Sankhyā : The Indian Journal of Statistics
1990, Volume 52, Series A, Pt. 1, pp. 103–114

ON EXPECTATIONS OF FUNCTIONS OF ORDER STATISTICS

By K. BALASUBRAMANIAN and M. I. BEG*
Indian Statistical Institute

SUMMARY. In this paper we describe a method of deriving linear relations among expectations of functions of order statistics. This unifies various adhoc methods used in deriving such relations. This method also sets up a one-to-one correspondence between these linear relations and a set of combinatorial identities.

1. Introduction

If h is a Borel measurable function from $\mathcal{R}^k$ to $\mathcal{R}$ and if $W_0, W_1, \ldots$ are all k-vectors of order statistics from a distribution, a relation of the form $C_0Eh(W_0) = \sum_r C_rEh(W_r)$ is termed linear if C_r's are constants, independent of the underlying distribution. Such relations are scattered in the literature, a large number of them finding mention in David (1981). By specializing (putting $h \equiv 1$) we get $C_0 = \sum_r C_r$ which, in general, is a combinatorial identity. It is remarkable that this combinatorial identity is equivalent to the linear relation in the sense that it can be used to derive the relation itself. We prove this equivalence and exploit it to prove a general theorem on linear relations. A large number of such relations are proved with the associated combinatorial identities. This paper, though in spirit is similar to that of Arnold (1977), goes beyond it.

The method is essentially using expectation under summation in identities involving terms of the form $p_0^{a_0}p_1^{a_1}, \ldots, p_k^{a_k}$ where $\sum_{i=0}^{k} p_i = 1$.

2. Main results

Suppose X has an arbitrary distribution with a continuous c.d.f. $F(x)$ and h is any Borel measurable function from $\mathcal{R}$ to $\mathcal{R}$ such that $E\{h(X)\}$ exists. Let $X_{r:n}$ denote the r-th order statistic in a random sample of size n from the distribution of X. It is well known that

$$E\{h(X_{r:n})\} = r\binom{n}{r}\int_{\mathcal{R}} h(x)F^{r-1}(x)\,(1-F(x))^{n-r}dF(x).$$

---

*Research supported by C.S.I.R. Presently at University of Hyderabad.

AMS (1980) *subject classifications:* Primary 62G30; Secondary 05A19.

Key words and phrases: Order statistics, recurrence relation, combinatorial identity.

We will write $r\binom{n}{r}$ as $\alpha(r, n)$. Then

$$E\{h(X_{r:n})F^a(X_{r:n})(1-F(X_{r:n}))^b\}$$
$$= \alpha(r, n) \int_{\mathcal{R}} h(x)F^{r+a-1}(x)(1-F(x))^{n+b-r}dF(x)$$
$$= (\alpha(r, n)/\alpha(r+a, n+a+b))E\{h(X_{r+a:n+a+b})\}, \quad \ldots \quad (2.1)$$

where a and b are integers such that $n+a+b \geqslant r+a > 0$. (2.1) is the fundamental result which we are going to exploit. In what follow we assume that all series considered are convergent absolutely and uniformly w.r.t. the parameters involved so that operations on them are justified.

Theorem 2.1 : *Let S be a subset of Z^2 (where Z is the set of all integers) with K, a mapping from S to $\mathcal{R}$ and δ, a real number. Then the following three statements are equivalent :*

(i) $\sum_{(a,b)\in S} K(a, b)p^aq^b = \delta,$

for all $p \in (0, 1)$, $q=1-p$.

(ii) $\delta E\{h(X_{r:n})\} = \sum_{(a,b)\in S} K(a, b)(\alpha(r, n)/\alpha(r+a, n+a+b))E\{h(X_{r+a:n+a+b})\}$

for all r and n such that $0 < r+a \leqslant n+a+b$.

(iii) $\sum_{(a,b)\in S} K(a, b)\,(\alpha(r, n)/\alpha(r+a, n+a+b)) = \delta,$

for all r and n such that $0 < r+a \leqslant n+a+b$. *for all* $(a, b) \in S$.

Proof : (i) $\Longrightarrow$ (ii) : If (i) is true then

$$\sum_{(a,b)\in S} K(a, b)F^a(X_{r:n})(1-F(X_{r:n}))^b = \delta$$

or

$$\sum_{(a,b)\in S} K(a, b)h(X_{r:n})F^a(X_{r:n})(1-F(X_{r:n}))^b = \delta\, h(X_{r:n}).$$

Taking expectation on both sides and using (2.1), we get

$$\sum_{(a,b)\in S} K(a, b)(\alpha(r, n)/\alpha(r+a, n+a+b))\, E\{h(X_{r+a:n+a+b}) = \delta\, E\{(h(X_{r:n})\},$$

which is (ii).

(ii) $\Longrightarrow$ (iii) : Take $h(\,.\,) \equiv 1$ in (ii) we get (iii).

(iii) $\Longrightarrow$ (i) : Allowing r and n to tend to ∞ in such a way that r/n tends to p, using Stirling's approximation for factorials it is easy to verify that

$$(\alpha(r, n)/\alpha(r+a, n+a+b)) \text{ tends to } p^aq^b$$

and the result (i) follows.

This completes the proof of Theorem 2.1.

Remark 1 : (ii) gives a recurrence relation between the expected values of functions of order statistics whereas (iii) gives a combinatorial identity.

Remark 2 : The recurrence relations between the moments, moment generating functions, characteristics functions, and distribution functions (nontruncated and truncated), whenever they exist can be got by setting $h(x) = x^k$, $h(x) = \exp(tx)$, $h(x) = \exp(itx)$ and $h(s) = I_{(-\infty,u)}(x)$, $h(x) = I_{(-\infty,u)}(x) I_{(a,b)}(x)$ respectively. From distribution functions we can pass on to density functions (whenever they exist).

We now obtain results based on joint distribution of two order statistics.

Suppose $X_{r:n}$ and $X_{s:n} (1 \leqslant r < s \leqslant n)$ are r-th and s-th order statistics from a random sample of size n from a distribution with a continuous c.d.f $F(x)$ and h is a Borel measurable function from $\mathcal{R}^2$ to $\mathcal{R}$. It is well known that, whenever it exists,

$$E\{h(X_{r:n}, X_{s:n})\}$$
$$= \alpha(r, s, n) \iint_{x<y} h(x, y) F^{r-1}(x)(F(y)-F(x))^{s-r-1}(1-F(y))^{n-s} dF(x)\, dF(y)$$

where $\alpha(r, s, n) = n!/[(r-1)!(s-r-1)!(n-s)!]$. Then

$$E\{h(X_{r:n}, X_{s:n}) F^a(X_{r:n})(F(X_{s:n})-F(X_{r:n}))^b(1-F(X_{s:n}))^c\}$$
$$= \alpha(r, s, n) \iint_{x<y} h(x, y) F^{r+a-1}(x)\,(F(y)-F(x))^{s+b-r-1}(1-F(y))^{n+c-s}\, dF(x) dF(y)$$
$$= (\alpha(r, s, n)/\alpha(r+a, s+a+b, n+a+b+c))\, E\{h(x_{r+a:n+a+b+c}, X_{s+a+b:n+a+b+c})\} \quad \ldots (2.2)$$

where a, b and c are integers such that $1 \leqslant r < s \leqslant n$, $1 \leqslant r+a < s+a+b \leqslant n+a+b+c$.

Theorem 2.2 : *Let $S \subset Z^3$ (where Z is the set of all integers) with K, a mapping from S to $\mathcal{R}$ and δ, a real number. Then the following three statements are equivalent :*

(i) $\sum_{(a,b,c)\epsilon s} K(a, b, c) p_1^a p_2^b p_3^c = \delta$, *for all* $p_1, p_2, p_3 \in (0, 1)$; $p_1+p_2+p_3 = 1$

(ii) $\delta E\{h(X_{r:n}, X_{s:n})\} = \sum_{(a,b,c)\epsilon s} K(a,b,c)(\alpha(r,s,n)/\alpha(r+a, s+a+b, n+a+b+c))$

$E\{h(x_{r+a:n+a+b+c}, X_{s+a+b:n+a+b+c})\}$

for all r, s, n, $1 \leqslant r < s \leqslant n$, $1 \leqslant r+a < s+a+b \leqslant n+a+b+c$, *for all* $(a, b, c) \in S$.

(iii) $\sum_{(a,b,c)\epsilon s} K(a, b, c)\, (\alpha(r, s, n)/\alpha(r+a, s+a+b, n+a+b+c)) = \delta$,

for all r, s, n, $1 \leqslant r < s \leqslant n$, $1 \leqslant r+a < s+a+b \leqslant n+a+b+c$, *for all* $(a, b, c) \in S$.

Proof of Theorem 2.2 is similar to that of Theorem 2.1.

Generalization of Theorem 2.2 is now clear. Suppose $X_{r_1:n}$, $X_{r_2:n}$ ..., $X_{r_k:n}$ are r_1-th, r_2-th, ..., r_k-th order statistics $(1 \leqslant r_1 < r_2 < \ldots < r_k \leqslant n)$ from a random sample of size n from a distribution with a continuous c.d.f $F(x)$ and h is a Borel measurable function from $\mathcal{R}^k$ to $\mathcal{R}$. It is well known that, whenever it exists,

$$E\{h(X_{r_1:n}, X_{r_2:n}, \ldots, X_{r_k:n})\} = \alpha(r_1, r_2, \ldots, r_k, n)$$

$$\underset{x_1 < \ldots < x_k}{\int \ldots \int} h(x_1, x_2 \ldots, x_k) \prod_{j=0}^{k} (F(x_{j+1}) - F(x_j))^{r_{j+1} - r_j - 1} dF(x_1) dF(x_2) \ldots dF(x_k)$$

where $x_0 = -\infty$, $x_{k+1} = +\infty$, $r_0 = 0$, $r_{k+1} = n+1$, and

$$\alpha(r_1, r_2, \ldots, r_k, n) = n! / \prod_{j=1}^{k} (r_{j+1} - r_j - 1)!$$

It is now easy to see that, for integers $a_0, a_1, \ldots, a_k$

$$E\{h(X_{r_1:n}, X_{r_2:n}, \ldots, X_{r_k:n}) \prod_{j=0}^{k} (F(X_{r_{j+1}:n}) - F(X_{r_j:n}))^{a_j}\}$$

$$= (\alpha(r_1, r_2, \ldots, r_k, n) / \alpha(r_1 + a_0, r_2 + a_0 + a_1, \ldots, r_k + a_0 + a_1 + \ldots + a_{k-1}, N))$$

$$E\{h(X_{r_1+a_0:N}, X_{r_2+a_0+a_1:N}, \ldots, X_{r_k+a_0+a_1+\ldots+a_{k-1}:N})\} \quad \ldots \quad (2.3)$$

where $N = n + a_0 + a_1 + \ldots + a_k$. If $b_j = a_0 + a_1 + \ldots + a_j$, the RHS of (2.3) may be written as

$$(\alpha(r_1, r_2, \ldots, r_k, n) / \alpha(r_1 + b_0, r_2 + b_1, \ldots, r_k + b_{k-1}, N))$$

$$E\{h(x_{r_1+b_0:N}, X_{r_2+b_1:N}, \ldots, x_{r_k+b_{k-1}:N})\},$$

provided $1 \leqslant r_1 + b_0 < r_2 + b_1 < \ldots < r_k + b_{k-1} \leqslant N$.

We now have the following generalization of Theorem 2.2.

Theorem 2.3 : *Let $S \subset z^{k+1}$ (where z is the set of all intergers) with K, a mapping from S to $\mathcal{R}$ nnd δ, a real number. Then the following three statements are equivalent :*

$$\text{(i)} \sum_{a \in S} k(a)\, p^{a} = \delta,$$

where $a = (a_0, a_1, \ldots, a_k)$, $p^{a} = p_0^{a_0} p_1^{a_1} \ldots p_k^{a_k}$, and $p_i \epsilon (0, 1)$ for all i with $\sum_{i=0}^{k} p_i = 1$.

(ii) $\delta E\{h(X_{r_1:n}, X_{r_2:n}, \ldots, X_{r_k:n})\}$

$$\sum_{a\in S} K(a)\,(\alpha(r_1, r_2, \ldots, r_k, n)/\alpha(r_1+b_0, r_2+b_1, \ldots, r_k+b_{k-1}, N))$$

$$E\{h(X_{r_1+b_0:N}, X_{r_2+b_1:N}, \ldots, X_{r_k+b_{k-1}:N})\},$$

where $b_j=a_0+a_1+\ldots+a_j$, $N=n+b_k$, *and* $1\leqslant r_1+b_0<r_2+b_1<\ldots<r_k+b_{k-1}\leqslant N$.

(iii) $\sum_{a\in S} K(a)\,(\alpha(r_1, r_2, \ldots, r_k, n)/\alpha(r_1+b_0, r_2+b_1, \ldots, r_k+b_{k-1}, N) = \delta,$

where $b_j=a_0+a_1+\ldots+a_j$, $N=n+b_k$, *and* $1\leqslant r_1+b_0<r_2+b_1<\ldots<r_k+b_{k-1}\leqslant N$.

Proof of Theorem 2.3 is similar to that of Theorem 2.1. Remark 1 and Remark 2 with obvious modifications are true for Theorems 2.2 and 2.3 also.

3. Applications

In this section we present some applications of each of our theorems of Section 2 separately. We notice that several known recurrence relations can be deduced from our results. But combinatorial identities are not emphasized and are treated only cursorily.

3.1. *Examples for Theorem* 1.1. *Example* 1 : Let $S = \{(a, b)\,|\,a \geqslant 0, b \geqslant 0\,;\, a+b = m > 0\}$ and $K(a, b) = \binom{m}{a}$. Then from binomial distribution we have

$$\sum_{(a,b)\in S} K(a, b)p^a q^b = \sum_{s=0}^{m} \binom{m}{s} p^s q^{m-s} = 1.$$

Hence

$$E\{h(X_{r:n})\} = \sum_{s=0}^{m} \binom{m}{s} (\alpha(r, n)/\alpha(r+s, n+m))E\{h(X_{r+s:n+m})\}, \quad \ldots \quad (3.1)$$

and

$$\sum_{s=0}^{m} \binom{m}{s} (\alpha(r, n)/\alpha(r+s, n+m)) = 1$$

or

$$\sum_{s=0}^{m} \left[\binom{m}{s} \Big/ (r+s) \binom{m+n}{r+s}\right] = 1/r \binom{n}{r} \qquad \ldots \quad (3.2)$$

for positive integers n, m and r such that $r \leqslant n$.

In particular if we take $m = 1$, then $S = \{(0, 1), (1, 0)\}$ and from (3.1) we get

$$E\{h(X_{r:n})\} = \alpha(r, n)[E\{h(X_{r:n+1})\}/\alpha(r, n+1)+E\{h(X_{r+1:n+1})\}/\alpha(r+1, n+1)]$$

or $\quad (n+1)\, E\{(h(X_{r:n})\} = (n+r+1)\, E\{h(X_{r:n+1})\}+rE\{h(X_{r+1:n+1})\}. \quad \ldots \quad (3.3)$

Replacing n by $(n-1)$ in (3.3), we get

$$nE\{h(X_{r:n-1})\} = (n-r)\, E\{h(X_{r:n})\}+rE\{h(X_{r+1:n})\},$$

which is due to Srikantan (1962). Taking $h(X_{r:n}) = X_{r:n}^k$, $k = 1, 2, \ldots$, and writing $\mu_{r:n}^{(k)} = E(X_{r:n}^k)$, we get

$$n\,\mu_{r:n-1}^{(k)} = (n-r)\,\mu_{r:n}^{(k)} + r\,\mu_{r+1:n}^{(k)}$$

which is due to Cole (1951). (3.3) can also be obtained by using the identity (Johnson, 1957, 1978),

$$\binom{N}{K}\,[K-NF(X_{r:n})] = N\left[\binom{N-1}{K-1}(1-F(X_{r:n})) - \binom{N-1}{K}F(X_{r:n})\right],$$

where N and K are positive integers, $1 \leqslant K \leqslant N$.

Example 2 : Using the identity $1 = (p^{-1}+1-p^{-1})^m$, we have

$$1 = \sum_{s=0}^{m}\binom{m}{s}p^{-s}(1-p^{-1})^{m-s} = \sum_{s=0}^{m}\binom{m}{s}(-1)^{m-s}p^{-m}(1-p)^{m-s}.$$

Thus

$$E\{h(X_{r:n})\} = \sum_{s=0}^{m}\binom{m}{s}(-1)^{m-s}(\alpha(r,n)/\alpha(r-m,n-s))E\{h(X_{r-m:n-s})\} \quad \ldots \quad (3.4)$$

for $m, n, r > 0$ and $m < r$.

Again starting with $1 = (q^{-1}+1-q^{-1})^m$, we get

$$E\{h(X_{r:n})\} = \sum_{s=0}^{m}(-1)^{m-s}\binom{m}{s}(\alpha(r,n)/\alpha(r+m-s,n-s))\,E\{h(X_{r+m-s:n-s})\} \ldots (3.5)$$

for $m, n, r > 0$ and $r+m > s$.

(3.4) and (3.5) are the recurrence relations given by Krishnaiah and Rizvi (1966).

Example 3 : Let $S = \{(a,b)\,|\,a = m > 0, b \geqslant 0\}$ and $K(a,b) = \binom{a+b-1}{b}$. We then have from negative binomial distribution,

$$\sum_{(a,b)\in s} K(a,b)p^a q^b = \sum_{s=0}^{\infty}\binom{m+s-1}{s}p^m q^s = 1.$$

Hence

$$E\{h(X_{r:n})\} = \sum_{s=0}^{\infty}\binom{m+s+1}{s}(\alpha(r,n)/\alpha(r+m,n+m+s))\,E\{h(X_{r+m:n+m+s})\} \quad \ldots \quad (3.6)$$

for positive integers m, n and r such that $r \leqslant n$.

Example 4 : From geometric distribution (a particular case of Example 3 with $m = 1$) we have

$$\sum_{s=0}^{\infty} p\,q^s = 1.$$

Thus $E\{h(X_{r:n})\} = \sum_{s=0}^{\infty}(\alpha(r,n)/\alpha(r+1,n+s+1))\,E\{h(X_{r+1:n+s+1})\}$, $\quad\ldots\quad (3.7)$

for positive integers r and n such that $r \leqslant n$.

Example 5: From (2.1) we can write

$$E\{h(X_{1:t})F^{r-1}(X_{1:t})\,(1-F(X_{1:t}))^{n-t-r+1}\} = (\alpha(1,\,t)/\alpha(r,\,n))\;E\{h(X_{r:n})\}$$

or
$$\sum_{r=1}^{n} C_r E\{h(X_{r:n})\}$$

$$= \sum_{r=1}^{n} C_r\left[r\binom{n}{r}\Big/t\right]E\{h(X_{1:t})F^{r-1}(X_{1:t})(1-F(X_{1:t}))^{n-t-r+1}\} \qquad \dots \quad (3.8)$$

where C_r is a function of r. We now consider the following particular cases.

Taking $C_r = 1/(n-r+1)$ in (3.8), we get

$$\sum_{r=1}^{n}(1/(n-r+1))\;E\{h(X_{r:n})\}$$

$$= \sum_{r=1}^{n}(1/t)[r\binom{n}{r}/(b-r+1)]E\{h(X_{1:t})F^{r-1}(X_{1:t})\;(1-F(X_{1:t}))^{n-t-r+1}\}$$

$$= \sum_{r=1}^{n}(1/t)\binom{n}{r-1}E\{h(X_{1:t})F^{r-1}(X_{1:t})(1-F(X_{1:t}))^{n-t-r+1}\}$$

$$= \sum_{r=0}^{n}(1/t)\binom{n}{r}E\{h(X_{1:t})F^{r}(X_{1:t})\;(1-F(X_{1:t}))^{n-r-t}\}$$

$$= [(1-F(X_{1:t}))^{-t}/t]\left[\sum_{r=0}^{n-1}\binom{n}{r}\;E\{h(X_{1:t})F^{r}(X_{1:t})(1-F(X_{1:t}))^{n-r}\}\right]$$

$$= [(1-F(X_{1:t}))^{-t}/t][E\{h(X_{1:t})\sum_{r=0}^{n-1}\binom{n}{r}\;F^{r}(X_{1:t})(1-F(X_{1:t}))^{n-r}\}].$$

Putting $t = 1$, we get

$$\sum_{r=1}^{n}(1/(n-r+1))\;E\{h(X_{r:n})\}$$

$$= \;E\{h(X_{1:1})[(1-F^{n}(X_{1:1}))/(1-F(X_{1:1}))]\}$$

$$= \;E\{h(X_{1:1})\sum_{u=1}^{n}F^{u-1}(X_{1:1})\}$$

$$= \sum_{u=1}^{n}E\{h(X_{1:1})F^{u-1}(X_{1:1})\}$$

$$= \sum_{u=1}^{n}(\alpha(1,\,1)/\alpha(u,\,u))\;E\{h(X_{u:u})\},\text{ using (2.1)}$$

$$= \sum_{u=1}^{n}(1/u)\;E\{h(X_{u:u})\} = \sum_{r=1}^{n}(1/r)E\{h(X_{r:r})\}. \qquad \dots \quad (3.9)$$

Similarly with $C_r = 1/r$ in (3.8), we get

$$\sum_{r=1}^{n} (1/r)E\{h(X_{r:n})\} = \sum_{r=1}^{n} (1/r)E\{h(X_{1:r})\}. \quad \ldots \quad (3.10)$$

(3.9) and (3.10) are due to Joshi (1973).

Taking $C_r = r$ in (3.8), we get

$$\sum_{r=1}^{n} rE\{h(X_{r:n})\} = (n/2)E\{h(X_{1:1})\}+(n^2/2)E\{h(X_{2:2})\}. \quad \ldots \quad (3.11)$$

With $C_r = (r-1)^{[d-1]}$, where $x^{[d]} = x(x-1), \ldots, (x-d+1)$, we then have from (3.8),

$$\sum_{r=1}^{n} (r-1)^{[d-1]}E\{h(X_{r:n})\} = (n^{[d]}/l) \sum_{r=1}^{d} s(d, r)E\{h(X_{r:r})\}/r. \quad \ldots \quad (3.12)$$

where $s(d, k)$'s are Stirling numbers of the first kind defined as

$$x^{[d]} = \sum_{k=1}^{d} s(d, k)x^k.$$

3.2 *Examples for Theorem* 2.2. *Example* 1 : Let $S=\{(a, b, c) \mid a, b, c \geqslant 0 ; a+b+c = m \geqslant 0\}$ and $K(a, b, c) = (m\,!/a\,!\,b\,!\,c\,!)$. Then from trinomial distribution we have,

$$\sum_{(a,b,c)\epsilon S} K(a, b, c)p_1^a p_2^b p_3^c = \sum_{(a,b,c)\epsilon S} (m\,!/a\,!\,b\,!\,c\,!)p_1^a p_2^b p_3^c = 1.$$

Hence

$$E\{h(X_{r:n}, X_{s:n})\} = \sum_{(a,b,c)\epsilon S} (m!/a!b!c!)(\alpha(r, s, n)/\alpha(r+a, s+a+c, n+a+b+c))$$

$$E\{h(X_{r+a:n+a+b+c}, X_{s+a+c:n+a+b+c})\}. \quad \ldots \quad (3.13)$$

In particular if we take $m = 1$, then $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ and from (3.13) we get

$$E\{h(X_{r:n}, X_{s:n})\} = \alpha(r, s, n) \, [(E\{h(X_{r+1:n+1}, X_{s+1:n+1})\}/\alpha(r+a, s+1, n+1))$$
$$+(E\{h(X_{r:n+1}, X_{s:n+1})\}/\alpha(r, s, n+1))+(E\{h(X_{r:n+1}, X_{s+1:n+1})\}/\alpha(r, s+1, n+1))]$$
$$= (1/(n+1)) \, [rE\{h(X_{r+1:n+1}, X_{s+a:n+1})\}+(n-s+1)E\{h(X_{r:n+1}, X_{s:n+1})\}+(s-r)$$
$$E\{h(X_{r:n+1}, X_{s+1:n+1})\}]. \quad \ldots \quad (3.14)$$

Taking $h(x, y) = x^j y^k$ and writing $\mu_{r,s:n}^{(j,k)} = E\{X_{r:n}^j X_{s:n}^k\}$ in (3.14) we get

$$(n+1)\mu_{r,s:n}^{(j,k)} = r\mu_{r+1,\,s+1:n+1}+(n-s+1)\mu_{r,s:n+1}^{(j,k)}+(s-r)\mu_{r,s+1:n+1}^{(j,k)}.$$

Govindarajulu (1963) obtained this recurrence relation for $j = k = 1$.

Let

$$A(n, r, s) = P_r\{x_{r:n} \leqslant \xi_p < \xi_q \leqslant x_{s:n}\}$$
$$= P_r\{x_{r:n} \leqslant \xi_p, x_{s:n} \geqslant \xi_q\}, \ \xi_p < \xi_q.$$

If $h(x, y) = I_{(-\infty,\xi_p]}(x)\, I_{[\xi_q,\infty)}(y)$, then

$$E\{h(x_{r:n}, x_{s:n})\} = E\{I_{(-\infty,\xi_p]}(x_{r:n})\, I_{[\xi_q,\infty)}(x_{s:n})\} = A(n, r, s),$$

and (3.14) gives

$$(n+1)\, A(n, r, s) = rA(n+1, r+1, s+1)+(s-r)\, A(n+1, r, s+1) +(n-s+1)\, A(n+1, r, s).$$

which is due to Reiss and Ruschendorf (1976).

If $h(x, y) = g(y-x)$, we get from (3.14)

$$(n+1)\, E\{g(X_{s:n}-X_{r:n})\} = rE\{g(X_{s+1:n+1}-X_{r+1:n+1})\} +(s-r)\, E\{g(X_{s+1:n+1}-X_{r:n+1})\} +(n-s+1)E\{g(X_{s:n+1}-X_{r:n+1})\}.$$

Taking $g(u) = u$ and $s = r+1$, we have

$$(n+1)E(X_{r+1:n}-X_{r:n}) = rE(X_{r+2:n+1}-X_{r+1:n+1})+(n-r)E)X_{r+1:n+1}-X_{r:n+1}) +E(X_{r+2:n+1}-X_{r:n+1})$$

or

$$(n+1)E(X_{r+1:n}-X_{r:n}) = rE(X_{r+2:n+1}-X_{r+1:n1})+(n-r)E(X_{r+1:n+1}-X_{r:n+1}) +E(X_{r+2:n+1}-X_{r+1:n+1}+X_{r+1:n+1}-X_{r:n+1}).$$

Let $\chi_{n+r} = E(X_{r+1:n}-X_{r:n})$, then the above reduces of the form

$$(n+1)\chi_{n:r} = r\chi_{n+1:r+1}+(n-r)\chi_{n+1:r}+(\chi_{n+1:r+1}+\chi_{n+1:r})$$

or

$$(n+1)\chi_{n:r} = (r+1)\chi_{n+1::r+1}+(n-r-1)\chi_{n+1:r},$$

which is due to Sillitto (1951). Other results of Sillitto (1951) can be easily deduced from (3.14).

Example 2 : Let $S = \{(a, b, c) \mid a \geqslant 0,\ b \geqslant 0,\ c = m > 0\}$ and $K(a, b, c) = (a+b+c-1)\,!/a\,!\,b\,!\,(c-1)\,!$. We then have from bivariate negative binomial distribution,

$$\sum_{(a,b,c)\epsilon S} K(c, b, c)p_1^a p_2^b p_3^c = \sum_{(a,b,c)\epsilon S} [(a+b+c-1)\,!/a!\,b!\,(c-1)!]\;\; p_1^a p_2^b p_3^c = 1,$$

$$\sum_{i=1}^{3} p_i = 1.$$

Hence

$$E\{h(X_{r:n}, X_{s:n})\} = \sum_{(a,b,c)\epsilon S} [(a+b+c-1)\,!/a\,!\,b\,!\,(c-1)\,!] \,(\alpha(r, s, n)/\alpha(r+a+c, s+a+c, n+a+b+c))\, E\{h(X_{r+a:n+a+b+c}, X_{s+a+c:n+a+b+c})\}. \quad \ldots \quad (3.15)$$

Example 3. From bivariate geometric distribution (a particular case of Example 2 with $m = 1$) we have

$$\sum_{(a,b)\in S} [(a+b)!/a!\,b!]p_1^a p_2^b p_3 = 1, \ \sum_{i=1}^{3} p_i = 1.$$

Hence

$$E\{h(X_{r:n}, X_{s:n})\} = \sum_{(a,b)\in S} \binom{a+b}{a} (\alpha(r, s, n)/\alpha(r+a, s+a+1, n+a+b+c))$$

$$E\{h(X_{r+a:n+a+b+1}, X_{s+a+1:n+a+b+1})\}. \qquad \ldots \quad (3.16)$$

3.3 *Examples for Theorem* 2.3. *Example* 1 : Let $S = \{(a_0, a_1, \ldots, a_k)$ $a_0, a_1, \ldots, a_k \geqslant 0;\ \sum_{i=0}^{k} a_i = m > 0\}$ and $K(a) = \binom{m}{a_0, a_1, \ldots, a_k}$. We then have from multinomial distribution

$$\sum_{a\in S} K(a)p^a = \sum_{a\in S} \binom{m}{a_0, a_1, \ldots, a_k} p_0^{a_0} p_1^{a_1}, \ldots, p_k^{a_k} = 1 \ ; \ \sum_{i=0}^{k} p_i = 1.$$

Hence

$$E\{h(X_{r_1:n}, X_{r_2:n}, \ldots, X_{r_k:n})\}$$

$$= \sum_{a\in S} \binom{m}{a_0, a_1, \ldots, a_k} (\alpha(r_1, r_2, \ldots, r_k, n)/\alpha(r_1+b_0, r_2+b_1, \ldots, r_k+b_{k-1}, N))$$

$$E\{h(X_{r_1+b_0:N}, X_{r_2+b_1:N}, \ldots, X_{r_k+b_{k-1}:N})\}. \qquad \ldots \quad (3.17)$$

Example 2 : A generalization of negative binomial distribution gives the identity

$$\sum_{r=0}^{\infty} \sum_{\substack{a_1+\ldots+a_k = r \\ a_0 = m}} \binom{m+r-1}{r} \binom{r}{a_1, a_2, \ldots, a_k} p_1^{a_1}\, p_2^{a_2}, \ldots, p_k^{a_k}\, p_0^m = 1$$

where p_i's are nonnegative and $\sum_{i=0}^{k} p_i = 1$.

Hence

$$E\{h(X_{r_1:n}, X_{r_2:n}, \ldots, X_{r_k:n})\}$$

$$\sum_{r=0}^{\infty} \sum_{\substack{a_1+a_2+\ldots+a_k = r \\ a_0 = m}} \binom{m+r-1}{r} \binom{r}{a_1, a_2, \ldots, a_k} (\alpha(r_1, r_2, \ldots, a_k, n)/$$

$$\alpha(r_1+b_0, r_2+b_1, \ldots, r_k+b_{k-1}, N))$$

$$E\{h(X_{r_1+b_0:N}, X_{r_2+b_1:N}, \ldots, X_{r_k+b_{k-1}:N})\}. \qquad \ldots \quad (3.18)$$

Example 3 : This is a particular case of Example 2 and is from a generalization of geometric distribution. This is got by setting $m = 1$. We state the result

$$E\{h(X_{r_1:n}, X_{r_2:n}, \ldots, X_{r_k:n})\}$$

$$= \sum_{r=0}^{\infty} \sum_{\substack{a_1+a_2+\ldots+a_k = r \\ a_0 = 1}} \binom{r}{a_1, a_2, \ldots, a_k} (\alpha(r_1, r_2, \ldots, r_k, n)/$$

$$\alpha(r_1+b_0, r_2+b_1, \ldots, r_k+b_{k-1}, N))\, E\{h(X_{r_1+b_0:N}, \ldots, X_{r_k+b_{k-1}:N})\}. \quad \ldots \quad (3.19)$$

Note : A general procedure for getting certain type of identities and recurrence relation for expectation of functions of order statistics can be given as follows. We give it only for 'one order' statistics, but the generalization is obvious.

(i) Whenever we have an identity of the form

$$\sum_{(a,b)\in S} K(a, b)p^a q^b = \delta \text{ for all } p \in (0, 1), \quad \ldots \quad (3.20)$$

we can get the recurrence relation

$$\delta E\{h(X_{r:n})\} = \sum_{(a,b)\in S} K(a,b)(\alpha(r, n)/\alpha(r+a; n+a+b))\, E\{h(X_{r+a:n+a+b})\} \ldots \quad (3.21)$$

(ii) Whenever we suspect a recurrence relation of the form (3.21) we can settle it by proving (3.20).

Conclusion : All the recurrence relations are essentially 'linear' in character. These are got by interchanging the order of summation and expectation. Such a method obviously will work for conditional expectations too. Authors of this paper are exploring in detail conditional expectations of order statistics in a separate paper.

References

Arnold, B. C. (1977) : Recurrence relations between expectations of functions of order statistics. *Scand. Actuar. J.* 169-174.

Cole, R. H. (1951) : Relations between moments of order statistics. *Ann. Math. Statist.*, 22, 308-310.

David, H. A. (1981) : *Order Statistics*, John Wiley and Sons, Inc., New York.

Govindarajulu, Z. (1963) : On moments of order statistics and quasi-ranges from normal populations. *Ann. Math. Statist.*, 34, 633-651.

Johnson, N. L. (1957, 1958) : A note on the mean deviation of the binomial distribution. *Biometrika*, 44, 532-533 and 45, 587.

JOSHI, P. C. (1973): Two identities involving order statistics. *Biometrika*, **60**, 428-429.

KRISHNAIAH, P. R. and RIZVI, M. H. (1966): A note on recurrence relations between expected values of functions of order statistics. *Ann. Math. Statist.*, **37**, 733-734.

REISS, R. D. and RUSCHENDORF, L. (1976): On Wilks' distribution-free confidence intervals for quantile intervals. *J. Amer. Statist. Ass.*, **71**, 940-944.

RIORDAN, J. (1968): *Combinatorial Identities*, John Wiley and Sons, Inc. New York.

SILLITTO, G. P. (1951): Interrelations between certain linear systematic statistics of samples from any continuous population. *Biometrika*, **38**, 377-382.

SRIKANTAN, K. S. (1962): Recurrence relations between the PDF's of order statistics and some applications. *Ann. Math. Statist.*, **33**, 169-177.

Paper received : *April*, 1988.
Revised : *June*, 1988.