathcal{R}^{\mathsf{IsoCont}} \subset \mathcal{R}^{\mathsf{Cont}}$ which further satisfies,

4. The induced map $\tilde{F} : T\Sigma/K|_x \to TM/\mathcal{D}|_y$ is injective.

As a special case, for $K = T\Sigma$, we shall denote the corresponding relation $\mathcal{R}^{\mathsf{Cont}}$ as $\mathcal{R}^{\mathsf{Hor}}$.

We shall refer to a section of $\mathcal{R}^{\mathsf{Cont}}$ as a *formal* Ω-regular, K-contact immersion $(\Sigma, K) \to (M, \mathcal{D})$. We shall be needing the following lemma later, in the proof of Lemma 3.2.6.

Lemma 3.1.8. *The following holds true for the relation $\mathcal{R}^{\mathsf{Cont}}$.*

1. *For each $(x, y) \in \Sigma \times M$, the subset $\mathcal{R}^{\mathsf{Cont}}_{(x,y)}$ is a submanifold of $J^1_{(x,y)}(\Sigma, M)$*
2. *$\mathcal{R}^{\mathsf{Cont}}$ is a submanifold of $J^1(\Sigma, M)$*
3. *The projection map $p = p^1_0 : J^1(\Sigma, M) \to J^0(\Sigma, M)$ restricts to a submersion on $\mathcal{R}^{\mathsf{Cont}}$*

Proof. Note that $J^1(\Sigma, M)$ and $\hom(K, TM/\mathcal{D})$ are both vector bundles overs $J^0(\Sigma, M) = \Sigma \times M$. Consider the bundle map,

$$\begin{tikzcd}
\Xi_1 : J^1(\Sigma, M) \arrow[rr] \arrow[dr] & & \hom(K, TM/\mathcal{D}) \arrow[dl] \\
& J^0(\Sigma, M) &
\end{tikzcd}$$

defined over $(x, y) \in J^0(\Sigma, M) = \Sigma \times M$ by,

$$\begin{aligned}
\Xi_1|_{(x,y)} : J^1_{(x,y)}(\Sigma, M) &\to \hom(K_x, TM/\mathcal{D}|_y) \\
(x, y, F) &\mapsto F^*\lambda|_{K_x} = \lambda \circ F|_{K_x}
\end{aligned}$$

From the commutative diagram,

$$\begin{tikzcd} K_x \arrow[r, "F|_{K_x}"] \arrow[dr, "\Xi_1(F)"'] & T_yM \arrow[d, two heads, "\lambda"] \\ & TM/\mathcal{D}|_y \end{tikzcd}$$

it is immediate that $\Xi_1|_{(x,y)}$ is in fact surjective, since λ is an epimorphism. Consequently, Ξ_1 is a bundle epimorphism; $\ker \Xi_1$ is a sub-bundle over $J^0(\Sigma, M)$, given as,

$$\ker \Xi_1|_{(x,y)} = \big\{(x, y, F) \;\big|\; F(K_x) \subset \mathcal{D}_y\big\}.$$

Let us now consider a fiber-preserving map, $\Xi_2 : \ker \Xi_1 \to \hom(\Lambda^2 K, TM/\mathcal{D})$, over $J^0(\Sigma, M)$, given by,

$$\begin{aligned} \Xi_2|_{(x,y)} : \ker \Xi_1|_{(x,y)} &\to \hom\left(\Lambda^2 K_x, TM/\mathcal{D}|_y\right) \\ F &\mapsto F^*\Omega|_{K_x} - \tilde{F} \circ \Omega_{K_x} := \Big(X \wedge Y \mapsto \Omega(FX, FY) - \tilde{F} \circ \Omega_{K_x}(X, Y)\Big) \end{aligned}$$

where $\tilde{F} : T\Sigma/K|_x \to TM/\mathcal{D}|_y$ is the induced map and $\Omega_K : \Lambda^2 K \to T\Sigma/K$ is the curvature 2-form. Note that the relation $\mathcal{R}^{\mathsf{Cont}}$ is then given as,

$$\mathcal{R}^{\mathsf{Cont}}_{(x,y)} = \Xi_2|^{-1}_{(x,y)}(0) \cap \{\Omega\text{-regular injective linear maps } T_x\Sigma \to T_yM\}.$$

In order to prove that $\mathcal{R}^{\mathsf{Cont}}_{(x,y)}$ is a manifold, we shall show that each point is a regular point of the map $\Xi_2|_{(x,y,)}$.

The derivative of $\Xi_2|_{(x,y)}$ at some $(x, y, F) \in \ker \Xi_1|_{(x,y)}$ is given by,

$$\begin{aligned} d_{(x,y,F)}\Xi_2|_{(x,y)} : \ker \Xi_1|_{(x,y)} &\to \hom\left(\Lambda^2 T_x\Sigma, TM/\mathcal{D}|_y\right) \\ G &\mapsto \left(X \wedge Y \mapsto \Omega(FX, GY) + \Omega(GX, FY) - \tilde{G} \circ \Omega_{K_x}(X, Y)\right) \end{aligned}$$

We then have the diagram,

$$\begin{tikzcd} \ker \Xi_1|_{(x,y)} \arrow[r, "G \mapsto G|_{K_x}"] \arrow[d, "G \mapsto \tilde{G}"'] & \hom(K_x, \mathcal{D}_y) \arrow[r, "\Omega_F"] \arrow[dr] & \hom\left(K_x, \hom(K_x, TM/\mathcal{D}|_y)\right) \arrow[d, two heads, "A"] \\ \hom(T\Sigma/K|_x, TM/\mathcal{D}|_y) \arrow[rr, "\_\circ\, \Omega^K_x"'] & & \hom\left(\Lambda^2 K_x, TM/\mathcal{D}|_y\right) \end{tikzcd} \tag{$*$}$$

where, $\Omega_F(G)(X)(Y) = \Omega(FX, GY)$and A is the skew-symmetrization map given by, $A(F)(X,Y) = F(X)(Y) - F(Y)(X)$, for $X, Y \in K_x$. Indeed,

$$\begin{aligned}(A \circ \Omega_F(G))(X \wedge Y) &= \Omega_F(G)(X)(Y) - \Omega_F(G)(Y)(X) \\ &= \Omega(FX, GY) - \Omega(FY, GX) \\ &= \Omega(FX, GY) + \Omega(GX, FY)\end{aligned}$$

and hence we see that,

$$d_F\Xi_2(G) = A \circ \Omega_F\big(G|_{K_x}\big) - \tilde{G} \circ \Omega_{K_x}.$$

Now suppose $(x, y, F) \in \ker \Xi_1|_{(x,y)}$ is such that $F : T_\sigma\Sigma \to T_yM$ is injective and Ω-regular, i.e,

$$\begin{aligned}\tilde{\Omega} : \mathcal{D}_y &\to \hom(K_x, TM/\mathcal{D}|_y) \\ \xi &\mapsto F^*\iota_\xi\Omega|_{K_x} = \big(X \mapsto \Omega(\xi, FX)\big)\end{aligned}$$

is surjective and $\tilde{F}$ is injective. Applying the $\hom(K_x, \_)$ functor we then have that the map Ω_F is surjective. But then from the diagram ($*$) above, it follows that for Ω-regular F, $d_F\Xi_2|_{(x,y)}$ is surjective. Indeed, given any $P : \Lambda^2K_x \to TM/\mathcal{D}|_y$, we can arbitrarily fix some injective linear map $G_2 : T\Sigma/K|_x \to TM/\mathcal{D}|_y$ and then solve $A \circ \Omega_F(G_1) = P + G_1 \circ \Omega_{K_x}$ for $G_1 : K_x \to \mathcal{D}_y$. Then we may get $G : T_x\Sigma \to T_yM$ so that $\tilde{G} = G_2$ and hence $d_F\Xi_2(G) = P$. Consequently $\mathcal{R}^{\mathsf{Cont}}_{(x,y)}$ is a submanifold of $J^1_{(x,y)}(\Sigma, M)$.

Recall, $\Xi_1 : \hom(T\Sigma, TM) \to \hom(K, TM/\mathcal{D})$ is an *epimorphism*, over the manifold $J^0(\Sigma, M) = \Sigma \times M$ and $\ker \Xi_1 = \big\{(x, y, F)\big|F(K_x) \subset \mathcal{D}_y\big\}$ is a subbundle of $\hom(T\Sigma, TM)$. Now,

$$\Xi_2 : \ker \Xi_1 \to \hom(\Lambda^2K, TM/\mathcal{D})$$

is a fiber-preserving map, so that Ξ_2 restricted to each fiber over $(x, y) \in J^0(\Sigma, M)$ is regular at each point of $\mathcal{R}^{\mathsf{Cont}}_{(x,y)}$. Since,

$$\mathcal{R}^{\mathsf{Cont}} = \Xi_2^{-1}(\mathbf{0}) \cap \mathcal{R}_\Omega$$

where $\mathbf{0} = \mathbf{0}_{\Sigma\times M} \hookrightarrow \hom(\Lambda^2K, TM/\mathcal{D})$ is the 0-section and $\mathcal{R}_\Omega$ is the space of Ω-regular linear maps, we have that Ξ_2 is a submersion at Ω-regular points. Consequently, $\mathcal{R}^{\mathsf{Cont}}$ is a submanifold.

Lastly, we have the commutative diagram,

$$\begin{tikzcd}
\mathcal{R}^{\text{Cont}} \subset \ker \Xi_1 \arrow[rr, "\Xi_2"] \arrow[dr, "p^1_0|_{\mathcal{R}^{\text{Cont}}}"'] & & \hom(\Lambda^2 K, TM/\mathcal{D}) \arrow[dl, "\pi"] \\
& J^0(\Sigma, M) &
\end{tikzcd}$$

Since Ξ_2 is a submersion at the Ω-regular points in $\Xi_2^{-1}(\mathbf{0})$, we have $p^1_0|_{\mathcal{R}^{\text{Cont}}}$ is a submersion. $\square$

3.2 A 1-jet h-Principle for Regular K-contact Immersions

We have seen that the operator $\mathfrak{D}^{\text{Cont}}$ is infinitesimally invertible, in the sense of Definition 2.2.21, on $\mathcal{A} \subset J^1(\Sigma, M)$, the space of $d_\nabla\lambda$-regular K-contact immersions; clearly, the inversion is of order 0 and defect 1. Following the discussion in the last chapter, we define the relations $\mathcal{R}^{\text{Cont}}_\alpha = \mathcal{R}^{\text{Cont}}_\alpha(\mathfrak{D}^{\text{Cont}}, \mathcal{A}, 0) \subset J^{\alpha+1}(\Sigma, M)$ for the operator $\mathfrak{D}^{\text{Cont}}$. Jets in $\mathcal{R}^{\text{Cont}}_\alpha$ are represented by Ω-regular, infinitesimal solutions of $\mathfrak{D}^{\text{Cont}}$, of order α (Definition 2.2.26). The relations $\mathcal{R}^{\text{Cont}}_\alpha$ have the same solution space for all $\alpha \geq 0$, namely, the space of Ω-regular, K-contact immersions $\Sigma \to M$. Let us denote this sheaf of solutions as $\Phi^{\text{Cont}} = \mathrm{Sol}(\mathcal{R}^{\text{Cont}}_\alpha)$ for any $\alpha \geq 0$. Now by appealing to the discussion in section 2.2.2, we conclude the following :

- The relations $\mathcal{R}^{\text{Cont}}_\alpha$ satisfy the *local* h-principle for $\alpha \geq 2$, i.e, the jet map

  $$j^{\alpha+1} : \Phi^{\text{Cont}} \to \Gamma\mathcal{R}^{\text{Cont}}_\alpha$$

  is a *local* weak homotopy equivalence, by Theorem 2.2.28.

- The solution sheaf Φ^{Cont} is microflexible, by Theorem 2.2.27.

Therefore, in order to conclude the existence of local K-contact immersions, we need at least a formal solution of $\mathcal{R}^{\text{Cont}}_2$, which is a relation of order 3. We shall now identify the image of $\mathcal{R}^{\text{Cont}}_\alpha$ under $p^{\alpha+1}_1 : J^{\alpha+1}(\Sigma, M) \to J^1(\Sigma, M)$, which will enable us to state the local h-principle in terms of the 1-jet map.

It is immediate from Definition 3.1.7 that $\mathcal{R}^{\text{Cont}}$ is a subrelation of $\mathcal{R}^{\text{Cont}}_0$. Note that

$$\mathrm{Sol}(\mathcal{R}^{\text{Cont}}) = \mathrm{Sol}(\mathcal{R}^{\text{Cont}}_\alpha), \qquad \text{for all } \alpha \geq 0$$

The following lemma relates $\mathcal{R}^{\text{Cont}}_\alpha$ with $\mathcal{R}^{\text{Cont}}$ for $\alpha \geq 1$.

Lemma 3.2.1. *For any $\alpha \geq 1$, the jet projection map $p = p_1^{\alpha+1} : J^{\alpha+1}(\Sigma, M) \to J^1(\Sigma, M)$ maps the relation $\mathcal{R}_\alpha^{\text{Cont}}$ surjectively onto $\mathcal{R}^{\text{Cont}}$. Furthermore, for each $(x, y) \in \Sigma \times M$, the map $p : \mathcal{R}_\alpha^{\text{Cont}}|_{(x,y)} \to \mathcal{R}^{\text{Cont}}|_{(x,y)}$ is a fiber bundle with contractible fiber and any section of $\mathcal{R}^{\text{Cont}}$ defined over a contractible chart in Σ can be lifted to $\mathcal{R}_\alpha^{\text{Cont}}$ along p.*

We postpone the proof of the above lemma to section 3.3. As a direct consequence, we get the following corollary.

Corollary 3.2.2. *For any $\alpha \geq 1$, the sheaf map $p : \Gamma\mathcal{R}_\alpha^{\text{Cont}} \to \Gamma\mathcal{R}^{\text{Cont}}$ induced by the jet projection map $p = p_1^{\alpha+1}$ is a weak homotopy equivalence.*

Proof. It is immediate from Lemma 3.2.1, that $p : \Gamma\mathcal{R}_\alpha^{\text{Cont}} \to \Gamma\mathcal{R}^{\text{Cont}}$ is a surjective sheaf map with *contractible* fiber. Consequently, the sheaf map is a *local* weak homotopy equivalence. Now both the sheaves are flexible. Hence an application of the homomorphism theorem (Theorem 2.2.6) gives us that p is indeed a weak homotopy equivalence. $\square$

Corollary 3.2.2, in conjunction with the earlier observation then implies that the relation $\mathcal{R}^{\text{Cont}}$ satisfies the local parametric h-principle. The same is true for $\mathcal{R}^{\text{IsoCont}} \subset \mathcal{R}^{\text{Cont}}$ and for $\mathcal{R}^{\text{Hor}}$ as well (Definition 3.1.7). At this point, we have an intermediate result for Ω-regular, $\mathcal{D}$-horizontal immersions $\Sigma \to M$, for Σ open.

Theorem 3.2.3. *If Σ is an open manifold, then the relation $\mathcal{R}^{\text{Hor}}$ satisfies the parametric h-principle.*

Proof. Let Φ^{Hor} denote the sheaf of Ω-regular, $\mathcal{D}$-horizontal immersions, i.e, $\Phi^{\text{Hor}} = \text{Sol}(\mathcal{R}^{\text{Hor}})$. It is easily seen to be invariant under the natural action of $\text{Diff}(\Sigma)$ on $C^\infty(\Sigma, M)$. Indeed, for any horizontal u and any diffeomorphism ζ, $u \circ \zeta$ is clearly horizontal. Furthermore, Ω-regularity is preserved, as it is apparent from the commutative diagram,

$$\begin{tikzcd} u^*\mathcal{D} \arrow[r, two heads] \arrow[dr, two heads] & \hom(T\Sigma, u^*TM/\mathcal{D}) \arrow[d, "\cong"', "(d\zeta)^*"] \\ & \hom(T\Sigma, u^*TM/\mathcal{D}) \end{tikzcd}$$

Now, we have already observed that Φ^{Hor} is a microflexible sheaf and $j^3 : \Phi^{\text{Hor}} \to \Gamma\mathcal{R}_2^{\text{Hor}}$ is a local weak homotopy equivalence. Hence, it follows from Remark 2.2.12 that j^3 is a weak homotopy equivalence. Now by Corollary 3.2.2, we have that $p_1^3 : \Gamma\mathcal{R}_\alpha^{\text{Hor}} \to \Gamma\mathcal{R}^{\text{Hor}}$ is a weak equivalence as well. Hence,

$$j^1 = p_1^3 \circ j^3 : \Phi^{\text{Hor}} \to \Gamma\mathcal{R}^{\text{Hor}}$$

is a weak homotopy equivalence. Thus, the relation $\mathcal{R}^{\mathsf{Hor}}$ satisfies the parametric h-principle on open manifolds. □

Extension h-principle

As before, Σ is a manifold with a given distribution $K \subset T\Sigma$. Let $\tilde{\Sigma} = \Sigma \times \mathbb{R}$ be the product manifold with the natural fibering $\pi : \Sigma \times \mathbb{R} \to \Sigma$. Consider the distribution $\tilde{K} = d\pi^{-1}(K)$ on $\tilde{\Sigma}$ so that corank of $\tilde{K}$ is the same as that of K.

We can now define an operator $\tilde{\mathfrak{D}}^{\mathsf{Cont}}$ for the pair $(\tilde{\Sigma}, \tilde{K})$, as we did in the case of (Σ, K). Let us denote the associated relations on $\tilde{\Sigma}$ by $\tilde{\mathcal{R}}^{\mathsf{Cont}}_{\alpha}$, $\alpha \geq 0$, and $\tilde{\mathcal{R}}^{\mathsf{Cont}} \subset \tilde{\mathcal{R}}^{\mathsf{Cont}}_0$. Let $\tilde{\Phi}^{\mathsf{Cont}}$ be the sheaf of Ω-regular, $\tilde{K}$-contact immersions. As noted earlier, $\tilde{\Phi}^{\mathsf{Cont}} = \mathrm{Sol}(\tilde{\mathcal{R}}^{\mathsf{Cont}}_{\alpha}) = \mathrm{Sol}(\tilde{\mathcal{R}}^{\mathsf{Cont}})$.

Note that the derivative of any fibre-preserving local diffeomorphism φ of $\Sigma \times \mathbb{R}$ takes $\tilde{K}$ isomorphically onto itself. Therefore, if u is $\tilde{K}$-contact then so is $u \circ \varphi$, for any $\varphi \in \mathrm{Diff}(\Sigma \times \mathbb{R}, \pi)$. Also Ω-regularity is $\mathrm{Diff}(\Sigma \times \mathbb{R}, \pi)$-invariant as well. This implies that the sheaf $\tilde{\Phi}^{\mathsf{Cont}}$ is invariant under the natural $\mathrm{Diff}(\Sigma \times \mathbb{R}, \pi)$ action on $\Sigma \times \mathbb{R}$.

Theorem 3.2.4. [Gro86, pg. 339] *The first order relation $\tilde{\mathcal{R}}^{\mathsf{Cont}}$ satisfies the parametric h-principle near $\Sigma \times \{0\}$.*

Proof. We have the following :

- The sheaf $\tilde{\Phi}^{\mathsf{Cont}} = \mathrm{Sol}\, \tilde{\mathcal{R}}^{\mathsf{Cont}} = \mathrm{Sol}\, \tilde{\mathcal{R}}^{\mathsf{Cont}}_{\alpha}$ is microflexible by Theorem 2.2.27
- The map $j^1 : \tilde{\Phi}^{\mathsf{Cont}} \to \Gamma \tilde{\mathcal{R}}^{\mathsf{Cont}}$ is a local weak homotopy equivalence, as argued in the proof of Theorem 3.2.3.
- The solution sheaf $\tilde{\Phi}^{\mathsf{Cont}}$ is invariant under the action of $\mathrm{Diff}(\tilde{\Sigma}, \pi)$

Hence an application of Theorem 2.2.10 gives us that the map $j^1 : \tilde{\Phi}^{\mathsf{Cont}}|_{\Sigma} \to \Gamma \tilde{\mathcal{R}}^{\mathsf{Cont}}|_{\Sigma}$ is a weak homotopy equivalence. In other words, we have the parametric h-principle for $\tilde{\mathcal{R}}^{\mathsf{Cont}}$ near $\Sigma \times 0$. □

Observe that $\tilde{\Sigma}$ is an *open* manifold and it admits a deformation retraction into an arbitrary small neighborhood of $\Sigma \times 0$, by an action of $\mathrm{Diff}(\tilde{\Sigma}, \pi)$. Hence, pullback of any $\tilde{K}$-contact, Ω-regular immersion near $\Sigma \times 0$, by a deformation retraction gives a *global* $\tilde{K}$-contact, Ω-regular immersion on $\tilde{\Sigma}$. Consequently, we get the following.

Corollary 3.2.5. *Ω-regular $\tilde{K}$-contact immersions $(\tilde{\Sigma}, \tilde{K}) \to (M, \mathcal{D})$ satisfy the parametric h-principle. In fact, $\tilde{\mathcal{R}}^{\mathsf{Cont}}$ satisfies the h-principle over $\tilde{\Sigma} = \Sigma \times \mathbb{R}$.*

We put forward the relation $\tilde{\mathcal{R}}^{\mathsf{Cont}} \subset J^1(\tilde{\Sigma}, M)$ as a suitable candidate for an extension of the relation $\mathcal{R}^{\mathsf{Cont}}$ (see Definition 2.2.13). Note that the natural restriction morphism $C^\infty(\tilde{\Sigma}, M) \to C^\infty(\Sigma, M)$, taking $\tilde{u} \mapsto \tilde{u}|_{\Sigma\times 0}$, gives rise to a map in the jet level, $ev : J^1(\tilde{\Sigma}, M)|_\Sigma \to J^1(\Sigma, M)$, given by,

$$\Big((x,t),\ y,\ F : T_{(x,t)}\tilde{\Sigma} \to T_y M\Big) \mapsto \Big(x,\ y,\ F|_{T_x\Sigma}\Big)$$

Now if $\tilde{u}$ is $\tilde{K}$-contact, let $u = \tilde{u}|_\Sigma$; then for any $(\sigma, 0) \in \Sigma \times 0$ we have $d_\sigma u = d_{(\sigma,0)}\tilde{u}|_{T_\sigma\Sigma\oplus 0}$ and hence,

$$d_\sigma u(K_\sigma) = d_{(\sigma,0)}\tilde{u}|_{T_\sigma\Sigma}\big(\tilde{K}_{(\sigma,0)} \cap T_\sigma\Sigma\big) \subset \mathcal{D}_{\tilde{u}(\sigma,0)} = \mathcal{D}_{u(\sigma)}$$

Thus, u is then K-contact. It follows that we have induced sheaf maps,

$$ev : \mathrm{Sol}\,\tilde{\mathcal{R}}^{\mathsf{Cont}}|_{\Sigma\times 0} \to \mathrm{Sol}\,\mathcal{R}^{\mathsf{Cont}}, \qquad ev : \Gamma\tilde{\mathcal{R}}^{\mathsf{Cont}}|_{\Sigma\times 0} \to \Gamma\mathcal{R}^{\mathsf{Cont}}$$

The next lemma justifies the hypothesis (3) of Theorem 2.2.15.

Lemma 3.2.6. *Let $O \subset \Sigma$ be a coordinate chart and $C \subset O$ is a compact subset. Suppose that $U \subset M$ is an open subset such that $\mathcal{D}|_U$ is trivial. Then, given any Ω-regular K-contact immersion $u : \mathrm{Op}\, C \to U \subset M$, the 1-jet map*

$$j^1 : ev^{-1}(u) \to ev^{-1}(F = j^1_u)$$

induces a surjective map between the set of path components. Furthermore, the homotopy can be kept C^0-small.

Proof. We have the commutative diagram

$$\begin{tikzcd}
ev^{-1}(u) \arrow[r, hook] \arrow[d, dashed, "j^1"'] & \tilde{\Phi}^{\mathsf{Cont}}|_{C\times 0} \arrow[r] \arrow[d] & \Phi^{\mathsf{Cont}}|_C \arrow[d] & u \arrow[d, mapsto] \\
ev^{-1}(F) \arrow[r, hook] & \tilde{\Psi}^{\mathsf{Cont}}|_{C\times 0} \arrow[r] & \Psi^{\mathsf{Cont}}|_C & F = j^1_u
\end{tikzcd}$$

with the sheaves,

$$\Phi^{\mathsf{Cont}} = \mathrm{Sol}\,\mathcal{R}^{\mathsf{Cont}}, \quad \Psi^{\mathsf{Cont}} = \Gamma\mathcal{R}^{\mathsf{Cont}}, \qquad \tilde{\Phi}^{\mathsf{Cont}} = \mathrm{Sol}\,\tilde{\mathcal{R}}^{\mathsf{Cont}}, \quad \tilde{\Psi}^{\mathsf{Cont}} = \Gamma\tilde{\mathcal{R}}^{\mathsf{Cont}}.$$

Fix some neighborhood V of C, with $C \subset V \subset O$, over which u is defined and then fix an arbitrarily small open neighborhood U_ϵ of $u(V)$. The proof now proceeds in a few steps.

Step 1 Given an arbitrary extension $\tilde{F}$ of F along ev, we construct a regular solution $\bar{u}$ on $\tilde{Op}C$, so that $j^1_{\bar{u}}|_{\mathrm{Op}\,C} = \tilde{F}|_{\mathrm{Op}\,C}$.

Step 2 We get an homotopy between $j^1_{\bar{u}}$ and $\tilde{F}$, in the affine bundle $J^1(W, U_\epsilon)$, which is constant on points of C.

Step 3 Using Lemma 3.1.8, we then push the homotopy obtained in Step 2 inside the relation $\tilde{\mathcal{R}}^{\mathsf{Cont}}$, thus completing the proof.

Proof of Step 1 **:** Suppose $\tilde{F} \in \tilde{\Psi}^{\mathsf{Cont}}|_{C\times 0}$ is some arbitrary extension of F. Using Lemma 3.2.1, we then get an arbitrary lift $\hat{F} \in \Gamma\tilde{\mathcal{R}}^{\mathsf{Cont}}_\alpha|_C$ of $\tilde{F}$, for α sufficiently large (in fact, $\alpha \geq 4$ will suffice). The formal maps are represented in the following diagram.

$$\begin{tikzcd}
& \tilde{\mathcal{R}}^{\mathsf{Cont}}_\alpha|_\Sigma \arrow[d, "p_1^{\alpha+1}"] \\
& \tilde{\mathcal{R}}^{\mathsf{Cont}}|_\Sigma \arrow[d, "ev"] \\
\Sigma \arrow[uur, "\hat{F}"] \arrow[ur, "\tilde{F}"'] \arrow[r, "F"'] & \mathcal{R}^{\mathsf{Cont}}
\end{tikzcd}$$

We can now define a map $\hat{u} : \tilde{\mathrm{Op}}(C) \to U_\epsilon$ so that $j^{\alpha+1}_{\hat{u}}(p,0) = \hat{F}(p,0)$, by applying a Taylor series argument. In particular, we have $\hat{u}|_{C\times 0} = u$ and $\hat{u}$ is regular on points of $\mathrm{Op}(C)\times 0$. Since C is a compact set and regularity is an open condition, we have that $\hat{u}$ is regular on some open set W satisfying, $C \subset W \subset \bar{W} \subset \tilde{\mathrm{Op}}(C)$. We also have $\hat{u}$ is a regular infinitesimal solution, along the set $W_0 = (V\times 0)\cap W \subset \tilde{\mathrm{Op}}(C)$, of order,

$$\alpha \geq 2.1 + 3.0 + \max\{1, 2.1 + 0\} = 4,$$

for the equation $\tilde{\mathfrak{D}} = 0$, where $\tilde{\mathfrak{D}} = \tilde{\mathfrak{D}}^{\mathsf{Cont}} : v \mapsto v^*\lambda^s|_{\tilde{K}}$ is defined over $C^\infty(W, U_\epsilon)$. Now, from Theorem 2.2.30, we can get an inversion $\tilde{\mathfrak{D}}^{-1}_{\hat{u}}$, on W; we have an Ω-regular map $\bar{u} : V \to U_\epsilon$ such that, $\tilde{\mathfrak{D}}(\bar{u}) = 0$ and furthermore,

$$j^1_{\bar{u}} = j^1_{\hat{u}} \quad \text{on points of } W_0.$$

In particular, $j^1_{\bar{u}}(p,0) = \tilde{F}(p,0)$ for $(p,0) \in W_0$ and so u on $\mathrm{Op}\,C$ is extended to $\bar{u}$ on W.

Proof of Step 2 **:** Let us denote $\tilde{u} = \operatorname{bs}\tilde{F}$ and define, $v_t(x,s) = \bar{u}(x,ts)$ for $(x,s) \in W$. Note that,

$$v_0(x,s) = \bar{u}(x,0) = \hat{u}(x,0) = \tilde{u}(x,0)$$

and so v_t is a homotopy between the maps $\bar{u}$ and $\pi^*(\tilde{u}|_{\operatorname{Op} C})|_W$, where $\pi : \Sigma \times \mathbb{R} \to \Sigma$ is the projection. Now, with the help of some auxiliary choice of parallel transport on the vector bundle $J^1(W, U_\epsilon)$, we can get isomorphisms,

$$\varphi(x,s) : J^1_{(x,0),\bar{u}(x,0)}(W,U_\epsilon) \to J^1_{(x,s),\bar{u}(x,s)}(W,U_\epsilon), \quad \text{for } (x,s) \in W, \text{ for } s \text{ sufficiently small,}$$

so that $\varphi(x,0) = \mathrm{Id}$. We then define the homotopy,

$$G_t|_{(x,s)} = (1-t)\cdot \varphi(x,ts)\circ \tilde{F}|_{(x,0)} + t\cdot j^1_{\bar{u}}(x,ts) \ \in J^1_{(x,ts),\bar{u}(x,ts)}(W,U_\epsilon).$$

Clearly G_t covers v_t; we have

$$G_0|_{(x,s)} = \varphi(x,0)\circ \tilde{F}|_{(x,0)} = \tilde{F}|_{(x,0)} = \tilde{F}|_{(x,s)} \qquad \text{and} \qquad G_1|_{(x,s)} = j^1_{\bar{u}}(x,s).$$

Thus we have obtained a homotopy G_t between $\pi^*(\tilde{F}|_{\operatorname{Op} C})|_{\tilde{O}pC}$ and $j^1_{\bar{u}}$. Similar argument produces a homotopy between $\tilde{F}$ and $\pi^*(\tilde{F}|_{\operatorname{Op} C})|_W$ as well. Concatenating the two homotopies, we have a homotopy H_t between $\tilde{F}$ and $j^1_{\bar{u}}$, in the affine bundle $J^1(W,U_\epsilon) \to W \times U_\epsilon$. However, H_t need not lie in $\tilde{\mathcal{R}}^{\mathsf{Cont}}$.

Proof of Step 3 **:** We now get an tubular neighborhood $\mathcal{N} \subset J^1(W,U_\epsilon)$ of $\operatorname{Im}\tilde{F}$, which *fiber-wise* deformation retracts onto $\tilde{\mathcal{R}}^{\mathsf{Cont}} \cap \mathcal{N}$. Indeed, this follows from Lemma 3.1.8. Suppose $\rho : \mathcal{N} \to \tilde{\mathcal{R}}^{\mathsf{Cont}}$ is such a retraction. Now, note that on points of C,

$$H_t|_{(x,0)} = (1-t)\cdot \tilde{F}|_{(x,0)} + t\cdot j^1_{\bar{u}}(x,0) = \tilde{F}|_{(x,0)}.$$

Since C is compact, we may get a neighborhood W', satisfying $C \subset W' \subset W$, such that the homotopy $H_t|_{W'}$ takes its values in the open neighborhood $\mathcal{N}$ of $\operatorname{Im}\tilde{F}$. Then composing with the retraction ρ, we can push this homotopy inside the relation $\tilde{\mathcal{R}}^{\mathsf{Cont}}$, obtaining a homotopy $\tilde{F}_t \in \tilde{\Psi}|_C$ joining $\tilde{F}$ to $j^1_{\bar{u}}$. Observe that the homotopy remains constant on points of C. In particular we have that, $ev(\tilde{F}_t) = F$ on points of C. Since U_ϵ is taken to be arbitrarily small, the homotopy in the base maps are always kept C^0-small. This concludes the proof. $\square$

In the light of Lemma 3.2.6, we see that in order to achieve the h-principle for Ω-regular K-contact immersions of an arbitrary manifold Σ, we only need to get a suitable extension of the relation $\mathcal{R}^{\text{Cont}}$. Then a direct application of Theorem 2.2.15 gives us the desired h-principle. We now state the main theorem of this Chapter.

Theorem 3.2.7. *If $ev : \tilde{\mathcal{R}}^{\text{Cont}}|_{\Sigma\times 0} \to \mathcal{R}^{\text{Cont}}$ is locally surjective, i.e, if $\tilde{\mathcal{R}}^{\text{Cont}}$ is an extension of $\mathcal{R}^{\text{Cont}}$, then $\mathcal{R}^{\text{Cont}}$ satisfies the C^0-dense h-principle.*

Proof. From the hypothesis we have that $\tilde{\mathcal{R}}^{\text{Cont}}$ is an extension of $\mathcal{R}^{\text{Cont}}$ in the sense of Definition 2.2.13. Furthermore,

- $\operatorname{Sol}\tilde{\mathcal{R}}^{\text{Cont}}$ is microflexible by Theorem 2.2.27 and the sheaf is invariant under the $\operatorname{Diff}(\tilde{\Sigma}, \pi)$ action. Hence the restricted sheaf $\operatorname{Sol}\tilde{\mathcal{R}}^{\text{Cont}}|_{\Sigma}$ is flexible by Theorem 2.2.9.
- $\tilde{\mathcal{R}}^{\text{Cont}}$ satisfies the local h-principle by Theorem 2.2.28 and Lemma 3.2.1.

Thus, the first two hypothesis of Theorem 2.2.15 are satisfied; Lemma 3.2.6 justifies the last hypothesis. Hence $\mathcal{R}^{\text{Cont}}$ satisfies the C^0-dense h-principle by a direct application of Theorem 2.2.15. □

Remark 3.2.8. The above theorem should be compared to the *approximation theorem* of Gromov ([Gro96, pg. 258]) for 'overregular maps'. Recall that Gromov defines *overregular maps*, in the context of $\mathcal{D}$-horizontal maps, as those formal Ω-regular, $\mathcal{D}$-horizontal maps $F : T\Sigma \to TM$, covering some $u : \Sigma \to M$, for which the subspace $F(T_x\Sigma) \subset \mathcal{D}_{u(\sigma)}$ is contained in an Ω-regular, Ω-isotropic subspace. In other words, F is *overregular* if it admits (point-wise) extension to formal Ω-regular maps $\tilde{\Sigma} \to M$. Gromov proceeds to state that : "Overregular maps satisfy the C^0-dense h-principle".

In the next chapter, we shall turn our focus on to a special class of distribution $\mathcal{D}$, known as fat distributions. We shall see that the local extensibility property is satisfied in many interesting situations. We end this chapter with the proof of Lemma 3.2.1.

3.3 Proof of Lemma 3.2.1

To simplify the notation, we assume that $K = T\Sigma$, i.e, we prove the statement for the relation $\mathcal{R}^{\text{Hor}}$. The argument for a general K is similar, albeit cumbersome. As the lemma is local in nature, without loss of generality we assume that $\mathcal{D}$ is cotrivializable and hence suppose that $\mathcal{D} = \bigcap_{s=1}^{p} \ker \lambda^s$. We denote the tuples,

$$\lambda = (\lambda^s) \in \Omega^1(M, \mathbb{R}^p) \text{ and } d\lambda = (d\lambda^s) \in \Omega^2(M, \mathbb{R}^p).$$

We need to consider the three operators :

- $u \mapsto u^*\lambda$,
- $u \mapsto u^* d\lambda$,
- $d : \Omega^1(\Sigma, \mathbb{R}^p) \to \Omega^2(\Sigma, \mathbb{R}^p)$, the exterior derivative operator,

and their respective symbols :

- we have the bundle map, $\Delta_\lambda : J^1(\Sigma, M) \to \Omega^1(\Sigma, \mathbb{R}^p)$ so that $\Delta_\lambda\big(j^1_u\big) = u^*\lambda = \big(u^*\lambda^s\big)$. Explicitly,
$$\Delta_\lambda(x, y, F : T_x\Sigma \to T_y M) = \big(x, F \circ \lambda|_y\big).$$
- we have the bundle map, $\Delta_{d\lambda} : J^1(\Sigma, M) \to \Omega^1(\Sigma, \mathbb{R}^p)$ so that $\Delta_{d\lambda}\big(j^1_u\big) = u^* d\lambda = \big(u^* d\lambda^s\big)$. Explicitly,
$$\Delta_{d\lambda}(x, y, F : T_x\Sigma \to T_y M) = \big(x, F^* d\lambda|_y\big).$$
- we have the bundle map, $\Delta_d : \Omega^1(\Sigma, \mathbb{R}^p)^{(1)} \to \Omega^2(\Sigma, M)$ so that $\Delta_d(j^1_\alpha) = d\alpha$. Explicitly,
$$\Delta_d\big(x, \alpha, F : T_x\Sigma \to \hom(T_x\Sigma, \mathbb{R}^p)\big) = \big(x, (X \wedge Y) \mapsto F(X)(Y) - F(Y)(X)\big).$$

Jet Prolongation of Symbols : Recall that given some arbitrary r^{th}-order operator $\mathfrak{D} : \Gamma X \to \Gamma G$ represented by the bundle map $\Delta : X^{(r)} \to G$ as, $\Delta(j^r_u) = \mathfrak{D}(u)$, we have the α-jet prolongation, $\Delta^{(\alpha)} : X^{(r+\alpha)} \to G^{(\alpha)}$ defined as,

$$\Delta^{(\alpha)}(j^{r+\alpha}_u(x)) = j^\alpha_{\mathfrak{D}(u)}(x)$$

We can immediately observe that the diagram,

$$\begin{tikzcd} X^{(r+\alpha)} \arrow[r, "\Delta^{(\alpha)}"] \arrow[d, "p^{r+\alpha}_{r+\beta}"'] & G^{(\alpha)} \arrow[d, "p^\alpha_\beta"] \\ X^{(r+\beta)} \arrow[r, "\Delta^{(\beta)}"'] & G^{(\beta)} \end{tikzcd}$$

commutes for any $\alpha \ge \beta$. Indeed, we have,

$$p^\alpha_\beta \circ \Delta^{(\alpha)}\big(j^{r+\alpha}_u(x)\big) = p^\alpha_\beta\big(j^\alpha_{\mathfrak{D}u}(x)\big) = j^\beta_{\mathfrak{D}x}(x) = \Delta^{(\beta)}\big(j^{\beta+r}_u(x)\big) = \Delta^{(\beta)} \circ p^{r+\alpha}_{r+\beta}\big(j^{r+\alpha}_u(x)\big)$$

We now observe the following interplay between the symbols of the operators introduced above.

- We have the commutative diagram,

$$\begin{tikzcd} J^{\alpha+1}(\Sigma, M) \arrow[r, "\Delta_\lambda^{(\alpha)}"] \arrow[d, "p_\alpha^{\alpha+1}"'] & \Omega^1(\Sigma, \mathbb{R}^p)^{(\alpha)} \arrow[d, "\Delta_d^{(\alpha-1)}"] \\ J^\alpha(\Sigma, M) \arrow[r, "\Delta_{d\lambda}^{(\alpha-1)}"'] & \Omega^2(\Sigma, \mathbb{R}^p)^{(\alpha-1)} \end{tikzcd}$$

Indeed, we observe,

$$\Delta_d^{(\alpha-1)} \circ \Delta_\lambda^{(\alpha)}\big(j_u^{\alpha+1}(x)\big) = \Delta_d^{(\alpha-1)}\big(j_{u^*\lambda}^\alpha(x)\big) = j^{\alpha-1}_{d\left(u^*\lambda\right)}(x) = j^{\alpha-1}_{u^*d\lambda}(x) = \Delta_{d\lambda}^{(\alpha-1)}\big(j_u^\alpha(x)\big),$$

and hence we get,

$$\Delta_d^{(\alpha-1)} \circ \Delta_\lambda^{(\alpha)} = \Delta_{d\lambda}^{(\alpha-1)} \circ p_\alpha^{\alpha+1}.$$

- We have the two commutative diagrams,

$$\begin{tikzcd} J^{\alpha+1}(\Sigma, M) \arrow[r, "\Delta_\lambda^{(\alpha)}"] \arrow[d, "p_\alpha^{\alpha+1}"'] & \Omega^1(\Sigma, \mathbb{R}^p)^{(\alpha)} \arrow[d, "p_{\alpha-1}^\alpha"] \\ J^\alpha(\Sigma, M) \arrow[r, "\Delta_\lambda^{(\alpha-1)}"'] & \Omega^1(\Sigma, \mathbb{R}^p)^{(\alpha-1)} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} J^{\alpha+1}(\Sigma, M) \arrow[r, "\Delta_{d\lambda}^{(\alpha)}"] \arrow[d, "p_\alpha^{\alpha+1}"'] & \Omega^2(\Sigma, \mathbb{R}^p)^{(\alpha)} \arrow[d, "p_{\alpha-1}^\alpha"] \\ J^\alpha(\Sigma, M) \arrow[r, "\Delta_{d\lambda}^{(\alpha-1)}"'] & \Omega^2(\Sigma, \mathbb{R}^p)^{(\alpha-1)} \end{tikzcd}$$

Now let us fix $\mathcal{R}_{d\lambda} \subset J^1(\Sigma, M)$ representing the $(d\lambda^s)$-regular immersions $\Sigma \to M$, i.e,

$$\mathcal{R}_{d\lambda} = \Big\{(x, y, F : T_x\Sigma \to T_yM) \;\Big|\; F \text{ is injective and } (d\lambda^s)\text{-regular}\Big\}.$$

Next recall that $\mathcal{R}_\alpha \subset J^{\alpha+1}(\Sigma, M)$ is given as,

$$\mathcal{R}_\alpha = \Big\{j_u^{\alpha+1}(x) \in J^{\alpha+1}(\Sigma, M)|_x \;\Big|\; j_{u^*\lambda}^\alpha(x) = 0 \text{ and } u \text{ is } (d\lambda^s)\text{-regular}\Big\}.$$

Hence we can identify $\mathcal{R}_\alpha$ as,

$$\mathcal{R}_\alpha = \ker\big(\Delta_\lambda^{(\alpha)}\big) \cap \big(p_1^{\alpha+1}\big)^{-1}(\mathcal{R}_{d\lambda}) \subset J^{\alpha+1}(\Sigma, M).$$

We denote a sub-relation,

$$\bar{\mathcal{R}}_\alpha = \mathcal{R}_\alpha \cap \ker\big(\Delta_{d\lambda}^{(\alpha)}\big) \subset \mathcal{R}_\alpha.$$

In particular, observe that $\bar{\mathcal{R}}_0$ is then precisely $\mathcal{R}^{\mathsf{Hor}}$, i.e, the relation of Ω-regular, horizontal immersions $\Sigma \to M$.

The proof of Lemma 3.2.1 follows from the next two results.

Sublemma 3.3.1. *For any $\alpha \geq 0$, we have, $\bar{\mathcal{R}}_\alpha = p^{\alpha+2}_{\alpha+1}(\mathcal{R}_{\alpha+1})$ and the fiber of $p^{\alpha+2}_{\alpha+1} : \mathcal{R}_{\alpha+1}|_{(x,y)} \to \bar{\mathcal{R}}_\alpha|_{(x,y)}$ is affine for each $(x, y) \in \Sigma \times M$. Furthermore, any section of $\bar{\mathcal{R}}_\alpha|_O$, over some contractible charts $O \subset \Sigma$, can be lifted to a section of $\mathcal{R}_{\alpha+1}|_O$ along $p^{\alpha+2}_{\alpha+1}$*

Sublemma 3.3.2. *For any $\alpha \geq 0$, the map $p^{\alpha+2}_{\alpha+1} : \bar{\mathcal{R}}_{\alpha+1}|_{(x,y)} \to \bar{\mathcal{R}}_\alpha|_{(x,y)}$ is surjective, with affine fibers, for each $(x, y) \in \Sigma\times M$. Furthermore, any section of $\bar{\mathcal{R}}_\alpha|_O$ over some contractible chart $O \subset \Sigma$ can be lifted to a section of $\bar{\mathcal{R}}_{\alpha+1}|_O$ along $p^{\alpha+2}_{\alpha+1}$.*

Assuming these, let us first give a proof of the jet lifting lemma.

Proof of Lemma 3.2.1. We have the following ladder-like schematic representation,

$$\begin{tikzcd}
J^{\alpha+1}(\Sigma, M) \arrow[r, "p^{\alpha+1}_\alpha"] & J^\alpha(\Sigma, M) \arrow[r] & \cdots \arrow[r] & J^2(\Sigma, M) \arrow[r, "p^2_1"] & J^1(\Sigma, M) \\
\cup & \cup & & \cup & \cup \\
\mathcal{R}_\alpha \arrow[r] \arrow[dr, two heads, "p^{\alpha+1}_\alpha"] & \mathcal{R}_{\alpha-1} \arrow[r] & \cdots \arrow[r] & \mathcal{R}_1 \arrow[r] \arrow[dr, two heads] & \mathcal{R}_0 \\
& \bar{\mathcal{R}}_{\alpha-1} \arrow[u, hook] \arrow[ul, dashed, bend left, "\text{lift using full rank of } \lambda \text{ (Sublemma 3.3.1)}"] & & & \mathcal{R}^{\text{Hor}} = \bar{\mathcal{R}}_0 \arrow[u, hook] \arrow[llll, dashed, "\text{lift inductively to } \bar{\mathcal{R}}_{\alpha-1}", "\text{using } \Omega\text{-regularity (Sublemma 3.3.2)}"']
\end{tikzcd}$$

For any $\alpha \geq 1$, we have,

$$p^{\alpha+1}_1 = p^\alpha_1 \circ p^{\alpha+1}_\alpha = p^2_1 \circ \cdots \circ p^{\alpha+1}_\alpha$$

From Sublemma 3.3.1 we have that $p^{\alpha+1}_\alpha$ maps $\mathcal{R}_\alpha$ surjectively onto $\bar{\mathcal{R}}_{\alpha-1}$. Also, using Sublemma 3.3.2 inductively, we have that $p^\alpha_1 : \bar{\mathcal{R}}_{\alpha-1} \to \mathcal{R}^{\text{Hor}}$ is a surjection as well. Combining the two, we have the claim.

Since at each step of the induction, we have contractible fiber, we see that the fiber of $p^{\alpha+1}_1$ is contractible as well. In fact, we are easily able to get lifts of sections over contractible charts as well. This concludes the proof. □

We now prove the above sublemmas.

Proof of Sublemma 3.3.1. We have the following commutative diagram,

$$\begin{tikzcd}
\mathcal{R}_{\alpha+1} \arrow[r, hook] & J^{\alpha+2}(\Sigma, M) \arrow[r, "\Delta^{(\alpha+1)}_\lambda"] \arrow[d, "p^{\alpha+2}_{\alpha+1}"'] & \Omega^1(\Sigma, \mathbb{R}^p)^{(\alpha+1)} \arrow[d, "p^{\alpha+1}_\alpha"', "\Delta^{(\alpha)}_d"] \\
\bar{\mathcal{R}}_\alpha \arrow[r, hook] & J^{\alpha+1}(\Sigma, M) \arrow[r, "\Delta^{(\alpha)}_\lambda{,}\ \Delta^{(\alpha)}_{d\lambda}"'] & \Omega^1(\Sigma, \mathbb{R}^p)^{(\alpha)} \oplus \Omega^2(\Sigma, \mathbb{R}^p)^{(\alpha)}
\end{tikzcd} \tag{$*$}$$

Since we have $\mathcal{R}_{\alpha+1} \subset \ker \Delta_\lambda^{(\alpha+1)}$, we get that,

$$p_{\alpha+1}^{\alpha+2}(\mathcal{R}_{\alpha+1}) \subset \ker \Delta_\lambda^{(\alpha)} \cap \ker \Delta_{d\lambda}^{(\alpha)}.$$

Also,

$$\mathcal{R}_{\alpha+1} \subset \left(p_1^{\alpha+2}\right)^{-1}(\mathcal{R}_{d\lambda}) \Rightarrow p_{\alpha+1}^{\alpha+2}\left(\mathcal{R}_{\alpha+1}\right) \subset \left(p_\alpha^{\alpha+1}\right)^{-1}(\mathcal{R}_{d\lambda}).$$

Hence we see that, $\left(p_{\alpha+1}^{\alpha+2}\right)(\mathcal{R}_{\alpha+1}) \subset \bar{\mathcal{R}}_\alpha$.

Conversely, let us assume that we are given a jet,

$$\left(x, y, P_i : \operatorname{Sym}^i T_x\Sigma \to T_y M,\ i = 1, \ldots, \alpha + 1\right) \in \bar{\mathcal{R}}_\alpha|_{(x,y)}.$$

We wish to find $Q : \operatorname{Sym}^{\alpha+2} T_x\Sigma \to T_y M$ so that,

$$(x, y, P_i, Q) \in \mathcal{R}_{\alpha+1}|_{(x,y)}.$$

Recall that $\Delta_\lambda(x, y, F : T_x\Sigma \to T_y M) = \left(x, \lambda|_y \circ F : T_x\Sigma \to \mathbb{R}^p\right)$. Then we may write,

$$\Delta_\lambda^{(\alpha+1)}\left(x, y, P_i, Q\right) = \left(x, \lambda \circ F, R_i : \operatorname{Sym}^i T_x\Sigma \to \hom(T_x\Sigma, \mathbb{R}^p),\ i = 1, \ldots, \alpha + 1\right),$$

so that $R_{\alpha+1} : \operatorname{Sym}^{\alpha+1} T_x\Sigma \to \hom(T_x\Sigma, \mathbb{R}^p)$ is the *only* symmetric tensor which involves Q. In fact we observe that $R_{\alpha+1}$ is given explicitly as,

$$R_{\alpha+1}\left(X_1, \ldots, X_{\alpha+1}\right)(Y) = \lambda \circ Q\left(X_1, \ldots, X_{\alpha+1}, Y\right) + \text{terms involving } P_i$$

Now from the commutative diagram $(*)$ we have,

$$\begin{aligned}
\left(x, \lambda \circ F, R_i\ i = 1, \ldots, \alpha\right) &= p_\alpha^{\alpha+1} \circ \Delta_\lambda^{(\alpha+1)}(x, y, P_i, Q) \\
&= \Delta_\lambda^{(\alpha)} \circ p_{\alpha+1}^{\alpha+2}(x, y, P_i, Q) \\
&= \Delta_\lambda^{(\alpha)}(x, y, P_i) \\
&= 0
\end{aligned}$$

That is we get that, $R_i = 0$ for $i = 1, \ldots, \alpha$. We need to find Q so that $R_{\alpha+1} = 0$ as well. We claim that the tensor,

$$R'_{\alpha+1} : \left(X_1, \ldots, X_{\alpha+1}, Y\right) \mapsto R_{\alpha+1}(X_1, \ldots, X_{\alpha+1})(Y),$$

is symmetric.

Let us write $\Delta_d^{(\alpha)}(x, y, \lambda \circ F, R_i) = \big(x, \omega, S_i : \operatorname{Sym}^i T_x\Sigma \to \hom(\Lambda^2 T_x\Sigma, \mathbb{R}^p), i = 1, \ldots, \alpha\big)$, we then get the *pure* α-jet as,

$$S_\alpha(X_1, \ldots, X_\alpha)(Y \wedge Z) = R_{\alpha+1}(X_1, \ldots, X_\alpha, Y)(Z) - R_{\alpha+1}(X_1, \ldots, X_\alpha, Z)(Y).$$

Again going back to the commutative diagram $(*)$ we have,

$$\begin{aligned} \Delta_d^{(\alpha)}\big(x, \lambda \circ F, R_i\big) &= \Delta_d^{(\alpha)} \circ \Delta_\lambda^{(\alpha+1)}(x, y, P_i, Q) \\ &= \Delta_{d\lambda}^{(\alpha)} \circ p_{\alpha+1}^{\alpha+2}(x, y, P_i, Q) \\ &= \Delta_{d\lambda}^{(\alpha)}(x, y, P_i) \\ &= 0 \end{aligned}$$

and so in particular, $S_\alpha = 0$. But then we readily have that $R'_{\alpha+1}$ is a symmetric tensor.

Let us now fix some basis $\{\partial_1, \ldots, \partial_{k+1}\}$ of $T_x\Sigma$ so that, $T_x\Sigma = \langle \partial_1, \ldots, \partial_{k+1}\rangle$, where $\dim \Sigma = k+1$. Then we have the standard basis for the symmetric space $\operatorname{Sym}^{\alpha+2} T_x\Sigma$, so that,

$$\operatorname{Sym}^{\alpha+2} T_x\Sigma = \Big\langle \partial_J = \partial_{j_1} \odot \cdots \odot \partial_{j_{\alpha+2}} \;\Big|\; J = (1 \le j_1 \le \cdots \le j_{\alpha+2} \le k+1) \Big\rangle.$$

Then for each tuple $J = (j_1, \ldots, j_{\alpha+2})$, we see that the *only* equation involving $Q(\partial_J)$ is,

$$0 = R_{\alpha+1}(\partial_1, \ldots, \partial_{j_{\alpha+1}})(\partial_{j_{\alpha+2}}) = \lambda \circ Q(\partial_J) + \text{terms with } P_i.$$

This is an affine equation in $Q(\partial_J) \in T_yM$, which admits solution since $\lambda|_y : T_yM \to \mathbb{R}^p$ has full rank. Thus we have solved Q.

This concludes the proof that $p_{\alpha+1}^{\alpha+2}(\mathcal{R}_{\alpha+1}) = \bar{\mathcal{R}}_\alpha$. Since Q is solved from an affine system of equation, it is immediate that the fiber $\big(p_{\alpha+1}^{\alpha+2}\big)^{-1}\big(x, y, P_i\big)$ is affine in nature. In fact, we see that the projection is an affine fiber bundle. Furthermore, since $\lambda = (\lambda^s)$ has full rank at each point, we are able to get lifts of sections over a fixed contractible chart $O \subset \Sigma$, where we may choose some coordinate vector fields as the basis for $T\Sigma|_O$. $\square$

Next we prove that $p_{\alpha+1}^{\alpha+2}(\bar{\mathcal{R}}_{\alpha+1}) = \bar{\mathcal{R}}_\alpha$ for any $\alpha \ge 0$.

Proof of Sublemma 3.3.2. We have the following commutative diagram,

$$\begin{tikzcd}
\bar{\mathcal{R}}_{\alpha+1} \arrow[r, hook] \arrow[d, dashed] & J^{\alpha+2}(\Sigma, M) \arrow[r, "\Delta_\lambda^{(\alpha+1)},\, \Delta_{d\lambda}^{(\alpha+1)}"] \arrow[d, "p^{\alpha+2}_{\alpha+1}"'] & \Omega^1(\Sigma, \mathbb{R}^p)^{(\alpha+1)} \oplus \Omega^2(\Sigma, \mathbb{R}^p)^{(\alpha+1)} \arrow[d, "p_\alpha^{\alpha+1}"', "p_\alpha^{\alpha+1}"] \\
\bar{\mathcal{R}}_\alpha \arrow[r, hook] & J^{\alpha+1}(\Sigma, M) \arrow[r, "\Delta_\lambda^{(\alpha)},\, \Delta_{d\lambda}^{(\alpha)}"'] & \Omega^1(\Sigma, \mathbb{R}^p)^{(\alpha)} \oplus \Omega^2(\Sigma, \mathbb{R}^p)^{(\alpha)}
\end{tikzcd} \tag{$**$}$$

We have already proved that $p^{\alpha+2}_{\alpha+1}$ maps $\mathcal{R}_{\alpha+1}$ surjectively onto $\bar{\mathcal{R}}_\alpha$; since $\bar{\mathcal{R}}_{\alpha+1} \subset \mathcal{R}_{\alpha+1}$ we have that $p^{\alpha+2}_{\alpha+1}$ maps $\bar{\mathcal{R}}_{\alpha+1}$ into $\bar{\mathcal{R}}_\alpha$. We show the surjectivity.

Suppose $\sigma = \big(x, y, P_i : \mathrm{Sym}^i\, T_x\Sigma \to T_yM,\ i = 1, \ldots, \alpha+1\big) \in \bar{\mathcal{R}}_\alpha|_(x, y)$ is a given jet. We need to find out $Q : \mathrm{Sym}^{\alpha+2}\, T_x\Sigma \to T_yM$ such that, $(x, y, P_i, Q) \in \bar{\mathcal{R}}_{\alpha+1}|_{(x,y)}$. We have seen that in order to find Q so that $(x, y, P_i, Q) \in \mathcal{R}_{\alpha+1}|_{(x,y)}$, we must solve the affine system,

$$\lambda \circ Q = \text{terms with } P_i.$$

which is indeed solvable since λ has full rank. Now in order to find $(x, y, P_i, Q) \in \bar{\mathcal{R}}_{\alpha+1} = \bar{\mathcal{R}}_\alpha \cap \ker \Delta^{(\alpha+1)}_{d\lambda}$, we need to figure out the equations involved in $\Delta^{(\alpha+1)}_{d\lambda}$. Let us write,

$$\Delta^{(\alpha+1)}_{d\lambda}(x, y, P_i, Q) = \big(x, P_1^* d\lambda, R_i : \mathrm{Sym}^i\, T_x\Sigma \to \hom(\Lambda^2 T_x\Sigma, \mathbb{R}^p), i = 1, \ldots, \alpha+1\big).$$

Then the *pure* $\alpha+1$-jet $R_{\alpha+1} : \mathrm{Sym}^{\alpha+1}\, T_x\Sigma \to \hom(\Lambda^2 T_x\Sigma, \mathbb{R}^p)$ is the only expression that involves Q. In fact we have that $R_{\alpha+1}$ is given as,

$$\begin{aligned}
R_{\alpha+1}(X_1, \ldots, X_{\alpha+1})(Y \wedge Z) = d\lambda\big(&Q(X_1, \ldots, X_{\alpha+1}, Y), P_1(Z)\big) \\
&+ d\lambda\big(P_1(Y), Q(X_1, \ldots, X_{\alpha+1}, Z)\big) \\
&+ \text{terms involving } P_i \text{ with } i \geq 2
\end{aligned}$$

Now looking at commutative diagram $(**)$, we have,

$$\begin{aligned}
(x, y, P_1^* d\lambda, R_i, i = 1, \ldots, \alpha) &= p_\alpha^{(\alpha+1)} \circ \Delta^{(\alpha+1)}_{d\lambda}(x, y, P_i, Q) \\
&= \Delta^{(\alpha)}_{d\lambda} \circ p^{\alpha+2}_{\alpha+1}(x, y, P_i, Q) \\
&= \Delta^{(\alpha)}_{d\lambda}(x, y, P_i) \\
&= 0
\end{aligned}$$

That is we have, $R_i = 0$ for $i = 1, \ldots, \alpha$. In order to find Q such that $R_{\alpha+1} = 0$, let us fix some basis $\{\partial_1, \ldots, \partial_{k+1}\}$ of $T_x\Sigma$, where $\dim \Sigma = k+1$. Then we have the standard basis for the symmetric space $\mathrm{Sym}^{\alpha+2} T_x\Sigma$, so that,

$$\mathrm{Sym}^{\alpha+2} T_x\Sigma = Span\Big\langle \partial_J = \partial_{j_1} \odot \cdots \odot \partial_{j_{\alpha+2}} \;\Big|\; J = (1 \le j_1 \le \cdots \le j_{\alpha+2} \le k+1)\Big\rangle.$$

Now for any tuple J and for any $1 \le a < b \le k+1$, we have the equation involving the tensor Q given as,

$$\begin{aligned} 0 = R_{\alpha+1}(\partial_J)(\partial_a \wedge \partial_b) = d\lambda\big(Q(\partial_{J+a}), P_1(\partial_b)\big) + d\lambda\big(P_1(\partial_a), Q(\partial_{J+b}\big) \\ + \text{terms with } P_i \text{ for } i \ge 2, \end{aligned}$$

where $J+a$ is the tuple obtained by ordering $(j_1, \ldots, j_{\alpha+2}, a)$. Now observe that,

$$a < b \Rightarrow J + a \prec J + b,$$

where $\prec$ is the lexicographic ordering on the set of all ordered $\alpha+2$ tuples. We then treat the above equation as,

$$\Big(\iota_{P_1(\partial_a)} d\lambda\Big) \circ Q(\partial_{J+b}) = \Big(\iota_{P_1(\partial_b)} d\lambda\Big) \circ Q(\partial_{J+a}) + \text{terms with } P_1.$$

Thus we have identified the defining system of equations for the tensor Q given as,

$$\begin{cases} \lambda \circ Q(\partial_I) = \text{terms with } P_i, \quad \text{for each } \alpha+2 \text{ tuple } I \\ \iota_{P_1(\partial_a)} d\lambda \circ Q(\partial_{J+b}) = \iota_{P_1(\partial_b)} d\lambda \circ Q(\partial_{I+a}) + \text{terms with } P_i, \\ \qquad\qquad \text{for each } \alpha+1\text{-tuple } J \text{ and } 1 \le a < b \le k+1 \end{cases} \tag{†}$$

But this system can be solved for each $Q(\partial_I) \in T_yM$ in a *triangular* fashion, using the ordering $\prec$ on the basis, since we have that $P_1 : T_x\Sigma \to T_yM$ is Ω-regular. Indeed, it follows from Ω-regularity, that for any collection of independent vectors $\{v_1, \ldots, v_r\}$ in $T_x\Sigma$, the collection of 1-forms,

$$\iota_{P_1(v_i)} d\lambda^s|_{\mathcal{D}_y}, \quad 1 \le i \le r,\ 1 \le s \le p,$$

are independent. As $\mathcal{D}$ is given as the kernel of $\lambda^1, \ldots, \lambda^p$, we see that this is equivalent to the following non-vanishing condition:

$$\Big(\bigwedge_{s=1}^{p} \lambda^s\Big) \wedge \bigwedge_{i=1}^{r} \Big(\iota_{v_1} d\lambda^1 \wedge \ldots \wedge \iota_{v_i} d\lambda^s\Big) \ne 0.$$

But then clearly, at each stage of the triangular system, we have a full rank affine system. Consequently, the solution space for Q is contractible.

We have thus proved that $p^{\alpha+2}_{\alpha+1} : \bar{\mathcal{R}}_{\alpha+1}|_{(x,y)} \to \bar{\mathcal{R}}_{\alpha}|_{(x,y)}$ is indeed surjective, with contractible fiber. In fact, the algorithmic nature of the solution shows that, if $O \subset \Sigma$ is a contractible chart, then we are able to obtain the lift of any section of $\bar{\mathcal{R}}_\alpha|_O$ to $\bar{\mathcal{R}}_{\alpha+1}$, along $p^{\alpha+2}_{\alpha+1}$. This concludes the proof. □

Remark 3.3.3. In the above proof of Sublemma 3.3.2, the full strength of Ω-regularity of F has not been utilized. Note that, with our *choice* of the ordered basis of $T_x\Sigma$, the vector $P_1(\partial_{k+1})$ does not appear in the above triangular system $(\dagger)$. In fact, we can prove Lemma 3.2.1 only under the milder assumption that $\operatorname{Im} F$ contains a codimension one Ω-regular subspace, which in our case is the subspace $\langle F(\partial_1), \ldots, F(\partial_k)\rangle \subset T_x\Sigma$. In Chapter 6, we shall come back to this observation.

Chapter 4

K-contact and Horizontal Immersions in Fat Distributions

This chapter concerns with h-principle and existence of K-contact maps into 'degree 2' fat distributions and Quaternionic contact distributions for some specific K on the domain manifold. The main results of this chapter are Theorem 4.2.4, Theorem 4.2.17 and Theorem 4.2.26, which can be found in section 4.2. We first recall the preliminaries of fat distributions and introduce an invariant, called 'degree', for corank 2 fat distributions.

4.1 Fat Distributions and their Degree

We have already come across a class of fat distributions, namely contact distributions. It is also known that that holomorphic and quaternionic counterparts of contact structures are fat as well. The primary goal of this section is to identify a class of C^∞-distributions which are the real analogue of holomorphic contact structures. But before delving into these, we discuss some algebraic notions, which will become the backbone for the rest of this chapter. The terminology introduced in this linear algebraic interlude will be made clear later in the chapter.

4.1.1 Distributions from a Purely Linear Algebraic Viewpoint

Unless mentioned otherwise, by a tuple (D, E, Ω) we will mean that D, E are real vector spaces and $\Omega : \Lambda^2 D \to E$ is a linear map, interpreted as an E-valued linear 2-form on D. Given two such tuples (D_i, E_i, Ω_i) for $i = 1, 2$, we consider a morphism between them as a pair of *monomorphisms* $(F, G) : (D_1, E_1) \to (D_2, E_2)$ such that the following diagram is commutative.

$$\begin{tikzcd} \Lambda^2 D_1 \arrow[r, "\Omega_1"] \arrow[d, "\wedge^2 F"'] & E_1 \arrow[d, "G"] \\ \Lambda^2 D_2 \arrow[r, "\Omega_2"'] & E_2 \end{tikzcd}$$

In other words, $F^*\Omega_2 = G \circ \Omega_1$, where $F^*\Omega_2 = \Omega_2 \circ \wedge^2 F$.

As a special case, consider the tuple (D', E', Ω') such that $E' = 0$ which implies that Ω' is the 0 map. Then for a morphism $(F, G) : (D', E', \Omega') \to (D, E, \Omega)$ we must have that, $F^*\Omega = G \circ \Omega' = 0$. This captures the isotropy condition for formal horizontal maps.

Dualizing the map Ω, we get the canonical map $\omega : E^* \to \Lambda^2 D^*$. Now for any ordered basis $\mathcal{B} = (e_1, \ldots, e_p)$ of E, where $p = \dim E$, we can associate skew-symmetric bilinear forms ω^i on D, defined by, $\omega^i = \omega(e^i)$, where (e^i) is the dual basis for E^*. Note that Ω then have a representation $\Omega = \omega^i e_i$. In particular, we see that the span

$$\langle \omega^1, \ldots, \omega^p \rangle$$

of these 2-forms on D is a well-defined subspace of $\Lambda^2 D^*$, that only depends on Ω.

Given a tuple (D, E, Ω), we have a linear map,

$$\begin{aligned} \Omega_V : D &\to \hom(V, E) \\ x &\mapsto \iota_x \Omega|_V = \big(v \mapsto \Omega(x, v)\big) \end{aligned}$$

For any subspace $V \subset D$, we define Ω-*dual* of V by,

$$V^\Omega = \ker \Omega_V = \Big\{ x \in D \;\Big|\; \Omega(x, v) = 0, \text{ for all } v \in V \Big\}.$$

In particular, for $V = D$, D^Ω is the kernel of the two form Ω given as,

$$D^\Omega = \ker \Omega = \Big\{ x \in D \;\Big|\; \Omega(x, v) = 0, \text{ for all } v \in D \Big\}.$$

Definition 4.1.1. Given a tuple (D, E, Ω) as above, a subspace $V \subset D$ is called,

- Ω-*regular* if the linear map Ω_V is surjective.
- Ω-*isotropic* if $\Omega(u, v) = 0$ for all $u, v \in V$. Hence, V is Ω-isotropic if and only if $V \subset \ker \Omega_V$.

A morphism $(F, G) : (D', E', \Omega') \to (D, E, \Omega)$ is called Ω-regular if $F(D') \subset D$ is Ω-regular.

Proposition 4.1.2. *Given a tuple (D, E, Ω), a subspace $V \subset D$ is Ω-regular if and only if* $\operatorname{codim} V^{\Omega} = \dim E \times \dim V$.

Proof. It follows from the first isomorphism theorem that the map Ω_V is surjective if and only if $\operatorname{codim} \ker \Omega_V = \dim \hom(V, E) = \dim E \times \dim V$. This completes the proof since $V^{\Omega} = \ker \Omega_V$. $\square$

Fat Tuple

We now introduce the notion of a *fat* tuple, parallel to the notion of 1-fatness introduced in [Gro96, pg. 255].

Definition 4.1.3. A tuple (D, E, Ω) is called *fat* if for every non-zero $v \in D$, the 1-dimensional subspace $\langle v \rangle$ generated by v, is Ω-regular.

One immediate example of a fat tuple is given as $(D, \mathbb{R}, \omega)$, where D is a symplectic vector space, with a symplectic 2-from $\omega : \Lambda^2 D \to \mathbb{R}$. We should note that if (D, E, Ω) is fat then $\dim D$ must be even. In fact, if $\dim E \geq 2$, then $\dim D$ must be divisible by 4 (Theorem 4.1.22).

Let (D, E, Ω) be any tuple such that $\dim E = p$. Choosing a basis $(e_1, \ldots, e_p)$ of E, we may write,

$$\Omega = \sum \omega^i e_i.$$

If the tuple is fat, then it is easy to see that each $\omega^i : \Lambda^2 D \to \mathbb{R}$ is a *nondegenerate* 2-form on D. Furthermore, these 2-forms are linearly independent, i.e, for any linear combination $\sum_{i=1}^{p} c_i \omega^i = 0$, we must have $c_i = 0$ for $i = 1, \ldots, p$. Now for each $1 \leq i, j \leq p$, we have an automorphism $A^{ij} : D \to D$ defined by,

$$\omega^i(x, A^{ij} y) = \omega^j(x, y), \qquad \text{for all } x, y \in D.$$

We can easily observe that,

$$A^{ij} = I_{\omega^i}^{-1} \circ I_{\omega^j},$$

where, $I_{\omega_i} : D \to D^*$ is the isomorphism defined by $I_{\omega_i}(v) = \omega_i(v, \_)$. We will refer to A^{ij} as the *connecting automorphism* for the pair of nondegenerate forms (ω^i, ω^j). We now make some rather trivial observations.

Proposition 4.1.4. *The automorphisms A^{ij} satisfy the cocycle conditions, that is,*

$$A^{ij} A^{jk} = A^{ik}, \quad A^{ii} = Id, \quad \left(A^{ij}\right)^{-1} = A^{ji}, \quad \text{for any } 1 \leq i, j, k \leq p$$

Recall that, for any $V \subset D$, the ω^i-symplectic complement of V is defined as,

$$V^{\perp_i} = \{x \in D \big| \omega^i(x, y) = 0, \text{ for all } y \in V\}.$$

Therefore, the Ω-dual of a subspace $V \subset D$ can be expressed as,

$$V^\Omega = \bigcap_{s=1}^{p} V^{\perp_i}.$$

We observe that, though the individual sets on the right-hand side depends on the choice of a basis for E, their *intersection* is independent of the choice. Let us now observe how these complements are related to each other.

Proposition 4.1.5. *For any subspace $V \subset D$ the following holds,*

- $V^{\perp_j} = \big(A^{ij}V\big)^{\perp_i}$
- $V^{\perp_i} = A^{ij}\big(V^{\perp_j}\big)$

Proof. For $V \subset D$ we have,

$$\begin{aligned} V^{\perp_j} &= \{x \in D \big| \omega^j(x, y) = 0, \forall y \in V\} \\ &= \{x \in D \big| \omega^i(x, A^{ij}y) = 0, \forall y \in V\} \\ &= \{x \in D \big| \omega^i(x, z) = 0, \forall z \in A^{ij}V\} \\ &= \big(A^{ij}V\big)^{\perp_i} \end{aligned}$$

Similar argument gives us $V^{\perp_i} = A^{ij}\big(V^{\perp_j}\big)$. □

The next proposition justifies why one should look at these automorphisms in the first place. This was first observed in [Dat11] for the case $p = 2$.

Proposition 4.1.6. *Let (D, E, Ω) be a fat tuple. A subspace $V \subset D$ is Ω-regular, if and only if, for any fixed i_0, $1 \le i_0 \le p$, the sum*

$$\sum_{j=1}^{p} A^{i_0 j} V$$

is a direct sum for any representation $\Omega = (\omega^s)$.

Proof. With a fixed choice $\Omega = (\omega^1, \ldots, \omega^p)$ we have, by Proposition 4.1.5, that,

$$V^\Omega = \bigcap_{j=1}^{p} V^{\perp_j} = \bigcap_{j=1}^{p} \big(A^{i_0 j}\big)^{\perp_{i_0}} = \Big(\sum_{j=1}^{p} A^{i_0 j} V\Big)^{\perp_{i_0}}, \quad \text{for some fixed } 1 \le i_0 \le p.$$

Since ω^{i_0} is nondegenerate, $\operatorname{codim} V^\Omega = \dim\big(\sum A^{i_0 j}V\big)$. Now, by Proposition 4.1.2, $V \subset D$ is Ω-regular if and only if $\operatorname{codim} V^\Omega = \dim E \times \dim V = p \dim V$. Clearly, $\dim\big(\sum_{j=1}^{p} A^{i_0 j}V\big) = p \dim V$ precisely when the sum is a direct sum. Hence, the proof follows. $\square$

Since for a fat tuple (D, E, Ω) every 1-dimensional subspace of D is Ω-regular, we have the following corollary.

Corollary 4.1.7. *For any non-zero vector $v \in D$, and for any fixed i_0, the vectors $\{A^{i_0 j}v, 1 \le j \le p\}$ are linearly independent.*

We now focus on the case when $p = \dim E = 2$.

Degree of a Fat Tuple (D, E, Ω) with $\dim E = 2$

In what follows below, (D, E, Ω) will denote a fat tuple, where $\dim E = 2$. Any choice of ordered basis $\mathcal{B}$ of E defines an ordered pair of 2-forms (ω^1, ω^2) on D, which represents Ω, in turn defines the connecting automorphism $A : D \to D$ given by $\omega^1(x, Ay) = \omega^2(x, y)$, for all $x, y \in D$.

Observation 4.1.8. Since (D, E, Ω) is, in particular, fat and has $\dim E = 2$, we have the following.

1. A subspace $V \subset D$ is Ω-regular if and only if $V \cap AV = 0$ (follows from Proposition 4.1.6).

2. For every $0 \neq v \in D$, the tuple of vectors $\{v, Av\}$ are linearly independent (follows from Corollary 4.1.7). In other words, A has *no real eigenvalue*.

3. $V^{\perp_2} = \big(AV\big)^{\perp_1}$ and $V^{\perp_1} = A\big(V^{\perp_2}\big)$ (follows from Proposition 4.1.5).

Let $L(\Omega)$ be the set of all *connecting automorphisms* of the triple (D, E, Ω). Then,

Lemma 4.1.9. *For any $A, B \in L(\Omega)$, B can be written as a polynomial in A and vice versa.*

Proof. Suppose (ω^1, ω^2) and $(\hat{\omega}^1, \hat{\omega}^2)$ be two representatives of Ω and $A, B \in L(\Omega)$ be the respective connecting morphisms. We have already noted that,

$$\langle \omega^1, \omega^2 \rangle = \langle \hat{\omega}^1, \hat{\omega}^2 \rangle.$$

Hence we must have some $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in Gl_2(\mathbb{R})$, so that we may write,

$$\hat{\omega}^1 = p\omega^1 + q\omega^2, \quad \hat{\omega}^2 = r\omega^1 + s\omega^2.$$

We now relate A and B. For any $x, y \in D$ we have,

$$\begin{aligned}
&\hat{\omega}^1(x, By) = \hat{\omega}^2(x, y) \\
\Rightarrow &p\omega^1(x, By) + q\omega^2(x, By) = r\omega^1(x, y) + s\omega^2(x, y) \\
\Rightarrow &\omega^1(x, pBy) + \omega^1(x, qABy) = \omega^1(x, ry) + \omega^1(x, sAy) \\
\Rightarrow &\omega^1\big(x, (pB + qAB - rI + sA)y\big) = 0
\end{aligned}$$

Since this holds for every $x, y \in D$ and since ω^1 is nondegenerate, we get,

$$pB + qAB - rI + sA = 0 \Rightarrow (pI + qA)B = rI - sA.$$

As A has no real eigenvalue, we have that $\det(pI + qA) \neq 0$ and hence,

$$B = (pI + qA)^{-1}(rI - sA).$$

But now we can write B as a polynomial in A. Indeed, any operator C must satisfy its characteristic polynomial, say,

$$C^n + c_{n-1}C^{n-1} + \ldots + c_1 C + c_0 I = 0, \quad \text{where } n = \dim D$$

If C is nonsingular, then $c_0 \neq 0$ and C^{-1} is then written as a polynomial in C. Hence, $(pI + qA)^{-1}$ and therefore B can be written as a polynomial in A as well. □

Proposition 4.1.10. *For any $A, B \in L(\Omega)$, $\deg \mu_A = \deg \mu_B$, where μ_A and μ_B are minimal polynomials of A and B respectively.*

Proof. Recall that for any linear map $T : D \to D$, the degree of the minimal polynomial is the *maximal* integer d such that the vectors $\{v, Tv, \ldots, T^{d-1}v\}$ are linearly independent for some $v \in V$. Now suppose $S = \sum_{i=1}^{k} a_i T^i$ for some scalars $a_i \in \mathbb{R}$. Then for every $v \in V$ we have that,

$$S^i v \in \langle v, Tv, \ldots, T^{d-1}v\rangle, \text{for any } i \geq 0,$$

where $d = \deg \mu_T(X)$. But then $\dim\langle v, Sv, \ldots, S^i v\rangle$ is bounded above by d and hence $\deg \mu_S(X) \leq d = \deg \mu_T(X)$. This observation with the preceding lemma completes the proof. □

We now associate a numerical invariant to a fat tuple with $\dim E = 2$.

Definition 4.1.11. Given a fat *tuple* (D, E, Ω) with $\dim E = 2$, we define the *degree* of (D, E, Ω) as the degree of the minimal polynomial of A, where $A \in L(\Omega)$.

Observation 4.1.12. We observe a few immediate properties of degree.

- If $A : D \to D$ is an automorphism, then the minimal polynomial μ_A must be non-zero. Therefore, $\deg(\Omega)$ is non-zero.
- The fatness condition on (D, E, Ω) implies that the operator A does not have any real eigenvalue. Hence, the minimal polynomial cannot be of odd degree.
- Given a fixed nondegenerate 2-form ω on some vector space V, an operator $T : V \to V$ is called *skew-Hamiltonian* (for ω) if we have that $(x, y) \mapsto \omega(x, Ty)$ is again skew-symmetric. In particular, we see that a connecting automorphism A between (ω^1, ω^2), is skew-Hamiltonian, with respect to ω^1. Hence, it follows that the degree of the minimal polynomial of A is bounded above by $\frac{1}{2}\dim D$ ([Wat05]).

Proposition 4.1.13. *Let (D, E, Ω) be a fat tuple with $\dim E = 2$. Let $V \subset D$ be any subspace. Then for $A, B \in L(\Omega)$,*

$$AV = V \quad \text{if and only if} \quad BV = V.$$

Proof. Suppose $V \subset D$ satisfies $V = AV$. If $B : D \to D$ is any other connecting automorphism then by Lemma 4.1.9, B can be written as a polynomial in A. That is we have, $B = b_0 I + b_1 A + \ldots + b_l A^l$ for some scalars $b_i \in \mathbb{R}$. But then,

$$BV = \left(\sum b_i A^i\right) V \subset \sum A^i V \subset V \;\Rightarrow\; BV = V,$$

since B is an automorphism. □

This leads to the following definition.

Definition 4.1.14. A subspace $V \subset D$ is called *invariant* if $V = AV$ for some connecting automorphism $A \in L(\Omega)$.

Proposition 4.1.15. *Suppose $W \subset D$ is invariant. Then,*

$$W^{\Omega} = W^{\perp_1} \cap W^{\perp_2} = W^{\perp_1} = W^{\perp_2},$$

for any representation (ω^1, ω^2) of Ω. Furthermore, $\operatorname{codim} W^{\Omega} = \dim W$.

Proof. If $A : D \to D$ is a connecting morphism, then by Proposition 4.1.5, we have,

$$W^{\perp_2} = (AW)^{\perp_1} = W^{\perp_1},$$

since W is invariant. It then follows that, $W^\Omega = W^{\perp_1} \cap W^{\perp_2} = W^{\perp_1} = W^{\perp_2}$. Since ω^i is nondegenerate, we have $\operatorname{codim} W^\Omega = \operatorname{codim} W^{\perp_i} = \dim W$. This concludes the proof. □

Let us now look at the special case when the degree of the fat tuple is 2.

Degree 2 Fat Tuple

The first immediate observation about a degree 2 fat tuple is the following.

Proposition 4.1.16. *Let (D, E, Ω) be a corank 2 fat tuple of degree 2. Then we can choose a representation $\Omega = (\omega^1, \omega^2)$ so that the relating automorphism $B : D \to D$ satisfies $B^2 = -I$.*

Proof. Let $E = \langle e_1, e_w\rangle$ and $\Omega = \omega^1 e_1 + \omega^2 e_2$, so that $A : D \to D$ is the connecting automorphism between for the pair (ω^1, ω^2). By the hypothesis, A satisfies $A^2 = \lambda A + \mu I$ for some scalars $\lambda, \mu \in \mathbb{R}$, such that $\lambda^2 + 4\mu < 0$. Now define,

$$p = -\lambda\sqrt{\frac{-1}{\lambda^2 + 4\mu}}, \quad q = -2\sqrt{\frac{-1}{\lambda^2 + 4\mu}}.$$

Then an easy computation shows that for $pI + qA$, we have $(pI + qA)^2 = -I$. Next consider the basis, $\{\hat{e}_1, \hat{e}_2\}$ of E given by,

$$\hat{e}_1 = qe_1, \quad \hat{e}_2 = pe_1 + e_2.$$

We write, $\Omega = \hat{\omega}^1\hat{e}_1 + \hat{\omega}^2\hat{e}_2$ and denote the connecting automorphism $B : D \to D$ between the pair $\hat{\omega}^1, \hat{\omega}^2$. Then we see that, $B = pI + qA$ and hence $B^2 = -I$. □

Next, let us list a few interesting properties of a degree 2 fat tuple.

Proposition 4.1.17. *Let (D, E, Ω) be a degree 2 fat tuple. Suppose $\Omega = (\omega^1, \omega^2)$ and $A : D \to D$ is the connecting automorphism. Then we have the following :*

1. *For any $V \subset D$, the subspace $V + AV$ is invariant.*
2. *For any $V \subset D$, we have $V^\Omega = (V + AV)^{\perp_1} = (V + AV)^{\perp_2} = (V + AV)^\Omega$.*
3. *For any $V \subset D$, the subspace $V + AV$ is independent of the choice of $A \in L(\Omega)$. Indeed, we have, $V + AV = (V^\Omega)^\Omega$.*

Proof. We have,

1. Since A has degree 2 minimal polynomial, let us assume that $A^2 = \lambda A + \mu I$ for some scalars $\lambda, \mu \in \mathbb{R}$. Now for any $V \subset D$ we have,

$$A(V + AV) = AV + A^2V = AV + (\lambda A + \mu I)V \subset AV + \lambda AV + \mu V \subset V + AV.$$

   Since A is an automorphism, we have $A(V + AV) = V + AV$. Thus $V + AV$ is invariant.

2. Since for any $V \subset D$, $V + AV$ is invariant, the claim follows by Proposition 4.1.15.

3. For any $V \subset D$ we have,

$$\begin{aligned}(V^\Omega)^\Omega &= \left(V^\Omega\right)^{\perp_1} \cap \left(V^\Omega\right)^{\perp_2} \\ &= \left((V + AV)^{\perp_1}\right)^{\perp_1} \cap \left((V + AV)^{\perp_2}\right)^{\perp_2} \\ &= (V + AV) \cap (V + AV) \\ &= V + AV\end{aligned}$$

   The claim then follows. □

The following proposition is interesting in its own right, as it characterizes invariant subspaces in terms of Ω for degree 2 fat tuples. However we shall not have any occasion to use it.

Proposition 4.1.18. *Let (D, E, Ω) be a degree 2 fat tuple. Then for any subspace $V \subset D$ the following hold :*

1. *V^Ω is invariant.*

2. *V is invariant if and only $V = V^{\Omega\Omega}$.*

Proof. We have,

1. For any $V \subset D$, we have $V^\Omega = (V + AV)^{\perp_1} = (V + AV)^{\perp_2}$, where $A \in L(\Omega)$. Hence,

$$AV^\Omega = A\left((V + AV)^{\perp_2}\right) = (V + AV)^{\perp_1} = V^\Omega.$$

   Thus V^Ω is invariant.

2. For any $V \subset D$, we have seen that $(V^\Omega)^\Omega = V + AV$. Then clearly, V is A-invariant if and only if $(V^\Omega)^\Omega = V$. □

Thus, we may redefine the notion of invariance of a subset $V \subset D$, for a degree 2 fat tuple, in view of Proposition 4.1.18(2).

The importance of the following proposition will be clear in the next sections.

Proposition 4.1.19 (Extension Property)**.** *Suppose (D, E, Ω) is a degree 2 fat tuple. For any Ω-regular subspace $V \subset D$, and for any*

$$\tau \notin \left(V^{\Omega}\right)^{\Omega},$$

we have that the subspace $V' = V + \langle\tau\rangle$ is Ω-regular.

Proof. Let $A \in L(\Omega)$. Recall that $V \subset D$ is Ω-regular if and only if $V + AV$ is a direct sum. Let $\tau \notin (V^{\Omega})^{\Omega} = V + AV$ and set, $V' = V + \langle\tau\rangle$. We need to show that $V' + AV'$ is a direct sum. If not, then $A\tau \in V + AV + \langle\tau\rangle$. This implies that,

$$(V + AV) \cap \langle\tau, A\tau\rangle = \langle A\tau\rangle.$$

But then $\langle A\tau\rangle$ is invariant, which contradicts that A has no real eigenvalue. Hence V' is regular. □

4.1.2 Fat Distributions

Definition 4.1.20. A distribution $\mathcal{D} \subset TM$ is called *fat* if for each $x \in M$, the tuple $\left(\mathcal{D}_x, TM/\mathcal{D}|_x, \Omega|_x\right)$ is a *fat tuple*, as defined in Definition 4.1.3.

In other words, every 1-dimensional subspace $\langle v\rangle$ in $\mathcal{D}_x$ is Ω-regular, where Ω is the curvature form of $\mathcal{D}$.

Example 4.1.21. Using the local framing we can easily see that contact structures are corank 1 fat distributions (Theorem 2.1.7). The converse is true as well, i.e, every corank 1 fat distribution is a contact structure.

Before discussing fat distributions of corank ≥ 2, let us first emphasize that fat distributions are not quite plentiful, or *generic*, even though fatness is an open condition. The next theorem gives us some numerical constraints for the existence of fat distributions.

Theorem 4.1.22 ([Ray68])**.** *If $\mathcal{D} \subset TM$ is a fat distribution of rank n and corank p, then we must have the following :*

- *n is even and $n \geq p + 1$.*

- *If $p \geq 2$, then 4 divides n.*
- *The $n-1$-sphere S^{n-1} admits p-many linearly independent vector fields.*

Conversely, for any pair of integers (n, p) satisfying the above, there is a fat distribution $\mathcal{D}$ on $\mathbb{R}^{n+p}$ of corank p.

Remark 4.1.23 (Constructing a Fat Distribution)**.** The proof of the above theorem can be found in [Mon02, pg. 71]. We discuss the converse statement which is of some interest in the present thesis. Suppose there is a pair of integers (n, p) such that S^{n-1} admits p-many linearly independent vector fields. Then it is well-known (see [Hus94]) that there are p-many linear maps $J^a : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the Clifford relations, namely,

$$J^a J^b + J^b J^a = -2\delta_{a,b} I, \quad \text{for } 1 \leq a, b \leq p.$$

Now consider $\mathbb{R}^{n+p} = \mathbb{R}^n \times \mathbb{R}^p$ with the coordinates $\{x^1, \ldots, x^n, z^1, \ldots, z^p\}$ and define the 1-forms λ^a on $\mathbb{R}^{n+p}$ by,

$$\lambda^a = dz^a - \sum_{i,j=1}^{n} J^a_{ij} x^i dx^j, \quad \text{for } a = 1, \ldots, p,$$

where we have $J^a_{ij} = J^a(\partial_{x_i}, \partial_{x_j})$. It may be verified that $\mathcal{D} = \bigcap_{s=1}^{p} \ker \lambda^s$ is a fat distribution on $\mathbb{R}^{n+p}$ of corank p.

We shall now discuss some properties of fat distributions. First, let us observe some equivalent criterion for a distribution to be fat.

Proposition 4.1.24. *Given a distribution $\mathcal{D} \subset TM$, the following are equivalent.*

1. *$\mathcal{D}$ is a fat distribution*
2. *Any non-zero local section $X \in \mathcal{D}$ defined on a neighborhood of $x \in M$, Lie bracket generates the tangent space $T_x M$ in 1-step, i.e,*

$$T_x M = \mathcal{D}_x + [X, \mathcal{D}]_x, \text{ for any } 0 \neq X \in \mathcal{D} \text{ about } x \in M.$$

3. *For any nonvanishing $\alpha \in \operatorname{Ann}(\mathcal{D})$ the 2-form $\omega(\alpha)$ is nondegenerate, where ω is the dual curvature map.*

Proof. Let us first show that 1 $\Leftrightarrow$ 2. Suppose $\mathcal{D}$ is fat. Then for any $0 \neq X \in \mathcal{D}_x$, the 1-dimensional subspace $\langle X \rangle$ is Ω-regular, i.e, the map $\mathcal{D}_y \ni Y \mapsto \Omega(X, Y) \in TM/\mathcal{D}|_x$ is surjective. Recall that the curvature form Ω is given as,

$$\Omega(X,Y) = -[\tilde{X},\tilde{Y}]_x \mod \mathcal{D}_x, \text{ for local extensions } \tilde{X},\tilde{Y} \in \mathcal{D} \text{ of } X,Y \in \mathcal{D}_x \text{ respectively.}$$

Now for any choice of extension $\tilde{X} \in \mathcal{D}$ of X we have,

$$[\tilde{X},\mathcal{D}]_x \mod \mathcal{D}_x = TM/\mathcal{D}|_x \Rightarrow \mathcal{D}_x + [\tilde{X},\mathcal{D}]_x = T_xM.$$

That is, $\tilde{X}$ Lie bracket generates T_xM in 1-step. The converse is true for the same reason.

Now we prove, 2 $\Leftrightarrow$ 3. Recall that the dual curvature form of $\mathcal{D}$ is given by the map,

$$\begin{aligned} \omega : \operatorname{Ann}(\mathcal{D}) &\to \Lambda^2\mathcal{D}^* \\ \alpha &\mapsto d\alpha|_{\mathcal{D}} \end{aligned}$$

which is in fact a bundle map. For any $X,Y \in \mathcal{D}_x$ and $\alpha \in \operatorname{Ann}\mathcal{D}_x$, respectively choose some arbitrary *local* extensions $\tilde{X},\tilde{Y} \in \mathcal{D}$ and $\tilde{\alpha} \in \operatorname{Ann}\mathcal{D}$ about x. We then have,

$$\omega(\alpha)(X,Y) = d\tilde{\alpha}(\tilde{X},\tilde{Y})|_x = -\tilde{\alpha}([\tilde{X},\tilde{Y}])|_x = -\alpha([\tilde{X},\tilde{Y}]|_x).$$

Now assume that $\mathcal{D}$ is fat. If possible, suppose for some $0 \neq \alpha \in \operatorname{Ann}(\mathcal{D})$ at the point $x \in M$, the 2-from $\omega(\alpha)$ is degenerate. Then, in particular, there exists a nonzero vector $X \in \mathcal{D}_x$ such that $\omega(\alpha)(X,Y) = 0$ for any $Y \in \mathcal{D}_x$. But then we have,

$$0 = d\alpha(X,Y) = -\alpha([\tilde{X},\tilde{Y}]_x), \quad \text{for local extension } \tilde{X},\tilde{Y} \in \mathcal{D} \text{ of } X,Y \text{ about } x,$$

and consequently we get,

$$[\tilde{X},\mathcal{D}]_x \subset \ker\alpha \;\Rightarrow\; \mathcal{D}_x + [\tilde{X},\mathcal{D}]_x \subset \ker\alpha \subsetneq T_xM.$$

That is, the nonzero field $\tilde{X} \in \mathcal{D}$ fails to bracket generate TM at the point x, which is a contradiction. Hence $\omega(\alpha)$ is nondegenerate. Since $\alpha \in \operatorname{Ann}\mathcal{D}_x$ is arbitrary, this proves the claim.

To prove the converse, if possible, suppose there is some local field $0 \neq X \in \mathcal{D}$ about $x \in M$, which fails to bracket generate T_xM. Then in particular we have that, $E_x = \mathcal{D}_x + [X,\mathcal{D}]_x \subset T_xM$ has positive codimension. Then we can get a local 1-form $0 \neq \alpha \in \operatorname{Ann}(\mathcal{D})$ such that

α_x vanishes over E_x. But then,

$$\omega(\alpha)(X,Y) = d\alpha(X,Y) = -\alpha([X,Y]_x) = 0, \quad \text{for any local section } Y \in \mathcal{D} \text{ about } x.$$

This contradicts the fact that $\omega(\alpha)$ is nondegenerate at x. Hence X bracket generates TM at x. Since $x \in M$ is arbitrary, this concludes the proof. □

Remark 4.1.25. In view of Proposition 4.1.24(2), fat distributions are also known as *strongly bracket generating* distributions.

As a direct consequence of Proposition 4.1.24(3), we get the following.

Corollary 4.1.26. *A corank 1 distribution $\xi \subset TM$ on M is fat if and only if ξ is a contact structure.*

Holomorphic Contact Structure

This is a direct holomorphic analogue of the contact structure.

Definition 4.1.27. Given a complex manifold M of (complex) dimension $2n+1$, a *holomorphic contact structure* is a corank 1 complex subbundle Ξ of the holomorphic tangent bundle $T^{(1,0)}M$, which is locally given as the kernel of some holomorphic 1-form α satisfying $\alpha \wedge (d\alpha)^n \neq 0$.

The standard example of a holomorphic contact structure on $\mathbb{C}^{2n+1}$ is given as,

$$\Xi = \ker\Big(dz - \sum_{i=1}^{n} y_i dx^i\Big),$$

where $\{z, x^i, y_i\}$ are the standard complex coordinates. Just as we saw in Theorem 2.1.7 for the contact structures, we have a holomorphic Darboux theorem.

Theorem 4.1.28 ([AFL17])**.** *Every holomorphic contact structure Ξ on a complex manifold of dimension $2n+1$, is locally (biholomorphically) equivalent to the standard holomorphic contact structure on $\mathbb{C}^{2n+1}$.*

Recall that the complex manifold M comes equipped with an *integrable* complex structure $J : TM \to TM$. The real tangent bundle TM is canonically isomorphic with the holomorphic tangent bundle $T^{(1,0)}M \subset TM_{\mathbb{C}} = TM \otimes \mathbb{C}$, by the map, $X \mapsto X + \iota JX$. Here ι is the complex structure on the complex vector bundle $TM_{\mathbb{C}}$. Hence a holomorphic contact structure Ξ on a complex manifold M, can be identified with a *real* corank 2 subbundle $\mathcal{D} \subset TM$. In other words, $\mathcal{D}$ is the underlying real bundle to Ξ.

Corollary 4.1.29. *The underlying real bundle $\mathcal{D} \subset TM$ of a holomorphic contact structure $\Xi \subset T^{(1,0)}M$ on a complex manifold M is a corank 2 fat distribution.*

Proof. By the complex Darboux theorem (Theorem 4.1.28), Ξ is locally given as the kernel of the 1-form $\alpha = dz - \sum_{j=1}^n y_j dx_j$, where $\{z, y_j, x_j\}$ are some complex coordinates. Writing $z = z_1 + \iota z_2$ etc, we have a real coordinate system $\{z_1, z_2, x_{j1}, x_{j2}, y_{j1}, y_{j2}\}$ on M. Writing, $\alpha = \alpha_1 + \iota\alpha_2$, where α_i are (real) 1-forms on M, we then have,

$$\alpha_1 = dz_1 - \sum_{j=1}^n \left(y_{j1}dx_{j1} - y_{j2}dx_{j2}\right), \qquad \alpha_2 = dz_2 - \sum_{j=1}^n \left(y_{j2}dx_{j1} + y_{j1}dx_{j2}\right).$$

Then the underlying real bundle, $\mathcal{D} \underset{loc.}{=} \ker\alpha_1 \cap \ker\alpha_2$ is further described as follows :

$$\mathcal{D} \underset{loc.}{=} \left\langle \partial_{y_{j1}},\ \partial_{y_{j2}},\ \partial_{x_{j1}} + y_{j1}\partial_{z_1} + y_{j2}\partial_{z_2},\ \partial_{x_{j2}} - y_{j2}\partial_{z_1} + y_{j2}\partial_{z_2} \right\rangle.$$

Since $TM \underset{loc.}{=} \mathcal{D} \oplus \langle \partial_{z_1}, \partial_{z_2}\rangle$, an easy computation shows that each vector in the above frame Lie bracket generates TM in 1-step. Thus $\mathcal{D}$ is indeed a corank 2 fat distribution. $\square$

We now observe that,

Proposition 4.1.30. *Let $\mathcal{D}$ be the underlying real distribution of a holomorphic contact structure Ξ on M and Ω be the curvature form of $\mathcal{D}$. Then a $\mathcal{D}$-horizontal immersion $u : \Sigma \to M$ is Ω-regular if and only if u is a totally real immersion.*

Let us recall the definition.

Definition 4.1.31. Given a complex manifold (M, J), where $J : TM \to TM$ is the almost complex structure, a *totally real submanifold* is a real submanifold $N \subset M$ such that $TN \cap J(TN) = 0$. More generally, a smooth immersion $u : \Sigma \to M$ is said to be *totally real immersion* if $\operatorname{Im} du \cap J(\operatorname{Im} du) = 0$.

Proof of Proposition 4.1.30. Since Ξ is a complex vector bundle, the underlying real vector bundle $\mathcal{D}$ must be J-invariant. Now, we have observed, $\mathcal{D} \underset{loc.}{=} \ker\alpha_1 \cap \ker\alpha_2$, where

$$\alpha_1 = dz_1 - \sum_{j=1}^n \left(y_{j1}dx_{j1} - y_{j2}dx_{j2}\right), \qquad \alpha_2 = dz_2 - \sum_{j=1}^n \left(y_{j2}dx_{j1} + y_{j1}dx_{j2}\right).$$

An easy computation then gives us that, $d\alpha_1(X, JY) = -d\alpha_2(X, Y)$ for any $X, Y \in \mathcal{D}$. Consequently we see that the connecting automorphism $A : \mathcal{D} \to \mathcal{D}$ for the tuple (ω^1, ω^2), where $\omega^i = d\alpha_i|_{\mathcal{D}}$, is given as $A = -J|_{\mathcal{D}}$. Since $\mathcal{D}$ is fat by Corollary 4.1.29, the proof is immediate from Observation 4.1.8 (1). $\square$

Quaternionic Contact Structure

Quaternionic contact structures, as introduced by Biquard in [Biq99], are generalization of the contact structures in the quaternionic setup. Before going into the definition, let us motivate the nomenclature.

Unlike in the case of holomorphic contact structures, the quaternionic counterpart, cannot be defined simply by replacing the base field with the Quaternions, as they are noncommutative. In order to understand what could be a possible way to generalize, we look at the contact structures from an algebraic point of view. Recall the *real* Heisenberg Lie algebra structure on $\mathbb{C}^n \oplus \mathbb{R}$, where the Lie bracket of two vectors in $\mathbb{C}^n$ is given by,

$$[(x_1, \ldots, x_n), (y_1, \ldots, y_n)] = \operatorname{Im} \sum_{i=1}^{n} \bar{x}_i y_i, \quad \text{for } (x_j), (y_j) \in \mathbb{C}^n$$

and $[\mathbb{R}, \mathbb{C}^n] = 0$. For a given contact structure $\xi = \ker \alpha$ on a manifold of dimension $2n+1$, we have that $\Omega = d\alpha|_\xi$ is a nondegenerate 2-form and we are able to get an almost complex structure $J : \xi \to \xi$, compatible with Ω, i.e, $\Omega(JX, JY) = \Omega(X, Y)$ for all $X, Y \in \xi$. Then for each $x \in M$, the vector space $\xi_x \oplus T\Sigma/\xi|_x$, equipped with the Lie bracket,

$$[X, Y] = \Omega(X, Y) \text{ and } [X, Z] = 0 \text{ for } X, Y \in \xi_x, Z \in T\Sigma/\xi|_x$$

given by the curvature form, is isomorphic (as Lie algebras) to the Heisenberg algebra $\mathbb{C}^n \oplus \mathbb{R}$.

Now following this approach, we have the *quaternionic* Heisenberg Lie algebra, where the underlying vector space is given to be, $\mathbb{H}^n \oplus \operatorname{Im} \mathbb{H}$, with the Lie bracket defined for a pair of quaternionic vectors in this case. A corank 3 distribution $\mathcal{D} \subset TM$ on a manifold M of dimension $4n+3$ is called quaternionic contact if the vector space $\mathcal{D}_x \oplus T_x M/\mathcal{D}_x$, equipped with the Lie bracket given by the curvature from, is isomorphic (as Lie algebras) to the quaternionic Heisenberg Lie algebra $\mathbb{H}^n \oplus \operatorname{Im} \mathbb{H}$.

Remark 4.1.32. Given a distribution $\mathcal{D} \subset TM$, this associated Lie algebra, $\mathcal{D}_x \oplus TM/\mathcal{D}|_x$, defined via the the Lie bracket of vector fields, is known as the *nilpotentization* of the distribution $\mathcal{D}$ at the point $x \in M$ ([Tan70, Mon02]).

Formally we define,

Definition 4.1.33. A *quaternionic contact structure* on a manifold M of dimension $4n+3$ is a corank 3 distribution $\mathcal{D} \subset TM$, given locally as the common kernel of 1-forms $(\lambda^1, \lambda^2, \lambda^3) \in \Omega^1(M, \mathbb{R}^3)$ such that there exists a Riemannian metric g on $\mathcal{D}$ and a Quaternionic structure

$(J_i, i = 1,2,3)$ on $\mathcal{D}$ satisfying,

$$d\lambda^i|_{\mathcal{D}} = g(J_{i}\_, \_).$$

By a Quaternionic structure we mean that J_i are (local) endomorphisms which satisfy the quaternionic relations: $J_1^2 = J_2^2 = J_3^2 = -1 = J_1J_2J_3$. Equivalently, there exist an S^2-bundle $Q \to M$ of triples of almost complex structures (J_1, J_2, J_3) on $\mathcal{D}$.

Quaternionic contact structures are interesting in themselves and they appear in many different contexts [BG08, IV10]. Let us first give one well-known example.

Example 4.1.34. Consider the unit sphere $S^{4n+3} \subset \mathbb{R}^{4n+4} \cong \mathbb{H}^{n+1}$. For each $x \in S^{4n+3}$, consider the quaternionic subspace of $\mathbb{H}^{n+1}$, orthogonal to x, namely,

$$\langle x\rangle^{\perp} = \{y \in \mathbb{H}^{n+1} \mid \langle x, y\rangle = 0\},$$

with respect to the inner product $\langle x, y\rangle = \sum x_i^{\dagger} y_i$ for $x = (x_i), y = (y_i) \in \mathbb{H}^n$. Here $(\cdot)^{\dagger}$ denotes the *quaternionic conjugate*. Now, this subspace has *real* dimension $4n$. We get a corank 3 distribution $\mathcal{D}$ on S^{4n+3} given as,

$$\mathcal{D}_x = T_x S^{4n+3} \cap \langle x\rangle^{\perp}, \quad x \in S^{4n+3}.$$

One can easily check that $\mathcal{D}$ is indeed a quaternionic contact structure.

Just as in the cases of contact and holomorphic contact structures, we have the following.

Proposition 4.1.35. *A quaternionic contact structure is a (corank 3) fat distribution.*

Proof. Let $\mathcal{D}$ be a quaternionic contact structure on a manifold, equipped with a Riemannian metric g on $\mathcal{D}$. Consider some local 1-forms $\lambda^i, i = 1,2,3$ defining $\mathcal{D}$ such that $d\lambda^i|_{\mathcal{D}}$ satisfies the relation, $d\lambda^i|_{\mathcal{D}} = g(J_{i}\_, \_)$, where $\{J_i \quad i = 1,2,3\}$ abide by the quaternionic relations. Now, for any tuple (p^1, p^2, p^3) we have,

$$\left(\sum p^i J_i\right)^2 = -\sum (p^i)^2 \mathrm{Id},$$

and hence $\sum p^i J_i$ is invertible whenever $(p^1, p^2, p^3) \neq 0$. But then for any such tuple we have that,

$$\sum p^i d\lambda^i|_{\mathcal{D}} = \sum p^i g(J_{i}\_, \_) = g(J\_, \_), \quad \text{where } J = \sum p^i J_i.$$

Since J is an automorphism and g is a Riemannian metric, $g(J\_, \_)$ is indeed nondegenerate. Now any local 1-form $0 \neq \alpha \in \operatorname{Ann}\mathcal{D}$ can be expressed as $\alpha = \sum p^i \lambda^i$, for some non-zero tuple (p^1, p^2, p^3). Therefor, $d\alpha|_{\mathcal{D}} = \sum p^i(x) d\lambda^i|_{\mathcal{D}}$ is a nondegenerate 2-form. Hence, $\mathcal{D}$ is fat by Proposition 4.1.24(3). $\square$

4.1.3 Degree of a Corank 2 Fat Distribution

Let $\mathcal{D} \subset TM$ be a corank 2 fat distribution with curvature form $\Omega : \Lambda^2\mathcal{D} \to TM/\mathcal{D}$. Choosing some trivialization of $TM/\mathcal{D}$ over some $U \subset M$, we can write, $\Omega|_U = (\omega^1, \omega^2)$, where ω^i are 2-forms on $\mathcal{D}|_U$. Since $\mathcal{D}$ is fat, by Proposition 4.1.24(3), we have that $\omega^i : \Lambda^2\mathcal{D}|_U \to \mathbb{R}$ are nondegenerate 2-forms. In particular, we can define a *local* automorphism $A : \mathcal{D}|_U \to \mathcal{D}|_U$ given as,

$$\omega^1(u, Av) = \omega^2(u, v), \quad \text{for any } u, v \in \mathcal{D}|_U.$$

As a consequence of Proposition 4.1.10, for any $x \in U$, the degree of the automorphism $A_x : \mathcal{D}_x \to \mathcal{D}_x$ is independent of our choice of trivialization of $TM/\mathcal{D}$. In particular, we can now define the notion of degree for fat distribution.

Given a corank 2 fat distribution $\mathcal{D} \subset TM$, we define $\deg(x, \mathcal{D})$ as the degree of the minimal polynomial of $A_x : \mathcal{D}_x \to \mathcal{D}_x$.

Proposition 4.1.36. *Given a corank 2 fat distribution $\mathcal{D} \subset TM$, the map $x \mapsto \deg(x, \mathcal{D})$ is lower-semicontinuous.*

Proof. Suppose at some $x \in M$, we have $d = \deg(x, \mathcal{D})$. Choose some trivialization of $\Omega \underset{loc.}{=} (\omega^1, \omega^2)$ on a neighborhood $x \in U \subset M$ and get the local automorphism $A : \mathcal{D}|_U \to \mathcal{D}|_U$. Choose some local nonvanishing section $X \in \mathcal{D}|_U$. Then, for $v = X_x$, we have $\{v, Av, \ldots, A^{d-1}v\}$ are linearly independent. But then, $\{V, AV, \ldots, A^{d-1}V\}$ must also be linearly independent on some open neighborhood $U' \subset U$ of x. Now, for any $y \in U'$ we have that A_y has minimal polynomial of degree at least d. This proves that $x \mapsto \deg(x, \mathcal{D})$ is a lower-semicontinuous map. $\square$

Definition 4.1.37. A corank 2 fat distribution $\mathcal{D} \subset TM$ is said to be of degree d, if $\deg(x, \mathcal{D}) = d$ for each $x \in M$.

Example 4.1.38. The underlying real distribution $\mathcal{D}$ of a holomorphic contact structure Ξ on a complex manifold of dimension $2n + 1$, is a degree 2 fat distribution.

Example 4.1.39. Any corank 2 fat distribution $\mathcal{D}$ on a manifold of dimension 6 is of degree 2, since $\deg(x, \mathcal{D})$ must be a non-zero even number and also $\deg(x, \mathcal{D}) \leq \frac{1}{2} \dim \mathcal{D}_x = 2$, at each $x \in M$ (Observation 4.1.12).

However, there are fat distributions of type $(4,6)$ which are not equivalent to holomorphic contact. We shall revisit this in a later chapter.

Example 4.1.40. We will now give an example of a fat distribution on $\mathbb{R}^{10}$, with degree 4. Consider, $\mathbb{R}^{10}$ with coordinates, $\{z, w, x_i, y_i, 1 \le i \le 4\}$. Take the 1-forms,

$$\alpha_1 = dz - \sum_{i=1}^{4} y_i dx_i, \quad \alpha_2 = dw - \big(y_1 dx_2 - y_2 dx_1\big) - 2\big(y_3 dx_4 - y_4 dx_3\big).$$

Clearly these are independent forms, and therefore $\mathcal{D} = \ker \alpha_1 \cap \ker \alpha_2$ is a corank 2 distribution. A framing $\langle Y_i, X_i \rangle$ of $\mathcal{D}$ can be given as follows :

$$\begin{aligned} X_1 &= \partial_{x_1} + y_1\partial_z - y_2\partial_w, & X_2 &= \partial_{x_2} + y_2\partial_z + y_1\partial_w, \\ X_3 &= \partial_{x_3} + y_3\partial_z - 2y_4\partial_w, & X_4 &= \partial_{x_4} + y_4\partial_z + 2y_3\partial_w, \end{aligned}$$
$$Y_i = \partial_{y_i}, \text{ for } i = 1, \ldots, 4.$$

Note that,

$$d\alpha_1 = \sum_{i=1} dx_i \wedge dy_i, \qquad d\alpha_2 = \big(dx_2 \wedge dy_1 - dx_1 \wedge dy_2\big) + 2\big(dx_4 \wedge dy_3 - dx_3 \wedge dy_4\big).$$

It is easy to see that $d\alpha_i|_{\mathcal{D}}$ are indeed nondegenerate. Let us consider the automorphism $A : \mathcal{D} \to \mathcal{D}$ given by, $d\alpha_2(x, y) = d\alpha_1(x, Ay)$ for any $x, y \in \mathcal{D}$. The action of A on the framing is then given by,

$$AX_1 = -X_2,\ AX_2 = X_1,\ AX_3 = -2X_4,\ AX_4 = 2X_3,$$
$$AY_1 = Y_2,\ AY_2 = -Y_1,\ AY_3 = 2Y_4,\ AY_4 = -2Y_3.$$

We can check that the minimal polynomial of this operator A is $(T^2+1)(T^2+4)$. Thus, $\mathcal{D} \subset TM$ is a degree 4 fat distribution. Consequently, germ of $\mathcal{D}$ is *not* equivalent to a holomorphic contact distribution.

4.2 h-Principle for Immersions into Fat Distributions

In this section we shall obtain some new applications of h-principle of K-contact maps to conclude existence of horizontal immersions in degree 2 fat and Quaternionic contact distributions. Furthermore, we shall prove the existence of K-contact maps into degree 2 fat distribution, where K is a contact structure on Σ. The proofs are based on the contents of Chapter 3.

4.2.1 Horizontal Immersions into Degree 2 Fat Distribution

Given $(M, \mathcal{D})$, recall from Definition 3.1.7, we have the relation $\mathcal{R}^{\text{Hor}} \subset J^1(\Sigma, M)$. Sections of $\mathcal{R}^{\text{Hor}}$ are monomorphisms $F : T\Sigma \to u^*TM$, covering some $u : \Sigma \to M$ such that $F(\mathcal{D}_\sigma) \subset \mathcal{D}_{u(\sigma)}$. Furthermore, F is Ω-regular and is Ω-isotropic, i.e, $F^*\Omega = 0$. Here Ω is the curvature for of $\mathcal{D}$.

We prove the following.

Theorem 4.2.1. *Suppose $\mathcal{D} \subset TM$ is a degree 2 fat distribution on a manifold M and Σ is an arbitrary manifold. Then $\mathcal{R}^{\text{Hor}}$ satisfies the C^0-dense h-principle provided,*

$$\operatorname{rk} \mathcal{D} \geq 4 \dim \Sigma + 4.$$

As seen in section 3.2, we have the relation $\tilde{\mathcal{R}}^{\text{Hor}} \subset J^1(\tilde{\Sigma}, M)$ associated to Ω-regular, $\mathcal{D}$-horizontal immersions of $\tilde{\Sigma} = \Sigma \times \mathbb{R}$ into M, which admits a natural morphism $ev : \tilde{\mathcal{R}}^{\text{Hor}}|_{\Sigma \times 0} \to \mathcal{R}^{\text{Hor}}$, induced by the restriction map $C^\infty(\tilde{\Sigma}, M) \to C^\infty(\Sigma, M)$. We prove the following.

Lemma 4.2.2. *Suppose $\mathcal{D}$ is a degree 2 fat distribution on M. If $\operatorname{rk} \mathcal{D} \geq 4 \dim \Sigma + 4$, then $\tilde{\mathcal{R}}^{\text{Hor}}$ is an extension of $\mathcal{R}^{\text{Hor}}$, i.e, $ev : \tilde{\mathcal{R}}^{\text{Hor}}|_{\Sigma \times 0} \to \mathcal{R}^{\text{Hor}}$ is locally surjective.*

Proof. Let (x, y, F) represent a jet in $\mathcal{R}^{\text{Hor}}$. Then $V = \text{Im } F$ is an Ω-regular, Ω-isotropic subspace of $\mathcal{D}_y$. Since V is Ω-isotropic, i.e, $V \subset V^\Omega$, we have $V^{\Omega\Omega} \subset V^\Omega$. Now, we use the hypothesis that $\mathcal{D}$ is degree 2 fat; in particular $(\mathcal{D}_y, TM/\mathcal{D}|_y, \Omega_y)$ is a fat tuple (Definition 4.1.3). Since V is Ω-regular, it follows from Proposition 4.1.17(3) and Observation 4.1.8(1), that,

$$\dim V^{\Omega\Omega} = 2 \dim V.$$

Also, from Definition 4.1.1, we have,

$$\operatorname{codim} V^\Omega = 2 \dim V.$$

Since $\operatorname{rk} \mathcal{D} \geq 4 \dim \Sigma + 4$ by hypothesis, it follows that the codimension of $V^{\Omega\Omega}$ in V^Ω is ≥ 4. Now, for any non-zero $\tau \in V^\Omega \setminus V^{\Omega\Omega}$, the subspace $V' = V \oplus \langle \tau \rangle$ is an Ω-regular subspace by Proposition 4.1.19. Clearly V' is Ω-isotropic. We can define a linear map $\tilde{F} : T_x\Sigma \times \mathbb{R} \to T_yM$ as follows:

$$\tilde{F}(v, t) = F(v) + t\tau, \quad \text{for all } v \in T_x\Sigma \text{ and } t \in \mathbb{R}.$$

Now suppose $(F, u) : T\Sigma \to TM$ is a bundle map representing a section of $\mathcal{R}^{\text{Hor}}$, with $u = \operatorname{bs} F : \Sigma \to M$ being the base map of F. Denote $V_x = \operatorname{Im} F_x = F(T_x\Sigma) \subset \mathcal{D}_{u(x)}$ for all

$x \in \Sigma$. It follows from the above discussion that we have two vector bundle over Σ defined as follows,

$$T\Sigma^\Omega := \bigcup_{x\in\Sigma} V_x^\Omega \quad \text{and} \quad T\Sigma^{\Omega^\Omega} := \bigcup_{x\in\Sigma} V_x^{\Omega^\Omega}$$

Since $T\Sigma^{\Omega^\Omega}$ is a subbundle of $T\Sigma^\Omega$ of positive codimension, using a local field τ in $T\Sigma^\Omega \setminus (T\Sigma^\Omega)^\Omega$ as discussed in the previous paragraph, F can be locally lifted over each contractible open set $O \subset \Sigma$ to a section of $\tilde{\mathcal{R}}^{\mathsf{Hor}}|_O$. Thus $ev : \Gamma\tilde{\mathcal{R}}^{\mathsf{Hor}} \to \Gamma\mathcal{R}^{\mathsf{Hor}}$ is surjective on such O and hence $\tilde{\mathcal{R}}^{\mathsf{Hor}}$ is indeed an extension of $\mathcal{R}^{\mathsf{Hor}}$. □

Remark 4.2.3. In fact, it follows from the above lemma that the relation $\tilde{\mathcal{R}}^{\mathsf{Hor}}$ is non-empty if $\operatorname{rk}\mathcal{D} \geq \dim\Sigma + 4$, i.e, $\dim M \geq 4\dim\Sigma + 6$.

Proof of Theorem 4.2.1 now follows from a direct application of Theorem 3.2.7.

Existence of Regular Horizontal Immersions

The main result of this section can be stated as follows.

Theorem 4.2.4 ([BD20])**.** *Suppose $\mathcal{D} \subset TM$ is a degree 2 fat distribution. Then any $u : \Sigma \to M$ can be homotoped to a Ω-regular, $\mathcal{D}$-horizontal map, provided*

$$\operatorname{rk}\mathcal{D} \geq \max\left\{4\dim\Sigma + 4,\ 5\dim\Sigma - 3\right\}.$$

Furthermore the homotopy can be made arbitrarily C^0-close to u.

In order to prove Theorem 4.2.4, it is enough to obtain a formal Ω-regular, $\mathcal{D}$-horizontal immersion, covering a given smooth map $u : \Sigma \to M$, which gives a *global* section of the relation $\mathcal{R}^{\mathsf{Hor}} \subset J^1(\Sigma, M)$. Then a direct application of Theorem 4.2.1 gives us the required C^0-small homotopy.

Consider the subbundle $\mathcal{F} \subset \hom(T\Sigma, u^*TM)$, where the fibers are given by,

$$\mathcal{F}_x = \left\{F : T_x\Sigma \to \mathcal{D}_{u(x)} \;\middle|\; F \text{ is injective, } \Omega\text{-regular and } \Omega\text{-isotropic}\right\}, \quad x \in \Sigma.$$

Clearly, $\Gamma\mathcal{F}$ consists of formal maps in $\Psi^{\mathsf{Hor}} = \Gamma\mathcal{R}^{\mathsf{Hor}}$ which covers the given $u : \Sigma \to M$. In order to get a global section of $\mathcal{F}$, we appeal to the obstruction theory ([Hus94]) for fiber bundles.

Recall that given a fibration $P \hookrightarrow E \to B$, with typical fiber P, and a section s defined over the n-skeleton $B^{(n)} \subset B$, the obstruction to extending this section to a section over the

$(n+1)$-skeleton lies in the cohomology group

$$H^{n+1}(B, \underline{\pi_n(P)})$$

with local coefficient system $\underline{\pi_n(P)}$. In particular, if the fiber is $(\dim B - 1)$-connected, then we have a global section of the bundle $E \to B$.

To determine the connectivity of the fibers of $\mathcal{F}$, let us consider a degree 2 fat tuple (D, E, Ω) and let $V_k(D)$ denote the space of k-frames in D. Define a subset $R(k)$ of $V_k(D)$ by,

$$R(k) = \left\{(v_1, \ldots, v_k) \in V_k(D) \;\middle|\; \text{the span } \langle v_1, \ldots, v_k\rangle \text{ is } \Omega\text{-regular and } \Omega\text{-isotropic}\right\}$$

We first observe that the connectivity of this space does not depend on our choice of degree 2 tuple. Indeed we have the following.

Lemma 4.2.5. *The space $R(k)$ is $4n-4k+2$-connected, where $\dim D = 4n$.*

Proof. The proof is by induction over k. For $k=1$, we have

$$R(1) = \{v \in D \mid v \neq 0 \text{ and } \langle v\rangle \text{ is } \Omega\text{-regular, } \Omega\text{-isotropic}\}.$$

But from Definition 4.1.3, *every* 1-dimensional subspace of D is Ω-regular as well as isotropic. Thus we get,

$$R(1) \equiv D \setminus \{0\} \simeq S^{4n-1}$$

Hence $R(1)$ is $4n-2$-connected. Note that, $4n-2 = 4n-4.1+2$.

Let $k \geq 2$ and assume that, $R(k-1)$ is $4n-4(k-1)+2 = 4n-4k+6$-connected. Observe that the projection map $p : V_k(D) \to V_{k-1}(D)$ given by $p(v_1, \ldots, v_k) = (v_1, \ldots, v_{k-1})$ maps, $R(k)$ into $R(k-1)$. We now identify the fibers of $p : R(k) \to R(k-1)$.

Let $b = (v_1, \ldots, v_{k-1}) \in R(k-1)$, so that $V = \langle v_1, \ldots, v_{k-1}\rangle$ is Ω-regular and Ω-isotropic. As we saw in the proof of Lemma 4.2.2, a tuple $(v_1, \ldots, v_{k-1}, \tau) \in R(k)$ if and only if $\tau \in V^\Omega \setminus V^{\Omega\Omega}$. We have thus identified, the fiber of p over b with,

$$p^{-1}(b) \equiv V^\Omega \setminus V^{\Omega\Omega}.$$

Note that, $\dim V^{\Omega\Omega} = 2\dim V = 2k$ and

$$\operatorname{codim} V^\Omega = 2\dim V \;\Rightarrow\; \dim V^\Omega = \dim D - 2\dim V = 4n - 2(k-1).$$

Hence the fiber $p^{-1}(b)$ may be identified with $F(k) := \mathbb{R}^{4n-2k+2} \setminus \mathbb{R}^{2(k-1)}$ and so it is $4n - 2k + 2 - (2k-2) - 2 = 4n - 4k + 2$-connected. Next consider the fibration long exact sequence associated to $p : R(k) \to R(k-1)$,

$$\cdots \to \pi_i(F(k)) \to \pi_i(R(k)) \to \pi_i(R(k-1)) \to \pi_{i-1}(F(k)) \to \cdots$$

Since $\pi_i(F(k)) = 0$ for $i \leq 4n - 4k + 2$, we get isomorphisms $\pi_i(R(k)) \cong \pi_i(R(k-1))$ for $i \leq 4n - 4k + 2$. But from the induction hypothesis, $\pi_i(R(k-1)) = 0$ for $i \leq 4n - 4k + 6$. Hence,

$$\pi_i(R(k)) \cong \pi_i(R(k-1)) = 0,$$

for $i \leq 4n - 4k + 2 < 4n - 4k + 6$. This concludes the induction step and hence the lemma is proved. □

Let us now give the proof of main theorem.

Proof of Theorem 4.2.4. For $x \in \Sigma$, we may easily identify the fiber $\mathcal{F}_x$ with $R(k)$ for the degree 2 fat tuple $\big(\mathcal{D}_{u(x)}, TM/\mathcal{D}|_{u(x)}, \Omega|_{u(x)}\big)$ and hence it follows from Lemma 4.2.5 that the fibers of $\mathcal{F}$ are $4n - 4k + 2$-connected. Now from the hypothesis we have,

$$\operatorname{rk}\mathcal{D} \geq 5\dim\Sigma - 3 \;\Leftrightarrow\; 4n \geq 5k - 3 \;\Leftrightarrow\; 4n - 4k + 2 \geq k - 1 = \dim\Sigma - 1.$$

Hence, $\mathcal{F}$ has a global section. We thus have a formal, Ω-regular, Ω-isotropic, $\mathcal{D}$-horizontal map F, covering the given $u : \Sigma \to M$. Furthermore since $\operatorname{rk}\mathcal{D} \geq 4\dim\Sigma + 4$, by Theorem 4.2.1, this formal map F can be homotoped to a holonomic section of $\mathcal{R}^{\text{Hor}}$. In particular, there exists an Ω-regular, $\mathcal{D}$-horizontal immersion $\Sigma \to M$, arbitrarily C^0-close to u. This concludes the proof. □

In particular, one may take $u : \Sigma \to M$ to be a constant map. Then as a direct corollary we get,

Corollary 4.2.6. *Let $\mathcal{D}$ be a degree 2 fat distribution on M. Then Σ admits an Ω-regular $\mathcal{D}$-horizontal immersion in an arbitrary small open subset in M, provided* $\operatorname{rk}\mathcal{D} \geq \max\{4\operatorname{rk}\mathcal{D} + 4,\ 5\dim\Sigma - 3\}$.

As a consequence of Proposition 4.1.30, we see that the Ω-regularity condition in the context of holomorphic contact structure translates into totally real condition. We thus have the immediate corollary of Theorem 4.2.4.

Corollary 4.2.7. *Given a holomorphic contact structure Ξ on a complex manifold M, any smooth map $u : \Sigma \to M$ can be homotoped to a totally real horizontal immersion, provided*

$$\mathrm{rk}_{\mathbb{R}}\, \Xi \geq \max\{4 \dim \Sigma + 4,\ 5 \dim \Sigma - 3\}.$$

Furthermore, the homotopy can be made arbitrarily C^0-small.

An Application to Symplectic Geometry

Let us now discuss an interesting consequence of the h-principle Theorem 4.2.1 in the symplectic geometry. Recall that given a manifold N with a symplectic form ω, an immersion $f : \Sigma \to N$ is called *Lagrangian* if $f^*\omega = 0$. Now, further assume that the symplectic form ω in question is exact, say, $\omega = d\mu$ for some 1-form μ on N. In this case, N must be an open manifold. Now, a Lagrangian immersion $f : \Sigma \to N$ is called *exact* if the closed form $f^*\mu$ is exact. The space of exact Lagrangian immersions depends on the choice of a primitive μ. We refer to [Gro86, EM02] for the h-principle for exact Lagrangian immersions.

Now, consider a manifold N with a pair of exact symplectic forms $(d\mu^1, d\mu^2)$ on it.

Definition 4.2.8. An immersion $f : \Sigma \to (N, d\mu^1, d\mu^2)$ is said to be an *exact* $(d\mu^1, d\mu^2)$-*Lagrangian* if $f^*\mu^1, f^*\mu^2$ are exact 1-forms.

Let $M = N \times \mathbb{R}^2$ and $\pi : M \to N$ be the canonical projection map onto N. Then on M we have the 1-forms,

$$\lambda^i = dz_i - \pi^*\mu^i,\ i = 1, 2,$$

where z_1, z_2 are the coordinates on $\mathbb{R}^2$. Clearly λ^1 and λ^2 are independent at each point of M and so we have a corank 2 distribution $\mathcal{D} = \ker \lambda^1 \cap \ker \lambda^2$. Note that $d\pi : \mathcal{D} \to TN$ is a fiberwise isomorphism. Furthermore, the curvature form $\Omega_{\mathcal{D}}$ is given as, $\Omega_{\mathcal{D}} = (d\lambda^1|_{\mathcal{D}}, d\lambda^2|_{\mathcal{D}}) = \big(\pi^* d\mu^1|_{\mathcal{D}}, \pi^* d\mu^2|_{\mathcal{D}}\big)$.

Now, identify $C^\infty(\Sigma, M)$ with $C^\infty(\Sigma, N) \times C^\infty(\Sigma) \times C^\infty(\Sigma)$ for an arbitrary manifold Σ. Suppose $u = (f, \phi^1, \phi^2) : \Sigma \to N \times \mathbb{R} \times \mathbb{R}$ is a C^∞-map. Then,

$$u^*\lambda_i = d\big(z_i \circ u\big) - \big(\pi \circ u\big)^* \mu_i = d\phi^i - f^*\mu^i, \quad i = 1, 2.$$

Therefore,

$$u^*\lambda^i = 0,\ \text{for } i = 1, 2 \ \Leftrightarrow\ f \text{ is exact } (d\mu^1, d\mu^2)\text{-Lagrangian}.$$

Hence, u is $\mathcal{D}$-horizontal if and only if $f = \pi \circ u$ is an exact $(d\mu^1, d\mu^2)$-Lagrangian. Using Theorem 4.2.1 we can now get an h-principle result for the regular exact $(d\mu^1, d\mu^2)$-Lagrangians in certain cases. This partially improves some of the results obtained in [Dat11].

For immersions $f : \Sigma \to N$, we have a similar notion of $(d\mu^1, d\mu^2)$-regularity. A subspace $V \subset T_x N$ is called $(d\mu^1, d\mu^2)$-*regular* if the map,

$$\begin{aligned}\psi : T_x N &\to \hom(V, \mathbb{R}^2)\\ \partial &\to \left(\iota_\partial d\mu^1|_V, \iota_\partial d\mu^2|_V\right)\end{aligned}$$

is surjective (compare Definition 4.1.1). Similarly, an immersion $f : \Sigma \to N$ is called $(d\mu^1, d\mu^2)$-*regular* if $V = \operatorname{Im} df_\sigma$ is $(d\mu^1, d\mu^2)$-regular for each $\sigma \in \Sigma$.

Definition 4.2.9. A monomorphism $F : T\Sigma \to TN$ is said to be a *formal* regular, $(d\mu^1, d\mu^2)$-Lagrangian if for each $\sigma \in \Sigma$,

- the subspace $V = \operatorname{Im} F_\sigma \subset T_{u(\sigma)} N$ is $(d\mu^1, d\mu^2)$-regular subspace, and
- $F^* d\mu^i = 0$, that is, V is $d\mu^i$-isotropic, for $i = 1, 2$.

Let us denote by $\mathcal{R}^{\mathsf{ExaLag}} \subset J^1(\Sigma, M)$ the underlying relation.

We then have the following.

Proposition 4.2.10. *Every (formal) regular, $(d\mu^1, d\mu^2)$-Lagrangian map immersion lifts to a (formal) Ω-regular $\mathcal{D}$-horizontal immersion. Conversely, any (formal) Ω-regular $\mathcal{D}$-horizontal immersion projects to a (formal) regular, exact $(d\mu^1, d\mu^2)$-Lagrangian immersion.*

Proof. Suppose $(F, f) : T\Sigma \to TN$ is a given formal, regular $(d\mu^1, d\mu^2)$-Lagrangian map. Set, $u = (f, 0, 0) : \Sigma \to M$. Then we can get a canonical lift $H : T\Sigma \to TM$ covering u, by using the fact that $d\pi : \mathcal{D}_{u(\sigma)} \to T_{f(\sigma)} N$ is an isomorphism. We have the commutative diagram,

$$\begin{tikzcd} & \mathcal{D} \arrow[d, "d\pi|_\mathcal{D}"] & \pi^* d\mu^i|_\mathcal{D} = d\lambda_i|_\mathcal{D} \\ T\Sigma \arrow[ur, "H"] \arrow[r, "F"'] & TN & d\mu_i \end{tikzcd}$$

Therefore, H is injective. We claim that H is Ω-regular and $d\lambda^i$-isotropic for $i = 1, 2$ (in other words Ω-isotropic). The isotropy condition follows easily, since,

$$H^* d\lambda^i|_\mathcal{D} = H^* \pi^* d\mu^i|_\mathcal{D} = (d\pi|_\mathcal{D} \circ H)^* d\mu^i = F^* d\mu^i = 0, \qquad i = 1, 2.$$

To deduce the Ω-regularity, observe that we have a commutative diagram,

$$\begin{tikzcd} \mathcal{D}_{u(\sigma)} \arrow[r, "\phi"] \arrow[d, "d\pi|_{u(\sigma)}"'] & \hom(\operatorname{Im} H_\sigma, \mathbb{R}^2) \\ T_{f(\sigma)}N \arrow[r, "\psi"'] & \hom(\operatorname{Im} F_\sigma, \mathbb{R}^2) \arrow[u, "\left(d\pi|_{u(\sigma)}\right)^*"'] \end{tikzcd}$$

where both the vertical maps are isomorphisms and the maps ϕ, ψ are given as,

$$\phi(v) = \left(\iota_v d\lambda^i|_{\operatorname{Im} H}\right)_{i=1,2} \qquad \text{and} \qquad \psi(u) = \left(\iota_u d\mu^i|_{\operatorname{Im} F}\right)_{i=1,2}.$$

Now, $(d\mu^1, d\mu^2)$-regularity of F is equivalent to surjectivity of ψ which implies surjectivity of ϕ. Thus, the lift H is a formal, regular, isotropic $\mathcal{D}$-horizontal map. A similar argument proves the converse statement as well. $\square$

Since $d\mu^i$ is symplectic for $i = 1, 2$, we can define an automorphism $A : TN \to TN$ by, $d\mu^1(u, Av) = d\mu^2(u, v)$ for $u, v \in TN$. We now restrict to pairs $(d\mu^1, d\mu^2)$ for which A has no real eigenvalue and the degree of the minimal polynomial of A is 2 (at every point). Clearly, this gives rise to the automorphism $\tilde{A} : \mathcal{D} \to \mathcal{D}$ satisfying, $d\lambda^1(u, \tilde{A}v) = d\lambda^2(u, v)$ for $u, v \in \mathcal{D}$; which enjoys similar properties. In particular, $\mathcal{D}$ is then a fat distribution of degree 2.

Example 4.2.11. As a concrete example, one may consider the exact symplectic forms,

$$\omega^1 = \sum_{i=1}^{2n} dx_i \wedge dy_i \quad \text{and} \quad \omega^2 = \sum_{i=1}^{n} \left(dx_{2i-1} \wedge dy_{2i} - dx_{2i} \wedge dy_{2i-1}\right),$$

on $\mathbb{R}^{4n}$, with the coordinates $(x_i, y_i; i = 1, \ldots, 2n)$. Then the automorphism A in this case satisfies, $A^2 = -I$. A very similar calculation as in Corollary 4.1.29 shows that any holomorphic symplectic form ω on a complex n-manifold (i.e $d\omega = 0$ and ω^n is nonvanishing), locally gives rise to such pairs, when written as $\omega = \omega^1 + \iota\omega^2$. Furthermore, the associated distribution $\mathcal{D}$ on $\mathbb{R}^{4n+2}$ is precisely the real distribution underlying a standard holomorphic contact structure, as observed in Corollary 4.1.29.

We now have the following h-principle.

Theorem 4.2.12 ([BD20])**.** *Suppose, $d\mu^1, d\mu^2$ are exact symplectic forms on N, related by an automorphism $A : TN \to TN$, such that A_x has no real eigenvalue and the minimal polynomial of A_x has degree 2, for all $x \in N$. Then the relation $\mathcal{R}^{\mathsf{ExaLag}}$ satisfies the C^0-dense h-principle, provided* $\dim N \geq 4 \dim \Sigma + 4$.

Proof. Suppose $F_0 \in \Gamma\mathcal{R}^{\mathsf{ExaLag}}$ is given. That is, $F_0 : T\Sigma \to TM$ is a formal, regular, exact $(d\mu^1, d\mu^2)$-Lagrangian, with $f_0 = \operatorname{bs} F_0$. Consider the lift, $\tilde{F}_0 : T\Sigma \to TM$ with $\operatorname{bs}\tilde{F}_0 = (f_0, 0, 0)$; which is a formal, Ω-regular, isotropic $\mathcal{D}$-horizontal map by Proposition 4.2.10. Now,

$$\dim N \geq 4\dim\Sigma + 4 \Leftrightarrow \dim M \geq 4\dim\Sigma + 6 \Leftrightarrow \operatorname{rk}\mathcal{D} \geq 4\dim\Sigma + 4.$$

Hence, by Theorem 4.2.1, we have a homotopy $\tilde{F}_t$ of formal, Ω-regular, $\mathcal{D}$-horizontal maps, so that $\tilde{F}_1 = d\tilde{f}_1$; where $\tilde{f}_t = \operatorname{bs}\tilde{F}_t$. Furthermore $\tilde{f}_t$ is arbitrarily C^0-close to $\tilde{f}_0$. Now consider the projected map, $F_t = d\pi \circ \tilde{F}_t$, which covers $f_t = \pi \circ \tilde{f}_t$. Again by Proposition 4.2.10, F_t is formal, regular, exact $(d\mu^1, d\mu^2)$-Lagrangian, i.e, $F_t \in \Gamma\mathcal{R}^{\mathsf{ExaLag}}$. Furthermore, $F_1 = df_1$. Hence, we have the required homotopy, proving the h-principle. Clearly, f_t is C^0-close to f_0 for all t. This concludes the C^0-dense h-principle for $\mathcal{R}^{\mathsf{ExaLag}}$. □

An obstruction-theoretic argument as in Theorem 4.2.4 gives us the following result,

Theorem 4.2.13. *Suppose $(N, d\mu^1, d\mu^2)$ is as in Theorem 4.2.12. If $\dim N \geq \max\{4\dim\Sigma + 4,\ 5\dim\Sigma - 3\}$, then any $f : \Sigma \to N$ can be homotoped to a regular exact $(d\mu^1, d\mu^2)$-Lagrangian, keeping the homotopy arbitrarily C^0-small.*

The above theorem improves the result in [Dat11], where the author proved the existence of regular, exact $(d\mu^1, d\mu^2)$-Lagrangian immersions $\Sigma \to N$, under the condition $\dim N \geq 6\dim\Sigma$.

4.2.2 Horizontal Immersions into Quaternionic Contact Manifolds

Given an arbitrary quaternionic contact structure $\mathcal{D} \subset TM$ (see Definition 4.1.33), we wish to study $\mathcal{D}$-horizontal immersions $u : \Sigma \to M$. The aim of this section is to prove the following h-principle.

Theorem 4.2.14. *Suppose $\mathcal{D} \subset TM$ is a quaternionic contact structure and Σ is an arbitrary manifold. Then $\mathcal{R}^{\mathsf{Hor}} \subset J^1(\Sigma, M)$ satisfies the C^0-dense h-principle, provided*

$$\operatorname{rk}\mathcal{D} \geq 4\dim\Sigma + 4$$

In fact, given any monomorphism $F : T\Sigma \to TM$, covering some $u : \Sigma \to M$ and satisfying the curvature condition $F^*\Omega = 0$, we can homotope F to a $\mathcal{D}$-horizontal immersion, keeping the homotopy arbitrary C^0-small, provided $\operatorname{rk}\mathcal{D} \geq 4\dim\Sigma + 4$.

Denoting the curvature form of $\mathcal{D}$ by Ω, we recall that,

- a subspace $V \subset \mathcal{D}_x$ is Ω-isotropic if we have $V \subset V^\Omega$.
- a subspace $V \subset \mathcal{D}_x$ is Ω-regular if and only if $\operatorname{codim} V^\Omega = \operatorname{cork} \mathcal{D} \times \dim V = 3 \dim V$.

The following justifies the *absence* of Ω-regularity in the h-principle statement above.

Proposition 4.2.15 ([Pan16]). *If $\mathcal{D}$ is a quaternionic contact structure, then any Ω-isotropic subspace of $\mathcal{D}_x$ is Ω-regular*

Proof. Since $(M, \mathcal{D})$ is a quaternionic contact manifold, we have a Riemannian metric g on $\mathcal{D}$ and local 1-forms λ^s on some open neighborhood U of x, defining $\mathcal{D}|_U = \bigcap_{s=1}^3 \ker \lambda^s$. Furthermore, the automorphisms $J_s : \mathcal{D}|_U \to \mathcal{D}|_U$ defined via the relation $d\lambda^s\big(u, v\big) = g(J_s u, v)$ for any vectors $u, v \in \mathcal{D}|_U$, satisfy the quaternionic relations,

$$J_1^2 = J_2^2 = J_3^2 = -1 = J_1 J_2 J_3.$$

Now, suppose $V \subset \mathcal{D}_x$ is an Ω-isotropic subspace. We show that $\sum J_s V$ is a direct sum. First note that $J_s^* = -J_s$, where J_s^* is the adjoint of J_s. Indeed, we have,

$$g(u, J_s^* v) = g(J_s u, v) = d\lambda^s(u, v) = -d\lambda^s(v, u) = -g(J_s v, u) = g(u, -J_s v),$$

for any $u, v \in \mathcal{D}|_U$. Since V is Ω-isotropic, i.e, $V \subset V^\Omega = \bigcap_{s=1}^3 V^{\perp_{d\lambda^s}}$, for any $u, v \in V$ we have,

$$g(J_1 u, J_2 v) = g(u, -J_1 J_2 v) = g(u, J_3 v) = d\lambda^s(u, v) = 0.$$

Similar arguments give us that, $g(J_i u, J_j v) = 0$ for any $i \neq j$. Thus, we see that $J_1 V, J_2 V, J_3 V$ are pairwise g-orthogonal subspaces. Hence $\sum_{s=1}^3 J_s V$ is a direct sum, so that $\dim\big(\sum J_s V\big) = 3 \dim V$.

Next we show that, $V^\Omega \cap \big(\sum J_s V\big) = 0$. In fact, for any $w \in V^\Omega$ and for any $z = J_1 v^1 + J_2 v^2 + J_3 v^3 \in \sum J_s V$, we have that

$$g(z, w) = \sum_{s=1}^3 g(J_s v^s, w) = \sum_{s=1}^3 d\lambda^s(v^s, w) = 0.$$

In other words, V^Ω and $\sum J_s V$ are g-orthogonal and hence they have zero intersection. But then we readily have that, $\operatorname{codim} V^\Omega \geq \dim\big(\sum J_s V\big) = 3 \dim V$. On the other hand, it is clear that $\operatorname{codim} V^\Omega \leq 3 \dim V$, which implies that $\operatorname{codim} V^\Omega = 3 \dim V$, proving that V is indeed Ω-regular. $\square$

We now proceed as in the previous section to prove the following.

Lemma 4.2.16. *If* $\operatorname{rk}\mathcal{D} \geq 4\dim\Sigma + 4$, *then* $\tilde{\mathcal{R}}^{\mathsf{Hor}}$ *is an extension of* $\mathcal{R}^{\mathsf{Hor}}$.

Proof. Let (x, y, F) represent a jet in $\mathcal{R}^{\mathsf{Hor}}$. Then $V = \operatorname{Im} F$ is an Ω-isotropic subspace of $\mathcal{D}_y$ and so $V \subset V^{\Omega}$. By Proposition 4.2.15, V is Ω-regular as well. Hence we have,

$$\operatorname{codim} V^{\Omega} = 3\dim V.$$

Now from the dimension condition we have,

$$\operatorname{rk}\mathcal{D} \geq 4\dim\Sigma + 4 \Rightarrow \dim V^{\Omega} = \operatorname{rk}\mathcal{D} - 3\dim\Sigma \geq \dim\Sigma + 4 = \dim V + 4.$$

Thus, we have that the codimension of V in V^{Ω} is ≥ 4. Now, for any $\tau \in V^{\Omega} \setminus V$, we have that $V' = V + \langle\tau\rangle$ is again isotropic. We may then define an extension $\tilde{F} : T_x\Sigma \oplus \mathbb{R} \to T_yM$ by $\tilde{F}(v, t) = F(v) + t\tau$ for all $v \in T_x\Sigma$ and $t \in \mathbb{R}$. Clearly $(x, y, \tilde{F})$ is then a jet in $\tilde{\mathcal{R}}^{\mathsf{Hor}}$. Proceeding just as in Lemma 4.2.2, we can now complete the proof. □

The proof of Theorem 4.2.14 now follows directly from Theorem 3.2.7.

Existence of Horizontal Immersions in Quaternionic Contact Structures

We conclude from the above h-principle, the following existence result.

Theorem 4.2.17. *Let $\mathcal{D}$ be a quaternionic contact structure on M. Then any map $u : \Sigma \to M$ can be homotoped to a $\mathcal{D}$-horizontal immersion provided,*

$$\operatorname{rk}\mathcal{D} \geq \max\{4\dim\Sigma + 4,\ 5\dim\Sigma - 3\}.$$

Furthermore, the homotopy can be made arbitrarily C^0-small.

The proof is similar to that of Theorem 4.2.4; in fact it is simpler since Ω-regularity is automatic by Proposition 4.2.15. Let us denote,

$$R(k) = \left\{(v_1, \dots, v_k) \in V_k(\mathcal{D}_x) \;\middle|\; \text{the span } \langle v_1, \dots, v_k\rangle \subset \mathcal{D}_x \text{ is } \Omega_x\text{- isotropic}\right\},$$

where $V_k(D)$ is the space of k-frames in a vector space D. We then have the following.

Lemma 4.2.18. *The space $R(k)$ is $4n - 4k + 2$-connected, where $\operatorname{rk}\mathcal{D} = 4n$.*

Proof. The proof is via induction on k. For $k = 1$, we have,

$$R(1) = \left\{v \in \mathcal{D}_x \;\middle|\; v \text{ is nonzero, } \langle v\rangle \text{ is isotropic}\right\}.$$

But every one dimensional subspace is isotropic. Hence we have,

$$R(1) = \mathcal{D}_x \setminus 0 \simeq S^{4n-1},$$

which is $4n-2$ connected. Note that, $4n-2 = 4n-4.1+1$.

Inductively, assume that for some $k \geq 2$, the space $R(k-1)$ is $4n-4(k-1)+2 = 4n-4k+6$-connected. We see that the projection map $p : V_k(\mathcal{D}_x) \to V_{k-1}(\mathcal{D}_x)$ given by, $p(v_1, \ldots, v_k) = (v_1, \ldots, v_{k-1})$ maps $R(k)$ into $R(k-1)$. In particular, we have a fiber bundle $p : R(k) \to R(k-1)$. Now, let $b = (v_1, \ldots, v_{k-1}) \in R(k-1)$, so that $V = \langle v_1, \ldots, v_{k-1} \rangle$ is Ω-isotropic, i.e, $V \subset V^\Omega$. Proceeding as in Lemma 4.2.16, we get that a tuple $(v_1, \ldots, v_k, \tau) \in R(k)$ if and only if $\tau \in V^\Omega \setminus V$. Now we see that,

$$\dim V^\Omega = \dim \mathcal{D}_x - 3 \dim V = 4n - 3(k-1) = 4n - 3k + 3.$$

Thus we have identified the fiber $p^{-1}(b)$ with,

$$F(k) := \mathbb{R}^{4n-3k+3} \setminus \mathbb{R}^{k-1},$$

which is $(4n-3k+3) - (k-1) - 2 = 4n-4k+2$-connected.

Next, consider the fibration long exact sequence for the fiber bundle $p : R(k) \to R(k-1)$,

$$\cdots \to \pi_i(F(k)) \to \pi_i(R(k)) \to \pi_i(R(k-1)) \to \pi_{i-1}(F(k)) \to \cdots$$

Since $\pi_i(F(k)) = 0$, for $i \leq 4n-4k+2$, we get isomorphism $\pi_i(R(k)) \cong \pi_i(R(k-1))$ for $i \leq 4n-4k+2$. But from the induction hypothesis, $\pi_i(R(k-1)) = 0$ for $i \leq 4n-4k+6$. Hence,

$$\pi_i(R(k)) = 0, \quad \text{for } i \leq 4n-4k+2 < 4n-4k+6.$$

This concludes the induction argument as we have proved that $R(k)$ is $4n-4k+2$-connected. $\square$

Proof of Theorem 4.2.17. Suppose $u : \Sigma \to M$ is an arbitrary map. Consider the subbundle $\mathcal{F} \subset \hom(T\Sigma, u^*\mathcal{D})$ where the fibers are given by,

$$\mathcal{F}_x = \left\{ F : T_x\Sigma \to \mathcal{D}_{u(x)} \,\middle|\, F \text{ is injective and } \Omega\text{-isotropic} \right\}, \quad x \in \Sigma.$$

Clearly, $\Gamma\mathcal{F}$ consists of formal maps in $\Psi^{\mathsf{Hor}} = \Gamma\mathcal{R}^{\mathsf{Hor}}$, that covers u. We can identify the fiber $\mathcal{F}_x$ with the space $R(k)$ and hence by Lemma 4.2.18, the fiber is $4n-4k+2$-connected. So, under the hypothesis,

$$\operatorname{rk}\mathcal{D} = 4n \geq 5\dim\Sigma - 3 = 5k - 3 \Leftrightarrow 4n - 4k + 2 \geq k - 1,$$

the fibers of $\mathcal{F}$ are $(\dim\Sigma - 1)$-connected. We then have a global section $F \in \Gamma\mathcal{F}$. The proof now follows directly from Theorem 4.2.14. $\square$

Applying Theorem 4.2.17 to the constant maps $\Sigma \to M$ we have the immediate corollary.

Corollary 4.2.19. *Given a quaternionic contact structure $\mathcal{D}$ on M, every manifold Σ admits a $\mathcal{D}$-horizontal immersion in some arbitrarily small coordinate neighborhood in M, provided* $\operatorname{rk}\mathcal{D} \geq \max\{4\dim\Sigma + 4,\ 5\dim\Sigma - 3\}$.

An Application to Symplectic Geometry

Just as in the previous section, let us now consider a *triple* of symplectic forms, say ω^i for $i = 1, 2, 3$, on a Riemannian manifold (N, g). Suppose, the endomorphisms J_i of TN defined by,

$$g(J_i\_, \_) = \omega^i(\_, \_), \quad i = 1, 2, 3,$$

satisfy the quaternionic relations : $J_1^2 = J_2^2 = J_3^2 = -1 = J_1 J_2 J_3$.

Example 4.2.20. Any hyperkähler manifold (N, g) gives rise to such a symplectic triple ([BG08]).

Further assume that the symplectic forms are exact, i.e, $\omega^i = d\mu^i$, for $i = 1, 2, 3$.

Definition 4.2.21. An immersion $u : \Sigma \to N$ is called an *exact* $(d\mu^1, d\mu^2, d\mu^3)$*-Lagrangian* if $u^*\mu^i$ is an exact 1-form for $i = 1, 2, 3$.

Now, given $(N, g, d\mu^i, i = 1, 2, 3)$ as above, there exists a corank 3 distribution $\mathcal{D}$ on $M = N \times \mathbb{R}^3$, given by,

$$\mathcal{D} = \bigcap_{i=1}^{3} \ker\left(dz_i - \pi^*\mu^i\right),$$

where $\pi : M \to N$ is the projection and $\{z^1, z^2, z^3\}$ are the coordinates along $\mathbb{R}^3$. Clearly, $\mathcal{D}$ is then a quaternionic structure on M. Now, arguing just as in Theorem 4.2.12, we have the following as a corollary to Theorem 4.2.14.

Corollary 4.2.22. *Let $(N, g, d\mu^i, i = 1, 2, 3)$ as above. Then, there exists an exact $(d\mu^i)$-Lagrangian immersion $\Sigma \to N$, provided* $\dim N \geq \max\{4\dim\Sigma + 4,\ 5\dim\Sigma - 3\}$.

4.2.3 Isocontact Immersions into Degree 2 Fat Distribution

So far we have only exhibited examples of *horizontal* immersions in specific fat distributions. In our next example, we will consider K-contact maps in degree 2 fat distributions, for K nontrivial. Given $K \subset T\Sigma$ and $\mathcal{D} \subset TM$, recall from Definition 3.1.7, the relation $\mathcal{R}^{\mathsf{IsoCont}} \subset J^1(\Sigma, M)$, representing the formal Ω-regular, K-isocontact maps. That is, $\Psi^{\mathsf{IsoCont}} = \Gamma\mathcal{R}^{\mathsf{IsoCont}}$ consists of monomorphisms $F : T\Sigma \to TM$, covering some $u : \Sigma \to M$, inducing $K = F^{-1}\mathcal{D}$. Furthermore, F is Ω-regular, i.e, the bundle map

$$\begin{aligned} u^*\mathcal{D} &\to \hom(K, u^*TM/\mathcal{D}) \\ \xi &\mapsto \big(X \mapsto \Omega(\xi, FX)\big) \end{aligned}$$

is surjective; and F satisfies the curvature condition, i.e, $F^*\Omega|_K = \Omega_K$. Here, Ω and Ω_K are the curvature forms of $\mathcal{D}$ and K respectively.

The goal of this section is to prove the following h-principle.

Theorem 4.2.23. *Suppose $\mathcal{D} \subset TM$ is a degree 2 fat distribution and $\xi \subset T\Sigma$ is a fixed contact structure. Then, $\mathcal{R}^{\mathsf{IsoCont}}$ satisfies the C^0-dense h-principle, provided*

$$\operatorname{rk}\mathcal{D} \geq 2\operatorname{rk}\xi + 4.$$

In fact, suppose we are given any monomorphism $F : T\Sigma \to TM$, covering some $u : \Sigma \to M$, inducing $K = F^{-1}\mathcal{D}$ and satisfying the curvature condition $F^*\Omega|_\xi = \tilde{F} \circ \Omega_\xi$. Then F can be homotoped to a K-isocontact immersion $\Sigma \to (M, \mathcal{D})$, provided $\operatorname{rk}\mathcal{D} \geq 2\operatorname{rk}\xi + 4$, while keeping the homotopy arbitrarily C^0-small.

Before proceeding any further, let us first explain the absence of "Ω-regularity" in the above statement.

Proposition 4.2.24. *Let $\mathcal{D} \subset TM$ be a degree 2 fat distribution and $\xi \subset T\Sigma$ be a contact structure. Then any formal isocontact immersion $F : (T\Sigma, \xi) \to (TM, \mathcal{D})$ satisfying the curvature condition is Ω-regular.*

Proof. Suppose, $F : T\Sigma \hookrightarrow TM$ induces $\xi = F^{-1}\mathcal{D}$ and satisfies, $F^*\Omega|_\xi = \tilde{F} \circ \Omega_\xi$; where $\tilde{F} : T\Sigma/\xi \to u^*TM/\mathcal{D}$ is the induced injective bundle map and Ω_ξ, $\Omega_\mathcal{D}$ are the curvature forms of ξ and $\mathcal{D}$ respectively. Fix some $x \in \Sigma$, and let $y \in M$, so that $F : T_x\Sigma \to T_yM$. We may choose some trivializations of $T\Sigma/\xi$ and $TM/\mathcal{D}$ about x and y, respectively. Then there exists local 2-forms $\sigma, \omega^1, \omega^2$, such that,

$$\Omega_\xi \underset{loc.}{=} \sigma|_\xi \quad \text{and} \quad \Omega_{\mathcal{D}} \underset{loc.}{=} (\omega^1|_{\mathcal{D}}, \omega^2|_{\mathcal{D}}),$$

with respect to the trivializations. Denote by $A : \mathcal{D}_y \to \mathcal{D}_y$ the connecting automorphism for the tuple $(\omega^1|_{\mathcal{D}_y}, \omega^2|_{\mathcal{D}_y})$. Since $\tilde{F}$ is injective, we then see that the curvature condition at the point x translates into,

$$F^*\omega^1|_\xi = b_1\sigma, \quad F^*\omega^2|_\xi = b_2\sigma, \qquad \text{for some scalars } (b_1, b_2) \neq (0, 0).$$

Without loss of generality, assume that $b_1 \neq 0$. Let us denote $V = F(\xi_x) \subset \mathcal{D}_y$, so that we may write,

$$\omega^2|_V = c\omega^1|_V, \quad \text{for } c = \frac{b_2}{b_1}.$$

Since σ is a nondegenerate 2-form on ξ_x and $b_1 \neq 0$, we observe that V is a symplectic subspace of $\mathcal{D}_y$, with respect to ω^1. Recall from Observation 4.1.8 (1) that, V is Ω-regular if and only if $V + AV$ is a direct sum. Now, if possible, let $z = Av \in V \cap AV$ for some $0 \neq v \in V$. Then, using Observation 4.1.8 (3), we have for any $u \in V$,

$$\omega^1(z, u) = \omega^1(Av, u) = \omega^2(v, u) = c\omega^1(v, u) \Rightarrow \omega^1(Av - cv, u) = 0.$$

Since V is ω^1-symplectic, we have that $Av = cv$, implying that c is an eigenvalue of A. But this is a contradiction by Observation 4.1.8 (2). Hence, we have that V is Ω-regular. Since $x \in \Sigma$ is arbitrary, we have that any formal isocontact map F satisfying the curvature condition is Ω-regular. □

The proof of Theorem 4.2.23 follows from Theorem 3.2.7, provided we can prove the 'local extensibility' property. Recall that we have the relation $\tilde{\mathcal{R}}^{\mathsf{IsoCont}} \subset J^1(\tilde{\Sigma}, M)$, whose sections are the formal, Ω-regular, $\tilde{K}$-isocontact immersions $\tilde{\Sigma} \to M$, where $\tilde{K} = d\pi^{-1}K$ and $\pi : \tilde{\Sigma} = \Sigma \times \mathbb{R} \to \Sigma$ is the projection. We also have a natural map $ev : \tilde{\mathcal{R}}^{\mathsf{IsoCont}}|_{\Sigma \times 0} \to \mathcal{R}^{\mathsf{IsoCont}}$ induced by the projection map $\tilde{\Sigma} \to \Sigma$. We show that $\tilde{\mathcal{R}}^{\mathsf{IsoCont}}$ is indeed an extension in the sense of Definition 2.2.13.

Lemma 4.2.25. *Let $\mathcal{D} \subset TM$ be degree 2 fat and $\xi \subset T\Sigma$ be a contact distribution. Then $\tilde{\mathcal{R}}^{\mathsf{IsoCont}}$ is an extension of $\mathcal{R}^{\mathsf{IsoCont}}$, provided $\operatorname{rk} \mathcal{D} \geq 2 \operatorname{rk} \xi + 4$.*

Proof. Suppose (x, y, F) is a jet in $\mathcal{R}^{\mathsf{IsoCont}}$ and denote, $V = F(\xi_x) \subset \mathcal{D}_y$. Since the induced map $\tilde{F} : T\Sigma/\Sigma|_x \to TM/\mathcal{D}|_y$ is injective, we may choose suitable trivialization of $T\Sigma/\xi$ and

$TM/\mathcal{D}$ around x and y respectively, so that we have a representation $\Omega \underset{loc.}{=} (\omega^1, \omega^2)$ and the curvature condition $F^*\Omega|_{\xi_x} = \tilde{F} \circ \Omega_{\xi_x}$ translates into,

$$V \subset \mathcal{D}_y \text{ is } \omega^1\text{-symplectic and } \omega^2\text{-isotropic.}$$

We show that $V^\Omega \cap V^{\Omega\Omega} = 0$.

Denote by $A : \mathcal{D}_y \to \mathcal{D}_y$ the connecting automorphism for the tuple (ω^1, ω^2). Since $\mathcal{D}$ is a degree 2 fat distribution, A must satisfy $A^2 = \lambda A + \mu I$ for some scalars $\lambda, \mu \in \mathbb{R}$, with $\mu \neq 0$. Now for any $u, v \in V$ we have,

$$\omega^1(u, Av) = \omega^2(u, v) = 0, \text{ as } V \text{ is } \omega^2\text{-isotropic.}$$

This further implies that for all $u, v \in V$,

$$\omega^1(Au, Av) = \omega^2(u, Av) = \omega^1(u, A^2 v) = \lambda\omega^1(u, Av) + \mu\omega^1(u, v) = \mu\omega^1(u, v).$$

As $\mu \neq 0$, we get that AV is ω^1-symplectic. But then $V + AV$ is ω^1-symplectic as well, since $\omega^1(V, AV) = 0$. We then have,

$$V^{\Omega\Omega} \cap V^\Omega = (V + AV) \cap (V + AV)^{\perp_1} = 0, \text{ by Proposition 4.1.17.}$$

Since V is an Ω-regular subspace, from Definition 4.1.1, it follows that the codimension of V^Ω in $\mathcal{D}_y$ is $2\dim V$. Hence, from the dimension condition, it follows that $\dim V^\Omega \geq 4$. Now, for any $\tau \in V^\Omega$ we have that $\tau \notin V^{\Omega\Omega}$ and so by Proposition 4.1.19, $V' = V + \langle\tau\rangle$ is an Ω-regular subspace of $\mathcal{D}_y$. Let us define an extension $\hat{F} : T_x\Sigma \times \mathbb{R} \to T_y M$ of F by $\hat{F}(v, t) = F(v) + t\tau$ for $t \in \mathbb{R}$ and $v \in T_x\Sigma$. It is then immediate that $\hat{F}^{-1}(\mathcal{D}_y) = \tilde{\xi}_x$ and $\hat{F}$ is Ω-regular. Furthermore, for any $(v, t) \in \tilde{\xi}_x = \xi_x \oplus \mathbb{R}$, we have that,

$$\Omega(\hat{F}(t), \hat{F}(v)) = \Omega(t\tau, F(v)) = 0, \quad \text{as } \tau \in V^\Omega = (F(\xi_x))^\Omega,$$

and so, $\Omega_{\tilde{\xi}}(t, v) = 0 = \Omega(\hat{F}(t), \hat{F}(v))$. In other words, $\hat{F}$ satisfies the curvature condition relative to $\Omega_{\tilde{\xi}}$ and $\Omega_{\mathcal{D}}$. Proceeding as in Lemma 4.2.2, we can now complete the proof. □

The proof of Theorem 4.2.23 is now immediate from Theorem 3.2.7.

Existence of Isocontact Immersions

In this section, we prove the following.

Theorem 4.2.26. *Suppose $\xi \subset T\Sigma$ is a contact structure on Σ and $\mathcal{D} \subset TM$ is a degree 2 fat distribution on M. Then any map $u : \Sigma \to M$ can be homotoped to an isocontact immersion $(\Sigma, \xi) \to (M, \mathcal{D})$ provided,*

- $\operatorname{rk} \mathcal{D} \geq \max\{2 \operatorname{rk} \xi + 4,\ 3 \operatorname{rk} \xi - 2\}$, *and*
- *one of the following two conditions holds true,*
  - *both ξ and $\mathcal{D}$ are cotrivializable.*
  - $H^2(\Sigma) = 0$.

Furthermore, the homotopy can be made arbitrary C^0-close to u.

Suppose $\dim M = 4n + 2$ and $\dim \Sigma = 2k + 1$. Let us first assume that we are given some injective bundle map $G : T\Sigma/\xi \to u^*TM/\mathcal{D}$, covering a map $u : \Sigma \to M$. Now, as in the previous two cases, we construct a sub-bundle $\mathcal{F} \subset \hom(\xi, u^*\mathcal{D})$, where the fibers are given by,

$$\mathcal{F}_x = \left\{F : \xi_x \to \mathcal{D}_{u(x)} \;\middle|\; F \text{ is injective and } F^*\Omega|_{\xi_x} = G_x \circ \Omega_\xi\right\}, \quad \text{for } x \in \Sigma.$$

We wish to get a global section of the bundle $\mathcal{F}$. Towards this end, we need to figure out the connectivity of the fibers $\mathcal{F}_x$.

Let us consider the following linear algebraic set up. Let (D, E, Ω) be a degree 2 fat tuple. Suppose Ω is represented by a pair of 2-forms (ω^1, ω^2) on D and $A : D \to D$ is the connecting automorphism for the pair (ω^1, ω^2). Consider the subspace $R(k) \subset V_{2k}(D)$ given as,

$$R(k) = \left\{b = (u_1, v_1, \ldots, u_k, v_k) \in V_{2k}(D) \middle|^{b \text{ is a symplectic basis for } \omega^1|_V \text{ and } V \text{ is } \omega^2\text{-isotropic,}}_{\text{where } V = \langle u_i, v_i,\ i = 1, \ldots, k\rangle}\right\}.$$

As we argued in Lemma 4.2.25, it is easy to see, by an application of Theorem 2.1.7 about the point x, that the connectivity question about $\mathcal{F}_x$ can be translated to that of $R(k)$, where $\operatorname{rk} \xi = 2k$.

Lemma 4.2.27. *The space $R(k)$ is $4n - 4k + 2$-connected, where $\dim D = 4n$*

Proof. We proceed by induction on k. For $k = 1$,

$$R(1) = \left\{(u, v) \in V_2(D) \;\middle|\; \omega^1(u, v) = 1 \text{ and } \omega^2(u, v) = 0\right\}.$$

If $(u, v) \in R(1)$, then

$$v \in u^{\perp_2} \setminus u^{\perp_1} = u^{\perp_2} \setminus \left(u^{\perp_1} \cap u^{\perp_2}\right),$$

As (D, E, Ω) is a fat tuple, every non-zero u is Ω-regular and hence in particular, $u^{\perp_1} \cap u^{\perp_2}$ is a codimension 1 hyperplane in $u^{\perp_2}$. Clearly the complement space is then disconnected. But since we also demand that $\omega^1(u, v) = 1$, the space of all such v is a codimension 1 affine subspace of D and hence is a contractible set. Thus, we have obtained that, $R(1)$ is homotopically equivalent to the space of nonzero vectors $u \in D$. But, $D \setminus 0 \cong \mathbb{R}^{4n} \setminus 0 \simeq S^{4n-1}$ is clearly $(4n-2)$-connected. Hence, $R(1)$ is $(4n-2)$-connected. Note that $4n - 2 = 4n - 4.1 + 2$.

Let us now assume that $R(k-1)$ is $4n-4(k-1)+2 = 4n-4k+6$-connected for some $k \geq 2$. Observe that the projection map $p : V_{2k}(D) \to V_{2k-2}(D)$ clearly maps $R(k)$ into $R(k-1)$. For a fixed tuple $b = (u_1, v_1, \ldots, u_{k-1}, v_{k-1}) \in R(k-1)$, the span $V = \langle u_1, \ldots, v_{k-1}\rangle$ is ω^1-symplectic and ω^2-isotropic. As argued in Lemma 4.2.25 we then have that $V + AV$ is ω^1-symplectic, i.e, $(V + AV) \cap (V + AV)^{\perp_1} = 0$. Since V is Ω-regular, we get that

$$\dim(V + AV)^{\perp_1} = \dim D - \dim(V + AV) = \dim D - 2\dim V = 4n - 4(k-1) = 4n - 4k + 4.$$

Since (D, E, Ω) is a degree 2 fat tuple, from Proposition 4.1.17 (2) we get that,

$$(V + AV)^{\perp_1} = (V + AV)^{\perp_2} = V^{\Omega}.$$

Thus it follows from the ω^1-symplecticity of $V + AV$ that the restriction of ω^1 and ω^2 to the space $\hat{D} = V^{\Omega}$ are symplectic. Moreover, since V^{Ω} is invariant, $(\hat{D}, E, \Omega|_{\hat{D}})$ is also a degree 2 fat tuple. So if we choose any $(u, v) \in V_2\hat{D}$, satisfying $\omega^1(u, v) = 1$ and $\omega^2(u, v) = 0$, it follows that $(u_1, \ldots, v_{k-1}, u, v) \in F(k)$. In fact, we may identify the fiber $p^{-1}(b)$ with the space,

$$\left\{(u, v) \in V_2(\hat{D}) \;\middle|\; \omega^1(u, v) = 1, \omega^2(u, v) = 0\right\},$$

which is $(\dim V^{\Omega} - 2)$-connected as it has been already noted above. Thus, $p^{-1}(b)$ is $\dim V^{\Omega} - 2 = (4n - 4k + 4) - 2 = 4n - 4k + 2$-connected.

Now an application of the homotopy long exact sequence to the bundle $p : R(k) \to R(k-1)$ gives us that,

$$\pi_i\big(R(k)\big) = \pi_i\big(R(k-1)\big), \quad \text{for } i \leq 4n - 4k + 2.$$

But then by induction hypothesis we have,

$$\pi_i\big(R(k)\big) = \pi_i\big(R(k-1)\big) = 0, \quad \text{for } i \leq 4n - 4k + 2.$$

Hence, $R(k)$ is $4n - 4k + 2$-connected. This concludes the proof. □

We now prove the existence.

Proof of Theorem 4.2.26. Suppose $u : \Sigma \to M$ is any given map. We first observe the implication of the second part of the hypothesis. If both ξ and $\mathcal{D}$ are given to be cotrivializable, it is easy to note that there exists an injective bundle morphism $G : T\Sigma/\xi \to u^*TM/\mathcal{D}$. In general, the obstruction to the existence of such a map G lies in $H^2(\Sigma)$ ([Hus94]). Hence, with $H^2(\Sigma) = 0$, we have the required bundle map.

Now for a fixed monomorphism G, we construct the fiber bundle $\mathcal{F} = \mathcal{F}(u, G) \subset \hom(\xi, u^*TM)$ as discussed above. By Lemma 4.2.27, the fibers of $\mathcal{F}$ are $4n - 4k + 2$ connected, where $\operatorname{rk}\mathcal{D} = 4n$ and $\operatorname{rk}\xi = 2k$. From the hypothesis we have,

$$\operatorname{rk}\mathcal{D} \geq 3\operatorname{rk}\xi - 2 = 6k - 2 \;\Leftrightarrow\; 4n - 4k + 2 \geq 2k = \dim\Sigma - 1$$

Hence we have a global section $\hat{F} \in \Gamma\mathcal{F}$.

Lastly we observe that for any such global section $\hat{F} \in \Gamma\mathcal{F}$, we may get a formal, ξ-isocontact immersion $F : T\Sigma \to u^*TM$ covering u, satisfying $F|_{\mathcal{D}} = \hat{F}$ and $\tilde{F} = G$, by choosing some splitting of $T\Sigma/\xi$ and $u^*TM/\mathcal{D}$. The proof now follows from a direct application of Theorem 4.2.23. $\square$

An application of Theorem 4.2.26 to constant maps $\Sigma \to M$ gives us the following corollary.

Corollary 4.2.28. *Let $\xi \subset T\Sigma, \mathcal{D} \subset TM$ be as in Theorem 4.2.26, satisfying the hypothesis. Then there exists an ξ-isocontact immersion $(\Sigma, \xi) \to (M, \mathcal{D})$ in any arbitrary small neighborhood of a point in M.*

Chapter 5

Partially Horizontal Maps

Throughout this chapter M will denote a smooth manifold with a fixed distribution $\mathcal{D}$ having the curvature form $\Omega = \Omega_{\mathcal{D}}$. We shall prove an h-principle (Theorem 5.1.17) for certain 'regular' class of smooth immersions $u : \Sigma \to (M, \mathcal{D})$, which induce distributions on Σ. Gromov defines such maps as *partially* horizontal maps. Application of the h-principle are contained in section 5.2.

5.1 A General Approach to m-Horizontal Immersions

In general, for an arbitrary immersion $u : \Sigma \to M$, the object $du^{-1}(\mathcal{D}) \subset T\Sigma$ need not be a distribution. We are thus naturally led to the following definition introduced by Gromov.

Definition 5.1.1. [Gro96, Pg.256] An immersion $u : \Sigma \to (M, \mathcal{D})$ is said to be *m-horizontal* if $du^{-1}(\mathcal{D})$ is a rank m distribution on Σ.

An m-horizontal immersion, where $m = \dim \Sigma$, is clearly a $\mathcal{D}$-horizontal immersion. This justifies the nomenclature.

Now, given an m-horizontal immersion $u : \Sigma \to M$, let us denote the distribution $du^{-1}(\mathcal{D})$ by G. Note that, $u : (\Sigma, G) \to (M, \mathcal{D})$ is then a G-isocontact immersion. Therefore, the induced map,

$$\tilde{du} : T\Sigma/G \to u^*TM/\mathcal{D}$$

is injective and we have the following numerical constraints:

$$\operatorname{rk} \mathcal{D} \geq m, \quad \operatorname{cork} \mathcal{D} \geq \dim \Sigma - m.$$

As discussed in Proposition 3.1.4, we have the following commutative diagram:

$$\begin{tikzcd}
\Lambda^2 G \arrow[r, "du"] \arrow[d, "\Omega_G"'] & \Lambda^2 u^*\mathcal{D} \arrow[d, "u^*\Omega_{\mathcal{D}}"] \\
T\Sigma/G \arrow[r, "\tilde{du}"'] & u^*TM/\mathcal{D}
\end{tikzcd}$$

where Ω_G denotes the curvature form of G.

For smooth maps $u : \Sigma \to M$ and distributions $G \subset T\Sigma$ of rank m, let us now consider the operator,

$$(u, G) \mapsto u^*\lambda|_G,$$

where $\lambda : TM \to TM/\mathcal{D}$ is the quotient map. We have that

$$u^*\lambda|_G = 0 \;\Rightarrow\; du(G) \subset \mathcal{D} \;\Rightarrow\; G \subset du^{-1}(\mathcal{D}).$$

Now, if we further assume that the map $u : \Sigma \to M$ is an immersion satisfying,

$$\mathrm{rk}(\lambda \circ du) \geq \dim \Sigma - m,$$

then a simple dimension counting argument gives us that $du^{-1}(\mathcal{D}) = G$, and hence u is then an m-horizontal immersion. Furthermore, it should be noted that in this case G is completely determined by u.

We now formalize this in the framework of differential operators as discussed in section 2.2.2. Note that, the space of m-distributions on Σ can be viewed as the space of sections of the m-Grassmannian bundle $\mathrm{Gr}_m T\Sigma$ over Σ. Let

$$\mathcal{B} = C^\infty(\Sigma, M) \times \Gamma \mathrm{Gr}_m T\Sigma.$$

For each $(u, G) \in \mathcal{B}$, consider the vector space,

$$\mathcal{E}_{(u,G)} = \Gamma \hom\big(G, u^*TM/\mathcal{D}\big).$$

We have $u^*\lambda|_G \in \mathcal{E}_{(u,G)}$. Thus the operator,

$$\mathfrak{D}^{m\text{-HorGr}} : (u, G) \mapsto u^*\lambda|_G$$

can be treated as a *section* of the infinite dimensional vector bundle $\mathcal{E} \to \mathcal{B}$. The solutions (u, G) of the equation $\mathfrak{D}^{m\text{-HorGr}} = 0$, for which $u : \Sigma \to M$ is an immersion with $\mathrm{rk}\,\mathrm{Im}(\lambda \circ du) \geq$

$\dim \Sigma - m$, are precisely the m-horizontal immersions $\Sigma \to M$. We should note that $\mathfrak{D}^{m\text{-HorGr}}$ is *not* a differential operator in the sense of Definition 2.2.19. Let us now introduce an auxiliary differential operator in this context.

5.1.1 Auxiliary Differential operator $\mathfrak{D} = \mathfrak{D}^{G,\rho}$

Let us first fix a rank m distribution $G \subset T\Sigma$ and a splitting ρ of the short exact sequence,

$$\begin{tikzcd} 0 \arrow[r] & G \arrow[r, hook] & T\Sigma \arrow[r, two heads] & T\Sigma/G \arrow[r] \arrow[l, bend right, "\rho"'] & 0 \end{tikzcd}$$

Since $\mathrm{Gr}_m(W)$ is locally parameterized by hom $(V, W/V)$, for any subspace V in a vector space W, we may identify a neighborhood of G in $\Gamma \mathrm{Gr}_m T\Sigma$ with the infinite dimensional vector space $\Gamma \hom(G, T\Sigma/G)$ as follows : For each morphism $\phi : G \to T\Sigma/G$, we have the injective map,

$$\begin{aligned} \bar{\phi} : G &\to T\Sigma \\ X &\mapsto X + \rho\phi X \end{aligned}$$

which defines a rank m distribution, namely,

$$G^\phi = \operatorname{Im} \bar{\phi} = \left\{ X + \rho\phi X \;\middle|\; X \in G \right\}.$$

Then $\left\{ G^\phi \;\middle|\; \phi \in \Gamma \hom(G, T\Sigma/G) \right\}$ is a neighborhood of G parameterized by $\Gamma \hom(G, T\Sigma/G)$.

Consider the subspace,

$$\mathcal{U} = \mathcal{U}(G, \rho) = C^\infty(\Sigma, M) \times \Gamma \hom(G, T\Sigma/G).$$

We can then identify $\mathcal{U} \hookrightarrow \mathcal{B}$ by $(u, \phi) \mapsto \left(u, G^\phi\right)$. For simplicity, let us assume that $\mathcal{D} = \bigcap_{s=1}^p \ker \lambda^s$ for global 1-forms $\lambda^s \in \Omega^1(M)$, so that the fiber over (u, G^ϕ) is $\Gamma \hom(G^\phi, \mathbb{R}^p)$. We then observe that the vector bundle $\mathcal{E} \to \mathcal{B}$ trivializes over $\mathcal{U}$ as,

$$\mathcal{E}|_{\mathcal{U}} \cong \mathcal{U} \times \Gamma \hom(G, \mathbb{R}^p).$$

We have the following diagram,

$$\begin{tikzcd}
\mathcal{U} \times \Gamma \hom(G, \mathbb{R}^p) \arrow[r, hook] \arrow[d] & \mathcal{E} \arrow[d] \\
\mathcal{U} = C^\infty(\Sigma, M) \times \Gamma \hom(G, T\Sigma/G) \arrow[r, hook] & \mathcal{B}
\end{tikzcd}$$

where, there is a canonical isomorphism of fibers,

$$\begin{aligned}
\Gamma \hom(G, \mathbb{R}^p) &\longrightarrow \Gamma \hom(G^\phi, \mathbb{R}^p) \\
\alpha &\longmapsto \alpha \circ \bar{\phi}^{-1}
\end{aligned}$$

Now, consider an auxiliary differential operator,

$$\begin{aligned}
\mathfrak{D} = \mathfrak{D}^{G,\rho} : C^\infty(\Sigma, M) \times \Gamma \hom(G, T\Sigma/G) &\longrightarrow \Gamma \hom(G, \mathbb{R}^p) \\
(u, \phi) &\longmapsto \bar{\phi}^* u^* \lambda^s = \big(X \mapsto u^* \lambda^s (X + \rho\phi X)\big)
\end{aligned}$$

It is clear that,

$$\mathfrak{D}(u, \phi) = 0 \ \Rightarrow \ u^* \lambda^s|_{\operatorname{Im} \bar{\phi}} = 0, \ s = 1, \dots, p \ \Rightarrow \ du\big(\operatorname{Im} \bar{\phi}\big) \subset \mathcal{D} \quad \text{i.e} \quad du(G^\phi) \subset \mathcal{D}.$$

Hence, if $(u, \phi) \in \mathcal{U}$ is a solution of $\mathfrak{D} = 0$, where u is an immersion and $\operatorname{rk} \operatorname{Im}(\lambda^s \circ du) \geq \dim \Sigma - m$, then $u : \Sigma \to M$ is indeed an m-horizontal immersion, inducing the rank m distribution G^ϕ.

Let us now determine the linearization operator of $\mathfrak{D}$ at some (u, ϕ),

$$\mathfrak{L}_{(u,\phi)} : \Gamma u^* TM \oplus \Gamma \hom(G, T\Sigma/G) \to \Gamma \hom(G, \mathbb{R}^p).$$

Suppose $\xi \in \Gamma u^* TM$ is represented by the family of maps $u_t : \Sigma \to M$ such that, $\xi_\sigma = \frac{d}{dt}|_{t=0} u_t(\sigma)$ for $\sigma \in \Sigma$ and $u_0 = u$. Then for any $\psi \in \Gamma \hom(G, T\Sigma/G)$ we have,

$$\begin{aligned}
\mathfrak{L}_{(u,\phi)}(\xi, \psi) &= \frac{d}{dt}\Big|_{t=0} \mathfrak{D}(u_t, \phi + t\psi) \\
&= \lim_{t \to 0} \frac{1}{t} \Big[u_t^* \lambda^s \circ \overline{\phi + t\psi} - u^* \lambda^s \circ \bar{\phi} \Big] \\
&= \lim_{t \to 0} \frac{1}{t} \Big[u_t^* \lambda^s - u^* \lambda^s \Big] \circ \bar{\phi} + \lim_{t \to 0} \frac{1}{t} u_t^* \lambda^s \circ \big(t\rho \circ \psi\big) \\
&= \Big(\iota_\xi d\lambda^s + d\iota_\xi \lambda^s \Big) \circ \bar{\phi} + u^* \lambda^s \circ \rho \circ \psi
\end{aligned}$$

Infinitesimal Inversion of $\mathfrak{D}$

Restricting the linearization operator to the subspace $\Gamma u^*\mathcal{D}$ we have,

$$\begin{aligned}\mathcal{L}_{(u,\phi)} : \Gamma u^*\mathcal{D} \oplus \Gamma\hom(G, T\Sigma/G) &\to \Gamma\hom(G, \mathbb{R}^p)\\ (\xi, \psi) &\mapsto \iota_\xi d\lambda^s \circ \bar{\phi} + u^*\lambda^s \circ \rho \circ \psi\end{aligned}$$

$\mathcal{L}_{(u,\phi)}$ is clearly $C^\infty(\Sigma)$-linear and therefore, is induced by a bundle map

$$u^*\mathcal{D} \oplus \hom(G, T\Sigma/G) \to \hom(G, \mathbb{R}^p).$$

Furthermore, if $\mathfrak{D}(u, \phi) = 0$, then it follows that

$$\mathcal{L}_{(u,\phi)}(\xi, \psi) = \iota_\xi \Omega \circ \bar{\phi} + u^*\lambda^s \circ \rho \circ \psi,$$

where $\Omega = \Omega_\mathcal{D}$ is the curvature form. As before, we identify the regularity condition on the solution tuples (u, ϕ).

Definition 5.1.2. A tuple (u, ϕ), where $u : \Sigma \to M$ is an m-horizontal immersion inducing $G^\phi = du^{-1}(\mathcal{D})$, is called Ω-*regular* if the bundle map $\mathcal{L}_{(u,\phi)}$ is surjective.

It is to be noted that the above notion of Ω-regularity is different from what was defined in the previous chapter. The notion of regularity is indeed independent of any choice of defining forms for $\mathcal{D}$ or the choice of a splitting map $\rho : T\Sigma/G \to T\Sigma$, as it can be seen from the proposition below.

Now, when u is an m-horizontal immersion inducing the chosen distribution G, that is if $G = du^{-1}(\mathcal{D})$, then the homomorphism $\phi = 0$. Since ρ is a splitting morphism in the diagram below,

$$\begin{tikzcd} T\Sigma \arrow[r, "du"] \arrow[d] & TM \arrow[d] \\ T\Sigma/G \arrow[r, "\tilde{du}"'] \arrow[u, bend left, "\rho"] & TM/\mathcal{D} \end{tikzcd}$$

we have that,

$$u^*\lambda^s \circ \rho\big(X \mod G\big) = \tilde{du}\big(\rho(X \mod G) \mod G\big) = \tilde{du}(X \mod G),$$

for any $X \in T\Sigma$. Thus, we have $\tilde{du} = u^*\lambda^s \circ \rho$ for any choice of splitting map ρ. Hence, the linearization operator becomes,

$$\mathcal{L}_{(u,0)}(\xi, \psi) = \iota_\xi \Omega|_G + \tilde{du} \circ \psi$$

In particular, the Ω-regularity of a solution tuple may now be defined as follows.

Definition 5.1.3. A tuple (u, H), where $u : \Sigma \to M$ is an m-horizontal immersion inducing the rank m distribution $H = du^{-1}(\mathcal{D})$, is called Ω-regular if the bundle map,

$$\begin{aligned} \mathcal{L}_{(u,H)} : u^*\mathcal{D} \oplus \hom(H, T\Sigma/H) &\to \hom(H, u^*TM/\mathcal{D}) \\ (\xi, \psi) &\mapsto \iota_\xi \Omega|_H + \tilde{du} \circ \psi \end{aligned}$$

is an epimorphism, where $\tilde{du} : T\Sigma/H \to u^*TM/\mathcal{D}$ is the induced monomorphism.

Let us now show that all these notions of regularity are in fact equivalent.

Proposition 5.1.4. *Suppose $G \subset T\Sigma$ is a k-dimensional distribution on Σ and $\rho : T\Sigma/G \hookrightarrow G$ is a splitting morphism. Let $u : \Sigma \to M$ be an m-horizontal immersion inducing the distribution $G^\phi = du^{-1}\mathcal{D}$, for some morphism $\phi : G \to T\Sigma/G$. Denote G^ϕ by H. Then the following are equivalent.*

1. *The operator $\mathcal{L}_{(u,H)}$ is an epimorphism.*

2. *The operator $\mathcal{L}_{(u,\phi)}$ is an epimorphism.*

3. *The bundle map,*

$$\begin{aligned} \Omega_\bullet : u^*\mathcal{D} &\to \hom\Big(H, u^*(TM/\mathcal{D}) \big/ \operatorname{Im} \tilde{du}\Big) \\ \xi &\mapsto q \circ \iota_\xi \Omega|_G \end{aligned}$$

is an epimorphism, where $q : u^*TM/\mathcal{D} \to u^*(TM/\mathcal{D}) \big/ \operatorname{Im} \tilde{du}$ is the quotient map.

Proof. Let us first prove, 1 $\Leftrightarrow$ 2. Recall that the map $\mathcal{L}_{(u,H)}$ is given as,

$$\begin{aligned} \mathcal{L}_{(u,H)} : u^*\mathcal{D} \oplus \hom(H, T\Sigma/H) &\to \hom(H, u^*TM/\mathcal{D}) \\ (\xi, \psi) &\mapsto \iota_\xi \Omega|_H + \tilde{du}^H \circ \psi = \Big(X \mapsto \Omega(\xi, u_* X) + \tilde{du}^H \circ \psi(X)\Big) \end{aligned}$$

where $\tilde{du}^H : T\Sigma/H \to u^*TM/\mathcal{D}$ is the induced monomorphism. We shall relate this map with $\mathcal{L}_{(u,\phi)}$.

First observe that the bundle map, $\beta : T\Sigma/G \to T\Sigma/H$ given as,

$$\beta(X \mod G) = \rho\big(X \mod G\big) \mod H,$$

is an isomorphism. Indeed we have that,

$$\beta(X \mod G) = 0 \;\Rightarrow \rho\big(X \mod G\big) \in H \;\Rightarrow \rho(X \mod G) = 0,$$

since $\operatorname{Im}\rho \cap H = \operatorname{Im}\rho \cap G^{\phi} = 0$. Now, ρ being injective, we conclude that $X \mod G = 0$, showing that β is injective. But, this further implies that β is an isomorphism for $\operatorname{rk} T\Sigma/G = \operatorname{rk} T\Sigma/H$.

We also have that $\bar{\phi} : G \to G^{\phi} = H$ is an isomorphism and hence get the isomorphism,

$$\begin{aligned} \alpha : \hom(G, T\Sigma/G) &\to \hom(H, T\Sigma/H) \\ \psi &\mapsto \beta \circ \psi \circ \bar{\phi}^{-1} \end{aligned}$$

Now for any $\psi : G \to T\Sigma/G$ and $X \in G$ we see,

$$u^*\lambda^s\big(\rho\psi(X)\big) = \tilde{du}^H\Big(\rho\psi(X) \mod H\Big) = \tilde{du}^H \circ \beta(\psi(X)).$$

Consequently we now have that,

$$\begin{aligned} \mathcal{L}_{(u,\phi)}\Big(\xi, \psi\Big)(X) &= \Omega\big(\xi, \bar{\phi}(X)\big) + u^*\lambda^s\big(\rho\psi(X)\big) \\ &= \Omega\big(\xi, \bar{\phi}(X)\big) + \tilde{du}^H \circ \beta\big(\psi(X)\big) \\ &= \Omega(\xi, \bar{\phi}X) + \tilde{du}^H \circ \alpha(\psi)(\bar{\phi}X) \\ &= \mathcal{L}_{(u,H)}\Big(\xi, \alpha(\psi)\Big)\big(\bar{\phi}X\big) \\ \Rightarrow \mathcal{L}_{(u,\phi)}(\xi, \psi) &= \mathcal{L}_{(u,H)}(\xi, \alpha(\psi)) \circ \bar{\phi} \end{aligned}$$

It is now immediate that $\mathcal{L}_{(u,\phi)}$ is surjective if and only if $\mathcal{L}_{(u,H)}$ is, since both α and $\bar{\phi}$ are isomorphisms. This concludes the proof that $1 \Leftrightarrow 2$.

Next, we show that $1 \Leftrightarrow 3$. For any $(\xi, \psi) \in u^*\mathcal{D} \oplus \hom(H, T\Sigma/H)$ we have,

$$\mathcal{L}_{(u,H)}(\xi, \psi) = \iota_\xi \Omega|_H + \tilde{du} \circ \psi.$$

We observe that $\mathcal{L}_{(u,H)}$ restricted to $\hom(H, T\Sigma/H)$ equals the morphism

$$(\tilde{du})_* : \hom(H, T\Sigma/H) \to \hom(H, u^*(TM/\mathcal{D})),$$

which is injective linear. Hence, the fiber wise surjectivity of $\mathcal{L}_{(u,H)}$ is equivalent to that of $\Omega_\bullet$. $\square$

The Microflexibility and Local h-Principle for the Sheaf of Ω-regular Tuples

For a given distribution $G \subset T\Sigma$, with a splitting map $\rho : T\Sigma/G \hookrightarrow T\Sigma$, we define a subspace $A \subset C^\infty(\Sigma, M) \times \Gamma(\hom(G, T\Sigma/G)$ as follows :

$$A = \left\{(u, \phi) \;\middle|\; u \text{ is an immersion, } \operatorname{rk}(\lambda^s \circ du) \geq \dim \Sigma - m \text{ and } \mathcal{L}_{(u,\phi)} \text{ is an epimorphism}\right\}$$

It is immediate that $A = \operatorname{Sol} \mathcal{A}$ for an *open* relation $\mathcal{A} \subset J^1(\Sigma, M) \times \hom(G, T\Sigma/G)^{(1)}$. Furthermore, we have observed that the operator $\mathfrak{D} : (u, \phi) \mapsto u^*\lambda^s|_{G^\phi}$ is infinitesimally invertible over the solution space A, with order of inversion 0.

As usual, we have the relation,

$$\mathcal{R}_\alpha^{G,\rho} = \mathcal{R}_\alpha^{G,\rho}(\mathfrak{D}, \mathcal{A}, 0) \subset J^{\alpha+1}(\Sigma, M) \times \hom(G, T\Sigma/G)^{(\alpha+1)},$$

consisting of $\mathcal{A}$-regular, α-infinitesimal solutions of $\mathcal{D}$. In other words, for $\alpha \geq 0$, the smooth solutions of $\mathcal{R}_\alpha$ are precisely the Ω-regular tuples (u, ϕ), where $u : \Sigma \to M$ is an m-horizontal immersion, inducing the distribution $G^\phi = du^{-1}(\mathcal{D})$. Let us denote the sheaves,

$$\Phi^{G,\rho} = \operatorname{Sol} \mathcal{R}_\alpha^{G,\rho} \quad \text{and} \quad \Psi_\alpha^{G,\rho} = \Gamma \mathcal{R}_\alpha^{G,\rho}.$$

Observation 5.1.5. We have the following.

1. The solution sheaf $\Phi^{G,\rho}$ is a microflexible sheaf, by Theorem 2.2.27
2. For $\alpha \geq 2$, the relation $\mathcal{R}_\alpha^{G,\rho}$ satisfies the local h-principle, i.e, the jet map $j^{\alpha+1} : \Phi^{G,\rho} \to \Psi_\alpha^{G,\rho}$ is a local weak homotopy equivalence, by Theorem 2.2.28

Just as before, let us now define the following first jet relation.

Definition 5.1.6. For a fixed $G \subset T\Sigma$ and a splitting map $\rho : T\Sigma/G \hookrightarrow T\Sigma$, define the relation $\mathcal{R}^{G,\rho} \subset J^(\Sigma, M) \times \hom(G, T\Sigma/G)^{(1)}$ consisting of jets $j^1_{u,\phi}(\sigma)$ satisfying the following :

- du_σ is an injective, inducing $G_\sigma^\phi = du^{-1}\mathcal{D}_{u(\sigma)}$ and the map $\mathcal{L}_{(u,\phi)}$ is surjective at σ
- the curvature condition $u^*\Omega|_{G_\sigma^\phi} = \tilde{du}^\phi \circ \Omega_{G_\sigma^\phi}$ holds, where Ω_{G^ϕ} is the curvature form of G^ϕ and $\tilde{du}^\phi : T\Sigma/G^\phi|_\sigma \to TM/\mathcal{D}|_{u(\sigma)}$ is the induced map.

It is immediate that $\mathcal{R}^{G,\rho} \subset \mathcal{R}_0^{G,\rho}$ and $\Phi^{G,\rho} = \operatorname{Sol}\mathcal{R}^{G,\rho}$. Let us now focus on a relation, independent of the choice of G and ρ.

5.1.2 The Relation $\mathcal{R}^{m\text{-HorGr}}$

Recall from Proposition 3.1.4 that for any tuple (u, H), where $u : \Sigma \to M$ is an m-horizontal immersion inducing the distribution $H = du^{-1}\mathcal{D}$, the curvature condition is understood as,

$$u^*\Omega|_H = \tilde{du} \circ \Omega_H,$$

where $\tilde{du} : T\Sigma/H \to u^*TM/\mathcal{D}$ is the induced map and $\Omega_H : \Lambda^2 H \to T\Sigma/H$ is the curvature 2-form. Now, the curvature form Ω_H at the point σ is determined by the first jet $j^1_H(\sigma)$. Thus, just as in Definition 3.1.7, we define a first order relation $\mathcal{R}^{m\text{-HorGr}}$ as follows.

Definition 5.1.7. $\mathcal{R}^{m\text{-HorGr}} \subset J^1(\Sigma, M) \times (\mathrm{Gr}_m T\Sigma)^{(1)}$ consists of jets $j^1_{u,H}(\sigma)$ satisfying,

- $P = du_\sigma$ is injective.
- $P^*\lambda^s|_{H_\sigma} = 0$ and the induced map $\tilde{P} : T\Sigma/H|_\sigma \to TM/\mathcal{D}|_{u(\sigma)}$ is injective.
- the linear map,

$$\begin{aligned} \mathcal{D}_{u(\sigma)} \oplus \hom(H, T\Sigma/H)|_\sigma &\to \hom\left(H_\sigma, TM/\mathcal{D}|_{u(\sigma)}\right) \\ (\xi, \psi) &\mapsto P^*\iota_\xi\Omega|_{H_\sigma} + \tilde{P} \circ \psi \end{aligned}$$

  is surjective.
- the curvature condition,

$$P^*\Omega|_{H_\sigma} = \tilde{P} \circ \Omega_{H_\sigma},$$

  holds at the point σ, where $\Omega_H : \Lambda^2 H \to T\Sigma/H$ is the curvature 2-form.

For each $\alpha \geq 0$, we have the relations $\mathcal{R}_\alpha^{m\text{-HorGr}} \subset J^{\alpha+1}(\Sigma, M) \times (\mathrm{Gr}_m T\Sigma)^{(\alpha+1)}$, consisting of jets $j^{\alpha+1}_{u,H}(\sigma)$ satisfying,

- $j^1_{u,H}(\sigma) \in \mathcal{R}^{m\text{-HorGr}}$, and

- $j^{\alpha}_{\mathfrak{D}^{m\text{-HorGr}}(u,H)}(\sigma) = 0$ in $\hom(H, u^*TM/\mathcal{D})^{(\alpha)}$.

It is immediate that $\mathcal{R}^{m\text{-HorGr}} \subset \mathcal{R}_0^{m\text{-HorGr}}$. We have the following jet lifting lemma, very similar to Lemma 3.2.1.

Lemma 5.1.8. *For any $\alpha \geq 1$, the jet projection map*

$$p = p_1^{\alpha+1} : J^{\alpha+1}(\Sigma, M) \times (Gr_m T\Sigma)^{(\alpha+1)} \to J^1(\Sigma, M) \times (Gr_m T\Sigma)^{(1)}$$

maps the relation $\mathcal{R}_\alpha^{m\text{-HorGr}}$ surjectively onto $\mathcal{R}^{m\text{-HorGr}}$. The fiber over each jet in $\mathcal{R}^{m\text{-HorGr}}$ is contractible. Furthermore, any section of $\mathcal{R}^{m\text{-HorGr}}$, defined over a contractible chart in Σ, can be lifted to $\mathcal{R}_\alpha^{m\text{-HorGr}}$ along p, and consequently, the induced sheaf map $p : \Gamma\mathcal{R}_\alpha^{m\text{-HorGr}} \to \Gamma\mathcal{R}^{m\text{-HorGr}}$ is a weak homotopy equivalence.

Proof. In order to proof the lemma, let us interpret the curvature condition in a different light. First observe that we have an operator,

$$\begin{aligned} C^\infty(\Sigma, M) \times \Omega^1(\Sigma, \mathbb{R}^q) \times C^\infty\big(\Sigma, \mathrm{Mat}_{q\times p}\big) &\to \Omega^1(\Sigma, \mathbb{R}^p) \\ \Big(u,\ \mu = \big(\mu^r\big),\ A\Big) &\mapsto \big(u^*\lambda^s\big) - A\mu \end{aligned}$$

where $q = \dim \Sigma - m$. It is then immediate that for any tuple (u, μ, A),

$$\big(u^*\lambda^s\big) = A\mu \quad \Rightarrow \quad u^*\lambda^s|_{\cap_{r=1}^q \ker \mu^r} = 0.$$

In particular, if we assume that the tuple $\mu = \big(\mu^r\big)$ of 1-forms is point-wise independent, and if u is an immersion with $\operatorname{rk} \operatorname{Im}(\lambda^s \circ du) \geq q$, then we have that

$$du^{-1}(\mathcal{D}) = H := \cap_{r=1}^q \ker \mu^r.$$

Now, applying the exterior derivative on both sides of the equation $\big(u^*\lambda^s\big) = A\mu$ and restricting to the common kernel H, we have,

$$\big(u^* d\lambda^s|_H\big) = A\big(d\mu^r|_H\big).$$

We note that this equation represents the curvature condition for the tuple (u, H), i.e, the equation

$$u^*\Omega|_H = \tilde{du} \circ \Omega_H.$$

Indeed, we have that the matrix A represents the linear map $\tilde{du} : T\Sigma/H \cong \mathbb{R}^q \to TM/\mathcal{D} \cong \mathbb{R}^p$ and the curvature 2-form for $H = \cap \ker \mu^r$ is given as, $\Omega_H = (d\mu^r|_H)$.

Now in order to represent jets in $\mathcal{R}^{m\text{-HorGr}}_\alpha$, we need to find out the α^{th}-order differentials of the equations $\big(u^*\lambda^s\big) = A\big(\mu^r\big)$. Since we are only interested in jets, we might as well work with some choice of local coordinates and consequently, we get system of equations,

$$\big(u^*\lambda^s(\partial_i)\big) = A\big(\mu^r(\partial_i)\big) \quad \text{for } \partial_i \equiv \partial_{x^i},\ 1 \leq i \leq \dim\Sigma.$$

Expanding we have,

$$\Big((\lambda^s_\nu \circ u)\partial_i u^\nu\Big)_{q\times k} = A\Big(\mu^r_i\Big)_{p\times n},$$

where $\lambda^s = \sum_{\nu=1}^n \lambda^s_\mu dy^\nu$ and $\mu^r = \sum_{i=1}^k \mu^r_i dx^i$, with respect to the coordinates, for $n = \dim M, k = \dim\Sigma$. For some arbitrary partial differential ∂_I, where the multi-index I is of order $|I| \leq \alpha$, we have, by the Leibniz rule,

$$\left(\partial_I\Big((\lambda^s_\nu \circ u)\partial_i u^\nu\Big)\right) = A\Big(\partial_I\mu^r_i\Big) + \text{terms involving higher order derivatives of } A \qquad (*)$$

Treating these as formal equations in the jet $j^\alpha_{u,\mu,A}(\sigma)$, we note that the higher order jets in A occur *linearly*. In particular, setting all the higher order jets of A at σ to identically zero, we see that equation $(*)$ transforms into,

$$\left(\partial_I\Big((\lambda^s_\nu \circ u)\partial_i u^\nu\Big)\right)\Big|_\sigma = A(\sigma)\Big(\partial_I\mu^r_i\Big)\Big|_\sigma \qquad (*')$$

Note that this system is identical to the equations defining the relation $\mathcal{R}_\alpha$ associated to the horizontal immersion relation $\mathcal{R}^{\text{Hor}}$, modulo $\operatorname{Im} A(\sigma)$.

Now, suppose we are given some jet $j^1_{u,H}(\sigma) \in \mathcal{R}^{m\text{-HorGr}}$, which is represented as the jet $j^1_{u,\mu,A}(\sigma)$. Then note that the Ω-regularity is satisfied modulo $\operatorname{Im} A(\sigma)$ (Definition 5.1.10). Also the curvature condition is given as,

$$\left(u^*d\lambda^s|_H\right)\Big|_\sigma \equiv 0 \mod \operatorname{Im} A(\sigma),$$

which is same as the curvature condition for horizontal immersions, modulo $\operatorname{Im} A(\sigma)$. Thus, following the proof of Lemma 3.2.1, we are able to formally solve for the jet $j^\alpha_u(\sigma)$ satisfying the system,

$$\left(\partial_I\Big((\lambda^s_\nu \circ u)\partial_i u^\nu\Big)\right)\Big|_\sigma \equiv 0 \mod \operatorname{Im} A(\sigma), \quad \text{for all } |I| \leq \alpha. \qquad (**)$$

Furthermore, the space of all such solutions is contractible.

Now, since $A(\sigma)$ represents the *injective* linear map $\tilde{du}_\sigma$, for any given jet $j^\alpha_u(\sigma)$ solving $(**)$, we are able to solve for the jet $j^\alpha_\mu(\sigma)$ for the tuple $\mu = (\mu^r)$, satisfying $(*')$ as well, which again has an affine solution space. Lastly, we get a jet $j^\alpha_{u,\mu,A}(\sigma)$ solving $(*)$ at σ, by arbitrarily solving the the higher jets of A satisfying the linear system given by $(*)$. In particular, setting all of them to zero gives a jet in $\mathcal{R}^{m\text{-HorGr}}_\alpha$, lifting the given jet $j^1_{u,H}(\sigma)$ in $\mathcal{R}^{m\text{-HorGr}}$.

As observed, the fiber over the jet in $\mathcal{R}^{m\text{-HorGr}}$ is contractible. Furthermore, the above argument can be performed over a contractible open set as well. This concludes the proof. □

5.1.3 h-principle on Open Manifolds

We have the following h-principle for open manifolds.

Theorem 5.1.9. *If Σ is an open manifold then the relation $\mathcal{R}^{m\text{-HorGr}}$ satisfies the parametric h-principle.*

Proof. Denote the solution sheaf of $\mathcal{R}^{m\text{-HorGr}}$ by $\Phi^{m\text{-HorGr}}$ and the sheaf of sections by $\Psi^{m\text{-HorGr}}$. We proceed with the proof in the following steps.

Step 1 We first show that $\Phi^{m\text{-HorGr}}$ is microflexible and $\mathcal{R}^{m\text{-HorGr}}$ satisfies the local h-principle, i.e, the sheaf map $j^1 : \Phi^{m\text{-HorGr}} \to \Psi^{m\text{-HorGr}}$ is a local weak homotopy equivalence.

Step 2 Next we show that $\Phi^{m\text{-HorGr}}$ is invariant under the natural $\mathrm{Diff}(\Sigma)$-action.

The proof is then immediate by appealing to Remark 2.2.12.

Proof of Step 1 : Observe that for a fixed $G \subset T\Sigma$ and a splitting map ρ, we have the fiber-preserving map,

$$\begin{aligned} \Xi : J^1(\Sigma, M) \times \hom(G, T\Sigma/G)^{(1)} &\to J^1(\Sigma, M) \times (\mathrm{Gr}_m T\Sigma)^{(1)} \\ \left(j^1_u(\sigma), j^1_\phi(\sigma)\right) &\mapsto \left(j^1_u(\sigma), j^1_{G^\phi}(\sigma)\right) \end{aligned}$$

where $G^\phi \subset T\Sigma$ is interpreted as a (local) section of $\mathrm{Gr}_m T\Sigma$. This map Ξ embeds $\mathcal{R}^{G,\rho}$ (Definition 5.1.6) as an *open* subset $\widetilde{\mathcal{R}^{G,\rho}} := \mathrm{Im}\,\Xi \subset \mathcal{R}^{m\text{-HorGr}}$ and it is easy to see that,

$$\mathcal{R}^{m\text{-HorGr}} = \bigcup_{G,\rho} \widetilde{\mathcal{R}^{G,\rho}}.$$

Now, we have the the sheaves $\Phi^{G,\rho} = \operatorname{Sol}\mathcal{R}^{G,\rho}$ and $\Psi_\alpha^{G,\rho} = \Gamma\mathcal{R}_\alpha^{G,\rho}$, as in Observation 5.1.5. We see that Ξ induces the sheaf maps,

$$\Phi^{G,\rho} \hookrightarrow \Phi^{m\text{-HorGr}}, \quad \Psi_\alpha^{G,\rho} \hookrightarrow \Gamma\mathcal{R}_\alpha^{m\text{-HorGr}}.$$

As $\Gamma\hom(G, T\Sigma/G)$ gives an open covering for the space $\Gamma\mathrm{Gr}_m T\Sigma$, we see that $\Phi^{G,\rho}$ embeds in $\Phi^{m\text{-HorGr}}$ as an *open* subsheaf; in fact, by varying $G \subset T\Sigma$ and the splitting map ρ, we can cover $\Phi^{m\text{-HorGr}}$ by the images of the sheaves $\Phi^{G,\rho}$.

Since the question of microflexibility is regarding lifting homotopies defined on pairs of *compact* sets in Σ, we see that any homotopy lifting diagram for $\Phi^{m\text{-HorGr}}$ can be transferred to some $\Phi^{G,\rho} = \operatorname{Sol}\mathcal{R}^{G,\rho}$, for some suitably chosen G, ρ. Now, $\Phi^{G,\rho}$ is microflexible by Observation 5.1.5 (1). Hence, we see that $\Phi^{m\text{-HorGr}}$ is microflexible as well.

By similar arguments, we also get $j^3 : \Phi^{m\text{-HorGr}} \to \Gamma\mathcal{R}_2^{m\text{-HorGr}}$ is a local weak homotopy equivalence by Observation 5.1.5 (2). Now, from Lemma 5.1.8 we have, $p_1^3 : \Gamma\mathcal{R}_2^{m\text{-HorGr}} \to \Psi^{m\text{-HorGr}}$ is a weak homotopy equivalence. Composing the maps we have, $j^1 = p_1^3 \circ j^3 : \Phi^{m\text{-HorGr}} \to \Psi^{m\text{-HorGr}}$ is a local weak homotopy equivalence.

Proof of Step 2 : Recall that, as a consequence of Proposition 5.1.4, an m-horizontal immersion u, inducing $H = du^{-1}\mathcal{D}$, is Ω-regular if the bundle map,

$$\begin{aligned} \Omega_\bullet : u^*\mathcal{D} &\to \hom\left(H, u^*TM/\mathcal{D}/\operatorname{Im}\tilde{du}\right) \\ \xi &\mapsto \left(X \mapsto \Omega(\xi, u_* X) \mod \operatorname{Im}\tilde{du}\right) \end{aligned}$$

is surjective. In particular, Ω-regularity of (u, H) is completely understood via the *image* of the differential map $du : T\Sigma \to TM$. Now $\operatorname{Im} du$ remains unchanged after an $\mathrm{Diff}(\Sigma)$-action. Indeed, suppose $\zeta \in \mathrm{Diff}(\Sigma)$ is some (local) diffeomorphism and denote $v = u \circ \zeta$. Then we see that

$$dv^{-1}\mathcal{D} = (du \circ d\zeta)^{-1}\mathcal{D} = d\zeta^{-1} du^{-1}\mathcal{D}$$

As u is an m-horizontal immersion and ζ is a diffeomorphism, we have that v is again an m-horizontal immersion. As for the regularity, we similarly observe that $\operatorname{Im}\tilde{du} = \operatorname{Im}\tilde{dv}$ and hence clearly the tuple $(v, dv^{-1}(\mathcal{D}))$ is then Ω-regular. Thus, the solution sheaf is $\mathrm{Diff}(\Sigma)$-invariant.

This completes the proof in view of Remark 2.2.12. $\square$

h-Principle of the Relation $\mathcal{R}^{m\text{-}\mathbf{Hor}} \subset J^1(\Sigma, M)$

Recall that in Proposition 5.1.4 (3), we have rephrased the Ω-regularity of the tuple (u, H) in terms the surjectivity of the following bundle map,

$$\begin{aligned} \Omega_\bullet : u^*\mathcal{D} &\to \hom\left(H, u^*TM/\mathcal{D}/\operatorname{Im}\tilde{du}\right) \\ \xi &\mapsto \left(X \mapsto \Omega(\xi, u_*X) \mod \operatorname{Im}\tilde{du}\right) \end{aligned}$$

which only involves the jets in $J^1(\Sigma, M)$. So, we may adopt the following definition for Ω-regularity of m-horizontal immersions due to Gromov.

Definition 5.1.10. [Gro96, pg. 256] An m-horizontal immersion $u : \Sigma \to M$, inducing $H = du^{-1}(\mathcal{D})$, is called $\Omega_\bullet$-*regular*, if the bundle map,

$$\begin{aligned} \Omega_\bullet : u^*\mathcal{D} &\longrightarrow \hom\left(H, u^*(TM/\mathcal{D})/\operatorname{Im}\tilde{du}\right) \\ \xi &\longmapsto q \circ \iota_\xi \Omega|_H \end{aligned}$$

is an epimorphism, where $\tilde{du} : T\Sigma/H \to u^*TM/\mathcal{D}$ is the induced monomorphism and $q : u^*TM/\mathcal{D} \to u^*(TM/\mathcal{D})/\operatorname{Im}\tilde{du}$ is the quotient map.

On the other hand the curvature condition on (u, H) involves first jet information from $(\mathrm{Gr}_m T\Sigma)^{(1)}$. Now, for a jet $j^1_{(u,G)}(\sigma) \in \mathcal{R}^{m\text{-}\mathsf{HorGr}}$, the curvature condition $u^*\Omega|_{G_\sigma} = \tilde{du} \circ \Omega_G|_\sigma$ gives us, $u^*\Omega|_{G_\sigma} \equiv 0 \mod \operatorname{Im}\tilde{du}$, and consequently we have that $d_\sigma u(G) \subset \ker \Omega_\bullet$. We then proceed to define a relation $\mathcal{R}^{m\text{-}\mathsf{Hor}} \subset J^1(\Sigma, M)$ as follows.

Definition 5.1.11. [Gro96, pg. 256] The relation $\mathcal{R}^{m\text{-}\mathsf{Hor}} \subset J^1(\Sigma, M)$ consists of 1-jets $(\sigma, y, F) \in J^1(\Sigma, M)$ satisfying the following.

- F is injective, such that $\dim(\operatorname{Im} F \cap \mathcal{D}_y) = m$.
- If $G = F^{-1}\mathcal{D}_y$ and $\tilde{F} : T_\sigma\Sigma/G \to TM/\mathcal{D}|_y$ is the induced map, then the map $\Omega_\bullet$ defined as,

$$\begin{aligned} \Omega_\bullet : \mathcal{D}_y &\to \hom\left(G, \left(TM/\mathcal{D}|_y\right)/\operatorname{Im}\tilde{F}\right) \\ \xi &\mapsto F^*\iota_\xi\Omega|_G \mod \operatorname{Im}\tilde{F} \end{aligned}$$

  is surjective.
- $F(G) \subset \ker \Omega_\bullet$.

The obvious projection map, $j^1_{u,G}(\sigma) \to j^1_u(\sigma)$ maps the relation $\mathcal{R}^{m\text{-}\mathsf{HorGr}}$ into $\mathcal{R}^{m\text{-}\mathsf{Hor}}$. Indeed, we should note the following schematic diagram of relations,

$$\begin{tikzcd}
\mathcal{R}^{G,\rho} \arrow[rr, hook, "j^1_{(u,\phi)}(\sigma)\mapsto j^1_{(u,G^\phi)}(\sigma)"] \arrow[dr, "j^1_{(u,\phi)}(\sigma)\mapsto j^1_u(\sigma)"'] & & \mathcal{R}^{m\text{-}\mathsf{HorGr}} \arrow[dl, "j^1_u(\sigma)\leftarrow\!\shortmid j^1_{(u,H)}(\sigma)"] \\
& \mathcal{R}^{m\text{-}\mathsf{Hor}} &
\end{tikzcd}$$

for some fixed $G \subset T\Sigma$ and a splitting map $\rho : T\Sigma/G \hookrightarrow T\Sigma$. Here, the horizontal map is an embedding. Clearly, we have induced diagram in the solutions sheaves :

$$\begin{tikzcd}
\Phi^{G,\rho} \arrow[rr, hook, "(u,\phi)\mapsto(u,G^\phi)"] \arrow[dr, "(u,\phi)\mapsto u"'] & & \Phi^{m\text{-}\mathsf{HorGr}} \arrow[dl, "u\leftarrow\!\shortmid(u,H)"] \\
& \Phi^{m\text{-}\mathsf{Hor}} &
\end{tikzcd}$$

Note that the right diagonal arrow is in fact an identification. We have similar diagram for map of sections as well. We summarize all the relations that we have encountered so far in a tabular form in Table 5.1.

TABLE 5.1: The Operators and the Relations and the Sheaves

Domain	$(u,\phi) \in C^\infty(\Sigma, M) \times \Gamma \hom(G, T\Sigma/G)$ *	$(u,H) \in C^\infty(\Sigma, M) \times \Gamma Gr_m T\Sigma$	$u \in C^\infty(\Sigma, M)$
Operator	$\mathfrak{D}^{G,\rho} : (u,\phi) \mapsto \bar{\phi}^* u^* \lambda^s$ †	$\mathfrak{D}^{m\text{-HorGr}} : (u,H) \mapsto u^*\lambda\|_H$	- ‡
Equation	$\mathfrak{D}^{G,\rho} = 0$	$\mathfrak{D}^{m\text{-HorGr}} = 0$	-
Regularity Condition	u is an immersion, inducing $G^\phi = du^{-1}\mathcal{D}$ § and $\mathcal{L}_{u,\phi} : u^*\mathcal{D} \oplus \hom(G, T\Sigma/G) \to \hom(G, u^*TM/\mathcal{D})$ $(\xi,\psi) \mapsto u^*(\iota_\xi \Omega) \circ \bar{\phi} + u^*\lambda \circ \rho \circ \psi$ is surjective	u is an immersion, inducing $H = du^{-1}\mathcal{D}$ and $\mathcal{L}_{u,H} : u^*\mathcal{D} \oplus \hom(H, T\Sigma/H) \to \hom(H, u^*TM/\mathcal{D})$ $(\xi,\psi) \mapsto u^*(\iota_\xi \Omega)\|_H + \tilde{du} \circ \psi$ is surjective, $\tilde{du} : T\Sigma/H \to u^*TM/\mathcal{D}$ is the induced map	u is an immersion, inducing $H = du^{-1}\mathcal{D}$ and $\Omega_\bullet : u^*\mathcal{D} \to \hom\left(H, u^*TM/\mathcal{D}/\operatorname{Im}\tilde{du}\right)$ $\xi \mapsto u^*(\iota_\xi\Omega)\|_H \mod \operatorname{Im}\tilde{du}$ is surjective, $\tilde{du} : T\Sigma/H \to u^*TM/\mathcal{D}$ is the induced map
Sheaf of Regular Solutions	$\Phi^{G,\rho}$	$\Phi^{m\text{-HorGr}}$	$\Phi^{m\text{-Hor}}$
Curvature Condition	$u^*\Omega\|_{G^\phi} = \tilde{du}^\phi \circ \Omega_{G^\phi}$, where Ω_{G^ϕ} is the curvature form of G^ϕ and $\tilde{du}^\phi : T\Sigma/G^\phi \to u^*TM/\mathcal{D}$ is the induced map	$u^*\Omega\|_H = \tilde{du} \circ \Omega_H$, where Ω_H is the curvature form of H and $\tilde{du} : T\Sigma/H \to u^*TM/\mathcal{D}$ is the induced map	$du(H) \subset \ker \Omega_\bullet$ where $H := du^{-1}\mathcal{D}$
$\alpha + 1$**-Jet Relation of Regular, Infinitesimal Solution of Order** α	$\mathcal{R}^{G,\rho}_\alpha$	$\mathcal{R}^{m\text{-HorGr}}_\alpha$	-
1**-Jet Relation, with Curvature Condition**	$\mathcal{R}^{G,\rho}$	$\mathcal{R}^{m\text{-HorGr}}$	$\mathcal{R}^{m\text{-Hor}}$
Sheaf of Formal Solutions	$\Psi^{G,\rho}$	$\Psi^{m\text{-HorGr}}$	$\Psi^{m\text{-Hor}}$

* For a fixed $G \subset T\Sigma$ and a splitting map $\rho : T\Sigma/G \hookrightarrow T\Sigma$
† Recall : $\bar{\phi}(X) = X + \rho\phi X$ for $X \in G$
‡ There is no associated operator
§ Recall : $G^\phi = \operatorname{Im}\bar{\phi} = \left\{X + \rho\phi X \;\middle|\; X \in G\right\}$

We have the following result that relates $\mathcal{R}^{m\text{-HorGr}}$ and $\mathcal{R}^{m\text{-Hor}}$.

Lemma 5.1.12. *The fiber-preserving map $\mathcal{R}^{m\text{-HorGr}} \to \mathcal{R}^{m\text{-Hor}}$ is surjective, with contractible fibers. Furthermore any section in $\Psi^{m\text{-Hor}}$ admits a lift to a section $\Psi^{m\text{-HorGr}}$, over contractible open sets in Σ and consequently, $\Psi^{m\text{-HorGr}} \underset{w.h.e}{\simeq} \Psi^{m\text{-Hor}}$.*

Proof. Suppose the jet $(\sigma, y, P) \in \mathcal{R}^{m\text{-Hor}}$ is represented as $j^1_u(\sigma)$ for some $u : \mathrm{Op}(\sigma) \to M$. Denote $H = P^{-1}\mathcal{D}$. Since $P(H) \subset \ker \Omega_\bullet$, we see that for all $X, Y \in H$ at σ,

$$0 = \Omega_\bullet(PX)(PY) = \Omega(PX, PY) \mod \mathrm{Im}\,\tilde{P}.$$

In other words we have,

$$\Omega(PX, PY) \in \mathrm{Im}\,\tilde{P}.$$

Since $\tilde{P}$ is an injective, we can define a 2-form $R : \Lambda^2 H \to T\Sigma/H$ by,

$$R(X, Y) = \tilde{P}^{-1} \circ \Omega(PX, PY), \quad \forall X, Y \in H_\sigma.$$

Then, $P^*\Omega|_H = \tilde{P} \circ R$.

Now, for $q = \mathrm{codim}\, H = \dim \Sigma - m$, consider a q-tuple of 1-forms $(\mu^1, \ldots, \mu^q)$ on $\mathrm{Op}(\sigma)$, which is linearly independent at the point σ. The surjectivity of the map $j^1_{(\mu^r)}(\sigma) \mapsto (\mu^r, d\mu^r)|_\sigma$ then implies the same for,

$$j^1_{(\mu^r)}(\sigma) \mapsto \big(d\mu^r|_{\cap_{r=1}^p \ker \mu^r}\big)\big|_\sigma.$$

Hence, we can get a jet $j^1_{(\mu^r)}(\sigma)$, so that,

$$H = \bigcap_{r=1}^{p} \ker \mu^r_\sigma \quad \text{and} \quad R = \big(d\mu^r|_H\big).$$

In other words, the jet $j^1_{(\mu^r)}(\sigma)$ can now be identified with a jet $j^1_G(\sigma) \in \mathrm{Gr}_m T\Sigma$, so that $H = G_\sigma$ and $\Omega_{G_\sigma} = R$. It then follows that $j^1_{u,G}(\sigma) \in \mathcal{R}^{m\text{-HorGr}}$ is the desired lift of $j^1_u(\sigma) = (\sigma, y, P)$.

It is clear from above that the space of all the lifts is affine and hence it is contractible. Also, the argument can easily be performed for sections over contractible open sets of Σ. This concludes the proof. $\square$

We then have the following corollary to Theorem 5.1.9

Corollary 5.1.13. *The relation $\mathcal{R}^{m\text{-Hor}}$ satisfies the parametric h-principle over open manifold Σ.*

Proof. Observe that, a solution $u \in \Phi^{m\text{-Hor}}$ uniquely determines the tuple $(u, G) \in \Phi^{m\text{-HorGr}}$, where $G = du^{-1}\mathcal{D}$. Consequently, we may identify $\Phi^{m\text{-Hor}}$ with $\Phi^{m\text{-HorGr}}$. We have the commutative diagram,

$$\begin{tikzcd}
(u, H) \arrow[d, mapsto] & \Phi^{m\text{-HorGr}} \arrow[r, "j^1"] \arrow[d] & \Psi^{m\text{-HorGr}} \arrow[d] & j^1_{u,H}(\sigma) \arrow[d, mapsto] \\
u & \Phi^{m\text{-Hor}} \arrow[r, "j^1"'] & \Psi^{m\text{-Hor}} & j^1_u(\sigma)
\end{tikzcd}$$

It then follows that,

- the sheaf map $\Psi^{m\text{-HorGr}} \to \Psi^{m\text{-Hor}}$ induced by the projection, is a weak homotopy equivalence (by Lemma 5.1.12).
- the map $j^1 : \Phi^{m\text{-HorGr}} \to \Psi^{m\text{-HorGr}}$ is a weak homotopy equivalence whenever Σ is an open manifold (by Theorem 5.1.9).

Hence we have that the map $j^1 : \Phi^{m\text{-Hor}} \to \Psi^{m\text{-Hor}}$ is a weak homotopy equivalence, provided Σ is open. $\square$

5.1.4 A Candidate for an Extension

In order to get any h-principle for closed manifolds, we need to discuss the extension problem for m-horizontal immersions. For a fixed $G \subset T\Sigma$ and a splitting morphism $\rho : T\Sigma/G \hookrightarrow T\Sigma$, we have a canonical choice of a distribution $\tilde{G}$ on $\tilde{\Sigma} = \Sigma \times \mathbb{R}$, namely,

$$\tilde{G} := d\pi^{-1}(G) \subset T\tilde{\Sigma},$$

where, $\pi : \tilde{\Sigma} \to \Sigma$ is the canonical projection. We see, $\operatorname{cork} \tilde{G} = \operatorname{cork} G$. Therefor the inclusion map $\Sigma \hookrightarrow \tilde{\Sigma}$ induces an isomorphism $T\Sigma/G \to T\tilde{\Sigma}/\tilde{G}|_\Sigma$ and moreover, $T\tilde{\Sigma}/\tilde{G} \cong \pi^*(T\Sigma/G)$. We have a canonical choice of splitting morphism $\tilde{\rho} : T\tilde{\Sigma}/\tilde{G} \to T\tilde{\Sigma}$ defined as,

$$\tilde{\rho}|_{\sigma,t} = \big(\rho|_\sigma, 0\big), \quad \text{for } (\sigma, t) \in \tilde{\Sigma}.$$

In other words, $\tilde{\rho} : (v, w) \mod \tilde{G} \mapsto \big(\rho(v \mod G), 0\big)$ for any $(v, w) \in T_\sigma\Sigma \oplus \mathbb{R}$.

Note that for any $\psi : \tilde{G} \to T\tilde{\Sigma}/\tilde{G}$, the distribution

$$\tilde{G}^\psi = \{X + \tilde{\rho}\psi X \mid X \in \tilde{G}\}$$

is transverse to $\pi^*T\Sigma \subset T\tilde{\Sigma}$ at each point. Indeed, for any $(0,c) \in T_\sigma\Sigma \oplus T_t\mathbb{R}$, we have that

$$(0,c) + \tilde{\rho}\psi(0,c) = (0,c) + \big(\rho\psi(0,c), 0\big) = \big(\rho\psi(0,c), c\big) \notin T_\sigma\Sigma \oplus 0, \quad \text{for } c \neq 0.$$

Hence, if $\tilde{G}^\psi = du^{-1}(\mathcal{D})$, then $v = u|_\Sigma$ is an m-horizontal distribution and,

$$(dv_\sigma)^{-1}\mathcal{D} = (du_\sigma)^{-1}\mathcal{D} \cap T_\sigma\Sigma = \tilde{G}^\psi_\sigma \cap T_\sigma\Sigma = G^\phi_\sigma,$$

where $\phi = \psi|_{G\oplus 0}$.

Let us denote by $\tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}} \subset J^1(\Sigma, M)$ the relation consisting of first jets $j^1_{u,\psi}(\Sigma) \in J^1(\tilde{\Sigma}, M) \times \hom(\tilde{G}, T\tilde{\Sigma}/\tilde{G})^{(1)}$ satisfying the following.

- du_σ is an injective morphism, inducing the $m+1$-dimensional subspace, $\tilde{G}^\psi|_\sigma = du_\sigma^{-1}(\mathcal{D}_{u(\sigma)})$
- the linear map,

$$\begin{aligned}\mathcal{D}_{u(\sigma)} \oplus \hom(\tilde{G}, T\tilde{\Sigma}/\tilde{G})|_\sigma &\to \hom(\tilde{G}_\sigma, TM/\mathcal{D}|_{u(\sigma)})\\ (\xi,\zeta) &\mapsto u^*\iota_\xi\Omega\big|_\sigma \circ \bar{\psi} + u^*\lambda\big|_\sigma \circ \rho \circ \zeta\end{aligned}$$

  is surjective
- the curvature condition,

$$u^*\Omega|_{\tilde{G}^\psi_\sigma} = \tilde{du}^\psi_\sigma \circ \Omega_{\tilde{G}^\psi_\sigma},$$

  holds at the point σ, where $\tilde{du}^\psi_\sigma : T\tilde{\Sigma}/\tilde{G}^\psi|_\sigma \to TM/\mathcal{D}|_{u(\sigma)}$ is the induced map and $\Omega_{\tilde{G}^\psi}$ is the curvature 2-form for $\tilde{G}^\psi$.

In other words,

$$\tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}} = \left\{ j^1_{u,\psi}(\sigma) \in J^1(\tilde{\Sigma}, M) \times \hom(\tilde{G}, T\tilde{\Sigma}/\tilde{G})^{(1)} \;\middle|\; j^1_{u,\tilde{G}^\psi}(\sigma) \in \tilde{\mathcal{R}}^{m+1\text{-HorGr}} \right\}.$$

In view of Remark 2.2.18, we put forth the relations $\tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}}$ as a collection of possible candidates for an extension of the relation $\mathcal{R}^{m\text{-Hor}}$ (see Definition 5.1.11).

The ev Map : We have the two natural bundles,

$$X = \big(\Sigma \times M\big) \to \Sigma, \qquad \tilde{X} = (\tilde{\Sigma} \times M) \times \hom(\tilde{G}, T\tilde{\Sigma}/G) \to \tilde{\Sigma}$$

Consider the fiber-preserving morphism,

$$ev : \Gamma\tilde{X}|_{\Sigma\times 0} \to \Gamma X$$
$$(u,\psi) \mapsto u|_\Sigma$$

It follows from Proposition 5.1.4 that if (u,ψ) is a Ω-regular tuple, $v = u|_\Sigma$ is $\Omega_\bullet$-regular. Consequently, the induced map in the jet maps $\tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}}|_\Sigma$ into $\mathcal{R}^{m\text{-Hor}}$.

Let us denote, $\tilde{\Phi} = \operatorname{Sol}\tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}}$. We then have that $\tilde{\Phi}$ consists of Ω-regular tuple (u,ψ), satisfying $\tilde{G}^\psi = du^{-1}\mathcal{D}$, and clearly it is *not* invariant under the natural $\operatorname{Diff}(\tilde{\Sigma},\pi)$ action. Indeed, for any $(u,\psi) \in \tilde{\Phi}$ and for some $\zeta \in \operatorname{Diff}(\tilde{\Sigma},\pi)$, the induced distribution $H = d(u\circ\zeta)^{-1}\mathcal{D} = d\zeta^{-1}\tilde{G}^\psi$, may fail to be transverse to $\pi^*T\Sigma$ everywhere. Thus we are unable to apply Theorem 2.2.9 to the sheaf $\tilde{\Phi}$ to get the flexibility of $\tilde{\Phi}|_\Sigma$. Instead, we prove it directly.

Proposition 5.1.14. *The restricted sheaf $\tilde{\Phi}|_\Sigma$ is flexible.*

Proof. We divide the proof into the following steps.

Step 1 We introduce an auxiliary differential operator $\hat{\mathfrak{D}}_0$ (see [Gro96, pg. 260]).

Step 2 We identify a suitable regularity condition for solutions of $\hat{\mathfrak{D}}_0$ so that the sheaf $\hat{\Phi}$ of regular solutions of $\hat{\mathfrak{D}}_0$ is microflexible.

Step 3 We get a $\operatorname{Diff}(\tilde{\Sigma},\pi)$-action on $\hat{\Phi}$ and consequently get the flexibility of $\hat{\Phi}|_\Sigma$.

Step 4 We deduce the flexibility of $\tilde{\Phi}|_\Sigma$ from that of $\hat{\Phi}|_\Sigma$.

Proof of Step 1 : First consider the operator,

$$\hat{\mathfrak{D}}_0 : C^\infty(\tilde{\Sigma},M)\times C^\infty(\tilde{\Sigma},\mathbb{R})\times\Gamma\hom(\tilde{G},T\tilde{\Sigma}/\tilde{G}) \to \Gamma\hom\left(\tilde{G},\mathbb{R}^p\right)$$
$$(u,h,\psi) \mapsto \left(u\circ\bar{h}\right)^*\lambda^s\circ\bar{\psi}$$

where the map $\bar{h}:\tilde{\Sigma}\to\tilde{\Sigma}$ is given as, $\bar{h}(\sigma,t) = \big(\sigma,h(\sigma,t)\big)$. In particular, observe that for the canonical projection map $\pi_2 : \tilde{\Sigma} = \Sigma\times\mathbb{R}\to\mathbb{R}$, we have $\bar{\pi}_2 = \mathrm{Id}_{\tilde{\Sigma}}$. Moreover, if $\partial_t h \neq 0$, then $\bar{h}$ becomes a fiber preserving diffeomorphism of $\Sigma\times\mathbb{R}$, i.e, $\bar{h}\in\operatorname{Diff}(\tilde{\Sigma},\pi)$.

It is now immediate that,

$$\hat{\mathfrak{D}}_0(u,h,\psi) = 0 \;\Rightarrow\; (u\circ\bar{h})^*\lambda^s|_{\operatorname{Im}\bar{\psi}} = 0,\; s = 1,\ldots,p \;\Rightarrow\; d(u\circ\bar{h})\big(\tilde{G}^\psi\big)\subset\mathcal{D},$$

where $\tilde{G}^\psi = \operatorname{Im}\bar{\psi}$. Furthermore, $u\circ\bar{h}$ is an $(m+1)$-horizontal immersion, inducing $\tilde{G}^\psi = d(u\circ\bar{h})^{-1}\mathcal{D}$, if the following conditions hold :

- u is an immersion, with $\operatorname{rk}(\lambda^s \circ du) \geq \dim \tilde{\Sigma} - (m+1) = \dim \Sigma - m$, and
- $\partial_t h \neq 0$, i.e, $\bar{h}$ is a diffeomorphism.

Proof of Step 2 : Next, we determine the linearization operator of $\hat{\mathfrak{D}}_0$ at some (u, h, ψ),

$$\mathfrak{L}_{(u,h,\psi)} : \Gamma u^*TM \oplus C^\infty(\tilde{\Sigma}, \mathbb{R}) \oplus \Gamma \hom(\tilde{G}, T\tilde{\Sigma}/\tilde{G}) \to \Gamma \hom(\tilde{G}, \mathbb{R}^p)$$

Restricting the operator to the subspace $\Gamma u^*TM \oplus 0 \oplus \Gamma \hom(\tilde{G}, T\tilde{\Sigma}/\tilde{G})$ we find out,

$$\begin{aligned}
\mathfrak{L}_{(u,h,\psi)}(\xi, 0, \zeta) &= \frac{d}{dt}\Big|_{t=0} \tilde{\mathfrak{D}}(u_t, h, \psi + t\zeta) \\
&= \lim_{t\to 0} \frac{1}{t}\Big[(u_t \circ \bar{h})^* \lambda^s \circ \overline{\psi + t\zeta} - (u \circ \bar{h})^* \lambda^s \circ \bar{\psi}\Big] \\
&= \lim_{t\to 0} \frac{1}{t}\Big[\bar{h}^*\Big(u_t^*\lambda^s - u^*\lambda^s\Big) \circ \overline{\psi + t\zeta} + (u \circ \bar{h})^* \lambda^s \circ \Big(\overline{\psi + t\zeta} - \bar{\psi}\Big)\Big] \\
&= \bar{h}^*\big(\iota_\xi d\lambda^s + d\iota_\xi \lambda^s\big) \circ \bar{\psi} + (u \circ \bar{h})^* \lambda^s \circ \tilde{\rho} \circ \zeta
\end{aligned}$$

Further restricting this to the subspace $\Gamma u^*\mathcal{D} \oplus 0 \oplus \Gamma \hom(\tilde{G}, T\tilde{\Sigma}/\tilde{G})$ we get the operator,

$$\begin{aligned}
\mathcal{L}_{(u,h,\psi)} : \Gamma u^*\mathcal{D} \oplus \Gamma \hom(\tilde{G}, T\tilde{\Sigma}/\tilde{G}) &\to \Gamma \hom(\tilde{G}, \mathbb{R}^p) \\
(\xi, \zeta) &\mapsto (u \circ \bar{h})^*(\iota_\xi d\lambda^s) \circ \bar{\psi} + (u \circ \bar{h})^* \lambda^s \circ \tilde{\rho} \circ \zeta
\end{aligned}$$

Clearly $\mathcal{L}_{(u,h,\psi)}$ is $C^\infty(\tilde{\Sigma})$-linear and hence it is given by a bundle map,

$$u^*\mathcal{D} \oplus \hom(\tilde{G}, T\tilde{\Sigma}/\tilde{G}) \to \hom(\tilde{G}, \mathbb{R}^p).$$

It follows that $\mathcal{L}_{(u,h,\psi)}$ is surjective precisely when the tuple $(u \circ \bar{h}, \psi)$ is Ω-regular (Definition 5.1.2).

Let $\hat{\Phi}$ be the sheaf of tuples $(u, h, \psi) \in C^\infty(\tilde{\Sigma}, M) \times C^\infty(\tilde{\Sigma}, \mathbb{R}) \times \Gamma \hom(\tilde{G}, T\tilde{\Sigma}/\tilde{G})$, satisfying the conditions below :

$$u \text{ is an immersion}, \quad \partial_t h \neq 0, \quad G^\psi = d(u \circ \bar{h})^{-1}\mathcal{D} \quad \text{and} \quad (u \circ \bar{h}, \psi) \text{ is an } \Omega\text{-regular tuple.}$$

Note that we have a sheaf morphism,

$$\begin{aligned}
\hat{\Phi} &\to \tilde{\Phi} \\
(u, h, \psi) &\mapsto \big(u \circ \bar{h}, \psi\big)
\end{aligned}$$

It follows from Theorem 2.2.27 that $\hat{\Phi}$ is microflexible.

Proof of Step 3 : As we have already noted, every smooth map $h : \tilde{\Sigma} \to \mathbb{R}$, satisfying $\partial_t h \neq 0$, defines a fiber preserving diffeomorphism $\bar{h} : \tilde{\Sigma} \to \tilde{\Sigma}$. In fact, every element of $\mathrm{Diff}(\tilde{\Sigma}, \pi)$ can be uniquely realized in this way for some map $\tilde{\Sigma} \to \mathbb{R}$. Now suppose $\theta \in \mathrm{Diff}(\tilde{\Sigma}, \pi)$ is given and consider a tuple $(u, h, \psi) \in \hat{\Phi}$. In particular, since $\partial_t h \neq 0$, we have $\theta \circ \bar{h} \in \mathrm{Diff}(\tilde{\Sigma}, \pi)$, whenever it is defined. Then for any such *compatible* tuple, there is a unique $\kappa \in C^\infty(\tilde{\Sigma}, \mathbb{R})$ so that, $\theta \circ \bar{h} = \bar{\kappa}$. We now define the action as,

$$\theta \cdot (u, h, \psi) \mapsto (u \circ \theta^{-1}, \kappa, \psi).$$

Observe that,

$$(u \circ \theta^{-1}) \circ \bar{\kappa} = (u \circ \theta^{-1}) \circ (\theta \circ \bar{h}) = u \circ \bar{h} \;\Rightarrow\; d(u \circ \theta \circ \bar{\kappa})^{-1}\mathcal{D} = d(u \circ \bar{h})^{-1}\mathcal{D} = \tilde{G}^{\psi}.$$

Hence we have $(u \circ \theta^{-1}, \kappa, \psi) \in \hat{\Psi}$. Consequently, it follows from Theorem 2.2.9 that $\hat{\Phi}|_\Sigma$ is flexible.

Proof of Step 4 : Now, fix some arbitrary pair of compact sets A, B with $A \subset B \subset \Sigma$ and consider a homotopy lifting diagram,

$$\begin{tikzcd} P \times 0 \arrow[r, "v_0^p,\, \varphi_0^p"] \arrow[d, hook] & \tilde{\Phi}|_B \arrow[d] \\ P \times [0,1] \arrow[r, "u_t^p,\, \psi_t^p"'] & \tilde{\Phi}|_A \end{tikzcd} \qquad (*)$$

where P is an arbitrary compact polyhedron. Observe that given any tuple $(u, \psi) \in \tilde{\Phi}$, the tuple $(u, \pi_2, \psi) \in \hat{\Phi}$, where $\pi_2 : \tilde{\Sigma} = \Sigma \times \mathbb{R} \to \mathbb{R}$ is the canonical projection, since $u \circ \bar{\pi}_2 = u \circ \mathrm{Id}_{\tilde{\Sigma}} = u$. Thus we can get a new homotopy lifting diagram from $(*)$ as follows.

$$\begin{tikzcd} P \times 0 \arrow[r, "v_0^p,\, \pi_2,\, \varphi_0^p"] \arrow[d, hook] & \hat{\Phi}|_B \arrow[d] \\ P \times [0,1] \arrow[r, "u_t^p,\, \pi_2,\, \psi_t^p"'] \arrow[ur, dashed] & \hat{\Phi}|_A \end{tikzcd} \qquad (**)$$

Since $\hat{\Phi}|_\Sigma$ is a flexible sheaf, we have a map,

$$(v_t^p,\ h_t^p,\ \varphi_t^p) : P \times [0,1] \to \hat{\Phi}|_B,$$

which solves the diagram ($**$). But then the tuple $(v_t^p \circ \bar{h}_t^p,\ \psi_t^p) : P \times [0,1] \to \tilde{\Phi}|_B$ solves the diagram ($*$). Consequently, we have that the sheaf $\tilde{\Phi}|_\Sigma$ is flexible. This concludes the proof. $\square$

Lastly, we state the following lemma, which justifies hypothesis (3) of Theorem 2.2.15.

Lemma 5.1.15. *Let $O \subset \Sigma$ be a coordinate chart and $C \subset O$ is a compact subset. Suppose that $U \subset M$ is an open subset such that $\mathcal{D}|_U$ is trivial. Then, given any $\Omega_\bullet$-regular, m-horizontal immersion $u : \operatorname{Op} C \to U \subset M$, the 1-jet map,*

$$j^1 : ev^{-1}(u) \to ev^{-1}(F = j^1_u)$$

in the commutative diagram below,

$$\begin{tikzcd}
ev^{-1}(u) \arrow[r, hook] \arrow[d, dashed, "j^1"'] & \tilde{\Phi}^{\tilde{G},\tilde{\rho}}|_{C\times 0} \arrow[r, "ev"] \arrow[d] & \Phi^{m\text{-Hor}}|_C \arrow[d] & u \arrow[d, mapsto] \\
ev^{-1}(F) \arrow[r, hook] & \tilde{\Psi}^{\tilde{G},\tilde{\rho}}|_{C\times 0} \arrow[r, "ev"'] & \Psi^{m\text{-Hor}}|_C & F = j^1_u
\end{tikzcd}$$

induces a surjective map between the set of path components, for some suitable choice of $G \subset TO$ and a splitting map $\rho : TO/G \hookrightarrow TO$, Furthermore, the homotopy can be kept C^0-small.

Proof. Since $u : O \to U$ is a given m-horizontal immersion, fix $G = du^{-1}\mathcal{D}$, a rank m distribution on $O \subset \Sigma$. Next, choose some splitting map $\rho : TO/G \hookrightarrow TO$. Recall the notations,

$$\Phi^{m\text{-Hor}} = \operatorname{Sol}\mathcal{R}^{m\text{-Hor}}, \quad \tilde{\Phi}^{\tilde{G},\tilde{\rho}} = \operatorname{Sol}\tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}}, \qquad \Psi^{m\text{-Hor}} = \Gamma\mathcal{R}^{m\text{-Hor}}, \quad \tilde{\Psi}^{\tilde{G},\tilde{\rho}} = \Gamma\tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}},$$

and the fiber preserving map $ev : \mathcal{R}^{\tilde{G},\tilde{\rho}}|_C \to \mathcal{R}^{m\text{-Hor}}$. Now, fix some neighborhood V of C, with $C \subset V \subset O$, over which u is defined and then fix an arbitrarily small open neighborhood U_ϵ of $u(V)$.

The proof follows in a similar fashion as Lemma 3.2.6. We only mention the main steps.

Step 1 Given an arbitrary extension $\tilde{F} \in \tilde{\Psi}|_{C\times 0}$ of F along ev, we construct a regular solution $(v, \psi) \in \tilde{\Phi}$ on $\tilde{O}pC$, so that $j^1_{v,\psi}|_{\operatorname{Op} C} = \tilde{F}|_{\operatorname{Op} C}$.

Step 2 We get an homotopy between $j^1_{v,\psi}$ and $\tilde{F}$, in the relation

$$\tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}} \subset J^1(W, U_\epsilon) \times \hom(\tilde{G}, T\tilde{\Sigma}/\tilde{G})^{(1)} \to J^1(W, U_\epsilon) = W \times U_\epsilon,$$

over $W \times U_\epsilon$, so that the homotopy is constant on points of C. In particular, the homotopy belongs to $ev^{-1}(F)$. Note that, here $W \subset \tilde{\mathrm{Op}}C$ is an open neighborhood of C, to be fixed in Step 1, as done in Lemma 3.2.6.

The first step can be done identically as in Lemma 3.2.6; the jet lifting argument in the context of $\tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}}$ is provided by Lemma 5.1.8, since $\tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}}$ embeds as an *open* subset of $\tilde{\mathcal{R}}^{m\,+\,1\text{-Hor}}$. Let us elaborate on step 2 now. We break it in few sub-steps.

Step $2a$ First we identify the image of $\tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}} \hookrightarrow \tilde{\mathcal{R}}^{m\,+\,1\text{-Hor}}$ under the map $j^1_{u,\phi}(\sigma) \mapsto j^1_u(\sigma)$. In fact, we consider the fiber-preserving map,

$$\begin{aligned} \Pi : \tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}} &\to J^1(\tilde{\Sigma}, M) \times \hom(\tilde{G}, T\tilde{\Sigma}/\tilde{G})^{(0)} \\ j^1_{u,\phi}(\sigma) &\mapsto \big(j^1_u(\sigma), j^0_\phi(\sigma)\big) \end{aligned}$$

which forgets the *pure* first jet data of $j^1_\phi(\sigma)$ and let us denote the image as, $\tilde{\mathcal{R}}_0^{\tilde{G},\tilde{\rho}} := \operatorname{Im}\Pi$. We note that, $\tilde{\mathcal{R}}_0^{\tilde{G},\tilde{\rho}}$ *embeds* into $\tilde{\mathcal{R}}^{m\,+\,1\text{-Hor}}$ via the map $\big(j^1_u(\sigma), j^0_\phi(\sigma)\big) \mapsto j^1_u(\sigma)$ and so we have the diagram,

$$\begin{tikzcd} \tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}} \arrow[r, "\Pi"] \arrow[rr, bend right, "j^1_{u,\phi}(\sigma)\mapsto j^1_u(\sigma)"'] & \tilde{\mathcal{R}}_0^{\tilde{G},\tilde{\rho}} \arrow[r, hook] & \tilde{\mathcal{R}}^{m\,+\,1\text{-Hor}} \end{tikzcd}$$

Identifying $\tilde{\mathcal{R}}_0^{\tilde{G},\tilde{\rho}}$ as a subset of $\tilde{\mathcal{R}}^{m\,+\,1\text{-Hor}}$, we note that, as a consequence of Lemma 5.1.12, any path in $\tilde{\mathcal{R}}_0^{\tilde{G},\tilde{\rho}}$ can be lifted to a path in $\tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}}$, via the map Π.

Step $2b$ We look at the image of the two formal sections of $j^1_{v,\psi}$ and $\tilde{F}$ from Step 1 under the map Π, and get a homotopy joining $\Pi\big(j^1_{v,\psi}\big)$ and $\Pi\big(\tilde{F}\big)$, say,

$$\hat{H}_t : \Pi\big(j^1_{v,\psi}\big) \sim \Pi\big(\tilde{F}\big), \text{ in the affine bundle } J^1(\tilde{\Sigma}, M) \times \hom(\tilde{G}, T\tilde{\Sigma}/\tilde{G})^{(0)},$$

which is constant on C. Again, this step can be performed in a similar fashion as presented in Lemma 3.2.6. Note that this homotopy need not be inside $\tilde{\mathcal{R}}_0^{\tilde{G},\tilde{\rho}}$.

Step $2c$ We claim that $\tilde{\mathcal{R}}_0^{\tilde{G},\rho}$ is a local neighborhood retract. Thus we can push the homotopy $\hat{H}_t$ from the previous step into $\tilde{\mathcal{R}}^{\tilde{G},\rho}$, while keeping the endpoints fixed. We now have a homotopy,

$$\tilde{H}_t : \Pi(j^1_{v,\psi}) \sim \Pi(\tilde{F}), \quad \text{in } \tilde{\mathcal{R}}_0^{\tilde{G},\tilde{\rho}},$$

which is fixed on points of C. This claim will be proved in Lemma 5.1.16.

Step $2d$ Lastly, by Lemma 5.1.12, we can lift the homotopy $\tilde{H}_t$ from the last step to a homotopy in $\tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}}$, joining $j^1_{v,\psi}$ and $\tilde{F}$. That is, we have a homotopy,

$$\tilde{F}_t : j^1_{v,\psi} \sim \tilde{F}, \quad \text{in } \tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}}$$

which, by construction, is constant on points of C. Hence $\tilde{F}_t$ is in fact a homotopy in $ev^{-1}(\tilde{F})$. This concludes the proof. □

Let us now prove that the subset $\tilde{\mathcal{R}}_0^{\tilde{G},\tilde{\rho}} \subset J^1(\tilde{\Sigma}, M) \times \hom(\tilde{G}, T\tilde{\Sigma}/\tilde{G})^{(0)}$, as considered in the above lemma, is a fiberwise local neighborhood retract. For notational simplicity, let us work with,

$$\mathcal{R}^{G,\rho} \subset J^1(\Sigma, M) \times \hom(G, T\Sigma/G)^{(1)}, \text{ for some fixed } G \subset T\Sigma \text{ and } \rho : T\Sigma/G \hookrightarrow T\Sigma.$$

We note that $\mathcal{R}_0^{G,\rho}$ consists of tuples,

$$\big(\sigma, y,\ P : T_x\Sigma \to T_y M,\ \varphi : G_\sigma \to T\Sigma/G|_\sigma\big)$$

satisfying,

- P is injective, inducing $G^\varphi = P^{-1}\mathcal{D}$.
- the map,

$$\begin{aligned} \Omega_\bullet : \mathcal{D}_y &\to \hom\left(G^\varphi, TM/\mathcal{D}|_y / \operatorname{Im} \tilde{P}\right) \\ \xi &\mapsto P^*(\iota_\xi \Omega)|_{G^\varphi} \mod \operatorname{Im} \tilde{P} \end{aligned}$$

  is surjective. By Proposition 5.1.4, this takes care of Ω-regularity.
- $P(G^\varphi) \subset \ker \Omega_\bullet$, or equivalently, $P^*\Omega|_{G^\varphi} = 0 \mod \operatorname{Im} \tilde{P}$.

We prove the following.

Lemma 5.1.16. *The following holds true for* $\mathcal{R}_0^{G,\rho}$.

- $\mathcal{R}_0^{G,\rho}|_{(x,y)}$ *is a submanifold of* $J^1_{(x,y)}(\Sigma, M) \times \hom(G, T\Sigma/G)|_x$*, for* $(x, y) \in J^0(\Sigma, M)$.

- *$\mathcal{R}_0^{G,\rho}$ is a submanifold of $J^1(\Sigma, M) \times \hom(G, T\Sigma/G)$.*
- *The projection map $p : J^1(\Sigma, M) \times \hom(G, T\Sigma/G) \to J^0(\Sigma, M)$ restricts to a submersion on $\mathcal{R}_0^{G,\rho}$.*

As a consequence, $\mathcal{R}_0^{G,\rho}$ is a fiber-wise, local neighborhood retract.

Proof. We prove it in a few steps.

Step 1 We consider the fiber-preserving map,

$$\begin{aligned}\Xi_1 : J^1(\Sigma, M) \times \hom(G, T\Sigma/G) &\to \hom(G, TM/\mathcal{D})\\ (x, y, P, \varphi) &\mapsto \lambda \circ P \circ \bar{\varphi}\end{aligned}$$

We show that $\Xi_1|_{(x,y)}$ over each fiber is a submersion and consequently, $\Xi_1|_{(x,y)}^{-1}(0) = \{(x, y, P, \varphi) \mid P^{-1}\mathcal{D}_y = G^\varphi\}$ is a submanifold of $J^1_{(x,y)}(\Sigma, M) \times \hom(G, T\Sigma/G)|_x$. Furthermore, $\Xi_1^{-1}(0)$ is a submanifold of $J^1(\Sigma, M) \times \hom(G, T\Sigma/G)$ as well.

Step 2 For fixed $(x, y) \in J^0(\Sigma, M)$, we aim to show $\mathcal{R}_0^{G,\rho}|_{(x,y)}$ is a submanifold. We break this into the following steps.

Step 2a First, we get a natural map,

$$\begin{aligned}\Theta = \Theta_{(x,y)} : \Xi_1|_{(x,y)}^{-1}(0) &\to \hom(T\Sigma/G|_x, TM/\mathcal{D}|_y)\\ (x, y, P, \varphi) &\mapsto \lambda \circ P \circ \rho\end{aligned}$$

and show that Θ is a submersion.

Step 2b Next, we define the map,

$$\begin{aligned}\Xi_2^A : \Theta^{-1}(A) &\to \hom\left(\Lambda^2 G_x,\ TM/\mathcal{D}|_y / \operatorname{Im} A\right)\\ (x, y, P, \varphi) &\mapsto \bar{\varphi}^* P^* \Omega \mod \operatorname{Im} A\end{aligned}$$

for a linear map $A : T\Sigma/G|_x \to TM/\mathcal{D}|_y$. Note that, $(\Xi_2^A)^{-1}(0)$ consists of tuples (x, y, P, φ) satisfying the formal curvature condition,

$$P(G^\varphi) \subset \ker \Omega_\bullet.$$

We show that, for A injective, the set $\mathcal{R}_0^{G,\rho}|_{(x,y)} \cap \Theta^{-1}(A)$ consists of regular points of Ξ_2^A. Consequently, $\mathcal{R}_0^{G,\rho}|_{(x,y)} \cap \Theta^{-1}(A)$ is a submanifold. $\Omega_\bullet$-regularity is crucial at this step.

Step 2c For a fixed monomorphism $A : T\Sigma/G|_x \hookrightarrow TM/\mathcal{D}|_y$, we identify an open set,

$$\mathcal{M}_A = \left\{B : T\Sigma/G|_x \hookrightarrow TM/\mathcal{D}|_y \;\middle|\; B \pitchfork \operatorname{Im} A\right\} \subset \hom(T\Sigma/G, TM/\mathcal{D})|_{(x,y)}.$$

Next we fix a choice of splitting map $\eta : (TM/\mathcal{D}|_y)\big/ \operatorname{Im} A \hookrightarrow TM/\mathcal{D}|_y$ and using it we get the isomorphisms,

$$\hat{\eta}(B) : (TM/\mathcal{D}|_y)\big/ \operatorname{Im} A \to (TM/\mathcal{D}|_y)\big/ \operatorname{Im} B.$$

We then define the fiber-preserving map,

$$\hat{\Xi}_2^A : \Theta^{-1}(\mathcal{M}_A) \to \mathcal{M}_A \times \hom\left(\Lambda^2 G_x, TM/\mathcal{D}|_y\big/ \operatorname{Im} A\right)$$

by,

$$\hat{\Xi}_2(x, y, P, \varphi) = \Big(B,\ \hat{\eta}(B)^{-1} \circ \Xi_2^B(x, y, P, \varphi)\Big),$$

for $(x, y, P, \varphi) \in \Theta^{-1}(B), B \in \mathcal{M}_A$. We note that $\mathcal{R}_0^{G,\rho}|_{(x,y)} \cap \Theta^{-1}(\mathcal{M}_A)$ are regular points of $\hat{\Xi}_2$ and consequently is a submanifold.

We conclude that $\mathcal{R}_0^{G,\rho}|_{(x,y)}$ is a submanifold of $\Xi_1|_{(x,y)}^{-1}(0)$ for fixed (x, y).

Step 3 Lastly, using local trivialization argument, we prove that $\mathcal{R}_0^{G,\rho}$ is a submanifold and furthermore, restriction of $p : J^1(\Sigma, M) \times \hom(G, T\Sigma/G) \to J^0(\Sigma, M)$ to $\mathcal{R}_0^{G,\rho}$ is a submersion.

Proof of Step 1 : We have the bundle map, $\Xi_1 : J^1(\Sigma, M) \times \hom(G, T\Sigma/G) \to \hom(G, TM/\mathcal{D})$ over $J^0(\Sigma, M)$, defined as,

$$\begin{aligned} \Xi_1|_{(x,y)} : J^1_{(x,y)}(\Sigma, M) \times \hom(G, T\Sigma/G)|_x &\to \hom(G_x, TM/\mathcal{D}|_y) \\ \big(x, y, P, \varphi\big) &\mapsto \lambda \circ P \circ \bar{\varphi} \end{aligned}$$

for $(x, y) \in J^0(\Sigma, M) = \Sigma \times M$. That is, $\Xi_1|_{(x,y)}(x, y, P, \varphi) = P^*\lambda|_{G^\varphi} = \bar{\varphi}^* P^* \lambda$. Note that $\Xi_1|_{(x,y)}$ is *not* a linear map. We have the derivative map of $\Xi_1|_{(x,y)}$ at some (x, y, P, φ) as,

$$\begin{aligned} d(\Xi_1|_{(x,y)})|_{(x,y,P,\varphi)} : \hom(T_x\Sigma, T_yM) \times \hom(G, T\Sigma/G)|_x &\to \hom(G_x, TM/\mathcal{D}|_y) \\ (P_1, \varphi_1) &\mapsto \lambda \circ P_1 \circ \bar{\varphi} + \lambda \circ P \circ \rho \circ \varphi_1 \end{aligned}$$

Now, for any given $Q : G_x \to TM/\mathcal{D}|_y$, first we get some $\tilde{Q} : G_x \to T_yM$ so that $Q = \lambda \circ \tilde{Q}$ and then we get a $P_1 : T_x\Sigma \to T_yM$, satisfying $P_1|_{G^\varphi} = \tilde{Q} \circ \bar{\varphi}^{-1}$. We have the diagram,

$$\begin{tikzcd}
T_x\Sigma \arrow[rr, dashed, "P_1"] & & T_yM \arrow[d, two heads, "\lambda"] \\
G^\varphi \arrow[u, hook] & G_x \arrow[l, "\cong"', "\bar{\varphi}"] \arrow[ur, "\tilde{Q}"] \arrow[r, "Q"'] & TM/\mathcal{D}|_y
\end{tikzcd}$$

Then, setting $\varphi_1 = 0$, we have,

$$d\big(\Xi_1|_{(x,y)}\big)|_{(x,y,P,\varphi)}\big(P_1, \varphi_1\big) = \lambda \circ P_1\bar{\varphi} + 0 = \lambda \circ \tilde{Q} \circ \bar{\varphi}^{-1} \circ \bar{\varphi} = \lambda \circ \tilde{Q} = Q.$$

Hence, $\Xi_1|_{(x,y)}$ is a submersion and so,

$$\big(\Xi_1|_{(x,y)}\big)^{-1}(0) = \Big\{\big(x, y, P, \varphi\big) \;\big|\; P^*\lambda|_{G^\varphi} = 0\Big\}$$

is a submanifold of $J^1_{(x,y)}(\Sigma, M) \times \hom(G, T\Sigma/G)|_x$. The tangent space at some (x, y, P, φ) is given as,

$$\ker\big(d\Xi_1|_{(x,y)}\big)|_{(x,y,P,\varphi)} = \Big\{(P_1, \varphi_1) \;\big|\; \lambda \circ P_1 \circ \bar{\varphi} + \lambda \circ P \circ \rho \circ \varphi_1 = 0\Big\}.$$

Proof of Step $2a$ **:** First observe that for a tuple (x, y, P, φ) satisfying $P^*\lambda|_{G^\varphi} = 0$, we have the composition of the two maps,

$$T\Sigma/G|_x \xrightarrow{\quad\cong\quad} T_x\Sigma/G^\varphi \xrightarrow{\quad\tilde{P}\quad} TM/\mathcal{D}|_y$$

$$Z \longmapsto \rho(Z) \mod G^\varphi \longmapsto P\rho(Z) \mod \mathcal{D}_y$$

Thus, we may consider the map,

$$\begin{aligned}
\Theta : \Xi_1|_{(x,y)}^{-1}(0) = \big\{(x, y, P, \varphi)\big|P^*\lambda|_{G^\varphi} = 0\big\} &\longrightarrow \hom(T\Sigma/G|_x, TM/\mathcal{D}|_y) \\
(x, y, P, \varphi) &\longmapsto \lambda \circ P \circ \rho = \Big(Z \mapsto P\rho(Z) \mod \mathcal{D}_y\Big)
\end{aligned}$$

We find out the derivative of Θ at (x, y, P, φ) as,

$$T_{(x,y,P,\varphi)}\Theta(P_1, \varphi_1) = \lambda \circ P_1 \circ \rho, \quad \text{for } (P_1, \varphi_1) \in T_{(x,y,P,\varphi)}\big(\Xi_1|_{(x,y)}^{-1}(0)\big).$$

We claim that, Θ is a submersion. Suppose, a linear map $Q : T\Sigma/G|_x \to TM/\mathcal{D}|_y$ is given. Let us define $P_1 : T_x\Sigma \to T_yM$ in two steps, using the splitting, $T_x\Sigma = G_x \oplus \operatorname{Im}\rho$.

- First, define $P_1|_{\operatorname{Im}\rho}$ so that, $\lambda \circ P_1|_{\operatorname{Im}\rho} = Q \circ \rho^{-1}$.
- Next, define $P_1|_{G_x}$ so that, $\lambda \circ P_1|_{G_x} = -Q \circ \varphi$.

This is expressed in the following two commutative diagrams :

$$\begin{tikzcd}
T\Sigma/G|_x \arrow[r, "Q"] \arrow[d, hook, "\rho"', "\cong"] & TM/\mathcal{D}|_y \\
\operatorname{Im}\rho \arrow[r, dashed, "P_1|_{\operatorname{Im}\rho}"'] \arrow[ur, "Q\circ\rho^{-1}"'] & T_yM \arrow[u, twoheadrightarrow, "\lambda"']
\end{tikzcd}
\qquad
\begin{tikzcd}
T\Sigma/G|_x \arrow[r, "-Q"] & TM/\mathcal{D}|_y \\
G_x \arrow[u, "\varphi"] \arrow[r, dashed, "P_1|_{G_x}"'] \arrow[ur, "-Q\circ\varphi"'] & T_yM \arrow[u, twoheadrightarrow, "\lambda"']
\end{tikzcd}$$

Then, first note that for any $X \in G_x$,

$$\lambda \circ P_1 \circ \bar{\varphi}(X) = \lambda P_1\big(X + \rho\varphi X\big) = \lambda P_1(X) + \lambda P_1 \rho\varphi X = -Q\varphi X + Q\varphi X = 0.$$

Now, setting $\varphi_1 = 0$, we have,

$$(P_1, \varphi_1) \in T_{(x,y,P,\varphi)}\big(\Xi_1|_{(x,y)}^{-1}(0)\big) = \Big\{(P_1, \varphi_1) \;\big|\; \lambda \circ P_1 \circ \bar{\varphi} + \lambda \circ P \circ \rho \circ \varphi_1 = 0\Big\}.$$

On the other hand,

$$T_{(x,y,P,\varphi)}\Theta(P_1, \varphi_1) = \lambda \circ P_1 \circ \rho = Q.$$

Thus, Θ is indeed a submersion.

Proof of Step $2b$: For some linear map $A : T\Sigma/G|_x \to TM/\mathcal{D}|_y$, consider the fiber,

$$O_A = \Theta^{-1}(A) = \Big\{(x, y, P, \varphi) \;\big|\; P^*\lambda|_{G^\varphi} = 0, \quad \tilde{P} \circ \hat{\varphi} = A\Big\} \subset \Xi_1|_{(x,y)}^{-1}(0),$$

where $\hat{\varphi} : T\Sigma/G|_x \to T_x\Sigma/G^\varphi$ is the isomorphism, $\hat{\varphi}(Z) = \rho Z \mod G^\varphi$. In particular, we then have $\operatorname{Im} A = \operatorname{Im}(\tilde{P} \circ \hat{\varphi}) = \operatorname{Im}\tilde{P}$ and hence,

$$O_A = \Big\{(x, y, P, \varphi) \;\big|\; P^*\lambda|_{G^\varphi} = 0, \quad \operatorname{Im}\tilde{P} = \operatorname{Im} A\Big\}.$$

Since Θ is a submersion, O_A is a submanifold.

Now, consider the map,

$$\begin{aligned}\Xi_2 = \Xi_2^A : O_A &\to \hom\left(\Lambda^2 G_x, TM/\mathcal{D}|_y / \operatorname{Im} A\right)\\ (x,y,P,\varphi) &\mapsto \bar{\varphi}^* P^* \Omega \mod \operatorname{Im} A\end{aligned}$$

Then, we have,

$$\Xi_2(x,y,P,\varphi) = 0 \quad \Rightarrow \quad \bar{\varphi}^* P^* \Omega \equiv 0 \mod \operatorname{Im} \tilde{P},$$

since, $\operatorname{Im} A = \operatorname{Im} \tilde{P}$. Hence, in particular, for an injective map $A : T\Sigma/G|_x \to TM/\mathcal{D}|_y$,

$$\mathcal{R}_0^{G,\rho}|_{(x,y)} \cap O_A = \Xi_2^{-1}(0) \cap \left\{P : T_x\Sigma \to T_yM \text{ is injective and } \Omega_\bullet\text{-regular}\right\}.$$

Let us now compute the derivative of Ξ_2 at some $(x,y,P,\varphi) \in O_A$,

$$d\Xi_2|_{(x,y,P,\varphi)} : T_{(x,y,P,\varphi)} O_A \to \hom\Big(\Lambda^2 G_x, TM/\mathcal{D}|_y / \operatorname{Im} A\Big).$$

We have for $X, Y \in G_x$,

$$\begin{aligned}
&d\Xi_2|_{(x,y,P,\varphi)}\big(P_1,\varphi_1\big)(X,Y)\\
&= \lim_{t\to 0} \frac{1}{t}\Big[\overline{\varphi + t\varphi_1}^* (P + tP_1)^* \Omega - \bar{\varphi}^* P^* \Omega\Big](X,Y) \mod \operatorname{Im} A\\
&= \lim_{t\to 0} \frac{1}{t}\Big[\Omega\Big((P + tP_1) \circ \overline{\varphi + t\varphi_1} X, (P + tP_1) \circ \overline{\varphi + t\varphi_1} Y\Big) - \Omega\big(P\bar{\varphi}X, P\bar{\varphi}Y\big)\Big]\\
&= \Omega\Big(P_1 \circ \bar{\varphi}X, P \circ \bar{\varphi}Y\Big) \mod \operatorname{Im} A + \Omega\Big(P \circ \bar{\varphi}X, P_1 \circ \bar{\varphi}Y\Big) \mod \operatorname{Im} A\\
&\qquad + \lim_{t\to 0} \frac{1}{t}\Big[\Omega\Big(P \circ \overline{\varphi + t\varphi_1} X, P \circ \overline{\varphi + t\varphi_1} Y\Big) - \Omega\Big(P\bar{\varphi}X, P\bar{\varphi}Y\Big)\Big] \mod \operatorname{Im} A\\
&= \Big[\Omega\Big(P_1 \circ \bar{\varphi}X, P \circ \bar{\varphi}Y\Big) + \Omega\Big(P \circ \bar{\varphi}X, P_1 \circ \bar{\varphi}Y\Big)\\
&\qquad + \Omega\Big(P \circ \rho\varphi_1 X, P \circ \bar{\varphi}Y\Big) + \Omega\Big(P \circ \bar{\varphi}X, P \circ \rho\varphi_1 Y\Big)\Big] \mod \operatorname{Im} A
\end{aligned}$$

Now suppose $(x,y,P,\varphi) \in O_A$ satisfies $\Xi_2(x,y,P,\varphi) = 0$ and it is $\Omega_\bullet$-regular, i.e, P is injective and the map,

$$\begin{aligned}\Omega_\bullet : \mathcal{D}_y &\to \hom(G^\varphi, TM/\mathcal{D}|_y / \operatorname{Im} \tilde{P})\\ \xi &\mapsto P^* \iota_\xi \Omega \mod \operatorname{Im} \tilde{P}\end{aligned}$$

is surjective. Recall that $\operatorname{Im} A = \operatorname{Im} \tilde{P}$ and so, we must have that $A : T\Sigma/G|_x \to TM/\mathcal{D}|_y$ is injective. Applying the $\hom(G^\varphi, \_)$ functor to $\Omega_\bullet$, we have the surjective map,

$$\hom(G^\varphi, \mathcal{D}_y) \to \hom\Big(G^\varphi, \hom\big(G^\varphi, TM/\mathcal{D}|_y / \operatorname{Im} \tilde{P}\big)\Big).$$

Composing with the alternating map, $Alt : F \mapsto \big(X \wedge Y \mapsto F(X)(Y) - F(Y)(X)\big)$ and then the isomorphism $\bar{\varphi} : G_x \to G^\varphi$, we have the following diagram,

$$\begin{tikzcd}
\hom(G^\varphi, \mathcal{D}_y) \arrow[r, two heads] \arrow[dr, dashed, bend right, two heads] & \hom\Big(G^\varphi, \hom\big(G^\varphi, TM/\mathcal{D}|_y / \operatorname{Im} \tilde{P}\big)\Big) \arrow[d, two heads, "\mathrm{Alt}"] \\
 & \hom\Big(\Lambda^2 G^\varphi, TM/\mathcal{D}|_y / \operatorname{Im} \tilde{P}\Big) \arrow[d, two heads, "\cong"] \\
 & \hom\Big(\Lambda^2 G_x, TM/\mathcal{D}|_y / \operatorname{Im} \tilde{P}\Big)
\end{tikzcd}$$

so that the diagonal arrow is surjective as well. That is we have the surjective map,

$$\begin{aligned}
\hom(G^\varphi, \mathcal{D}_y) &\to \hom\Big(\Lambda^2 G_x, TM/\mathcal{D}|_y / \operatorname{Im} \tilde{P}\Big) \\
Q &\mapsto \left(X \wedge Y \mapsto \Big(\Omega(Q\bar{\varphi}X, P\bar{\varphi}Y) + \Omega(P\bar{\varphi}X, Q\bar{\varphi}Y)\Big) \mod \operatorname{Im} \tilde{P}\right)
\end{aligned}$$

Suppose $R : \Lambda^2 G_x \to TM/\mathcal{D}|_y / \operatorname{Im} \tilde{P}$ is some arbitrary map. Then, we can find $Q : G^\varphi \to \mathcal{D}_y$ so that,

$$\Big(\Omega(Q\bar{\varphi}X, P\bar{\varphi}Y) + \Omega(P\bar{\varphi}X, Q\bar{\varphi}Y)\Big) \mod \operatorname{Im} \tilde{P} = R(X, Y), \quad X, Y \in G_x.$$

Now, we have the tangent space,

$$\begin{aligned}
T_{(x,y,P,\varphi)} O_A &= \ker T_{(x,y,P,\varphi)} \Theta \\
&= \Big\{(P_1, \varphi_1) \mid \lambda \circ P_1 \circ \bar{\varphi} + \lambda \circ P \circ \rho \circ \varphi_1 = 0, \quad \lambda \circ P_1 \circ \rho = 0\Big\} \\
&= \Big\{(P_1, \varphi_1) \mid \lambda \circ P_1|_{G_x} + \lambda \circ P \circ \rho \circ \varphi_1 = 0, \quad \lambda \circ P_1 \circ \rho = 0\Big\}
\end{aligned}$$

Let $P_1 : T_x\Sigma \to T_yM$ be an arbitrary extension of $Q : G^\varphi \to \mathcal{D}_y$ so that $\operatorname{Im} P_1 \subset \mathcal{D}_y$ and $\varphi_1 = 0$. Then, note that $\lambda \circ Q = 0$ and hence,

$$\lambda \circ P_1 \circ \bar{\varphi} + \lambda \circ P \circ \rho \circ \varphi_1 = \lambda \circ Q \circ \bar{\varphi} + 0 = 0, \quad \text{and} \quad \lambda \circ P_1 \circ \rho = 0.$$

Thus (P_1, φ_1) is in the tangent space. We also observe that for any $X, Y \in G_x$,

$$\begin{aligned}
&d\Xi_2|_{(x,y,P,\varphi)}(P_1, \varphi_1)(X, Y) \\
&= \Big[\Omega\Big(P_1 \circ \bar{\varphi}X, P \circ \bar{\varphi}Y\Big) + \Omega\Big(P \circ \bar{\varphi}X, P_1 \circ \bar{\varphi}Y\Big) \\
&\quad + \Omega\Big(P \circ \rho\varphi_1 X, P \circ \bar{\varphi}Y\Big) + \Omega\Big(P \circ \bar{\varphi}X, P \circ \rho\varphi_1 Y\Big)\Big] \mod \operatorname{Im}\tilde{P} \\
&= \Big[\Omega\Big(Q \circ \bar{\varphi}X, P \circ \bar{\varphi}Y\Big) + \Omega\Big(P \circ \bar{\varphi}X, Q \circ \bar{\varphi}Y\Big) + 0\Big] \mod \operatorname{Im}\tilde{P}, \quad \text{as } P_1|_{G^\varphi} = Q \\
&= R(X, Y), \quad \text{by our choice of } Q
\end{aligned}$$

So, the $\Omega_\bullet$-regular points in $\Xi_2^{-1}(0)$ are regular points of Ξ_2 and hence,

$$\Xi_2^{-1}(0) \cap \{\Omega_\bullet\text{-regular points}\}$$

is a submanifold of O_A. Clearly, this subset is precisely, $\mathcal{R}_0^{G,\rho}|_{(x,y)} \cap O_A$.

Proof of Step $2c$ **:** Now, let us fix a monomorphism $A : T\Sigma/G|_x \hookrightarrow TM/\mathcal{D}|_y$ and a choice of a splitting map, $\eta : TM/\mathcal{D}|_y/\operatorname{Im} A \hookrightarrow TM/\mathcal{D}|_y$. Next consider the subset,

$$\mathcal{M}_A = \Big\{B : T\Sigma/G|_x \hookrightarrow TM/\mathcal{D}|_y \;\Big|\; B \pitchfork \operatorname{Im}\eta \Leftrightarrow \operatorname{Im} B \cap \operatorname{Im}\eta = 0\Big\} \subset \hom\big(T\Sigma/G, TM/\mathcal{D}\big)\big|_{(x,y)}$$

Clearly $\mathcal{M}_A$ is an open subset, as it is defined via a transversality condition. Now, we have the submersion $\Theta : \Xi_1|_{(x,y)}^{-1}(0) \to \hom(T\Sigma/G, TM/\mathcal{D})|_{(x,y)}$ and let us now consider the restriction of Θ,

$$\Theta^{-1}(\mathcal{M}_A) \to \mathcal{M}_A.$$

Note that for any $B \in \mathcal{M}_A$ we have an isomorphism,

$$\begin{aligned}
\hat{\eta}(B) : TM/\mathcal{D}|_y/\operatorname{Im} A &\to TM/\mathcal{D}|_y/\operatorname{Im} B \\
Z &\mapsto \eta(Z) \mod \operatorname{Im} B
\end{aligned}$$

Next we define a smooth fiber-preserving map,

$$\begin{tikzcd}
\Theta^{-1}(\mathcal{M}_A) \arrow[rr, "\hat{\Xi}_2"] \arrow[dr, "\Theta"'] & & \mathcal{M}_A \times \hom\big(\Lambda^2 G_x,\ TM/\mathcal{D}|_y/\operatorname{Im} A\big) \arrow[dl] \\
& \mathcal{M}_A &
\end{tikzcd}$$

as follows : for $(x,y,P,\varphi) \in \Theta^{-1}(B)$, where $B \in \mathcal{M}_A$, we define,

$$\hat{\Xi}_2(x,y,P,\varphi) = \Big(B,\ \hat{\eta}(B)^{-1} \circ \Xi_2^B(x,y,P,\varphi)\Big) = \Big(B,\ \hat{\eta}(B)^{-1} \circ \big(\bar{\varphi}^* P^* \Omega \mod \operatorname{Im} A\big)\Big).$$

Clearly $\hat{\Xi}_2^{-1}(0)$ consists of those tuples in $\Theta^{-1}(\mathcal{M}_A)$, for which the formal curvature condition holds. Also, it follows from the previous step that, in the fiber O_B over $B \in \mathcal{M}_A$, the set $\mathcal{R}_0^{G,\rho}|_{(x,y)} \cap O_B$ are regular points of $\hat{\Xi}_2|_{O_B}$, since

$$\hat{\Xi}_2|_B = \eta(B)^{-1} \circ \Xi_2^B, \quad \text{where } \eta(B) \text{ is a linear isomorphism.}$$

It then follows from the above diagram that,

$$\mathcal{R}_0^{G,\rho}|_{(x,y)} \cap \Theta^{-1}(\mathcal{M}_A) \text{ are regular points in } \hat{\Xi}_2^{-1}(0).$$

Consequently, this is a submanifold of $\Theta^{-1}(\mathcal{M}_A)$.

Now, by construction, $\mathcal{M}_A$ forms an *open* cover of the base manifold $\operatorname{Mon}(T\Sigma/G, TM/\mathcal{D})|_{(x,y)}$ of monomorphisms $T\Sigma|_x \hookrightarrow TM/\mathcal{D}|_y$. Hence, $\mathcal{R}_0^{G,\rho}|_{(x,y)}$ is a submanifold of $\Xi_1|_{(x,y)}^{-1}(0)$. This concludes the proof of Step 2.

Proof of Step 3 **:** Performing the previous steps more generally, we can prove that $\mathcal{R}_0^{G,\rho}$ is in fact submanifold of $J^1(\Sigma,M) \times \hom(G, T\Sigma/G)$. Note that we have the diagram,

$$\begin{tikzcd}
J^1(\Sigma,M) \times \hom(G,T\Sigma/G) \arrow[drr] & & \Xi_1^{-1}(0) \arrow[ll, hook'] \arrow[d] \arrow[dr, "\Theta"] & \\
& & & \hom(T\Sigma/G, TM/\mathcal{D}) \arrow[dl] \\
& & J^0(\Sigma,M) &
\end{tikzcd}$$

where, $\Xi_1(x,y,P,\varphi) = \lambda \circ P \circ \bar{\varphi}$ and Θ is a submersion.

Now choose contractible neighborhoods $U \subset \Sigma, V \subset M$ around some points $x_0 \in \Sigma$ and $y_0 \in M$, and fix the bundle isomorphisms,

$$\varepsilon_1 : G|_U \to U \times G_{x_0}, \quad \varepsilon_2 : TM/\mathcal{D}|_V \to V \times TM/\mathcal{D}|_{y_0}, \quad \varepsilon_3 : T\Sigma/G|_U \to U \times T\Sigma/G|_{x_0}.$$

We then have the bundle isomorphism,

$$\begin{aligned}\Psi : \hom\big(T\Sigma/G, TM/\mathcal{D}\big)|_{U\times V} &\longrightarrow (U\times V)\times \hom\big(T\Sigma/G, TM/\mathcal{D}\big)|_{(x_0,y_0)}\\ \big(\psi : T\Sigma/G|_x \to TM/\mathcal{D}|_y\big) &\longmapsto \Big(x, y,\ \varepsilon_2|_y \circ \psi \circ \varepsilon_3|_x^{-1}\Big)\end{aligned}$$

Just as in the previous step, for some fixed $A : T\Sigma/G|_{x_0} \hookrightarrow TM/\mathcal{D}|_{y_0}$ and a splitting $\eta : TM/\mathcal{D}|_{y_0}\big/\operatorname{Im} A \hookrightarrow TM/\mathcal{D}|_{y_0}$, we have the open set,

$$\mathcal{M}_A = \Big\{B : T\Sigma/G|_{y_0} \hookrightarrow TM/\mathcal{D}|_{y_0}\ \big|\ B \pitchfork \operatorname{Im}\eta\Big\} \subset \hom(T\Sigma/G, TM/\mathcal{D})|_{(x_0,y_0)}.$$

Note that, for any,

$$B : T\Sigma/G|_x \to TM/\mathcal{D}|_y \quad \text{in} \quad \Psi^{-1}\big(\{(x,y)\}\times \mathcal{M}_A\big)$$

we have an *isomorphism*,

$$\begin{aligned}\tilde{\eta}(x,y,B) : TM/\mathcal{D}|_{y_0}\big/\operatorname{Im} A &\to TM/\mathcal{D}|_y\big/\operatorname{Im} B\\ Z &\mapsto \Big(\varepsilon_2|_y^{-1}\eta(Z)\Big) \mod \operatorname{Im} B\end{aligned}$$

Indeed, it follows from the definition of Ψ that, $B \in \Psi^{-1}\mathcal{M}_A \Leftrightarrow (\varepsilon_2 \circ B) \pitchfork \operatorname{Im}\eta$.

Now, consider the subset,

$$\mathcal{O} = \mathcal{O}(x_0, y_0, A, \rho) := \Theta^{-1}\circ \Psi^{-1}\Big(\big(U\times V\big)\times \mathcal{M}_A\Big),$$

which is clearly *open* in $\Xi_1^{-1}(0)$. We proceed to define a fiber-preserving map $\tilde{\Xi}_2$,

$$\begin{tikzcd} \mathcal{O} \arrow[rr, "\tilde{\Xi}_2"] \arrow[dr] & & (U\times V)\times \mathcal{M}_A \times \hom\big(\Lambda^2 G_{x_0},\ TM/\mathcal{D}|_{y_0}\big/\operatorname{Im} A\big) \arrow[dl] \\ & U\times V & \end{tikzcd}$$

as follows. For $(x,y)\in U\times V$ and $(x,y,P,\varphi)\in\mathcal{O}$, denote $B = \Theta(x,y,P,\varphi)$. Now, $\tilde{\Xi}_2(x,y,P,\varphi)$ can be defined so that the following diagram is commutative :

$$\begin{tikzcd} \Lambda^2 G_{x_0} \arrow[r, dashed, "\tilde{\Xi}_2(x,y,P,\varphi)"] \arrow[d, "\varepsilon_1|_x"', "\cong"] & TM/\mathcal{D}|_{y_0}\big/\operatorname{Im} A \arrow[d, "\cong"', "\tilde{\eta}(x,y,B)"] \\ \Lambda^2 G_x \arrow[r, "\Xi_2^B(x,y,P,\varphi)"'] & TM/\mathcal{D}|_y\big/\operatorname{Im} B \end{tikzcd}$$

It then follows that, $\tilde{\Xi}_2^{-1}(0)$ consists of those tuples $(x, y, P, \varphi) \in \mathcal{O}$, which satisfy the *formal* curvature condition. Arguing as before, we have that,

$$\mathcal{R}_0^{G,\rho}|_{U\times V} \cap \mathcal{O} \text{ are regular points of } \tilde{\Xi}_2^{-1}(0).$$

Thus $\mathcal{R}_0^{G,\rho}$ is locally a manifold. Hence we conclude that $\mathcal{R}_0^{G,\rho}$ is a submanifold of $J^1(\Sigma, M) \times \hom(G, T\Sigma/G)$. In fact, the same argument shows that the restriction of $p : J^1(\Sigma, M) \times \hom(G, T\Sigma/G) \to J^0(\Sigma, M)$ to $\mathcal{R}_0^{G,\rho}$ is a submersion. This concludes the proof. $\square$

We end this section by stating the following h-principle, similar to Theorem 3.2.7.

Theorem 5.1.17. *Suppose, for any given $F \in \Psi^{m\text{-Hor}}$ and any contractible open set $O \subset \Sigma$, there exists a suitable rank m distribution $G \subset TO$ and a splitting map $\rho : TO/G \hookrightarrow TO$, so that $F|_O$ is in the image of $ev : \Gamma\tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}}|_O \to \Psi^{m\text{-Hor}}$. Then the relation $\mathcal{R}^{m\text{-Hor}}$ satisfies the C^0-dense h-principle.*

Proof. For any G and ρ fixed over some $O \subset \Sigma$, we have the following :

- $\tilde{\Phi}^{\tilde{G},\tilde{\rho}}|_\Sigma = \operatorname{Sol}\tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}}|_\Sigma$ is flexible by Proposition 5.1.14.
- We observed in Theorem 5.1.9 that $\tilde{\mathcal{R}}^{m+1\text{-HorGr}}$ enjoys the local h-principle. Since the relation $\tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}}$ can be embedded as an *open* set in the relation $\tilde{\mathcal{R}}^{m+1\text{-HorGr}}$, it satisfies the same.

Lastly, Lemma 5.1.15 justifies the more general version of hypothesis (3) of Theorem 2.2.15, as observed in Remark 2.2.18. The proof is now immediate from the remark, i.e, $\mathcal{R}^{m\text{-Hor}}$ satisfies the C^0-dense h-principle. $\square$

In the next section, we shall see that the "surjectivity hypothesis" in the above h-principle is indeed satisfied in many interesting situations.

5.2 h-Principle for Partially Horizontal Immersions into Fat Distributions

In this section, we shall obtain h-principle of m-horizontal immersions of a general manifold Σ in $(M, \mathcal{D})$, for some fat distribution $\mathcal{D}$. Suppose $\operatorname{cork}\mathcal{D} = p$. In the context of m-horizontal immersions of Σ in $(M, \mathcal{D})$, if we set $q := \dim\Sigma - m$, then

$$0 \le q \le p = \operatorname{cork}\mathcal{D}.$$

The left endpoint $q = 0$ gives us $m = \dim \Sigma$ and so, the m-horizontal immersions are just the $\mathcal{D}$-horizontal immersions. The other end, i.e, $q = p$ gives rise to transverse immersions.

5.2.1 Immersions Transverse to a Corank p Distribution

Let us recall from Definition 5.1.11, the relation $\mathcal{R}^{m\text{-Hor}} \subset J^1(\Sigma, M)$, whose sections are monomorphisms $F : T\Sigma \to TM$, covering some $u : \Sigma \to M$, inducing the rank m subbundle $H := F^{-1}\mathcal{D}$. Moreover, F satisfies,

- the $\Omega_\bullet$-regularity : the bundle map

$$\begin{aligned} \Omega_\bullet : u^*\mathcal{D} &\to \hom\left(H, u^*TM/\mathcal{D}/\operatorname{Im}\tilde{F}\right) \\ \xi &\mapsto \left(X \mapsto \Omega(\xi, FX) \mod \operatorname{Im}\tilde{F}\right) \end{aligned}$$

  is an epimorphism, where we have the induced map $\tilde{F} : T\Sigma/H \hookrightarrow u^*TM/\mathcal{D}$.

- the curvature condition : $F(H) \subset \ker \Omega_\bullet$.

First we observe the following.

Proposition 5.2.1. *For $m = \dim \Sigma - \operatorname{cork} \mathcal{D}$, the relation $\mathcal{R}^{m\text{-Hor}} \subset J^1(\Sigma, M)$ can be given as,*

$$\mathcal{R}^{m\text{-Hor}} = \Big\{(\sigma, y, F) \;\big|\; F : T_\sigma\Sigma \to T_yM \text{ is injective linear and } T_yM = \operatorname{Im} F + \mathcal{D}_y\Big\}.$$

Proof. Let $F : T_\sigma\Sigma \to T_yM$ be an injective linear map such that $\operatorname{Im} F + \mathcal{D}_y = T_yM$ and suppose $m = \dim F^{-1}\mathcal{D}_y$. Then $\dim \Sigma + \dim D_y - \dim F^{-1}\mathcal{D}_y = \dim T_yM$, i.e, $\dim \Sigma = m + p$. Denote, $F^{-1}\mathcal{D}_y$ by H. Then,

$$\begin{aligned} \operatorname{Im} F + \mathcal{D}_y = T_yM &\Rightarrow \operatorname{codim} H = \text{codim } F^{-1}\mathcal{D}_y = \text{ codim } \mathcal{D}_y \\ &\Rightarrow \text{the induced map } \tilde{F} : T_\sigma\Sigma/H \to u^*T_yM/\mathcal{D}_y \text{ is an isomorphism} \\ &\Rightarrow \hom\left(H, u^*TM/\mathcal{D}/\operatorname{Im}\tilde{F}\right) = 0 \end{aligned}$$

The last condition clearly implies that $\Omega_\bullet : \mathcal{D} \to \hom\left(H, u^*TM/\mathcal{D}/\operatorname{Im}\tilde{F}\right)$ is surjective and $F(H) \subset \ker \Omega_\bullet$. Therefore, $(\sigma, y, F) \in \mathcal{R}^{m\text{-Hor}}$, where $\dim \Sigma = m + p$.

Conversely, suppose that $(\sigma, y, F) \in \mathcal{R}^{m\text{-Hor}}$. Then, in particular $\dim F^{-1}D = m = \dim \Sigma - \operatorname{cork} \mathcal{D}$. Denoting $F^{-1}\mathcal{D}_y$ by H, we have $\operatorname{codim} H = \operatorname{codim} \mathcal{D}_y$ and therefore, $\tilde{F} : T_\sigma\Sigma/H \to T_yM/\mathcal{D}_y$ is an isomorphism. This implies that $T_\sigma\Sigma \to T_\sigma\Sigma/H \to T_yM/\mathcal{D}_y$ is surjective. In other words, $\operatorname{Im} F + \mathcal{D}_y = T_yM$. This completes the proof. ☐

We recall the definition.

Definition 5.2.2. Given a distribution $\mathcal{D}$ on M, a smooth map $u : \Sigma \to M$ is said to be transverse to $\mathcal{D}$ if the composition map $T\Sigma \xrightarrow{du} u^*TM \to u^*TM/\mathcal{D}$ is surjective.

In view of the above proposition, we can now identify the solution space of $\mathcal{R}^{m\text{-Hor}}$ with the space of immersions transverse to $\mathcal{D}$, whenever $m = \dim \Sigma - \operatorname{cork} \mathcal{D}$. We prove the following.

Theorem 5.2.3. *Let $\mathcal{D}$ be a corank p distribution on M and $m = \dim \Sigma - p$. Then, the relation $\mathcal{R}^{m\text{-Hor}}$ satisfies the C^0-dense h-principle, provided*

$$\operatorname{rk} \mathcal{D} > m \quad (\textit{or}, \ \dim M > \dim \Sigma).$$

In particular, given a monomorphism $F : T\Sigma \to TM$, such that $F^{-1}\mathcal{D}$ is a subbundle of corank p, we can homotope F to an m-horizontal immersion $\Sigma \to M$, provided $\dim M > \dim \Sigma$.

The above theorem can now be restated as follows.

Theorem. *Let $\mathcal{D}$ be a corank p distribution on M. Then a monomorphism $F : T\Sigma \to TM$, such that $F^{-1}\mathcal{D}$ is a subbundle of corank p, can be homotoped to a transverse immersion $\Sigma \to (M, \mathcal{D})$, provided $\dim M > \dim \Sigma$.*

Remark 5.2.4. In [Gro86, pg. 84], Gromov conjectured that given a bracket-generating distribution $\mathcal{D} \subset TM$, smooth maps $\Sigma \to M$ transverse to $\mathcal{D}$, abide by the C^0-dense, parametric h-principle. In [EM02, pg 131], the h-principle is proved for contact distributions; furthermore the authors indicate a possible way to prove the general conjecture as well. In a recent article [IdPS20], the conjecture is proved for *analytic* manifold M equipped with an *analytic*, bracket-generating distribution $\mathcal{D} \subset TM$. Note that Theorem 5.2.3 is applicable for any distribution $\mathcal{D}$, but assumes that the maps under considerations are additionally immersions.

We now prove the h-principle.

Proof of Theorem 5.2.3. We only need to justify a suitable "surjectivity hypothesis" is true, as in the statement of Theorem 5.1.17.

Suppose we are given formal section $F \in \Psi^{m\text{-Hor}}$, covering some $u : \Sigma \to M$ and inducing the subbundle $H = F^{-1}\mathcal{D}$. Fix some contractible sets $O \subset \Sigma$ and $U \subset M$ satisfying $O \subset u^{-1}U$. Now, we choose an *arbitrary* non-zero section τ over O such that,

$$\tau(\sigma) \in \mathcal{D}_{u(\sigma)} \setminus F(H_\sigma), \quad \sigma \in O,$$

which exists, since $\operatorname{rk}\mathcal{D} > m$ by hypothesis.

Let us now extend $F|_O$ to a bundle map $P : T(O \times \mathbb{R}) \to TU$ defined by,

$$P(v, c\partial_t) = F(v) + c\tau, \quad \text{for } v \in T_\sigma\Sigma \text{ and any } c \in \mathbb{R},$$

where ∂_t is the coordinate vector field along $\mathbb{R}$. Note that P is a monomorphism and it is transverse to $\mathcal{D}$ since $F \pitchfork \mathcal{D}$, covering the map $\tilde{u} = u \circ \pi : O \times \mathbb{R} \to U$, i.e, we have

$$\operatorname{Im} P|_{(\sigma,t)} + \mathcal{D}_{u(\sigma)} = T_{u(\sigma)}U, \quad \text{for } (\sigma, t) \in O \times \mathbb{R}.$$

Then from Proposition 5.2.1, P is a *formal* $m+1$-horizontal immersion $O \times \mathbb{R} \to U$, i.e, $P \in \tilde{\Psi}^{m+1\text{-Hor}}$.

Now, let us fix some $G \subset TO$ and a splitting map $\rho : TO/G \hookrightarrow TO$, so that the distribution $H|_O$ can be realized as $H|_O = G^\phi$ for some $\phi : G \to TO/G$. Recall that for $\tilde{O} = O \times \mathbb{R}$, we have the relations,

$$\tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}} \subset J^1(\tilde{O}, U) \times \hom(\tilde{G}, T\tilde{\Sigma}/\tilde{G})^{(1)} \quad \text{and} \quad \tilde{\mathcal{R}}^{m+1\text{-Hor}} \subset J^1(\tilde{O}, U \times \mathbb{R}) \times \hom(\tilde{G}, T\tilde{O}/\tilde{G})^{(1)},$$

and the map, $ev : \tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}}|_O \to \mathcal{R}^{m\text{-Hor}}$ given by, $j^1_{v,\psi}(\sigma) \mapsto j^1_{v|_\Sigma}(\sigma)$. It is clear that the section P takes its value in the image of $\tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}}$ in $\tilde{\mathcal{R}}^{m+1\text{-Hor}}$, under the map, $j^1_{v,\psi}(\sigma) \mapsto j^1_v(\sigma)$. Now, $\tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}}$ has been identified as an open set of $\tilde{\mathcal{R}}^{m+1\text{-HorGr}}$ and hence, by an application of Lemma 5.1.12, we can get a formal section $\tilde{P} \in \Gamma\tilde{\mathcal{R}}^{\tilde{G},\tilde{\rho}}$, extending the section P. It is immediate that, $ev(\tilde{P}|_{O\times 0}) = P|_{T\Sigma} = F$. The h-principle now follows directly from Theorem 5.1.17. □

5.2.2 Partially Horizontal Immersions into Fat Distribution

Our goal in this section is to prove the following theorem.

Theorem 5.2.5. *Let $\mathcal{D}$ be a corank p fat distribution on M and $m = \dim\Sigma - (p-1)$. Then, $\mathcal{R}^{m\text{-Hor}}$ satisfies the C^0-dense h-principle.*

In other words, we are considering immersions in a corank p fat distribution $\mathcal{D} \subset TM$, which induces a corank $p-1$ distribution on Σ. The proof of this h-principle is in the same vein as in Theorem 4.2.1. Before we proceed to prove the theorem, we make the following observation.

Proposition 5.2.6. *If $\mathcal{D} \subset TM$ is a corank p fat distribution, then any monomorphism $F : T\Sigma \to TM$, inducing a corank $p-1$ distribution $G = F^{-1}(\mathcal{D}) \subset T\Sigma$, is $\Omega_\bullet$-regular.*

Proof. Recall, the $\Omega_\bullet$-regularity F is understood as the surjectivity of the map,

$$\begin{aligned}\Omega_\bullet : u^*\mathcal{D} &\to \hom\left(G, u^*(TM/\mathcal{D})/\operatorname{Im}\tilde{F}\right)\\ \partial &\mapsto F^*\iota_\partial\Omega|_G \mod \operatorname{Im}\tilde{F}\end{aligned}$$

where $\tilde{F} : T\Sigma/G \to u^*TM/\mathcal{D}$ is the induced bundle map. Since $\tilde{F}$ is injective, it follows that

$$rk\left(u^*(TM/\mathcal{D})/\operatorname{Im}\tilde{F}\right) = \operatorname{rk} TM/\mathcal{D} - \operatorname{rk}\operatorname{Im}\tilde{F} = \operatorname{rk} TM/\mathcal{D} - \operatorname{rk} T\Sigma/G = p - (p-1) = 1.$$

Thus, we have $u^*(TM/\mathcal{D})/\operatorname{Im}\tilde{F} \underset{loc.}{\cong} \mathbb{R}$. We may now choose a (local) trivialization of $u^*TM/\mathcal{D}$ so that, $u^*\Omega = (\omega^1, \ldots, \omega^p)$ and the map $\Omega_\bullet$ is given as,

$$\Omega_\bullet(\xi) = F^*\iota_\xi\omega^p|_G \quad \text{for } \xi \in \mathcal{D}.$$

Since $\mathcal{D}$ is a fat distribution, ω^p must be a nondegenerate 2-form on $\mathcal{D}$. Consequently, $\Omega_\bullet$ is surjective, proving the regularity. □

In view of the lemma, the above theorem can now be restated as follows.

Theorem. *Let $\mathcal{D}$ be a corank p fat distribution on M. Then a monomorphism $F : T\Sigma \to TM$ such that $F^{-1}\mathcal{D}$ is a subbundle of corank $p-1$, can be homotoped to a partially horizontal immersion inducing a corank $p-1$ distribution, provided $\operatorname{rk}\mathcal{D} > 2m$, where $m = \dim\Sigma - (p-1)$.*

Let us now prove the h-principle.

Proof of Theorem 5.2.5. In view of Theorem 5.1.17, we only need to show that the (local) extensibility criteria is satisfied. Suppose $F \in \mathcal{R}^{m\text{-Hor}}$ is a *formal* m-horizontal map, with $u = \operatorname{bs} F$, inducing the subbundle $H = F^{-1}\mathcal{D}$. Fix some contractible open sets $U \subset M$ and $O \subset \Sigma$ satisfying, $O \subset u^{-1}(U)$. Now, F satisfies the formal curvature condition, $F(H) \subset \ker\Omega_\bullet$, where,

$$\Omega_\bullet : u^*\mathcal{D} \to \hom\left(G, u^*TM/\mathcal{D}/\operatorname{Im}\tilde{F}\right).$$

Since $u^*(TM/\mathcal{D})/\operatorname{Im}\tilde{F}$ is of rank 1, there exists a trivialization of $u^*TM/\mathcal{D}$ over O such that, $u^*\Omega \underset{loc.}{=} (\omega^1, \ldots, \omega^p)$ and the map $\Omega_\bullet$ is locally given as,

$$\Omega_\bullet(\xi) = F^*\iota_\xi\omega^p|_H, \quad \text{for } \xi \in \mathcal{D}|_O.$$

The curvature condition implies that $F(H)$ is isotropic with respect to ω^p. Since $\operatorname{rk}\mathcal{D} > 2m = 2\dim F(H)$ and ω^p is a *nondegenerate* 2-form (which is a consequence of fatness of $\mathcal{D}$), we can get a non-vanishing field $\tau \in u^*\mathcal{D} \setminus F(H)$ over the contractible open set O, so that,

$$\omega^p(\tau, F(H)) = 0, \text{ on points of } O.$$

Let us now extend $F|_O$ to a bundle map $P : T(O \times \mathbb{R}) \to TU$ by setting, $P(v, c\partial_t) = F(v) + c\partial_t$ for $v \in T_\sigma O$ and $c \in \mathbb{R}$, where ∂_t is the coordinate vector field along $\mathbb{R}$. It is clear from our choice of τ that P a monomorphism, covering the map $\tilde{u} = u \circ \pi : O \times \mathbb{R} \to U$, inducing an rank $m+1$ subbundle on $O \times \mathbb{R}$. Furthermore, P satisfies the formal curvature condition. By Proposition 5.2.6, it is $\Omega_\bullet$-regular. Consequently, $P \in \tilde{\Psi}^{m+1\text{-Hor}}$. We can now proceed as in the proof of Theorem 5.2.3 to conclude the h-principle. □

Corollary 5.2.7. *Suppose $\mathcal{D} \subset TM$ is a corank p fat distribution and $\dim M \geq 3\dim\Sigma - p + 1$. Then any map $u : \Sigma \to M$ can be C^0-approximated by an m-horizontal immersion $\Sigma \to M$, where $m = \dim\Sigma - (p-1)$, provided, there exists a rank m sub-bundle $G \subset T\Sigma$ along with a bundle monomorphism $\tilde{F} : T\Sigma/G \to u^*TM/\mathcal{D}$.*

Proof. Under the hypothesis of the theorem, we only need to produce an injective bundle map $F : G \to \mathcal{D}$, covering u, such that $F(G) \subset \ker\Omega_\bullet$, where

$$\begin{aligned} \Omega_\bullet : u^*\mathcal{D} &\to \hom(G, u^*(TM/\mathcal{D})/\operatorname{Im}\tilde{F}) \\ \xi &\mapsto F^*\iota_\xi\Omega|_G \mod \operatorname{Im}\tilde{F} \end{aligned}$$

Consider the bundle $\mathcal{F} \subset \hom(G, u^*\mathcal{D})$, where the fibers are given as,

$$\mathcal{F}_\sigma = \left\{F : T_\sigma\Sigma \to \mathcal{D}_{u(\sigma)} \middle| F \text{ is injective and } \operatorname{Im} F \subset \ker\Omega_\bullet\right\}, \quad \text{for } \sigma \in \Sigma.$$

We are looking for a global section of this $\mathcal{F}$.

Just as we did in Theorem 5.2.5, let us choose a suitable trivialization of $u^*TM/\mathcal{D}$, so that $\Omega_\bullet$ can be locally represented as,

$$\Omega_\bullet(\xi)(X) = \omega(\xi, FX), \quad \xi \in u^*\mathcal{D}, X \in G,$$

for some (local) 2-form ω on $\mathcal{D}$. The curvature condition is then understood as $F(G)$ being ω-isotropic. Since $\mathcal{D}$ is a fat distribution, ω must be nondegenerate. An argument very similar to that in Lemma 4.2.5 then shows that the fiber $\mathcal{F}_\sigma$ is $(\operatorname{rk}\mathcal{D} - 2m)$-connected. Now, observe that the dimension condition $\dim M \geq 3 \dim \Sigma - (p-1)$ is equivalent to,

$$\operatorname{rk} \mathcal{D} \geq 3 \dim \Sigma - 2p + 1 \Leftrightarrow \operatorname{rk} \mathcal{D} - 2m \geq \dim \Sigma - 1, \quad \text{since } \dim \Sigma = m + (p-1).$$

Therefor, $\mathcal{F}$ admits a global section, say, $\hat{F} : G \to u^*\mathcal{D}$.

Lastly, by choosing some isomorphisms $T\Sigma \cong G \oplus T\Sigma/G$ and $TM \cong \mathcal{D} \oplus TM/\mathcal{D}$, we can define a morphism $F : T\Sigma \to u^*TM$ given as,

$$F = \hat{F} + \tilde{F}.$$

It is immediate that F is a formal m-horizontal immersion, covering $u : \Sigma \to M$ and inducing the bundle G, which satisfies the formal curvature condition. Now Theorem 5.2.5 applies, since we have from the hypothesis,

$$\dim M \geq 3 \dim \Sigma - p + 1 \Rightarrow \operatorname{rk} \mathcal{D} > 2m.$$

Hence, the map u can be homotoped to an m-horizontal immersion $\Sigma \to M$, while keeping the homotopy arbitrarily C^0-small. $\square$

5.2.3 Partially Horizontal Immersions into Quaternionic Contact Distribution

For a corank 3 distribution $\mathcal{D} \subset TM$ and a manifold Σ, there are exactly 4 possible values of m for which m-horizontal immersions $\Sigma \to M$ are defined, namely,

$$0 \leq \dim \Sigma - m \leq 3, \quad \text{i.e,} \quad m = \dim \Sigma - q, \; q = 0, 1, 2, 3.$$

When $\mathcal{D}$ is a quaternionic contact structure, the h-principle in all these cases, except for $m = \dim \Sigma - 1$, have already been addressed in Theorem 4.2.14, Theorem 5.2.5 and Theorem 5.2.3. We now prove the following.

Theorem 5.2.8. *Given $\mathcal{D} \subset TM$ is a quaternionic contact structure on M and Σ is any manifold. Then, for $m = \dim \Sigma - 1$, the relation $\mathcal{R}^{m\text{-Hor}} \subset J^1(\Sigma, M)$ satisfies the C^0-dense h-principle, provided*

$$\operatorname{rk} \mathcal{D} \geq 4m + 4.$$

Proof. Suppose we have a monomorphism, $F : T\Sigma \to TM$, with base map $u : \Sigma \to M$, inducing an m-dimensional subbundle $H := F^{-1}\mathcal{D}$. Furthermore suppose that the bundle map,

$$\begin{aligned} \Omega_\bullet : u^*\mathcal{D} &\to \hom\left(H, u^*TM/\mathcal{D}/\operatorname{Im}\tilde{F}\right) \\ \xi &\mapsto \left(F^*\iota_\xi\Omega\right)|_H \mod \operatorname{Im}\tilde{F} \end{aligned}$$

is surjective and $F(H) \subset \ker\Omega_\bullet$. We shall find a formal extension of F over contractible open subsets $O \subset \Sigma$.

As $\mathcal{D}$ is a quaternionic contact structure, it follows from Definition 4.1.33 that we have a Riemannian metric g on $\mathcal{D}$ and a trivialization $TM/\mathcal{D}|_O = \mathbb{R}^3$, so that the automorphisms $J_i : \mathcal{D} \to \mathcal{D}$, defined over O by,

$$d\lambda^i|_{\mathcal{D}} = g(J_i\_, \_), \quad i = 1, 2, 3,$$

satisfy the quaternionic relations, where $\lambda = \lambda^i \otimes e_i$ for the standard basis (e_1, e_2, e_3) of $\mathbb{R}^3$. Now consider a nonvanishing section R of $T\Sigma/H$ over O and let,

$$\hat{e}_3 := \tilde{F}(R) \in \mathbb{R}^3.$$

Suitably scaling R if necessary, we can extend $\hat{e}_3$ to a orthonormal framing $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ of $\mathbb{R}^3$, so that $\hat{e}_i = Be_i$ for some $B \in SO(3)$. Then it follows that the automorphisms $\hat{J}_i$ defined by,

$$d\hat{\lambda}^i|_{\mathcal{D}} = g(\hat{J}_i\_, \_), \quad i = 1, 2, 3,$$

satisfy the quaternionic relations as well, where $\lambda = \hat{\lambda}^i \otimes \hat{e}_i$.

Now, from our choice above, over O we have,

$$\left(u^*TM/\mathcal{D}/\operatorname{Im}\tilde{F}\right) \cong \left\langle[\hat{e}_1], [\hat{e}_2]\right\rangle \cong \mathbb{R}^2, \quad \text{where } [\hat{e}_i] := \hat{e}_i \mod \operatorname{Im}\tilde{F}.$$

Under this isomorphism, the map $\Omega_\bullet$ is then simply given as,

$$\begin{aligned} \Omega_\bullet : u^*\mathcal{D} &\to \hom\left(H, \mathbb{R}^2\right) \\ \xi &\mapsto \Big(X \mapsto \left(\hat{\omega}^1(\xi, FX), \hat{\omega}^2(\xi, FX)\right)\Big) \end{aligned}$$

where $\hat{\omega}^i = d\hat{\lambda}^i|_{\mathcal{D}}$. For each $\sigma \in O$, consider the tuple

$$T_\sigma := \Big(\mathcal{D}_{u(\sigma)},\ \mathbb{R}^2,\ \hat{\Omega}_\sigma = \big(\hat{\omega}^1,\ \hat{\omega}^2\big)_{u(\sigma)}\Big).$$

The connecting morphism $A : \mathcal{D}_{u(\sigma)} \to \mathcal{D}_{u(\sigma)}$ for the pair $(\hat{\omega}^1, \hat{\omega}^2)$ is given as, $A = -\hat{J}_1^{-1}\hat{J}_2$. Indeed, for $u, v \in \mathcal{D}$ we have,

$$\hat{\omega}^1(u, Av) = g(\hat{J}_1 u, -\hat{J}_1^{-1}\hat{J}_2 v) = g(u, \hat{J}_2 v) = \hat{\omega}^2(u, v),$$

since the adjoint $\hat{J}_s^* = -\hat{J}_s$ (as observed in Proposition 4.2.15). Therefore,

$$A = -\hat{J}_1^{-1}\hat{J}_2 = \hat{J}_1\hat{J}_2 = \hat{J}_3 \ \Rightarrow\ A^2 = \hat{J}_3^2 = -I,$$

and so, T_σ is a degree 2 fat tuple. We also observe that,

- the surjectivity of $\Omega_\bullet$ is equivalent to the $\hat{\Omega}_x$-regularity of $F(H_\sigma)$, while
- the curvature condition $F(H) \subset \ker \Omega_\bullet$ means that $F(H_\sigma)$ is $\hat{\Omega}_x$-isotropic.

But then, just as we argued in Lemma 4.2.2, under the dimension condition, we can get a continuous section $\tau \in u^*\mathcal{D}$ over O such that,

$$\tau \in \big(V^{\hat{\Omega}}\big)^{\hat{\Omega}} \setminus V^{\hat{\Omega}}, \quad \text{where } V = F(H)|_O \text{ is a vector bundle.}$$

The rest of the proof now follows as in Theorem 5.2.5. □

Remark 5.2.9. It may be noted that to prove the h-principle for m-horizontal immersions, for $m = \dim\Sigma - 1$, into quaternionic contact distributions (Theorem 5.2.8), we reduced the underlying algebraic problem of extension to the extension problem for horizontal immersions in degree 2 fat distirbutions (see Chapter 4). Similarly, the h-principle for m-horizontal immersions, for $m = \dim\Sigma - (p-1)$, into corank p fat distributions was reduced to the extensibility problem of Legendrian immersions in contact distributions.

Chapter 6

Germs of Horizontal 2-Submanifold in Fat Distribution of Type $(4,6)$

Our goal here is to prove the existence of germs of 2-dimensional horizontal submanifolds for a certain class of corank 2 fat distribution $\mathcal{D}$ on $\mathbb{R}^6$, which admit a pair of Reeb like vector fields (see Definition 6.2.1). Holomorphic contact distributions are the best known examples in this class of fat corank 2 distributions. We may recall that the holomorphic contact manifolds are modeled on the holomorphic 1-jet space $J^1(\mathbb{C}^n,\mathbb{C})$ and just like their real counterparts, as explained in Example 2.1.20, 1-jet prolongation of any holomorphic map $\mathbb{C}^n \to \mathbb{C}$ is a holomorphic Legendrian embedding. So there are plenty of holomorphic horizontal submanifolds in any holomorphic contact manifold. In [FL18b] the authors have shown that holomorphic Legendrian embeddings of an open Riemann surface Σ into the standard holomorphic contact manifold $\left(\mathbb{C}^{2n+1}, dz - \sum_i y_i dx_i\right)$ satisfy the parametric *Oka principle*. In particular, they prove that the space of Legendrian holomorphic embeddings $\Sigma \hookrightarrow \mathbb{C}^{2n+1}$ has the same homotopy type as the space of continuous maps $\Sigma \to \mathbb{S}^{4n-1}$. The authors further observe that such a global h-principle type result may not be true for a general holomorphic contact manifold.

Since $\mathcal{D}$ in our case is a corank 2 fat distribution on $\mathbb{R}^6$, it is necessarily of degree 2. Though we have studied horizontal immersions in a degree 2 fat distribution in Chapter 4, we may note that this particular case is not covered there, since $\mathcal{D}$ in the present case can not admit an isotropic 2-subspace which is Ω-regular (see Remark 3.1.6). However, this does not rule out the possibility of obtaining a germ of horizontal 2-submanifold as the operator is still underdetermined. In fact, the results of [FL18b] supports this in the special case of standard holomorphic contact distribution on $\mathbb{C}^{2n+1}$.

As we shall be working in a setup where Ω-regularity is impossible to achieve, we need to appeal to a different flavor of implicit function theorem in lieu of Theorem 2.2.24, namely, Hamilton's implicit function theorem.

6.1 Hamilton's Implicit Function Theorem

Nash's Implicit Function Theorem [Nas56] in the context of C^∞-isometric immersions has been generalized by several authors, we have already encountered one of its variation in Theorem 2.2.24. Here we recall Hamilton's formalism of infinite dimensional implicit function theorem that works for smooth differential operators between Fréchet spaces. This theorem is used crucially in order to get the local h-principle of horizontal maps into corank 2 fat distributions which admit Reeb directions. To begin with, we discuss the basic notion of tame spaces and tame operators from the exposition by Hamilton ([Ham82]).

Definition 6.1.1. [Ham82, pg. 67] A *Fréchet space* is a complete, Hausdorff, metrizable, locally convex topological vector space.

In particular the topology of a Fréchet space F is given by a countable collection of seminorms $\{|\cdot|_n\}$, such that a sequence $f_j \to f$ if and only if $|f_j - f|_n \to 0$ for all n, as $j \to \infty$. A choice of this collection of norms is called a grading on the space and we say $(F, \{|\cdot|_n\})$ is a *graded* Fréchet space.

Example 6.1.2. Many naturally occurring spaces are in fact Fréchet spaces.

1. Every Banach space $(X, |\cdot|_X)$ is a Fréchet space. It may also be graded if we set $|\cdot|_n = |\cdot|_X$ for all n ([Ham82, pg. 68]).

2. Given a compact manifold X, possibly with boundary, the function space $C^\infty(X)$ is a graded Fréchet space. More generally, given any vector bundle $E \to X$, the space of sections $\Gamma(E)$ is also a graded Fréchet space. The C^k-norms on the sections give a possible grading ([Ham82, pg. 68]).

3. Given a Banach space $(X, |\cdot|_X)$, denote by $\Sigma(X)$ the space of exponentially decreasing sequences of X, which consists of sequences $\{x_k\}$ of elements of X, such that,

$$|\{x_k\}|_n = \sum_{k=0}^{\infty} e^{nk} |x_k|_X < \infty, \quad \forall n \geq 0.$$

   Then $\Sigma(X)$ is a graded Fréchet space with the norms defined above ([Ham82, pg. 134]).

Definition 6.1.3. [Ham82, pg. 135] A linear map $L : F \to G$ between Fréchet spaces F, G is said to satisfy *tame estimates* of degree r and base b if there exists a constant $c = c(n)$ such that,

$$|Lf|_n \leq C|f|_{n+r}, \quad \forall n \geq b, \quad \forall f \in F$$

L is said to be tame if it satisfies the tame estimates for some n and r.

Example 6.1.4. We have that a large class of operators are in fact tame.

1. A linear partial differential operator $L : C^\infty(X) \to C^\infty(X)$ of order r satisfies the tame estimate $|Lu|_n \leq |u|_{n+r}$ for all $n \geq 0$ and hence L is tame of degree r ([Ham82, pg. 135]).

2. Inverses of elliptic, parabolic, hyperbolic and sub-elliptic operators are tame maps ([Ham82, pg. 67]). In particular, the solution of elliptic boundary value problem is tame ([Ham82, pg. 161]).

3. Composition of two tame maps is again tame ([Ham82, pg. 136]).

Definition 6.1.5. [Ham82, pg. 136] Given graded Fréchet spaces F, G, we say F is a *tame direct summand* of G if there are tame linear maps $L : F \to G$ and $M : G \to F$ such that the composition $ML : F \to F$ is the identity.

We now define tame Fréchet spaces.

Definition 6.1.6. [Ham82, pg. 136] A Fréchet space F is said to be *tame* if F is a tame direct summand of $\Sigma(X)$, for some Banach space X.

Example 6.1.7. Given a compact manifold X, possibly with boundary, and a vector bundle $E \to X$, the section space $\Gamma(E)$ is a *tame* Fréchet space ([Ham82, pg. 139]).

Let us also define, tame smooth maps.

Definition 6.1.8. [Ham82, pg. 143] A map $P : U \subset F \to G$, between Fréchet spaces F, G, defined over some open set $U \subset F$ is said to be a *smooth tame map*, if P is smooth and all the derivatives $D^k P$ are tame linear maps.

We now state the inverse function theorem.

Theorem 6.1.9. [Ham82, pg. 171] *Given tame Fréchet spaces F, G and a tame smooth map $P : U \to G$, where $U \subset F$ is open. Suppose that for the derivative $DP(u)$ at $u \in U$, the equation $DP(u)h = k$ admits unique solution $h = VP(u)k$ for each $k \in G$. Furthermore, assume that $VP : U \times G \to F$ is a smooth tame map. Then P is locally invertible and each local inverse P^{-1} is smooth tame.*

Remark 6.1.10. Unlike the inverse function theorem for Banach spaces, one needs to have that the derivative DP is invertible on an *open* set $U \subset F$.

6.2 Fat Distributions with Reeb-like Vector Fields

Suppose Ξ is a given holomorphic contact structure on a complex manifold M and $\mathcal{D} \subset TM$ is the isomorphic real distribution under the canonical isomorphism $TM \cong T_{(1,0)}M$. It follows from the proof of Corollary 4.1.29, there exists local 1-forms $\alpha_1, \alpha_2 \in \Omega^1(M)$ and local vector fields Z_1, Z_2, such that $\mathcal{D}$ can be written as $\mathcal{D} \underset{loc.}{=} \ker\alpha_1 \cap \ker\alpha_2$ and the tangent bundle TM locally as the direct sum $TM \underset{loc.}{=} \mathcal{D} \oplus \langle Z_1, Z_2\rangle$. Furthermore, α_i and Z_i satisfy the relations below:

$$[Z_1, Z_2] = 0, \quad \alpha_i(Z_j) = \delta_{ij}, \quad \iota_{Z_i} d\alpha_j|_{\mathcal{D}} = 0, \quad i, j = 1, 2.$$

Motivated by this, we consider the following.

Definition 6.2.1. A corank 2 distribution $\mathcal{D}$ on M is said to admit *(local) Reeb directions* Z_1, Z_2, if $\mathcal{D} \underset{loc.}{=} \ker\alpha_1 \cap \ker\alpha_2$ and $TM = \mathcal{D} \oplus \langle Z_1, Z_2\rangle$ such that,

1. $\alpha_1(Z_1) = 1, \alpha_1(Z_2) = 0,$
2. $\alpha_2(Z_1) = 0, \alpha_2(Z_2) = 1,$
3. $\iota_{Z_i} d\alpha_j|_{\mathcal{D}} = 0$ for $i, j = 1, 2,$
4. $[Z_1, Z_2] = 0.$

As observed, the real distribution underlying any holomorphic contact structures, admits (local) Reeb directions.

Now, given any corank 2 fat distribution $\mathcal{D}$ on a manifold M of dimension $4n+2$, one can find ([Ge92]) a coordinate system $(x_1, \ldots, x_{4n}, z_1, z_2)$ and 1-forms,

$$\alpha_i = dz_i - \sum_{j,k} \Gamma^i_{jk} x_j dx_k + R_i, \quad i = 1, 2,$$

such that $\mathcal{D} \underset{loc.}{=} \ker\alpha_1 \cap \ker\alpha_2$. Here $R_i = \sum_{j=1}^{2} f_{ij} dz_j + \sum_{j=1}^{4n} g_{ij} dx_j$ is a 1-form such that, $f_{ij}, g_{ij} \in O(|x|^2 + |z|^2)$ and $\{\Gamma^i_{jk}\}$ constitute the structure constants of some nilpotent Lie algebra, known as the nilpotentization (Remark 4.1.32), associated to the distribution $\mathcal{D}$. In particular $\Gamma^i_{jk} = -\Gamma^i_{kj}$. Observe that, if we take $f_{ij} = 0$ and g_{ij} to be functions of x_k's only, then any such tuple of forms (α_1, α_2) above gives a distribution, which admits local Reeb directions $\langle \partial_{z_1}, \partial_{z_2}\rangle$.

From the classification results of [CFS05], we see that the only possible Lie algebra that can arise as the nilpotentization of a corank 2 fat distribution on a 6 dimensional manifold is the complex Heisenberg Lie algebra.

Question 6.2.2. Is every (germ of) corank 2 fat distribution on $\mathbb{R}^6$, which admits local Reeb directions, diffeomorphic to the germ of the distribution underlying a holomorphic contact structure?

For a general corank 2 fat distribution, the answer is clearly no. From the result of Montgomery (Theorem 2.1.10), it follows that a *generic* distribution germ of type $(4,6)$ cannot admit a local frame, which generates a finite dimensional Lie algebra. Whereas, a holomorphic contact distribution admits such a frame, as observed in Corollary 4.1.29, generating the complex Heisenberg Lie algebra. Since the set of germs of fat distributions of type $(4,6)$ is open, there are plenty of fat distributions, non-isomorphic to the contact holomorphic one. But it is not clear whether any of these fat distributions admit (local) Reeb directions.

Now suppose $\mathcal{D}$ is a corank 2 fat distribution on a manifold M of dimension 6, defined by a pair of 1-forms α_1, α_2. Hence $\omega_i = d\alpha_i|_{\mathcal{D}}$ are nondegenerate and the connecting homomorphism $A : \mathcal{D} \to \mathcal{D}$ defined by

$$\omega_2(u,v) = \omega_1(u, A_x v), \quad \forall u, v \in \mathcal{D}_x, x \in M,$$

has no real eigenvalue. We further assume that the distribution $\mathcal{D}$ admits *local Reeb directions*.

Remark 6.2.3. Any corank 2 fat distribution on a six dimensional manifold is in fact of degree 2, as we have already seen in Example 4.1.39. But it is clear that Theorem 4.2.1 is not applicable due to dimension constraints.

Now for a fixed manifold $\Sigma = \mathbb{D}^2$, consider the partial differential operator,

$$\begin{aligned} \mathfrak{D} : C^\infty(\Sigma, M) &\to \Omega^1(\Sigma, \mathbb{R}^2) \\ u &\mapsto (u^*\alpha_1, u^*\alpha_2) \end{aligned}$$

The C^∞-solutions of $\mathfrak{D}(u) = 0$ are precisely the $\mathcal{D}$-horizontal maps $\Sigma \to M$. Furthermore, horizontality implies the isotropy condition, $u^*d\alpha_1 = 0 = u^*d\alpha_2$; implying that $du_x : T_x\Sigma \to \mathcal{D}_{u(x)}$ is an isotropic map with respect to both the forms $\omega_i = d\alpha_i|_{\mathcal{D}}$ on $\mathcal{D}$ for every $x \in \Sigma$.

6.3 Local Inversion of $\mathfrak{D}$

Linearizing $\mathfrak{D}$ at an $u \in C^\infty(\Sigma, M)$ we have the linear differential operator $\mathfrak{L}_u$ as follows:

$$\mathfrak{L}_u : \Gamma u^*TM \to \Omega^1(\Sigma, \mathbb{R}^2)$$

$$\partial \mapsto \Big(d(\alpha_i \circ \partial) + u^*\iota_\partial d\alpha_i\Big)_{i=1,2}$$

Since $\Sigma = \mathbb{D}^2$ is a compact manifold with boundary, we have (see Example 6.1.7),

Observation 6.3.1. The spaces $\Gamma(u^*TM)$ and $\Omega^1(\Sigma,\mathbb{R}^2)$ are tame Fréchet spaces for any $u:\Sigma \to M$.

Since the linearization $\mathfrak{L}_u$ at $u:\Sigma \to M$ is a linear partial differential operator of order 1, we have (see Example 6.1.4),

Observation 6.3.2. $\mathfrak{L}_u$ is a tame linear map of order 1 for any $u:\Sigma \to M$. Consequently, $\mathfrak{D}$ is a smooth tame map.

This sets the problem into the framework of the differential operator between Fréchet spaces for studying the existence of local inversion. We first prove the following result.

Proposition 6.3.3. *If u is a smooth horizontal immersion then $\mathfrak{L}_u$ admits a tame inverse $\mathfrak{M}_u$.*

Note that we are assuming the existence of $\mathcal{D}$-horizontal immersions here. We first observe the following linear algebraic result.

Lemma 6.3.4. *If $V \subset \mathcal{D}_x$ is common isotropic with respect to $\omega_i = d\alpha_i|_{\mathcal{D}}$ and $\dim V = 2$, then $V = AV$*

Proof. Since V is common isotropic,

$$V \subset V^{\perp_1} \cap V^{\perp_2} = (V + AV)^{\perp_1} \;\Rightarrow\; \dim(V+AV)^{\perp_1} \geq \dim V = 2$$

and so, $\dim(V+AV) \leq \dim \mathcal{D}_x - 2 = 2$. On the other hand, $\dim(V+AV) \geq \dim V = 2$. Hence, $\dim(V+AV) = 2 = \dim V$, which is possible only if $V = AV$. $\square$

Remark 6.3.5. In view of Definition 4.1.14, the last lemma can be restated as follows : every Ω-isotropic subspace of a corank 2 fat distribution on 6-dimensional manifold is invariant.

Proposition 6.3.6. *If u is a smooth $\mathcal{D}$-horizontal immersion, then given any $(P,Q) \in \Omega^1(\Sigma,\mathbb{R}^2)$, the equation $\mathfrak{L}_u(\partial) = (P,Q)$ admits a unique solution $\partial = \mathfrak{M}_u(P,Q)$, subject to a boundary condition. The process of obtaining the solution depends on a choice of complex structure J on $\mathcal{D}$.*

Proof. We first choose an almost complex structure J on $\mathcal{D}$, compatible with $\omega_1 = d\alpha_1|_{\mathcal{D}}$. Since u is $\mathcal{D}$-horizontal we have,

$$u^*\alpha_i = 0 \Rightarrow u^*d\alpha_i = 0, \quad \text{for } i = 1,2.$$

Thus $\operatorname{Im} du_\sigma$ is common isotropic with respect to both $\omega_i = d\alpha_i|_{\mathcal{D}}$ and so in particular, $\operatorname{Im} du_\sigma$ is J-totally real, since J is ω_1-compatible. Also since u is an immersion, $\dim \operatorname{Im} du_\sigma = 2$. Then by Lemma 6.3.4, we also have that,

$$A(\operatorname{Im} du_\sigma) = \operatorname{Im} du_\sigma, \quad \text{for } \sigma \in \Sigma.$$

Let us denote, $X = u_*(\partial_x)$, $Y = u_*(\partial_y)$, where ∂_x, ∂_y are the coordinate vector fields on $\Sigma = \mathbb{D}^2$. We thus have

$$\langle AX, AY \rangle = \langle X, Y \rangle.$$

Hence, A restricts to an automorphism on $\langle X, Y \rangle$:

$$A_0 = A|_{\langle X,Y \rangle}.$$

Let us write,

$$AX = pX + qY, \quad AY = rX + sY \tag{$*$}$$

for some functions $p, q, r, s \in C^\infty(\Sigma)$. Then we have that $A_0 = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ with respect to the basis (X, Y). Since A has no real eigenvalue, A_0 also has no real eigenvalue. This means that the characteristic polynomial

$$\lambda^2 - (p+s)\lambda + (ps - qr)$$

of A_0 has negative discriminant, i.e.,

$$(p+s)^2 - 4(ps - qr) = (p-s)^2 + 4qr < 0.$$

Now, let us consider the equation,

$$\mathfrak{L}_u(\partial) = (P, Q),$$

where $P, Q \in \Omega^1(\Sigma)$. We write,

$$\partial = \partial_0 + aZ_1 + bZ_2,$$

where $\partial_0 \in u^*\mathcal{D}$ and Z_1, Z_2 are the Reeb directions associated to (α_1, α_2), pulled back along u. Using properties (1), (2) and (3) of Definition 6.2.1, we then have,

$$\mathfrak{L}_u(\partial) = \Big(da + u^*\iota_{\partial_0} d\alpha_1,\; db + u^*\iota_{\partial_0} d\alpha_2\Big).$$

Now, let us write,

$$P = P_1 dx + P_2 dy, \quad Q = Q_1 dx + Q_2 dy$$

Evaluating both sides of $\mathfrak{L}_u(\partial) = (P, Q)$ on ∂_x, ∂_y, we then have the system,

$$\begin{cases} \partial_x a + d\alpha_1(\partial_0, X) = P_1 \\ \partial_y a + d\alpha_1(\partial_0, Y) = P_2 \end{cases} \tag{1}$$

$$\begin{cases} \partial_x b + d\alpha_2(\partial_0, X) = Q_1 \\ \partial_y b + d\alpha_2(\partial_0, Y) = Q_2 \end{cases} \tag{2}$$

We also consider an auxiliary system of equations:

$$\begin{cases} d\alpha_1(\partial_0, JX) = 0 \\ d\alpha_1(\partial_0, JY) = 0 \end{cases} \tag{3}$$

Now from $(*)$ we have,

$$d\alpha_2(\partial_0, X) = d\alpha_1(\partial_0, AX) = p\, d\alpha_1(\partial_0, X) + q\, d\alpha_1(\partial_0, Y)$$

$$d\alpha_2(\partial_0, Y) = d\alpha_1(\partial_0, AY) = r\, d\alpha_1(\partial_0, X) + s\, d\alpha_1(\partial_0, Y)$$

This transforms (2) into the following system of PDEs:

$$\begin{cases} \partial_x b + p\, d\alpha_1(\partial_0, X) + q\, d\alpha_1(\partial_0, Y) = Q_1 \\ \partial_y b + r\, d\alpha_1(\partial_0, X) + s\, d\alpha_1(\partial_0, Y) = Q_2 \end{cases} \tag{2'}$$

Using (1) we eliminate ∂_0 from (2′) and get,

$$\begin{cases} \partial_x b - p\partial_x a - q\partial_y a = Q_1 - pP_1 - qP_2 \\ \partial_y b - r\partial_x a - s\partial_y a = Q_2 - rP_1 - sP_2 \end{cases} \tag{2''}$$

Since $(p-s)^2 + 4qr < 0$, the system of PDEs given by ($2''$) is elliptic. Hence, the Dirichlet problem ($2''$) with the boundary condition

$$a|_{\partial\Sigma} = a_0, \quad b|_{\partial\Sigma} = b_0, \tag{4}$$

will have a unique solution,

$$(a, b) = M_u(P, Q, a_0, b_0).$$

Lastly, using the solution $(a, b) = M_u(P, Q, a_0, b_0)$ we get from (1), (3),

$$\begin{cases} d\alpha_1(\partial_0, X) = P_1 - \partial_x a \\ d\alpha_1(\partial_0, Y) = P_2 - \partial_y a \\ d\alpha_1(\partial_0, JX) = 0 \\ d\alpha_1(\partial_0, JY) = 0 \end{cases} \tag{5}$$

Since $\operatorname{Im} du_\sigma$ is J-totally real, $\mathcal{D} = \langle X, Y, JX, JY \rangle$ is a local framing, and since $d\alpha_1|_{\mathcal{D}}$ is nondegenerate, (5) can be uniquely solved for ∂_0. Thus, $\mathfrak{L}_u(\partial) = (P, Q)$ has a unique solution

$$\partial = \mathfrak{M}_u(P, Q, a_0, b_0).$$

subject to satisfying the auxiliary system (3) and the boundary condition (4). □

We can now prove Proposition 6.3.3

Proof of Proposition 6.3.3. From Proposition 6.3.6 we have that $\mathfrak{L}_u$ admits unique solution $\mathfrak{M}_u$, whenever u is $\mathcal{D}$-horizontal immersion. As in Proposition 6.3.6, M_u is obtained as a solution to a Dirichlet problem and hence it is tame (see Example 6.1.4). Then $\mathfrak{M}_u$ is obtained from M_u by solving a linear system, which is again tame. Hence the inverse $\mathfrak{M}_u$ is tame, being composition of two tame maps (see (3) of Example 6.1.4). □

Tame Inversion of $\mathfrak{D}$

From Proposition 6.3.3 we see that the linearization $\mathfrak{L}_u$ admits right inverse $\mathfrak{M}_u$, provided u is $\mathcal{D}$-horizontal immersion. But in order to apply the Implicit Function Theorem due to Hamilton (Theorem 6.1.9), we need to show that there is an *open* set of maps $\mathfrak{U} \subset C^\infty(\Sigma, M)$ such that the family $\{\mathfrak{L}_u \mid u \in \mathfrak{U}\}$ admits a smooth tame inverse. We now identify this set $\mathfrak{U}$.

We first restrict ourselves to a collection $\mathfrak{U}_0$ of maps $u : \Sigma \to M$ satisfying the following two conditions:

- u is an immersion
- $\operatorname{Im} du$ is transverse to $\langle Z_1, Z_2 \rangle$

This collection $\mathfrak{U}_0 \subset C^\infty(\Sigma, M)$ is clearly open, since it is defined by open conditions. Now, we have a canonical projection,

$$\pi_{\mathcal{D}} : TM = \mathcal{D} \oplus \langle Z_1, Z_2 \rangle \to \mathcal{D}.$$

For any $u \in \mathfrak{U}_0$ we see that the image $\pi_{\mathcal{D}}(\operatorname{Im} du)$ has dimension 2 at each point of Σ. Let us choose an almost complex structure $J : \mathcal{D} \to \mathcal{D}$, compatible with $d\alpha_1|_{\mathcal{D}}$, as in Proposition 6.3.6. Then the set

$$\left\{(X, Y) \in \mathrm{Fr}_2\mathcal{D} \;\middle|\; V = \langle X, Y \rangle \text{ is } J\text{-totally real}\right\}$$

is open in the 2-frame bundle $\mathrm{Fr}_2\mathcal{D}$, since the totally real condition $V \cap JV = 0$ is open. For any tuple (X, Y) we have the framing (X, Y, JX, JY) of $\mathcal{D}$ and we can write,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}_{4\times 4}$$

with respect to this basis. Let, $\mathcal{O}_x \subset \mathrm{Fr}_2\mathcal{D}_x$ be the set of those $(X, Y) \in \mathrm{Fr}_2\mathcal{D}_x$ such that,

- $V = \langle X, Y \rangle$ is J-totally real
- The matrix A_{11} as above is negative definite

Since both are open conditions, we see that $\mathcal{O}_x$ is open in $\mathrm{Fr}_2\mathcal{D}_x$.

We now define,

Definition 6.3.7. A map $u : \Sigma \to M$ is said to be *admissible* if it satisfies the following.

- $u \in \mathfrak{U}_0$, i.e, f is an immersion with $\operatorname{Im} du \pitchfork \langle Z_1, Z_2 \rangle$
- $\operatorname{Im} du_\sigma = \langle u_*\partial_x, u_*\partial_y \rangle \in \pi_{\mathcal{D}}^{-1}\big(\mathcal{O}_{u(\sigma)}\big)$ for each $\sigma \in \Sigma$

Denote by $\mathfrak{U} \subset C^\infty(\Sigma, M)$ the set of admissible maps.

In fact we have defined an *open* relation $\mathcal{A} \subset J^1(\Sigma, M)$ such that $\mathfrak{U} = \operatorname{Sol} \mathcal{A}$. Since $\mathcal{A}$ is an open relation, we have that $\mathfrak{U}$ is open in $C^\infty(\Sigma, M)$. It is apparent that any $\mathcal{D}$-horizontal immersion is admissible. We now prove the following.

Theorem 6.3.8. *The linearization $\mathfrak{L}_u$ admits a smooth tame inverse $\mathfrak{M}_u$ for every $u \in \mathfrak{U}$*

Proof. Suppose $u \in \mathfrak{U}$. We have, $\operatorname{Im} du = \langle u_*\partial_x, u_*\partial_y \rangle$. Let us write,

$$u_*\partial_x = X + a_1 Z_1 + a_2 Z_2, \quad u_*\partial_y = Y + b_1 Z_1 + b_2 Z_2$$

where $X = \pi_{\mathcal{D}}(u_*\partial_x), Y = \pi_{\mathcal{D}}(u_*\partial_y)$. By assumption, $(X, Y) \in Fr_2\mathcal{D}$ is totally real and so we have a basis (X, Y, JX, JY) of $\mathcal{D}$. Hence, we can write,

$$\begin{cases} AX = pX + qY + p'JX + q'JY \\ AY = rX + sY + r'JX + s'JY \end{cases} \tag{6}$$

The matrix of A has the form,

$$\begin{pmatrix} p & r & * & * \\ q & s & * & * \\ p' & r' & * & * \\ q' & s' & * & * \end{pmatrix}$$

and by the hypothesis on $\mathfrak{U}$, $A_{11} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is negative definite, which is equivalent to,

$$(p - s)^2 + 4qr < 0.$$

Now we wish to solve $\mathfrak{L}_u(\partial) = (P, Q)$, as we did in Proposition 6.3.6, where

$$\begin{aligned} \mathfrak{L}_u : \Gamma u^*TM &\to \Omega^1(\Sigma, \mathbb{R}^2) \\ \partial &\mapsto \Big(d\big(\alpha_i \circ \partial\big) + u^*\iota_\partial d\alpha_i\Big)_{i=1,2} \end{aligned}$$

Let $\partial = \partial_0 + aZ_1 + bZ_2$, where $\partial_0 \in u^*\mathcal{D}$. Since $[Z_1, Z_2] = 0$ (by (4) of Definition 6.2.1), we have,

$$d\alpha_1(Z_1, Z_2) = Z_1(\alpha_1(Z_2)) - Z_2(\alpha_1(Z_1)) - \alpha_1([Z_1, Z_2]) = Z_1(0) - Z_2(1) - \alpha_1(0) = 0,$$

and similarly, $d\alpha_2(Z_1, Z_2) = 0$. Hence,

$$d\alpha_1(\partial, u_*\partial_x) = d\alpha_1(\partial_0 + aZ_1 + bZ_2, X + a_1 Z_1 + a_2 Z_2) = d\alpha_1(\partial_0, X),$$

and similarly the remaining ones. Thus, we get a system as before,

$$\begin{cases} \partial_x a + d\alpha_1(\partial_0, X) = P_1 \\ \partial_y a + d\alpha_1(\partial_0, Y) = P_2 \end{cases} \tag{7}$$

$$\begin{cases} \partial_x b + d\alpha_1(\partial_0, AX) = Q_1 \\ \partial_y b + d\alpha_1(\partial_0, AY) = Q_2 \end{cases} \tag{8}$$

We add the linear equations,

$$d\alpha_1(\partial_0, JX) = 0 = d\alpha_1(\partial_0, JY) \tag{9}$$

to (7),(8). Then using (6) and (9), the system (8) becomes,

$$\begin{cases} \partial_x b + p\, d\alpha_1(\partial_0, X) + q\, d\alpha_1(\partial_0, Y) = Q_1 \\ \partial_y b + r\, d\alpha_1(\partial_0, X) + s\, d\alpha_1(\partial_0, Y) = Q_2 \end{cases} \tag{8$'$}$$

Using (7) we can eliminate ∂_0 in (8$'$) and get,

$$\begin{cases} \partial_x b - p\partial_x a - q\partial_y a = Q_1 - pP_1 - qP_2 \\ \partial_y b - r\partial_x a - s\partial_y a = Q_2 - rP_1 - sP_2 \end{cases} \tag{8$''$}$$

Since $(p-s)^2 + 4qr < 0$, we have that (8$''$) is elliptic. Hence given any arbitrary boundary condition $a|_{\partial\Sigma} = a_0,\ b|_{\partial\Sigma} = b_0$, we have the unique solution,

$$(a, b) = M_u(P, Q, a_0, b_0).$$

Then, as in Proposition 6.3.6, we obtain unique solution

$$\partial = \mathfrak{M}_u(P, Q, a_0, b_0)$$

to the system given by (7), (8) and (9). Thus whenever $u \in \mathfrak{U}$, we have a solution $\mathfrak{M}_u$ for the linearized equation $\mathfrak{L}_u = (P, Q)$. As argued in the proof of Proposition 6.3.3, both $\mathfrak{L}_u$ and $\mathfrak{M}_u$ are tame operators. □

Since $\mathfrak{L}_u$ is surjective for every $u \in \mathfrak{U}$ and the family of right inverses $\mathfrak{M} : \mathfrak{U} \times \Omega^1(\Sigma, \mathbb{R}^2) \to C^\infty(\Sigma, M)$ is a smooth tame map, we obtain the following by an application of Theorem 6.1.9.

Theorem 6.3.9. *The operator $\mathfrak{D}$ restricted to $\mathfrak{U}$ is locally right invertible. Given any $u_0 \in \mathfrak{U}$, there exists an open neighborhood U of u_0 and a smooth tame map $\mathfrak{D}_{u_0}^{-1} : \mathfrak{D}(U) \to U$ such that $\mathfrak{D} \circ \mathfrak{D}_{u_0}^{-1} = Id$.*

The proof of the Implicit Function Theorem, in fact, implies that there exists a positive integer r_0 such that the following holds true.

Theorem 6.3.10. *Let $u_0 \in \mathfrak{U}$ and $g_0 = \mathfrak{D}(u_0)$. Let $\epsilon > 0$ be any positive number. Then there exists a $\delta > 0$ and an integer r_0, such that for any $\alpha \geq r_0$ and for every $g \in \Omega^1(\Sigma, \mathbb{R}^2)$ with $|g|_\alpha < \delta$, there is an $u = \mathfrak{D}_{u_0}^{-1}(g_0 + g) \in \mathfrak{U}$ satisfying the following conditions:*

$$\mathfrak{D}(u) = g_0 + g \quad \text{and} \quad |u - u_0|_{\alpha+2} < \epsilon.$$

6.4 Existence of Horizontal Germs and the Local h-Principle

Since we are only interested in germs, without loss of generality, we assume that $M = \mathbb{R}^6$ and $\Sigma = \mathbb{R}^2$. Suppose, we have a corank 2 fat distribution $\mathcal{D}$ on M, which admits Reeb directions (Definition 6.2.1). Consider the (open) relation $\mathcal{A} \subset J^1(\Sigma, M)$, as in the previous section, so that the set of admissible maps $\mathfrak{U}$ are precisely the smooth holonomic sections of $\mathcal{A}$. We have shown that the operator, $\mathfrak{D} : u \mapsto \big(u^*\alpha_1, u^*\alpha_2\big)$ is locally invertible over $\mathfrak{U}$.

Now following Gromov([Gro86]), we can get the (parametric) local h-principle. One crucial thing to observe is that unlike Theorem 2.2.24, the inversion of $\mathfrak{D}$ as we have obtained in Theorem 6.3.9, does not conform to the notion of *locality* as considered by Gromov. Yet we observe that the proof of local h-principle goes through, without the locality property of $\mathfrak{D}^{-1}$. For the sake of completeness, we reproduce the proof.

Recall from Definition 2.2.26 that a germ of a map $u : \Sigma \to M$ at $\sigma \in \Sigma$ is an *infinitesimal solution* of order α of $\mathfrak{D}(u) = 0$ if,

$$j^\alpha_{\mathfrak{D}(u)}(\sigma) = 0.$$

Now since $\mathfrak{D}$ has order 1, the property that u is an infinitesimal solution of order α, only depends on the jet $j^{\alpha+1}_u(\sigma)$. Consider the relation,

$$\mathcal{R}_\alpha = \mathcal{R}_\alpha(\mathfrak{D}, 0, \mathcal{A}) \subset J^{\alpha+1}(\Sigma, M),$$

consisting of jets $j^{\alpha+1}_u(\sigma)$, represented by $u : \mathrm{Op}(\sigma) \to M$, so that,

$$u \in \mathrm{Sol}\,\mathcal{A} = \mathfrak{U} \qquad \text{and} \qquad j^{\alpha+1}_{\mathfrak{D}(u)} = 0.$$

We note that, for any $\alpha \geq 0$, the smooth solutions of $\mathcal{R}_\alpha$ are precisely the $\mathcal{A}$-regular solutions, i.e, admissible solutions of $\mathfrak{D} = 0$. We then prove the following.

Theorem 6.4.1. *If α is sufficiently large, then every infinitesimal solution $u : \mathrm{Op}(\sigma) \to M$ of order α admits a homotopy $u_t : \mathrm{Op}(\sigma) \to M$, such that $u_0 = u$ on some $\mathrm{Op}\,\sigma$ and u_1 is a $\mathcal{D}$-horizontal admissible solution, i.e, $\mathfrak{D}(u_1) = 0$. Furthermore, the jets $j^{\alpha+1}_{u_t}(\sigma)$ belongs to $\mathcal{R}_\alpha$, for all $t \in [0,1]$.*

Proof. Suppose u is defined on an open ball $V \subset \Sigma$ about σ. Since $u \in \mathrm{Sol}\,\mathcal{A}$ and $\mathcal{A}$ is open, we can get a neighborhood V_0 of σ, such that $\sigma \in V_0 \subset V$ and $u|_{V_0}$ is a solution of $\mathcal{A}$. In other words, $u|_{V_0}$ is admissible. Denote, $g_0 = \mathfrak{D}(u|_{V_0})$.

Since $j^{\alpha+1}_u(\sigma) \in \mathcal{R}_\alpha$, we have $j^\alpha_{g_0}(\sigma) = j^\alpha_{\mathfrak{D}(u)}(\sigma) = 0$. Hence for any given $\epsilon > 0$, there exists a neighborhood $W \subset V_0$ of σ such that $|g_0|_\alpha < \epsilon$ on W. We can get some g_ϵ on V_0 so that,

- $g_\epsilon = -g_0$ on some neighborhood $W \subset V_0$ of σ, and
- g_ϵ is ϵ-small in C^α-norm, i.e, $|g_\epsilon|_\alpha < \epsilon$ on V_0.

Now let us apply Theorem 6.3.10 for the domain V_0. Since $y_0 := u|_{V_0}$ is admissible, we have that $\mathfrak{D}_{y_0}$ admits a local inverse. In particular, there exists some $\epsilon, \delta > 0$ such that for any $|g|_\alpha < \epsilon$ we have unique y such that $\mathfrak{D}(y) = \mathfrak{D}(y_0) + g$ and $|y - y_0|_{\alpha+1} < \delta$. Here we require that α to be sufficiently large. Now, in particular, for this $\epsilon = \epsilon(y_0, r)$, we can get W and g_ϵ as above. Then we have unique solutions,

$$u_t = \mathfrak{D}^{-1}_{y_0}(tg_\epsilon),$$

over V_0, satisfying $|u_t - y_0|_{\alpha+1} < \epsilon$ for $t \in [0,1]$. Now,

$$\mathfrak{D}(u_t) = \mathfrak{D}(y_0) + tg_\epsilon = \mathfrak{D}(u|_{V_0}) + tg_\epsilon = g_0 + tg_\epsilon,$$

In particular we have, $\mathfrak{D}(u_0) = g_0$ and hence $u_0 = u|_{V_0}$ from uniqueness. On the other hand, over W,

$$\mathfrak{D}(u_1) = g_0 + g_\epsilon = g_0 - g_0 = 0.$$

Thus u_1 is a solution $\mathfrak{D}(u_1) = 0$, over W. Furthermore the jet $j^{\alpha+1}_{u_t}(\sigma) \in \mathcal{R}_\alpha$ for all $t \in [0,1]$. ☐

Thus we have a (parametric) *local* h-principle for $\mathcal{R}_\alpha$ (see [Gro86, pg. 119]).

Corollary 6.4.2. *For α large enough, The jet map $j^{\alpha+1} : \operatorname{Sol}\mathcal{R}_\alpha \to \Gamma\mathcal{R}_\alpha$ is a local weak homotopy equivalence at any $\sigma \in \Sigma$.*

In order to prove the existence of a horizontal germ, we need to prove that $\mathcal{R}_\alpha \neq \emptyset$ at some σ. One issue with Theorem 6.4.1 is that we do not specify the higher jet order α, that is crucial in order to get a local solution. We now show that in fact we can get a lift to any arbitrary higher jet from the first jet relation of isotropic horizontal maps. Recall that given any map u satisfying $u^*\alpha_i = 0$ we have, taking derivatives, that $u^*d\alpha_i = 0$. That is, $\operatorname{Im} du$ is $\omega_i = d\alpha_i|_\mathcal{D}$-isotropic. Now from Proposition 6.3.3, we have that every solution is automatically admissible. Consider the relation $\mathcal{R} \subset \mathcal{R}_0 \subset J^1(\Sigma, M)$ consisting of $(x, y, F : T_x\Sigma \to T_yM)$ such that, $F^*d\lambda^s = 0$ for $s = 1, 2$. In other words, $F : T\Sigma \to TM$ is a formal isotropic $\mathcal{D}$-horizontal immersion. We have the following result.

Lemma 6.4.3. *For any $\alpha \geq 1$, the jet projection map $p = p_1^{\alpha+1} : J^{\alpha+1}(\Sigma, M) \to J^1(\Sigma, M)$ maps $\mathcal{R}_\alpha|_{(x,y)}$ surjectively onto $\mathcal{R}|_{(x,y)}$, for any $(x, y) \in \Sigma \times M$.*

Proof. The proof is similar to that of Lemma 3.2.1 for Ω-regular horizontal immersions. Note that since the distribution $\mathcal{D}$ in question is fat, by Definition 4.1.20, *every* 1-dimensional subspace $\langle v\rangle$ of $\mathcal{D}_y$ is Ω-regular. Now, suppose we have fixed some coordinate (x_1, x_2) about $\sigma \in \Sigma$. Then for any Ω-isotropic, injective map $F : T_\sigma\Sigma \to T_yM$, $\operatorname{Im} F$ admits a codimension 1 Ω-regular subspace, namely, $\langle F(\partial_{x_1})\rangle$. Consequently, the proof now follows immediately from Remark 3.3.3. ☐

Note that we have $\operatorname{Sol}\mathcal{R} = \operatorname{Sol}\mathcal{R}_\alpha$ for any $\alpha \geq 0$. Then as a direct consequence of Corollary 6.4.2 and Lemma 6.4.3, we have the following local h-principle.

Corollary 6.4.4 ([Bho20]). *The relation $\mathcal{R} \subset J^1(\Sigma, M)$ satisfies the (parametric) local h-principle, i.e, the jet map $j^1 : \operatorname{Sol}\mathcal{R} \to \Gamma\mathcal{R}$ is a local weak homotopy equivalence at any $\sigma \in \Sigma$.*

We can now prove the existence of germs of $\mathcal{D}$-horizontal submanifolds of dimension 2.

Theorem 6.4.5 ([Bho20]). *Given $\mathcal{D}$ is corank 2 fat distribution on a manifold M of dimension 6, admitting local Reeb directions. Then there exists a germ of a $\mathcal{D}$-horizontal submanifold of dimension 2.*

Proof. Suppose $\mathcal{D} = \ker\alpha_1 \cap \ker\alpha_2$ for some local 1-forms α_i around some $y \in M$. Pick some arbitrary $0 \neq v \in \mathcal{D}_y$ and set $u = Av$, where A is the (local) automorphism. Then, observe that,

$$d\alpha_1(u,v) = d\alpha_1(Av,v) = d\alpha_2(v,v) = 0 \quad \text{and,} \quad d\alpha_2(u,v) = d\alpha_1(u,Av) = d\alpha_1(u,u) = 0.$$

In other words, $\langle u, v\rangle \subset \mathcal{D}_y$ is Ω-isotropic. Now, consider the jet $\sigma = (0, y, F : T_x\mathbb{D}^2 \to T_yM) \in J^1(\mathbb{D}^2, M)$ given by $F(\partial_x) = u, F(\partial_y) = v$. Then clearly, we have $\sigma \in \mathcal{R}|_{(0,y)}$ by construction. But then by Corollary 6.4.4, we have a $\mathcal{D}$-horizontal immersion $u : \mathrm{Op}(0) \to M$. Since u is an immersion, it is a local diffeomorphism and so we have a germ of $\mathcal{D}$-horizontal submanifold of dimension 2. □

We conclude with the following Conjecture:

Conjecture. If Σ is an open 2-manifold then horizontal immersions of Σ in $(\mathbb{R}^6, \mathcal{D})$, with some higher order regularity condition, satisfy the h-principle.

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List of Publications

1. Aritra Bhowmick, Mahuya Datta
   On the h-principle for Horizontal Immersions in Certain Corank 2 Fat Distributions
   Preprint : https://arxiv.org/abs/2007.02058

2. Aritra Bhowmick
   On Horizontal Immersions of Discs in Fat Distributions of Type $(4, 6)$
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3. Aritra Bhowmick
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