Commuting tuples of operators and functions in the Schur-Agler class

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Dedicated to my teachers

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Contents

Notations & Abbreviations

$\mathbb{N}$	Set of all Natural numbers.
$\mathbb{Z}_+$	$\mathbb{N} \cup \{0\}$.
$\mathbb{Z}_+^n$	$\{\boldsymbol{t} = (t_1, \ldots, t_n) : t_i \in \mathbb{Z}_+, i = 1, \ldots, n\}$.
$\boldsymbol{z}$	$(z_1, \ldots, z_n) \in \mathbb{C}^n$.
$\boldsymbol{z}^{\boldsymbol{k}}$	$z_1^{k_1} \ldots z_n^{k_n}$.
$\lvert\boldsymbol{k}\rvert$	$k_1 + \ldots + k_n$.
$(T_1, \ldots, T_n)$	n-tuple of commuting operators on Hilbert spaces.
$T^{\boldsymbol{k}}$	$T_1^{k_1} \ldots T_n^{k_n}$.
$\mathbb{D}^n$	$\{\boldsymbol{z} : \lvert z_i\rvert < 1, i = 1, \ldots, n\}$.
$\mathbb{B}^n$	$\{\boldsymbol{z} : \sum_{i=1}^n \lvert z_i\rvert^2 < 1\}$.
$\mathcal{E}, \mathcal{E}_*$	Hilbert spaces.
$\mathcal{B}(\mathcal{E})$	Set of all bounded linear operators on $\mathcal{E}$.
$\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$	Set of all bounded linear operators from $\mathcal{E}$ to $\mathcal{E}_*$.
$P^\perp = I - P$	Where P is a orthogonal projection on a Hilbert space.

Introduction

The subject of linear operators evolved rapidly since the work of Riesz and von Neumann and their coauthors starting from 1916. Evidently, function theory profoundly impacts the development of operator theory, operator algebras, mathematical physics, and many other subjects of linear analysis. For instance, the theory of essentially bounded functions (following the Lebesgue measure theory) plays a crucial role in unifying the theory of normal operators, whereas the theory of bounded analytic functions plays a decisive role in formulating and solving several problems related to non-normal operators.

The principal goal of this thesis is to provide a further insight into the underlying connection between several variables analytic function theory and the structure of commuting tuples of bounded linear operators on Hilbert spaces. The broader approach is in line with the earlier studies of shift invariant subspaces of Beurling [29], Lax [81], and Halmos [63], the Sz.-Nagy and Foias [90] model theory, the de Branges' point of view of [37, 38] invariant subspaces that are contractively contained in the Hardy space, etc. Our contribution also follows Agler's modern approach to Schur functions [1, 2], and the line of research of Sarason in the generalization of Beurling theorem [101]. Some of our results again bring out the known contrast between the theory of single operators and the theory of multivariable operator theory.

More specifically, we study factorizations of Schur functions and Schur-Agler functions of various kinds. We relate the structure of inner functions with isometric colligations and $C_{0\cdot}$ operators. We also characterize Beurling quotient modules on Hardy space over polydisc in terms of model operators. We present several results concerning de Branges-Rovnyak kernels and Agler kernels. The main contributions of this thesis are:

1. *Factorizations of Schur-Agler functions:* We present algorithms to factorize Schur functions and Schur-Agler class functions in terms of colligation matrices. More precisely, we isolate checkable conditions on colligation matrices that ensure the existence of Schur (Schur-Agler class) factors of a Schur (Schur-Agler class) function and vice versa. This study also seeks to contribute to the understanding of the delicate structure of bounded analytic functions in several complex variables.

2. *Schur functions and inner functions on the bidisc:* We study representations of inner functions on the bidisc from a fractional linear transformation point of view.

We provide sufficient conditions, in terms of colligation matrices, for the existence of two-variable inner functions, and we present classification results of de Branges-Rovnyak kernels on the polydisc and on the open unit ball of $\mathbb{C}^n$, $n \geq 1$. We also classify, in terms of Agler kernels, two-variable Schur functions that admit a one variable factor.

3. *Beurling quotient modules on the polydisc:* We present a complete characterization of Beurling quotient modules of $H^2(\mathbb{D}^n)$ in terms of model operators. We provide two applications, first, we obtain a dilation theorem for Brehmer n-tuples of commuting contractions, and, second, we relate joint invariant subspaces with factorizations of inner functions.

Let us now elaborate on the above content chapter-wise. Our thesis comprises three chapters, excluding a preliminary chapter. Here is the detailed outline of the main chapters:

Chapter 2: Factorizations of Schur-Agler functions

The goal of this chapter is to clarify the link between isometric colligations and factors of Schur functions. Let Ω be an open connected set in $\mathbb{C}^n$, $n \geq 1$. By definition, the *Schur class* $\mathcal{S}(\Omega)$ consists of complex-valued analytic functions mapping from Ω into the closed unit disk $\overline{\mathbb{D}}$, that is

$$\mathcal{S}(\Omega) = \{\varphi : \Omega \to \mathbb{C} : \varphi \text{ is analytic and } \|\varphi\|_\infty \leq 1\},$$

where $\|\cdot\|_\infty$ denotes the supremum norm over Ω. In other words, $\mathcal{S}(\Omega)$ is the closed unit ball of the commutative Banach algebra $H^\infty(\Omega)$, the set of all bounded analytic functions on Ω under the supremum norm. The elements in the set $\mathcal{S}(\Omega)$ are called Schur functions [104, 105]. In this thesis, we will only focus on Ω as the open unit disc $\mathbb{D}^n$ and the open unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$.

Now we formulate the necessary set of notations for the main results of this chapter. Given $1 \leq m < p \leq n$ and Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$, we set

$$\mathcal{H}_m^p = \mathcal{H}_m \oplus \mathcal{H}_{m+1} \oplus \cdots \oplus \mathcal{H}_p.$$

In particular, $\mathcal{H}_1^n = \bigoplus_{i=1}^n \mathcal{H}_i$. Moreover, with respect to the orthogonal decomposition $\mathcal{H}_1^n = \mathcal{H}_1^m \oplus \mathcal{H}_{m+1}^n$, we represent an operator $D \in \mathcal{B}(\mathcal{H}_1^n)$ as

$$D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1^m \oplus \mathcal{H}_{m+1}^n).$$

Similarly, if $\mathcal{E}$ and $\mathcal{E}_*$ are Hilbert spaces, $B \in \mathcal{B}(\mathcal{H}_1^n, \mathcal{E})$ and $C \in \mathcal{B}(\mathcal{E}_*, \mathcal{H}_1^n)$, then we write

$$B = \begin{bmatrix} B_1 & B_2 \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1^m \oplus \mathcal{H}_{m+1}^n, \mathcal{E}) \quad \text{and} \quad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \in \mathcal{B}(\mathcal{E}_*, \mathcal{H}_1^m \oplus \mathcal{H}_{m+1}^n).$$

We are ready to introduce the relevant isometric colligations:

Definition 0.0.1. *Let $1 \leq m < n$. We say that an isometry $V \in \mathcal{B}(\mathcal{H})$ satisfies property $\mathcal{F}_m(n)$ if there exist Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_n$ such that $\mathcal{H} = \mathbb{C} \oplus \mathcal{H}_1^n$, and representing V as*

$$V = \begin{bmatrix} a & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1^m \oplus \mathcal{H}_{m+1}^n),$$

one has $D_{21} = 0$ and $aD_{12} = C_1 B_2$.

The *Schur-Agler class* $\mathcal{SA}(\mathbb{D}^n)$ [2] consists of scalar-valued analytic functions φ on $\mathbb{D}^n$ such that φ satisfies the n-variables von Neumann inequality, that is

$$\|\varphi(T_1, \dots, T_n)\|_{\mathcal{B}(\mathcal{H})} \leq 1,$$

for any n-tuples of commuting strict contractions on a Hilbert space $\mathcal{H}$. The elements of $\mathcal{SA}(\mathbb{D}^n)$ are called *Schur-Agler class functions*. If $\varphi \in \mathcal{SA}(\mathbb{D}^n)$, then we also say that φ is a *function in the Schur-Agler class* $\mathcal{SA}(\mathbb{D}^n)$. The following theorem of Jim Agler [2] then obtains:

Theorem 0.0.2 (Agler)**.** *Let φ be a function on $\mathbb{D}^n$. Then $\varphi \in \mathcal{SA}(\mathbb{D}^n)$ if and only if there exist Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_n$ and an isometric colligation*

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1^n),$$

such that $\varphi = \tau_V$ where

$$\tau_V(\boldsymbol{z}) = a + B(I_{\mathcal{H}_1^n} - E_{\mathcal{H}_1^n}(\boldsymbol{z})D)^{-1} E_{\mathcal{H}_1^n}(\boldsymbol{z})C,$$

$\mathcal{H}_1^n = \bigoplus_{i=1}^n \mathcal{H}_i$ *and* $E_{\mathcal{H}_1^n}(\boldsymbol{z}) = \bigoplus_{i=1}^n z_i I_{\mathcal{H}_i}$ *for all* $\boldsymbol{z} \in \mathbb{D}^n$.

Our first main result concerns factorizations of Schur-Agler class functions in $\mathcal{SA}(\mathbb{D}^n)$, $n > 1$, into Schur-Agler class factors with fewer variables.

Theorem 0.0.3. *Let $1 \leq m < n$, and let $\theta \in \mathcal{SA}(\mathbb{D}^n)$. If $\theta(0) \neq 0$, then*

$$\theta(\boldsymbol{z}) = \phi(z_1, \dots, z_m)\psi(z_{m+1}, \dots, z_n) \qquad (\boldsymbol{z} \in \mathbb{D}^n),$$

for some $\phi \in \mathcal{SA}(\mathbb{D}^m)$ *and* $\psi \in \mathcal{SA}(\mathbb{D}^{n-m})$ *if and only if*

$$\theta(\boldsymbol{z}) = \tau_V(\boldsymbol{z}) \qquad (\boldsymbol{z} \in \mathbb{D}^n),$$

for some isometric colligation V *satisfying property* $\mathcal{F}_m(n)$.

Next we investigate general n-variables ($n \in \mathbb{N}$) Schur-Agler class factors of Schur-Agler class functions in $\mathcal{SA}(\mathbb{D}^n)$. More specifically, for a given $\theta \in \mathcal{SA}(\mathbb{D}^n)$, we give a set of necessary and sufficient conditions on isometric colligations ensuring the existence of φ and ψ in $\mathcal{SA}(\mathbb{D}^n)$ such that $\theta = \varphi\psi$. Our classification is related to the following class of isometric colligations:

Definition 0.0.4. *We say that an isometry* $V \in \mathcal{B}(\mathcal{H})$ *satisfies property* $\mathcal{F}(n)$ *if there exist Hilbert spaces* $\{\mathcal{M}_i\}_{i=1}^n$ *and* $\{\mathcal{N}_i\}_{i=1}^n$ *such that*

$$\mathcal{H} = \mathbb{C} \oplus \Big(\bigoplus_{i=1}^n (\mathcal{M}_i \oplus \mathcal{N}_i) \Big),$$

and representing V *as*

$$V = \left[\begin{array}{c|ccc} a & B_1 & \cdots & B_n \\ \hline C_1 & D_{11} & \cdots & D_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ C_n & D_{n1} & \cdots & D_{nn} \end{array}\right] \in \mathcal{B}\Big(\mathbb{C} \oplus \Big(\bigoplus_{i=1}^n (\mathcal{M}_i \oplus \mathcal{N}_i) \Big)\Big),$$

and B_i, C_i *and* D_{ij} *as*

$$B_i = \begin{bmatrix} B_i(1) & B_i(2) \end{bmatrix} \in \mathcal{B}(\mathcal{M}_i \oplus \mathcal{N}_i, \mathbb{C}), \quad C_i = \begin{bmatrix} C_i(1) \\ C_i(2) \end{bmatrix} \in \mathcal{B}(\mathbb{C}, \mathcal{M}_i \oplus \mathcal{N}_i),$$

and

$$D_{ij} = \begin{bmatrix} D_{ij}(1) & D_{ij}(12) \\ D_{ij}(21) & D_{ij}(2) \end{bmatrix} \in \mathcal{B}(\mathcal{M}_j \oplus \mathcal{N}_j, \mathcal{M}_i \oplus \mathcal{N}_i),$$

one has

$$D_{ij}(21) = 0, \quad \textit{and} \quad aD_{ij}(12) = C_i(1)B_j(2),$$

for all $i, j = 1, \ldots, n$.

The second factorization result of this chapter states:

Theorem 0.0.5. *Suppose* $\theta \in \mathcal{SA}(\mathbb{D}^n)$, *and suppose that* $\theta(0) \neq 0$. *Then* $\theta = \phi\psi$ *for some* $\phi, \psi \in \mathcal{SA}(\mathbb{D}^n)$ *if and only if* $\theta = \tau_V$ *for some isometric colligation* V *satisfying property* $\mathcal{F}(n)$.

Now we consider factorizations of contractive multipliers (denoted by $\mathcal{M}_1(H_n^2)$) on the Drury-Arveson space H_n^2. Recall that H_n^2 is the reproducing kernel Hilbert space corresponding to the kernel

$$\mathbb{B}^n \times \mathbb{B}^n \ni (\boldsymbol{z}, \boldsymbol{w}) \mapsto \frac{1}{1 - \langle \boldsymbol{z}, \boldsymbol{w} \rangle}.$$

Theorem 0.0.6. *Suppose $\theta \in \mathcal{M}_1(H^2_n)$ and $\theta(0) \neq 0$. Then the following are equivalent:*

(1) There exist multipliers $\phi \in \mathcal{M}_1(H^2_m)$ and $\psi \in \mathcal{M}_1(H^2_{n-m})$ such that

$$\theta(\boldsymbol{z}) = \phi(z_1, \ldots, z_m)\psi(z_{m+1}, \ldots, z_n) \qquad (\boldsymbol{z} \in \mathbb{D}^n).$$

(2) There exist Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and isometric colligation

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} : \mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2) \to \mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2)^n,$$

such that writing $B = \begin{bmatrix} B(1) & B(2) \end{bmatrix}$, $C = \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}$ and $D = \begin{bmatrix} D_1 \\ \vdots \\ D_n \end{bmatrix}$, one has

$$C_j = \begin{cases} \begin{bmatrix} C_j(1) \\ 0 \end{bmatrix} & \text{if } 1 \leq j \leq m \\ \\ \begin{bmatrix} 0 \\ C_j(2) \end{bmatrix} & \text{if } m+1 \leq j \leq n, \end{cases}$$

and

$$D_j = \begin{cases} \begin{bmatrix} D_j(1) & D_j(2) \\ 0 & 0 \end{bmatrix} & \text{if } 1 \leq j \leq m \\ \\ \begin{bmatrix} 0 & 0 \\ 0 & D_j(3) \end{bmatrix} & \text{if } m+1 \leq j \leq n, \end{cases}$$

and

$$aD_i(2) = C_i(1)B(2),$$

for all $i = 1, \ldots, m$, and

$$\theta(\boldsymbol{z}) = a + B(I_{\mathcal{H}_1 \oplus \mathcal{H}_2} - E_{(\mathcal{H}_1 \oplus \mathcal{H}_2)^n}(\boldsymbol{z})D)^{-1}E_{(\mathcal{H}_1 \oplus \mathcal{H}_2)^n}(\boldsymbol{z})C \qquad (\boldsymbol{z} \in \mathbb{D}^n).$$

In the setting of the Drury-Arveson space, we prove the following analog of Theorem 2.3.4.

Theorem 0.0.7. *Suppose $\theta \in \mathcal{M}_1(H^2_n)$ and $\theta(0) \neq 0$. Then the following are equivalent:*

(1) There exist ϕ and ψ in $\mathcal{M}_1(H^2_n)$ such that $\theta = \phi\psi$.

(2) There exist Hilbert spaces $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *and isometric colligation*

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} : \mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2) \to \mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2)^n,$$

such that

$$\theta(\boldsymbol{z}) = a + B(I_{\mathcal{H}_1 \oplus \mathcal{H}_2} - E_{(\mathcal{H}_1 \oplus \mathcal{H}_2)^n}(\boldsymbol{z})D)^{-1} E_{(\mathcal{H}_1 \oplus \mathcal{H}_2)^n}(\boldsymbol{z})C \qquad (\boldsymbol{z} \in \mathbb{B}^n),$$

and writing $B = \begin{bmatrix} B(1) & B(2) \end{bmatrix}$, $C = \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}$, $D = \begin{bmatrix} D_1 \\ \vdots \\ D_n \end{bmatrix}$ *and*

$$C_i = \begin{bmatrix} C_i(1) \\ C_i(2) \end{bmatrix} \quad \text{and} \quad D_i = \begin{bmatrix} D_i(1) & D_i(12) \\ D_i(21) & D_i(2) \end{bmatrix},$$

one has

$$D_i(21) = 0 \quad \text{and} \quad aD_i(12) = C_i(1)B(2),$$

for all $i = 1, \ldots, n$.

Finally, we present a complete description of Schur-Agler class factors of Schur-Agler class functions on $\mathbb{D}^n$ vanishing at the origin.

Theorem 0.0.8. *Suppose* $\theta \in \mathcal{AS}(\mathbb{D}^n)$ *and* $\theta(0) = 0$. *Then*

(1) $\theta = \phi\psi$ *for some* $\phi, \psi \in \mathcal{SA}(\mathbb{D}^n)$ *and* $\psi(0) \neq 0$ *if and only if there exist Hilbert spaces* $\{\mathcal{H}_i\}_{i=1}^n$, $\{\mathcal{M}_i\}_{i=1}^n$ *and* $\{\mathcal{N}_i\}_{i=1}^n$ *and an isometric colligation*

$$V = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} = \left[\begin{array}{c|ccc} 0 & B_1 & \cdots & B_n \\ \hline C_1 & D_{11} & \cdots & D_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ C_n & D_{n1} & \cdots & D_{nn} \end{array}\right] \in \mathcal{B}(\mathbb{C} \oplus (\bigoplus_{i=1}^n \mathcal{H}_i)),$$

such that $\theta = \tau_V$ *and* $\mathcal{H}_k = \mathcal{M}_k \oplus \mathcal{N}_k$, $k = 1, \ldots, n$, *and representing* B_i, C_i *an* D_{ij} *as*

$$B_i = [B_i(1), B_i(2)] \in \mathcal{B}(\mathcal{M}_i \oplus \mathcal{N}_i, \mathbb{C}), \quad C_i = \begin{bmatrix} C_i(1) \\ C_i(2) \end{bmatrix} \in \mathcal{B}(\mathbb{C}, \mathcal{M}_i \oplus \mathcal{N}_i),$$

and $D_{ij} = \begin{bmatrix} D_{ij}(1) & D_{ij}(12) \\ D_{ij}(21) & D_{ij}(2) \end{bmatrix} \in \mathcal{B}(\mathcal{M}_j \oplus \mathcal{N}_j, \mathcal{M}_i \oplus \mathcal{N}_i)$, *one has* $B_i(2) = 0$, $D_{ij}(21) = 0$, *and*

$$C(1)C(1)^* D(12) = C(1)^* C(1) D(12) \quad \text{and} \quad C(1)^* C(1) > 0,$$

where $i, j = 1, \ldots, n$, *and* $C(1) = \begin{bmatrix} C_1(1) \\ \vdots \\ C_n(1) \end{bmatrix}$ *and* $D(12) = \begin{bmatrix} D_{ij}(12) \end{bmatrix}_{i,j=1}^n$.

(2) $\theta = \phi\psi$ for some $\phi, \psi \in \mathcal{SA}(\mathbb{D}^n)$ and $\phi(0) = 0 = \psi(0)$ if and only if there exist Hilbert spaces $\{\mathcal{H}_i\}_{i=1}^n$, $\{\mathcal{M}_i\}_{i=1}^n$ and $\{\mathcal{N}_i\}_{i=1}^n$, an isometry $X \in \mathcal{B}(\mathbb{C}, \bigoplus\limits_{i=1}^n \mathcal{M}_i)$, a bounded linear operator $Y \in \mathcal{B}(\bigoplus\limits_{i=1}^n \mathcal{N}_i, \mathbb{C})$ and an isometric colligation

$$V = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} = \left[\begin{array}{c|ccc} 0 & B_1 & \cdots & B_n \\ \hline C_1 & D_{11} & \cdots & D_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ C_n & D_{n1} & \cdots & D_{nn} \end{array}\right] \in \mathcal{B}\Big(\mathbb{C} \oplus (\bigoplus_{i=1}^n \mathcal{H}_i)\Big),$$

such that $\theta = \tau_V$ and $\mathcal{H}_k = \mathcal{M}_k \oplus \mathcal{N}_k$, $k = 1, \ldots, n$, and representing B_i, C_i an D_{ij} as

$$B_i = [B_i(1), B_i(2)] \in \mathcal{B}\Big(\mathcal{M}_i \oplus \mathcal{N}_i, \mathbb{C}\Big), \quad C_i = \begin{bmatrix} C_i(1) \\ C_i(2) \end{bmatrix} \in \mathcal{B}\Big(\mathbb{C}, \mathcal{M}_i \oplus \mathcal{N}_i\Big),$$

and $D_{ij} = \begin{bmatrix} D_{ij}(1) & D_{ij}(12) \\ D_{ij}(21) & D_{ij}(2) \end{bmatrix} \in \mathcal{B}(\mathcal{M}_j \oplus \mathcal{N}_j, \mathcal{M}_i \oplus \mathcal{N}_i)$, one has $B_i(2) = 0$, $C_i(1) = 0$, and

$$D_{ij}(21) = 0, \quad D(12) = XY \quad \textit{and} \quad X^* D(1) = 0,$$

where

$$D(1) = [D_{ij}(1)]_{i,j=1}^n \in \mathcal{B}\Big(\bigoplus_{p=1}^n \mathcal{M}_p\Big),$$

and

$$D(12) = \Big[D_{ij}(12)\Big]_{i,j=1}^n \in \mathcal{B}\Big(\bigoplus_{p=1}^n \mathcal{N}_p, \bigoplus_{p=1}^n \mathcal{M}_p\Big).$$

Similar results also hold for contractive multipliers on the Drury-Arveson space.

Chapter 3: Schur functions and inner functions on the bidisc

In this chapter, we study representations of inner functions on the bidisc from a fractional linear transformation point of view. We provide sufficient conditions, in terms of colligation matrices, for the existence of two-variable inner functions. Our sufficient conditions are not necessary in general, and we prove a weak converse for rational inner functions that admit a one variable factorization. We also present a classification of de Branges-Rovnyak kernels on the bidisc (which equally works in the setting of $\mathbb{D}^n$ and $\mathbb{B}^n$, $n \geq 1$), and classify, in terms of Agler kernels, two-variable Schur functions that admit a one variable factor.

Recall that $C_{0\cdot}$ denotes the set of all contractions T on Hilbert spaces such that $T^n \to 0$ in the strong operator topology (that is, $\|T^m h\| \to 0$ as $m \to \infty$ for all $h \in \mathcal{H}$). The first main theorem of this chapter states:

Theorem 0.0.9. *Let $\varphi \in \mathcal{S}(\mathbb{D}^2)$. If $\varphi = \tau_V$ for some isometric colligation*

$$V = \begin{bmatrix} a & B_1 & B_2 \\ C_1 & D_1 & D_2 \\ C_2 & 0 & D_3 \end{bmatrix} : \mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2) \to \mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2),$$

with $D_1, D_3 \in C_{0\cdot}$, then φ is an inner function.

Now we turn to de Branges-Rovnyak kernels on $\mathbb{D}^n$. Let $\Theta \in \mathcal{SA}(\mathbb{D}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$. Then M_Θ is a contraction from $H^2_{\mathcal{E}}(\mathbb{D}^n)$ into $H^2_{\mathcal{E}_*}(\mathbb{D}^n)$. It is now easy to check that $K_\Theta \geq 0$, where

$$K_\Theta(\boldsymbol{z}, \boldsymbol{w}) = \mathbb{S}_n(\boldsymbol{z}, \boldsymbol{w})^{-1}(I - \Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^*) \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n).$$

We say that K_Θ is a ($\mathcal{B}(\mathcal{E}_*)$-valued) de Branges-Rovnyak kernel on $\mathbb{D}^n$. In the following, we do not assume a priori that K is analytic in $\{z_1, \ldots, z_n\}$.

Theorem 0.0.10. *Let $K : \mathbb{D}^n \times \mathbb{D}^n \to \mathcal{B}(\mathcal{E}_*)$ be a kernel on $\mathbb{D}^n$. Then $K = K_\Theta$ for some Schur-Agler function $\Theta \in \mathcal{SA}(\mathbb{D}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ and a Hilbert space $\mathcal{E}$ if and only if there exist $\mathcal{B}(\mathcal{E}_*)$-valued kernels $K_1, \ldots, K_n$ on $\mathbb{D}^n$ such that*

$$K(\boldsymbol{z}, \boldsymbol{w}) = \sum_{i=1}^{n} \frac{1}{\prod\limits_{j \neq i} (1 - z_j \bar{w}_j)} K_i(\boldsymbol{z}, \boldsymbol{w}),$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n$, and $I_{\mathcal{E}_*} - \mathbb{S}_n^{-1} \cdot K \geq 0$.

In the last part of this chapter we study factorizations of two-variable Schur functions in terms of Agler kernels. We prove:

Theorem 0.0.11. *Let $\varphi \in \mathcal{S}(\mathbb{D}^2)$ and suppose $\varphi(\boldsymbol{0}) \neq 0$. The following assertions are equivalent:*

(1) There exist φ_1 and φ_2 in $\mathcal{S}(\mathbb{D})$ such that

$$\varphi(\boldsymbol{z}) = \varphi_1(z_1)\varphi_2(z_2) \qquad (\boldsymbol{z} \in \mathbb{D}^2).$$

(2) There exist Agler kernels $\{K_1, K_2\}$ of φ such that K_1 depends only on z_1 and $\bar{w}_1$, and

$$\overline{\varphi(\boldsymbol{0})}\, K_2(\cdot, (w_1, 0)) = \overline{\varphi(w_1, 0)}\, K_2(\cdot, \boldsymbol{0}) \qquad (w_1 \in \mathbb{D}).$$

(3) There exist Agler kernels $\{L_1, L_2\}$ of φ such that all the functions in $\mathcal{H}_{L_1}$ depends only on z_1, and

$$\varphi(\boldsymbol{0}) f(\cdot, 0) = \varphi(\cdot, 0)\, f(\boldsymbol{0}) \qquad (f \in \mathcal{H}_{L_2}).$$

(4) $\varphi = \tau_V$ for some co-isometric colligation

$$V = \begin{bmatrix} \varphi(\boldsymbol{0}) & B_1 & B_2 \\ C_1 & D_1 & D_2 \\ C_2 & 0 & D_4 \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2)),$$

with $\varphi(\mathbf{0})D_2 = C_1 B_2$.

Chapter 4: Beurling quotient modules on the polydisc

In this chapter, we present a complete characterization of Beurling quotient modules of vector-valued Hardy space over $\mathbb{D}^n$. We present two applications: first, we obtain a dilation theorem for Brehmer n-tuples of commuting contractions, and, second, we relate joint invariant subspaces with factorizations of inner functions.

Let $n \geq 1$ and let $\mathcal{E}$ be a Hilbert space. The $\mathcal{E}$-valued Hardy space over the polydisc $\mathbb{D}^n$, denoted by $H^2_{\mathcal{E}}(\mathbb{D}^n)$, is the Hilbert space of all $\mathcal{E}$-valued analytic functions f on $\mathbb{D}^n$ such that

$$\|f\| := \Big(\sup_{0\leq r<1} \int_{\mathbb{T}^n} \|f(rz_1,\ldots,rz_n)\|^2_{\mathcal{E}} dm(\boldsymbol{z})\Big)^{\frac{1}{2}} < \infty,$$

where $dm(\boldsymbol{z})$ is the normalized Lebesgue measure on the n-torus $\mathbb{T}^n$. Given another Hilbert space $\mathcal{E}_*$, we denote by $H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D}^n)$ the Banach space of all $\mathcal{B}(\mathcal{E}_*,\mathcal{E})$-valued bounded analytic functions on $\mathbb{D}^n$. A function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D}^n)$ is called *inner* if $f \mapsto \Theta f$ defines an isometry $M_\Theta : H^2_{\mathcal{E}_*}(\mathbb{D}^n) \to H^2_{\mathcal{E}}(\mathbb{D}^n)$. Finally, recall that a closed subspace $\mathcal{Q} \subseteq H^2_{\mathcal{E}}(\mathbb{D}^n)$ is said to be a *Beurling quotient module* (and $\mathcal{Q}^\perp$ is a *Beurling submodule*) if

$$\mathcal{Q} = H^2_{\mathcal{E}}(\mathbb{D}^n) \ominus \Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n) \cong H^2_{\mathcal{E}}(\mathbb{D}^n)/\Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n),$$

for some Hilbert space $\mathcal{E}_*$ and inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D}^n)$. In the context of Beurling theorem and the classical Sz.-Nagy and Foias dilation theory, it appears natural to raise the following question:

Question 1. *Which quotient modules of $H^2_{\mathcal{E}}(\mathbb{D}^n)$ admit Beurling representations?*

Given a quotient module $\mathcal{Q} \subseteq H^2_{\mathcal{E}}(\mathbb{D}^n)$, define the n-tuple of compression operators $C_z = (C_{z_1},\ldots,C_{z_n})$ on $\mathcal{Q}$ by

$$C_{z_i} = P_{\mathcal{Q}} M_{z_i}|_{\mathcal{Q}} \qquad (i = 1,\ldots,n),$$

where $P_{\mathcal{Q}} \in \mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}^n))$ is the orthogonal projection onto $\mathcal{Q}$. In the following theorem, we give the answer to Question 1:

Theorem 0.0.12. *Let $\mathcal{E}$ be a Hilbert space and let $\mathcal{Q}$ be a quotient module of $H^2_{\mathcal{E}}(\mathbb{D}^n)$. Then $\mathcal{Q}$ is a Beurling quotient module if and only if*

$$(I_{\mathcal{Q}} - C^*_{z_i}C_{z_i})(I_{\mathcal{Q}} - C^*_{z_j}C_{z_j}) = 0 \qquad (i \neq j).$$

Now we turn to the analytic model theory in several variables. Let $T = (T_1,\ldots,T_n)$ be a commuting tuple of contractions on a Hilbert space $\mathcal{H}$. We say that T is a *Brehmer tuple* (cf. [9], [39], [79]) if

$$\sum_{F\subseteq G} (-1)^{|F|} T_F T_F^* \geq 0,$$

for every $G \subseteq \{1, \ldots, n\}$ where $|F|$ denotes the cardinality of F and $T_F = \prod_{j \in F} T_j$ for all $F \subseteq \{1, \ldots, n\}$. We set, by convention, that $T_\emptyset = I_{\mathcal{H}}$ and $|\emptyset| = 0$. A contraction X on $\mathcal{H}$ is called pure if $X^{*n} \to 0$ as $n \to \infty$ in the strong operator topology. A Brehmer tuple T is called a pure Brehmer tuple if T_i is pure for all $i = 1, \ldots, n$.

We define model operators $T_{z_i,\Theta} = P_{\mathcal{Q}_\Theta} M_{z_i}|_{\mathcal{Q}_\Theta}$ for all $i = 1, \ldots, n$, and set

$$T_\Theta = (T_{z_1,\Theta}, \ldots, T_{z_n,\Theta}).$$

It is a natural question to ask which n-tuples of commuting contractions are unitarily equivalent to T_Θ on Beurling quotient modules $\mathcal{Q}_\Theta$. The following result (a refinement of Theorem 4.1.1) gives a complete answer to this question.

Theorem 0.0.13. *Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting contractions on $\mathcal{H}$. The following are equivalent.*

(a) *$T \cong T_\Theta$ for some Beurling quotient module $\mathcal{Q}_\Theta$.*

(b) *T is a pure Brehmer tuple and $(I_{\mathcal{H}} - T_i^* T_i)(I_{\mathcal{H}} - T_j^* T_j) = 0$ for all $i \neq j$.*

In the final section of this chapter we classify factorizations of inner functions in terms of existence of certain invariant subspaces.

Theorem 0.0.14. *Let $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D}^n)$ be an inner function. The following are equivalent.*

1. *There exist a Hilbert space $\mathcal{F}$ and inner functions Ψ and Φ in $H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{F})}(\mathbb{D}^n)$ and $H^\infty_{\mathcal{B}(\mathcal{F},\mathcal{E})}(\mathbb{D}^n)$, respectively, such that $\Theta = \Phi\Psi$.*

2. *There exists a T_Θ-invariant subspace $\mathcal{M} \subseteq \mathcal{Q}_\Theta$ such that $\mathcal{M} \oplus \mathcal{S}_\Theta$ is a Beurling submodule of $H^2_{\mathcal{E}}(\mathbb{D}^n)$.*

3. *There exists a T_Θ-invariant subspace $\mathcal{M} \subseteq \mathcal{Q}_\Theta$ such that*

$$(I - C_i^* C_i)(I - C_j^* C_j) = 0 \qquad (i \neq j),$$

where $C_s = P_{\mathcal{Q}_\Theta \ominus \mathcal{M}} T_{z_s,\Theta}|_{\mathcal{Q}_\Theta \ominus \mathcal{M}}$ for all $s = 1, \ldots, n$.

Chapter 1

Preliminaries

In this chapter, we recall the necessary definitions and basic facts from classical operator theory and function theory. We also present the background ideas of Schur functions, Schur-Agler functions, and reproducing kernel Hilbert spaces. For more about Schur functions and Schur-Agler functions we refer the reader to the book by Agler and McCarthy (see [6]) and the lecture notes by W.J. Helton (see [65]). We refer the reader to the book by Paulsen and Raghupathi [92] and Aronszajn [14] on reproducing kernel Hilbert spaces, and the classic by Sz.-Nagy and Foias (see [90]) on the theory of dilations and structure of contractions.

1.1 Reproducing Kernel Hilbert spaces

We begin by recalling the definition of kernel functions.

Definition 1.1.1. *Let $\mathcal{E}$ be a Hilbert space, and let Ω be a non-empty set. A function $K : \Omega \times \Omega \to \mathcal{B}(\mathcal{E})$ is called a positive kernel if*

$$\sum_{i,j=1}^{m} \langle K(\boldsymbol{z}_i, \boldsymbol{z}_j)\eta_j, \eta_i\rangle_{\mathcal{E}} \geq 0,$$

for all $\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_m\} \subseteq \Omega$, $\{\eta_1, \ldots, \eta_m\} \subseteq \mathcal{E}$ and $m \geq 1$.

We often say that K is a *kernel* and denote it by $K \geq 0$.

Let K be a $\mathcal{B}(\mathcal{E})$-valued kernel function, and let $\mathcal{H}_K$ be the closure of the linear space $\left\{\sum\limits_{i=1}^{m} K(\cdot, \boldsymbol{z}_i)\eta_i : \boldsymbol{z} \in \Omega, \;\; \eta \in \mathcal{E} \;\text{ and }\; m \in \mathbb{N}\right\}$ with respect to the inner product

$$\langle K(\cdot, \boldsymbol{w})\eta, K(\cdot, \boldsymbol{z})\zeta\rangle := \langle K(\boldsymbol{z}, \boldsymbol{w})\eta, \zeta\rangle_{\mathcal{E}},$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \Omega$ and $\eta, \zeta \in \mathcal{E}$. Then $\mathcal{H}_K$ forms a Hilbert space of $\mathcal{E}$-valued functions on Ω and

$$\mathcal{H}_K = \overline{\text{span}}\{K(\cdot, \boldsymbol{w})\eta : \eta \in \mathcal{E} \;\text{ and }\; \boldsymbol{w} \in \Omega\}.$$

The following equality, follows from the above inner product, is known as the reproducing property:

$$\langle f, K(\cdot, \boldsymbol{z})\eta\rangle = \langle f(z), \eta\rangle_{\mathcal{E}},$$

for all $\boldsymbol{z} \in \Omega$, $f \in \mathcal{H}_K$ and $\eta \in \mathcal{E}$. Let $\mathcal{H}_K$ be $\mathcal{E}$-valued reproducing kernel Hilbert space corresponding to a $\mathcal{B}(\mathcal{E})$-valued kernel function K. Suppose $\boldsymbol{w} \in \Omega$ and $ev(\boldsymbol{w}) : \mathcal{H}_K \to \mathcal{E}$ is the evaluation, that is

$$ev(\boldsymbol{w})(f) = f(\boldsymbol{w}) \qquad (f \in \mathcal{H}_K).$$

Then

$$K(\boldsymbol{z}, \boldsymbol{w}) = ev(\boldsymbol{z})ev(\boldsymbol{w})^* \qquad (\boldsymbol{z}, \boldsymbol{w} \in \Omega).$$

Now let $\Omega \subseteq \mathbb{C}^n$ be a domain and let $\mathcal{H}_K$ be a reproducing kernel Hilbert space of analytic functions on Ω. In this case, K is analytic in the first variables and we call it as an *analytic kernel*. We also call $\mathcal{H}_K$ as an *analytic reproducing kernel Hilbert space*. Suppose $\mathcal{H}_K$ is an analytic reproducing kernel Hilbert space, and let $z_i\mathcal{H}_K \subseteq \mathcal{H}_K$ for all $i = 1, \ldots, n$. Then

$$(M_{z_i} f)(\boldsymbol{w}) = w_i f(\boldsymbol{w}) \qquad (\boldsymbol{w} \in \Omega, f \in \mathcal{H}_K),$$

for all $i = 1, \ldots, n$, defines a commuting tuple of bounded linear operators $(M_{z_1}, \cdots, M_{z_n})$ on $\mathcal{H}_K$. It is easy to verify that

$$M_{z_i}^*(K(\cdot, \boldsymbol{w})\eta) = \overline{w_i} K(\cdot, \boldsymbol{w})\eta,$$

for all $\boldsymbol{w} \in \Omega$, $\eta \in \mathcal{E}$ and $i = 1, \ldots, n$. The following is a list of examples for analytic reproducing kernel Hilbert spaces:

Example 1.1.2. 1. *Let $n \in \mathbb{N}$. The Hardy space $H^2(\mathbb{D}^n)$ on the unit polydisc $\mathbb{D}^n$ is a reproducing kernel Hilbert space with the Szegö kernel*

$$\mathbb{S}_n(\boldsymbol{z}, \boldsymbol{w}) = \prod_{i=1}^{n} (1 - z_i \overline{w_i})^{-1} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n).$$

2. *Let $n \in \mathbb{N}$. The Drury-Arveson space H_n^2 on the unit ball $\mathbb{B}^n$ is a reproducing kernel Hilbert space with the Drury-Arveson kernel*

$$k_n(\boldsymbol{z}, \boldsymbol{w}) = (1 - \langle \boldsymbol{z}, \boldsymbol{w}\rangle_{\mathbb{C}^n})^{-1} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

3. *Let $n \in \mathbb{N}$ and let $\alpha > n$. The weighted Bergman space $L^2_{a,\alpha}(\mathbb{B}^n)$ on the unit ball $\mathbb{B}^n$ is a reproducing kernel Hilbert space with the weighted Bergman kernel*

$$K_{L^2_{a,\alpha}(\mathbb{B}^n)}(\boldsymbol{z}, \boldsymbol{w}) = (1 - \langle \boldsymbol{z}, \boldsymbol{w}\rangle_{\mathbb{C}^n})^{-\alpha} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

when $\alpha = n$, $L^2_{a,\alpha}(\mathbb{B}^n)$ is the usual Hardy space $H^2(\mathbb{B}^n)$.

4. *Let $n \in \mathbb{N}$. The Dirichlet space $\mathcal{D}(\mathbb{B}^n)$ on the unit ball $\mathbb{B}^n$ is a reproducing kernel Hilbert space with the Dirichlet kernel*

$$K_{\mathcal{D}(\mathbb{B}^n)}(\boldsymbol{z}, \boldsymbol{w}) = 1 + log\frac{1}{1 - \langle \boldsymbol{z}, \boldsymbol{w} \rangle} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

5. *Let $\lambda = (\lambda_1, \dots, \lambda_n)$, where $\lambda_i > -1$ and $i = 1, \dots, n$. The weighted Bergman space $L^2_{a,\lambda}(\mathbb{D}^n)$ on the unit polydisc $\mathbb{D}^n$ is a reproducing kernel Hilbert space with kernel*

$$K_{L^2_{a,\lambda}(\mathbb{D}^n)}(\boldsymbol{z}, \boldsymbol{w}) = \prod_{i=1}^{n} \frac{1}{(1 - z_i \overline{w_i})^{2+\lambda_i}} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n).$$

Now we turn to multipliers on reproducing kernel Hilbert spaces. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two Hilbert spaces. Let $\mathcal{H}_{K_1}$ and H_{K_2} be two reproducing kernel Hilbert spaces where $K_j : \Omega \times \Omega \to \mathcal{B}(\mathcal{E}_j)$. A function $\Theta : \Omega \to \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ is said to be a multiplier if $\Theta f \in H_{K_2}$ for all $f \in H_{K_1}$. The set of all multipliers is denoted by $\mathcal{M}(H_{K_1}, H_{K_2})$, that is

$$\mathcal{M}(H_{K_1}, H_{K_2}) = \{\Theta : \Omega \to \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2) : \Theta H_{K_1} \subseteq H_{K_2}\}.$$

By an application of the closed graph theorem, a multiplier Θ in $\mathcal{M}(H_{K_1}, H_{K_2})$ defines a bounded multiplication operator $M_\Theta : H_{K_1} \to H_{K_2}$ as

$$(M_\Theta f)(\omega) = (\Theta f)(\omega) = f(\omega)\Theta(\omega),$$

for all $f \in H_{K_1}$ and $\omega \in \Omega$. The multiplication operator M_Θ has the following property

$$M_\Theta^*(K_2(\cdot, \omega)\eta) = K_2(\cdot, \omega)\Theta(\omega)^*\eta,$$

for all $w \in \Omega$ and $\eta \in \mathcal{E}$. Indeed, for $f \in H_{K_1}$, we have

$$\begin{aligned}
\langle f, M_\Theta^*(K_2(\cdot, \omega)\eta)\rangle &= \langle \Theta f, K_2(\cdot, \omega)\eta\rangle \\
&= \langle \Theta(\omega) f(\omega), \ \ \eta\rangle \\
&= \langle f(\omega), \ \ \Theta(\omega)^*\eta\rangle \\
&= \langle f, K_2(\cdot, \omega)\Theta(\omega)^*\eta\rangle.
\end{aligned}$$

Moreover, $\mathcal{M}(H_{K_1}, H_{K_2})$ is a Banach space with respect to the norm

$$\|\Theta\|_{\mathcal{M}(H_{K_1}, H_{K_2})} := \|M_\Theta\|_{\mathcal{B}(H_{K_1}, H_{K_2})},$$

for all $\Theta \in \mathcal{M}(H_{K_1}, H_{K_2})$. If $K_1 = K_2$, we denote $\mathcal{M}(H_{K_1}, H_{K_1})$ simply by $\mathcal{M}(H_{K_1})$ is a Banach algebra with the norm $\|\cdot\|_{\mathcal{M}(H_{K_1})}$. The following result characterizes multipliers in terms of reproducing kernels.

Theorem 1.1.3. *Let $K_j : \Omega \times \Omega \to \mathcal{B}(\mathcal{E}_j)$, $j = 1, 2$, be analytic kernels. Suppose H_{K_1} and H_{K_2} are the corresponding analytic reproducing kernel Hilbert spaces, and suppose*

$\Theta : \Omega \to \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ *is a function. Then the following are equivalent:*

(i) $\Theta \in \mathcal{M}(H_{K_1}, H_{K_2})$.

(ii) *There exists a constant* $c > 0$ *such that*

$$(\mu, \nu) \mapsto c^2 K_2(\mu, \nu) - \Theta(\mu) K_1(\mu, \nu) \Theta(\nu)^* \quad (\mu, \nu \in \Omega),$$

is a kernel.

An analytic kernel $K : \Omega \times \Omega \to \mathcal{B}(\mathcal{E})$ is said to be *quasi-scalar kernel* if

$$K(z, w) = k(z, w) I_{\mathcal{E}} \qquad (z, w \in \Omega),$$

for some scalar kernel k on Ω. In this case, we have $\mathcal{H}_K = H_k \otimes \mathcal{E}$.

Theorem 1.1.4 ([6])**.** *Let* H_k *be a reproducing kernel Hilbert space with quasi-scalar kernel* k *on* Ω. *Let* $\Theta : \Omega \to \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ *be a function. Then the following are equivalent*

(i) $\Theta \in \mathcal{M}(H_k \otimes \mathcal{E}_1, H_k \otimes \mathcal{E}_2)$.

(ii) The operator

$$T(k(\cdot, \omega) \otimes \eta) = k(\cdot, \omega) \otimes \Theta(\omega)^* \eta \quad (\omega \in \Omega,\ \eta \in \mathcal{E}_2),$$

defines a bounded linear operator $T : H_k \otimes \mathcal{E}_1 \to H_k \otimes \mathcal{E}_2$.

In this case, T is the adjoint of M_Θ. Below we give a list of examples of multiplier algebras:

Example 1.1.5.

1. *Let* $H^2_{\mathcal{E}}(\mathbb{D}^n)$ *be the* $\mathcal{E}$*-valued Hardy space on* $\mathbb{D}^n$. *Then* $\mathcal{M}(H^2_{\mathcal{E}}(\mathbb{D}^n)) = H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D}^n)$, *the Banach algebra of* $\mathcal{B}(\mathcal{E})$*-valued bounded analytic functions on* $\mathbb{D}^n$.

2. *Let* $L^2_a(\mathbb{D}^n, \mathcal{E})$ *be the* $\mathcal{E}$*-valued Bergman space on* $\mathbb{D}^n$. *Then* $\mathcal{M}(L^2_a(\mathbb{D}^n, \mathcal{E})) = H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D}^n)$.

3. *Let* $H^2_{\mathcal{E}}(\mathbb{B}^n)$ *be the* $\mathcal{E}$*-valued Hardy space on* $\mathbb{B}^n$. *Then* $\mathcal{M}(H^2_{\mathcal{E}}(\mathbb{B}^n)) = H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{B}^n)$, *the Banach algebra of* $\mathcal{B}(\mathcal{E})$*-valued bounded analytic functions on* $\mathbb{B}^n$.

4. *Let* $L^2_a(\mathbb{B}^n, \mathcal{E})$ *be the* $\mathcal{E}$*-valued Bergman space on* $\mathbb{B}^n$. *Then* $\mathcal{M}(L^2_a(\mathbb{B}^n, \mathcal{E})) = H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{B}^n)$.

5. *Let* $H^2_n(\mathcal{E})$ *be the* $\mathcal{E}$*-valued Drury-Arveson space on* $\mathbb{B}^n$. *Then*

$$\mathcal{M}(H^2_n(\mathcal{E})) = \{\Phi : \mathbb{B}^n \to \mathcal{B}(\mathcal{E}) \text{ analytic and } \sup \|\Phi(rT)\| < \infty\},$$

where the supremum is taken over $r \in (0,1)$ *and commuting* n*-tuples* $T = (T_1, \dots, T_n)$ *on some Hilbert space* $\mathcal{H}$ *such that* $\sum\limits_{j=1}^{n} T_j T_j^* \leq I_{\mathcal{H}}$ *(see [16]).*

It is worthwhile to note that the multiplier algebra $\mathcal{M}(\mathcal{D}(\mathbb{D}))$ of the Dirichlet space $\mathcal{D}(\mathbb{D})$ is a proper subalgebra of $H^\infty(\mathbb{D})$. Also, note that if $n \geq 2$, than $\mathcal{M}(H^2_n) \subsetneq H^\infty(\mathbb{B}^n)$ the inclusion is contractive, and the multiplier norm and the supremum norm on the ball are not equivalent (see [64]).

1.2 Schur functions

We begin with the definition of Schur functions. Let $n \in \mathbb{N}$, $\Omega = \mathbb{D}^n$ or $\mathbb{B}^n$, and let $\mathcal{E}$ and $\mathcal{E}_*$ be Hilbert spaces. Let $H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D}^n)$ denote the Banach space of all $\mathcal{B}(\mathcal{E}_*,\mathcal{E})$-valued bounded analytic functions on $\mathbb{D}^n$ with the supremum norm

$$\|\Phi\|_\infty := \sup\{\|\Phi(\boldsymbol{z})\| : \boldsymbol{z} \in \mathbb{D}^n\} \qquad (\Phi \in H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D}^n)).$$

A function $\Phi \in H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D}^n)$ is said to be *Schur function* if $\|\Phi\|_\infty \leq 1$. We denote by $\mathcal{S}(\mathbb{D}^n, \mathcal{B}(\mathcal{E}_*,\mathcal{E}))$ the set of all *Schur functions* defined on $\mathbb{D}^n$, that is

$$\mathcal{S}(\mathbb{D}^n, \mathcal{B}(\mathcal{E}_*,\mathcal{E})) = \{\Phi : \mathbb{D}^n \to \mathbb{C} : \Phi \text{ is analytic and } \|\Phi\|_\infty \leq 1\}.$$

When $\mathcal{E}_* = \mathcal{E} = \mathbb{C}$, we will simply denote the set of all scalar valued *Schur functions* on $\mathbb{D}$ by $\mathcal{S}(\mathbb{D})$, that is,

$$\mathcal{S}(\mathbb{D}) = \{\varphi : \mathbb{D} \to \mathbb{C} : \varphi \text{ is analytic and } \|\varphi\|_\infty := \sup\{|\varphi(\boldsymbol{z})| : \boldsymbol{z} \in \mathbb{D}\} \leq 1\}.$$

We introduce the Schur functions on $\Omega = \mathbb{B}^n$ in subsection 1.2.6. The structure of Schur functions is closely related to operators, namely, colligation operators, on Hilbert spaces.

1.2.1 Colligation operators and transfer functions

A *colligation* (or *scattering operator matrix*) [40] is any bounded linear operator V of the form

$$V = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{E} \oplus \mathcal{H} \to \mathcal{E}_* \oplus \mathcal{H},$$

where $\mathcal{H}$, $\mathcal{E}$ and $\mathcal{E}_*$ are Hilbert spaces. The colligation is said to be isometric colligation (unitary colligation) if V is isometry (unitary). Now, let $\mathcal{H}$ be a Hilbert space and let

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} : \mathbb{C} \oplus \mathcal{H} \to \mathbb{C} \oplus \mathcal{H}, \tag{1.2.1}$$

be an isometry. The *transfer function realization* of the isometric colligation V is defined by

$$\tau_V(z) = a + zB(I_{\mathcal{H}} - zD)^{-1}C \qquad (z \in \mathbb{D}).$$

Since V is an isometry, it is easy to see that D is a contraction, and hence $\|zD\| = |z|\|D\| < 1$ for all $z \in \mathbb{D}$. It follows that τ_V defined above is analytic on $\mathbb{D}$. Since

$V^*V = I$, we have

$$\begin{bmatrix} |a|^2 + C^*C & \overline{a}B + C^*D \\ aB^* + D^*C & B^*B + D^*D \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & I_{\mathcal{H}} \end{bmatrix}.$$

Using this, we calculate

$$\begin{aligned} 1 - \overline{\tau_V(z)}\tau_V(z) &= 1 - |a|^2 - z\overline{a}B(I - zD)^{-1}C - a\overline{z}C^*(I - \overline{z}D^*)^{-1}B^* \\ &\quad - |z|^2 C^*(I - \overline{z}D^*)^{-1}B^*B(I - zD)^{-1}C \\ &= C^*C + zC^*D(I - zD)^{-1}C + \overline{z}C^*(I - \overline{z}D^*)^{-1}D^*C \\ &\quad - |z|^2(I - \overline{z}D^*)^{-1}(I - D^*D)(I - zD)^{-1}C \\ &= C^*(I - \overline{z}D^*)^{-1}[(I - \overline{z}D^*)(I - zD) + z(I - \overline{z}D^*)D \\ &\quad + \overline{z}D^*(I - zD) - |z|^2 I + |z|^2 D^*D](I - zD)^{-1}C \\ &= C^*(I - \overline{z}D^*)^{-1}[(1 - |z|^2)I](I - zD)^{-1}C. \end{aligned}$$

In particular, $|\tau_V(z)| \le 1$ for all $z \in \mathbb{D}$, and hence $\tau_V \in \mathcal{S}(\mathbb{D})$. Now we prove the converse:

Theorem 1.2.1 (I. Schur)**.** *Let φ be a function on $\mathbb{D}$. Then $\varphi \in \mathcal{S}(\mathbb{D})$ if and only if there exist a Hilbert space $\mathcal{H}$ and an isometric colligation*

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}),$$

such that $\varphi = \tau_V$.

Proof. $(\Rightarrow)$ This part follows from the above discussion.

$(\Leftarrow)$ Suppose $\varphi \in \mathcal{S}(\mathbb{D})$. Then M_φ defines a contraction on $H^2(\mathbb{D})$. That is, $I - M_\varphi M_\varphi^* \ge 0$, and hence, a simple calculation shows that

$$\langle (I - M_\varphi M_\varphi^*)\mathbb{S}(\cdot, w), \mathbb{S}(\cdot, z)\rangle = \frac{1 - \varphi(z)\overline{\varphi(w)}}{1 - z\overline{w}} = (1 - \varphi(z)\overline{\varphi(w)})\mathbb{S}(z, w).$$

Therefore, $(1 - \varphi(z)\overline{\varphi(w)})\mathbb{S}(z, w)$ is a kernel on $\mathbb{D}$ (see Lemma 1.2.2). Since a kernel can always be realized as a Grammian, it follows that there exists a Hilbert space $\mathcal{H}$ and a function $f : \mathbb{D} \to \mathcal{H}$ such that

$$(1 - \varphi(z)\overline{\varphi(w)})\mathbb{S}(z, w) = \langle f(z), f(w)\rangle_{\mathcal{H}}.$$

Since $\mathbb{S}(z, w) = (1 - z\bar{w})^{-1}$, it follows that

$$1 + \langle zf(z), wf(w)\rangle_{\mathcal{H}} = \varphi(z)\overline{\varphi(w)} + \langle f(z), f(w)\rangle_{\mathcal{H}}. \tag{1.2.2}$$

Therefore, the map

$$V : \begin{pmatrix} 1 \\ zf(z) \end{pmatrix} \mapsto \begin{pmatrix} \varphi(z) \\ f(z) \end{pmatrix}, \tag{1.2.3}$$

extends linearly to an isometry on

$$\overline{\text{span}}\left\{\begin{pmatrix} 1 \\ zf(z) \end{pmatrix} : z \in \mathbb{D}\right\}.$$

Adding an infinite dimensional summand to $\mathcal{H}$, if necessary, V can then be extended to an isometry from $\mathbb{C} \oplus \mathcal{H}$ to $\mathbb{C} \oplus \mathcal{H}$. We again denote the extension by V. Set

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}).$$

Note that (1.2.2) implies in particular that

$$\begin{aligned} &a + zBf(z) = \varphi(z), \quad \text{and} \\ &C + zDf(z) = f(z). \end{aligned}$$

By the second equality, we have

$$f(z) = (I - zD)^{-1}C,$$

and then, by the first equality above, we have

$$\varphi(z) = a + zB(I - zD)^{-1}C \qquad (z \in \mathbb{D}).$$

□

1.2.2 Two variables Schur functions

We continue the discussion by presenting a transfer function realization of two variables Schur functions (see [5] and also page 171, [6]). We begin with a lemma (which is also valid for Schur functions on $\mathbb{B}^n$).

Lemma 1.2.2. *If $\varphi \in \mathcal{S}(\mathbb{D}^n)$, then*

$$K_\varphi(\boldsymbol{z}, \boldsymbol{w}) = \frac{1 - \varphi(\boldsymbol{z})\overline{\varphi(\boldsymbol{w})}}{\prod_{i=1}^n (1 - z_i \overline{w_i})} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n),$$

is a kernel on $\mathbb{D}^n$.

Proof. Let $\varphi \in \mathcal{S}(\mathbb{D}^n)$. Then $M_\varphi \in \mathcal{B}(H^2(\mathbb{D}^n))$ is a contraction, or, equivalently, $I - M_\varphi M_\varphi^* \geq 0$ on $H^2(\mathbb{D}^n)$. A simple calculation shows that

$$\langle (I - M_\varphi M_\varphi^*)\mathbb{S}_n(\cdot, \boldsymbol{w}), \mathbb{S}_n(\cdot, \boldsymbol{z}) \rangle = \frac{1 - \varphi(\boldsymbol{z})\overline{\varphi(\boldsymbol{w})}}{\prod_{i=1}^n (1 - z_i \overline{w_i})},$$

where

$$\mathbb{S}_n(\boldsymbol{z}, \boldsymbol{w}) = \prod_{i=1}^{n} \frac{1}{1 - z_i \bar{w}_i} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n),$$

is the Szegö kernel on $\mathbb{D}^n$. Let $\{\boldsymbol{z}_1, \ldots, \boldsymbol{z}_m\} \subseteq \mathbb{D}^n$, $\{c_1, \ldots, c_m\} \subseteq \mathbb{C}^n$, and $m \in \mathbb{N}$. Then

$$\begin{aligned}\sum_{i=1}^{m}\sum_{i=1}^{m} c_i\overline{c_j}K_\varphi(\boldsymbol{z}_j, \boldsymbol{z}_i) &= \sum_{i=1}^{m}\sum_{i=1}^{m} c_i\overline{c_j}\langle (I - M_\varphi M_\varphi^*)\mathbb{S}_n(\cdot, \boldsymbol{z}_i), \mathbb{S}_n(\cdot, \boldsymbol{z}_j)\rangle \\ &= \langle (I - M_\varphi M_\varphi^*)\sum_{i=1}^{m} c_i\mathbb{S}_n(\cdot, \boldsymbol{z}_i), \sum_{j=1}^{m} c_j\mathbb{S}_n(\cdot, \boldsymbol{z}_j)\rangle \\ &= \|(I - M_\varphi M_\varphi^*)^{\frac{1}{2}}\sum_{i=1}^{m} c_i\mathbb{S}_n(\cdot, \boldsymbol{z}_i)\|^2 \geq 0.\end{aligned}$$

Therefore, K_φ is a kernel on $\mathbb{D}^n$. □

The following is one of the most influential and useful results in multivariable operator theory and we refer the reader to [2, 19, 20, 21, 22, 23, 31, 33, 34, 73, 80] and the references therein for a wide application and several connecting results.

Theorem 1.2.3. *Let $\varphi : \mathbb{D}^2 \to \mathbb{C}$ be a function. The following are equivalent:*
(1) $\varphi \in \mathcal{S}(\mathbb{D}^2)$.
(2) Then there exist kernels $K_1, K_2 : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{C}$ (known as Agler kernels) such that

$$1 - \varphi(\boldsymbol{z})\overline{\varphi(\boldsymbol{w})} = (1 - z_1\overline{w_1})K_1(\boldsymbol{z}, \boldsymbol{w}) + (1 - z_2\overline{w_2})K_2(\boldsymbol{z}, \boldsymbol{w}) \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^2).$$

(3) There exist Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and a unitary/isometric/co-isometric/contractive colligation operator

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2)),$$

such that $\varphi = \tau_V$, where

$$\tau_V(\boldsymbol{z}) = a + B(I_{\mathcal{H}_1 \oplus \mathcal{H}_2} - E_{\mathcal{H}_1 \oplus \mathcal{H}_2}(\boldsymbol{z})D)^{-1}E_{\mathcal{H}_1 \oplus \mathcal{H}_2}(\boldsymbol{z})C,$$

and $E_{\mathcal{H}_1 \oplus \mathcal{H}_2}(\boldsymbol{z}) = z_1 I_{\mathcal{H}_1} \oplus z_2 I_{\mathcal{H}_2}$ for all $\boldsymbol{z} \in \mathbb{D}^2$.

Proof. (1) ⇒ (2) First we assume that φ is an inner function. Let $\mathcal{S}_1$ be the maximal M_{z_1} invariant subspace of $\mathcal{Q}_\varphi$ (where $\mathcal{Q}_\varphi = H^2(\mathbb{D}) \ominus \varphi H^2(\mathbb{D})$), that is,

$$\mathcal{S}_1 = \{f \in \mathcal{Q}_\varphi : z_1^n f \in \mathcal{Q}_\varphi, \quad \forall n \in \mathbb{Z}_+\}.$$

Set $\mathcal{S}_2 = \mathcal{Q}_\varphi \ominus \mathcal{S}_1$. Since $\mathcal{S}_1$ and $\mathcal{S}_2$ are closed subspace of $\mathcal{Q}_\varphi$, then they are reproducing kernel Hilbert spaces with kernels

$$K_{\mathcal{S}_j}(\boldsymbol{z}, \boldsymbol{w}) = P_{\mathcal{S}_j}(K_{\mathcal{Q}_\varphi}(\cdot, \boldsymbol{w}))(\boldsymbol{z}),$$

where

$$K_{\mathcal{Q}_\varphi}(\boldsymbol{z},\boldsymbol{w}) = \frac{1-\varphi(\boldsymbol{z})\overline{\varphi(\boldsymbol{w})}}{(1-z_1\overline{w_1})(1-z_2\overline{w_2})},$$

for all $\boldsymbol{z},\boldsymbol{w} \in \mathbb{D}^2$. From the definition of $\mathcal{S}_1$, it is clear that $z_1\mathcal{S}_1 \subseteq \mathcal{S}_1$. Since $M_{z_2}^*\mathcal{Q}_\varphi \subseteq \mathcal{Q}_\varphi$ and $\{M_{z_1}, M_{z_2}\}$ are doubly commuting, it follows that $M_{z_2}^*\mathcal{S}_1 \subseteq \mathcal{S}_1$, and consequently, $z_2\mathcal{S}_2 \subseteq \mathcal{S}_2$. Note that $\mathcal{S}_j \subseteq \mathcal{Q}_\varphi \subseteq H^2(\mathbb{D}^2)$, hence, $\|M_{z_j}|_{\mathcal{S}_j}\|_{\mathcal{S}_j} = 1$ for $j = 1,2$. It follows that

$$K_j(\boldsymbol{z},\boldsymbol{w}) = (1-z_j\overline{w_j})K_{\mathcal{S}_j}(\boldsymbol{z},\boldsymbol{w})$$

is a kernel for $j = 1,2$. Now

$$\begin{aligned}\frac{1-\varphi(\boldsymbol{z})\overline{\varphi(\boldsymbol{w})}}{(1-z_1\overline{w_1})(1-z_2\overline{w_2})} &= K_{\mathcal{Q}_\varphi}(\boldsymbol{z},\boldsymbol{w})\\ &= K_{\mathcal{S}_1}(\boldsymbol{z},\boldsymbol{w}) + K_{\mathcal{S}_2}(\boldsymbol{z},\boldsymbol{w})\\ &= \frac{K_1(\boldsymbol{z},\boldsymbol{w})}{1-z_1\overline{w_1}} + \frac{K_2(\boldsymbol{z},\boldsymbol{w})}{1-z_2\overline{w_2}}.\end{aligned}$$

For the general case, assume that φ is a non-inner function. Then by a well known result of Rudin [97, Theorem 5.5.1], there exists a sequence of inner functions $\{\varphi_n\}_{n\geq 1}$ such that $\varphi_n \to \varphi$ uniformly on compact subsets of $\mathbb{D}^2$. Let $\{K_n^1, K_n^2\}$ be Agler kernels associated with φ_n for each $n \in \mathbb{N}$. Since $\mathcal{H}_{K_n^j}$ is contractively contained in $H^2(\mathbb{D}^2)$, we have

$$|K_n^j(\boldsymbol{z},\boldsymbol{w})| \leq \|K_n^j(\cdot,\boldsymbol{w})\|\|K_n^j(\cdot,\boldsymbol{z})\| \leq \frac{1}{\sqrt{(1-|w_1|^2)(1-|w_2|^2)}}\frac{1}{\sqrt{(1-|z_1|^2)(1-|z_2|^2)}},$$

for all $\boldsymbol{z},\boldsymbol{w} \in \mathbb{D}^2$, $n \in \mathbb{N}$ and $j = 1,2$. Since K_n^j is locally uniformly bounded, then it forms a normal family for each $j = 1,2$. Therefore by Montel's theorem there exists a subsequence $\{\varphi_{n_k}\}$ of $\{\varphi_n\}$ such that $\{K_{n_k}^j\}$ converges to kernel K^j uniformly on compact subsets of $\mathbb{D}^2$ for each $j = 1,2$. And finally, we have

$$\frac{1-\varphi(\boldsymbol{z})\overline{\varphi(\boldsymbol{w})}}{(1-z_1\overline{w_1})(1-z_2\overline{w_2})} = \frac{K^1(\boldsymbol{z},\boldsymbol{w})}{1-z_1\overline{w_1}} + \frac{K^2(\boldsymbol{z},\boldsymbol{w})}{1-z_2\overline{w_2}}.$$

(2) $\Rightarrow$ (3) Let $\{K_1, K_2\}$ be Agler kernels such that

$$1-\varphi(\boldsymbol{z})\overline{\varphi(\boldsymbol{w})} = (1-z_1\overline{w_1})K_1(\boldsymbol{z},\boldsymbol{w}) + (1-z_2\overline{w_2})K_2(\boldsymbol{z},\boldsymbol{w}) \qquad (\boldsymbol{z},\boldsymbol{w} \in \mathbb{D}^2).$$

By the reproducing property, we have

$$1-\varphi(\boldsymbol{z})\overline{\varphi(\boldsymbol{w})} = (1-z_1\bar{w}_1)\langle K_1(\cdot,\boldsymbol{w}), K_1(\cdot,\boldsymbol{z})\rangle_{\mathcal{H}_{K_1}} + (1-z_2\overline{w_2})\langle K_2(\cdot,\boldsymbol{w}), K_2(\cdot,\boldsymbol{z})\rangle_{\mathcal{H}_{K_2}},$$

and hence, by rearranging terms

$$\begin{aligned}1 + \langle \overline{w_1} K_1(\cdot, \boldsymbol{w}), \overline{z_1} K_1(\cdot, \boldsymbol{z})\rangle_{\mathcal{H}_{K_1}} + \langle \overline{w_2} K_2(\cdot, \boldsymbol{w}), \overline{z_2} K_2(\cdot, \boldsymbol{z})\rangle_{\mathcal{H}_{K_2}} &= \varphi(\boldsymbol{z})\overline{\varphi(\boldsymbol{w})} \\ &+ \langle K_1(\cdot, \boldsymbol{w}), K_1(\cdot, \boldsymbol{z})\rangle_{\mathcal{H}_{K_1}} + \langle K_2(\cdot, \boldsymbol{w}), K_2(\cdot, \boldsymbol{z})\rangle_{\mathcal{H}_{K_2}},\end{aligned}$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^2$. Therefore

$$V : \begin{pmatrix} 1 \\ \overline{w_1} K_1(\cdot, \boldsymbol{w}) \\ \overline{w_2} K_2(\cdot, \boldsymbol{w}) \end{pmatrix} \mapsto \begin{pmatrix} \overline{\varphi(\boldsymbol{w})} \\ K_1(\cdot, \boldsymbol{w}) \\ K_2(\cdot, \boldsymbol{w}) \end{pmatrix} \qquad (\boldsymbol{w} \in \mathbb{D}^2), \tag{1.2.4}$$

defines an isometry from $\mathcal{D}$ onto $\mathcal{R}$, where

$$\mathcal{D} = \overline{\text{span}} \left\{ \begin{pmatrix} 1 \\ \overline{w_1} K_1(\cdot, \boldsymbol{w}) \\ \overline{w_2} K_2(\cdot, \boldsymbol{w}) \end{pmatrix} : \boldsymbol{w} \in \mathbb{D}^2 \right\} \subseteq \mathbb{C} \oplus \mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2},$$

and

$$\mathcal{R} = \overline{\text{span}} \left\{ \begin{pmatrix} \overline{\varphi(\boldsymbol{w})} \\ K_1(\cdot, \boldsymbol{w}) \\ K_2(\cdot, \boldsymbol{w}) \end{pmatrix} : \boldsymbol{w} \in \mathbb{D}^2 \right\} \subseteq \mathbb{C} \oplus \mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2}.$$

Clearly V extends to a contractive map on $\mathbb{C} \oplus \mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2}$. By adding an infinite dimensional Hilbert space $\mathcal{H}$, if necessary, V extends to an isometry (unitary) from $\mathbb{C} \oplus \mathcal{H}_{K_1} \oplus (\mathcal{H}_{K_2} \oplus \mathcal{H})$ into itself. Let

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_{K_1} \oplus (\mathcal{H}_{K_2} \oplus \mathcal{H})).$$

From (1.2.4) we get

$$\begin{aligned} A + BE^*_{\mathcal{H}_1 \oplus \mathcal{H}_2}(\boldsymbol{w}) \begin{pmatrix} K_1(\cdot, \boldsymbol{w}) \\ K_2(\cdot, \boldsymbol{w}) \end{pmatrix} &= \overline{\varphi(\boldsymbol{w})}, \\ C + DE^*_{\mathcal{H}_1 \oplus \mathcal{H}_2}(\boldsymbol{w}) \begin{pmatrix} K_1(\cdot, \boldsymbol{w}) \\ K_2(\cdot, \boldsymbol{w}) \end{pmatrix} &= \begin{pmatrix} K_1(\cdot, \boldsymbol{w}) \\ K_2(\cdot, \boldsymbol{w}) \end{pmatrix}, \end{aligned}$$

and hence

$$\varphi(\boldsymbol{z}) = A^* + C^*(I - E_{\mathcal{H}_1 \oplus \mathcal{H}_2}(\boldsymbol{z}) D^*)^{-1} E_{\mathcal{H}_1 \oplus \mathcal{H}_2}(\boldsymbol{z}) B^*.$$

The proof of (3) $\Rightarrow$ (1) is similar to the above computation (or see the proof of Theorem 1.2.1). $\square$

1.2.3 Schur-Agler functions

Let $\mathcal{E}$ and $\mathcal{E}_*$ be Hilbert spaces. The *Schur-Agler class* $\mathcal{SA}(\mathbb{D}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ consists of $\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$-valued analytic functions φ on $\mathbb{D}^n$ such that φ satisfies the n-variables von

Neumann inequality

$$\|\varphi(T_1, \dots, T_n)\|_{\mathcal{B}(\mathcal{H})} \leq 1,$$

for arbitrary n-tuples of commuting strict contractions $T = (T_1, \dots, T_n)$ on Hilbert spaces arbitrary $\mathcal{H}$. Here

$$\varphi(T_1, \dots, T_n) = \sum_{\boldsymbol{k} \in \mathbb{Z}_+^n} \varphi_{\boldsymbol{k}} \otimes T^{\boldsymbol{k}},$$

where $\varphi = \sum_{\boldsymbol{k} \in \mathbb{Z}_+^n} \varphi_{\boldsymbol{k}} \boldsymbol{z}^{\boldsymbol{k}}$, $\varphi_{\boldsymbol{k}} \in \mathcal{B}(\mathcal{E}, \mathcal{E}_*)$, and $T^{\boldsymbol{k}} = T_1^{k_1} \cdots T_n^{k_n}$ for all $\boldsymbol{k} = (k_1, \dots, k_n) \in \mathbb{Z}_+^n$. The elements of $\mathcal{SA}(\mathbb{D}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ are called *Schur-Agler functions.* We denote by $\mathcal{SA}(\mathbb{D}^n)$ the scalar valued Schur-Agler functions on $\mathbb{D}^n$.

Following the classical (one variable) von Neumann inequality, Ando [12] proved that the von Neumann inequality also holds for commuting pairs of contractions. On the other hand, the von Neumann inequality does not hold in general for n-tuples, $n > 2$, of commuting contractions [45, 111]. It follows then that

$$\mathcal{S}(\mathbb{D}) = \mathcal{SA}(\mathbb{D}) \quad \text{and} \quad \mathcal{S}(\mathbb{D}^2) = \mathcal{SA}(\mathbb{D}^2),$$

but $\mathcal{S}(\mathbb{D}^n) \supsetneq \mathcal{SA}(\mathbb{D}^n)$ for all $n > 2$.

Example 1.2.4 (Kaijser–Varopoulos polynomial)**.** *Consider the following degree* $(2, 2, 2)$ *homogeneous polynomial in* z_1, z_2 *and* z_3 *variables*

$$p(z_1, z_2, z_3) = \frac{1}{5}(z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_2z_3 - 2z_3z_1).$$

It can be easily show that $\|p\|_\infty = 1$*. However, there exist commuting contractions* T_1, T_2 *and* T_3 *such that* $\|p(T_1, T_2, T_3)\| = \frac{6}{5} > 1$*. Hence* $p \in \mathcal{S}(\mathbb{D}^3)$*, but* $p \notin \mathcal{SA}(\mathbb{D}^3)$*.*

The following result is due to Agler [2]:

Theorem 1.2.5 (Agler)**.** *Let* φ *be a function on* $\mathbb{D}^n$*. Then* $\varphi \in \mathcal{SA}(\mathbb{D}^n)$ *if and only if there exist Hilbert spaces* $\mathcal{H}_1, \dots, \mathcal{H}_n$*, and an isometric colligation*

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1^n),$$

such that $\varphi = \tau_V$ *where*

$$\tau_V(\boldsymbol{z}) = a + B(I_{\mathcal{H}_1^n} - E_{\mathcal{H}_1^n}(\boldsymbol{z})D)^{-1}E_{\mathcal{H}_1^n}(\boldsymbol{z})C,$$

$\mathcal{H}_1^n = \bigoplus_{i=1}^n \mathcal{H}_i$ *and* $E_{\mathcal{H}_1^n}(\boldsymbol{z}) = \bigoplus_{i=1}^n z_i I_{\mathcal{H}_i}$ *for all* $\boldsymbol{z} \in \mathbb{D}^n$*.*

We also have the following analog of Theorem 1.2.3:

Theorem 1.2.6. *Given a function $\theta : \mathbb{D}^n \to \mathbb{C}$, the following are equivalent:*

(1) $\theta \in \mathcal{SA}(\mathbb{D}^n)$.

(2) *There exist kernels $K_1, \dots, K_n$ (known as Agler kernels) on $\mathbb{D}^n$ such that*

$$1 - \theta(\boldsymbol{z})\overline{\theta(\boldsymbol{w})} = \sum_{i=1}^{n} (1 - z_i \overline{w_i}) K_i(\boldsymbol{z}, \boldsymbol{w}), \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n).$$

We now define the Schur-Agler norm of a function $f \in \mathcal{SA}(\mathbb{D}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$:

$$\|f\|_{sa} = \sup_{T} \|f(T)\|,$$

where the supremum is taken over all n-tuples $T = (T_1, \dots, T_n)$ of commuting strict contractions on some Hilbert space $\mathcal{H}$. Clearly, supremum norm is dominated by Schur-Agler norm, that is $\|f\|_\infty \leq \|f\|_{sa}$ for all $f \in \mathcal{SA}(\mathbb{D}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$.

Theorem 1.2.7 (see [61] and [67])**.** *Let $\varphi \in \mathcal{S}(\mathbb{D}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$. Then $\varphi \in \mathcal{SA}(\mathbb{D}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ if and only if*

$$\|f\|_{\mathcal{SA}} = \|f\|_\infty.$$

In general, the condition $z_1 f \in \mathcal{SA}(\mathbb{D}^n)$, $n \geq 3$, for an analytic function f on $\mathbb{D}^n$ does not force $f \in \mathcal{SA}(\mathbb{D}^n)$ (see [60]).

1.2.4 Inner functions

Let $\mathcal{E}$ be a Hilbert space. Recall that the $\mathcal{E}$-valued Hardy space over the polydisc $\mathbb{D}^n$, denoted by $H^2_{\mathcal{E}}(\mathbb{D}^n)$, is the Hilbert space of all $\mathcal{E}$-valued analytic functions f on $\mathbb{D}^n$ such that

$$\|f\| := \Big(\sup_{0 \leq r < 1} \int_{\mathbb{T}^n} \|f(rz_1, \dots, rz_n)\|^2_{\mathcal{E}} dm(z) \Big)^{\frac{1}{2}} < \infty,$$

where $z = (z_1, \dots, z_n)$ and $dm(z)$ is the normalized Lebesgue measure on the n-torus $\mathbb{T}^n$. A function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_*, \mathcal{E})}(\mathbb{D}^n)$ is called *inner* if $f \mapsto \Theta f$ defines an isometry $M_\Theta : H^2_{\mathcal{E}_*}(\mathbb{D}^n) \to H^2_{\mathcal{E}}(\mathbb{D}^n)$ or equivalently $\Theta(z)^*\Theta(z) = I_{\mathcal{E}_*}$ a.e. $z \in \mathbb{T}^n$. The simplest example is $\Theta(z) = z_i I_{\mathcal{E}}$, $i = 1, \dots, n$, whenever $\mathcal{E}_* = \mathcal{E}$. Therefore, $(M_{z_1}, \dots, M_{z_n})$ is a commuting tuple of isometries on $H^2_{\mathcal{E}}(\mathbb{D}^n)$.

In the scalar case, a function $\varphi \in \mathcal{S}(\mathbb{D}^n)$ is said to be *inner* if

$$\lim_{r \nearrow 1} |\varphi(re^{it_1}, \cdots, re^{it_n})| = |\varphi(e^{it_1}, \cdots, e^{it_n})| = 1,$$

almost everywhere on the distinguished boundary $\mathbb{T}^n$ of $\mathbb{D}^n$. Now it can be easily proved that a function $\varphi \in \mathcal{S}(\mathbb{D}^n)$ is inner if and only if M_φ on $H^2(\mathbb{D}^n)$ is an isometry (in fact, if φ is non-constant then M_φ on $H^2(\mathbb{D}^n)$ is a pure isometry if and only if φ is inner).

For example, a function $\varphi \in \mathcal{S}(\mathbb{D}^n)$ is rational inner function if and only if

$$\varphi(z) = z_1^{k_1} \cdots z_n^{k_n} \frac{\overline{p(\frac{1}{\bar{z}_1}, \ldots, \frac{1}{\bar{z}_n})}}{p(z)},$$

where p is a polynomial with no zeros in $\mathbb{D}^n$ and $(k_1, \ldots, k_n)$ is the multi-degree of p (see Rudin [97, Theorem 5.2.5]). Some useful references on rational inner functions are [33, 35, 74, 75, 76]

1.2.5 Inner functions and isometric colligations

A function $\varphi : \mathbb{D} \to \mathcal{B}(\mathbb{C}^n)$ is called matrix valued rational inner function if φ is an inner function and entries of the matrix φ are rational functions with poles off $\mathbb{D}$. Let φ be a $n \times n$ matrix valued rational inner function. Now consider the following subset of $\mathbb{D}^2$

$$\mathcal{V} = \{(z, w) \in \mathbb{D}^2 : \det(\varphi(z) - wI) = 0\}.$$

Then

$$\overline{\mathcal{V}} \cap \partial(\mathbb{D}^2) = \overline{\mathcal{V}} \cap \mathbb{T}^2, \tag{1.2.5}$$

where $\partial\mathbb{D}^2$ is the topological boundary of $\mathbb{D}^2$ and the closure of $\mathcal{V}$ is taken in $\overline{\mathbb{D}^2}$. That is, $\mathcal{V}$ exits the bidisc through the distinguished boundary $\mathbb{T}^2$ of $\mathbb{D}^2$. A non-empty set $\mathcal{V} \subseteq \mathbb{C}$ is called a distinguished variety if there is a polynomial $p \in \mathbb{C}[z, w]$ such that $\mathcal{V} = \{(z, w) \in \mathbb{D}^2 : p(z, w) = 0\}$ and $\mathcal{V}$ exits the bidisc through the distinguished boundary.

Theorem 1.2.8. *Let $\mathcal{V}$ be a distinguished variety of $\mathbb{D}^2$. Then there is a matrix valued rational inner function φ such that*

$$\mathcal{V} = \{(z, w) \in \mathbb{D}^2 : \det(\varphi(z) - wI) = 0\}.$$

We refer [3, 4, 47, 72, 98] for details on this. The following result is a characterization of rational inner functions in terms of dimensions of state spaces.

Theorem 1.2.9. *Let $\varphi : \mathbb{D} \to \mathcal{B}(\mathbb{C}^n)$ be a function. Then φ is rational inner if and only if there exists a Hilbert space $\mathcal{H}$ with dim$(\mathcal{H}) < \infty$ and a unitary colligation*

$$U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathbb{C}^n \oplus \mathcal{H}),$$

such that

$$\varphi(z) = A + zB(I - zD)^{-1}C, \quad (z \in \mathbb{D}).$$

A similar result exists for rational inner functions on $\mathbb{D}^2$ (see [21]). Now we turn to characterizations of inner Schur functions. Recall that $C_{0\cdot}$ denotes the set of all contractions T on Hilbert spaces such that $T^m \to 0$ as $m \to \infty$ in the strong operator topology.

Theorem 1.2.10. *Let $\varphi \in \mathcal{S}(\mathbb{D})$ be a Schur function. Then φ is inner if and only if $\varphi = \tau_V$ for some isometric colligation*

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}),$$

with $D \in C_{0\cdot}$.

Proof. Let $\varphi \in \mathcal{S}(\mathbb{D})$ be an inner function, and suppose

$$\varphi = \sum_{m=0}^{\infty} a_m z^m \qquad (z \in \mathbb{D}),$$

is the power series representation of φ on $\mathbb{D}$. We clearly have

$$a_m = P_{\mathbb{C}} M_z^{*m} M_\varphi|_{\mathbb{C}} \qquad (m \geq 0),$$

where $P_{\mathbb{C}}$ denotes the orthogonal projection onto the space of constant functions in $H^2(\mathbb{D})$. Note that $M_\varphi 1 = \varphi$. Note that M_φ is an isometry on $H^2(\mathbb{D})$. Then for the model space $\mathcal{Q}_\varphi = H^2(\mathbb{D}) \ominus \varphi H^2(\mathbb{D})$, we have $M_z^* \mathcal{Q}_\varphi \subseteq \mathcal{Q}_\varphi$ and $M_z^* \varphi \in \mathcal{Q}_\varphi$ (indeed, $\langle M_z^* \varphi, \varphi f \rangle = \langle M_z^* 1, f \rangle = 0$ for all $f \in H^2(\mathbb{D})$). Therefore

$$\varphi(w) = P_{\mathbb{C}} M_\varphi|_{\mathbb{C}} + w P_{\mathbb{C}}|_{\mathcal{Q}_\varphi} (I_{\mathcal{Q}_\varphi} - w M_z^*|_{\mathcal{Q}_\varphi})^{-1} M_z^* M_\varphi|_{\mathbb{C}} \qquad (w \in \mathbb{D}). \tag{1.2.6}$$

Clearly

$$V = \begin{bmatrix} \varphi(0) & P_{\mathbb{C}}|_{\mathcal{Q}_\varphi} \\ M_z^* M_\varphi|_{\mathbb{C}} & M_z^*|_{\mathcal{Q}_\varphi} \end{bmatrix},$$

defines a unitary colligation operator on $\mathbb{C} \oplus \mathcal{Q}_\varphi$. And, of course, we have $M_z^*|_{\mathcal{Q}_\varphi} \in C_{0\cdot}$ and $\varphi = \tau_V$. For the converse part we refer [57, Theorem 10.1, page 122]. □

Note that the representation of φ in (1.2.6) reduces to a more compact form as

$$\varphi(w) = P_{\mathbb{C}} (I_{H^2(\mathbb{D})} - w M_z^*)^{-1} M_\varphi|_{\mathbb{C}} \qquad (w \in \mathbb{D}).$$

1.2.6 Realizations of Drury-Arveson multipliers

The *Drury-Arveson space*, denoted by H^2_n, is the Hilbert space of holomorphic functions on $\mathbb{B}^n$ corresponding to the reproducing kernel (cf. [16], [108])

$$k(\boldsymbol{z}, \boldsymbol{w}) = \frac{1}{1 - \langle \boldsymbol{z}, \boldsymbol{w} \rangle} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n),$$

where $\langle \boldsymbol{z}, \boldsymbol{w} \rangle$ is the usual scalar product in $\mathbb{C}^n$. We denote by $\mathcal{M}(H_n^2)$ the commutative Banach algebra of multipliers of H_n^2 equipped with the operator norm $\|\varphi\| := \|M_\varphi\|_{\mathcal{B}(H_n^2)}$. Also we define

$$\mathcal{M}_1(H_n^2) = \{\varphi \in \mathcal{M}(H_n^2) : \|\varphi\| \leq 1\}.$$

Set $\mathcal{M}_1(H_n^2)$ is called the Schur class on $\mathbb{B}^n$. Given a Hilbert space $\mathcal{H}$, we denote by $\mathcal{H}^n$ the n-copies of $\mathcal{H}$, that is

$$\mathcal{H}^n = \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n},$$

and $E_{\mathcal{H}^n} : \mathbb{B}^n \to \mathcal{B}(\mathcal{H}^n, \mathcal{H})$ the row operator

$$E_{\mathcal{H}^n}(\boldsymbol{z}) = (z_1 I_{\mathcal{H}}, \ldots, z_n I_{\mathcal{H}}) \qquad (\boldsymbol{z} \in \mathbb{B}^n).$$

The following characterization of multipliers parallel to the transfer function realizations of Schur-Agler class functions on $\mathbb{D}^n$ is our starting point:

Theorem 1.2.11 ([16, 56])**.** *Suppose φ is a complex-valued function on $\mathbb{B}^n$. Then the following are equivalent:*

(1) $\varphi \in \mathcal{M}_1(H_n^2)$.

(2) There exist a Hilbert space $\mathcal{H}$ and an isometric colligation

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} : \mathbb{C} \oplus \mathcal{H} \to \mathbb{C} \oplus \mathcal{H}^n,$$

such that $\varphi = \tau_V$ where

$$\tau_V(\boldsymbol{z}) = a + B(I_{\mathcal{H}} - E_{\mathcal{H}^n}(\boldsymbol{z})D)^{-1} E_{\mathcal{H}^n}(\boldsymbol{z})C \qquad (\boldsymbol{z} \in \mathbb{B}^n).$$

1.2.7 de Branges-Rovnyak kernels

In this section, we briefly discuss the basics of contractively contained shift invariant (and not necessarily closed) subspaces of the Hardy space. We refer to the survey article by Ball and Bolotnikov [18] for a detailed discussion about de Branges-Rovnyak spaces. Also see [24, 25, 37, 38, 54, 55, 99, 110].

Let T be a bounded operator on a Hilbert space $\mathcal{H}$. We define the range space $\mathrm{ran}\mathcal{H}$ with the inner product

$$\langle Th, Tk \rangle_{\mathrm{ran}\mathcal{H}} = \langle h, k \rangle_{\mathcal{H}}, \quad (h, k \in \mathcal{H} \ominus \ker(T)).$$

The space $\mathrm{ran}\mathcal{H}$ is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathrm{ran}\mathcal{H}}$. Moreover, T is a partial isometry on $\mathcal{H}$ with respect to the range space norm.

Let $\varphi \in \mathcal{S}(\mathbb{D})$ be Schur function. The de Branges-Rovnyak space $\mathcal{H}_\varphi$ corresponding to φ is the range space

$$\mathcal{H}_\varphi = (I - M_\varphi M_\varphi^*)^{\frac{1}{2}} H^2(\mathbb{D}).$$

which is a reproducing kernel Hilbert space with kernel (we call it *de Branges-Rovnyak kernel*)

$$K_\varphi(z,w)=\frac{1-\varphi(z)\overline{\varphi(w)}}{1-z\overline{w}}\qquad (z,w\in\mathbb{D}).$$

Observe that $\mathcal{H}_\varphi$ is the model space $\mathcal{Q}_\varphi=H^2(\mathbb{D})\ominus\varphi H^2(\mathbb{D})$ whenever φ is inner.

We now present the definition of the de Branges-Rovnyak space associated with an operator-valued Schur function. Given a Schur function $\Phi\in\mathcal{S}(\mathbb{D},\mathcal{B}(\mathcal{E},\mathcal{E}_*))$, the space

$$\mathcal{H}_\Phi=\left\{f\in H^2_{\mathcal{E}_*}(\mathbb{D}):\|f\|^2_{\mathcal{H}_\Phi}:=\sup\left\{\|f+\Phi g\|^2_{H^2_{\mathcal{E}_*}(\mathbb{D})}-\|g\|^2_{H^2_{\mathcal{E}}(\mathbb{D})}:g\in H^2_{\mathcal{E}}(\mathbb{D})\right\}<\infty\right\}$$

is called the de Branges-Rovnyak space associated with Φ. The space $\mathcal{H}_\Phi$ is invariant under the *backward shift operator*

$$B:f(z)\mapsto\frac{f(z)-f(0)}{z}.$$

Let $\mathcal{H}_{K_1}$ and $\mathcal{H}_{K_2}$ be reproducing kernels on a set X. Then $\mathcal{H}_{K_1}$ is said to be contractively contained in $\mathcal{H}_{K_2}$, if $\mathcal{H}_{K_1}$ is a vector subspace of $\mathcal{H}_{K_2}$ and

$$\|h\|_{\mathcal{H}_{K_2}}\le\|h\|_{\mathcal{H}_{K_1}}\qquad (h\in\mathcal{H}_{K_1}).$$

Notice that $\mathcal{H}_{K_1}$ is contractively contained in $\mathcal{H}_{K_2}$ if and only if

$$K_2-K_1\ge 0.$$

Now it is easy to observe that $\mathcal{H}_\Phi$ is contractively contained in $H^2_{\mathcal{E}_*}(\mathbb{D})$.

We now turn to de Branges-Rovnyak kernels corresponding to Schur-Agler functions on $\mathbb{D}^n$. Suppose $\Theta\in\mathcal{SA}(\mathbb{D}^n,\mathcal{B}(\mathcal{E},\mathcal{E}_*))$. Since $M_\Theta:H^2_{\mathcal{E}}(\mathbb{D}^n)\to H^2_{\mathcal{E}_*}(\mathbb{D}^n)$ is a contraction, we have $K_\Theta\ge 0$, where

$$K_\Theta(\boldsymbol{z},\boldsymbol{w})=\mathbb{S}_n(\boldsymbol{z},\boldsymbol{w})^{-1}(I-\Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^*)\qquad (\boldsymbol{z},\boldsymbol{w}\in\mathbb{D}^n).$$

In this case, we say that K_Θ is a ($\mathcal{B}(\mathcal{E}_*)$-valued) de Branges-Rovnyak kernel on $\mathbb{D}^n$, and the corresponding reproducing kernel Hilbert space as the de Branges-Rovnyak space associate with Θ.

Similarly, in the setting of the Drury-Arveson space, for a $\mathcal{B}(\mathcal{E},\mathcal{E}_*)$-valued contractive $\Theta\in\mathcal{M}(H^2_n\otimes\mathcal{E},H^2_n\otimes\mathcal{E}_*)$, the de Branges-Rovnyak kernel K_Θ corresponding to Θ is defined by

$$K_\Theta(\boldsymbol{z},\boldsymbol{w})=\frac{I-\Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^*}{1-\langle\boldsymbol{z},\boldsymbol{w}\rangle}\qquad (\boldsymbol{z},\boldsymbol{w}\in\mathbb{B}^n).$$

The reproducing kernel Hilbert space corresponding to K_Θ is called the de Branges-Rovnyak space on $\mathbb{B}^n$.

1.3 Structure of contractions

In this section, we briefly discuss some basic dilation theory in both one and severable variables. Along the way, we recall the classical Beurling-Lax-Halmos theorem and in the final part of this section, we review dilations of Brehmer tuples.

1.3.1 Isometries and von Neumann and Wold decomposition

Let $\mathcal{H}$ be a Hilbert space. An operator V on $\mathcal{H}$ is called an isometry if $\|Vh\| = \|h\|$ for all h in $\mathcal{H}$. An isometry V is called a pure isometry or a *shift* if $V^{*m} \to 0$ in the strong operator topology. The following classical result is due to von Neumann and Wold (cf [90]):

Theorem 1.3.1 (von Neumann and Wold decomposition)**.** *Let V be an isometry on $\mathcal{H}$. Then there exist unique reducing subspaces $\mathcal{H}_0$ and $\mathcal{H}_1$ of $\mathcal{H}$ such that*

1. *$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$,*
2. *$V|_{\mathcal{H}_0}$ is a unitary, and*
3. *$V|_{\mathcal{H}_1}$ is a pure isometry.*

In the above case, $\mathcal{H}_1 = \oplus_{m\geq 0} V^m \mathcal{W}$, where $\mathcal{W} = \ker V^*$ is the cyclic wandering subspace of $V|_{\mathcal{H}_1}$, and $\mathcal{H}_0 = \cap_{m\geq 0} V^m \mathcal{H}$. The following result is now immediate:

Corollary 1.3.2. *Let V be a pure isometry on $\mathcal{H}$. Then V is unitary equivalent to M_z on the Hardy space $H^2_{\mathcal{W}}(\mathbb{D})$, where $\mathcal{W} = \ker V^*$.*

As already pointed out, $M_\Theta(M_z \otimes I_{\mathcal{E}}) = (M_z \otimes I_{\mathcal{E}_*})M_\Theta$ for all $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D})$. In fact the converse is also true (see [102] for a more general result):

Theorem 1.3.3. *Let $X \in \mathcal{B}(H^2(\mathbb{D}) \otimes \mathcal{E}, H^2(\mathbb{D}) \otimes \mathcal{E}_*)$ such that*

$$X(M_z \otimes I_{\mathcal{E}}) = (M_z \otimes I_{\mathcal{E}_*})X.$$

Then there exists $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{E}_*)}(\mathbb{D})$ such that $X = M_\Theta$.

Proof. Let $w \in \mathbb{D}$ and $\eta \in \mathcal{E}_*$. Then

$$(M_z \otimes I_{\mathcal{E}})^*(X^*(\mathbb{S}(\cdot, w) \otimes \eta)) = X^*(M_z \otimes I_{\mathcal{E}})^*(\mathbb{S}(\cdot, w) \otimes \eta) = \overline{w} X^*(\mathbb{S}(\cdot, w) \otimes \eta),$$

implies that

$$X^*(\mathbb{S}(\cdot, w) \otimes \eta) \in \ker(M_z^* \otimes I_{\mathcal{E}} - \overline{w} I).$$

Since $\ker(M_z^* - \overline{w} I) = \mathbb{C}\mathbb{S}(\cdot, w)$, where $\mathbb{C}\mathbb{S}(\cdot, w) = \{\lambda \mathbb{S}(\cdot, w) : \lambda \in \mathbb{C}\}$, there exists a linear map $Y(w) : \mathcal{E}_* \to \mathcal{E}$ such that

$$X^*(\mathbb{S}(\cdot, w) \otimes \eta) = \mathbb{S}(\cdot, w) \otimes Y(w)\eta,$$

for all $\eta \in \mathcal{E}_*$ and $w \in \mathbb{D}$. Note that

$$\|Y(w)\eta\| = \frac{\|X^*(\mathbb{S}(\cdot, w) \otimes \eta)\|}{\|\mathbb{S}(\cdot, w)\|} \leq \|X\|\|\eta\|,$$

and hence, $Y(w)$ is a bounded operator for all $w \in \mathbb{D}$. Set $\Theta(w) = Y(w)^*$. Then

$$X^*(\mathbb{S}(\cdot, w) \otimes \eta) = \mathbb{S}(\cdot, w) \otimes \Theta(w)^*\eta.$$

By the reproducing property

$$\langle \Theta(w)\eta_1, \eta_2 \rangle = \langle X(\mathbb{S}(\cdot, 0) \otimes \eta_1), \mathbb{S}(\cdot, w) \otimes \eta_2 \rangle,$$

for all $\eta_1, \eta_2 \in \mathcal{E}_*$ and $w \in \mathbb{D}$. Since $w \mapsto \mathbb{S}(\cdot, w)$ is a co-analytic, the above equality implies that $w \mapsto \Theta(w)$ is an analytic on $\mathbb{D}$. □

Now we turn to analytic representations of shift invariant subspaces of vector-valued Hardy spaces. This is due to Beurling [29], Lax [81], and Halmos [63].

Theorem 1.3.4 (Beurling-Lax-Halmos). *Let $\mathcal{E}$ be a Hilbert space and let $\mathcal{S}$ be a non-zero closed subspace of $H^2_{\mathcal{E}}(\mathbb{D})$. Then $\mathcal{S}$ is invariant under M_z if and only if there exist a Hilbert space $\mathcal{E}_*$ and an inner function $\Theta \in H^2_{\mathcal{B}(\mathcal{E}_*, \mathcal{E})}(\mathbb{D})$ such that*

$$\mathcal{S} = \Theta H^2_{\mathcal{E}_*}(\mathbb{D}).$$

Moreover, Θ is unique up to a constant unitary right factor.

Proof. Let $\mathcal{W} = \mathcal{S} \ominus z\mathcal{S}$. Consider the operator $V : H^2_{\mathcal{W}}(\mathbb{D}) \to H^2_{\mathcal{E}}(\mathbb{D})$ defined by

$$V(z^n\eta) = M_z^n\eta \qquad (\eta \in \mathcal{W}, n \geq 0).$$

Then V extends to an isometry. Since M_z is a pure isometry, $M_z|_{\mathcal{S}}$ is also a pure isometry. Applying Wold decomposition to $M_z|_{\mathcal{S}}$, we get

$$\mathcal{S} = \bigoplus_{n=0}^{\infty} M_z^n \mathcal{W} = \mathcal{R}(V),$$

where $\mathcal{R}(V)$ is the range of V. Thus $VV^* = P_{\mathcal{S}}$. Also it is easy to observe that

$$VM_z^{\mathcal{W}} = M_z^{\mathcal{E}}V.$$

Now from Theorem 1.3.3, we get

$$V = M_\Theta,$$

for some $\Theta \in H^\infty_{\mathcal{B}(\mathcal{W}, \mathcal{E})}(\mathbb{D})$. Since V is an isometry, it follows that Θ is an inner function, and hence $\mathcal{S} = \Theta H^2_{\mathcal{W}}(\mathbb{D})$ for some inner function Θ.

For the uniqueness part, suppose

$$\Theta H^2_{\mathcal{E}_*}(\mathbb{D}) = \Phi H^2_{\mathcal{F}_*}(\mathbb{D}),$$

for some inner function $\Phi \in H^\infty_{\mathcal{B}(\mathcal{F}_*,\mathcal{E})}(\mathbb{D})$. Then by Douglas lemma, there exists a contraction X such that

$$M_\Theta = M_\Phi X.$$

We now observe that

$$M_\Phi X M_z = M_\Theta M_z = M_z M_\Theta = M_z M_\Phi X = M_\Phi M_z X.$$

Since M_Φ is an isometry, we get

$$X M_z = M_z X.$$

Thus from Theorem 1.3.3 we obtain $X = M_\Psi$ for some $\Psi \in H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{F}_*)}(\mathbb{D})$. Since M_Θ and M_Φ are isometries, it follows that M_Ψ is an isometry and consequently Ψ is an inner function. Again we observe that

$$\Phi\Psi H^2_{\mathcal{E}_*}(\mathbb{D}) = \Phi H^2_{\mathcal{F}_*}(\mathbb{D}).$$

This implies that $\Psi H^2_{\mathcal{E}_*}(\mathbb{D}) = H^2_{\mathcal{F}_*}(\mathbb{D})$, and hence Ψ is a unitary constant (cf. Lemma 1.3.9). □

It is worthwhile to note that the wandering subspace $\mathcal{W} := \mathcal{S} \ominus z\mathcal{S}$ has the following expression:

$$\mathcal{W} = \Theta\mathcal{E}_*.$$

As a corollary to the above theorem, we have the classical Beurling theorem:

Corollary 1.3.5 (Beurling)**.** *Let $\mathcal{S}$ be a nonzero closed subspace of $H^2(\mathbb{D})$. Then $\mathcal{S}$ is invariant under M_z if and only if there exists an inner function $\theta \in H^\infty(\mathbb{D})$ such that $\mathcal{S} = \theta H^2(\mathbb{D})$. Moreover, θ is unique up to a unimodular constant.*

However, Beurling type representations of shift invariant subspaces of $H^2(\mathbb{D}^n)$, $n \geq 2$, fails in general:

Example 1.3.6. *Consider the M_z invariant subspace*

$$\mathcal{S} = \{f \in H^2(\mathbb{D}^2) : f(0,0) = 0\}$$

of $H^2(\mathbb{D}^2)$. Since $\mathcal{S} = z_1(H^2(\mathbb{D}) \otimes \mathbb{C}) \oplus z_2(\mathbb{C} \otimes H^2(\mathbb{D})) \oplus z_1 z_2 H^2(\mathbb{D}^2)$, and $\mathcal{S}$ is a reproducing kernel Hilbert space, the kernel function k of $\mathcal{S}$ is given by

$$k(z,w) = \frac{z_1\bar{w}_1}{1 - z_1\bar{w}_1} + \frac{z_2\bar{w}_2}{1 - z_2\bar{w}_2} + z_1 z_2 \mathbb{S}_2(z,w)\bar{w}_1\bar{w}_2 \qquad (z, w \in \mathbb{D}^2),$$

where

$$\mathbb{S}_2(z,w) = (1 - z_1\bar{w}_1)^{-1}(1 - z_2\bar{w}_2)^{-1} \qquad (z, w \in \mathbb{D}^2),$$

is the Szegö kernel of $\mathbb{D}^2$. *A simple calculation shows that*

$$k(z,w) = (z_1(1 - z_2\bar{w}_2)\bar{w}_1 + z_2\bar{w}_2)\mathbb{S}_2(z,w) \qquad (z, w \in \mathbb{D}^2).$$

If possible, suppose that $\mathcal{S}$ *is a Beurling type invariant subspace, that is,* $\mathcal{S} = \theta H^2(\mathbb{D}^2)$ *for some inner function* $\theta \in H^\infty(\mathbb{D}^2)$. *Then* $k(z,w) = \theta(z)\overline{\theta(w)}\mathbb{S}_2(z,w)$, *from which it immediately follows that*

$$\theta(z)\overline{\theta(w)} = z_1(1 - z_2\bar{w}_2)\bar{w}_1 + z_2\bar{w}_2 \qquad (z, w \in \mathbb{D}^2).$$

Clearly, the left side is a positive definite function while the right side is not. This proves that $\mathcal{S}$ *is not a Beurling type invariant subspace.*

In fact, the following result characterizes Beurling type invariant subspaces of vector-valued Hardy spaces. A closed subspace $\mathcal{S}$ of $H^2_{\mathcal{E}}(\mathbb{D}^n)$ is said to be of Beurling type if there exist a Hilbert space $\mathcal{E}_*$ and an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D}^n)$ such that $\mathcal{S} = \Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n)$.

Theorem 1.3.7 ([85], [103])**.** *Let* $\mathcal{S}$ *be a non-zero closed subspace of* $H^2_{\mathcal{E}}(\mathbb{D}^n)$. *Then* $\mathcal{S}$ *is of Beurling type if and only if*

$$R_{z_i}R^*_{z_j} = R^*_{z_j}R_{z_i} \quad (i \neq j),$$

where $R_{z_i} = M_{z_i}|_{\mathcal{S}}$ *for* $i = 1, \dots, n$.

A subspace $\mathcal{S}$ of $H^2_{\mathcal{E}}(\mathbb{D}^n)$ is called reducing if $\mathcal{S}$ is invariant under $(M_{z_1}, \dots, M_{z_n})$ and $(M^*_{z_1}, \dots, M^*_{z_n})$. The following result characterizes all reducing subspaces of vector-valued Hardy spaces:

Theorem 1.3.8 ([103])**.** *Let* $\mathcal{S}$ *be a non-zero closed subspace of* $H^2_{\mathcal{E}}(\mathbb{D}^n)$. *Then* $\mathcal{S}$ *is a reducing subspace if and only if there exists a subspace* $\mathcal{F}$ *of* $\mathcal{E}$ *such that*

$$\mathcal{S} = H^2_{\mathcal{F}}(\mathbb{D}^n).$$

In particular, $(M_{z_1}, \dots, M_{z_n})$ on $H^2(\mathbb{D})$ is irreducible.

We will identify as usual M_{z_i} on $H^2_{\mathcal{E}}(\mathbb{D}^n)$ with $M_{z_i} \otimes I_{\mathcal{E}}$ on $H^2(\mathbb{D}^n) \otimes \mathcal{E}$, $i = 1, \dots, n$, and write

$$M_z \otimes I_{\mathcal{E}} = (M_{z_1} \otimes I_{\mathcal{E}}, \dots, M_{z_n} \otimes I_{\mathcal{E}}).$$

We know, for each $i = 1, \dots, n$, that

$$M^*_{z_i}(\mathbb{S}(\cdot, w) \otimes \eta) = \bar{w}_i(\mathbb{S}(\cdot, w) \otimes \eta),$$

and hence $M_{z_i}M_{z_i}^*(\mathbb{S}(\cdot,w)\otimes\eta) = z_i\bar{w}_i(\mathbb{S}(\cdot,w)\otimes\eta)$ for all $w\in\mathbb{D}^n$ and $\eta\in\mathcal{E}$. It is now easy to see that $D^2_{M_z^*\otimes I_\mathcal{E}} = P_\mathbb{C}\otimes I_\mathcal{E}$ (for the definition of defect operator see 1.3.13), where $P_\mathbb{C}$ denotes the orthogonal projection of $H^2(\mathbb{D}^n)$ onto the one-dimensional subspace of constant functions.

Lemma 1.3.9. *Let $\Theta\in H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D}^n)$ be an inner function. If $\Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n) = H^2_\mathcal{E}(\mathbb{D}^n)$, then Θ is an unitary constant.*

Proof. Since, by hypothesis, $M_\Theta : H^2_{\mathcal{E}_*}(\mathbb{D}^n)\to H^2_\mathcal{E}(\mathbb{D}^n)$ is unitary and $(M_{z_i}\otimes I_\mathcal{E})M_\Theta = M_\Theta(M_{z_i}\otimes I_{\mathcal{E}_*})$, it follows that $(M_{z_i}^*\otimes I_\mathcal{E})M_\Theta = M_\Theta(M_{z_i}^*\otimes I_{\mathcal{E}_*})$ for all $i=1,\dots,n$. Then

$$D^2_{M_z\otimes I_\mathcal{E}}M_\Theta = M_\Theta D^2_{M_z\otimes I_{\mathcal{E}_*}},$$

and hence

$$(P_\mathbb{C}\otimes I_\mathcal{E})M_\Theta = M_\Theta(P_\mathbb{C}\otimes I_{\mathcal{E}_*}).$$

Thus, for any $\eta\in\mathcal{E}_*$, we have $\Theta(z)\eta = \Theta(0)\eta$, $z\in\mathbb{D}^n$, that is, $\Theta\equiv\Theta(0)$ is a constant function. This completes the proof of the lemma. $\square$

The $n=1$ case of the above lemma can be found in [28, Chapter 5, Proposition 1.17]. Moreover, the present proof is slightly simpler.

1.3.2 Dilations

We begin with the definition of dilations:

Definition 1.3.10. *Let $T=(T_1,\dots,T_n)$ and $V=(V_1,\dots,V_n)$ be commuting tuples of contractions and isometries on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. We say that V is an isometric dilation of T if there exists an isometry $\Pi:\mathcal{H}\to\mathcal{K}$ such that $\Pi T_i^* = V_i^*\Pi$ for all $i=1,\dots,n$.*

It is easy to observe that $\Pi\mathcal{H}$ is invariant under $(V_1^*,\dots,V_n^*)$ and $(T_1,\dots,T_n)$ is jointly unitary equivalent to $(P_{\Pi\mathcal{H}}V_1|_{\Pi\mathcal{H}},\dots P_{\Pi\mathcal{H}}V_n|_{\Pi\mathcal{H}})$. The existence and uniqueness of isometric (or unitary) dilation of a Hilbert space contraction was first proved by Sz.-Nagy (see [89], [90]). Also, see [106] for the alternative proof of Sz.-Nagy dilation theorem using infinite matrix representation due to Schäffer.

Theorem 1.3.11 (Sz.-Nagy)**.** *A contraction always dilates to an isometry (and hence, to a unitary).*

We refer to the recent survey article by Shalit [107] for the introduction to dilation theory and its applications. Sz.-Nagy dilation was extended for pairs of commuting contractions by Andô (see [12]):

Theorem 1.3.12 (Andô)**.** *Any pair of commuting contractions can be dilated to a pair of commuting isometries (and hence, unitaries).*

In the above case, unlike dilations of single contractions, minimal isometric dilations are not necessarily unique. Also one can easily prove that the existence of isometric dilations of n-tuples of commuting contractions implies the n-variable von-Neumann inequality. However, in sharp contrast, for 3 or more than 3 variables, neither the von Neumann inequality nor the existence of isometric dilations holds in general(see [93], [111]). The fundamental result of Arveson[13] (also see [91], [95]) says that an n-tuple of commuting contractions admits an isometric dilation if and only if it satisfies the von Neumann inequality for matrix-valued polynomials of all matrix sizes. For more information on several variables dilations and von-Neumann inequality, we refer the reader to [9],[26], [27], [44], [53], [61],[68], [77], [78], [84] and the references therein.

On the other hand, if a commuting tuple satisfies the Brehmer positivity condition, then it admits an isometric dilation. We recall:

Definition 1.3.13. *Let $T = (T_1, \ldots, T_n)$ be a commuting tuple of contractions on a Hilbert space $\mathcal{H}$. We say that T is Brehmer if*

$$\sum_{F \subseteq G} (-1)^{|F|} T_F T_F^* \geq 0, \tag{1.3.1}$$

for every $G \subseteq \{1, \ldots, n\}$, where $|F|$ denotes the cardinality of F and $T_F = \prod_{j \in F} T_j$ for all $F \subseteq \{1, \ldots, n\}$. We set, by convention, that $T_\emptyset = I_{\mathcal{H}}$ and $|\emptyset| = 0$.

A commuting tuple of isometries $V = (V_1, \ldots, V_n)$ is said to be doubly commuting if $V_i V_j^* = V_j^* V_i$ for all $i \neq j$. Note that a commuting tuple of unitaries $U = (U_1, \ldots, U_n)$ is automatically doubly commuting (thanks to Fuglede-Putnam theorem). The following theorem concerns isometric dilations of Brehmer tuples (see [10, 46, 86]).

Theorem 1.3.14. *Let $T = (T_1, \ldots, T_n)$ be a commuting tuple of contractions on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

1. *T satisfies the Brehmer positivity.*
2. *T dilates to doubly commuting isometries $V = (V_1, \ldots, V_n)$ on $\mathcal{K}$.*

Definition 1.3.15. *Consider a commuting tuple $T = (T_1, \ldots, T_n)$ on some Hilbert space $\mathcal{H}$. We say that T is a*

(i) Szegö tuple if T satisfy (1.3.1) for $G = \{1, \ldots, n\}$, and

(ii) pure tuple if T_i is pure for all $i = 1, \ldots, n$.

Note that the compressed tuple $P_{\mathcal{Q}} M_z|_{\mathcal{Q}} = (P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}, \ldots, P_{\mathcal{Q}} M_{z_n}|_{\mathcal{Q}})$ is a pure Szegö tuple whenever $\mathcal{Q}$ is a joint $(M_{z_1}^*, \ldots, M_{z_n}^*)$ invariant closed subspace of $H^2_{\mathcal{E}}(\mathbb{D}^n)$. The converse is also true, which will be useful in our study.

Theorem 1.3.16. *Let $T = (T_1, \ldots, T_n)$ be a commuting tuple of contractions on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

1. *T is a pure Szegö tuple.*

2. *T dilates to $M_z = (M_{z_1}, \ldots, M_{z_n})$ on $H^2_{\mathcal{E}}(\mathbb{D}^n)$ for some Hilbert space $\mathcal{E}$.*

Chapter 2

Factorizations of Schur-Agler functions

2.1 Introduction

The primary goal of this chapter is to clarify the link between isometric colligations and factors of Schur functions.

We state one of our main results specializing to the $n = 1$ case (see Theorem 2.3.4): Suppose $\varphi \in \mathcal{S}(\mathbb{D})$. If $\varphi = \varphi_1\varphi_2$ for some φ_1 and φ_2 in $\mathcal{S}(\mathbb{D})$, then there exist Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and an (explicit) isometric colligation

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} := \left[\begin{array}{c|cc} a & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] : \mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2) \to \mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2), \tag{2.1.1}$$

such that

$$D_{21} = 0 \quad \text{and} \quad aD_{12} = C_1B_2, \tag{2.1.2}$$

and $\varphi = \tau_V$, where $\tau_V(z) = a + zB(I_{\mathcal{H}_1\oplus\mathcal{H}_2} - zD)^{-1}C$, $z \in \mathbb{D}$.
The converse is true under an additional assumption that $\varphi(0) \neq 0$ (see Theorem 2.5.1, Section 2.5, for the case $\varphi(0) = 0$): If $\varphi = \tau_V$ for some isometric colligation V as in (2.1.1) satisfying (2.1.2) and $a := \varphi(0) \neq 0$, then $\varphi = \varphi_1\varphi_2$ for some φ_1 and φ_2 in $\mathcal{S}(\mathbb{D})$. Moreover, in this case, ϕ and ψ are explicitly given by $\varphi_1 = \tau_{V_1}$ and $\varphi_2 = \tau_{V_2}$ where

$$V_1 = \begin{bmatrix} \alpha & B_1 \\ \frac{1}{\beta}C_1 & D_{11} \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1) \quad \text{and} \quad V_2 = \begin{bmatrix} \beta & \frac{1}{\alpha}B_2 \\ C_2 & D_{22} \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_2),$$

are isometric colligations and α and β are non-zero scalars which satisfy the following conditions

$$|\beta|^2 = |a|^2 + C_1^*C_1 \quad \text{and} \quad \alpha = \frac{a}{\beta}.$$

We also remark that the above one-variable factorization of Schur functions also relates to factorizations of Sz.-Nagy and Foias characteristic functions [66] as well as Brodskiĭ colligations [40] in terms of invariant subspaces of certain operators [40, Theorem 2.6]. More specifically, see the idea of the product of colligations (as well as for a similar result as above, but in one direction) in [8, Theorem 1.2.1] and [40, Theorem 2.8]. However, here out results are different in the following sense: (i) we are interested in scalar-valued (unlike operator-valued functions in [8, 40]) Schur functions, (ii) our isometric colligations are explicit, (iii) our method is reversible (see Subsection 2.6.4), and (perhaps most importantly) (iv) our ideas works in the setting of n-variable Schur(-Agler) functions.

Needless to say, transfer function realizations and isometric colligation matrices corresponding to Schur-Agler class functions in n-variables, $n \geq 1$, are among the most frequently used techniques in problems in function theory, operator theory and interdisciplinary subjects such as Nevanlinna-Pick interpolation [5], commutant lifting theorem and analytic model theory [100, 57, 58], scattering theory [15], interpolation and Toeplitz corona theorem [16], electrical network theory [65, 66], signal processing [71, 59], linear systems [70, 50, 109], operator algebras [87, 88] and image processing [96] (just to name a few). In this context and for deeper studies, we refer the reader to a number of classic work such as Livšic [82, 83], Brodskiĭ [40], Brodskiĭ and M. Livšic [41] and Pavlov [94]. Also see [11], [42] and [61] and the references therein.

From this point of view, along with a question of interest in its own right, here we aim at finding necessary and sufficient conditions on isometric colligations which guarantee that a Schur-Agler class function factors into a product of Schur-Agler class functions. More precisely, we aim to solve the following problem: Given $\theta \in \mathcal{SA}(\mathbb{D}^n)$, find a set of necessary and sufficient conditions on isometric colligations V which ensures that

$$\theta = \tau_V = \phi\psi,$$

for some (explicit) ϕ and ψ in $\mathcal{SA}(\mathbb{D}^n)$.

In this chapter we give a complete answer to this question by identifying checkable conditions on isometric colligations. Our results and approach are new even in the case of one variable and two-variable Schur functions. In this context, it is also worth noting that the structure of bounded analytic functions in several variables is much more complicated than the structure of Schur functions on the unit disc (for instance, consider the existence of inner-outer factorizations of bounded analytic functions in one variable). From this point of view, our approach is also focused on providing an understanding of the complex area of bounded analytic functions of two or more variables (as the transfer function realization technique has already proven to be extremely useful in proving many classical results like Nevanlinna-Pick interpolation theorem and Carathéodory interpolation theorem etc. in several variables).

Our main results, specializing to the $n = 2$ case, yields the following: Suppose $\theta \in \mathcal{S}(\mathbb{D}^2)$ and $a := \theta(0) \neq 0$. Then:

(1) *Theorem 2.2.4 implies that:* $\theta(\boldsymbol{z}) = \varphi_1(z_1)\varphi_2(z_2)$, $\boldsymbol{z} \in \mathbb{D}^2$, for some φ_1 and φ_2 in $\mathcal{S}(\mathbb{D})$ if and only if $\theta = \tau_V$ for some isometric colligation

$$V = \left[\begin{array}{c|cc} a & B_1 & B_2 \\ \hline C_1 & D_{11} & \frac{1}{a}C_1B_2 \\ C_2 & 0 & D_{22} \end{array}\right] \in \mathcal{B}(\mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2)).$$

(2) Theorem 2.3.4 implies that: $\theta = \varphi\psi$ for some φ and ψ in $\mathcal{S}(\mathbb{D}^2)$ if and only if there exist Hilbert spaces $\{\mathcal{M}_i\}_{i=1}^2$ and $\{\mathcal{N}_i\}_{i=1}^2$ and isometric colligation

$$V = \left[\begin{array}{c|cc} a & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \in \mathcal{B}\Big(\mathbb{C} \oplus ((\mathcal{M}_1 \oplus \mathcal{N}_1) \oplus (\mathcal{M}_2 \oplus \mathcal{N}_2))\Big),$$

such that $\theta = \tau_V$, and representing B_i, C_i and D_{ij} as

$$B_i = \begin{bmatrix} B_i(1) & B_i(2) \end{bmatrix} \in \mathcal{B}(\mathcal{M}_i \oplus \mathcal{N}_i, \mathbb{C}) \quad \text{and} \quad C_i = \begin{bmatrix} C_i(1) \\ C_i(2) \end{bmatrix} \in \mathcal{B}(\mathbb{C}, \mathcal{M}_i \oplus \mathcal{N}_i),$$

and $D_{ij} = \begin{bmatrix} D_{ij}(1) & D_{ij}(12) \\ D_{ij}(21) & D_{ij}(2) \end{bmatrix} \in \mathcal{B}(\mathcal{M}_j \oplus \mathcal{N}_j, \mathcal{M}_i \oplus \mathcal{N}_i)$, respectively, one has $D_{ij}(21) = 0$ and $aD_{ij}(2) = C_i(1)B_j(2)$, $i, j = 1, 2$.

Moreover, in the case of (1) (see Theorem 2.2.3): $\phi_1(z) = \tau_{\tilde{V}_1}(z)$ and $\phi_2(z) = \tau_{\tilde{V}_2}(z)$, $z \in \mathbb{D}$, where

$$\tilde{V}_1 = \begin{bmatrix} \alpha & B_1 \\ \frac{1}{\beta}C_1 & D_{11} \end{bmatrix} \quad \text{and} \quad \tilde{V}_2 = \begin{bmatrix} \beta & \frac{1}{\alpha}B_2 \\ C_2 & D_{22} \end{bmatrix},$$

and α and β are non-zero scalars satisfying the conditions $|\beta|^2 = 1 - C_2^*C_2$ and $\alpha = \frac{a}{\beta}$; and in the case of (2) (see Theorem 2.3.3): $\varphi(\boldsymbol{z}) = \tau_{V_1}(\boldsymbol{z})$ and $\psi(\boldsymbol{z}) = \tau_{V_2}(\boldsymbol{z})$, $\boldsymbol{z} \in \mathbb{D}^2$, where

$$V_1 = \begin{bmatrix} \alpha & B(1) \\ \frac{1}{\beta}C(1) & D(1) \end{bmatrix} \quad \text{and} \quad V_2 = \begin{bmatrix} \beta & \frac{1}{\alpha}B(2) \\ C(2) & D(2) \end{bmatrix},$$

and

$$D(1) = \Big[D_{kl}(1)\Big]_{k,l=1}^2, \; D(2) = \Big[D_{kl}(2)\Big]_{k,l=1}^2, \; B(i) = \begin{bmatrix} B_1(i) & B_2(i) \end{bmatrix} \text{ and } C(i) = \begin{bmatrix} C_1(i) \\ C_2(i) \end{bmatrix},$$

for all $i = 1, 2$, and α and β are non-zero scalars satisfying the conditions $|\beta|^2 = |a|^2 + C(1)^*C(1)$ and $\alpha = \frac{a}{\beta}$.

Remark 2.1.1. *The assumption that $\theta(0) \neq 0$ is not essential for the necessary parts of the above results (and Theorems 2.2.4 and 2.3.4) and the case of $\theta(0) = 0$ will be treated* *separately in Section 2.5. As we will see there, functions vanishing at the origin reveals more detailed properties of corresponding isometric colligations.*

The rest of this chapter is organized as follows. Section 2.2 contains the definition of $\mathcal{F}_m(n)$ class of isometric colligations, $1 \leq m < n$, and a classification of factorizations of functions in the Schur-Agler class $\mathcal{SA}(\mathbb{D}^n)$, $n > 1$, into Schur-Agler class factors with fewer variables. Section 2.3 introduces the $\mathcal{F}(n)$ class of isometric colligations, which connects the representation of a Schur-Agler class function to its Schur-Agler class factors. Section 2.4 deals with similar factorization results in the setting of the unit ball in $\mathbb{C}^n$. In Section 2.5, we will discuss factorizations of Schur-Agler class functions vanishing at the origin. The concluding Section 2.6 outlines some concrete examples and presents results concerning one variable factors of Schur-Agler class functions and a remark on the reversibility of our method of factorizations.

This chapter is based on the published paper [48].

2.2 Factorizations and Property $\mathcal{F}_m(n)$

In this section, we present results concerning factorizations of Schur-Agler class functions in $\mathcal{SA}(\mathbb{D}^n)$, $n > 1$, into Schur-Agler class factors with fewer variables. More specifically, our interest here is to identify (and then classify) isometric colligations V such that $\tau_V \in \mathcal{SA}(\mathbb{D}^n)$ and

$$\tau_V(\boldsymbol{z}) = \phi(z_1, \dots, z_m)\psi(z_{m+1}, \dots, z_n) \qquad (\boldsymbol{z} \in \mathbb{D}^n),$$

for some (canonical, in terms of V) $\phi \in \mathcal{SA}(\mathbb{D}^m)$ and $\psi \in \mathcal{SA}(\mathbb{D}^{n-m})$. Throughout this section we will always assume that $1 \leq m < n$.

Recall that, given $1 \leq m < p \leq n$ and Hilbert spaces $\mathcal{H}_1, \dots, \mathcal{H}_n$, we set

$$\mathcal{H}_m^p = \mathcal{H}_m \oplus \mathcal{H}_{m+1} \oplus \cdots \oplus \mathcal{H}_p.$$

In particular, $\mathcal{H}_1^n = \bigoplus_{i=1}^n \mathcal{H}_i$. Moreover, with respect to the orthogonal decomposition $\mathcal{H}_1^n = \mathcal{H}_1^m \oplus \mathcal{H}_{m+1}^n$, we represent an operator $D \in \mathcal{B}(\mathcal{H}_1^n)$ as

$$D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1^m \oplus \mathcal{H}_{m+1}^n).$$

Similarly, if $\mathcal{E}$ and $\mathcal{E}_*$ are Hilbert spaces, $B \in \mathcal{B}(\mathcal{H}_1^n, \mathcal{E})$ and $C \in \mathcal{B}(\mathcal{E}_*, \mathcal{H}_1^n)$, then we write

$$B = \begin{bmatrix} B_1 & B_2 \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1^m \oplus \mathcal{H}_{m+1}^n, \mathcal{E}) \quad \text{and} \quad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \in \mathcal{B}(\mathcal{E}_*, \mathcal{H}_1^m \oplus \mathcal{H}_{m+1}^n).$$

Now we are ready to introduce the central object of this section.

Definition 2.2.1. *Let $1 \leq m < n$. We say that an isometry $V \in \mathcal{B}(\mathcal{H})$ satisfies property $\mathcal{F}_m(n)$ if there exist Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ such that $\mathcal{H} = \mathbb{C} \oplus \mathcal{H}_1^n$, and representing V as*

$$V = \begin{bmatrix} a & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1^m \oplus \mathcal{H}_{m+1}^n),$$

one has $D_{21} = 0$ and $aD_{12} = C_1 B_2$.

More specifically, an isometry $V \in \mathcal{B}(\mathcal{H})$ satisfies property $\mathcal{F}_m(n)$ if there exist Hilbert spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ such that $\mathcal{H} = \mathbb{C} \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$, and writing V as

$$V = \left[\begin{array}{c|ccc} a & B_1 & \cdots & B_n \\ \hline C_1 & D_{11} & \cdots & D_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ C_n & D_{n1} & \cdots & D_{nn} \end{array}\right],$$

on $\mathbb{C} \oplus (\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n)$, one has

$$D_{ij} = 0,$$

for all $i = m+1, \ldots, n$ and $j = 1, \ldots, m$, and

$$aD_{ij} = C_i B_j,$$

for all $i = 1, \ldots, m$ and $j = m+1, \ldots, n$. By way of example, we consider the two variables situation. We say that an isometry V satisfies property $\mathcal{F}_1(2)$ if there exist Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ such that

$$V = \begin{bmatrix} a & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & 0 & D_{22} \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2)$$

and $aD_{12} = C_1 B_2$.

Let us introduce some more notation. Let $1 \leq m < p \leq n$. We set

$$E_{\mathcal{H}_m^p}(\boldsymbol{z}) = z_m I_{\mathcal{H}_m} \oplus \cdots \oplus z_p I_{\mathcal{H}_p} \qquad (\boldsymbol{z} \in \mathbb{C}^n).$$

Also for $X \in \mathcal{B}(\mathcal{H}_m^p)$, $\|X\| \leq 1$, define

$$R_m^p(\boldsymbol{z}, X) = \Big(I_{\mathcal{H}_m^p} - E_{\mathcal{H}_m^p}(\boldsymbol{z})X\Big)^{-1} \qquad (\boldsymbol{z} \in \mathbb{D}^n).$$

Note that $R_m^p(\boldsymbol{z}, X)$ is a function of $\{z_m, \ldots, z_p\}$ variables. Moreover, we will denote $R_1^n(\boldsymbol{z}, X)$ simply by $R(\boldsymbol{z}, X)$.

Now we proceed to prove that a pair of isometric colligations is naturally associated with an isometric colligation satisfying property $\mathcal{F}_m(n)$. More specifically, given $\tau_{V_1} \in$

$\mathcal{SA}(\mathbb{D}^m)$ and $\tau_{V_2} \in \mathcal{SA}(\mathbb{D}^{n-m})$ for some isometric colligations V_1 and V_2, we aim to construct an explicit isometric colligation V such that V satisfies property $\mathcal{F}_m(n)$ and

$$\tau_V(\boldsymbol{z}) = \tau_{V_1}(z_1, \dots, z_m)\tau_{V_2}(z_{m+1}, \dots, z_n) \qquad (\boldsymbol{z} \in \mathbb{D}^n).$$

To this end, let $\mathcal{H}_1, \dots, \mathcal{H}_n$ be Hilbert spaces. Suppose

$$V_1 = \begin{bmatrix} a_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1^m), \quad \text{and} \quad V_2 = \begin{bmatrix} a_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_{m+1}^n),$$

are isometric colligations. Define $\tilde{V}_1$ and $\tilde{V}_2$ in $\mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1^m \oplus \mathcal{H}_{m+1}^n)$ by

$$\tilde{V}_1 = \begin{bmatrix} a_1 & B_1 & 0 \\ C_1 & D_1 & 0 \\ 0 & 0 & I \end{bmatrix} \quad \text{and} \quad \tilde{V}_2 = \begin{bmatrix} a_2 & 0 & B_2 \\ 0 & I & 0 \\ C_2 & 0 & D_2 \end{bmatrix},$$

and set $V = \tilde{V}_1\tilde{V}_2$. It is easy to check, by swapping rows and columns (of $\tilde{V}_2$), that $\tilde{V}_1$ and $\tilde{V}_2$ are isometries and thus the isometric colligation

$$V = \left[\begin{array}{c|cc} a_1a_2 & B_1 & a_1B_2 \\ \hline a_2C_1 & D_1 & C_1B_2 \\ C_2 & 0 & D_2 \end{array}\right] \in \mathcal{B}\Big(\mathbb{C} \oplus (\mathcal{H}_1^m \oplus \mathcal{H}_{m+1}^n)\Big),$$

satisfies property $\mathcal{F}_m(n)$. Let $\boldsymbol{z} \in \mathbb{D}^n$. Clearly

$$\tau_V(\boldsymbol{z}) = a_1a_2 + \begin{bmatrix} B_1 & a_1B_2 \end{bmatrix} R(\boldsymbol{z}, \begin{bmatrix} D_1 & C_1B_2 \\ 0 & D_2 \end{bmatrix}) E_{\mathcal{H}_1^n}(\boldsymbol{z}) \begin{bmatrix} a_2C_1 \\ C_2 \end{bmatrix},$$

where

$$\begin{aligned} R\Big(\boldsymbol{z}, \begin{bmatrix} D_1 & C_1B_2 \\ 0 & D_2 \end{bmatrix}\Big)^{-1} &= I_{\mathcal{H}_1^n} - \begin{bmatrix} E_{\mathcal{H}_1^m}(\boldsymbol{z}) & 0 \\ 0 & E_{\mathcal{H}_{m+1}^n}(\boldsymbol{z}) \end{bmatrix} \begin{bmatrix} D_1 & C_1B_2 \\ 0 & D_2 \end{bmatrix} \\ &= \begin{bmatrix} I_{\mathcal{H}_1^m} - E_{\mathcal{H}_1^m}(\boldsymbol{z})D_1 & -E_{\mathcal{H}_1^m}(\boldsymbol{z})C_1B_2 \\ 0 & I_{\mathcal{H}_{m+1}^n} - E_{\mathcal{H}_{m+1}^n}(\boldsymbol{z})D_2 \end{bmatrix}. \end{aligned}$$

By the inverse formula of an invertible upper triangular matrix, it follows that

$$R\Big(\boldsymbol{z}, \begin{bmatrix} D_1 & C_1B_2 \\ 0 & D_2 \end{bmatrix}\Big) = \begin{bmatrix} R_1^m(\boldsymbol{z}, D_1) & R_1^m(\boldsymbol{z}, D_1)E_{\mathcal{H}_1^m}(\boldsymbol{z})C_1B_2R_{m+1}^n(\boldsymbol{z}, D_2) \\ 0 & R_{m+1}^n(\boldsymbol{z}, D_2) \end{bmatrix}.$$

We now infer, in view of the above equality, that

$$\begin{aligned}
\tau_V(\boldsymbol{z}) &= a_1a_2 + \begin{bmatrix} B_1 & a_1B_2 \end{bmatrix} R\Big(\boldsymbol{z}, \begin{bmatrix} D_1 & C_1B_2 \\ 0 & D_2 \end{bmatrix}\Big) E_{\mathcal{H}_1^n}(\boldsymbol{z}) \begin{bmatrix} a_2C_1 \\ C_2 \end{bmatrix} \\
&= a_1a_2 + \begin{bmatrix} B_1 & a_1B_2 \end{bmatrix} \begin{bmatrix} R_1^m(\boldsymbol{z}, D_1) & R_1^m(\boldsymbol{z}, D_1)E_{\mathcal{H}_1^m}(\boldsymbol{z})C_1B_2R_{m+1}^n(\boldsymbol{z}, D_2) \\ 0 & R_{m+1}^n(\boldsymbol{z}, D_2) \end{bmatrix} \\
&\qquad \times \begin{bmatrix} a_2E_{\mathcal{H}_1^m}(\boldsymbol{z})C_1 \\ E_{\mathcal{H}_{m+1}^n}(\boldsymbol{z})C_2 \end{bmatrix} \\
&= a_1a_2 + a_2B_1R_1^m(\boldsymbol{z}, D_1)E_{\mathcal{H}_1^m}(\boldsymbol{z})C_1 + a_1B_2R_{m+1}^n(\boldsymbol{z}, D_2)E_{\mathcal{H}_{m+1}^n}(\boldsymbol{z})C_2 \\
&\quad + B_1R_1^m(\boldsymbol{z}, D_1)E_{\mathcal{H}_1^m}(\boldsymbol{z})C_1B_2R_{m+1}^n(\boldsymbol{z}, D_2)E_{\mathcal{H}_{m+1}^n}(\boldsymbol{z})C_2 \\
&= \Big(a_1 + B_1R_1^m(\boldsymbol{z}, D_1)E_{\mathcal{H}_1^m}(\boldsymbol{z})C_1\Big)\Big(a_2 + B_2R_{m+1}^n(\boldsymbol{z}, D_2)E_{\mathcal{H}_{m+1}^n}(\boldsymbol{z})C_2\Big) \\
&= \tau_{V_1}(z_1, \ldots, z_m)\tau_{V_2}(z_{m+1}, \ldots, z_n),
\end{aligned}$$

for all $\boldsymbol{z} \in \mathbb{D}^n$. We have therefore proved the following result:

Theorem 2.2.2. *Let $1 \leq m < n$, and let $\mathcal{H}_1, \ldots, \mathcal{H}_n$ be Hilbert spaces. Suppose*

$$V_1 = \begin{bmatrix} a_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus (\bigoplus_{i=1}^m \mathcal{H}_i)) \quad \textit{and} \quad V_2 = \begin{bmatrix} a_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus (\bigoplus_{i=m+1}^n \mathcal{H}_i)),$$

are isometric colligations. Define $\tilde{V}_1$, $\tilde{V}_2$ and V in $\mathcal{B}\Big(\mathbb{C} \oplus \Big((\bigoplus_{i=1}^m \mathcal{H}_i) \oplus (\bigoplus_{i=m+1}^n \mathcal{H}_i)\Big)\Big)$ by

$$\tilde{V}_1 = \begin{bmatrix} a_1 & B_1 & 0 \\ C_1 & D_1 & 0 \\ 0 & 0 & I \end{bmatrix} \quad \textit{and} \quad \tilde{V}_2 = \begin{bmatrix} a_2 & 0 & B_2 \\ 0 & I & 0 \\ C_2 & 0 & D_2 \end{bmatrix},$$

and $V = \tilde{V}_1\tilde{V}_2$, respectively. Then

$$V = \left[\begin{array}{c|cc} a_1a_2 & B_1 & a_1B_2 \\ \hline a_2C_1 & D_1 & C_1B_2 \\ C_2 & 0 & D_2 \end{array}\right] \in \mathcal{B}\Big(\mathbb{C} \oplus \Big((\bigoplus_{i=1}^m \mathcal{H}_i) \oplus (\bigoplus_{i=m+1}^n \mathcal{H}_i)\Big)\Big),$$

is an isometric colligation, V satisfies property $\mathcal{F}_m(n)$ and

$$\tau_V(\boldsymbol{z}) = \tau_{V_1}(z_1, \ldots, z_m)\tau_{V_2}(z_{m+1}, \ldots, z_n) \qquad (\boldsymbol{z} \in \mathbb{D}^n).$$

Now to prove the reverse direction, *we assume in addition that* $\tau_V(0) \neq 0$ (for the case of transfer functions vanishing at the origin, see Section 2.5) : Suppose $\mathcal{H}_1, \ldots, \mathcal{H}_n$ are Hilbert spaces and

$$V = \begin{bmatrix} a & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & 0 & D_{22} \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1^m \oplus \mathcal{H}_{m+1}^n), \tag{2.2.1}$$

is an isometric colligation satisfying property $\mathcal{F}_m(n)$. Thus

$$aD_{12} = C_1 B_2. \tag{2.2.2}$$

Suppose $a := \tau_V(0) \neq 0$. Since $V^*V = I$, we have

$$|a|^2 + C_1^* C_1 + C_2^* C_2 = 1,$$

implies that

$$1 - C_2^* C_2 = |a|^2 + C_1^* C_1 > 0,$$

as $a \neq 0$. Then there exists a scalar β, $0 < |\beta| \leq 1$, such that

$$|\beta|^2 = 1 - C_2^* C_2.$$

It now follows that

$$C_1^* C_1 = |\beta|^2 - |a|^2, \tag{2.2.3}$$

and

$$\alpha := \frac{a}{\beta}, \tag{2.2.4}$$

is a non-zero scalar. Define

$$V_1 = \begin{bmatrix} \alpha & B_1 & 0 \\ \frac{1}{\beta} C_1 & D_{11} & 0 \\ 0 & 0 & I \end{bmatrix} \quad \text{and} \quad V_2 = \begin{bmatrix} \beta & 0 & \frac{1}{\alpha} B_2 \\ 0 & I & 0 \\ C_2 & 0 & D_{22} \end{bmatrix},$$

on $\mathbb{C} \oplus \mathcal{H}_1^m \oplus \mathcal{H}_{m+1}^n$. It follows from (2.2.3) and (2.2.4) that

$$|\alpha|^2 + \frac{1}{|\beta|^2} C_1^* C_1 = |\alpha|^2 + \frac{1}{|\beta|^2}(|\beta|^2 - |a|^2) = 1 + |\alpha|^2 - \frac{|a|^2}{|\beta|^2} = 1,$$

that is

$$|\alpha|^2 + \frac{1}{|\beta|^2} C_1^* C_1 = 1. \tag{2.2.5}$$

Also, we see that $B_1^* B_1 + D_{11}^* D_{11} = I$, and

$$\bar{\alpha} B_1 + \frac{1}{\bar{\beta}} C_1^* D_{11} = \frac{1}{\bar{\beta}}(\bar{\alpha}\bar{\beta} B_1 + C_1^* D_{11}) = \frac{1}{\bar{\beta}}(\bar{a} B_1 + C_1^* D_{11}) = 0,$$

and hence $V_1^* V_1 = I$. We now proceed to prove that V_2 is also an isometry. First, it easy to see that $\bar{a} B_2 + C_1^* D_{12} + C_2^* D_{22} = 0$, and hence, by (2.2.2), we have

$$0 = \bar{a} B_2 + C_1^* D_{12} + C_2^* D_{22} = \bar{a} B_2 + \frac{1}{a} C_1^* C_1 B_2 + C_2^* D_{22} = \frac{\bar{a}_2}{\alpha}(|\alpha|^2 + \frac{1}{|\beta|^2} C_1^* C_1) B_2 + C_2^* D_{22}.$$

Then (2.2.5) implies that $\frac{\bar{\beta}}{\alpha} B_2 + C_2^* D_{22} = 0$. Finally, again from $V^*V = I$ we get

$$B_2^* B_2 + D_{12}^* D_{12} + D_{22}^* D_{22} = I.$$

Now again by (2.2.2) we have

$$B_2^*B_2 + D_{12}^*D_{12} + D_{22}^*D_{22} = B_2^*(1 + \frac{1}{|a|^2}C_1^*C_1)B_2 + D_{22}^*D_{22}$$
$$= \frac{1}{|\alpha|^2}B_2^*(|\alpha|^2 + \frac{1}{|\beta|^2}C_1^*C_1)B_2 + D_{22}^*D_{22},$$

so that $\frac{1}{|\alpha|^2}B_2^*B_2 + D_{22}^*D_{22} = I$, by (2.2.5), from which we conclude that $V_2^*V_2 = I$. Finally, notice that

$$V_1V_2 = \begin{bmatrix} \alpha\beta & B_1 & B_2 \\ C_1 & D_{11} & \frac{1}{\alpha\beta}C_1B_2 \\ C_2 & 0 & D_{22} \end{bmatrix} = \begin{bmatrix} a & B_1 & B_2 \\ C_1 & D_{11} & \frac{1}{a}C_1B_2 \\ C_2 & 0 & D_{22} \end{bmatrix},$$

and hence $V = V_1V_2$, by (2.2.2). Then, by Theorem 2.2.2, we have

$$\tau_V(\boldsymbol{z}) = \tau_{\tilde{V}_1}(z_1, \ldots, z_m)\tau_{\tilde{V}_2}(z_{m+1}, \ldots, z_n),$$

for all $\boldsymbol{z} \in \mathbb{D}^n$ where $\tilde{V}_1 = \begin{bmatrix} \alpha & B_1 \\ \frac{1}{\beta}C_1 & D_{11} \end{bmatrix}$ and $\tilde{V}_2 = \begin{bmatrix} \beta & \frac{1}{\alpha}B_2 \\ C_2 & D_{22} \end{bmatrix}$. Thus we have proved the following statement:

Theorem 2.2.3. *Suppose $\mathcal{H}_1, \ldots, \mathcal{H}_n$ are Hilbert spaces and a be a non-zero scalar. If*

$$V = \left[\begin{array}{c|cc} a & B_1 & B_2 \\ \hline C_1 & D_{11} & \frac{1}{a}C_1B_2 \\ C_2 & 0 & D_{22} \end{array}\right] \in \mathcal{B}\Big(\mathbb{C} \oplus \big((\bigoplus_{i=1}^{m}\mathcal{H}_i) \oplus (\bigoplus_{i=m+1}^{n}\mathcal{H}_i)\big)\Big),$$

is an isometric colligation, then

$$\tilde{V}_1 = \begin{bmatrix} \alpha & B_1 \\ \frac{1}{\beta}C_1 & D_{11} \end{bmatrix} \quad \text{and} \quad \tilde{V}_2 = \begin{bmatrix} \beta & \frac{1}{\alpha}B_2 \\ C_2 & D_{22} \end{bmatrix}.$$

are isometric colligations in $\mathcal{B}\Big(\mathbb{C} \oplus (\bigoplus_{i=1}^{m}\mathcal{H}_i)\Big)$ and $\mathcal{B}\Big(\mathbb{C} \oplus (\bigoplus_{i=m+1}^{n}\mathcal{H}_i)\Big)$, respectively, and

$$\tau_V(\boldsymbol{z}) = \tau_{\tilde{V}_1}(z_1, \ldots, z_m)\tau_{\tilde{V}_2}(z_{m+1}, \ldots, z_n) \qquad (\boldsymbol{z} \in \mathbb{D}^n),$$

where α and β are non-zero scalars and satisfies the following conditions

$$|\beta|^2 = |a|^2 + C_1^*C_1 \quad \text{and} \quad \alpha = \frac{a}{\beta}.$$

Summing up the results of Theorems 2.2.2 and 2.2.3, we conclude the following factorization theorem on Schur-Agler class functions in $\mathcal{SA}(\mathbb{D}^n)$, $n \geq 2$:

Theorem 2.2.4. *Let $1 \leq m < n$, and let $\theta \in \mathcal{SA}(\mathbb{D}^n)$. If $\theta(0) \neq 0$, then*

$$\theta(\boldsymbol{z}) = \phi(z_1, \ldots, z_m)\psi(z_{m+1}, \ldots, z_n) \qquad (\boldsymbol{z} \in \mathbb{D}^n),$$

for some $\phi \in \mathcal{SA}(\mathbb{D}^m)$ and $\psi \in \mathcal{SA}(\mathbb{D}^{n-m})$ if and only if

$$\theta(\boldsymbol{z}) = \tau_V(\boldsymbol{z}) \qquad (\boldsymbol{z} \in \mathbb{D}^n),$$

for some isometric colligation V satisfying property $\mathcal{F}_m(n)$.

2.3 Factorizations and Property $\mathcal{F}(n)$

In this section we investigate general n-variables Schur-Agler class factors of Schur-Agler class functions in $\mathcal{SA}(\mathbb{D}^n)$. More specifically, for a given $\theta \in \mathcal{SA}(\mathbb{D}^n)$, we give a set of necessary and sufficient conditions on isometric colligations ensuring the existence of φ and ψ in $\mathcal{SA}(\mathbb{D}^n)$ such that $\theta = \varphi\psi$. We identify a new class of isometric colligations, namely $\mathcal{F}(n)$, and prove that the (Schur-Agler class) factors of Schur-Agler class functions are completely determined by isometric colligations satisfying property $\mathcal{F}(n)$. Here we do not set any restriction on n, that is, we will assume that $n \geq 1$.

We first identify the relevant isometric colligations:

Definition 2.3.1. *We say that an isometry $V \in \mathcal{B}(\mathcal{H})$ satisfies property $\mathcal{F}(n)$ if there exist Hilbert spaces $\{\mathcal{M}_i\}_{i=1}^n$ and $\{\mathcal{N}_i\}_{i=1}^n$ such that*

$$\mathcal{H} = \mathbb{C} \oplus \Big(\bigoplus_{i=1}^n (\mathcal{M}_i \oplus \mathcal{N}_i) \Big),$$

and representing V as

$$V = \left[\begin{array}{c|ccc} a & B_1 & \cdots & B_n \\ \hline C_1 & D_{11} & \cdots & D_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ C_n & D_{n1} & \cdots & D_{nn} \end{array}\right] \in \mathcal{B}\Big(\mathbb{C} \oplus \Big(\bigoplus_{i=1}^n (\mathcal{M}_i \oplus \mathcal{N}_i) \Big)\Big),$$

and B_i, C_i and D_{ij} as

$$B_i = \begin{bmatrix} B_i(1) & B_i(2) \end{bmatrix} \in \mathcal{B}(\mathcal{M}_i \oplus \mathcal{N}_i, \mathbb{C}), \quad C_i = \begin{bmatrix} C_i(1) \\ C_i(2) \end{bmatrix} \in \mathcal{B}(\mathbb{C}, \mathcal{M}_i \oplus \mathcal{N}_i),$$

and

$$D_{ij} = \begin{bmatrix} D_{ij}(1) & D_{ij}(12) \\ D_{ij}(21) & D_{ij}(2) \end{bmatrix} \in \mathcal{B}(\mathcal{M}_j \oplus \mathcal{N}_j, \mathcal{M}_i \oplus \mathcal{N}_i),$$

one has

$$D_{ij}(21) = 0, \quad \text{and} \quad aD_{ij}(12) = C_i(1)B_j(2),$$

for all $i, j = 1, \ldots, n$.

As in Section 2.2, here we also first prove that a pair of isometric colligations is naturally associated with an isometric colligation satisfying property $\mathcal{F}(n)$. Let $\{\mathcal{M}_i\}_{i=1}^n$ and $\{\mathcal{N}_i\}_{i=1}^n$ be Hilbert spaces, and let

$$V_1 = \begin{bmatrix} \alpha & B \\ C & D \end{bmatrix} = \left[\begin{array}{c|ccc} \alpha & B_1 & \cdots & B_n \\ \hline C_1 & D_{11} & \cdots & D_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ C_n & D_{n1} & \cdots & D_{nn} \end{array}\right] \in \mathcal{B}(\mathbb{C} \oplus \mathcal{M}_1^n),$$

and

$$V_2 = \begin{bmatrix} \beta & F \\ G & H \end{bmatrix} = \left[\begin{array}{c|ccc} \beta & F_1 & \cdots & F_n \\ \hline G_1 & H_{11} & \cdots & H_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ G_n & H_{n1} & \cdots & H_{nn} \end{array}\right] \in \mathcal{B}(\mathbb{C} \oplus \mathcal{N}_1^n),$$

be isometric colligations. Given $i = 1, \ldots, n$, we define $\mathcal{H}_i = \mathcal{M}_i \oplus \mathcal{N}_i$, and bounded linear operators $\tilde{B}_i$, $\tilde{C}_i$ and $\tilde{D}_{ij}$ as

$$\tilde{B}_i = \begin{bmatrix} B_i & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{H}_i, \mathbb{C}),\; \tilde{C}_i = \begin{bmatrix} C_i \\ 0 \end{bmatrix} \in \mathcal{B}(\mathbb{C}, \mathcal{H}_i),\; \text{and } \tilde{D}_{ij} = \begin{bmatrix} D_{ij} & 0 \\ 0 & \delta_{ij} I \end{bmatrix} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i),$$

for all $i, j = 1, \ldots, n$. Set

$$\tilde{V}_1 = \left[\begin{array}{c|ccc} \alpha & \tilde{B}_1 & \cdots & \tilde{B}_n \\ \hline \tilde{C}_1 & \tilde{D}_{11} & \cdots & \tilde{D}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{C}_n & \tilde{D}_{n1} & \cdots & \tilde{D}_{nn} \end{array}\right] \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1^n). \tag{2.3.1}$$

On the other hand, let

$$\tilde{V}_2 = \left[\begin{array}{c|ccc} \beta & \tilde{F}_1 & \cdots & \tilde{F}_n \\ \hline \tilde{G}_1 & \tilde{H}_{11} & \cdots & \tilde{H}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{G}_n & \tilde{H}_{n1} & \cdots & \tilde{H}_{nn} \end{array}\right] \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1^n), \tag{2.3.2}$$

where

$$\tilde{F}_i = \begin{bmatrix} 0 & F_i \end{bmatrix} \in \mathcal{B}(\mathcal{H}_i, \mathbb{C}),\; \tilde{G}_i = \begin{bmatrix} 0 \\ G_i \end{bmatrix} \in \mathcal{B}(\mathbb{C}, \mathcal{H}_i),\; \text{and } \tilde{H}_{ij} = \begin{bmatrix} \delta_{ij} I & 0 \\ 0 & H_{ij} \end{bmatrix} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i),$$

for all $i, j = 1, \ldots, n$. Define $V = \tilde{V}_1 \tilde{V}_2$. It then follows that $V \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1^n)$ is an isometry and

$$V = \left[\begin{array}{c|ccc} \alpha\beta & \hat{B}_1 & \cdots & \hat{B}_n \\ \hline \hat{C}_1 & \hat{D}_{11} & \cdots & \hat{D}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{C}_n & \hat{D}_{n1} & \cdots & \hat{D}_{nn} \end{array}\right] := \begin{bmatrix} \alpha\beta & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}, \tag{2.3.3}$$

where

$$\hat{B}_i = \begin{bmatrix} B_i & \alpha F_i \end{bmatrix} \in \mathcal{B}(\mathcal{H}_i, \mathbb{C}), \quad \hat{C}_i = \begin{bmatrix} \beta C_i \\ G_i \end{bmatrix} \in \mathcal{B}(\mathbb{C}, \mathcal{H}_i), \tag{2.3.4}$$

and

$$\hat{D}_{ij} = \begin{bmatrix} D_{ij} & C_i F_j \\ 0 & H_{ij} \end{bmatrix} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i), \tag{2.3.5}$$

for all $i, j = 1, \ldots, n$. Define $X(\boldsymbol{z}) : \mathbb{C} \to \mathbb{C}$, $\boldsymbol{z} \in \mathbb{D}^n$, by

$$X(\boldsymbol{z}) = \hat{B}(I_{\mathcal{H}_1^n} - E_{\mathcal{H}}(\boldsymbol{z})\hat{D})^{-1} E_{\mathcal{H}}(\boldsymbol{z})\hat{C}.$$

Then $\tau_V(\boldsymbol{z}) = \alpha\beta + X(\boldsymbol{z})$, $\boldsymbol{z} \in \mathbb{D}^n$. Next, define the flip operator $\eta : \mathcal{H}_1^n \to \mathcal{M}_1^n \oplus \mathcal{N}_1^n$, by

$$\eta\Big(\bigoplus_{i=1}^n (f_i \oplus g_i)\Big) = (\bigoplus_{i=1}^n f_i) \oplus (\bigoplus_{i=1}^n g_i), \tag{2.3.6}$$

for all $f_i \in \mathcal{M}_i$ and $g_i \in \mathcal{N}_i$, $i = 1, \ldots, n$. Then η is a unitary operator and so

$$X(\boldsymbol{z}) = (\hat{B}\eta^*)\Big(I_{\mathcal{M}_1^n \oplus \mathcal{N}_1^n} - (\eta E_{\mathcal{H}_1^n}(\boldsymbol{z})\eta^*)(\eta\hat{D}\eta^*)\Big)^{-1} (\eta E_{\mathcal{H}_1^n}(\boldsymbol{z})\eta^*)(\eta\hat{C}).$$

On the other hand, the definition of the flip operator η reveals that

$$\hat{B}\eta^* = \begin{bmatrix} B & \alpha F \end{bmatrix}, \ \eta\hat{C} = \begin{bmatrix} \beta C \\ G \end{bmatrix}, \ \eta\hat{D}\eta^* = \begin{bmatrix} D & CF \\ 0 & H \end{bmatrix},$$

and

$$\eta E_{\mathcal{H}_1^n}(\boldsymbol{z})\eta^* = \begin{bmatrix} E_{\mathcal{M}_1^n}(\boldsymbol{z}) & 0 \\ 0 & E_{\mathcal{N}_1^n}(\boldsymbol{z}) \end{bmatrix}.$$

In particular, this yields

$$I_{\mathcal{M}_1^n \oplus \mathcal{N}_1^n} - (\eta E_{\mathcal{H}_1^n}(\boldsymbol{z})\eta^*)(\eta\hat{D}\eta^*) = \begin{bmatrix} I - E_{\mathcal{M}_1^n}(\boldsymbol{z})D & -E_{\mathcal{M}_1^n}(\boldsymbol{z})CF \\ 0 & I - E_{\mathcal{N}_1^n}(\boldsymbol{z})H \end{bmatrix} \qquad (\boldsymbol{z} \in \mathbb{D}^n).$$

In order to further ease the notation, for Hilbert spaces $\{\mathcal{S}_i\}_{i=1}^n$ and $\boldsymbol{z} \in \mathbb{D}^n$, we set

$$E_{\mathcal{S}}(\boldsymbol{z}) = \bigoplus_{i=1}^n z_i I_{\mathcal{S}_i},$$

and, for $Y \in \mathcal{B}(\bigoplus_{i=1}^{n} \mathcal{S}_i)$, $\|Y\| \leq 1$, define $r(\boldsymbol{z}, Y) = \Big(I_{\mathcal{S}_1^n} - E_{\mathcal{S}}(\boldsymbol{z})Y\Big)^{-1}$.

Continuing the above computation, for each $\boldsymbol{z} \in \mathbb{D}^n$, we now have

$$\Big(I_{\mathcal{M}_1^n \oplus \mathcal{N}_1^n} - (\eta E_{\mathcal{H}_1^n}(\boldsymbol{z})\eta^*)(\eta \hat{D} \eta^*)\Big)^{-1} = \begin{bmatrix} r(\boldsymbol{z}, D) & r(\boldsymbol{z}, D)E_{\mathcal{M}_1^n}(\boldsymbol{z})CFr(\boldsymbol{z}, H) \\ 0 & r(\boldsymbol{z}, H) \end{bmatrix}.$$

Moreover, since $(\eta E_{\mathcal{H}_1^n}(\boldsymbol{z})\eta^*)(\eta \hat{C}) = \begin{bmatrix} \beta E_{\mathcal{M}_1^n}(\boldsymbol{z})C \\ E_{\mathcal{N}_1^n}(\boldsymbol{z})G \end{bmatrix}$, it follows that

$$X(\boldsymbol{z}) = \begin{bmatrix} B & \alpha F \end{bmatrix} \begin{bmatrix} \beta r(\boldsymbol{z}, D)E_{\mathcal{M}_1^n}(\boldsymbol{z})C + r(\boldsymbol{z}, D)E_{\mathcal{M}_1^n}(\boldsymbol{z})CFr(\boldsymbol{z}, H)E_{\mathcal{N}_1^n}(\boldsymbol{z})G \\ r(\boldsymbol{z}, H)E_{\mathcal{N}_1^n}(\boldsymbol{z})G \end{bmatrix}$$
$$= \beta Br(\boldsymbol{z}, D)E_{\mathcal{M}_1^n}(\boldsymbol{z})C + Br(\boldsymbol{z}, D)E_{\mathcal{M}_1^n}(\boldsymbol{z})CFr(\boldsymbol{z}, H)E_{\mathcal{N}_1^n}(\boldsymbol{z})G + \alpha Fr(\boldsymbol{z}, H)E_{\mathcal{N}_1^n}(\boldsymbol{z})G,$$

and so $\tau_V(\boldsymbol{z}) = \tau_{V_1}(\boldsymbol{z})\tau_{V_2}(\boldsymbol{z})$, $\boldsymbol{z} \in \mathbb{D}^n$. We have therefore proved:

Theorem 2.3.2. *Suppose* $V_1 = \begin{bmatrix} \alpha & B \\ C & D \end{bmatrix} \in \mathcal{B}\Big(\mathbb{C} \oplus (\bigoplus_{i=1}^{n} \mathcal{M}_i)\Big)$ *and* $V_2 = \begin{bmatrix} \beta & F \\ G & H \end{bmatrix} \in \mathcal{B}\Big(\mathbb{C} \oplus (\bigoplus_{i=1}^{n} \mathcal{N}_i)\Big)$ *are isometric colligations, and let* $V = \tilde{V}_1\tilde{V}_2$*, where* $\tilde{V}_1$ *and* $\tilde{V}_2$ *are as in* (2.3.1) *and* (2.3.2)*, respectively. Then the isometric colligation* $V \in \mathcal{B}\Big(\mathbb{C} \oplus (\bigoplus_{i=1}^{n}(\mathcal{M}_i \oplus \mathcal{N}_i))\Big)$ *as in* (2.3.3) *satisfies property* $\mathcal{F}(n)$ *and* $\tau_V = \tau_{V_1}\tau_{V_2}$.

We have the following interpretations of the above theorem: Let $\theta, \phi, \psi \in \mathcal{SA}(\mathbb{D}^n)$, and suppose $\theta = \phi\psi$. Suppose $V_1 = \begin{bmatrix} \alpha & B \\ C & D \end{bmatrix}$ and $V_2 = \begin{bmatrix} \beta & F \\ G & H \end{bmatrix}$ are isometric colligations on $\mathbb{C} \oplus \mathcal{M}_1^n$ and $\mathbb{C} \oplus \mathcal{N}_1^n$, respectively, and $\phi = \tau_{V_1}$, and $\psi = \tau_{V_2}$. Then the isometric colligation $V = \tilde{V}_1\tilde{V}_2$, as constructed in Theorem 2.3.2, satisfies property $\mathcal{F}(n)$ and $\tau_V(\boldsymbol{z}) = \tau_{V_1}(\boldsymbol{z})\tau_{V_2}(\boldsymbol{z})$ for all $\boldsymbol{z} \in \mathbb{D}^n$, that is, $\theta = \tau_V$.

Now we proceed to treat the converse of Theorem 2.3.2. Let $V \in \mathcal{B}(\mathcal{H})$ be an isometric colligation, and let V satisfies property $\mathcal{F}(n)$. As in Theorem 2.2.3, here also we assume that $a := \tau_V(0) \neq 0$. Now

$$\mathcal{H} = \mathbb{C} \oplus \Big(\bigoplus_{i=1}^{n}(\mathcal{M}_i \oplus \mathcal{N}_i)\Big),$$

for some Hilbert spaces $\{\mathcal{M}_i\}_{i=1}^n$ and $\{\mathcal{N}_i\}_{i=1}^n$, and

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} = \left[\begin{array}{c|ccc} a & B_1 & \cdots & B_n \\ \hline C_1 & D_{11} & \cdots & D_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ C_n & D_{n1} & \cdots & D_{nn} \end{array}\right], \tag{2.3.7}$$

where

$$B_i = \begin{bmatrix} B_i(1) & B_i(2) \end{bmatrix} \in \mathcal{B}(\mathcal{M}_i \oplus \mathcal{N}_i, \mathbb{C}), \quad C_i = \begin{bmatrix} C_i(1) \\ C_i(2) \end{bmatrix} \in \mathcal{B}(\mathbb{C}, \mathcal{M}_i \oplus \mathcal{N}_i), \tag{2.3.8}$$

and

$$D_{ij} = \begin{bmatrix} D_{ij}(1) & \frac{1}{a} C_i(1) B_j(2) \\ 0 & D_{ij}(2) \end{bmatrix} \in \mathcal{B}(\mathcal{M}_j \oplus \mathcal{N}_j, \mathcal{M}_i \oplus \mathcal{N}_i), \tag{2.3.9}$$

for all $i, j = 1, \ldots, n$. Set

$$D(1) = \begin{bmatrix} D_{ij}(1) \end{bmatrix}_{i,j=1}^n \in \mathcal{B}(\bigoplus_{i=1}^n \mathcal{M}_i), \quad D(2) = \begin{bmatrix} D_{ij}(2) \end{bmatrix}_{i,j=1}^n \in \mathcal{B}(\bigoplus_{i=1}^n \mathcal{N}_i), \tag{2.3.10}$$

and

$$D(12) = \begin{bmatrix} D_{ij}(12) \end{bmatrix}_{i,j=1}^n \in \mathcal{B}\Big(\bigoplus_{i=1}^n \mathcal{M}_i, \bigoplus_{i=1}^n \mathcal{N}_i\Big),$$

and consider the flip operator $\eta : \Big(\bigoplus_{i=1}^n (\mathcal{M}_i \oplus \mathcal{N}_i)\Big) \to (\bigoplus_{i=1}^n \mathcal{M}_i) \oplus (\bigoplus_{i=1}^n \mathcal{N}_i)$ (see (2.3.6)). Then

$$\eta D \eta^* = \begin{bmatrix} D(1) & D(12) \\ 0 & D(2) \end{bmatrix} \in \mathcal{B}\Big((\bigoplus_{i=1}^n \mathcal{M}_i) \oplus (\bigoplus_{i=1}^n \mathcal{N}_i)\Big).$$

If we define $V_\eta := \begin{bmatrix} 1 & 0 \\ 0 & \eta \end{bmatrix} V \begin{bmatrix} 1 & 0 \\ 0 & \eta \end{bmatrix}^*$, it then follows that $V_\eta = \begin{bmatrix} a & B\eta^* \\ \eta C & \eta D \eta^* \end{bmatrix}$ is an isometry on $\bigoplus_{i=1}^n (\mathcal{M}_i \oplus \mathcal{N}_i)$. Moreover, since $B\eta^* = \begin{bmatrix} B(1) & B(2) \end{bmatrix}$ and $\eta C = \begin{bmatrix} C(1) & C(2) \end{bmatrix}^t$, we see that

$$V_\eta = \begin{bmatrix} a & B(1) & B(2) \\ C(1) & D(1) & \frac{1}{a} C(1) B(2) \\ C(2) & 0 & D(2) \end{bmatrix} \in \mathcal{B}\Big(\mathbb{C} \oplus (\bigoplus_{i=1}^n \mathcal{M}_i) \oplus (\bigoplus_{i=1}^n \mathcal{N}_i)\Big),$$

where

$$B(i) = \begin{bmatrix} B_1(i) & B_2(i) \end{bmatrix} \quad \text{and} \quad C(i) = \begin{bmatrix} C_1(i) \\ C_2(i) \end{bmatrix}, \tag{2.3.11}$$

for all $i = 1, 2$. We have now arrived at the setting of the proof of Theorem 2.2.4 (more specifically, compare V_η with V in (2.2.1)). Following the constructions of V_1 and V_2 in the proof of Theorem 2.2.4, we set

$$\begin{cases} V_1 = \begin{bmatrix} \alpha & B(1) \\ \frac{1}{\beta} C(1) & D(1) \end{bmatrix} \in \mathcal{B}\Big(\mathbb{C} \oplus (\bigoplus_{i=1}^n \mathcal{M}_i)\Big) \\[2ex] V_2 = \begin{bmatrix} \beta & \frac{1}{\alpha} B(2) \\ C(2) & D(2) \end{bmatrix} \in \mathcal{B}\Big(\mathbb{C} \oplus (\bigoplus_{i=1}^n \mathcal{N}_i)\Big), \end{cases} \tag{2.3.12}$$

where

$$\begin{aligned}|\beta|^2 &= |a|^2 + C(1)^*C(1)\\ &= 1 - C(2)^*C(2),\end{aligned} \tag{2.3.13}$$

and

$$\alpha = \frac{a}{\beta}. \tag{2.3.14}$$

Since $a \neq 0$, it follows that α (and β too) is a non-zero scalars. One may now proceed, similarly as in the proof of Theorem 2.2.4, to see that V_1 and V_2 are isometries. Then, applying Theorem 2.3.2 to the pair of isometries V_1 and V_2, we get the canonical pair of isometries $\tilde{V}_1$ and $\tilde{V}_2$ such that $\tau_{\tilde{V}_1\tilde{V}_2} = \tau_{V_1}\tau_{V_2}$. On the other hand, it follows directly from the construction of $\tilde{V}_1$ and $\tilde{V}_2$ (see (2.3.3)) that $V = \tilde{V}_1\tilde{V}_2$ and consequently, $\tau_V = \tau_{\tilde{V}_1\tilde{V}_2} = \tau_{V_1}\tau_{V_2}$. We have therefore proved the following counterpart of Theorem 2.2.3 for isometric colligations satisfying property $\mathcal{F}(n)$.

Theorem 2.3.3. *Let $V \in \mathcal{B}(\mathbb{C} \oplus (\bigoplus_{i=1}^{n}(\mathcal{M}_i \oplus \mathcal{N}_i)))$ be an isometric colligation, and let V satisfies property $\mathcal{F}(n)$. If $\tau_V(0) \neq 0$ and V admits the representation as in (2.3.7) with B, C and D as in (2.3.8) and (2.3.9), respectively, then*

$$V_1 = \begin{bmatrix} \alpha & B(1) \\ \frac{1}{\beta}C(1) & D(1)\end{bmatrix} \in \mathcal{B}\Big(\mathbb{C}\oplus(\bigoplus_{i=1}^{n}\mathcal{M}_i)\Big) \text{ and } V_2 = \begin{bmatrix} \beta & \frac{1}{\alpha}B(2) \\ C(2) & D(2)\end{bmatrix} \in \mathcal{B}\Big(\mathbb{C}\oplus(\bigoplus_{i=1}^{n}\mathcal{N}_i)\Big),$$

are isometric colligations where $B(i), C(i)$ and $D(i)$ are as in (2.3.10) and (2.3.11) and α and β are non-zero scalars and satisfies the following conditions

$$|\beta|^2 = |a|^2 + C(1)^*C(1) \quad \textit{and} \quad \alpha = \frac{a}{\beta}.$$

Moreover, $\tau_V = \tau_{V_1}\tau_{V_2}$.

This along with Theorem 2.3.2 yields the following classification of Schur-Agler class factors of Schur-Agler class functions in $\mathcal{SA}(\mathbb{D}^n)$, $n \geq 1$:

Theorem 2.3.4. *Suppose $\theta \in \mathcal{SA}(\mathbb{D}^n)$, and suppose that $\theta(0) \neq 0$. Then $\theta = \phi\psi$ for some $\phi, \psi \in \mathcal{SA}(\mathbb{D}^n)$ if and only if $\theta = \tau_V$ for some isometric colligation V satisfying property $\mathcal{F}(n)$.*

Given $\theta = \tau_V$ for some isometric colligation V satisfying property $\mathcal{F}(n)$, as presented above, we now know that $\theta = \varphi\psi$ for some $\phi, \psi \in \mathcal{SA}(\mathbb{D}^n)$. If V admits the representation as in (2.3.7), then it follows moreover from (2.3.12) that

$$\begin{cases} \phi(\boldsymbol{z}) = \alpha + \frac{1}{\beta}B(1)(I_{\mathcal{M}_1^n} - E_{\mathcal{M}_1^n}(\boldsymbol{z})D(1))^{-1}E_{\mathcal{M}_1^n}(\boldsymbol{z})C(1) \\ \psi(\boldsymbol{z}) = \beta + \frac{1}{\alpha}B(2)(I_{\mathcal{N}_1^n} - E_{\mathcal{N}_1^n}(\boldsymbol{z})D(2))^{-1}E_{\mathcal{N}_1^n}(\boldsymbol{z})C(2) \qquad (\boldsymbol{z} \in \mathbb{D}^n). \end{cases} \tag{2.3.15}$$

The assumption that $\theta(0) \neq 0$ in the proof of the sufficient part will be discussed in Section 2.5. Also see Subsection 2.6.3 for a natural connection between $\mathcal{F}_m(n)$ and $\mathcal{F}(n)$, $1 \leq m < n$.

2.4 Factorizations of multipliers on the ball

In this section, we continue with our study of factorizations of bounded analytic functions. Here we are interested in factorizations of multipliers of the Drury-Arveson space on the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ [16]. However (and curiously, if not surprisingly), the techniques involved in representing multiplier factors of multipliers of the Drury-Arveson space seems relatively simpler than that of the Schur-Agler class functions on $\mathbb{D}^n$. Moreover, since the proofs of the following results are often similar (in spirit) to the case of $\mathcal{SA}(\mathbb{D}^n)$, we will be rather sketchy.

The following simple observation on isometric colligations is quite useful: Suppose $\mathcal{H}$ is a Hilbert space and

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} : \mathbb{C} \oplus \mathcal{H} \to \mathbb{C} \oplus \mathcal{H}^n,$$

is a bounded linear operator. Then V is an isometry (that is $V^*V = I_{\mathbb{C}\oplus\mathcal{H}}$) if and only if

$$\begin{cases} |a|^2 + C^*C = 1 \\ \bar{a}B + C^*D = 0 \\ B^*B + D^*D = I_{\mathcal{H}}. \end{cases} \tag{2.4.1}$$

Before proceeding we make a brief remark concerning notation: Given Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, in what follows, we represent an operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K}^n)$ as

$$A = \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix}.$$

We now proceed to prove the first factorization result for contractive multipliers of the Drury-Arveson space. Suppose $1 \leq m < n$, $\theta \in \mathcal{M}_1(H^2_n)$, $\varphi \in \mathcal{M}_1(H^2_m)$ and $\psi \in \mathcal{M}_1(H^2_{n-m})$. Suppose that

$$\theta(\boldsymbol{z}) = \varphi(z_1, \ldots, z_m)\phi(z_{m+1}, \ldots, z_n) \qquad (\boldsymbol{z} \in \mathbb{B}^n).$$

Then there exist Hilbert spaces $\mathcal{M}$ and $\mathcal{N}$ and isometric colligations

$$V_1 = \begin{bmatrix} \alpha & E \\ F & G \end{bmatrix} \in \mathcal{B}(\mathbb{C}\oplus\mathcal{M}, \mathbb{C}\oplus\mathcal{M}^m) \qquad \text{and} \qquad V_2 = \begin{bmatrix} \beta & Y \\ Z & W \end{bmatrix} \in \mathcal{B}(\mathbb{C}\oplus\mathcal{N}, \mathbb{C}\oplus\mathcal{N}^{n-m}).$$

such that $\phi = \tau_{V_1}$ and $\psi = \tau_{V_2}$. Define

$$\mathcal{H} = \mathcal{M} \oplus \mathcal{N},$$

and

$$V = \begin{bmatrix} \alpha\beta & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}, \mathbb{C} \oplus \mathcal{H}^n), \tag{2.4.2}$$

where $B = \begin{bmatrix} E & \alpha Y \end{bmatrix}$ and

$$C_j = \begin{cases} \begin{bmatrix} \beta F_j \\ 0 \end{bmatrix} & \text{if } 1 \le j \le m \\ \\ \begin{bmatrix} 0 \\ Z_{j-m} \end{bmatrix} & \text{if } m+1 \le j \le n, \end{cases}$$

and

$$D_j = \begin{cases} \begin{bmatrix} G_j & F_j Y \\ 0 & 0 \end{bmatrix} & \text{if } 1 \le j \le m \\ \\ \begin{bmatrix} 0 & 0 \\ 0 & W_{j-m} \end{bmatrix} & \text{if } m+1 \le j \le n. \end{cases}$$

By taking into account of isometric properties of V_1 and V_2 and using property (2.4.1) repeatedly, we conclude that V is an isometry in $\mathcal{B}(\mathbb{C} \oplus \mathcal{H}, \mathbb{C} \oplus \mathcal{H}^n)$. We now compute

$$\begin{aligned}
(I_{\mathcal{H}} - E_{\mathcal{H}^n}(\boldsymbol{z})D)^{-1} &= (I_{\mathcal{H}} - \sum_{i=1}^{n} z_i D_i)^{-1} \\
&= (I_{\mathcal{H}} - \sum_{j=1}^{m} z_j D_j - \sum_{j=m+1}^{n} z_j D_j)^{-1} \\
&= \left(\begin{bmatrix} I_{\mathcal{M}} & 0 \\ 0 & I_{\mathcal{N}} \end{bmatrix} - \sum_{j=1}^{m} z_j \begin{bmatrix} G_j & F_j Y \\ 0 & 0 \end{bmatrix} - \sum_{j=m+1}^{n} z_j \begin{bmatrix} 0 & 0 \\ 0 & W_{j-m} \end{bmatrix} \right)^{-1} \\
&= \begin{bmatrix} I_{\mathcal{M}} - \sum_{j=1}^{m} z_j G_j & -\sum_{j=1}^{m} z_j F_j Y \\ 0 & I_{\mathcal{N}} - \sum_{j=m+1}^{n} z_j W_{j-m} \end{bmatrix}^{-1} \\
&= \begin{bmatrix} (I_{\mathcal{M}} - \sum_{j=1}^{m} z_j G_j)^{-1} & T \\ 0 & (I_{\mathcal{N}} - \sum_{j=m+1}^{n} z_j W_{j-m})^{-1} \end{bmatrix},
\end{aligned}$$

where
$$T = (I_{\mathcal{M}} - \sum_{j=1}^{m} z_j G_j)^{-1}\Big(\sum_{j=1}^{m} z_j F_j\Big) Y (I_{\mathcal{N}} - \sum_{j=m+1}^{n} z_j W_{j-m})^{-1}.$$
Moreover, since
$$E_{\mathcal{H}^n}(\boldsymbol{z})C = \begin{bmatrix} \beta \sum\limits_{j=1}^{m} z_j F_j \\ \sum\limits_{j=m+1}^{n} z_j Z_{j-m} \end{bmatrix},$$
it follows that
$$\begin{aligned}
\tau_V(\boldsymbol{z}) &= \alpha\beta + B(I_{\mathcal{H}} - E_{\mathcal{H}^n}(\boldsymbol{z})D)^{-1} E_{\mathcal{H}^n}(\boldsymbol{z})C \\
&= \alpha\beta + \begin{bmatrix} E & \alpha Y \end{bmatrix} (I_{\mathcal{H}} - E_{\mathcal{H}^n}(\boldsymbol{z})D)^{-1} E_{\mathcal{H}^n}(\boldsymbol{z})C \\
&= \alpha\beta + \beta E (I_{\mathcal{M}} - \sum_{j=1}^{m} z_j G_j)^{-1}\Big(\sum_{j=1}^{m} z_j F_j\Big) + ET \sum_{j=m+1}^{n} z_j Z_{j-m} \\
&\quad + \alpha Y (I_{\mathcal{N}} - \sum_{j=m+1}^{n} z_j W_{j-m})^{-1} \sum_{j=m+1}^{n} z_j Z_{j-m} \\
&= \tau_{V_1}(z_1, \dots, z_m)\tau_{V_2}(z_{m+1}, \dots, z_n),
\end{aligned}$$
for all $\boldsymbol{z} \in \mathbb{D}^n$. Since $\theta = \tau_{V_1}\tau_{V_2}$, it then follows that
$$\theta = \tau_V,$$
where V is the isometric colligation as in (2.4.2). We have thus proved part of the following theorem.

Theorem 2.4.1. *Suppose $\theta \in \mathcal{M}_1(H^2_n)$ and $\theta(0) \neq 0$. Then the following are equivalent:*

(1) There exist multipliers $\phi \in \mathcal{M}_1(H^2_m)$ and $\psi \in \mathcal{M}_1(H^2_{n-m})$ such that
$$\theta(\boldsymbol{z}) = \phi(z_1, \dots, z_m)\psi(z_{m+1}, \dots, z_n) \qquad (\boldsymbol{z} \in \mathbb{B}^n).$$

(2) There exist Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and isometric colligation
$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} : \mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2) \to \mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2)^n,$$
such that writing $B = \begin{bmatrix} B(1) & B(2) \end{bmatrix}$, $C = \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}$ and $D = \begin{bmatrix} D_1 \\ \vdots \\ D_n \end{bmatrix}$, one has

$$C_j = \begin{cases} \begin{bmatrix} C_j(1) \\ 0 \end{bmatrix} & \text{if } 1 \leq j \leq m \\ \\ \begin{bmatrix} 0 \\ C_j(2) \end{bmatrix} & \text{if } m+1 \leq j \leq n, \end{cases}$$

and

$$D_j = \begin{cases} \begin{bmatrix} D_j(1) & D_j(2) \\ 0 & 0 \end{bmatrix} & \text{if } 1 \leq j \leq m \\ \\ \begin{bmatrix} 0 & 0 \\ 0 & D_j(3) \end{bmatrix} & \text{if } m+1 \leq j \leq n, \end{cases}$$

and

$$aD_i(2) = C_i(1)B(2),$$

for all $i = 1, \ldots, m$, *and*

$$\theta(\boldsymbol{z}) = a + B(I_{\mathcal{H}_1 \oplus \mathcal{H}_2} - E_{(\mathcal{H}_1 \oplus \mathcal{H}_2)^n}(\boldsymbol{z})D)^{-1} E_{(\mathcal{H}_1 \oplus \mathcal{H}_2)^n}(\boldsymbol{z})C \qquad (\boldsymbol{z} \in \mathbb{B}^n).$$

Proof. We only need to show that (2) implies (1). Note that $V^*V = I$ implies, in particular, that

$$|a|^2 + \sum_{j=1}^{m} C_j(1)^* C_j(1) + \sum_{j=m+1}^{n} C_j(2)^* C_j(2) = 1.$$

Let

$$|\beta|^2 = |a|^2 + \sum_{j=1}^{m} C_j(1)^* C_j(1).$$

Since $a = \theta(0) \neq 0$, it follows that $\beta \neq 0$. Set

$$\alpha = \frac{a}{\beta},$$

and define

$$V_1 = \begin{bmatrix} \alpha & B(1) \\ \frac{1}{\beta} C_1(1) & D_1(1) \\ \vdots & \vdots \\ \frac{1}{\beta} C_m(1) & D_m(1) \end{bmatrix} \quad \text{and} \quad V_2 = \begin{bmatrix} \beta & \frac{1}{\alpha} B(2) \\ C_{m+1}(2) & D_{m+1}(3) \\ \vdots & \vdots \\ C_n(2) & D_n(3) \end{bmatrix}.$$

Clearly $V_1 \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1, \mathbb{C} \oplus (\mathcal{H}_1)^n)$ and $V_2 \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_2, \mathbb{C} \oplus (\mathcal{H}_2)^n)$. Now in view of (2.4.1), V_1 is an isometry if and only if

$$\begin{cases} |\alpha|^2 + \sum_{j=1}^{m} \frac{1}{|\beta|^2} C_j(1)^* C_j(1) = 1 \\ \bar{\alpha} B(1) + \sum_{j=1}^{m} \frac{1}{\beta} C_j(1)^* D_j(1) = 0 \\ B(1)^* B(1) + \sum_{j=1}^{m} D_j(1)^* D_j(1) = I. \end{cases}$$

Here the first equality is a simple consequence of the definitions of α and β. The second and the third equalities follows from the isometric property of V applied to the equality (2.4.1). The proof of the fact that V_2 is an isometry is similar (but requires the fact that $aD_j(2) = C_j(2)B(2)$ for all $1 \leq j \leq m$) and left to the reader. Then

$$\tau_{V_1} \in \mathcal{M}_1(H^2_m) \quad \text{and} \quad \tau_{V_2} \in \mathcal{M}_1(H^2_{n-m}).$$

We set

$$\varphi(z_1, \ldots, z_m) = \tau_{V_1}(z_1, \ldots, z_m) \quad \text{and} \quad \psi(z_{m+1}, \ldots, z_n) = \tau_{V_2}(z_{m+1}, \ldots, z_n),$$

for all $\boldsymbol{z} \in \mathbb{B}^n$. Then

$$\phi(z_1, \ldots, z_m) = \alpha + \frac{1}{\beta} B(1)\Big(1 - \sum_{j=1}^{m} z_j D_j(1)\Big)^{-1}\Big(\sum_{j=1}^{m} z_j C_j(1)\Big),$$

and

$$\psi(z_{m+1}, \ldots, z_n) = \beta + \frac{1}{\alpha} B(2)\Big(1 - \sum_{j=m+1}^{n} z_j D_j(3)\Big)^{-1}\Big(\sum_{j=m+1}^{n} z_j C_j(2)\Big),$$

for all $\boldsymbol{z} \in \mathbb{B}^n$. It is now routine to check that this indeed defines the required factors of θ, that is

$$\theta(\boldsymbol{z}) = \varphi(z_1, \ldots, z_m)\psi(z_{m+1}, \ldots, z_n) \qquad (\boldsymbol{z} \in \mathbb{B}^n),$$

which completes the proof that (1) and (2) are equivalent. □

The Drury-Arveson multipliers analog of Theorem 2.3.4, as stated below, also holds. We leave the similar verification, following the line of the proof of Theorem 2.3.4, to the reader.

Theorem 2.4.2. *Suppose $\theta \in \mathcal{M}_1(H^2_n)$ and $\theta(0) \neq 0$. Then the following are equivalent:*

(1) There exist ϕ and ψ in $\mathcal{M}_1(H^2_n)$ such that $\theta = \phi\psi$.

(2) There exist Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ and isometric colligation

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} : \mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2) \to \mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2)^n,$$

such that

$$\theta(\boldsymbol{z}) = a + B(I_{\mathcal{H}_1\oplus\mathcal{H}_2} - E_{(\mathcal{H}_1\oplus\mathcal{H}_2)^n}(\boldsymbol{z})D)^{-1}E_{(\mathcal{H}_1\oplus\mathcal{H}_2)^n}(\boldsymbol{z})C \qquad (\boldsymbol{z}\in\mathbb{B}^n),$$

and writing $B = \begin{bmatrix} B(1) & B(2) \end{bmatrix}$, $C = \begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix}$, $D = \begin{bmatrix} D_1 \\ \vdots \\ D_n \end{bmatrix}$ *and*

$$C_i = \begin{bmatrix} C_i(1) \\ C_i(2) \end{bmatrix} \quad \text{and} \quad D_i = \begin{bmatrix} D_i(1) & D_i(12) \\ D_i(21) & D_i(2) \end{bmatrix},$$

one has

$$D_i(21) = 0 \quad \text{and} \quad aD_i(12) = C_i(1)B(2),$$

for all $i = 1, \dots, n$.

2.5 Functions vanishing at the origin

As pointed out in Remark 2.1.1, factorizations of functions vanishing at the origin reveals more detailed structural properties of associated colligation matrices. To this end, in this section, we present a complete description of the connection between isometric colligations and Schur-Agler factors of Schur-Agler class functions vanishing at the origin. The case of one variable Schur functions will serve well to illustrate the notation scheme for functions in several variables that we adopt.

Suppose $\theta \in \mathcal{S}(\mathbb{D})$, $\theta(0) = 0$ and $\theta = \varphi\psi$ for some ϕ and ψ in $\mathcal{S}(\mathbb{D})$. The following two cases can arise:

Case (i) $\phi(0) = 0$ and $\psi(0) \neq 0$: Let $\varphi = \tau_{V_1}$ and $\psi = \tau_{V_2}$ where

$$V_1 = \begin{bmatrix} 0 & Q \\ R & S \end{bmatrix} \in \mathcal{B}(\mathbb{C}\oplus\mathcal{H}_1) \quad \text{and} \quad V_2 = \begin{bmatrix} x & Y \\ Z & W \end{bmatrix} \in \mathcal{B}(\mathbb{C}\oplus\mathcal{H}_2).$$

Therefore

$$\tilde{V}_1 = \left[\begin{array}{c|cc} 0 & Q & 0 \\ \hline R & S & 0 \\ 0 & 0 & I \end{array}\right] \quad \text{and} \quad \tilde{V}_2 = \left[\begin{array}{c|cc} x & 0 & Y \\ \hline 0 & I & 0 \\ Z & 0 & W \end{array}\right],$$

are isometries in $\mathcal{B}(\mathbb{C}\oplus(\mathcal{H}_1\oplus\mathcal{H}_2))$. On defining $V := \tilde{V}_1\tilde{V}_2$, we have the isometry

$$V = \left[\begin{array}{c|cc} 0 & B_1 & 0 \\ \hline C_1 & D_1 & D_2 \\ C_2 & 0 & D_4 \end{array}\right] \in \mathcal{B}(\mathbb{C}\oplus(\mathcal{H}_1\oplus\mathcal{H}_2)), \tag{2.5.1}$$

where

$$\left[\begin{array}{c|cc} 0 & B_1 & 0 \\ \hline C_1 & D_1 & D_2 \\ C_2 & 0 & D_4 \end{array}\right] = \left[\begin{array}{c|cc} 0 & Q & 0 \\ \hline xR & S & RY \\ Z & 0 & W \end{array}\right].$$

We then have $C_1 = xR$ and $D_2 = RY$, and consequently the condition $R^*R = 1$ yields

$$C_1C_1^*D_2 = |x|^2RR^*D_2 = |x|^2RR^*RY = |x|^2RY = |x|^2D_2 = C_1^*C_1D_2,$$

as $C_1^*C_1 = |x|^2(> 0)$. Moreover, with V as in (2.5.1), we compute τ_V as:

$$\begin{aligned}
\tau_V(z) &= \begin{bmatrix} B_1 & 0 \end{bmatrix} \left(I - z \begin{bmatrix} D_1 & D_2 \\ 0 & D_4 \end{bmatrix} \right)^{-1} z \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \\
&= z \begin{bmatrix} B_1 & 0 \end{bmatrix} \begin{bmatrix} (I - zD_1)^{-1} & (I - zD_1)^{-1} zD_2 (I - zD_4)^{-1} \\ 0 & (I - zD_4)^{-1} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \\
&= z \begin{bmatrix} B_1(I - zD_1)^{-1} & zB_1(I - zD_1)^{-1}D_2(I - zD_4)^{-1} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},
\end{aligned}$$

and so

$$\tau_V(z) = \left(zB_1(I - zD_1)^{-1} \right) \left(C_1 + zD_2(I - zD_4)^{-1}C_2 \right) \qquad (z \in \mathbb{D}). \tag{2.5.2}$$

Substituting the values of B_1, C_i, and D_j, $i = 1, 2$ and $j = 2, 4$, we have

$$\begin{aligned}
\tau_V(z) &= \left(zB_1(I - zD_1)^{-1} \right) \left(C_1 + zD_2(I - zD_4)^{-1}C_2 \right) \\
&= \left(zQ(I - zD_1)^{-1} \right) \left(xR + zRY(I - zW)^{-1}Z \right) \\
&= \left(zQ(I - zS)^{-1}R \right) \left(x + zY(I - zW)^{-1}Z \right),
\end{aligned}$$

for all $z \in \mathbb{D}$, which implies that $\theta = \tau_V$. Thus, we have collected together all the necessary properties of the isometric colligation V as:

$$C_1C_1^*D_2 = C_1^*C_1D_2 \quad \text{and} \quad C_1^*C_1 > 0. \tag{2.5.3}$$

Conversely, suppose V is an isometric colligation as in (2.5.1), let $\theta = \tau_V$ and let V satisfies the conditions in (2.5.3). Let x be a non-zero scalar such that

$$|x|^2 = C_1^*C_1.$$

Define $V_1 \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1)$ and $V_2 \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_2)$ by

$$V_1 = \begin{bmatrix} 0 & B_1 \\ \frac{1}{x}C_1 & D_1 \end{bmatrix} \quad \text{and} \quad V_2 = \begin{bmatrix} x & \frac{1}{\bar{x}}C_1^*D_2 \\ C_2 & D_4 \end{bmatrix}.$$

Note that $|x|^2 = 1 - C_2^* C_2 = C_1^* C_1$. A simple computation, following (2.4.1), then shows that V_1 and V_2 are isometric colligations. Now we compute

$$\tau_V(z) = zB_1(1 - zD_1)^{-1}C_1 + z^2 B_1(1 - zD_1)^{-1} D_2(1 - zD_4)^{-1} C_2,$$

and

$$\tau_{V_1}(z)\tau_{V_2}(z) = zB_1(1 - zD_1)^{-1}C_1 + z^2 B_1(1 - zD_1)^{-1}\{\frac{1}{|x|^2} C_1 C_1^* D_2\}(1 - zD_4)^{-1} C_2.$$

Thus, $\tau_V = \tau_{V_1}\tau_{V_2}$ where $\tau_V(0) = \tau_{V_1}(0) = 0$ and $\tau_{V_2}(0) \neq 0$.

Case (ii) $\phi(0) = \psi(0) = 0$: Suppose $\phi = \tau_{V_1}$ and $\psi = \tau_{V_2}$ where

$$V_1 = \begin{bmatrix} 0 & Q \\ R & S \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1) \quad \text{and} \quad V_2 = \begin{bmatrix} 0 & Y \\ Z & W \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_2),$$

are isometric colligations. We associate with V_1 and V_2 the isometric colligation

$$V = \begin{bmatrix} 0 & Q & 0 \\ R & S & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & 0 & Y \\ 0 & I & 0 \\ Z & 0 & W \end{bmatrix} = \left[\begin{array}{c|cc} 0 & Q & 0 \\ \hline 0 & S & RY \\ Z & 0 & W \end{array}\right],$$

in $\mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2)$ and set

$$V = \left[\begin{array}{c|cc} 0 & B_1 & 0 \\ \hline 0 & D_1 & D_2 \\ C_2 & 0 & D_4 \end{array}\right]. \tag{2.5.4}$$

Then, in view of (2.5.2), it follows that $\theta = \tau_V$. Also we pick the essential properties of the isometric colligation V as

$$X^* X = I, \quad X^* D_1 = 0, \quad \text{and} \quad D_2 = XY, \tag{2.5.5}$$

where $X = R$. Note that the first two equalities follows from the fact that V_1 is an isometry.

To prove the converse, suppose V is an isometric colligation as in (2.5.4), $\theta = \tau_V$, $X \in \mathcal{B}(\mathbb{C}, \mathcal{H}_2)$ is an isometry, $Y \in \mathcal{B}(\mathcal{H}_2, \mathbb{C})$ and the conditions in (2.5.5) hold. Since $V^* V = I$, we have

$$\begin{bmatrix} C_2^* C_2 & 0 & C_2^* D_4 \\ 0 & B_1 B_1^* + D_1^* D_1 & D_1^* D_2 \\ D_4^* C_2 & D_2^* D_1 & D_2^* D_2 + D_4 D_4^* \end{bmatrix} = I_{\mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2},$$

and hence $V_1 := \begin{bmatrix} 0 & B_1 \\ X & D_1 \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1)$ is an isometric colligation. Since $D_2 = XY$, $D_2^* D_2 = Y^* Y$, and hence $D_2^* D_2 + D_4^* D_4 = I$ yields $Y^* Y + D_4^* D_4 = I$. Thus $V_2 :=$

$\begin{bmatrix} 0 & Y \\ C_2 & D_4 \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_2)$ is an isometric colligation. Notice that

$$\tau_{V_1}(z)\tau_{V_2}(z) = z^2 B_1(1 - zD_1)^{-1} XY(1 - zD_4)^{-1} C_2,$$

and, on the other hand, in view of (2.5.2), we have

$$\tau_V(z) = z^2 B_1(1 - zD_1)^{-1} D_2(1 - zD_4)^{-1} C_2,$$

for all $z \in \mathbb{D}$. This and $XY = D_2$ implies that $\theta = \tau_V = \tau_{V_1}\tau_{V_2}$.

Thus we have proved the following:

Theorem 2.5.1. *Suppose $\theta \in \mathcal{S}(\mathbb{D})$ and $\theta(0) = 0$. Then:*

(1) $\theta = \phi\psi$ for some $\phi, \psi \in \mathcal{S}(\mathbb{D})$ and $\psi(0) \neq 0$ if and only if there exists an isometric colligation

$$V = \left[\begin{array}{c|cc} 0 & B_1 & 0 \\ \hline C_1 & D_1 & D_2 \\ C_2 & 0 & D_4 \end{array}\right] \in \mathcal{B}(\mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2)),$$

such that $\theta = \tau_V$ and

$$C_1 C_1^* D_2 = C_1^* C_1 D_2 \quad \textit{and} \quad C_1^* C_1 > 0.$$

(2) $\theta = \phi\psi$ for some $\phi, \psi \in \mathcal{S}(\mathbb{D})$ and $\phi(0) = 0 = \psi(0)$ if and only if there exists an isometric colligation

$$V = \left[\begin{array}{c|cc} 0 & B_1 & 0 \\ \hline 0 & D_1 & D_2 \\ C_2 & 0 & D_4 \end{array}\right] \in \mathcal{B}(\mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2)),$$

such that $\theta = \tau_V$ and

$$X^* D_1 = 0 \quad \textit{and} \quad D_2 = XY,$$

for some isometry $X \in \mathcal{B}(\mathbb{C}, \mathcal{H}_2)$ and bounded linear operator $Y \in \mathcal{B}(\mathcal{H}_2, \mathbb{C})$.

The general case of functions vanishing at the origin in several variables (in $\mathcal{SA}(\mathbb{D}^n)$ or $\mathcal{M}_1(H^2_n)$) can be studied using the technique developed in the proof of Theorem 2.5.1. In particular, similar arguments allow us to obtain also a similar classification of factorizations for functions in $\mathcal{SA}(\mathbb{D}^n)$ vanishing at the origin. We only state the result in the setting of Section 2.3 and leave out the details to the reader.

Theorem 2.5.2. *Suppose $\theta \in \mathcal{AS}(\mathbb{D}^n)$ and $\theta(0) = 0$. Then:*

(1) $\theta = \phi\psi$ for some $\phi, \psi \in \mathcal{SA}(\mathbb{D}^n)$ and $\psi(0) \neq 0$ if and only if there exist Hilbert spaces $\{\mathcal{H}_i\}_{i=1}^n$, $\{\mathcal{M}_i\}_{i=1}^n$ and $\{\mathcal{N}_i\}_{i=1}^n$ and an isometric colligation

$$V = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} = \left[\begin{array}{c|ccc} 0 & B_1 & \cdots & B_n \\ \hline C_1 & D_{11} & \cdots & D_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ C_n & D_{n1} & \cdots & D_{nn} \end{array}\right] \in \mathcal{B}(\mathbb{C} \oplus (\bigoplus_{i=1}^n \mathcal{H}_i)),$$

such that $\theta = \tau_V$ and $\mathcal{H}_k = \mathcal{M}_k \oplus \mathcal{N}_k$, $k = 1, \ldots, n$, and representing B_i, C_i an D_{ij} as

$$B_i = [B_i(1), B_i(2)] \in \mathcal{B}(\mathcal{M}_i \oplus \mathcal{N}_i, \mathbb{C}), \quad C_i = \begin{bmatrix} C_i(1) \\ C_i(2) \end{bmatrix} \in \mathcal{B}(\mathbb{C}, \mathcal{M}_i \oplus \mathcal{N}_i),$$

and

$$D_{ij} = \begin{bmatrix} D_{ij}(1) & D_{ij}(12) \\ D_{ij}(21) & D_{ij}(2) \end{bmatrix} \in \mathcal{B}(\mathcal{M}_j \oplus \mathcal{N}_j, \mathcal{M}_i \oplus \mathcal{N}_i),$$

one has

$$B_i(2) = 0, \quad D_{ij}(21) = 0, \quad C(1)C(1)^*D(12) = C(1)^*C(1)D(12) \quad \text{and} \quad C(1)^*C(1) > 0,$$

where $i, j = 1, \ldots, n$, and $C(1) = \begin{bmatrix} C_1(1) \\ \vdots \\ C_n(1) \end{bmatrix} \in \mathcal{B}(\mathbb{C}, \bigoplus_{i=1}^n \mathcal{M}_i)$ *and*

$$D(12) = \Big[D_{ij}(12)\Big]_{i,j=1}^n \in \mathcal{B}\Big(\bigoplus_{p=1}^n \mathcal{N}_p, \bigoplus_{p=1}^n \mathcal{M}_p\Big).$$

(2) $\theta = \phi\psi$ for some $\phi, \psi \in \mathcal{SA}(\mathbb{D}^n)$ and $\phi(0) = 0 = \psi(0)$ if and only if there exist Hilbert spaces $\{\mathcal{H}_i\}_{i=1}^n$, $\{\mathcal{M}_i\}_{i=1}^n$ and $\{\mathcal{N}_i\}_{i=1}^n$, an isometry $X \in \mathcal{B}(\mathbb{C}, \bigoplus_{i=1}^n \mathcal{M}_i)$, a bounded linear operator $Y \in \mathcal{B}(\bigoplus_{i=1}^n \mathcal{N}_i, \mathbb{C})$ and an isometric colligation

$$V = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix} = \left[\begin{array}{c|ccc} 0 & B_1 & \cdots & B_n \\ \hline C_1 & D_{11} & \cdots & D_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ C_n & D_{n1} & \cdots & D_{nn} \end{array}\right] \in \mathcal{B}\Big(\mathbb{C} \oplus (\bigoplus_{i=1}^n \mathcal{H}_i)\Big),$$

such that $\theta = \tau_V$ and $\mathcal{H}_k = \mathcal{M}_k \oplus \mathcal{N}_k$, $k = 1, \ldots, n$, and representing B_i, C_i an D_{ij} as

$$B_i = [B_i(1), B_i(2)] \in \mathcal{B}\Big(\mathcal{M}_i \oplus \mathcal{N}_i, \mathbb{C}\Big), \quad C_i = \begin{bmatrix} C_i(1) \\ C_i(2) \end{bmatrix} \in \mathcal{B}\Big(\mathbb{C}, \mathcal{M}_i \oplus \mathcal{N}_i\Big),$$

and

$$D_{ij} = \begin{bmatrix} D_{ij}(1) & D_{ij}(12) \\ D_{ij}(21) & D_{ij}(2) \end{bmatrix} \in \mathcal{B}(\mathcal{M}_j \oplus \mathcal{N}_j, \mathcal{M}_i \oplus \mathcal{N}_i),$$

one has

$$B_i(2) = 0, \quad C_i(1) = 0, \quad D_{ij}(21) = 0,$$

and

$$D(12) = XY \quad \textit{and} \quad X^* D(1) = 0,$$

where

$$D(1) = [D_{ij}(1)]_{i,j=1}^n \in \mathcal{B}\Big(\bigoplus_{p=1}^n \mathcal{M}_p\Big),$$

and

$$D(12) = \Big[D_{ij}(12)\Big]_{i,j=1}^n \in \mathcal{B}\Big(\bigoplus_{p=1}^n \mathcal{N}_p, \bigoplus_{p=1}^n \mathcal{M}_p\Big).$$

The case of contractive multipliers of the Drury-Arveson space vanishing at the origin can be stated and proved in a similar way.

2.6 Examples and remarks

This section is devoted to some concrete examples, further results and general remarks concerning Schur-Agler class functions in $\mathcal{SA}(\mathbb{D}^n)$.

2.6.1 One variable factors

Our interest here is to analyze Schur-Agler class functions in $\mathcal{SA}(\mathbb{D}^n)$ which can be factored as a product of n Schur functions. More specifically, let $\varphi \in \mathcal{SA}(\mathbb{D}^n)$ and let $\varphi(0) \neq 0$. Suppose

$$\varphi(\boldsymbol{z}) = \prod_{i=1}^n \varphi_i(z_i) \qquad (\boldsymbol{z} \in \mathbb{D}^n),$$

for some $\varphi_i \in \mathcal{S}(\mathbb{D})$, $i = 1, \ldots, n$. Then there exist isometric colligations

$$V_i = \begin{bmatrix} a_1 & \hat{B}_i \\ \hat{C}_i & \hat{D}_i \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_i),$$

such that

$$\varphi_i = \tau_{V_i},$$

for all $i = 1, \ldots, n$. Let

$$a = \prod_{i=1}^n a_i,$$

and define

$$\hat{V}_1 = \begin{bmatrix} a_1 & \hat{B}_1 & 0 \\ \hat{C}_1 & \hat{D}_1 & 0 \\ 0 & 0 & I \end{bmatrix} \quad \text{and} \quad \tilde{V}_n = \begin{bmatrix} a_n & 0 & \hat{B}_n \\ 0 & I & 0 \\ \hat{C}_n & 0 & \hat{D}_n \end{bmatrix},$$

in $\mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2^n)$ and $\mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1^{n-1} \oplus \mathcal{H}_n)$, respectively, and

$$\hat{V}_i = \begin{bmatrix} a_i & 0 & \hat{B}_i & 0 \\ 0 & I & 0 & 0 \\ \hat{C}_i & 0 & \hat{D}_i & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

in $\mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1^{i-1} \oplus \mathcal{H}_i \oplus \mathcal{H}_{i+1}^n)$ for all $1 < i < n$. Then

$$V = \prod_{i=1}^{n} \hat{V}_i,$$

is an isometry in $\mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1^n)$. Moreover, it follows that

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} = \left[\begin{array}{c|ccc} a & B_1 & \cdots & B_n \\ \hline C_1 & D_{11} & \cdots & D_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ C_n & D_{n1} & \cdots & D_{nn} \end{array}\right], \tag{2.6.1}$$

where

$$B_i = (\prod_{k=1}^{i-1} a_k)\hat{B}_i, \quad C_i = (\prod_{k=i+1}^{n} a_k)\hat{C}_i,$$

and

$$D_{ij} = \begin{cases} \hat{D}_i & \text{if } i = j \\ 0 & \text{if } i > j \\ (a_{i+1} \cdots a_{j-1})\hat{C}_i\hat{B}_j & \text{if } i < j. \end{cases}$$

Hence

$$aD_{i,j} = C_iB_j,$$

for all $1 \leqslant i < j \leqslant n$. Then by repeated application of Theorem 2.2.2, we have

$$\varphi = \tau_V.$$

The converse, as stated below, follows directly from repeated applications of Theorem 2.2.3. We have thus proved the following theorem.

Theorem 2.6.1. *Suppose $\theta \in \mathcal{SA}(\mathbb{D}^n)$ and $\theta(0) \neq 0$. Then*

$$\theta(\boldsymbol{z}) = \prod_{i=1}^{n} \theta_i(z_i) \qquad (\boldsymbol{z} \in \mathbb{D}^n),$$

for some Schur functions $\{\theta_i\}_{i=1}^n \subseteq \mathcal{S}(\mathbb{D})$ *if and only if there exist Hilbert spaces* $\mathcal{H}_1, \dots, \mathcal{H}_n$ *and an isometric colligation*

$$\begin{bmatrix} a & B \\ C & D \end{bmatrix} = \left[\begin{array}{c|ccc} a & B_1 & \cdots & B_n \\ \hline C_1 & D_{11} & \cdots & D_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ C_n & D_{n1} & \cdots & D_{nn} \end{array}\right]$$

on $\mathbb{C} \oplus \left(\bigoplus_{i=1}^n \mathcal{H}_i \right)$ *such that*

$$D_{ij} = \begin{cases} D_i & \text{if } i = j \\ 0 & \text{if } i > j \\ \frac{1}{a} C_i B_j & \text{if } i < j, \end{cases}$$

and

$$\theta(z) = a + B\Big(I_{\mathcal{H}_1^n} - E_{\mathcal{H}_1^n}(\boldsymbol{z}) D\Big)^{-1} E_{\mathcal{H}_1^n}(\boldsymbol{z}) C \qquad (\boldsymbol{z} \in \mathbb{D}^n).$$

2.6.2 Examples

Here we aim at applying our results to some concrete examples.

Example 1: First, we let $\phi \in \mathcal{S}(\mathbb{D})$ and $\phi = \tau_{V_0}$ for some isometric colligation

$$V_0 = \begin{bmatrix} a & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}).$$

Now we consider $\psi(z) = z^m$, $z \in \mathbb{D}$ and $m \in \mathbb{N}$. One then shows that

$$V_m = \left[\begin{array}{c|cccc} 0 & 1 & 0 & \cdots & 0 \\ \hline 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{array}\right] \in \mathcal{B}(\mathbb{C} \oplus \mathbb{C}^m),$$

is an isometric colligation and

$$\psi = \tau_{V_m}.$$

Set $\theta = \phi\psi = \tau_{V_0}\tau_{V_m}$. Then by Theorem 2.3.2 (or more specifically, by (2.3.3)) it follows that

$$\tau_V(z) = z^m \varphi(z) \qquad (z \in \mathbb{D}),$$

where $V \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H} \oplus \mathbb{C}^m)$ is an isometric colligation with the following representation

$$V = \left[\begin{array}{c|cc|cccc} 0 & B & a & 0 & 0 & \cdots & 0 \\ \hline 0 & D & C & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \end{array}\right] \in \mathcal{B}(\mathbb{C} \oplus (\mathcal{H} \oplus \mathbb{C}) \oplus \mathbb{C}^{m-1}).$$

Example 2: Our second example concerns Blaschke factors: If $\lambda \in \mathbb{D}$, then the *Blaschke factor* $b_\lambda \in \mathrm{Aut}(\mathbb{D})$ is defined by

$$b_\lambda(z) = \frac{z - \lambda}{1 - \bar{\lambda} z} \qquad (z \in \mathbb{D}).$$

Now observe that, for each $\lambda \in \mathbb{D}$, the matrix

$$V_\lambda = \begin{bmatrix} -\lambda & \sqrt{1 - |\lambda|^2} \\ \sqrt{1 - |\lambda|^2} & \bar{\lambda} \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathbb{C}),$$

is an isometric colligation and

$$b_\lambda = \tau_{V_\lambda}.$$

Now, suppose $\alpha, \beta \in \mathbb{D}$ and

$$\theta(\boldsymbol{z}) = b_\alpha(z_1) b_\beta(z_2) \qquad (\boldsymbol{z} \in \mathbb{D}^2).$$

Then Theorem 2.2.2 implies that

$$\theta = \tau_V,$$

where

$$V = \begin{bmatrix} \alpha\beta & \sqrt{1 - |\alpha|^2} & -\alpha\sqrt{1 - |\beta|^2} \\ -\beta\sqrt{1 - |\alpha|^2} & \bar{\alpha} & \sqrt{1 - |\alpha|^2}\sqrt{1 - |\beta|^2} \\ \sqrt{1 - |\beta|^2} & 0 & \bar{\beta} \end{bmatrix},$$

is an isometric colligation in $M_3(\mathbb{C})$.

2.6.3 On $\mathcal{F}_m(n)$ and $\mathcal{F}(n)$

Let $1 \leq m < n$. Suppose $V \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1^m \oplus \mathcal{H}_{m+1}^n)$ satisfies property $\mathcal{F}_m(n)$. On account of Theorem 2.2.3, we have

$$\tau_V(\boldsymbol{z}) = \tau_{V_1}(z_1, \ldots, z_m) \tau_{V_2}(z_{m+1}, \ldots, z_n) \qquad (\boldsymbol{z} \in \mathbb{D}^n),$$

for some isometric colligations $V_1 \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1^m)$ and $V_2 \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_{m+1}^n)$. Note that $\tau_{V_1} \in \mathcal{SA}(\mathbb{D}^m)$ and $\tau_{V_2} \in \mathcal{SA}(\mathbb{D}^{n-m})$. The above factorization and Theorem 2.3.4 further implies that

$$\tau_V = \tau_{\tilde{V}},$$

for some isometric colligation $\tilde{V} \in \mathcal{B}(\mathbb{C} \oplus \Big(\bigoplus_{i=1}^{n}(\mathcal{M}_i \oplus \mathcal{N}_i)\Big))$ satisfying property $\mathcal{F}(n)$. It is then natural to ask to what extent one can recover $\tilde{V}$ from V. To determine the isometric colligation $\tilde{V}$, we proceed as follows: First, we let

$$V = \left[\begin{array}{c|ccc} a & B_1 & \cdots & B_n \\ \hline C_1 & D_{11} & \cdots & D_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ C_n & D_{n1} & \cdots & D_{nn} \end{array}\right] \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1^n), \tag{2.6.2}$$

where $D_{ij} = 0$ for $i = m+1, \ldots, n$ and $j = 1, \ldots, m$; $aD_{ij} = C_i B_j$ for $i = 1, \ldots, m$ and $j = m+1, \ldots, n$. Let $\mathcal{L}$ be a Hilbert space. Set

$$\mathcal{K}_i = \begin{cases} \mathcal{H}_i \oplus \mathcal{L} & \text{if } 1 \le i \le m \\ \mathcal{L} \oplus \mathcal{H}_i & \text{if } m+1 \le i \le n. \end{cases}$$

We now define

$$Y_i = \begin{cases} \begin{bmatrix} B_i & 0 \end{bmatrix} & \text{if } 1 \le i \le m \\ \\ \begin{bmatrix} 0 & B_i \end{bmatrix} & \text{if } m+1 \le i \le n, \end{cases} \qquad Z_i = \begin{cases} \begin{bmatrix} C_i \\ 0 \end{bmatrix} & \text{if } 1 \le i \le m \\ \\ \begin{bmatrix} 0 \\ C_i \end{bmatrix} & \text{if } m+1 \le i \le n, \end{cases}$$

and

$$W_{ij} = \begin{cases} \begin{bmatrix} D_{ij} & 0 \\ 0 & \delta_{ij} I_{\mathcal{L}} \end{bmatrix} & \text{if } 1 \le i,j \le m \\ \\ \begin{bmatrix} \delta_{ij} I_{\mathcal{L}} & 0 \\ 0 & D_{ij} \end{bmatrix} & \text{if } m+1 \le i,j \le n, \end{cases}$$

and

$$W_{ij} = \begin{cases} \begin{bmatrix} 0 & D_{ij} \\ 0 & 0 \end{bmatrix} & \text{if } 1 \le i \le m,\ m+1 \le j \le n \\ \\ \begin{bmatrix} 0 & 0 \\ D_{ij} & 0 \end{bmatrix} & \text{if } m+1 \le i \le n, 1 \le j \le n. \end{cases}$$

Then, after some manipulations, it follows that the isometric colligation

$$\tilde{V} := \left[\begin{array}{c|ccc} a & Y_1 & \cdots & Y_n \\ \hline Z_1 & W_{11} & \cdots & W_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ Z_n & W_{n1} & \cdots & W_{nn} \end{array}\right] \in \mathcal{B}(\mathbb{C} \oplus \mathcal{K}_1^n), \tag{2.6.3}$$

satisfies property $\mathcal{F}(n)$ and $\tau_V = \tau_{\tilde{V}}$. More specifically, we have proved the following:

Theorem 2.6.2. *Suppose $1 \leq m < n$ and let V satisfies property $\mathcal{F}_m(n)$. If the representation of V is given by (2.6.2), then*

$$\tau_V = \tau_{\tilde{V}},$$

where $\tilde{V}$ is given by (2.6.3) and satisfies property $\mathcal{F}(n)$.

2.6.4 Reversibility of factorizations

A natural question to ask in connection with Theorem 2.3.4 is whether the canonical constructions of the colligation V (out of a pair of isometric colligations V_1 and V_2) satisfying property $\mathcal{F}(n)$ as in (2.3.3) and V_1 and V_2 (out of an isometric colligation V satisfying property $\mathcal{F}(n)$) as in (2.3.12) are reversible.

To answer this, we proceed as follows: Given $n \in \mathbb{N}$, we let $C(n)$ denote the set of all isometric colligations of the form $\begin{bmatrix} a & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \Big(\bigoplus\limits_{i=1}^{n} \mathcal{H}_i\Big))$ for some Hilbert spaces $\{\mathcal{H}_i\}_{i=1}^n$, and let $F(n)$ denote the set of all isometric colligations satisfying property $\mathcal{F}(n)$. Define $\pi : C(n) \times C(n) \to F(n)$ by

$$\pi(V_1, V_2) = V \qquad (V_1, V_2 \in C(n)),$$

where V is as in (2.3.3) (or Theorem 2.3.2). Also define $\kappa : F(n) \to C(n) \times C(n)$ by

$$\kappa(V) = (V_1, V_2) \qquad (V \in F(n)),$$

where V_1 and V_2 are as in (2.3.12). Given V_1 and V_2 in $C(n)$, the aim here is to compare $\kappa(\pi(V_1, V_2))$ with (V_1, V_2). Suppose

$$V_1 = \begin{bmatrix} \alpha & B \\ C & D \end{bmatrix} \in \mathcal{B}\Big(\mathbb{C} \oplus (\bigoplus_{i=1}^{n} \mathcal{M}_i)\Big) \quad \text{and} \quad V_2 = \begin{bmatrix} \beta & F \\ G & H \end{bmatrix} \in \mathcal{B}\Big(\mathbb{C} \oplus (\bigoplus_{i=1}^{n} \mathcal{N}_i)\Big)$$

are isometric colligations and $a = \alpha\beta \neq 0$. Then by (2.3.3), it follows that

$$\pi(V_1, V_2) \in \mathcal{B}\Big(\mathbb{C} \oplus (\bigoplus_{i=1}^{n} (\mathcal{M}_i \oplus \mathcal{N}_i))\Big),$$

and

$$\pi(V_1, V_2) = \begin{bmatrix} \alpha\beta & \hat{B}_1 & \cdots & \hat{B}_n \\ \hat{C}_1 & \hat{D}_{11} & \cdots & \hat{D}_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{C}_n & \hat{D}_{n1} & \cdots & \hat{D}_{nn} \end{bmatrix},$$

where $\hat{B}_i$, $\hat{C}_i$ and $\hat{D}_{ij}$, $i, j = 1, \ldots, n$, are given by as in (2.3.4) and (2.3.5). Since $\pi(V_1, V_2)$ satisfies property $\mathcal{F}(n)$, in view of (2.3.12), it follows that

$$\kappa(\pi(V_1, V_2)) = (\tilde{V}_1, \tilde{V}_2),$$

where

$$\tilde{V}_1 = \begin{bmatrix} \tilde{\alpha} & B_1 & \cdots & B_n \\ \frac{\beta}{\tilde{\beta}} C_1 & D_{11} & \cdots & D_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta}{\tilde{\beta}} C_n & D_{n1} & \cdots & D_{nn} \end{bmatrix} \quad \text{and} \quad \tilde{V}_2 = \begin{bmatrix} \tilde{\beta} & \frac{\alpha}{\tilde{\alpha}} F_1 & \cdots & \frac{\alpha}{\tilde{\alpha}} F_n \\ G_1 & H_{11} & \cdots & H_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ G_n & H_{n1} & \cdots & H_{nn} \end{bmatrix},$$

and $\tilde{\alpha}$ and $\tilde{\beta}$ are non-zero scalars satisfying the following relations (see (2.3.13) and (2.3.14))

$$|\tilde{\beta}|^2 = |\alpha|^2|\beta|^2 + |\beta|^2\Big(\sum_{i=1}^{n} C_i^* C_i\Big) \quad \text{and} \quad \tilde{\alpha} = \frac{\alpha\beta}{\tilde{\beta}}.$$

But we know from $V_1^* V_1 = I$ that $|\alpha|^2 + C^*C = 1$, that is

$$|\alpha|^2 + \sum_{i=1}^{n} C_i^* C_i = 1.$$

So $\tilde{\beta} = \bar{\varepsilon}\beta$ and $\tilde{\alpha} = \varepsilon\alpha$ for some unimodular constant ε. Hence

$$\kappa \circ \pi\Big(\begin{bmatrix} \alpha & B \\ C & D \end{bmatrix}, \begin{bmatrix} \beta & F \\ G & H \end{bmatrix} \Big) = \Big(\begin{bmatrix} \varepsilon\alpha & B \\ \varepsilon C & D \end{bmatrix}, \begin{bmatrix} \bar{\varepsilon}\beta & \bar{\varepsilon}F \\ G & H \end{bmatrix} \Big),$$

where ε is an unimodular constant.

One could equally consider the same question for Theorem 2.2.4. The answer is similar.

Chapter 3

Schur functions and inner functions on the bidisc

3.1 Introduction

The principle aim of this chapter is threefold: (1) Provide representations of inner functions in $\mathcal{S}(\mathbb{D}^2)$ in terms of the isometric colligation operators (a certain class of 2×2 block operator matrices). (2) Establish a classification of de Branges-Rovnyak kernels on $\mathbb{D}$ (which also works in the setting of $\mathbb{D}^n$ and the open unit ball of $\mathbb{C}^n$, $n \geq 1$). (3) Provide a classification, in terms of Agler kernels, of Schur functions in $\mathcal{S}(\mathbb{D}^2)$ which admit a one variable factorization.

Our first aim of this chapter is to provide sufficient (as well as necessary, for reducible rational functions) conditions in terms of isometric colligation for a function in $\mathcal{S}(\mathbb{D}^2)$ to be inner. Our presentation here, needless to say, is based on Agler's realization formula and Agler kernels for functions in $\mathcal{S}(\mathbb{D}^2)$ [2].

We now return to the topic of representations of inner functions in $\mathcal{S}(\mathbb{D}^2)$. Finding an analog of Theorem 1.2.10 for inner functions in $\mathcal{S}(\mathbb{D}^2)$ seems to be a subtle and unattended problem. Here the main difficulty is to deal with the 2×2 block operator matrix $D \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, or more specifically, with the resolvent part of τ_V which involves the inverse of the 2×2 block operator matrix. Instead, in Theorem 3.2.1 we prove that a function $\varphi \in \mathcal{S}(\mathbb{D}^2)$ is inner whenever $\varphi = \tau_V$ for some isometric colligation

$$V = \begin{bmatrix} a & B_1 & B_2 \\ C_1 & D_1 & D_2 \\ C_2 & 0 & D_3 \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2)),$$

with $D_1, D_3 \in C_{0\cdot}$. This is the main content of Section 3.2.

The converse of the above fact is not true in general (see Example 3.5.1). However, a weak converse holds for rational inner functions that admit a one variable factorization (see Theorem 3.5.3). This is the main content of Section 3.5.

Now we turn to our second goal of this chapter: Classification of de Branges-Rovnyak kernels on $\mathbb{D}^n$ and the open unit ball of $\mathbb{C}^n$. Here we explain the idea in the setting of operator-valued Schur functions on $\mathbb{D}$. Let $\Theta \in \mathcal{S}(\mathbb{D}, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$. We call the kernel

$$K_\Theta(z, w) := \frac{I - \Theta(z)\Theta(w)^*}{1 - z\overline{w}}, \qquad (z, w \in \mathbb{D}).$$

the *de Branges-Rovnyak kernel* corresponding to Θ. The classical de Branges-Rovnyak theory says that the de Branges-Rovnyak space, $\mathcal{H}_{K_\Theta}$ is contractively contained (not necessarily closed) subspace of the Hardy space $H^2_{\mathcal{E}_*}(\mathbb{D})$ and invariant under backward shift operator.

Section 3.3 focuses on the following question: How can we recognize when a kernel admits a de Branges-Rovnyak kernel representation?

The following is our answer to this question (see Theorem 3.3.1): Let $K \geq 0$ be a $\mathcal{B}(\mathcal{E}_*)$-valued kernel (which is not a priori analytic in its first variable). Then $K = K_\Theta$ for some $\Theta \in \mathcal{S}(\mathbb{D}, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ and Hilbert space $\mathcal{E}$ if and only if

$$I_{\mathcal{E}_*} - (1 - z\overline{w}) \cdot K \geq 0,$$

where $\cdot$ denotes the Hadamard product. This also covers a (variation of the) classical result due to de Branges and Rovnyak (see Theorem 3.3.2 and the discussion preceding it).

In the setting of Schur-Agler functions on $\mathbb{D}^n$ (see more details in Section 3.3), in Theorem 3.3.3 we prove the following: Let $K : \mathbb{D}^n \times \mathbb{D}^n \to \mathcal{B}(\mathcal{E}_*)$ be a kernel on $\mathbb{D}^n$ (again, K is not a priori analytic in $z_1, \ldots, z_n$). Then there exist a Hilbert space $\mathcal{E}$ and a $\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$-valued Schur-Agler function Θ (in notation, $\Theta \in \mathcal{SA}(\mathbb{D}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$) such that

$$K = K_\Theta,$$

where

$$K_\Theta(\boldsymbol{z}, \boldsymbol{w}) := \frac{I - \Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^*}{\prod_{i=1}^{n}(1 - z_i\bar{w}_i)} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n),$$

if and only if there exist $\mathcal{B}(\mathcal{E}_*)$-valued kernels $K_1, \ldots, K_n$ (called *Agler kernels* of φ) on $\mathbb{D}^n$ such that

$$K(\boldsymbol{z}, \boldsymbol{w}) = \sum_{i=1}^{n} \frac{1}{\prod_{j\neq i}(1 - z_j\bar{w}_j)} K_i(\boldsymbol{z}, \boldsymbol{w}) \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n),$$

and

$$I_{\mathcal{E}_*} - \Big(\prod_{i=1}^{n}(1 - z_i\bar{w}_i)\Big) \cdot K \geq 0.$$

An analogous but somewhat simpler statement also holds in the setting of multipliers of the Drury-Arveson space (see Theorem 3.3.4).

The final goal of this chapter is to describe those two-variable Schur functions that admit a one variable Schur factor. This is the main content of Section 3.4. More specifically (see Theorem 3.4.1): Let $\varphi \in \mathcal{S}(\mathbb{D}^2)$ and suppose $\varphi(\mathbf{0}) \neq 0$, where $\mathbf{0} = (0,0)$ (see Remark 3.4.1 on the assumption $\varphi(\mathbf{0}) \neq 0$). The following assertions are equivalent:

(1) There exist φ_1 and φ_2 in $\mathcal{S}(\mathbb{D})$ such that $\varphi(\boldsymbol{z}) = \varphi_1(z_1)\varphi_2(z_2)$, $\boldsymbol{z} \in \mathbb{D}^2$.

(2) There exist Agler kernels $\{K_1, K_2\}$ of φ such that K_1 depends only on z_1 and $\bar{w}_1$, and

$$\overline{\varphi(\mathbf{0})}\, K_2(\cdot, (w_1, 0)) = \overline{\varphi(w_1, 0)}\, K_2(\cdot, \mathbf{0}) \qquad (w_1 \in \mathbb{D}).$$

(3) There exist Agler kernels $\{L_1, L_2\}$ of φ such that all the functions in $\mathcal{H}_{L_1}$ depend only on z_1, and $\varphi(\mathbf{0}) f(\cdot, 0) = \varphi(\cdot, 0)\, f(\mathbf{0})$, $f \in \mathcal{H}_{L_2}$.

(4) $\varphi = \tau_V$ for some co-isometric colligation

$$V = \begin{bmatrix} \varphi(\mathbf{0}) & B_1 & B_2 \\ C_1 & D_1 & D_2 \\ C_2 & 0 & D_4 \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2)),$$

with $\varphi(\mathbf{0}) D_2 = C_1 B_2$.

We remark that, given the importance of the rich structure, inner functions on the bidisc have been considered in many occasions previously in different contexts. For instance, see [41, 113] and the references therein.

It is worthwhile to point out that our main motivation for considering colligation matrices, as in part (4) above and the one following Theorem 1.2.3, comes from the recent paper [48].

This chapter is based on the published paper [49].

3.2 Inner Functions and Realizations

Our purpose here is to prove a statement analogous to the sufficient part of Theorem 1.2.10. We will again return to this topic in Section 3.5 with some counterexamples and a weak converse.

It will be convenient, to begin with, to introduce some terminology and basic observations. The following construction also could be of some independent interest. We write $\oplus l^2 = l^2 \oplus l^2 \oplus \cdots$, that is

$$\oplus l^2 = \Big\{ \{a_{ij}\} := \Big\{ \{\{a_{0j}\}_{j\geq 0}, \{a_{1j}\}_{j\geq 0}, \{a_{2j}\}_{j\geq 0}, \ldots\} : \sum_{i,j=0}^{\infty} |a_{ij}|^2 < \infty \Big\}.$$

One can easily verify that
$$\tau(\{a_{ij}\}) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij} z_1^i z_2^j,$$
defines a unitary $\tau : \oplus l^2 \to H^2(\mathbb{D}^2)$, and $M_{z_1}\tau = \tau S$, where S denotes the *shift* on $\oplus l^2$, that is
$$S\Big(\{a_{ij}\}\Big) = \Big\{\{0\}, \{a_{0j}\}_{j\geq 0}, \{a_{1j}\}_{j\geq 0}, \dots\Big\}.$$
Here $\{0\} \in l^2$ is the zero sequence. Now, let $\varphi = \sum\limits_{i=0}^{\infty}(\sum\limits_{j=0}^{\infty} \varphi_{ij} z_2^j) z_1^i \in H^\infty(\mathbb{D}^2)$. We define the *block Toeplitz operator* with symbol φ to be the bounded linear operator T_φ on $\oplus l^2$ defined by
$$\Big(T_\varphi\Big(\{a_{ij}\}\Big)\Big)_{ij} = \sum_{k=0}^{i}\sum_{l=0}^{j} \varphi_{i-k,j-l} a_{kl} \qquad (i,j \geq 0),$$
which in matrix notation becomes
$$T_\varphi = \begin{bmatrix} \Phi_0 & 0 & 0 & 0 & \cdots \\ \Phi_1 & \Phi_0 & 0 & 0 & \cdots \\ \Phi_2 & \Phi_1 & \Phi_0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$
where
$$\Phi_k = \begin{bmatrix} \varphi_{k0} & 0 & 0 & 0 & \cdots \\ \varphi_{k1} & \varphi_{k0} & 0 & 0 & \cdots \\ \varphi_{k2} & \varphi_{k1} & \varphi_{k0} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$
is a Toeplitz operator on l^2 for all $k \geq 0$. More specifically, we have
$$M_\varphi \tau = \tau T_\varphi \qquad (\varphi \in H^\infty(\mathbb{D}^2)),$$
where M_φ denotes the multiplication operator on $H^2(\mathbb{D}^2)$ with analytic symbol φ, that is, $M_\varphi f = \varphi f$ for all $f \in H^2(\mathbb{D}^2)$. Indeed, if $\varphi = \sum\limits_{i=0}^{\infty}(\sum\limits_{j=0}^{\infty} \varphi_{ij} z_2^j) z_1^i$ and $\{a_{ij}\} \in \oplus l^2$, then
$$\begin{aligned} M_\varphi \tau(\{a_{ij}\}) &= \left(\sum_{i,j=0}^{\infty} \varphi_{ij} z_1^i z_2^j\right)\left(\sum_{k,l=0}^{\infty} a_{kl} z_1^k z_2^l\right) = \sum_{j,l=0}^{\infty}\sum_{i,k=0}^{\infty} \varphi_{ij} a_{kl} z_1^{i+k} z_2^{l+j} \\ &= \sum_{j,l=0}^{\infty}\left\{\sum_{i=0}^{\infty}\left(\sum_{k=0}^{i} \varphi_{i-k,j} a_{kl}\right) z_1^i\right\} z_2^{l+j} = \sum_{i=0}^{\infty}\sum_{k=0}^{i}\left\{\sum_{j,l=0}^{\infty} \varphi_{i-k,j} a_{kl} z_2^{l+j}\right\} z_1^i \\ &= \sum_{i=0}^{\infty}\sum_{k=0}^{i}\left\{\sum_{j=0}^{\infty}\left(\sum_{l=0}^{j} \varphi_{i-k,j-l} a_{kl}\right) z_2^j\right\} z_1^i = \sum_{i,j=0}^{\infty}\left(\sum_{k,l=0}^{i} \varphi_{i-k,j-l} a_{kl}\right) z_1^i z_2^j, \end{aligned}$$
and hence $M_\varphi \tau(\{a_{ij}\}) = \tau T_\varphi(\{a_{ij}\})$. In particular, we have

$$T_{z_2} = \begin{bmatrix} S_{l^2} & 0 & 0 & 0 & \cdots \\ 0 & S_{l^2} & 0 & 0 & \cdots \\ 0 & 0 & S_{l^2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where S_{l^2} denotes the shift on l^2, that is, $S_{l^2}(\{a_0, a_1, \ldots\}) = \{0, a_0, a_1, \ldots\}$ for all $\{a_m\}_{m\geq 0} \in l^2$. Continuing with the above notation, we set

$$Y_0 = \begin{bmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \vdots \end{bmatrix}, \quad \text{and} \quad Y_j = S^j Y_0, \tag{3.2.1}$$

for all $j \geq 1$. Then $T_\varphi = \begin{bmatrix} Y_0 & Y_1 & Y_2 & \ldots \end{bmatrix}$. Since $T_\varphi^* T_\varphi = (Y_i^* Y_j)$, it follows that M_φ on $H^2(\mathbb{D}^2)$ is an isometry if and only if T_φ on $\oplus l^2$ is an isometry, which is also equivalent to

$$Y_i^* Y_j = \delta_{ij} I_{l^2}. \tag{3.2.2}$$

We are now ready to present the main theorem of this section.

Theorem 3.2.1. *Let $\varphi \in \mathcal{S}(\mathbb{D}^2)$. If $\varphi = \tau_V$ for some isometric colligation*

$$V = \begin{bmatrix} a & B_1 & B_2 \\ C_1 & D_1 & D_2 \\ C_2 & 0 & D_3 \end{bmatrix} : \mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2) \to \mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2),$$

with $D_1, D_3 \in C_{0\cdot}$, then φ is an inner function.

Proof. Since $\varphi = \tau_V$, and

$$\tau_V(\boldsymbol{z}) = a + \begin{bmatrix} B_1 & B_2 \end{bmatrix} \left(I_{\mathcal{H}_1 \oplus \mathcal{H}_2} - E_{\mathcal{H}_1 \oplus \mathcal{H}_2}(\boldsymbol{z}) \begin{bmatrix} D_1 & D_2 \\ 0 & D_3 \end{bmatrix} \right)^{-1} E_{\mathcal{H}_1 \oplus \mathcal{H}_2}(\boldsymbol{z}) \begin{bmatrix} C_1 \\ C_2 \end{bmatrix},$$

we have

$$\varphi(\boldsymbol{z}) = a + \sum_{i=1}^{\infty} B_1 D_1^{i-1} C_1 z_1^i + \sum_{j=1}^{\infty} B_2 D_3^{j-1} C_2 z_2^j + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} B_1 D_1^{i-1} D_2 D_3^{j-1} C_2 z_1^i z_2^j,$$

for all $\boldsymbol{z} \in \mathbb{D}^2$. Using the same notation preceding the statement, we set

$$\Phi_0 = \begin{bmatrix} a & 0 & 0 & 0 & \cdots \\ B_2 C_2 & a & 0 & 0 & \cdots \\ B_2 D_3 C_2 & B_2 C_2 & a & 0 & \cdots \\ B_2 D_3^2 C_2 & B_2 D_3 C_2 & B_2 C_2 & a & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

and

$$\Phi_j = \begin{bmatrix} B_1D_1^{j-1}C_1 & 0 & 0 & 0 & \cdots \\ B_1D_1^{j-1}D_2C_2 & B_1D_1^{j-1}C_1 & 0 & 0 & \cdots \\ B_1D_1^{j-1}D_2D_3C_2 & B_1D_1^{j-1}D_2C_2 & B_1D_1^{j-1}C_1 & 0 & \cdots \\ B_1D_1^{j-1}D_2D_3^2C_2 & B_1D_1^{j-1}D_2D_3C_2 & B_1D_1^{j-1}D_2C_2 & B_1D_1^{j-1}C_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

for $j \geq 1$. We first claim that $Y_0 = \begin{bmatrix} \Phi_0 \\ \Phi_1 \\ \Phi_2 \\ \vdots \end{bmatrix}$ is an isometry. In fact, since $Y_0^*Y_0 = \sum_{m=0}^{\infty} \Phi_m^*\Phi_m$, there exists a sequence of scalars $\{y_m\}_{m\geq 0}$ such that

$$Y_0^*Y_0 = \begin{bmatrix} y_0 & y_1 & y_2 & \cdots \\ \overline{y_1} & y_0 & y_1 & \cdots \\ \overline{y_2} & \overline{y_1} & y_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

We need to show that $y_0 = 1$ and $y_k = 0$ for all $k \geq 1$. Note that

$$\begin{aligned} y_0 = |a|^2 &+ C_1^* \left(\sum_{j=0}^{\infty} D_1^{*j} B_1^* B_1 D_1^j \right) C_1 \\ &+ C_2^* \left[\sum_{k=0}^{\infty} D_3^{*k} \left\{ B_2^*B_2 + D_2^* \left(\sum_{l\geq 0} D_1^{*l} B_1^* B_1 D_1^l \right) D_2 \right\} D_3^k \right] C_2. \end{aligned}$$

Since $V^*V = I$, it follows that

$$\begin{bmatrix} |a|^2 + C_1^*C_1 + C_2^*C_2 & \bar{a}B_1 + C_1^*D_1 & \bar{a}B_2 + C_1^*D_2 + C_2^*D_3 \\ aB_1^* + D_1^*C_1 & B_1^*B_1 + D_1^*D_1 & B_1^*B_2 + D_1^*D_2 \\ aB_2^* + D_2^*C_1 + D_3^*C_2 & B_2^*B_1 + D_2^*D_1 & B_2^*B_2 + D_2^*D_2 + D_3^*D_3 \end{bmatrix} = I. \tag{3.2.3}$$

In particular

$$\begin{aligned} I &= B_1^*B_1 + D_1^*D_1 \\ &= B_1^*B_1 + D_1^*(B_1^*B_1 + D_1^*D_1)D_1 \\ &= B_1^*B_1 + D_1^*B_1^*B_1D_1 + D_1^{*2}D_1^2, \end{aligned}$$

and hence $I = \sum_{j=0}^{m} D_1^{*j}(B_1^*B_1)D_1^j + D_1^{*(m+1)}(B_1^*B_1)D_1^{(m+1)}$ for all $m \geq 1$. Using the fact that $D_1 \in C_{0\cdot}$, we have

$$\sum_{j=0}^{\infty} D_1^{*j}(B_1^*B_1)D_1^j = I, \tag{3.2.4}$$

in the strong operator topology. Similarly, $B_2^*B_2 + D_2^*D_2 + D_3^*D_3 = I$ and $D_3 \in C_{0\cdot}$ implies that

$$\sum_{j=0}^{\infty} D_3^{*j}(B_2^*B_2 + D_2^*D_2)D_3^j = I, \tag{3.2.5}$$

in the strong operator topology. This with the condition $|a|^2 + C_1^*C_1 + C_2^*C_2 = 1$ in (3.2.3) implies that

$$y_0 = |a|^2 + C_1^*C_1 + C_2^*C_2 = 1.$$

Next we consider

$$\begin{aligned} y_1 = {} & aC_2^*B_2^* + C_2^*D_2^*\left(\sum_{j=0}^{\infty} D_1^{*j}B_1^*B_1D_1^j\right)C_1 \\ & + C_2^*D_3^*\left[\sum_{k=0}^{\infty} D_3^{*k}\left\{B_2^*B_2 + D_2^*\left(\sum_{l=0}^{\infty} D_1^{*l}B_1^*B_1D_1^l\right)D_2\right\}D_3^k\right]C_2 \,. \end{aligned}$$

Thus by (3.2.4) and (3.2.5), it follows that

$$y_1 = aC_2^*B_2^* + C_2^*D_2^*C_1 + C_2^*D_3^*C_2 = C_2^*(aB_2^* + D_2^*C_1 + D_3^*C_2) = 0,$$

as $aB_2^* + D_2^*C_1 + D_3^*C_2 = 0$ follows from (3.2.3). Similarly

$$y_j = C_2^*D_3^{*j}(aB_2^* + D_2^*C_1 + D_3^*C_2) = 0,$$

for all $j \geq 2$. This proves that Y_0 is an isometry.

Since the shift S on $\oplus l^2$ is an isometry, $Y_j := S^jY_0$, $j \geq 1$, is also an isometry (see the construction in (3.2.1)). Our final goal is to prove that $T_\varphi := \begin{bmatrix} Y_0 & Y_1 & Y_2 & \dots \end{bmatrix}$ is an isometry, or equivalently, by virtue of (3.2.2) and $Y_m^*Y_m = I$ for all $m \geq 0$,

$$Y_p^*Y_q = 0 \qquad (p > q \geq 0).$$

Since $Y_p^*Y_q = Y_0^*S^{*p}S^qY_0 = Y_0^*S^{*(p-q)}Y_0$ for all $p > q \geq 0$, it actually suffices to check that

$$Y_0^*S^{*(j+1)}Y_0 = 0 \qquad\qquad (j \geq 0).$$

So we fix $j \geq 0$ and observe

$$S^jY_0 = \begin{bmatrix} \underbrace{0 \ \dots \ 0}_{(j+1)} & \Phi_0 & \Phi_1 & \Phi_2 & \dots \end{bmatrix}^t.$$

Hence

$$Y_0^* S^{*(j+1)} Y_0 = \Phi_0^* \Phi_{j+1} + \Phi_1^* \Phi_{j+2} + \cdots .$$

Therefore there exists a sequence $\{c_m\}_{m\in\mathbb{Z}}$ such that

$$Y_0^* S^{*(j+1)} Y_0 = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots \\ c_{-1} & c_0 & c_1 & \cdots \\ c_{-2} & c_{-1} & c_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It is suffices to prove that $c_k = 0$ for all $k \in \mathbb{Z}$. A simple calculation shows that

$$c_0 = (\bar{a}B_1 + C_1^* D_1) D_1^{j+1} C_1 + C_2^* \left[\sum_{m=0}^{\infty} D_3^{*m} \left\{ B_2^* B_1 + D_2^* D_1 \right\} D_1^{j+1} D_2 D_3^m \right] C_2.$$

By (3.2.3), $\bar{a}B_1 + C_1^* D_1 = 0$ and $B_2^* B_1 + D_2^* D_1 = 0$, and hence $c_0 = 0$. Now let $k > 0$. Then

$$c_k = C_2^* D_3^{*(k-1)} (B_2^* B_1 + D_2^* D_1) D_1^{j+1} C_1 + C_2^* D_3^{*k} \left[\sum_{m=0}^{\infty} D_3^{*m} \left\{ B_2^* B_1 + D_2^* D_1 \right\} D_1^{j+1} D_2 D_3^m \right] C_2,$$

and hence $c_k = 0$. Finally, since

$$c_{-k} = (\bar{a}B_1 + C_1^* D_1) D_1^{j+1} D_2 D_3^{k-1} C_2 + C_2^* \left[\sum_{m=0}^{\infty} D_3^{*m} \left\{ B_2^* B_1 + D_2^* D_{1,1} \right\} D_1^{j+1} D_2 D_3^m \right] D_3^k C_2,$$

it again follows that $c_{-k} = 0$. This implies that T_φ or, equivalently, M_φ is an isometry, and completes the proof. □

Remark 3.2.1. *Let $V \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H})$ be an isometric colligation, and let $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ for Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Suppose $D := P_{\mathcal{H}} V|_{\mathcal{H}}$ and suppose that $D\mathcal{H}_1 \subseteq \mathcal{H}_1$. Set*

$$D = \begin{bmatrix} D_1 & D_2 \\ 0 & D_3 \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2).$$

It is easy to see that if $D \in C_{0\cdot}$, then D_1 and D_3 are also in $C_{0\cdot}$. Consequently, Theorem 3.2.1 also holds for those Schur functions φ such that $\varphi = \tau_V$ with V as above. Of course, if D_1 and D_3 are in $C_{0\cdot}$, then D is not necessarily in $C_{0\cdot}$.

3.3 de Branges-Rovnyak kernels

The goal of this section is to study de Branges-Rovnyak kernels on $\mathbb{D}^n$ and the open unit ball of $\mathbb{C}^n$, $n \geq 1$. Specifically, we seek characterizations of analytic kernels that admit certain factorizations involving Schur(-Agler) functions. Our investigation is partly motivated by a classical result of de Branges and Rovnyak (see the Theorem 3.3.2 for more details).

In the following, we characterize de Branges-Rovnyak kernels defined on the disc $\mathbb{D}$. The proof uses the standard and commonly used "lurking-isometry" techniques. Therefore, our proof is fairly standard and, perhaps, it can also be achieved using existing results about Schur(-Agler) functions([7]). Note also that the theorem below does not assume a priori that K is analytic in its first variable.

Theorem 3.3.1. *Let $K : \mathbb{D} \times \mathbb{D} \to \mathcal{B}(\mathcal{E}_*)$ be a kernel on $\mathbb{D}$. Then $K = K_\Theta$ for some $\Theta \in \mathcal{S}(\mathbb{D}, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ and Hilbert space $\mathcal{E}_*$ if and only if*

$$I_{\mathcal{E}_*} - (1 - z\bar{w}) \cdot K \geq 0.$$

Proof. If $K = K_\Theta$, then

$$I_{\mathcal{E}_*} - (1 - z\bar{w})K(z, w) = \Theta(z)\Theta(w)^* \geq 0 \qquad (z, w \in \mathbb{D}).$$

Conversely, if $I_{\mathcal{E}_*} - (1 - z\overline{w}) \cdot K \geq 0$, then there exist a Hilbert space $\mathcal{F}$ and a function (a priori not necessarily analytic) $F : \mathbb{D} \to \mathcal{B}(\mathcal{F}, \mathcal{E}_*)$ such that

$$I_{\mathcal{E}} - (1 - z\bar{w})K(z, w) = F(z)F(w)^* \qquad (z, w \in \mathbb{D}).$$

Clearly, F is a contractive function on $\mathbb{D}$. Again, since $K \geq 0$, there exist a Hilbert space $\mathcal{G}$ and a function $G : \mathbb{D} \to \mathcal{B}(\mathcal{G}, \mathcal{E}_*)$ such that $K(z, w) = G(z)G(w)^*$, $z, w \in \mathbb{D}$. Then

$$I_{\mathcal{E}_*} - G(z)G(w)^* + z\bar{w}G(z)G(w)^* = F(z)F(w)^*,$$

and hence

$$I_{\mathcal{E}_*} + z\bar{w}G(z)G(w)^* = G(z)G(w)^* + F(z)F(w)^*,$$

for all $z, w \in \mathbb{D}$. Therefore

$$V : \begin{bmatrix} I_{\mathcal{E}_*} \\ \bar{w}G(w)^* \end{bmatrix} \eta \mapsto \begin{bmatrix} F(w)^* \\ G(w)^* \end{bmatrix} \eta \qquad (w \in \mathbb{D}, \eta \in \mathcal{E}_*),$$

defines an isometry from a subspace of $\mathcal{E}_* \oplus \mathcal{G}$ to $\mathcal{F} \oplus \mathcal{G}$. Then, adding an infinite-dimensional summand to $\mathcal{G}$ if necessary, V can then be extended to an isometry, denoted by V again, from $\mathcal{E}_* \oplus \mathcal{G}$ to $\mathcal{F} \oplus \mathcal{G}$. Set

$$V = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{E}_* \oplus \mathcal{G} \to \mathcal{F} \oplus \mathcal{G}.$$

Then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \eta \\ \bar{w}G(w)^*\eta \end{bmatrix} = \begin{bmatrix} F(w)^*\eta \\ G(w)^*\eta \end{bmatrix},$$

for all $\eta \in \mathcal{E}$ and $w \in \mathbb{D}$, which implies that

$$A + \bar{w}BG(w)^* = F(w)^* \text{ and } C + \bar{w}DG(w)^* = G(w)^*,$$

for all $w \in \mathbb{D}$. The latter equality implies that $G(w)^* = (I - \overline{w}D)^{-1}C$, and hence, the first equality yields

$$F(w)^* = A + \overline{w}B(I - \overline{w}D)^{-1}C,$$

for all $w \in \mathbb{D}$. Hence

$$F(z) = A^* + zC^*(I - zD^*)^{-1}B^* \qquad (z \in \mathbb{D}),$$

that is, $F = \tau_{V^*}$ is analytic on $\mathbb{D}$ and bounded by 1, where

$$V^* = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix},$$

is a co-isometric colligation. Consequently, $\Theta := F \in \mathcal{S}(\mathbb{D}, \mathcal{B}(\mathcal{F}, \mathcal{E}_*))$, and hence

$$I_{\mathcal{E}_*} - (1 - z\bar{w})K(z,w) = \Theta(z)\Theta(w)^*,$$

that is, $K(z,w) = \dfrac{I_{\mathcal{E}_*} - \Theta(z)\Theta(w)^*}{1 - z\bar{w}}$ for all $z, w \in \mathbb{D}$. This completes the proof. □

We denote by $\mathbb{S}_n$ the Szegö kernel on $\mathbb{D}^n$, that is

$$\mathbb{S}_n(\boldsymbol{z}, \boldsymbol{w}) = \prod_{i=1}^{n} \frac{1}{1 - z_i \bar{w}_i} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n).$$

Also we denote $\mathbb{S}_1$ simply by $\mathbb{S}$. The following is a variation of a result due to de Branges and Rovnyak [36, 37]. Also, we refer the reader to the classic Sz.-Nagy and Foias [90, Section 8, page 231] for a detailed proof and some historical notes. The proof below follows the proof of the previous theorem. Again, a priori we do not assume (in contrast to Sz.-Nagy and Foias) that K is analytic in its first variable.

Theorem 3.3.2. *Let $K : \mathbb{D} \times \mathbb{D} \to \mathcal{B}(\mathcal{E}_*)$ be a kernel. Then*

$$0 \leq K \leq \mathbb{S} \text{ and } \mathbb{S}^{-1} \cdot K \geq 0,$$

if and only if there exist a Hilbert space $\mathcal{E}$ and an operator-valued Schur function $\Theta \in \mathcal{S}(\mathbb{D}, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ such that

$$K(z,w) = \frac{\Theta(z)\Theta(w)^*}{1 - z\bar{w}} \qquad (z, w \in \mathbb{D}).$$

Proof. Suppose $0 \leq K \leq \mathbb{S}$ and $\mathbb{S}^{-1} \cdot K \geq 0$. Now $0 \leq K \leq \mathbb{S}$ implies that

$$\frac{1}{1 - z\bar{w}} I - K(z,w) \geq 0.$$

As in the proof of the previous theorem, there exist a Hilbert space $\mathcal{F}$ and a function $G : \mathbb{D} \to \mathcal{B}(\mathcal{F}, \mathcal{E}_*)$ such that

$$I - (1 - z\bar{w})K(z, w) = (1 - z\bar{w})G(z)G(w)^*.$$

Again, since $\mathbb{S}^{-1} \cdot K \geq 0$, there exist a Hilbert space $\mathcal{G}$ and a function $F : \mathbb{D} \to \mathcal{B}(\mathcal{G}, \mathcal{E}_*)$ such that

$$I - F(z)F(w)^* = (1 - z\bar{w})G(z)G(w)^*$$

The remaining argument is similar to that of the proof of the previous theorem. □

We now turn to de Branges-Rovnyak kernels on $\mathbb{D}^n$. Suppose $\Theta \in \mathcal{SA}(\mathbb{D}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$. Since M_Θ is a contraction from $H^2_{\mathcal{E}}(\mathbb{D}^n)$ into $H^2_{\mathcal{E}_*}(\mathbb{D}^n)$, it is easy to check (as also pointed out earlier) that $K_\Theta \geq 0$, where

$$K_\Theta(\boldsymbol{z}, \boldsymbol{w}) = \mathbb{S}_n(\boldsymbol{z}, \boldsymbol{w})^{-1}(I - \Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^*) \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n).$$

Here we say that K_Θ is a ($\mathcal{B}(\mathcal{E}_*)$-valued) de Branges-Rovnyak kernel on $\mathbb{D}^n$. In the following, we do not assume a priori that K is analytic in $z_1, \ldots, z_n$.

Theorem 3.3.3. *Let $K : \mathbb{D}^n \times \mathbb{D}^n \to \mathcal{B}(\mathcal{E}_*)$ be a kernel on $\mathbb{D}^n$. Then $K = K_\Theta$ for some Schur-Agler function $\Theta \in \mathcal{SA}(\mathbb{D}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$ and a Hilbert space $\mathcal{E}$ if and only if there exist $\mathcal{B}(\mathcal{E}_*)$-valued kernels $K_1, \ldots, K_n$ on $\mathbb{D}^n$ such that*

$$K(\boldsymbol{z}, \boldsymbol{w}) = \sum_{i=1}^{n} \frac{1}{\prod\limits_{j \neq i}(1 - z_j\bar{w}_j)} K_i(\boldsymbol{z}, \boldsymbol{w}),$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n$, and $I_{\mathcal{E}_*} - \mathbb{S}_n^{-1} \cdot K \geq 0$.

Proof. The "only if" part of this statement is easy, and the proof of the "if" part is similar to the proof of Theorem 3.3.1. We give only a sketch: Suppose $K_1, \ldots, K_n$ are $\mathcal{B}(\mathcal{E}_*)$-valued kernels on $\mathbb{D}^n$, $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n$, and suppose

$$K(\boldsymbol{z}, \boldsymbol{w}) = \sum_{i=1}^{n} \frac{1}{\prod\limits_{j \neq i}(1 - z_j\bar{w}_j)} K_i(\boldsymbol{z}, \boldsymbol{w}).$$

Then

$$\mathbb{S}_n^{-1}(\boldsymbol{z}, \boldsymbol{w})K(\boldsymbol{z}, \boldsymbol{w}) = \sum_{i=1}^{n}(1 - z_i\bar{w}_i)K_i(\boldsymbol{z}, \boldsymbol{w}).$$

Since $I_{\mathcal{E}_*} - \mathbb{S}_n^{-1} \cdot K \geq 0$, there exist a Hilbert space $\mathcal{G}$ and a function $G : \mathbb{D}^n \to \mathcal{B}(\mathcal{G}, \mathcal{E}_*)$ such that

$$I_{\mathcal{E}} - \mathbb{S}_n^{-1}(\boldsymbol{z}, \boldsymbol{w})K(\boldsymbol{z}, \boldsymbol{w}) = G(\boldsymbol{z})G(\boldsymbol{w})^*.$$

Again, since $K_i \geq 0$, there exist Hilbert spaces $\mathcal{F}_1, \ldots, \mathcal{F}_n$, and functions $F_i : \mathbb{D}^n \to \mathcal{B}(\mathcal{F}_i, \mathcal{E}_*)$, $i = 1, \ldots, n$, such that $K_i(\boldsymbol{z}, \boldsymbol{w}) = F_i(\boldsymbol{z})F_i(\boldsymbol{w})^*$ for all $i = 1, \ldots, n$. Hence

$$\mathbb{S}_n^{-1}(\boldsymbol{z}, \boldsymbol{w})K(\boldsymbol{z}, \boldsymbol{w}) = \sum_i^n (1 - z_i\bar{w}_i)F_i(\boldsymbol{z})F_i(\boldsymbol{w})^*,$$

which implies

$$I_{\mathcal{E}_*} + \sum_{i=1}^n z_i\bar{w}_i F_i(\boldsymbol{z})F_i(\boldsymbol{w})^* = G(\boldsymbol{z})G(\boldsymbol{w})^* + \sum_{i=1}^n F_i(\boldsymbol{z})F_i(\boldsymbol{w})^*,$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^n$. Now one can proceed with the lurking-isometry method, as in the proof of Theorem 3.3.1, to complete the proof of the theorem. □

An analogous statement also holds in the case of multipliers of the *Drury-Arveson space* H_n^2. In this setting, the de Branges-Rovnyak kernel K_Θ corresponding to $\Theta \in \mathcal{M}_d(\mathcal{E}, \mathcal{E}_*)$ is defined by

$$K_\Theta(\boldsymbol{z}, \boldsymbol{w}) = \frac{I - \Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^*}{1 - \langle \boldsymbol{z}, \boldsymbol{w} \rangle} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

The proof of the following theorem is completely analogous to the proof of Theorems 3.3.1 and 3.3.3. We leave details to the reader.

Theorem 3.3.4. *Let $\mathcal{E}_*$ be a Hilbert space and $K : \mathbb{B}^n \times \mathbb{B}^n \to \mathcal{B}(\mathcal{E})$ be a kernel. Then $K = K_\Theta$ for some $\Theta \in \mathcal{M}_n(\mathcal{E}, \mathcal{E}_*)$ and Hilbert space $\mathcal{E}$ if and only if*

$$I_{\mathcal{E}_*} - (1 - \langle \boldsymbol{z}, \boldsymbol{w} \rangle) \cdot K(\boldsymbol{z}, \boldsymbol{w}) \geq 0.$$

In the above theorem, we do not assume a priori that K is analytic in $z_1, \ldots, z_n$.

3.4 Agler Kernels and Factorizations

In this section we investigate factorizations of two-variable Schur functions in terms of Agler kernels. We shall be particularly interested in the case of one variable factors and Agler kernels of functions in $\mathcal{S}(\mathbb{D}^2)$.

Here and in what follows, $\mathcal{H}_K$ will denote the reproducing kernel Hilbert space corresponding to the kernel K. Moreover, if $K : \mathbb{D}^2 \times \mathbb{D}^2 \to \mathbb{C}$, then $K(\cdot, \boldsymbol{w}) \in \mathcal{H}_K$ will denote the kernel function at $\boldsymbol{w} \in \mathbb{D}^2$, that is

$$\Big(K(\cdot, \boldsymbol{w})\Big)(\boldsymbol{z}) = K(\boldsymbol{z}, \boldsymbol{w}) \qquad (\boldsymbol{z} \in \mathbb{D}^2),$$

and

$$f(\boldsymbol{w}) = \langle f, K(\cdot, \boldsymbol{w}) \rangle_{\mathcal{H}_K},$$

for all $f \in \mathcal{H}_K$ and $\boldsymbol{w} \in \mathbb{D}^2$. For notational convenience we write $\mathbf{0} = (0,0)$.

We are now ready for the main result of this section (see Remark 3.4.1 on the assumption $\varphi(\mathbf{0}) \neq 0$):

Theorem 3.4.1. *Let $\varphi \in \mathcal{S}(\mathbb{D}^2)$ and suppose $\varphi(\mathbf{0}) \neq 0$. The following assertions are equivalent:*

(1) There exist φ_1 and φ_2 in $\mathcal{S}(\mathbb{D})$ such that

$$\varphi(\boldsymbol{z}) = \varphi_1(z_1)\varphi_2(z_2) \qquad (\boldsymbol{z} \in \mathbb{D}^2).$$

(2) There exist Agler kernels $\{K_1, K_2\}$ of φ such that K_1 depends only on z_1 and $\bar{w}_1$, and

$$\overline{\varphi(\mathbf{0})}\, K_2(\cdot, (w_1, 0)) = \overline{\varphi(w_1, 0)}\, K_2(\cdot, \mathbf{0}) \qquad (w_1 \in \mathbb{D}).$$

(3) There exist Agler kernels $\{L_1, L_2\}$ of φ such that all the functions in $\mathcal{H}_{L_1}$ depends only on z_1, and

$$\varphi(\mathbf{0}) f(\cdot, 0) = \varphi(\cdot, 0)\, f(\mathbf{0}) \qquad (f \in \mathcal{H}_{L_2}).$$

(4) $\varphi = \tau_V$ for some co-isometric colligation

$$V = \begin{bmatrix} \varphi(\mathbf{0}) & B_1 & B_2 \\ C_1 & D_1 & D_2 \\ C_2 & 0 & D_4 \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus (\mathcal{H}_1 \oplus \mathcal{H}_2)),$$

with $\varphi(\mathbf{0}) D_2 = C_1 B_2$.

Proof. Suppose first that $\varphi(\boldsymbol{z}) = \varphi_1(z_1)\varphi_2(z_2)$, $\boldsymbol{z} \in \mathbb{D}^2$, for some φ_1 and φ_2 in $\mathcal{S}(\mathbb{D})$. Then

$$1 - \varphi(\boldsymbol{z})\overline{\varphi(\boldsymbol{w})} = 1 - \varphi_1(z_1)\overline{\varphi_1(w_1)} + \varphi_1(z_1)(1 - \varphi_2(z_2)\overline{\varphi_2(w_2)})\,\overline{\varphi_1(w_1)},$$

and hence

$$1 - \varphi(\boldsymbol{z})\overline{\varphi(\boldsymbol{w})} = (1 - z_1\bar{w}_1)K_1(\boldsymbol{z}, \boldsymbol{w}) + (1 - z_2\bar{w}_2)K_2(\boldsymbol{z}, \boldsymbol{w}),$$

where

$$K_1(\boldsymbol{z}, \boldsymbol{w}) = \frac{1 - \varphi_1(z_1)\overline{\varphi_1(w_1)}}{1 - z_1\bar{w}_1} \quad \text{and} \quad K_2(\boldsymbol{z}, \boldsymbol{w}) = \frac{\varphi_1(z_1)(1 - \varphi_2(z_2)\overline{\varphi_2(w_2)})\,\overline{\varphi_1(w_1)}}{1 - z_2\bar{w}_2},$$

and $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^2$. Then $\{K_1, K_2\}$ are Agler kernels of φ and satisfies the conditions of (2). This proves (1)$\Rightarrow$(2).

(2)$\Rightarrow$ (3): Set $L_i = K_i$, $i = 1, 2$, and suppose $f \in \mathcal{H}_{K_1}$. Since

$$\boldsymbol{w} \mapsto f(w) = \langle f, L_1(\cdot, \boldsymbol{w})\rangle = \langle f, K_1(\cdot, \boldsymbol{w})\rangle,$$

and K_1 depends only on z_1 and $\bar{w}_1$, it follows that all the functions in $\mathcal{H}_{L_1}$ depend only on z_1. On the other hand, if $f \in \mathcal{H}_{L_2}$, then

$$\begin{aligned}\varphi(\mathbf{0})f(w_1,0) &= \langle f, \overline{\varphi(\mathbf{0})}L_2(\cdot,(w_1,0))\rangle_{\mathcal{H}_{L_2}}\\ &= \langle f, \overline{\varphi(\mathbf{0})}K_2(\cdot,(w_1,0))\rangle_{\mathcal{H}_{K_2}}\\ &= \langle f, \overline{\varphi(w_1,0)}K_2(\cdot,(\mathbf{0}))\rangle_{\mathcal{H}_{K_2}}\\ &= \varphi(w_1,0)\langle f, K_2(\cdot,(\mathbf{0}))\rangle_{\mathcal{H}_{K_2}},\end{aligned}$$

and hence $\varphi(\mathbf{0})g(w_1,0) = \varphi(w_1,0)f(\mathbf{0})$ for all $w_1 \in \mathbb{D}$.

(3)$\Rightarrow$ (2): This is just the reverse of the argument in the above proof.

(2)$\Rightarrow$ (4): Suppose $\{K_1, K_2\}$ are Agler kernels of φ, and suppose that K_1 depends only on z_1 and $\bar{w}_1$, and

$$\overline{\varphi(\mathbf{0})}\, K_2(\cdot,(w_1,0)) = \overline{\varphi(w_1,0)}\, K_2(\cdot,\mathbf{0}) \qquad (w_1 \in \mathbb{D}). \tag{3.4.1}$$

Now

$$1-\varphi(\boldsymbol{z})\overline{\varphi(\boldsymbol{w})} = (1-z_1\bar{w}_1)\langle K_1(\cdot,\boldsymbol{w}), K_1(\cdot,\boldsymbol{z})\rangle_{\mathcal{H}_{K_1}} + (1-z_2\bar{w}_2)\langle K_2(\cdot,\boldsymbol{w}), K_2(\cdot,\boldsymbol{z})\rangle_{\mathcal{H}_{K_2}},$$

implies that

$$\begin{aligned}1+z_1\bar{w}_1\langle K_1(\cdot,\boldsymbol{w}), K_1(\cdot,\boldsymbol{z})\rangle_{\mathcal{H}_{K_1}} + z_2\bar{w}_2\langle K_2(\cdot,\boldsymbol{w}), K_2(\cdot,\boldsymbol{z})\rangle_{\mathcal{H}_{K_2}} &= \varphi(\boldsymbol{z})\overline{\varphi(\boldsymbol{w})}\\ &+ \langle K_1(\cdot,\boldsymbol{w}), K_1(\cdot,\boldsymbol{z})\rangle_{\mathcal{H}_{K_1}} + \langle K_2(\cdot,\boldsymbol{w}), K_2(\cdot,\boldsymbol{z})\rangle_{\mathcal{H}_{K_2}},\end{aligned}$$

for all $\boldsymbol{z},\boldsymbol{w} \in \mathbb{D}^2$. Therefore

$$V : \begin{bmatrix} 1 \\ \bar{w}_1 K_1(\cdot,\boldsymbol{w}) \\ \bar{w}_2 K_2(\cdot,\boldsymbol{w}) \end{bmatrix} \mapsto \begin{bmatrix} \overline{\varphi(\boldsymbol{w})} \\ K_1(\cdot,\boldsymbol{w}) \\ K_2(\cdot,\boldsymbol{w}) \end{bmatrix} \qquad (\boldsymbol{w} \in \mathbb{D}^2),$$

defines an isometry from $\mathcal{D}$ onto $\mathcal{R}$, where

$$\mathcal{D} = \overline{\operatorname{span}}\left\{ \begin{bmatrix} 1 \\ \bar{w}_1 K_1(\cdot,\boldsymbol{w}) \\ \bar{w}_2 K_2(\cdot,\boldsymbol{w}) \end{bmatrix} : \boldsymbol{w} \in \mathbb{D}^2 \right\} \subseteq \mathbb{C} \oplus \mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2},$$

and

$$\mathcal{R} = \overline{\operatorname{span}}\left\{ \begin{bmatrix} \overline{\varphi(\boldsymbol{w})} \\ K_1(\cdot,\boldsymbol{w}) \\ K_2(\cdot,\boldsymbol{w}) \end{bmatrix} : \boldsymbol{w} \in \mathbb{D}^2 \right\} \subseteq \mathbb{C} \oplus \mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2}.$$

Note that

$$\mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2} = \overline{\operatorname{span}}\left\{ \begin{bmatrix} \bar{w}_1 K_1(\cdot,\boldsymbol{w}) \\ \bar{w}_2 K_2(\cdot,\boldsymbol{w}) \end{bmatrix} : \boldsymbol{w} \in \mathbb{D}^2 \right\}.$$

Indeed, if

$$\begin{bmatrix} f \\ g \end{bmatrix} \in [\mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2}] \ominus \operatorname{span}\left\{ \begin{bmatrix} \bar{w}_1 K_1(\cdot, \boldsymbol{w}) \\ \bar{w}_2 K_2(\cdot, \boldsymbol{w}) \end{bmatrix} : \boldsymbol{w} \in \mathbb{D}^2 \right\},$$

then

$$0 = \left\langle \begin{bmatrix} f \\ g \end{bmatrix}, \begin{bmatrix} \bar{w}_1 K_1(\cdot, \boldsymbol{w}) \\ \bar{w}_2 K_2(\cdot, \boldsymbol{w}) \end{bmatrix} \right\rangle_{\mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2}} = \langle f, \bar{w}_1 K_1(\cdot, \boldsymbol{w}) \rangle_{\mathcal{H}_{K_1}} + \langle g, \bar{w}_2 K_2(\cdot, \boldsymbol{w}) \rangle_{\mathcal{H}_{K_2}},$$

that is, $w_1 f(\boldsymbol{w}) + w_2 g(\boldsymbol{w}) = 0$ for all $\boldsymbol{w} \in \mathbb{D}^2$. Since K_1 depends only on z_1 and $\bar{w}_1$, all the functions in $\mathcal{H}_{K_1}$ depend only on z_1. Therefore, if $w_2 = 0$, then the above equality implies that $w_1 f((w_1, 0)) = 0$, and hence $f = 0$. Consequently, $w_2 g(\boldsymbol{w}) = 0$, $\boldsymbol{w} \in \mathbb{D}^2$, and hence $g = 0$, proves our claim. In particular, $V \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2})$ is an isometry. The above proof also implies that

$$\mathcal{H}_{K_i} = \overline{\operatorname{span}}\{\bar{w}_i K_i(\cdot, \boldsymbol{w}) : \boldsymbol{w} \in \mathbb{D}^2\},$$

for $i = 1, 2$. Now we consider the co-isometry V^* and set

$$V^* = \begin{bmatrix} \varphi(\boldsymbol{0}) & B \\ C & D \end{bmatrix} = \begin{bmatrix} \varphi(\boldsymbol{0}) & B_1 & B_2 \\ C_1 & D_1 & D_2 \\ C_2 & D_3 & D_4 \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus (\mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2})).$$

Since

$$\begin{bmatrix} \overline{\varphi(\boldsymbol{0})} & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} 1 \\ \bar{w}_1 K_1(\cdot, \boldsymbol{w}) \\ \bar{w}_2 K_2(\cdot, \boldsymbol{w}) \end{bmatrix} = \begin{bmatrix} \overline{\varphi(\boldsymbol{w})} \\ K_1(\cdot, \boldsymbol{w}) \\ K_2(\cdot, \boldsymbol{w}) \end{bmatrix} \qquad (\boldsymbol{w} \in \mathbb{D}^2),$$

it follows that

$$\overline{\varphi(\boldsymbol{0})} + C^* \begin{bmatrix} \bar{w}_1 K_1(\cdot, \boldsymbol{w}) \\ \bar{w}_2 K_2(\cdot, \boldsymbol{w}) \end{bmatrix} = \overline{\varphi(\boldsymbol{w})},$$

and

$$B^* + D^* \begin{bmatrix} \bar{w}_1 K_1(\cdot, \boldsymbol{w}) \\ \bar{w}_2 K_2(\cdot, \boldsymbol{w}) \end{bmatrix} = \begin{bmatrix} K_1(\cdot, \boldsymbol{w}) \\ K_2(\cdot, \boldsymbol{w}) \end{bmatrix},$$

for all $\boldsymbol{w} \in \mathbb{D}^2$. Now plug $\boldsymbol{w} = \boldsymbol{0}$ into the identity above to see that

$$B^* = \begin{bmatrix} K_1(\cdot, \boldsymbol{0}) \\ K_2(\cdot, \boldsymbol{0}) \end{bmatrix},$$

and hence

$$D^* \begin{bmatrix} \bar{w}_1 K_1(\cdot, \boldsymbol{w}) \\ \bar{w}_2 K_2(\cdot, \boldsymbol{w}) \end{bmatrix} = \begin{bmatrix} K_1(\cdot, \boldsymbol{w}) - K_1(\cdot, \boldsymbol{0}) \\ K_2(\cdot, \boldsymbol{w}) - K_2(\cdot, \boldsymbol{0}) \end{bmatrix}.$$

Since $D^* = \begin{bmatrix} D_1^* & D_3^* \\ D_2^* & D_4^* \end{bmatrix}$, it follows that

$$\bar{w}_1 D_1^* K_1(\cdot, \boldsymbol{w}) + \bar{w}_2 D_3^* K_2(\cdot, \boldsymbol{w}) = K_1(\cdot, \boldsymbol{w}) - K_1(\cdot, \boldsymbol{0}),$$

and

$$\bar{w}_1 D_2^* K_1(\cdot, \boldsymbol{w}) + \bar{w}_2 D_4^* K_2(\cdot, \boldsymbol{w}) = K_2(\cdot, \boldsymbol{w}) - K_2(\cdot, \boldsymbol{0}). \tag{3.4.2}$$

Plugging $w_2 = 0$ into the first identity, we get

$$\bar{w}_1 D_1^* K_1(\cdot, (w_1, 0)) = K_1(\cdot, (w_1, 0)) - K_1(\cdot, \boldsymbol{0}),$$

for all $w_1 \in \mathbb{D}$. Again, noting that K_1 depends only on z_1 and $\bar{w}_1$, we deduce

$$\bar{w}_1 D_1^* K_1(\cdot, \boldsymbol{w}) = K_1(\cdot, \boldsymbol{w}) - K_1(\cdot, \boldsymbol{0}) \qquad (\boldsymbol{w} \in \mathbb{D}^2),$$

and consequently $D_3^* \Big(\bar{w}_2 K_2(\cdot, \boldsymbol{w}) \Big) = 0$, $\boldsymbol{w} \in \mathbb{D}^2$. This, along with the fact that $\{\bar{w}_2 K_2(\cdot, \boldsymbol{w}) : \boldsymbol{w} \in \mathbb{D}^2\}$ is dense in $\mathcal{H}_{K_2}$, implies $D_3 = 0$. We next plug $w_2 = 0$ into (3.4.2) to get

$$D_2^* (\bar{w}_1 K_1(\cdot, (w_1, 0))) = K_2(\cdot, (w_1, 0)) - K_2(\cdot, \boldsymbol{0}).$$

Now we turn to compute C_1^*. Since $C^* \begin{bmatrix} \bar{w}_1 K_1(\cdot, \boldsymbol{w}) \\ \bar{w} K_2(\cdot, \boldsymbol{w}) \end{bmatrix} = \overline{\varphi(\boldsymbol{w})} - \overline{\varphi(\boldsymbol{0})}$, we have

$$C_1^*(\bar{w}_1 K_1(\cdot, \boldsymbol{w})) + C_2^*(\bar{w}_2 K_2(\cdot, \boldsymbol{w})) = \overline{\varphi(\boldsymbol{w})} - \overline{\varphi(\boldsymbol{0})} \qquad (\boldsymbol{w} \in \mathbb{D}^2).$$

In particular, if $w_2 = 0$, then

$$C_1^*(\bar{w}_1 K_1(\cdot, \boldsymbol{w})) = \overline{\varphi((w_1, 0))} - \overline{\varphi(\boldsymbol{0})} \qquad (w_1 \in \mathbb{D}).$$

Finally, we compute B_2. Observe that

$$B_2^* + D_2^*(\bar{w}_1 K_1(\cdot, \boldsymbol{w})) + D_4^*(\bar{w}_2 K_2(\cdot, \boldsymbol{w})) = K_2(\cdot, \boldsymbol{w}),$$

for all $\boldsymbol{w} \in \mathbb{D}^2$. If $w_2 = 0$, then

$$B_2^* + D_2^*(\bar{w}_1 K_1(\cdot, (w_1, 0))) = K_2(\cdot, (w_1, 0)),$$

which implies that $B_2^* = K_2(\cdot, \boldsymbol{0})$. Finally, if we let $\boldsymbol{w} \in \mathbb{D}^2$, then

$$B_2^* C_1^*(\bar{w}_1 K_1(\cdot, \boldsymbol{w})) = (\overline{\varphi(w_1, 0)} - \overline{\varphi(\boldsymbol{0})}) K_2(\cdot, \boldsymbol{0}) = \overline{\varphi(\boldsymbol{0})} K_2(\cdot, (w_1, 0)) - \overline{\varphi(\boldsymbol{0})} K_2(\cdot, \boldsymbol{0}),$$

by assumption (3.4.1), and hence

$$B_2^* C_1^*(\bar{w}_1 K_1(\cdot, \boldsymbol{w})) = \overline{\varphi(\boldsymbol{0})}(K_2(\cdot, (w_1, 0)) - K_2(\cdot, \boldsymbol{0})) = \overline{\varphi(\boldsymbol{0})} D_2^*(\bar{w}_1 K_1(\cdot, \boldsymbol{w})).$$

This proves that $\varphi(\boldsymbol{0}) D_2 = C_1 B_2$.

(4)⇒ (1) is essentially along the lines of [48, Theorem 2.3]. However, for the sake of completeness, we sketch the proof. Let $a = \varphi(\mathbf{0})$. Since $VV^* = I$, it follows that

$$I = \begin{bmatrix} |a|^2 + B_1B_1^* + B_2B_2^* & aC_1^* + B_1D_1^* + B_2D_2^* & aC_2^* + B_2D_4^* \\ \overline{a}C_1 + D_1B_1^* + D_2B_2^* & C_1C_1^* + D_1D_1^* + D_2D_2^* & C_1C_2^* + D_2D_4^* \\ \overline{a}C_2 + D_4B_2^* & C_2C_1^* + D_4D_2^* & C_2C_2^* + D_4D_4^* \end{bmatrix}.$$

Then there exists $y \in \mathbb{C}$ such that

$$|y|^2 = |a|^2 + B_2B_2^* = 1 - B_1B_1^* > 0,$$

as $a \neq 0$. Let $x = \frac{a}{y}$, and

$$V_1 = \begin{bmatrix} y & B_1 \\ \frac{1}{x}C_1 & D_1 \end{bmatrix} \quad \text{and} \quad V_2 = \begin{bmatrix} x & \frac{1}{y}B_2 \\ C_2 & D_4 \end{bmatrix}.$$

Clearly, $x \neq 0$. We first claim that V_1 and V_2 are co-isometries. Indeed

$$V_2V_2^* = \begin{bmatrix} |x|^2 + \frac{1}{|y|^2}B_2B_2^* & xC_2^* + \frac{1}{y}B_2D_4^* \\ \bar{x}C_2 + \frac{1}{\bar{y}}D_4B_2^* & C_2C_2^* + D_4D_4^* \end{bmatrix} = \begin{bmatrix} 1 & xC_2^* + \frac{1}{y}B_2D_4^* \\ \bar{x}C_2 + \frac{1}{\bar{y}}D_4B_2^* & C_2C_2^* + D_4D_4^* \end{bmatrix}.$$

as $|y|^2 = |a|^2 + B_2B_2^*$ and $a = xy$. Also note that, since $aC_2^* + B_2D_4^* = 0$, we have that $xC_2^* + \frac{1}{y}B_2D_4^* = 0$, which implies that V_2 is a co-isometry. Next, we compute

$$V_1V_1^* = \begin{bmatrix} |y|^2 + B_1B_1^* & \frac{y}{\bar{x}}C_1^* + B_1D_1^* \\ \frac{\bar{y}}{x}C_1 + D_1B_1^* & \frac{1}{|x|^2}C_1C_1^* + D_1D_1^* \end{bmatrix}.$$

Since $C_1C_1^* + D_1D_1^* + D_2D_2^* = 1$, $aD_2 = C_1B_2$, $a = xy$ and $|y|^2 - |a|^2 = B_2B_2^*$, we have

$$\frac{1}{|x|^2}C_1C_1^* + D_1D_1^* = 1.$$

Moreover, since $aC_1^* + B_1D_1^* + B_2D_2^* = 0$ implies that $\frac{y}{\bar{x}}C_1^* + B_1D_1^* = 0$, we have that V_1 is also a co-isometry. Finally, set $\varphi_1(\boldsymbol{z}) = \tau_{V_1}(z_1)$ and $\varphi_2(\boldsymbol{z}) = \tau_{V_2}(z_2)$, $\boldsymbol{z} \in \mathbb{D}^2$. It is then easy to check that

$$\varphi(\boldsymbol{z}) = \tau_V(z) = \tau_{V_1}(z_1)\tau_{V_2}(z_2) = \varphi_1(\boldsymbol{z})\varphi_2(\boldsymbol{z}),$$

for all $\boldsymbol{z} \in \mathbb{D}^2$. This completes the proof. □

In the setting of Theorem 3.4.1, one can also explicitly compute the entries of the block operator matrix V in part (4). The technique involved in the computation is standard and quite well known (cf. [17, Remark 3.6]). However, we outline some details for the sake of making this chapter self-contained. We already know that

$$B_2^* = K_2(\cdot, \mathbf{0}) \quad \text{and} \quad C_1^*(\bar{w}_1K_1(\cdot, \boldsymbol{w})) = \overline{\varphi((w_1, 0))} - \overline{\varphi(\mathbf{0})},$$

and

$$D_2^*(\bar{w}_1 K_1(\cdot,(w_1,0))) = K_2(\cdot,(w_1,0)) - K_2(\cdot,\mathbf{0}),$$

for all $\boldsymbol{w} \in \mathbb{D}^2$. Now let $g \in \mathcal{H}_{K_2}$ and $\boldsymbol{w} \in \mathbb{D}^2$. Then

$$(z_1 D_2 g)(\boldsymbol{w}) = \langle g, K_2(\cdot,(w_1,0)) - K_2(\cdot,\mathbf{0})\rangle = g((w_1,0)) - g(\mathbf{0}),$$

and hence

$$(D_2 g)(\boldsymbol{w}) = \frac{g((w_1,0)) - g(\mathbf{0})}{w_1} \qquad (\boldsymbol{w} \in \mathbb{D}^2).$$

for all $g \in \mathcal{H}_{K_2}$. Similarly, if $w_1 = 0$, then (3.4.2) implies that

$$\bar{w}_2 D_4^* K_2(\cdot,(w_1,0)) = K_2(\cdot,\boldsymbol{w}) - K_2(\cdot,(w_1,0)),$$

and hence, in a similar way we have

$$(D_4 g)(\boldsymbol{w}) = \frac{g(\boldsymbol{w}) - g((w_1,0))}{w_2} \qquad (g \in \mathcal{H}_{K_2}, \boldsymbol{w} \in \mathbb{D}^2),$$

as well as

$$(D_1 f)(\boldsymbol{w}) = \frac{f(\boldsymbol{w}) - f(\mathbf{0})}{w_1} \qquad (f \in \mathcal{H}_{K_1}, \boldsymbol{w} \in \mathbb{D}^2).$$

Now we turn to compute C_1 and C_2. Since $C_1^*(\bar{w}_1 K_1(\cdot,\boldsymbol{w})) = \overline{\varphi((w_1,0))} - \overline{\varphi(\mathbf{0})}$, we have

$$(z_1 C_1 1)(\boldsymbol{w}) = \langle C_1 1, \bar{w}_1 K_1(\cdot,\boldsymbol{w})\rangle = \varphi((w_1,0)) - \varphi(\mathbf{0}),$$

and hence

$$(C_1 1)(\boldsymbol{w}) = \frac{\varphi(w_1,0) - \varphi(\mathbf{0})}{w_1} \quad \text{and} \quad (C_2 1)(\boldsymbol{w}) = \frac{\varphi(\boldsymbol{w}) - \varphi(w_1,0)}{w_2},$$

for all $\boldsymbol{w} \in \mathbb{D}^2$. Finally, we note that $(B_1 f)(\boldsymbol{w}) = f(\mathbf{0})$ and $(B_2 g)(\boldsymbol{w}) = g(\mathbf{0})$ for all $f \in \mathcal{H}_{K_1}$ and $g \in \mathcal{H}_{K_2}$.

In particular, if φ is inner, then we have the following:

Example 3.4.2. *Given an inner function $\varphi \in \mathcal{S}(\mathbb{D}^2)$ satisfying one of the equivalent conditions of Theorem 3.4.1, we have $\varphi(\boldsymbol{z}) = \varphi_1(z_1)\varphi_2(z_2)$, $\boldsymbol{z} \in \mathbb{D}^2$, for some φ_1 and φ_2 in $\mathcal{S}(\mathbb{D})$. Then*

$$1 = |\varphi(\boldsymbol{z})| = |\varphi_1(z_1)||\varphi_2(z_2)| \le |\varphi_1(z_1)| \le 1 \qquad (\boldsymbol{z} \in \mathbb{T}^2 \text{ a.e.})$$

from which we see that φ_1, as well as φ_2, are inner functions. Moreover, for $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{D}^2$, we have

$$1 - \varphi(\boldsymbol{z})\overline{\varphi(\boldsymbol{w})} = 1 - \varphi_1(z_1)\overline{\varphi_1(w_1)} + \varphi_1(z_1)(1 - \varphi_2(z_2)\overline{\varphi_2(w_2)})\overline{\varphi_1(w_1)}.$$

Hence $\{K_1, K_2\}$ are Agler kernels of φ, where

$$K_1(\boldsymbol{z},\boldsymbol{w}) = \frac{1-\varphi_1(z_1)\overline{\varphi_2(w_1)}}{1-z_1\bar{w}_1} \quad \textit{and} \quad K_2(\boldsymbol{z},\boldsymbol{w}) = \frac{\varphi_1(z_1)(1-\varphi_2(z_2)\overline{\varphi_2(w_2)})\overline{\varphi_1(w_1)}}{1-z_2\bar{w}_2}.$$

In this case the corresponding reproducing kernel Hilbert spaces are given by

$$\mathcal{H}_{K_1} = \mathcal{Q}_{\varphi_1} \otimes \mathbb{C} \quad \textit{and} \quad \mathcal{H}_{K_2} = \varphi_1\mathbb{C} \otimes \mathcal{Q}_{\varphi_2},$$

where $\mathcal{Q}_{\varphi_1} = H^2(\mathbb{D})/\varphi_1 H^2(\mathbb{D})$ and $\mathcal{Q}_{\varphi_2} = H^2(\mathbb{D})/\varphi_2 H^2(\mathbb{D})$ are model spaces. Moreover, the co-isometric (unitary) colligation operator V with state space $\mathcal{H}_{K_1} \oplus \mathcal{H}_{K_2}$ is given by

$$V = \begin{bmatrix} \varphi(\mathbf{0}) & P_{\mathbb{C}}|_{\mathcal{Q}_{\varphi_1}} & \varphi(\mathbf{0})P_{\mathbb{C}}M_{\varphi_1}^* \otimes P_{\mathbb{C}}|_{\mathcal{Q}_{\varphi_2}} \\ \varphi_2(0)M_z^*M_{\varphi_1}|_{\mathbb{C}} & M_z^*|_{\mathcal{Q}_{\varphi_1}} & M_z^*M_{\varphi_1}P_{\mathbb{C}}M_{\varphi_1}^* \otimes P_{\mathbb{C}}|_{\mathcal{Q}_{\varphi_2}} \\ M_{\varphi_1}|_{\mathbb{C}} \otimes M_z^*M_{\varphi_2}|_{\mathbb{C}} & 0 & I_{\varphi_1\mathbb{C}} \otimes M_z^*|_{\mathcal{Q}_{\varphi_2}} \end{bmatrix}.$$

Finally, we comment on the assumption that $\varphi(\mathbf{0}) \neq 0$ in Theorem 3.4.1.

Remark 3.4.1. *In the proof of Theorem 3.4.1, $\varphi(\mathbf{0}) \neq 0$ has been used only for the implication (4)$\Rightarrow$ (1). In the $\varphi(\mathbf{0}) = 0$ case, one can easily modify the argument of the aforementioned case to prove a similar statement. Here is a sample statement:*

Let $\varphi \in \mathcal{S}(\mathbb{D}^2)$ be a non-zero function and suppose $\varphi(\mathbf{0}) = 0$. Then the following are equivalent:

(1) $\varphi(\boldsymbol{z}) = \varphi_1(z_1)\varphi_2(z_2)$ for some $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{D})$ such that $\varphi_2(0) \neq 0$.

(2) $\varphi(\boldsymbol{z}) = z_1^p\varphi_1(z_1)\varphi_2(z_2)$ for some $p \geq 1$ and $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{D})$ such that $\varphi_1(0) \neq 0$ and $\varphi_2(0) \neq 0$.

(3) There exists $p \geq 1$ such that $\tilde{\varphi}(\boldsymbol{z}) = z_1^{-p}\varphi(\boldsymbol{z}) \in \mathcal{S}(\mathbb{D}^2)$, $\tilde{\varphi}(\mathbf{0}) \neq 0$, and there exist Agler kernels $\{K_1, K_2\}$ of $\tilde{\varphi}$ such that K_1 depends only on z_1 and $\bar{w}_1$, and

$$\overline{\tilde{\varphi}(\mathbf{0})}K_2(\cdot,(w_1,0)) = \overline{\tilde{\varphi}(w_1,0)}K_2(\cdot,\mathbf{0}) \qquad (w_1 \in \mathbb{D}).$$

(4) There exists $p \geq 1$ such that $\tilde{\varphi}(\boldsymbol{z}) = z_1^{-p}\varphi(\boldsymbol{z}) \in \mathcal{S}(\mathbb{D}^2)$, $\tilde{\varphi}(\mathbf{0}) \neq 0$, and $\tilde{\varphi} = \tau_V$ for some co-isometric colligation

$$V = \begin{bmatrix} \tilde{\varphi}(\mathbf{0}) & B_1 & B_2 \\ C_1 & D_1 & D_2 \\ C_2 & 0 & D_4 \end{bmatrix},$$

such that $\tilde{\varphi}(\mathbf{0})D_2 = C_1B_2$.

3.5 Counterexamples and a converse

We now return to two-variable inner functions, which we encountered in Section 3.2. The aim of this section is to further analyze Theorem 3.2.1. We begin by exhibiting counterexamples to the converse of Theorem 3.2.1. Then, in Theorem 3.5.3, we present a weak converse to Theorem 3.2.1.

Example 3.5.1. *Fix $t \in (0,1)$, and define*

$$\varphi_t(\boldsymbol{z}) = \frac{z_1 z_2 - t}{1 - t z_1 z_2} \qquad (\boldsymbol{z} \in \mathbb{D}^2).$$

It is fairly easy to verify that

$$|\varphi_t(\boldsymbol{z})| = 1 \qquad (\boldsymbol{z} \in \mathbb{T}^2),$$

and hence, φ_t is a rational inner function. Contrary to what we are proving, let us assume that there are Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, an operator $D_1 \in C_{0\cdot}$, and an isometric colligation

$$V_t = \begin{bmatrix} -t & B_1 & B_2 \\ C_1 & D_1 & D_2 \\ C_2 & 0 & D_3 \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2),$$

such that $\tau_{V_t} = \varphi_t$. Since

$$\varphi_t(\boldsymbol{z}) + t = \frac{(1-t^2) z_1 z_2}{1 - t z_1 z_2},$$

the preceding equality yields

$$\frac{(1-t^2)z_1 z_2}{1 - t z_1 z_2} = \begin{bmatrix} B_1 & B_2 \end{bmatrix} \left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix} \begin{bmatrix} D_1 & D_2 \\ 0 & D_3 \end{bmatrix} \right)^{-1} \begin{bmatrix} z_1 & 0 \\ 0 & z_2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

Now the left side is equal to

$$(1-t^2) z_1 z_2 (1 + t z_1 z_2 + t^2 z_1^2 z_2^2 + \cdots),$$

and the right side is equal to

$$z_1 B_1 (I - z_1 D_1)^{-1} C_1 + z_2 B_2 (I - z_2 D_4)^{-1} C_2 + z_1 z_2 B_1 (I - z_1 D_1)^{-1} D_2 (I - z_2 D_3)^{-1} C_2.$$

Comparing the coefficients of z_1, we see that $B_1 D_1^n C_1 = 0$, $n \geq 0$. Since $V_t^* V_t = I$, we have

$$\begin{bmatrix} -t & C_1^* & C_2^* \\ B_1^* & D_1^* & 0 \\ B_2^* & D_2^* & D_3^* \end{bmatrix} \begin{bmatrix} -t & B_1 & B_2 \\ C_1 & D_1 & D_2 \\ C_2 & 0 & D_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

In particular $B_1^* B_1 + D_1^* D_1 = I$ and $-t B_1 + C_1^* D_1 = 0$. The first equality implies (see the proof of the equality in (3.2.4)) that

$$\sum_{n=0}^{\infty} D_1^{*n} B_1^* B_1 D_1^n = I,$$

in the strong operator topology as $D_1 \in C_{0\cdot}$. Therefore

$$\sum_{n=0}^{\infty} \|B_1 D_1^n h\|^2 = \|h\|^2,$$

for all $h \in \mathcal{H}_1$. In particular, if we choose $h = C_1(1)$, then

$$\sum_{n=1}^{\infty} \|B_1 D_1^n C_1(1)\|^2 = \|C_1(1)\|^2.$$

Since $B_1 D_1^n C_1 = 0$ for all $n \geq 0$, we deduce $C_1 = 0$. Then $-tB_1 + C_1^* D_1 = 0$ implies that $B_1 = 0$, and hence $D_1^* D_1 = I$. However, this and the fact that $D_1 \in C_{0\cdot}$ are mutually contradictory. This shows that $\varphi_t \neq \tau_{V_t}$ for any isometric colligation V_t and $D_1 \in C_{0\cdot}$.

Now we turn to a weak converse of Theorem 3.2.1 in the setting of rational inner functions. We need the following inverse formula for 2×2 block matrices [69, page 18]:

Theorem 3.5.2. *Let $X = \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \in \mathcal{B}(\mathbb{C}^m \oplus \mathbb{C}^n)$, and suppose that P is invertible. Then X is invertible if and only if $\Delta := S - RP^{-1}Q$ is invertible. In this case, the inverse of X is given by*

$$X^{-1} = \begin{bmatrix} P^{-1} + P^{-1}Q\Delta^{-1}RP^{-1} & -P^{-1}Q\Delta^{-1} \\ -\Delta^{-1}RP^{-1} & \Delta^{-1} \end{bmatrix}.$$

We are now ready to establish the promised weak converse of Theorem 3.2.1.

Theorem 3.5.3. *Let $\varphi \in \mathcal{S}(\mathbb{D}^2)$ be a rational inner function and suppose $\varphi(\mathbf{0}) \neq 0$. Then the following are equivalent:*

(1) $\varphi = \tau_V$ for some isometric colligation

$$V = \begin{bmatrix} a & B \\ C & D \end{bmatrix} = \begin{bmatrix} a & B_1 & B_2 \\ C_1 & D_1 & D_2 \\ C_2 & 0 & D_3 \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2),$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite-dimensional Hilbert spaces and $D_1, D_3 \in C_{0\cdot}$.

(2) $\varphi(\boldsymbol{z}) = \varphi_1(z_1)\varphi(z_2)$, $\boldsymbol{z} \in \mathbb{D}^2$, for some rational inner functions φ_1 and φ_2 (in $\mathcal{S}(\mathbb{D})$).

Proof. (1)$\Rightarrow$ (2): Since $V \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2)$ is an isometry and $\dim \mathcal{H}_i < \infty$, $i = 1, 2$, V is onto, that is, V is a unitary operator. In particular, V is invertible. Since

$$a = \varphi(\mathbf{0}) \neq 0,$$

by the above theorem, we conclude that $aD - CB$ is invertible and

$$V^{-1} = \begin{bmatrix} a^{-1} + a^{-1}B(aD-CB)^{-1}C & -B(aD-CB)^{-1} \\ -(aD-CB)^{-1}C & a^{-1}(aD-CB)^{-1} \end{bmatrix}.$$

Since $V^* = V^{-1}$, in particular, we have

$$D^* = \begin{bmatrix} D_1^* & 0 \\ D_2^* & D_3^* \end{bmatrix} = a^{-1}(aD-CB)^{-1} = a^{-1}\begin{bmatrix} aD_1 - C_1B_1 & aD_2 - C_1B_2 \\ -C_2B_1 & aD_3 - C_2B_2 \end{bmatrix}^{-1},$$

and hence

$$a\begin{bmatrix} aD_1 - C_1B_1 & aD_2 - C_1B_2 \\ -C_2B_1 & aD_3 - C_2B_2 \end{bmatrix}\begin{bmatrix} D_1^* & 0 \\ D_2^* & D_3^* \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

But then this implies $(aD_2 - C_1B_2)D_3^* = 0$. Note that the invertibility of D immediately implies that D_3 is also invertible. Then $aD_2 - C_1B_2 = 0$, and hence, by Theorem 3.4.1, there exist rational inner functions φ_1 and φ_2 (here $\mathcal{H}_1$ and $\mathcal{H}_2$ are finite dimensional Hilbert spaces) such that $\varphi(\boldsymbol{z}) = \varphi_1(z_1)\varphi(z_2)$, $\boldsymbol{z} \in \mathbb{D}^2$.

(2)$\Rightarrow$ (1): Since φ_i $(\in \mathcal{S}(\mathbb{D}))$ is a rational inner function, there exists an isometric colligation

$$V_i = \begin{bmatrix} a_i & B_i \\ C_i & D_i \end{bmatrix} \in \mathcal{B}(\mathbb{C} \oplus \mathcal{H}_i),$$

such that $\dim(\mathcal{H}_i) < \infty$, $D_i \in C_{0\cdot}$, and $\varphi_i = \tau_{V_i}$ for all $i = 1, 2$. We define

$$V = \begin{bmatrix} a_1a_2 & B_1 & a_1B_2 \\ a_2C_1 & D_1 & C_1B_2 \\ C_2 & 0 & D_2 \end{bmatrix}.$$

Then a somewhat careful computation (or see the proof of [48, Theorem 2.2]) yields that $\varphi = \tau_V$. □

In this connection, and also in the context of Remark 3.2.1, it is probably worth mentioning that in the finite dimensional case we have the following: If $\begin{bmatrix} D_1 & D_2 \\ 0 & D_3 \end{bmatrix} \in \mathcal{B}(\mathbb{C}^p \oplus \mathbb{C}^q)$ for some $p, q \geq 1$, then

$$\sigma\left(\begin{bmatrix} D_1 & D_2 \\ 0 & D_3 \end{bmatrix}\right) = \sigma(D_1) \cup \sigma(D_3),$$

and in particular, $\begin{bmatrix} D_1 & D_2 \\ 0 & D_3 \end{bmatrix} \in C_{0\cdot}$ if and only if $D_1, D_3 \in C_{0\cdot}$.

Finally, we point out that part (1) of Theorem 3.4.1 and part (2) of Theorem 3.5.3 are related (in a different direction) to essential normality of Beurling type quotient modules of $H^2(\mathbb{D}^2)$ [62].

Chapter 4

Beurling quotient module on the polydisc

4.1 Introduction

Let $\mathcal{L} \subseteq H^2_{\mathcal{E}}(\mathbb{D}^n)$ be a closed subspace. Then $\mathcal{L}$ is said to be a *quotient module* if $M^*_{z_i}\mathcal{L} \subseteq \mathcal{L}$ for all $i = 1, \ldots, n$. The subspace $\mathcal{L}$ is called a *submodule* if $\mathcal{L}^{\perp}$ is a quotient module [52].

We pause for a brief aside to remark that if $n = 1$, then a closed subspace $\mathcal{Q} \subseteq H^2_{\mathcal{E}}(\mathbb{D})$ is a quotient module if and only if there exist a Hilbert space $\mathcal{E}_*$ and an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D})$ such that $\mathcal{Q}^{\perp} = \Theta H^2_{\mathcal{E}_*}(\mathbb{D})$, or equivalently

$$\mathcal{Q} = H^2_{\mathcal{E}}(\mathbb{D}) \ominus \Theta H^2_{\mathcal{E}_*}(\mathbb{D}) \cong H^2_{\mathcal{E}}(\mathbb{D})/\Theta H^2_{\mathcal{E}_*}(\mathbb{D}).$$

This follows from the classical Beurling-Lax-Halmos theorem [90]. Therefore, the preceding statement gives a satisfactory description of quotient modules of $H^2_{\mathcal{E}}(\mathbb{D})$. It is also worthwhile to emphasize that there is an inseparable alliance between quotient modules and bounded linear operators on Hilbert spaces. For instance, if $\mathcal{Q}$ is a quotient module of $H^2_{\mathcal{E}}(\mathbb{D})$, then the *module operator* (also known as *model operator*) $M_{\mathcal{Q}} := P_{\mathcal{Q}} M_z|_{\mathcal{Q}}$ is a pure contraction on $\mathcal{Q}$. The classical Sz.-Nagy and Foias theory says that, up to unitary equivalence, these are all pure contractions on Hilbert spaces.

Therefore, quotient modules of $H^2_{\mathcal{E}}(\mathbb{D}^n)$, $n \geq 1$, are of interest in operator theory, function theory, and operator algebras. However, in sharp contrast, the situation changes dramatically in the case when $n > 1$: In general, a quotient module of $H^2_{\mathcal{E}}(\mathbb{D}^n)$ does not necessarily admit a Beurling-type representation. In fact, concrete description of quotient modules of $H^2_{\mathcal{E}}(\mathbb{D}^n)$ is commonly regarded as one of the most difficult and important problems in modern operator theory and function theory [43, 52, 97, 112].

In this chapter, our interest is in comparing the variability of the classical Beurling representations of quotient modules in several variables. For a Hilbert space $\mathcal{E}$ and a closed subspace $\mathcal{Q} \subseteq H^2_{\mathcal{E}}(\mathbb{D}^n)$, we say that $\mathcal{Q}$ is a *Beurling quotient module* (and $\mathcal{Q}^\perp$ is a *Beurling submodule*) if

$$\mathcal{Q} = H^2_{\mathcal{E}}(\mathbb{D}^n) \ominus \Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n) \cong H^2_{\mathcal{E}}(\mathbb{D}^n)/\Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n),$$

for some Hilbert space $\mathcal{E}_*$ and inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D}^n)$. Since $M_{z_i} M_\Theta = M_\Theta M_{z_i}$ for all $i = 1, \dots, n$, it follows, in particular, that $\mathcal{Q}$ ($\mathcal{Q}^\perp$) is also a quotient module (submodule) of $H^2_{\mathcal{E}}(\mathbb{D}^n)$. In the context of the above discussion, it appears natural to raise the following question:

Question 2. *Which quotient modules of $H^2_{\mathcal{E}}(\mathbb{D}^n)$ admit Beurling representations?*

Curiously, despite its natural appeal and all possible applications, the above question remained fairly untouched. It is also the one variable work of Beurling [29] which stirred our interest in this question. Evidently, this has a lot to do with the module (or model) operators associated with quotient modules. Given a quotient module $\mathcal{Q} \subseteq H^2_{\mathcal{E}}(\mathbb{D}^n)$, define the n-tuple of commuting contractions $C_z = (C_{z_1}, \dots, C_{z_n})$ (we call it the *tuple of module operators* or *module operators* in short) on $\mathcal{Q}$ by

$$C_{z_i} = P_{\mathcal{Q}} M_{z_i}|_{\mathcal{Q}} \qquad (i = 1, \dots, n),$$

where $P_{\mathcal{Q}}$ is the orthogonal projection from $H^2_{\mathcal{E}}(\mathbb{D}^n)$ onto $\mathcal{Q}$. Therefore, $\mathcal{Q}$ is a contractive Hilbert module over $\mathbb{C}[z_1, \dots, z_n]$ in the following sense (see [43, 52]):

$$(p, h) \in \mathbb{C}[z_1, \dots, z_n] \times \mathcal{Q} \longrightarrow p(C_{z_1}, \dots, C_{z_n})h \in \mathcal{Q}.$$

The following theorem provides the answer to Question 2:

Theorem 4.1.1. *Let $\mathcal{E}$ be a Hilbert space and let $\mathcal{Q}$ be a quotient module of $H^2_{\mathcal{E}}(\mathbb{D}^n)$. Then $\mathcal{Q}$ is a Beurling quotient module if and only if*

$$(I_{\mathcal{Q}} - C^*_{z_i} C_{z_i})(I_{\mathcal{Q}} - C^*_{z_j} C_{z_j}) = 0 \qquad (i \neq j).$$

The proof of Theorem 4.1.1 depends on several lemmas, some of which are of independent interest and related to the delicate structure of submodules and quotient modules of (vector-valued) Hardy space over $\mathbb{D}^n$. This is the content of Section 4.2.

In section 4.3, we apply the above framework to dilations of n-tuples of commuting contractions. Let us explain this when $n = 2$. A pair of commuting contractions $T = (T_1, T_2)$ on $\mathcal{H}$ is called *Brehmer pair* if $D^2_{T^*} \geq 0$, where

$$D^2_{T^*} := I_{\mathcal{H}} - T_1 T_1^* - T_2 T_2^* + T_1 T_2 T_1^* T_2^*.$$

It is known [46, 86] that a pure Brehmer pair dilates to (M_{z_1}, M_{z_2}) on a vector-valued Hardy space, or, equivalently, (M_{z_1}, M_{z_2}) on vector-valued Hardy spaces are analytic models of a pure Brehmer pair. More specifically, if (T_1, T_2) is a pure Brehmer pair, then there exist a Hilbert space $\mathcal{D}$ (which is actually $\overline{\text{ran}}D_{T^*}$) and a quotient module $\mathcal{Q} \subseteq H^2_{\mathcal{D}}(\mathbb{D}^2)$ such that

$$(T_1, T_2) \cong (P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}, P_{\mathcal{Q}} M_{z_2}|_{\mathcal{Q}}).$$

Since $\mathcal{Q}$ is not necessarily a Beurling quotient module, this model is not completely comparable with the classical Sz.-Nagy and Foias analytic models of pure contractions. The missing piece is precisely a paraphrase of Theorem 4.1.1: Let (T_1, T_2) be a pair of commuting contractions on $\mathcal{H}$, and let $\mathcal{D}_{T^*} = \overline{ran} D_{T^*}$. Then there exist a Hilbert space $\mathcal{E}$ and an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{D}_{T^*})}(\mathbb{D}^n)$ such that

$$(T_1, T_2) \cong (P_{\mathcal{Q}_\Theta} M_{z_1}|_{\mathcal{Q}_\Theta}, P_{\mathcal{Q}_\Theta} M_{z_2}|_{\mathcal{Q}_\Theta}),$$

if and only if the pair (T_1, T_2) is a pure Brehmer pair and

$$(I_{\mathcal{H}} - T_1^* T_1)(I_{\mathcal{H}} - T_2^* T_2) = 0.$$

Here $\mathcal{Q}_\Theta := H^2_{\mathcal{D}_{T^*}}(\mathbb{D}^n)/\Theta H^2_{\mathcal{E}}(\mathbb{D}^n)$ is the Beurling quotient module of $H^2_{\mathcal{D}_{T^*}}(\mathbb{D}^2)$ corresponding to the inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E},\mathcal{D}_{T^*})}(\mathbb{D}^n)$. This is the main content of Theorem 4.3.3.

Section 4.4 deals with factorizations of inner functions in $H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D}^n)$ and invariant subspaces of tuples of module operators. We briefly explain the main content of Section 4.4 when $\mathcal{E} = \mathbb{C}$ and $n = 2$. The starting point is the following one variable result (see Sz.-Nagy and Foias, and Bercovici [28, 90]), which connects invariant subspaces of module operators with factorizations of the corresponding inner functions:

Let $\theta \in H^\infty(\mathbb{D})$ be an inner function. Then $T_\theta := P_{\mathcal{Q}_\theta} M_z|_{\mathcal{Q}_\theta}$ has an invariant subspace if and only if there exist inner functions φ and ψ in $H^\infty(\mathbb{D})$ such that

$$\theta = \varphi\psi.$$

However, in the case of $H^\infty(\mathbb{D}^2)$, the existence of joint invariant subspaces is not sufficient to ensure factorizations of inner functions (see Example 4.4.2). Theorem 4.4.3 deals with this missing link: Let $\theta \in H^\infty(\mathbb{D}^2)$ be an inner function, $\mathcal{Q}_\theta = H^2(\mathbb{D}^2)/\theta H^2(\mathbb{D}^2)$, and let $T_\theta = (P_{\mathcal{Q}_\theta} M_{z_1}|_{\mathcal{Q}_\theta}, P_{\mathcal{Q}_\theta} M_{z_2}|_{\mathcal{Q}_\theta})$ denote the pair of module operators. The following are equivalent.

1. $\theta = \varphi\psi$ for some inner functions $\varphi, \psi \in H^\infty(\mathbb{D}^2)$.

2. There exists a joint T_θ-invariant subspace $\mathcal{M} \subseteq \mathcal{Q}_\theta$ such that $\mathcal{M} \oplus \theta H^2(\mathbb{D}^2)$ is a Beurling submodule of $H^2(\mathbb{D}^2)$.

3. There exists a joint T_θ-invariant subspace $\mathcal{M} \subseteq \mathcal{Q}_\theta$ such that

$$(I - C_1^* C_1)(I - C_2^* C_2) = 0,$$

where $C_i = P_{\mathcal{Q}_\theta \ominus \mathcal{M}} M_{z_i}|_{\mathcal{Q}_\theta \ominus \mathcal{M}}$ and $i = 1, 2$.

In Corollary 4.4.4, we prove that nontrivial factorizations is equivalent to the existence of nontrivial invariant subspaces of tuples of module operators.

We say that two n-tuples $T = (T_1, \dots, T_n)$ on $\mathcal{H}$ and $R = (R_1, \dots, R_n)$ on $\mathcal{K}$ are unitarily equivalent (which we denote by $T \cong R$) if there exists a unitary $U \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $UT_i = R_i U$ for all $i = 1, \dots, n$.

This chapter is based on the published paper [30].

4.2 Proof of Theorem 4.1.1

Throughout this section we fix a Hilbert space $\mathcal{E}$ and a quotient module $\mathcal{Q}$ of $H^2_{\mathcal{E}}(\mathbb{D}^n)$. We denote by $\mathcal{S}$ the submodule $\mathcal{Q}^\perp$, that is

$$\mathcal{S} = H^2_{\mathcal{E}}(\mathbb{D}^n) \ominus \mathcal{Q} \cong H^2_{\mathcal{E}}(\mathbb{D}^n)/\mathcal{Q}.$$

In order to shorten some of our computations, we will use the standard notation of cross-commutators: $[T_1, T_2] := T_1T_2 - T_2T_1$ whenever T_1 and T_2 are bounded linear operators on some Hilbert space.

Now, by definition, $\mathcal{Q}$ is a Beurling quotient module if and only if there exist a Hilbert space $\mathcal{E}_*$ and an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_*, \mathcal{E})}(\mathbb{D}^n)$ such that $\mathcal{S} = \Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n)$, which, by [85, 103], equivalent to the condition that $[R^*_{z_j}, R_{z_i}] = 0$ for all $i \neq j$, where $R_{z_r} = M_{z_r}|_{\mathcal{S}}$, and $r = 1, \dots, n$. Then we have the following interpretation of Theorem 4.1.1:

Lemma 4.2.1. *For each i and j in $\{1, \dots, n\}$, define*

$$X_{ij} = P_{\mathcal{S}} M_{z_i} P_{\mathcal{Q}} M^*_{z_j} P_{\mathcal{S}}.$$

Then $\mathcal{Q}$ is a Beurling quotient module if and only if $X_{ij} = 0$ for all $i \neq j$.

Proof. Suppose $i \neq j$. Since $M_{z_i} M^*_{z_j} = M^*_{z_j} M_{z_i}$ and $I_{H^2_{\mathcal{E}}(\mathbb{D}^n)} - P_{\mathcal{S}} = P_{\mathcal{Q}}$, it follows that

$$[R^*_{z_j}, R_{z_i}] = R^*_{z_j} R_{z_i} - R_{z_i} R^*_{z_j} = P_{\mathcal{S}} M^*_{z_j} M_{z_i}|_{\mathcal{S}} - P_{\mathcal{S}} M_{z_i} P_{\mathcal{S}} M^*_{z_j}|_{\mathcal{S}} = P_{\mathcal{S}} M_{z_i} P_{\mathcal{Q}} M^*_{z_j}|_{\mathcal{S}}.$$

Therefore, $[R^*_{z_j}, R_{z_i}] = 0$ if and only if $(P_{\mathcal{S}} M_{z_i} P_{\mathcal{Q}} M^*_{z_j})|_{\mathcal{S}} = 0$, which is equivalent to $X_{ij} = (P_{\mathcal{S}} M_{z_i} P_{\mathcal{Q}} M^*_{z_j}) P_{\mathcal{S}} = 0$. $\square$

It is often convenient to work with $P_{\mathcal{Q}} M_{z_i} P_{\mathcal{Q}} \in \mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}^n))$, which we will denote by C_i, that is

$$C_i = P_{\mathcal{Q}} M_{z_i} P_{\mathcal{Q}} \in \mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}^n)) \qquad (i = 1, \dots, n).$$

Observe that $C = (P_{\mathcal{Q}} M_{z_1} P_{\mathcal{Q}}, \ldots, P_{\mathcal{Q}} M_{z_n} P_{\mathcal{Q}})$ is an n-tuple of commuting contractions on $H^2_{\mathcal{E}}(\mathbb{D}^n)$ (or, equivalently, C defines a contractive $\mathbb{C}[z_1, \ldots, z_n]$-Hilbert module structure on $H^2_{\mathcal{E}}(\mathbb{D}^n)$), and

$$C_i|_{\mathcal{Q}} = C_{z_i} \text{ and } C_i^*|_{\mathcal{Q}} = C_{z_i}^*,$$

for all $i = 1, \ldots, n$. Finally, to shorten notation we set $T^k = T_1^{k_1} \cdots T_n^{k_n}$ whenever $T = (T_1, \ldots, T_n)$ is a commuting tuple on some Hilbert space and $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$.

For the rest of this section, we fix i and j from $\{1, \ldots, n\}$, and assume that $i \neq j$. In what follows, we will use $\hat{k}_i$ ($\hat{k}_j$) to denote multi-indices in $\mathbb{Z}_+^n$ whose i-th (j-th) slot has zero entry. The following lemma will play a key role.

Lemma 4.2.2. *$[C_i, C^{*\hat{k}_i}] = P_{\mathcal{Q}} M_z^{*\hat{k}_i} P_{\mathcal{S}} M_{z_i} P_{\mathcal{Q}}$ for all $\hat{k}_i \in \mathbb{Z}_+^n \setminus \{0\}$.*

Proof. First notice that $C^{*l} = M_z^{*l} P_{\mathcal{Q}}$ and $C^l = P_{\mathcal{Q}} M_z^l$ for all $l \in \mathbb{Z}_+^n$. Since $[C_i, C^{*\hat{k}_i}] = C_i C^{*\hat{k}_i} - C^{*\hat{k}_i} C_i$, it follows that

$$[C_i, C^{*\hat{k}_i}] = P_{\mathcal{Q}} M_{z_i} M_z^{*\hat{k}_i} P_{\mathcal{Q}} - P_{\mathcal{Q}} M_z^{*\hat{k}_i} P_{\mathcal{Q}} M_{z_i} P_{\mathcal{Q}}.$$

Then, writing $P_{\mathcal{Q}} = I_{H^2_{\mathcal{E}}(\mathbb{D}^n)} - P_{\mathcal{S}}$ into the middle of the second term on the right side and using $M_z^{*\hat{k}_i} M_{z_i} = M_{z_i} M_z^{*\hat{k}_i}$, we get the desired equality. $\square$

For each $t = 1, \ldots, n$, we set $D_{C_t} = (P_{\mathcal{Q}} - C_t^* C_t)^{\frac{1}{2}}$. Since

$$C_t^* C_t = P_{\mathcal{Q}} M_{z_t}^* P_{\mathcal{Q}} M_{z_t} P_{\mathcal{Q}} = P_{\mathcal{Q}} M_{z_t}^* (I_{H^2_{\mathcal{E}}(\mathbb{D}^n)} - P_{\mathcal{S}}) M_{z_t} P_{\mathcal{Q}} = P_{\mathcal{Q}} - P_{\mathcal{Q}} M_{z_t}^* P_{\mathcal{S}} M_{z_t} P_{\mathcal{Q}},$$

that D_{C_t} is well defined follows from the fact that

$$P_{\mathcal{Q}} - C_t^* C_t = P_{\mathcal{Q}} M_{z_t}^* P_{\mathcal{S}} M_{z_t} P_{\mathcal{Q}} \geq 0. \tag{4.2.1}$$

We now recall a classical result due to R. Douglas [51]. Let A and B be contractions on a Hilbert space $\mathcal{H}$. The Douglas's range and inclusion theorem then says that $AA^* \leq BB^*$ if and only if there exists a contraction X such that $A = BX$. We are now ready for the third key lemma of this section.

Lemma 4.2.3. *Suppose $\hat{k}_i \in \mathbb{Z}_+^n \setminus \{0\}$. There exist contractions $X_{\hat{k}_i}$ and $Y_{\hat{k}_i}$ in $\mathcal{B}(\mathcal{Q})$ such that*

$$[C_i, C^{*\hat{k}_i}] = X_{\hat{k}_i} D_{C_i} \text{ and } [C^{\hat{k}_i}, C_i^*] = D_{C_i} Y_{\hat{k}_i}.$$

Proof. We already know that $D_{C_i}^2 = P_{\mathcal{Q}} - C_t^* C_t = P_{\mathcal{Q}} M_{z_i}^* P_{\mathcal{S}} M_{z_i} P_{\mathcal{Q}}$. Then, by Lemma 4.2.2, we have

$$\begin{aligned} D_{C_i}^2 - [C_i, C^{*\hat{k}_i}]^* [C_i, C^{*\hat{k}_i}] &= P_{\mathcal{Q}} M_{z_i}^* P_{\mathcal{S}} M_{z_i} P_{\mathcal{Q}} - (P_{\mathcal{Q}} M_{z_i}^* P_{\mathcal{S}} M_z^{\hat{k}_i}) P_{\mathcal{Q}} (M_z^{*\hat{k}_i} P_{\mathcal{S}} M_{z_i} P_{\mathcal{Q}}) \\ &= P_{\mathcal{Q}} M_{z_i}^* P_{\mathcal{S}} (I_{H^2_{\mathcal{E}}(\mathbb{D}^n)} - M_z^{\hat{k}_i} P_{\mathcal{Q}} M_z^{*\hat{k}_i}) P_{\mathcal{S}} M_{z_i} P_{\mathcal{Q}} \\ &= (P_{\mathcal{Q}} M_{z_i}^* P_{\mathcal{S}})(I_{H^2_{\mathcal{E}}(\mathbb{D}^n)} - M_z^{\hat{k}_i} P_{\mathcal{Q}} M_z^{*\hat{k}_i})(P_{\mathcal{Q}} M_{z_i}^* P_{\mathcal{S}})^*. \end{aligned}$$

Since $M_z^{\hat{k}_i} P_{\mathcal{Q}}$ is a contraction, it follows that $I_{H^2_{\mathcal{E}}(\mathbb{D}^n)} - M_z^{\hat{k}_i} P_{\mathcal{Q}} M_z^{*\hat{k}_i} \geq 0$, and hence

$$D_{C_i}^2 - [C_i, C^{*\hat{k}_i}]^*[C_i, C^{*\hat{k}_i}] \geq 0.$$

Then the first equality is an immediate consequence of the Douglas's range and inclusion theorem. Finally, since $[C_i, C^{*\hat{k}_i}]^* = [C^{\hat{k}_i}, C_i^*]$, the second equality follows from the first. □

The final ingredient is the following result. Again recall that $X_{ij} = P_{\mathcal{S}} M_{z_i} P_{\mathcal{Q}} M_{z_j}^* P_{\mathcal{S}}$ (see Lemma 4.2.1).

Lemma 4.2.4. *If $D_{C_i} D_{C_j} = 0$, then, for each $\hat{k}_i, \hat{l}_j \in \mathbb{Z}_+^n \setminus \{0\}$,*

1. $P_{\mathcal{Q}} M_z^{*\hat{k}_i} X_{ij} M_z^{\hat{l}_j} P_{\mathcal{Q}} = 0$,
2. $P_{\mathcal{Q}} M_{z_i}^* X_{ij} M_z^{\hat{l}_j} P_{\mathcal{Q}} = 0$, *and*
3. $P_{\mathcal{Q}} M_z^{*\hat{k}_i} X_{ij} M_{z_j} P_{\mathcal{Q}} = 0$.

Proof. By Lemma 4.2.3, we have on one hand $[C_i, C^{*\hat{k}_i}][C_j, C^{*\hat{l}_j}]^* = 0$, and on the other hand, by Lemma 4.2.2,

$$\begin{aligned}
[C_i, C^{*\hat{k}_i}][C_j, C^{*\hat{l}_j}]^* &= (P_{\mathcal{Q}} M_z^{*\hat{k}_i} P_{\mathcal{S}} M_{z_i}) P_{\mathcal{Q}} (M_{z_j}^* P_{\mathcal{S}} M_z^{\hat{l}_j} P_{\mathcal{Q}}) \\
&= P_{\mathcal{Q}} M_z^{*\hat{k}_i} X_{ij} M_z^{\hat{l}_j} P_{\mathcal{Q}}.
\end{aligned}$$

This proves (1). To verify (2), first observe that (4.2.1) implies

$$(P_{\mathcal{Q}} - C_i^* C_i)[C^{\hat{l}_j}, C_j^*] = (P_{\mathcal{Q}} M_{z_i}^* P_{\mathcal{S}} M_{z_i} P_{\mathcal{Q}})[C_j, C^{*\hat{l}_j}]^*.$$

By Lemma 4.2.2, we can write $[C_j, C^{*\hat{l}_j}]^* = P_{\mathcal{Q}} M_{z_j}^* P_{\mathcal{S}} M_z^{\hat{l}_j} P_{\mathcal{Q}}$, where, on the other hand, Lemma 4.2.3 implies that

$$(P_{\mathcal{Q}} - C_i^* C_i)[C^{\hat{l}_j}, C_j^*] = 0.$$

Therefore

$$0 = (P_{\mathcal{Q}} M_{z_i}^* P_{\mathcal{S}} M_{z_i} P_{\mathcal{Q}})(P_{\mathcal{Q}} M_{z_j}^* P_{\mathcal{S}} M_z^{\hat{l}_j} P_{\mathcal{Q}}) = P_{\mathcal{Q}} M_{z_i}^* (P_{\mathcal{S}} M_{z_i} P_{\mathcal{Q}} M_{z_j}^* P_{\mathcal{S}}) M_z^{\hat{l}_j} P_{\mathcal{Q}},$$

which proves (2). The proof of (3) is similar to that of (2): We first observe that

$$[C_i, C^{*\hat{k}_i}](P_{\mathcal{Q}} - C_j^* C_j) = P_{\mathcal{Q}} M_z^{*\hat{k}_i} X_{ij} M_{z_j} P_{\mathcal{Q}}$$

whereas Lemma 4.2.3 implies that $[C_i, C^{*\hat{k}_i}](P_{\mathcal{Q}} - C_j^* C_j) = 0$. □

We also need the following simple observation: $\mathcal{Q}$ reduces $(M_{z_1}^* P_{\mathcal{S}} M_{z_1}, \ldots, M_{z_n}^* P_{\mathcal{S}} M_{z_n})$, that is

$$P_{\mathcal{Q}}(M_{z_t}^* P_{\mathcal{S}} M_{z_t}) = (M_{z_t}^* P_{\mathcal{S}} M_{z_t}) P_{\mathcal{Q}} \qquad (t = 1, \ldots, n). \tag{4.2.2}$$

Indeed, for a fixed t in $\{1,\dots,n\}$, writing $P_{\mathcal{Q}} = I_{H^2_{\mathcal{E}}(\mathbb{D}^n)} - P_{\mathcal{S}}$, we see that

$$P_{\mathcal{Q}}(M_{z_t}^* P_{\mathcal{S}} M_{z_t}) P_{\mathcal{Q}} = M_{z_t}^* P_{\mathcal{S}} M_{z_t} P_{\mathcal{Q}} - P_{\mathcal{S}} M_{z_t}^* P_{\mathcal{S}} M_{z_t} P_{\mathcal{Q}},$$

and, on the other hand, $P_{\mathcal{S}} M_{z_t}^* P_{\mathcal{S}} = P_{\mathcal{S}} M_{z_t}^*$ and $M_{z_t}^* M_{z_t} = I_{H^2_{\mathcal{E}}(\mathbb{D}^n)}$ implies that

$$P_{\mathcal{S}} M_{z_t}^* P_{\mathcal{S}} M_{z_t} P_{\mathcal{Q}} = P_{\mathcal{S}} M_{z_t}^* M_{z_t} P_{\mathcal{Q}} = P_{\mathcal{S}} P_{\mathcal{Q}} = 0.$$

That is, $P_{\mathcal{Q}}(M_{z_t}^* P_{\mathcal{S}} M_{z_t}) P_{\mathcal{Q}} = (M_{z_t}^* P_{\mathcal{S}} M_{z_t}) P_{\mathcal{Q}}$. Then the claim follows from the fact that $M_{z_t}^* P_{\mathcal{S}} M_{z_t}$ is a self-adjoint operator.

Now we are ready to plunge into the main body of the proof of Theorem 4.1.1.

Proof of Theorem 4.1.1. Suppose $\mathcal{Q}$ is a Beurling quotient module. Then there exist a Hilbert space $\mathcal{E}_*$ and an inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D}^n)$ such that $\mathcal{S} = \Theta H^2_{\mathcal{F}}(\mathbb{D}^n)$ (see the discussion preceding Lemma 4.2.1). Then $P_{\mathcal{S}} = M_\Theta M_\Theta^*$. Now (4.2.1) and (4.2.2) implies that

$$P_{\mathcal{Q}} - C_t^* C_t = M_{z_t}^* P_{\mathcal{S}} M_{z_t} P_{\mathcal{Q}} \qquad (t = 1,\dots,n).$$

Therefore by applying (4.2.2) again we obtain

$$(P_{\mathcal{Q}} - C_i^* C_i)(P_{\mathcal{Q}} - C_j^* C_j) = (M_{z_i}^* P_{\mathcal{S}} M_{z_i}) P_{\mathcal{Q}} (M_{z_j}^* P_{\mathcal{S}} M_{z_j}) P_{\mathcal{Q}} = M_{z_i}^* P_{\mathcal{S}} M_{z_i} M_{z_j}^* P_{\mathcal{S}} M_{z_j} P_{\mathcal{Q}}.$$

We know by $M_\Theta^* M_\Theta = I_{H^2_{\mathcal{E}}(\mathbb{D}^n)}$ and $M_{z_t} M_\Theta = M_\Theta M_{z_t}$ for all $t = 1,\dots,n$, that $M_\Theta^* M_{z_j}^* M_{z_i} M_\Theta = M_{z_j}^* M_{z_i}$. Then $P_{\mathcal{S}} = M_\Theta M_\Theta^*$ implies that $P_{\mathcal{S}} M_{z_j}^* M_{z_i} P_{\mathcal{S}} = M_\Theta M_{z_j}^* M_{z_i} M_\Theta^* = M_\Theta M_{z_i} M_{z_j}^* M_\Theta^*$, and hence

$$M_{z_i}^* P_{\mathcal{S}} M_{z_i} M_{z_j}^* P_{\mathcal{S}} M_{z_j} P_{\mathcal{Q}} = M_{z_i}^* M_\Theta M_{z_i} M_{z_j}^* M_\Theta^* M_{z_j} P_{\mathcal{Q}} = M_\Theta M_\Theta^* P_{\mathcal{Q}} = 0,$$

which yields $(P_{\mathcal{Q}} - C_i^* C_i)(P_{\mathcal{Q}} - C_j^* C_j) = 0$. Thus we obtain

$$(I_{\mathcal{Q}} - C_{z_i}^* C_{z_i})(I_{\mathcal{Q}} - C_{z_j}^* C_{z_j}) = (P_{\mathcal{Q}} - C_i^* C_i)(P_{\mathcal{Q}} - C_j^* C_j)|_{\mathcal{Q}} = 0.$$

Now we turn to the converse. By taking into account Lemma 4.2.1, what we have to show is that $X_{ij} = 0$. We now describe a multi-step reduction process that reduces this claim to Lemma 4.2.4. First observe that $\overline{\text{span}}\{z^k \mathcal{Q} : k \in \mathbb{Z}_+^n\}$ reduces M_{z_t} for all $t = 1,\dots,n$. Then there exists a closed subspace $\mathcal{E}_1$ of $\mathcal{E}$ such that

$$\overline{\text{span}}\{z^k \mathcal{Q} : k \in \mathbb{Z}_+^n\} = H^2_{\mathcal{E}_1}(\mathbb{D}^n).$$

By setting $\mathcal{E}_0 = \mathcal{E} \ominus \mathcal{E}_1$, it follows that

$$H^2_{\mathcal{E}}(\mathbb{D}^n) = \overline{\text{span}}\{z^k \mathcal{Q} : k \in \mathbb{Z}_+^n\} \oplus H^2_{\mathcal{E}_0}(\mathbb{D}^n).$$

Since $\mathcal{Q}^{\perp} = \mathcal{S} \supseteq H^2_{\mathcal{E}_0}(\mathbb{D}^n)$, for each $f \in H^2_{\mathcal{E}_0}(\mathbb{D}^n)$, we have $P_{\mathcal{Q}} M_{z_i}^* P_{\mathcal{S}} f = P_{\mathcal{Q}} M_{z_i}^* f = 0$, as $H^2_{\mathcal{E}_0}(\mathbb{D}^n)$ reduces M_{z_i}. This proves that

$$X_{ij}|_{H^2_{\mathcal{E}_0}(\mathbb{D}^n)} = 0.$$

So we only need to check that $X_{ij}|_{\overline{\text{span}}\{z^k \mathcal{Q}: k \in \mathbb{Z}_+^n\}} = 0$, or, equivalently

$$X_{ij} M_z^l P_{\mathcal{Q}} = 0 \qquad (l \in \mathbb{Z}_+^n).$$

Since $X_{ij} = P_{\mathcal{S}} M_{z_i} P_{\mathcal{Q}} M_{z_j}^* P_{\mathcal{S}}$ (see the definition of X_{ij} in Lemma 4.2.1), we only need to consider $l \in \mathbb{Z}_+^n \setminus \{0\}$. Moreover, for each $f_0 \in H^2_{\mathcal{E}_0}(\mathbb{D}^n)$, since $M_{z_i}^* f_0 \in \mathcal{S}$, it follows that

$$\langle X_{ij} M_z^l f, f_0 \rangle = \langle P_{\mathcal{S}} M_{z_i} P_{\mathcal{Q}} M_{z_j}^* P_{\mathcal{S}} M_z^l f, f_0 \rangle = \langle P_{\mathcal{Q}} M_{z_j}^* P_{\mathcal{S}} M_z^l f, M_{z_i}^* f_0 \rangle = 0,$$

for all $f \in \mathcal{Q}$ and $l \in \mathbb{Z}_+^n \setminus \{0\}$. Therefore, it suffices to prove that

$$X_{ij} M_z^l \mathcal{Q} \perp M_z^k \mathcal{Q} \qquad (l \in \mathbb{Z}_+^n \setminus \{0\}, k \in \mathbb{Z}_+^n).$$

Note that the case $k = 0$ is trivial since $\text{ran} X_{ij} \subseteq \mathcal{S}$. Hence, we are reduced to showing that

$$P_{\mathcal{Q}} M_z^{*k} X_{ij} M_z^l P_{\mathcal{Q}} = 0 \qquad (k, l \in \mathbb{Z}_+^n \setminus \{0\}). \tag{4.2.3}$$

To prove this in full generality, we start with $k = e_i$ and $l = e_j$, where e_i and e_j are the multiindices with 1 in the i- th and j-th slot, respectively, and zero elsewhere. In this case, we prove a little bit more, namely

$$M_{z_i}^* X_{ij} M_{z_j} = 0.$$

We proceed as follows: By applying (4.2.1) twice we obtain

$$\begin{aligned}(P_{\mathcal{Q}} - C_i^* C_i)(P_{\mathcal{Q}} - C_j C_j^*) &= (P_{\mathcal{Q}} M_{z_i}^* P_{\mathcal{S}} M_{z_i} P_{\mathcal{Q}})(P_{\mathcal{Q}} M_{z_j}^* P_{\mathcal{S}} M_{z_j} P_{\mathcal{Q}}) \\ &= P_{\mathcal{Q}} (M_{z_i}^* P_{\mathcal{S}} M_{z_i}) P_{\mathcal{Q}} (M_{z_j}^* P_{\mathcal{S}} M_{z_j}) P_{\mathcal{Q}}\end{aligned}$$

Since $(P_{\mathcal{Q}} - C_i^* C_i)(P_{\mathcal{Q}} - C_j C_j^*) = 0$, by assumption, (4.2.2) implies that

$$0 = P_{\mathcal{Q}} (M_{z_i}^* P_{\mathcal{S}} M_{z_i}) P_{\mathcal{Q}} (M_{z_j}^* P_{\mathcal{S}} M_{z_j}) P_{\mathcal{Q}} = (M_{z_i}^* P_{\mathcal{S}} M_{z_i}) P_{\mathcal{Q}} (M_{z_j}^* P_{\mathcal{S}} M_{z_j}) = M_{z_i}^* X_{ij} M_{z_j},$$

which proves the desired identity. In particular, (4.2.3) holds whenever $k, l \in \mathbb{Z}_+^n \setminus \{0\}$ and $k_i, l_j \neq 0$. Now let us consider the remaining cases: $k, l \in \mathbb{Z}_+^n \setminus \{0\}$, where

Case 1: $k_i = l_j = 0$,

Case 2: $k_i \neq 0$ and $l_j = 0$, and

Case 3: $k_i = 0$ and $l_j \neq 0$.

The first case simply follows from part (1) of Lemma 4.2.4. For the remaining cases, we fix $k, l \in \mathbb{Z}_+^n \setminus \{0\}$. By (4.2.2) we have

$$P_{\mathcal{Q}} M_z^{*k} M_{z_i}^* (P_{\mathcal{S}} M_{z_i} P_{\mathcal{Q}}) = P_{\mathcal{Q}} M_z^{*k} (M_{z_i}^* P_{\mathcal{S}} M_{z_i}) P_{\mathcal{Q}} = P_{\mathcal{Q}} M_z^{*k} P_{\mathcal{Q}} (M_{z_i}^* P_{\mathcal{S}} M_{z_i}) P_{\mathcal{Q}}.$$

Therefore, $P_{\mathcal{Q}} M_z^{*k} M_{z_i}^* (P_{\mathcal{S}} M_{z_i} P_{\mathcal{Q}}) = (P_{\mathcal{Q}} M_z^{*k}) P_{\mathcal{Q}} M_{z_i}^* (P_{\mathcal{S}} M_{z_i} P_{\mathcal{Q}})$, from which it immediately follows that

$$P_{\mathcal{Q}} M_z^{*k} M_{z_i}^* X_{ij} M_z^l P_{\mathcal{Q}} = (P_{\mathcal{Q}} M_z^{*k})(P_{\mathcal{Q}} M_{z_i}^* X_{ij} M_z^l P_{\mathcal{Q}}),$$

and similarly

$$P_{\mathcal{Q}} M_z^{*k} X_{ij} M_{z_j} M_z^l P_{\mathcal{Q}} = (P_{\mathcal{Q}} M_z^{*k} X_{ij} M_{z_j} P_{\mathcal{Q}})(M_z^l P_{\mathcal{Q}}).$$

Then Case 2 and Case 3 follows from part (2) and part (3), respectively, of Lemma 4.2.4. This completes the proof that $\mathcal{Q}$ is a Beurling quotient module. $\square$

4.3 Isometric dilations

This section is meant to complement the dilation theory of (a concrete class of) n-tuples of commuting contractions.

We begin with the definition of isometric dilations. Let $T = (T_1, \dots, T_n)$ and $V = (V_1, \dots, V_n)$ be commuting tuples of contractions and isometries on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. We say that V is an *isometric dilation* of T (or T *dilates* to V) if there exists an isometry $\Pi : \mathcal{H} \to \mathcal{K}$ such that $\Pi T_i^* = V_i^* \Pi$ for all $i = 1, \dots, n$.

We will mostly restrict attention here to the case when V is $(M_{z_1}, \dots, M_{z_n})$ on $H^2_{\mathcal{E}}(\mathbb{D}^n)$ for some Hilbert space $\mathcal{E}$. In fact, if $n = 1$, then $T = (T)$ dilates to M_z on $H^2_{\mathcal{E}}(\mathbb{D})$ for some Hilbert space $\mathcal{E}$ if and only if T is a pure contraction (recall again that an operator X is pure if the sequence $\{X^{*m}\}_{m\geq 0}$ converges to 0 in the strong operator topology). This deep result is due to Sz.-Nagy and C. Foias [90]. However, in sharp contrast, if $n = 2$ ($n > 2$), then general n-tuples of pure commuting contractions do not dilate to (M_{z_1}, M_{z_2}) on vector-valued Hardy space over $\mathbb{D}^2$ (commuting tuples of isometries) (see [46, 86]).

However, the multivariable situation is completely favorable in the case of Brehmer tuples (see 1.3.13 for definition).

Theorem 4.3.1. *Every commuting n-tuple $(n \geq 2)$ of contractions satisfying Brehmer positivity co-extends to an n-tuple of doubly commuting isometries.*

We also refer [46, 86] for proof of above dilation results. In particular, the pure case is the following:

Theorem 4.3.2. *Every commuting n-tuple $(n \geq 2)$ of pure contractions satisfying Szegö positivity co-extends to $M_z = (M_{z_1}, \dots, M_{z_n})$ on $H^2_{\mathcal{D}}(\mathbb{D}^n)$, for some Hilbert space $\mathcal{D}$.*

In particular, there exists an isometry $\Pi : \mathcal{H} \to H^2_{\mathcal{D}}(\mathbb{D}^n)$ such that $\Pi T_i^* = M_{z_i}^* \Pi$ for all $i = 1, \ldots, n$. Then $\mathcal{Q} := \Pi\mathcal{H}$ is a quotient module of $H^2_{\mathcal{D}}(\mathbb{D}^n)$, and hence

$$T \cong (P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}, \ldots, P_{\mathcal{Q}} M_{z_n}|_{\mathcal{Q}}),$$

on $\mathcal{Q}$. Note again that, if $n = 1$, then $\mathcal{Q}$ is a Beurling quotient module, and hence

$$T \cong P_{(\Theta H^2_{\mathcal{E}_*}(\mathbb{D}))^\perp} M_z|_{(\Theta H^2_{\mathcal{E}_*}(\mathbb{D}))^\perp},$$

for some Hilbert space $\mathcal{E}_*$ and inner function $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D})$. This inner function Θ and the Beurling quotient module

$$\mathcal{Q}_\Theta = (\Theta H^2_{\mathcal{E}_*}(\mathbb{D}))^\perp,$$

are popularly known as the *characteristic function* of T and the *model space* corresponding to T, respectively [90]. In summary, pure contractions are unitarily equivalent to compressions of M_z to model spaces.

We now study an analog of the above analytic model theorem for n-tuples of commuting contractions. First we set up some notation. Let $\mathcal{E}$ and $\mathcal{E}_*$ be Hilbert spaces, and let $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D}^n)$ be an inner function. Let us denote by $\mathcal{Q}_\Theta = H^2_{\mathcal{E}}(\mathbb{D}^n) \ominus \Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n)$ and $\mathcal{S}_\Theta = \Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n)$ the Beurling quotient module and the Beurling submodule, respectively, corresponding to Θ. We also define $T_{z_i,\Theta} = P_{\mathcal{Q}_\Theta} M_{z_i}|_{\mathcal{Q}_\Theta}$ for all $i = 1, \ldots, n$, and set

$$T_\Theta = (T_{z_1,\Theta}, \ldots, T_{z_n,\Theta}).$$

One can now ask which n-tuples of commuting contractions are unitarily equivalent to T_Θ on Beurling quotient modules (or, model spaces) $\mathcal{Q}_\Theta$. The following result (a refinement of Theorem 4.1.1) yields a complete answer to this question.

Theorem 4.3.3. *Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting contractions on $\mathcal{H}$. The following are equivalent.*

(a) *$T \cong T_\Theta$ for some Beurling quotient module $\mathcal{Q}_\Theta$.*

(b) *T is a pure Brehmer tuple and $(I_{\mathcal{H}} - T_i^* T_i)(I_{\mathcal{H}} - T_j^* T_j) = 0$ for all $i \neq j$.*

The proof directly follows from Theorem 4.1.1 and Theorem 4.3.2.

4.4 Factorizations and invariant subspaces

The main goal of this section is to classify factorizations of inner functions in terms of invariant subspaces of tuples of module operators. Our observation will also bring out a key difference between n-tuples of operators, $n > 1$, and single operators.

The structure of invariant subspaces of bounded linear operators has been traditionally related to the theory of (nontrivial) factorizations of one variable inner functions.

For instance, the following result (see [90, Chapter VI], and more specifically [28, Chapter 5, Proposition 1.21]) connects invariant subspaces of module (or model) operators with factorizations of the corresponding inner functions. Here we follow the same notation as in the discussion preceding Theorem 4.3.3.

Theorem 4.4.1 (Sz.-Nagy and Foias, and Bercovici). *Let $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D})$ be an inner function. Then T_Θ has an invariant subspace if and only if there exist a Hilbert space $\mathcal{F}$ and inner functions $\Phi \in H^\infty_{\mathcal{B}(\mathcal{F},\mathcal{E})}(\mathbb{D})$ and $\Psi \in H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{F})}(\mathbb{D})$ such that*

$$\Theta = \Phi\Psi.$$

In the above, the corresponding T_Θ-invariant subspace is given by $\mathcal{M} = \mathcal{S}_\Phi \ominus \mathcal{S}_\Psi$ [28, Chapter 5, Proposition 1.21]. Here we are interested in the polydisc version of the above theorem. However, the following example shows that in the case when $n > 1$ the existence of joint invariant subspaces is not sufficient to ensure factorizations of inner functions.

Example 4.4.2. *Consider the submodule $\mathcal{S} = \{f \in H^2(\mathbb{D}^2) : f(0,0) = 0\}$ of $H^2(\mathbb{D}^2)$. Since*

$$\mathcal{S} = z_1(H^2(\mathbb{D}) \otimes \mathbb{C}) \oplus z_2(\mathbb{C} \otimes H^2(\mathbb{D})) \oplus z_1z_2H^2(\mathbb{D}^2),$$

and $\mathcal{S}$ is a reproducing kernel Hilbert space, the kernel function k of $\mathcal{S}$ is given by

$$k(z,w) = \frac{z_1\bar{w}_1}{1 - z_1\bar{w}_1} + \frac{z_2\bar{w}_2}{1 - z_2\bar{w}_2} + z_1z_2\mathbb{S}_2(z,w)\bar{w}_1\bar{w}_2 \qquad (z, w \in \mathbb{D}^2),$$

where

$$\mathbb{S}_2(z,w) = (1 - z_1\bar{w}_1)^{-1}(1 - z_2\bar{w}_2)^{-1} \qquad (z, w \in \mathbb{D}^2),$$

is the Szegö kernel of $\mathbb{D}^2$. From Example 1.3.6, it is clear that $\mathcal{S}$ is not a Beurling submodule. Now observe

$$\varphi(z) = \frac{2z_1z_2 - z_1 - z_2}{2 - z_1 - z_2} \qquad (z \in \mathbb{D}^2),$$

defines an inner function in $H^\infty(\mathbb{D}^2)$. We have $\varphi(0,0) = 0$, and $\varphi H^2(\mathbb{D}^2) \subsetneqq \mathcal{S} \subsetneqq H^2(\mathbb{D}^2)$. Set $\mathcal{M} = \mathcal{S} \ominus \varphi H^2(\mathbb{D}^2)$. Then $\mathcal{M}$ is a non-trivial $(P_{\mathcal{Q}_\varphi}M_{z_1}|_{\mathcal{Q}_\varphi}, P_{\mathcal{Q}_\varphi}M_{z_2}|_{\mathcal{Q}_\varphi})$-invariant subspace of $\mathcal{Q}_\varphi$, but φ is not factorable.

The missing component in the polydisc analog of Theorem 4.4.1 will be determined in Theorem 4.4.3.

We are now ready for the polydisc analog of Theorem 4.4.1. We will use the same notation as in the discussion preceding Theorem 4.3.3.

Theorem 4.4.3. *Let $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D}^n)$ be an inner function. The following are equivalent.*

1. *There exist a Hilbert space $\mathcal{F}$ and inner functions Ψ and Φ in $H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{F})}(\mathbb{D}^n)$ and $H^\infty_{\mathcal{B}(\mathcal{F},\mathcal{E})}(\mathbb{D}^n)$, respectively, such that $\Theta = \Phi\Psi$.*
2. *There exists a T_Θ-invariant subspace $\mathcal{M} \subseteq \mathcal{Q}_\Theta$ such that $\mathcal{M} \oplus \mathcal{S}_\Theta$ is a Beurling submodule of $H^2_{\mathcal{E}}(\mathbb{D}^n)$.*
3. *There exists a T_Θ-invariant subspace $\mathcal{M} \subseteq \mathcal{Q}_\Theta$ such that*

$$(I - C_i^* C_i)(I - C_j^* C_j) = 0 \qquad (i \neq j),$$

where $C_s = P_{\mathcal{Q}_\Theta \ominus \mathcal{M}} T_{z_s,\Theta}|_{\mathcal{Q}_\Theta \ominus \mathcal{M}}$ for all $s = 1, \ldots, n$.

Proof. (1) $\Rightarrow$ (2): Since $M_\Theta = M_\Phi M_\Psi$, we have $\Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n) \subseteq \Phi H^2_{\mathcal{F}}(\mathbb{D}^n)$. Define

$$\mathcal{M} := \Phi H^2_{\mathcal{F}}(\mathbb{D}^n) \ominus \Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n) = \mathcal{S}_\Phi \ominus \mathcal{S}_\Theta.$$

Clearly, $\mathcal{M}$ is a closed subspace of $\mathcal{Q}_\Theta$. Also note that

$$\mathcal{Q}_\Theta \ominus \mathcal{M} = (H^2_{\mathcal{E}}(\mathbb{D}^n) \ominus \Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n)) \ominus (\Phi H^2_{\mathcal{F}}(\mathbb{D}^n) \ominus \Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n)),$$

and hence, $\mathcal{Q}_\Theta \ominus \mathcal{M} = \mathcal{Q}_\Phi$. Since $T^*_{z_i,\Theta} = M^*_{z_i}|_{\mathcal{Q}_\Theta}$ and $\mathcal{Q}_\Phi \subseteq \mathcal{Q}_\Theta$, it follows that $\mathcal{Q}_\Phi$ is $T^*_{z_i,\Theta}$-invariant for all $i = 1, \ldots, n$. Consequently, $\mathcal{M}$ is a T_Θ-invariant subspace. For the second part, observe that

$$\mathcal{M} \oplus \mathcal{S}_\Theta = \mathcal{S}_\Phi,$$

is a Beurling submodule of $H^2_{\mathcal{E}}(\mathbb{D}^n)$.

(2) $\Rightarrow$ (1): Let $\mathcal{M}$ is a T_Θ-invariant subspace of $\mathcal{Q}_\Theta$ and suppose $\mathcal{M} \oplus \mathcal{S}_\Theta$ is a Beurling submodule of $H^2_{\mathcal{E}}(\mathbb{D}^n)$. Then (see the discussion preceding Lemma 4.2.1) there exist a Hilbert space $\mathcal{F}$ and an inner function $\Phi \in H^\infty_{\mathcal{B}(\mathcal{F},\mathcal{E})}(\mathbb{D}^n)$ such that

$$\mathcal{M} \oplus \mathcal{S}_\Theta = \Phi H^2_{\mathcal{F}}(\mathbb{D}^n).$$

In particular, $\Theta H^2_{\mathcal{E}}(\mathbb{D}^n) \subseteq \Phi H^2_{\mathcal{F}}(\mathbb{D}^n)$, and hence, by Douglas's range and inclusion theorem, there exists a contraction $X : H^2_{\mathcal{E}_*}(\mathbb{D}^n) \to H^2_{\mathcal{F}}(\mathbb{D}^n)$ such that $M_\Theta = M_\Phi X$. But now, since M_Φ is an isometry and

$$M_\Phi X M_{z_i} = M_\Theta M_{z_i} = M_{z_i} M_\Theta = M_{z_i} M_\Phi X = M_\Phi M_{z_i} X,$$

we find $X M_{z_i} = M_{z_i} X$ for all $i = 1, \ldots, n$. Then there exists $\Psi \in H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{F})}(\mathbb{D}^n)$ such that $X = M_\Psi$. Finally, since M_Θ and M_Φ are isometries, we obtain

$$\|f\| = \|M_\Theta f\| = \|M_\Phi M_\Psi f\| = \|M_\Psi f\| \qquad (f \in H^2_{\mathcal{E}}(\mathbb{D}^n)),$$

and hence, M_Ψ is an isometry.

(1) $\Rightarrow$ (3): As in the proof of (1) $\Rightarrow$ (2), if we set $\mathcal{M} = \Phi H^2_{\mathcal{F}}(\mathbb{D}^n) \ominus \Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n)$, then $\mathcal{Q}_\Theta \ominus \mathcal{M} = \mathcal{Q}_\Phi$, which implies $C_s = P_{\mathcal{Q}_\Phi} M_{z_s}|_{\mathcal{Q}_\Phi}$ for all $s = 1, \ldots, n$. Then the desired equality immediately follows from Theorem 4.1.1 applied to $(C_1, \ldots, C_n)$ on the Beurling quotient module $\mathcal{Q}_\Phi$.

(3) $\Rightarrow$ (2): Since $T^*_{z_i,\Theta} = M^*_{z_i}|_{\mathcal{Q}_\Theta}$, $i = 1, \ldots, n$, it follows that $\mathcal{Q}_\Theta \ominus \mathcal{M}$ is a quotient module of $H^2_{\mathcal{E}}(\mathbb{D}^n)$. This says $\mathcal{Q}_\Theta \ominus \mathcal{M}$ is a Beurling quotient module, taking into account the hypothesis and Theorem 4.1.1. Finally, we observe

$$H^2_{\mathcal{E}}(\mathbb{D}^n) \ominus (\mathcal{M} \oplus \mathcal{S}_\Theta) = \mathcal{Q}_\Theta \ominus \mathcal{M},$$

which implies that $\mathcal{M} \oplus \mathcal{S}_\Theta$ is a Beurling submodule. This completes the proof of the theorem. $\square$

It is now worthwhile to observe that the subspace $\mathcal{M} \oplus \varphi H^2(\mathbb{D}^2)$ in Example 4.4.2 is not a Beurling submodule.

Finally, let us concentrate on the trivial cases of the above theorem, namely, $\mathcal{M} = \{0\}$ and $\mathcal{M} = \mathcal{Q}_\Theta$. Recall that $\mathcal{M} = \Phi H^2_{\mathcal{F}}(\mathbb{D}^n) \ominus \Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n)$. Then $\mathcal{M} = \{0\}$ if and only if $\Phi H^2_{\mathcal{F}}(\mathbb{D}^n) = \Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n)$, which, since $\Theta = \Phi\Psi$, equivalent to $H^2_{\mathcal{F}}(\mathbb{D}^n) = \Psi H^2_{\mathcal{E}_*}(\mathbb{D}^n)$. By Lemma 1.3.9, the latter condition is equivalent to the condition that Ψ is a unitary constant. For the second case, we note that $\mathcal{M} = \mathcal{Q}_\Theta$ if and only if

$$\Phi H^2_{\mathcal{F}}(\mathbb{D}^n) \ominus \Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n) = H^2_{\mathcal{E}}(\mathbb{D}^n) \ominus \Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n),$$

which is equivalent to $H^2_{\mathcal{E}}(\mathbb{D}^n) = \Phi H^2_{\mathcal{F}}(\mathbb{D}^n)$. Therefore, we note, again by Lemma 1.3.9, that $\mathcal{M} = \mathcal{Q}_\Theta$ if and only if Φ is a unitary constant. This proves that $\mathcal{M}$ is a nontrivial T_Θ-invariant subspace of $\mathcal{Q}_\Theta$ if and only if the inner functions Φ and Ψ are not unitary constant.

In fact, something more can be said. We continue to use the setting and conclusion of Theorem 4.4.3.

Corollary 4.4.4. *Let $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E}_*,\mathcal{E})}(\mathbb{D}^n)$ be a nonconstant inner function. Then the inner functions Φ and Ψ are nonconstant if and only if the following holds:*

1. *$\mathcal{M}$ is a nontrivial T_Θ-invariant subspace of $\mathcal{Q}_\Theta$,*
2. *$\mathcal{M}$ is not a Beurling submodule of $H^2_{\mathcal{E}}(\mathbb{D}^n)$, and*
3. *$\mathcal{Q}_\Theta \ominus \mathcal{M}$ does not reduce $M_z \otimes I_{\mathcal{E}}$.*

Proof. We have already seen that $\mathcal{M}$ is a nontrivial subspace of $\mathcal{Q}_\Theta$ if and only if both the inner functions Φ and Ψ are not unitary constant. In particular, if Φ and Ψ are nonconstant, then $\mathcal{M}$ is a nontrivial subspace of $\mathcal{Q}_\Theta$. Now suppose that $\mathcal{M}$ is a Beurling submodule. Then there exist a Hilbert space $\mathcal{F}_1$ and an inner function $\Phi_1 \in H^\infty_{\mathcal{B}(\mathcal{F}_1,\mathcal{E})}(\mathbb{D}^n)$ such that $\Phi H^2_{\mathcal{F}}(\mathbb{D}^n) \ominus \Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n) = \Phi_1 H^2_{\mathcal{F}_1}(\mathbb{D}^n)$. In particular, $\Phi_1 H^2_{\mathcal{F}_1}(\mathbb{D}^n) \subseteq \Phi H^2_{\mathcal{F}}(\mathbb{D}^n)$, which implies that $\Phi_1 = \Phi\Phi_2$ for some inner function $\Phi_2 \in H^\infty_{\mathcal{B}(\mathcal{F}_1,\mathcal{F})}(\mathbb{D}^n)$. This yields $\Phi_1 H^2_{\mathcal{F}_1}(\mathbb{D}^n) = \Phi\Phi_2 H^2_{\mathcal{F}_1}(\mathbb{D}^n)$, and hence

$$\Phi\Phi_2 H^2_{\mathcal{F}_1}(\mathbb{D}^n) = \Phi H^2_{\mathcal{F}}(\mathbb{D}^n) \ominus \Phi\Psi H^2_{\mathcal{E}_*}(\mathbb{D}^n) = \Phi\mathcal{Q}_\Psi,$$

from which we obtain $\mathcal{Q}_\Psi = \Phi_2 H^2_{\mathcal{F}_1}(\mathbb{D}^n)$. Thus $\mathcal{Q}_\Psi$, or equivalently, $\mathcal{S}_\Psi$ reduces $M_z \otimes I_{\mathcal{E}}$, which implies that Ψ is a constant. This is a contradiction.

Finally, suppose towards a contradiction that $\mathcal{Q}_\Theta \ominus \mathcal{M}$ reduces $M_z \otimes I_{\mathcal{E}}$. Then

$$\mathcal{M} \oplus \mathcal{S}_\Theta = H^2_{\mathcal{E}}(\mathbb{D}^n) \ominus (\mathcal{Q}_\Theta \ominus \mathcal{M}),$$

also reduces $M_z \otimes I_{\mathcal{E}}$. On the other hand, since $\mathcal{M} \oplus \mathcal{S}_\Theta = \Phi H^2_{\mathcal{F}}(\mathbb{D}^n)$, it follows that $\Phi H^2_{\mathcal{F}}(\mathbb{D}^n) = H^2_{\mathcal{F}_2}(\mathbb{D}^n)$, and hence that Φ is a constant, which is a contradiction.

Now we turn to the converse part. Suppose $\mathcal{M}$ is a nontrivial T_Θ-invariant subspace of $\mathcal{Q}_\Theta$. Since $\Theta = \Phi\Psi$ and Θ is nonconstant, both Φ and Ψ cannot be constant. Moreover, since $\mathcal{M}$ is nontrivial, Φ and Ψ cannot be unitary constants (see the discussion preceding the statement of the corollary). It remains to show that Φ and Ψ cannot be constant isometry operators. First, let us assume that $\Phi \equiv V_1$ for some non-unitary isometry V_1 and that Ψ is nonconstant. Then

$$\mathcal{M} \oplus \Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n) = \Phi H^2_{\mathcal{F}}(\mathbb{D}^n) = V_1 H^2_{\mathcal{F}}(\mathbb{D}^n) = H^2_{V_1\mathcal{F}}(\mathbb{D}^n),$$

and hence $\mathcal{M} \oplus \Theta H^2_{\mathcal{E}_*}(\mathbb{D}^n)$ reduces $M_z \otimes I_{\mathcal{E}}$, which is a contradiction. On the other hand, if $\Psi \equiv V_2$ and Φ is nonconstant, where V_2 is a non-unitary isometry, then

$$\mathcal{M} = \Phi H^2_{\mathcal{F}}(\mathbb{D}^n) \ominus \Phi\Psi H^2_{\mathcal{E}_*}(\mathbb{D}^n) = \Phi(H^2_{\mathcal{F}}(\mathbb{D}^n) \ominus V_2 H^2_{\mathcal{E}_*}(\mathbb{D}^n)) = \Phi H^2_{\mathcal{F}\ominus V_2\mathcal{E}^*}(\mathbb{D}^n),$$

is a Beurling submodule of $H^2_{\mathcal{E}}(\mathbb{D}^n)$, which is a contradiction. This completes the proof of the corollary. $\square$

We refer the reader to the papers [32, 62, 113] and the survey [112] for other results (mostly in two variables) on Beurling quotient modules.

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List of Publications

1. Ramlal Debnath, Jaydeb Sarkar,
   Factorizations of Schur functions,
   Complex Analysis and Operator Theory.

2. Ramlal Debnath, Jaydeb Sarkar,
   Schur functions and inner functions on the bidisc,
   To appear in *Computational Methods and Function Theory (CMFT).*

3. Monojit Bhattacharjee, B. Krishna Das, Ramlal Debnath, Jaydeb Sarkar,
   Beurling quotient modules on the polydisc,
   Journal of Functional Analysis.