Projective modules and complete intersection ideals over affine algebras

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Projective modules and complete intersection ideals over affine algebras

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In memory of my grandmother
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List of Notations

$\mathbb{N}$	The set of all natural numbers $\{1, 2, ...\}$
$\mathbb{Z}$	The set of all integers $\{0, \pm 1, \pm 2, ...\}$
e_1	The vector $(1, ..., 0)$
$\mathsf{Um}_n(A)$	The set of all unimodular rows of length n over the ring A.
$M_n(A)$	The set of all $n \times n$ matrices over the ring A.
$\mathsf{GL}_n(A)$	The set of all invertible $n \times n$ matrices over the ring A.
$\mathsf{SL}_n(A)$	The set of all invertible $n \times n$ matrices with determinant 1 over the ring A.
$e_{i,j}(\lambda)$	The matrix with only possible non-zero entry is λ at the (i, j)-th position.
I_n	The $n \times n$ identity matrix.
$E_{i,j}(\lambda)$	$I_n + e_{i,j}(\lambda)$.
$E_n(A)$	The subgroup of $\mathsf{SL}_n(A)$, generated by the set $\{E_{i,j}(\lambda) : i \neq j, \lambda \in A\}$.
$\mathrm{Aut}\,(P)$	The group of all automorphism on P, where P is a projective module.
$\mathsf{E}(P)$	The subgroup of $\mathrm{Aut}\,(P)$, generated by all transvections on P, where P is a projective module.
$c.d._p(-)$	The p-th co-homological dimension of $-$.
$m\mathsf{Spec}(A)$	The set of all maximal ideals of the ring A.
$\mu(-)$	The minimum number of a generating set of $-$.
A^*	The units of the ring A.

Chapter 1

Introduction

Conventions

Unless otherwise stated, throughout the thesis, all rings are commutative Noetherian containing $1(\neq 0)$ having finite (Krull) dimension. All projective modules are finitely generated having constant rank.

Splitting problem

The birth of the subject was due to a conjecture (now a theorem) by J. P. Serre [58]. Serre conjectured the following:

Conjecture 1.0.1. *Let k be a field and $R = k[T_1, ..., T_d]$. Then every finitely generated projective $R-$module is free.*

D. Quillen [52] and A. A. Suslin [61] gave an affirmative answer to the conjecture independently. More generally, D. Quillen showed that k can be taken as a principal ideal domain. Their solutions to the conjecture opened up several other directions of research in the study of projective modules. One of the interesting studies occurred when the base field is replaced by an arbitrary ring. This is one of the primary themes of the thesis. The following question came into the literature naturally from their study:

Splitting problem:- Let A be a ring and P be a projective $A[T]-$module. Does there exist projective $A[T]-$modules Q such that $P \cong Q \oplus A[T]$?

Although the motivation came from the Quillen - Suslin theorem, the question makes sense for arbitrary rings. Thus we will not restrict ourselves to the polynomial rings only. Let A be a commutative Noetherian ring and P be a projective $A-$module. Investigating the splitting problem for P is nothing but investigating the existence of surjective $A-$linear maps from P to A. If there exists such a map we shall call P has a unimodular element and in that case an inverse image of 1 in P is a unimodular element. Whenever P is a free module of rank

n, we shall call an inverse image of 1, a unimodular row of length n. By a classical result of J. P. Serre [59] it follows that whenever the rank of the projective module P is strictly bigger than the (Krull) dimension of the ring A, it splits off a free module of rank one. Due to a well-known example of the projective module corresponding to the tangent bundle of an even dimensional real sphere, this result is the best possible in general. Whenever $R = A[T]$ and Q is a projective $R-$module, by a result of B. Plumstead [51], Q splits off a free module of rank one if the rank of Q is strictly bigger than the dimension of the ring A. Again by taking polynomial extension of the similar example discussed earlier one can show that this result is the best possible in general as well. Hence studying obstruction to split off a free module of rank one from a projective module of rank equal to the dimension of the ring (and rank equal to the dimension of the base ring in the case of polynomial extensions) has been interesting.

Lifting problem

We begin this section with an open question due to M. P. Murthy in [47], which is known as *Murthy's complete intersection conjecture*.

Conjecture 1.0.2. *Let k be a field and let $A = k[T_1, ..., T_d]$ be the polynomial ring in d variables. Let $n \in \mathbb{N}$ and $I \subset A$ be an ideal such that ht$(I) = n = \mu(I/I^2)$. Then $\mu(I) = n$.*

The conjecture is still open in general. The best known result on this conjecture is due to N. M. Kumar ([36], Theorem 5). In fact he proved the following more general result.

Theorem 1.0.3. *[36] Let R be a commutative Noetherian ring and $I \subset R[T]$ be an ideal containing a monic polynomial. Let $\mu(I/I^2) = n \geq \dim(R[T]/I) + 2$. Then there exists a projective $R[T]-$module P of rank n and a surjection $\phi : P \twoheadrightarrow I$.*

Since projective $k[T_1, ..., T_d]$-modules are free by the Quillen-Suslin Theorem, N. M. Kumar solved Murthy's complete intersection conjecture for the bound $2n \geq d + 2$. Later, S. Mandal improved the above result in [40] showing that P can actually be taken free. A closer inspection of S. Mandal's proof showed that he essentially proved that in the above set-up used in Theorem 1.0.3, any set of generators of I/I^2 can be lifted to a set of generators of the ideal I. Later M. K. Das [20] improved the bound of the Murthy's complete intersection conjecture over the base field $\overline{\mathbb{F}}_p$. Their solutions gave a sturdy indication towards a stronger version of the Murthy's complete intersection conjecture. Namely the study of a lifting property of a set of generators of I/I^2, whenever the ideal satisfies $\mu(I/I^2) = \text{ht}(I)$. However this stronger version of the Murthy's complete intersection conjecture no longer holds in general, as it is evidenced by the example due to S. M. Bhatwadekar and R. Sridharan ([12], Example 3.15). Although this lifting problem received a negative answer in the set-up of Murthy's complete intersection conjecture, investigating the lifting property became a recurrent theme in the literature. In particular, one can ask the following question:

Lifting Problem:- Let A be a ring and $I \subset A[T]$ be an ideal such that $\mu(I/I^2) = n$. Moreover, it is given that $I =< f_1, ..., f_n > +I^2$. Does there exist $F_i \in I$ such that $I =< F_1, ..., F_n >$, with $F_i - f_i \in I^2$, for all $i = 1, ..., n$?

As earlier we will not restrict ourselves to the polynomial rings only. Let A be a ring of dimension d and I be an ideal in A. By a result of N. M. Kumar [36] the lifting problem has an affirmative answer whenever $\mu(I/I^2) > d$. Let $R = A[T]$ and $I \subset A[T]$ be an ideal with $\mu(I/I^2) = \text{ht}(I) = \dim(R) = d + 1$. Using Suslin's monic polynomial theorem the ideal I contains a monic polynomial. Therefore by the results of N. M. Kumar and S. Mandal (discussed earlier) any set of generators of I/I^2 lifts to a set of generators of I. Similarly as before in the case with the splitting problem discussed earlier, the conditions $\mu(I/I^2) > d$, and $\text{ht}(I) = \mu(I/I^2) > \dim(A)$, whenever $I \subset A[T]$ are the best possible for arbitrary Noetherian rings.

A bridge

Let A be a ring and P be a projective $A-$module. A remarkable result of Eisenbud-Evans [25, the remark following Theorem A] gives us a leverage that most of the $A-$linear maps $P \to A$ has the property that the image ideal has height at least the rank of P. Such an ideal $I = \phi(P)$, is called a generic section of P whenever $\text{ht}(I) = \text{rank}(P)$. It was N. M. Kumar [37], who first noticed that there exists a possible connection in between these two problems, namely, the splitting problem and the lifting problem. In particular, he proved the following:

Theorem 1.0.4. *[37] Let A be an affine algebra of dimension $d \geq 2$ over an algebraically closed field k. Let P be a projective A-module of rank d. Then P splits off a free summand of rank one if and only if there exists an $A-$linear $\phi : P \to A$ such that the image ideal $I = \phi(P)$ has the property that ht$(I) = \mu(I) = d$.*

Although the statement of N. M. Kumar does not deal with the lifting problem directly, but it was proved later that over algebraically closed fields the lifting problem and the condition that $\text{ht}(I) = \mu(I/I^2) = \mu(I)$ are equivalent whenever $\text{ht}(I) = \dim(A)$. Further, the splitting problem was studied by M. P. Murthy in [48]. One of the seminal works in the literature was M. P. Murthy's idea of an obstruction group, which governed the splitting problem. In particular he proved the following:

Theorem 1.0.5. *[48] Let X = Spec(A) be a $d-$dimensional smooth affine variety over an algebraically closed field k. Let P be a projective $A-$module of rank d. Then, P splits off a free summand of rank one if and only if its top Chern class $c_d(P)$ vanishes in the Chow group $CH^d(X)$.*

If P splits off a free summand of rank one then it easily follows that $c_d(P) = 0$. To prove the reverse implication, M. P. Murthy showed that if $c_d(P) = 0$ then there exists a generic section I which is generated by d many elements, and then he appealed to the result of N. M.

Kumar stated above. However this no longer holds if the ground field is not algebraically closed, as evidenced by the same example on the tangent bundle of the real 2-sphere, mentioned earlier.

It was M. V. Nori who envisioned to replace the top Chern class by the Euler class whenever the field is not algebraically closed, which was extensively studied by S. M. Bhatwadekar and R. Sridharan in a series of papers ([11], [12], [13], [14], [16]) and by M. K. Das in ([19] and [21]). The main philosophy behind their studies was to establish the fact that: Lifting of an appropriate set of generators of I/I^2 to a set of generators of I is the precise obstruction for the splitting problem of P, where I is a generic section of P and the set of generators of I/I^2 is induced by the generic section $P \twoheadrightarrow I$. One of the main themes of the thesis is to study this connecting path whenever the ring is an affine algebra over various bases.

The thesis is divided into two parts. In the first part, the study is mainly restricted on the affine algebras over algebraically closed fields. We have also been able to prove some results over finitely generated $\mathbb{Z}-$algebras, which is also included in this part. In the second part of the thesis we mainly focus on real affine algebras.

On a question of Nori and its applications

Another interesting question in the literature was due to M. V. Nori. Let $X = \text{Spec}(A)$, be a smooth affine variety of dimension d. Let P be a projective A-module of rank n, and $\phi_0 : P \twoheadrightarrow I_0$ be a surjective homomorphism. Assume that the zero set of I_0, $V(I_0) = Y$ be a smooth affine sub-variety of X of dimension $d - n$ and $Z = V(I)$ be a smooth closed sub-variety of $X \times \mathbb{A}^1 = \text{Spec}(A[T])$, such that Z intersects $X \times \{0\}$ transversally in $Y \times \{0\}$. In this set up M. V. Nori asked the following question:

Question 1.0.6 Does there exist a surjective map $\phi : P[T] \twoheadrightarrow I/I^2$, which is compatible with ϕ_0, have a surjective lift $\psi : P[T] \twoheadrightarrow I$, such that
(i) $\psi|_{T=0} = \phi_0$ and
(ii) $\psi|_Z = \psi$?

This question has been answered affirmatively in the following cases:

- A is a ring and I contains a monic polynomial such that $\mu(I/I^2) \geq \dim(A[T]/I) + 2$ ([41], Theorem 2.1).
- A is a smooth local ring ([44], Theorem 4).
- A is a smooth affine domain of dimension $d \geq 3$ over an infinite perfect field k and $I \subset A[T]$ is an ideal of height d ([12], Theorem 3.8).
- A is a regular domain of dimension d which is essentially of finite type over an infinite perfect field k and $I \subset A[T]$ is an ideal of height n such that $2n \geq d + 3$ ([10], Theorem 4.13).

If A is an affine algebra over $\overline{\mathbb{F}}_p$, then we improve the bound imposed by S. Mandal (see, Theorem 3.2.1). In particular, we prove the following:

Theorem 1.0.7. *Let A be an affine algebra over $\overline{\mathbb{F}}_p$ and $I \subset A[T]$ be any proper ideal containing a monic polynomial. Suppose that P is a projective A-module of rank n, where $n \geq \max\{(\dim A[T]/I + 1), 2\}$. Then any surjective map $\overline{\phi} : P[T] \twoheadrightarrow I/I^2T$ lifts to a surjective map $\phi(T) : P[T] \twoheadrightarrow I$.*

We improve another version of the question asked by M. V. Nori (see, Theorem 3.3.2). In particular we improve a result of S. Mandal and R. Sridharan [43]. This particular result is crucial to the subtraction principle, which we use to develop the Euler class theory throughout the thesis. We have the following:

Theorem 1.0.8. *Let A be an affine algebra over $\overline{\mathbb{F}}_p$. $I = I_1 \cap I_2 \subset A[T]$, is an ideal and P is a projective A-module such that,*

(i) *I_1 contains a monic polynomial.*

(ii) *$I_2 = I_2(0)A[T]$ is an extended ideal.*

(iii) $I_1 + I_2 = A[T]$.

(iv) *rank*$(P) = n \geq \max\{(\dim(A[T]/I_1) + 1), 2\}$.

Suppose that there exist surjections $\rho : P \twoheadrightarrow I(0)$ and $\overline{\delta} : P[T]/I_1P[T] \twoheadrightarrow I_1/I_1^2$ such that $\overline{\delta} = \rho \otimes A/I_1(0)$. Then there exists a surjection $\eta : P[T] \twoheadrightarrow I$ such that $\eta(0) = \rho$.

The case of dimension two

If $\dim(A) = \text{ht}(I) = \mu(I/I^2) = 2$, then the question asked by M. V. Nori does not have an affirmative answer, even over the field of complex numbers ([12], Example 3.15). However, if the base field is $\overline{\mathbb{F}}_p$, then we show that (see, Theorem 3.4.6) such an example can not exist. In particular, we prove the following:

Theorem 1.0.9. *Let R be an affine domain of dimension two over $\overline{\mathbb{F}}_p$. Let $I \subset R[T]$ be an ideal such that $\mu(I/(I^2T)) = \text{ht}(I) = 2$ and $R/(I \cap R)$ is smooth. Let $I = (f_1, f_2) + (I^2T)$ be given. Then there exist $F_1, F_2 \in I$ such that $I = (F_1, F_2)$ and $F_i - f_i \in (I^2T)$ for $i = 1, 2$.*

Precise obstruction

As mentioned earlier, in the appendix of a paper by S. Mandal [41], M. V. Nori asked the following question, which is motivated by certain results in topology. For the convenience of understanding, we state the “free” version of the question below.

Question 1.0.10 Let R be a smooth affine domain of dimension d over a field k and $I \subset R[T]$ be an ideal of height n such that $\mu(I/I^2T) = n$, where $2n \geq d+3$. Assume that $I = (f_1, \cdots, f_n) + (I^2T)$ is given. Then, do there exist $F_i \in I$ ($i = 1, \cdots, n$), such that $I = (F_1, \cdots, F_n)$ where $F_i - f_i \in (I^2T)$ for $i = 1, \cdots, n$?

Here we shall focus on the case when I does not contain monic in the above question. As mentioned above, Nori's question has been answered comprehensively.

On the other hand, Bhatwadekar-Mohan Kumar-Srinivas gave an example in [12, Example 6.4] to show that Nori's question will have a negative answer if R is not smooth (even when R is local). They constructed an example of a normal affine $\mathbb{C}$-domain R of dimension 3 which has an isolated singularity at the origin, and an ideal $I \subset R[T]$ of height 3 such that a given set of generators of $I/(I^2T)$ cannot be lifted to a set of generators of I.

The results and the example stated above had profound impact on the development of the theory in understanding the behaviour of projective modules and local complete intersection ideals in past twenty years. Among recent instances, the Bhatwadekar-Sridharan solution played a crucial role in computing the group of isomorphism classes of oriented stably free R-modules of rank d where R is a smooth affine domain of dimenson d over $\mathbb{R}$ ([23], see also [24]). Further, Asok-Fasel [1] used it successfully to establish the isomorphism between the d-th Euler class group and the d-th Chow-Witt group (also the isomorphism between the weak Euler class group and the Chow group) — thus establishing a long standing conjecture.

In this context, we delve deep into this phenomenon and pose the following rephrased question.

Question 1.0.11 Let R be an affine domain of dimension d over a field k and $I \subset R[T]$ be an ideal of height n such that $\mu(I/I^2T) = n$, where $2n \geq d+3$. Assume that $I = (f_1, \cdots, f_n) + (I^2T)$ is given. Then, what is the precise obstruction for I to have a set of generators $F_1, \cdots, F_n$ such that $F_i - f_i \in (I^2T)$ for $i = 1, \cdots, n$?

Obviously we have left out the case when I contains a monic polynomial. We prove that the obstruction lies in the fact as to whether $I \cap R$ is contained in only smooth maximal ideals or not. More precisely, we prove the following result (Theorem 3.5.5).

Theorem 1.0.12. *Let R be an affine domain of dimension $d \geq 3$ over an infinite perfect field k and $I \subset R[T]$ be an ideal of height d such that $J := I \cap R$ is contained only in smooth maximal ideals. Let P be a projective $R[T]$-module of rank d such that there is a surjection*

$$\overline{\varphi} : P \twoheadrightarrow I/(I^2T).$$

Then, there is a surjection $\Phi : P \twoheadrightarrow I$ which lifts $\overline{\varphi}$.

Some applications

As an application of the Theorem 1.0.8, we improve the bound of some addition and subtraction principles, imposed by S. M. Bhatwadekar and R. Sridharan [11] over the ground field $\overline{\mathbb{F}}_p$. Let R be a $d-$dimensional affine $\overline{\mathbb{F}}_p-$algebra. We define the "$n-$th Euler class" group $E^n(R)$ for $2n \geq d+2$. Moreover, taking R to be smooth, we show that $E^n(R)$ is the precise obstruction group for the splitting problem of an $1-$stably free $R-$module of rank n.

Here is an interesting application of Theorem 1.0.12. For a commutative Noetherian $\mathbb{Q}$-algebra R of dimension $d \geq 3$, the d-th Euler class group $E^d(R[T])$ was defined in [19]. It was further proved that the canonical map $\phi : E^d(R) \longrightarrow E^d(R[T])$ is injective. The morphism ϕ is an isomorphism if R is smooth but it may not be surjective if R is not smooth (see [19] for the details). In this context, we may ask, precisely which Euler cycles $(I, \omega_I) \in E^d(R[T])$ have a preimage in $E^d(R)$? We answer this question in the following form (Theorem 3.5.8).

Theorem 1.0.13. *Let R be an affine domain of dimension $d \geq 3$ over a field k of characteristic zero. Let $(I, \omega_I) \in E^d(R[T])$ be such that $I \cap R$ is contained only in smooth maximal ideals. Then (I, ω_I) is in the image of the canonical morphism $\phi : E^d(R) \longrightarrow E^d(R[T])$.*

Another interesting application is the following Monic inversion principle (Theorem 3.5.10).

Theorem 1.0.14. *Let R be a domain of dimension d containing a field k (no restriction on k). Let $I \subset R[T]$ be an ideal such that ht$(I) = n = \mu(I/I^2T)$, where $2n \geq d+3$. Let $I = (f_1, \cdots, f_n) + (I^2T)$ be given. Assume that there exist $F_1, \cdots, F_n \in IR(T)$ such that $IR(T) = (F_1, \cdots, F_n)$ where $F_i - f_i \in I^2R(T)$. Assume further that $I \cap R$ is contained only in smooth maximal ideals of R. Then there are $g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots, g_n)$ with $g_i - f_i \in (I^2T)$.*

Monic inversion principle

We begin this section with a theorem by D. Quillen [52], which was crucial in his proof of Quillen-Suslin Theorem. For local rings, the result is due to G. Horrocks [30]. This theorem of D. Quillen is known as Affine Horrocks' Theorem. Before that let us recall that the ring $A(T)$ is obtained from $A[T]$ by inverting all monic polynomials in T.

Theorem 1.0.15. *[52] Let A be a ring and P be a projective $A[T]-$module. Suppose that $P \otimes A(T)$ is free. Then P is free.*

Let A be a ring. For any two projective $A-$modules P and Q, we shall call Q is a decomposition of P if there exists $n > 0$, such that $P \cong Q \oplus A^n$. One can think of a free module F in the way that every decomposition Q of F has a further decomposition. This point of view gave another direction to grasp the Theorem 1.0.15 in a desire to ask a more general question. It was M. Roitman who studied such projective modules over polynomial extensions which split off after inverting some monic polynomials. This philosophy in the literature is known as a monic inversion principle. In particular, M. Roitman posed the following question:

Question 1.0.16 Let A be a commutative Noetherian ring of dimension d and P be a projective $A[T]$-module of rank d. Suppose that there exists a surjection $\phi : P \twoheadrightarrow I$, where $I \subset A[T]$ is an ideal containing a monic polynomial. Then does P have a unimodular element?

In other words, this question asked that, if the projective $A(T)$-module $P \otimes A(T)$ has a unimodular element then does P have a unimodular element? In general the question is still open. It has an affirmative answer in the following cases:

- A is a local ring and $d = 1$ [30].
- A is a ring and $d = 1$ ([52], see Theorem 1.0.15).
- A is a local ring [56].
- A is a ring and $d = 2$ [8].
- A is a ring containing an infinite field [15].

We show that the following versions of the above question have affirmative answers:

Theorem 1.0.17. *(Theorem 5.1.4) Let A be a finite $\mathbb{Z}$-algebra of dimension $d \geq 1$. Moreover assume that there exists an integer $n \geq 2$ such that $n \in A^*$. Let P be a projective $A[T]-$module with trivial determinant of rank d and $I \subset A[T]$ be an ideal of height d containing a monic polynomial in T. Suppose that there exists a surjection $\alpha : P \twoheadrightarrow I$. Then P has a unimodular element.*

Theorem 1.0.18. *(Theorem 5.2.1) Let R be a $d-$dimensional affine algebra over $\overline{\mathbb{F}}_p$. Let P be a stably free $R[T]-$module of rank $d-1$. Assume that $P \otimes R(T)$ has a unimodular element. Then P has a unimodular element.*

Continuing with the philosophy built in the literature, it is now became customary to rephrase the Question 1.0.16, in terms of the lifting problem. To be precise one can ask the following question:

Question 1.0.19 Let A be a commutative Noetherian ring of dimension $d \geq 2$ and $I \subset R[T]$ be an ideal such that ht$(I) = \mu(I/I^2) = d$. Moreover assume that $I =< f_1, ..., f_d > +I^2$. Suppose that there exists $F_i \in IA(T)$ such that $IA(T) =< F_1, ..., F_d >$, with $F_i - f_i \in IA(T)^2$. Then does there exist $g_i \in I$, such that $I =< g_1, ..., g_d >$, with $g_i - f_i \in I^2$ for all $i = 1, ..., d$?

Question 1.0.19 has a negative answer for $d = 2$ ([12], Example 3.15). Whenever $d \geq 3$ this question has an affirmative answer in the following cases :

- A is a local ring ([19], Proposition 5.8(1)).
- A is an affine domain over an algebraically closed field of characteristic zero ([19], Proposition 5.8(2)).

- A is a regular domain which is essentially of finite type over an infinite perfect field k of characteristic unequal to 2 ([24], Theorem 5.11).

We shall focus on M. K. Das's proof [19] over the algebraically closed field of characteristic 0. We observe that with the machineries available today, the same proof goes on whenever the characteristic of the algebraically closed field is strictly bigger than d, that is $d! \in A^*$. The reason of this hypothesis coming into the picture is due to a remarkable result by R. A. Rao [54]. Tracking back to R. A. Rao's proof, a crucial step was to show a given unimodular row belongs to the same SL-orbit space with the *factorial rows*. And to do this one needs the hypothesis on $d!$. In the literature, this factorial row concept is due to A. A. Suslin (in unimodular rows of length three it is due to R. G. Swan and J. Towber). The above factorial row technique is fundamental in the sense that, till now this is the only technique available in the literature, when it comes to dealing with the cancellation problem. In the proof of ([24], Theorem 5.11) the regularity assumption is used rigorously throughout.

However, if the base field is $\overline{\mathbb{F}}_p$, we have been able to give a direct proof (Theorem 6.1.1) of the same, without the assumptions that either $d! \in A^*$ or the ring A is smooth. In particular, we have the following:

Theorem 1.0.20. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$ and $I \subset R[T]$ be an ideal such that ht$(I) = \mu(I/I^2) = d$. Moreover assume that $I =< f_1, ..., f_d > +I^2$. Suppose that there exists $F_i \in IR(T)$ such that $IR(T) =< F_1, ..., F_d >$, with $F_i - f_i \in IR(T)^2$. Then there exists $g_i \in I$, such that $I =< g_1, ..., g_d >$, where $g_i - f_i \in I^2$, for all $i = 1, ..., d$.*

As an application of the above result we have been able to define the $d-$th Euler class group $E^d(R[T])$ of $R[T]$, where R is an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$. Let P be a projective $R[T]-$module of rank d with a trivial determinant and $\chi : R[T] \cong \wedge^d P$ be an isomorphism. We then assign a *"local orientation"* $(I, \omega_I) \in E^d(R[T])$ to the pair (P, χ) and show that the vanishing of (I, ω_I) in the group $E^d(R[T])$ is sufficient for P to have a unimodular element (Theorem 7.3.4). Moreover assume that $(d-1)! \in R^*$. In this set up we show that the local orientation $e(P, \chi)$ induced by the pair (P, χ) is the precise obstruction for the splitting problem of P (see Theorem 7.3.3 and Theorem 7.3.6).

A splitting criterion on polynomial algebras over algebraically closed fields

In this section we shall discuss an analogue of N. M. Kumar's result (Theorem 1.0.4) for the polynomial algebras over algebraically closed field. Let A be a ring of dimension d and P be a projective $A[T]-$module. If the rank$(P) > d$ then, as mentioned earlier P splits off without any further conditions. Also recall that, by taking the polynomial extension of even dimensional real sphere one can establish the fact that: This result is the best possible. Therefore, investigating criteria for the splitting problem of P became interesting, whenever rank$(P) = d = \dim(A)$.

This study on polynomial algebras was initiated by S. M. Bhatwadekar and R. Sridharan ([15], Theorem 4.5). In particular, they proved the following:

Theorem 1.0.21. *[15] Let A be a $d(\geq 2)$-dimensional affine domain over an algebraically closed field k of characteristic 0. Let P be a projective $R[T]$-module of rank d with trivial determinant and $I \subset R[T]$ be an ideal of height d. Suppose that there exists a surjection $\phi : P \twoheadrightarrow I$. If $\mu(I) = d$ then P has a unimodular element.*

Note that, in their statement the hypothesis "algebraically closed field" is necessary. We show that their result can be achieved without the "characteristic zero" assumption (Theorem 8.2.2 and Remark 8.2.3). We also observe that the hypothesis "domain" is not crucial. In particular we prove the following:

Theorem 1.0.22. *Let R be a $d(\geq 2)$-dimensional affine algebra over an algebraically closed field k of $char(k) \neq 2$. Let P be a projective $R[T]$-module of rank d. Moreover, assume that there exists an ideal $I \subset R[T]$ of height d such that $\phi : P \twoheadrightarrow I$ is a surjection. If $\mu(I) = d$ then P has a unimodular element.*

With a suitable Cancellation result on a surface, over certain C_1-fields, one can establish the above result. The Cancellation result required in our case was proved by A. A. Suslin [64] for smooth surfaces. We remove the smoothness assumption (see Theorem 8.1.1) of A. A. Suslin's result.

Some miscellaneous results on $\overline{\mathbb{F}}_p$

In this section we shall discuss a few more results which we are able to prove, on affine $\overline{\mathbb{F}}_p$-algebras.

On Laurent polynomial algebras

Chapter 9 is devoted to studying the Laurent polynomial algebra $R[T, \frac{1}{T}]$, where R is a affine $\overline{\mathbb{F}}_p-$algebra. The theme of this chapter is to investigate the questions we tackled for polynomial algebras. Recall that a Laurent polynomial is said to be a doubly monic if the coefficients of the highest and the lowest degree terms are units. We improve the bounds of some analogous questions (Theorem 9.1.2 and Theorem 9.1.3) similar to the Murthy's complete intersection conjecture and Nori's question on Laurent polynomial rings over the base field $\overline{\mathbb{F}}_p$. In particular, we prove the followings:

Theorem 1.0.23. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ and $I \subset R[T, T^{-1}]$ be an ideal containing a doubly monic Laurent polynomial. Moreover assume $I =< f_1, ..., f_n > +I^2$, with $n \geq \max\{(\dim R[T, T^{-1}]/I + 1), 2\}$. Then there exists $g_i \in I$, for $i = 1, ..., n$, such that $I = (g_1, ..., g_n)$, with $g_i - f_i \in I^2$.*

Theorem 1.0.24. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ and $I \subset R[T, T^{-1}]$ be an ideal containing a doubly monic Laurent polynomial. Moreover assume that $I =< f_1, ..., f_n > +(I^2(T-1))$, with $n \geq \max\{(\dim R[T, T^{-1}]/I + 1), 2\}$. Then there exist $g_i \in I$, for $i = 1, ..., n$, such that $I = (g_1, ..., g_n)$, with $g_i - f_i \in I^2(T-1)$.*

To develop something similar to a Monic inversion principle in the Laurent polynomial rings, it is not enough to invert all the monic polynomials, as it is evidenced by an example due to S. M. Bhatwadekar. It also gave a hint towards the fact that: One needs to invert all doubly monic Laurent polynomials instead. Let $\mathfrak{R} = S^{-1}R[T, T^{-1}]$, where $S \subset R[T, T^{-1}]$ be the multiplicatively closed set consisting all doubly monic Laurent polynomials. In this set up we prove the following version of a Monic inversion principle (Theorem 9.2.2):

Theorem 1.0.25. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$ and $I \subset R[T, T^{-1}]$ be an ideal such that $ht(I) = \mu(I/I^2) = d$. Moreover assume that $I =< f_1, ..., f_d > +I^2$. Suppose that there exists $F_i \in I\mathfrak{R}$ be such that $I\mathfrak{R} =< F_1, ..., F_d >$, with $F_i - f_i \in (I\mathfrak{R})^2$. Then there exists $g_i \in I$ be such that $I =< g_1, ..., g_d >$, where $g_i - f_i \in I^2$.*

As an application of the above theorem we have been able to present a comprehensive account of the $d-$th Euler class group $E^d(R[T, \frac{1}{T}])$ which is absent in the literature, even when R is an affine algebra over any algebraically closed field. Let P be a projective $R[T, \frac{1}{T}]-$module of rank d with a trivial determinant and $\chi : R[T, \frac{1}{T}] \cong \wedge^d P$ be an isomorphism. We then assign a *"local orientation"* $(I, \omega_I) \in E^d(R[T, \frac{1}{T}])$ to the pair (P, χ) and show that the vanishing of (I, ω_I) in the group $E^d(R[T, \frac{1}{T}])$ is sufficient for P to have a unimodular element (Theorem 9.2.10). Moreover assume that $(d-1)! \in R^*$. In this set up we show that the local orientation $e(P, \chi)$ induced by the pair (P, χ) is the precise obstruction for the splitting problem of P (see Theorem 9.2.9 and Theorem 9.2.12).

Segre class of an ideal

In Chapter 10 we studied the Segre class of an ideal over polynomial and Laurent polynomial algebras over $\overline{\mathbb{F}}_p$. This study is motived from the work done by M. K. Das and R. Sridharan [22]. They gave an algebraic interpretation of the M. P. Murthy's idea of Segre class defined in ([48], Section 5). Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$. Let I be an ideal either of $R[T]$ or of $R[T, \frac{1}{T}]$ such that $\mu(I/I^2) = d$. Moreover assume that ht$(I) \geq 2$. Then we have assign a pair (I, ω_I) to an element $s(I, \omega_I)$ in the $d-$th Euler class group $E^d(R[T])$ or $E^d(R[T, \frac{1}{T}])$ respectively, where ω_I is a local orientation of I. We shall call the Segre class of the pair (I, ω_I) is $s(I, \omega_I)$. We have proved that the Segre class is the precise obstruction for the lifting problem of the pair (I, ω_I) (Theorem 10.1.6 and Theorem 10.2.6). In particular, we have the following:

Theorem 1.0.26. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$. Let $I \subset R[T]$ (or $I \subset R[T, T^{-1}]$) be an ideal of height ≥ 2 such that $\mu(I/I^2) = d$. Let ω_I be a local orientation of I. Then $s(I, \omega_I) = 0$ if and only if ω_I is a global orientation of I.*

Equivalence of two conjectures

In Chapter 11 we investigate any possible connections between the question asked by M. V. Nori on homotopy sections and the M. P. Murthy's complete intersection conjecture, of a curve in polynomial extensions over $\overline{\mathbb{F}}_p$. This particular study becomes interesting when the ideal does not have any finiteness condition (such as the ideal containing a monic polynomial). We shall begin with M. P. Murthy's complete intersection conjecture and another question closely related to it, followed by M. V. Nori's question in some favorable set-up. Although M. P. Murthy's complete intersection conjecture is on polynomial rings over a field, but here we shall call the following version as Murthy's complete intersection conjecture.

Question 1.0.27 Let A be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension d and $I \subset A[T]$ be an ideal such that $\mu(I/I^2) = \text{ht}(I) = d$. Then is $\mu(I) = n$?

Question 1.0.28 Let A be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension d and $I \subset A[T]$ be an ideal such that $\mu(I/I^2) = \text{ht}(I) = d$. Further assume that $I =< f_1, ..., f_d > +I^2$. Then can we lift f_i's to a set of generators of I ?

Question 1.0.29 Let A be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension d and $I \subset A[T]$ be an ideal of height d. Further assume that $I =< f_1, ..., f_d > +I^2T$, then does there exists $g_i \in I$, such that $I =< g_1, ..., g_d >$, with $f_i - g_i \in I^2T$?

We observe the fact that: In some favorable cases all the above three questions are equivalent over $\overline{\mathbb{F}}_p$. In particular, we have the following (Theorem 11.1.7):

Theorem 1.0.30. *Let A be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 3$, $(d-1)! \in A^*$. Let $I \subset A[T]$, be an ideal such that ht$(I) = \mu(I/I^2) = d$. Then the following assertions are equivalent*
(i) $\mu(I) = d$.
(ii) *If $I =< f_1, ..., f_d > +I^2$, then it has a lift to a set of generators of I.*
(iii) *If $I =< f_1, ..., f_d > +I^2T$, then it has a lift to a set of generators of I.*

On real affine algebras

In the second part of the thesis we studied projective modules and complete intersection ideals over some real affine algebras. In the geometric set-up, it is a well-known phenomenon that more often than not algebraically closed fields behave "nicely". The study of projective and stably free modules is no exception to that. For example, due to Suslin [62], we know that any stably free module over a complex (affine) algebra is free whenever the rank is equal to the dimension of the algebra. But this is not true in the case of real algebras (see [65]). Motivated from this particular example, one can look for some sufficient conditions on a real (affine) algebra which might guarantee results similar to the case of complex algebras. One of the goals in this study is to showcase a class of real algebras and to show that the projective and stably free modules over those real algebras behave exactly like complex algebras. Throughout this chapter (unless explicitly stated otherwise), we shall study real affine algebras under one of the following conditions:

- there are no real maximal ideals;
- the intersection of all real maximal ideals has height at least 1.

For the remaining part of the introduction we fix the notation R for such a real affine algebra of dimension d.

Let A be a ring of dimension d. Then by the Bass-Schanuels' cancellation theorem, the study of (finitely generated) stably free modules of rank d may be reduced to the study of unimodular rows of length $d+1$. One can easily check that a stably free module of rank d is free if and only if there is a corresponding unimodular row, which is, the first row of a matrix in $\mathrm{SL}_{d+1}(A)$. This observation encourages one to study the natural $\mathrm{SL}_{d+1}(A)$ action on $\mathrm{Um}_{d+1}(A)$, the set of all unimodular rows of length $d+1$. In [62], A. A. Suslin proved that the above action is trivial on affine algebras over the field of complex numbers. Recall that there is this normal subgroup $E_{d+1}(A)$ (see Definition 12.1.6) of $\mathrm{SL}_{d+1}(A)$. Due to a result of A. A. Suslin [63, Lemma 8.5], we know that there exists a unimodular row of length $d+1$ which is not elementarily completable over the field of complex numbers. Hence the study of $E_{d+1}(A)$ action on $\mathrm{Um}_{d+1}(A)$ becomes interesting over the field of complex numbers. In ([73], Section 5), L. N. Vaserstein defined an abelian group structure on the orbit set $\mathrm{Um}_3(A)/E_3(A)$ of unimodular rows of length 3 modulo elementary action by producing a bijection between $\mathrm{Um}_3(A)/E_3(A)$ and the elementary symplectic Witt group $W_E(A)$ for a commutative ring A of dimension 2. Later on, W. van der Kallen [69] inductively defined an abelian group structure on $\mathrm{Um}_{d+1}(A)/E_{d+1}(A)$ for higher dimensional rings.

Recall that, by the Bass-Kubota theorem ([69], Theorem 2.12), $SK_1(A)$ is isomorphic to the universal Mennicke symbols $MS_2(A)$, when A is a ring of dimension 1. Note that, in the general setup, the product formula of the abelian group $\mathrm{Um}_{d+1}(A)/E_{d+1}(A)$ can not be lifted inductively from the dimension 1 case. A necessary condition for the lifting is that the universal weak Mennicke symbol should coincide with the universal Mennicke symbol. In this scenario, we will say the van der Kallen group structure is nice, but this is not true in general (see [75], Example 2.2(c)). However, this phenomenon is true whenever the base field k is perfect and satisfying the following:

1. $char(k) \neq 2$ and $c.d._2(k) \leq 1$ (due to A. S. Garge and R. A. Rao [27], Theorem 3.9);
2. $c.d._2(k) \leq 2$ and the algebra is smooth of dimension bigger than 2 (due to J. Fasel [26], Theorem 2.1).

In Chapter 13 we show (Theorem 13.1.1) that, the van der Kallen group is nice over the real affine algebras belonging to the class mentioned earlier. To be precise we prove the following:

Theorem 1.0.31. *Let $d \geq 2$. Then the abelian group $Um_{d+1}(R)/E_{d+1}(R)$ has a nice group structure. That is for any $(a, a_1, ..., a_d)$ and $(b, a_1, ..., a_d) \in Um_{d+1}(R)$ we have*

$$[(a, a_1, ..., a_d)] \star [(b, a_1, ..., a_d)] = [(ab, a_1, ..., a_d)].$$

In particular, $WMS_{d+1}(R) \cong MS_{d+1}(R)$.

As a corollary of this result, we show that the van der Kallen group is a divisible group (Corollary 13.2.1).

In Chapter 14, we study a K_1 analogue of Suslin's result [62] over real affine algebras. In particular we show (Theorem 14.1.1 and 14.2.4) that the injective stability of SK_1 and K_1Sp can be improved in view towards Bass-Milnor-Serre and Vaserstein over real affine algebras belonging to the class mentioned earlier. In particular, we prove the followings results:

Theorem 1.0.32. *Let $I =< a > \subset R$, be a principal ideal. Let $\sigma \in SL_{d+1}(R, I)$ be a stably elementary matrix. Then σ is isotopic to identity. Moreover if R is nonsingular, then $E_{d+2}(R, I) \cap SL_{d+1}(R, I) = E_{d+1}(R, I)$, for $d \geq 3$. In other words $SK_1(R, I) = \frac{SL_{d+1}(R,I)}{E_{d+1}(R,I)}$.*

Theorem 1.0.33. *Let R be nonsingular. Let $d \geq 4$ and $I =< a > \subset R$ be a principal ideal. Moreover assume that if d is even then $4|d$. Let $n = 2[\frac{d+1}{2}]$, where $[-]$ denotes the smallest integer less than or equals to $-$. Then $K_1Sp(R, I) = \frac{Sp_n(R,I)}{Ep_n(R,I)}$.*

In the remaining chapters of this part, we study projective modules over real affine algebras. One of the main themes is to study various obstruction groups for the splitting problem of a projective $R-$module of rank d. M. P. Murthy studied the splitting problem (in [48]) for projective modules having rank equal to the dimension of the ring. In particular, for a smooth reduced complex affine domain A of dimension d, M. P. Murthy showed (see [48], Remark 2.13 and Theorem 3.8) that $F^dK_0(A)$ is the precise obstruction group. In [14] S. M. Bhatwadekar and R. Sridharan defined the 'Euler class group' and the 'weak Euler class group' of commutative Noetherian rings containing rationals. They showed that over smooth affine complex algebras all these groups are isomorphic. In a recent work [34], A. Krishna showed that, all these groups are isomorphic to the Levine–Weibel Chow group of 0-cycles $CH^d(A)$, for arbitrary reduced affine algebras over algebraically closed fields. In fact in the same paper A. Krishna solved the Murthy's conjecture on the absence of torsion in $F^dK_0(A)$, by showing that $CH^d(A)$ is torsion-free.

Let A be a ring. Recall that an ideal $I \subset A$ is said to be projectively generated if there exists a finitely generated projective $A-$module P of rank equals to $\mu(I/I^2)$ such that $P \twoheadrightarrow I$ is a surjection. In Chapter 15, we show that the '$d-$th Euler class group' (denoted as $E^d(R)$ or $E(R)$) is uniquely divisible on real algebra R of dimension d, which belongs to the class we mentioned earlier. Using this result we prove in Chapter 16 the following result:

Theorem 1.0.34. *Let $I \subset R$ be an ideal such that $ht(I) = \mu(I/I^2) = d \geq 3$. Then there exists a projective $R-$module of rank d with trivial determinant such that P maps surjectively onto I.*

As a corollary, we give a necessary and sufficient condition for a locally complete intersection ideal I of height d, such that I/I^2 is generated by d elements, to become a complete intersection ideal (Corollary 16.1.4). In Theorem 16.2.1, we show over the polynomial algebra $R[T]$, any local complete intersection ideal $I \subset R[T]$ such that I/I^2 is generated by d elements, is projectively generated. To be precise, we have the following:

Theorem 1.0.35. *Any local complete intersection ideal $I \subset R[T]$ with ht$(I) = \mu(I/I^2) = d \geq 3$, is projectively generated.*

In Theorem 16.2.2 we prove a Monic inversion principle over real affine algebras mentioned earlier. In particular, we prove the following:

Theorem 1.0.36. *Let $I \subset R[T]$ be an ideal such that ht$(I) = \mu(I/I^2) = d \geq 3$. Suppose that $I =< f_1, ..., f_d > +I^2$. Moreover, assume that there exist $F_i \in IR(T)$ such that $I =< F_1, ..., F_d >$, with $F_i - f_i \in I^2R(T)$. Then there exist $g_i \in I$ such that $I =< g_1, ..., g_d >$, with $g_i - f_i \in I^2$ for all $i = 1, ..., d$.*

This concludes the introductory part of the thesis.

Part I

On various base rings and its polynomial and Laurent polynomial extensions

Chapter 2

Preliminaries

2.1 Projective modules and locally complete intersection ideals

The purpose of this section is to recall some basic definitions and facts related to projective modules and complete intersection ideals. We begin with the following definitions.

Definition 2.1.1 Let A be a ring.

(i) A sequence of elements $a_1, ..., a_n \in A$ is called a regular sequence if a_i is a nonzero divisor on $A/ < a_1, ..., a_{i-1} >$, for $i = 1, ..., n$.

(ii) An ideal $I \subset A$ is called a complete intersection ideal of height n if I is generated by a regular sequence $a_1, ..., a_n$ of length n.

(iii) An ideal $I \subset A$ is called a locally complete intersection ideal of height n if the ideal $I_p \subset A_p$ is a complete intersection ideal of height n, for all prime ideals p such that $p \supset I$.

Remark 2.1.2 Note that I/I^2 is generated by n number of elements as an A/I-module is a necessary condition for a locally complete intersection ideal I to become a complete intersection ideal.

The next lemma is due to N. M. Kumar [45], recast slightly to suit our requirements.

Lemma 2.1.3. *Let A be a Noetherian ring and I be an ideal of A. Let J, K be ideals of A contained in I such that $K \subset I^2$ and $I = J + K$. Then there exists $e \in K$ such that $e(1-e) \in J$ and $I =< J, e >$.*

Proof Note that $(I/J)^2 = (I^2 + J)/J = I/J$ (as $K \subset I^2$ and $I = J + K$) hence I/J is an idempotent ideal of a Noetherian ring A/J. Let 'bar' denote going modulo J. Since the image of K maps sujectively onto I/J, we get $\overline{KI} = \overline{I}$. By Nakayama lemma there exists $e \in K$ such that $(\overline{1} - \overline{e})\overline{I} = \overline{0}$. Therefore $(1-e)I = J$ that is $I + eI = J$. Thus going modulo e we get $I = J$, hence $J+ < e >= I$. Since $e \in K \subset I$ and $(1-e)I = J$, we get $e(1-e) \in J$. $\square$

The next theorem is a consequence of a result by Eisenbud and Evans [25] to suit our requirements. For a proof of this version one can see ([14], Corollary 2.13).

Theorem 2.1.4. *Let A be ring and P be a projective $A-$module of rank n. Let $(\alpha, a) \in (P^* \oplus A)$, where P^* is the dual of P. Then there exists $\beta \in P^*$ such that ht$(I_a) \geq n$, where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $< \alpha(P), a >$ has height $\geq n$ then ht$(I) \geq n$. Further, if $(\alpha(P), a)$ is an ideal of height $\geq n$ and I is a proper ideal of A, then ht$(I) = n$.*

The next result is due to N. M. Kumar. Here we will prove it as a corollary of Theorem 2.1.4.

Lemma 2.1.5. *Let A be a commutative Noetherian ring of dimension d and $I \subset A$ be an ideal such that $I =< a_1, ..., a_n > +I^2$, where $n \geq d+1$. Then there exists $b_i \in I$, such that $I =< b_1, ..., b_n >$, with $a_i - b_i \in I^2$.*

Proof Applying Lemma 2.1.3 there exists $e \in I^2$ such that $I =< a_1, ..., a_n, e >$ and $e(1-e) \in< a_1, ..., a_n >$. Using Theorem 2.1.4 we can find $\lambda_i \in A$ for $i = 1, ..., n$, such that ht$(< b_1, ..., b_n >_e) \geq n$, where $b_i = a_i + \lambda_i e$. Since $n \geq d + 1 \geq \dim(A_e) + 1$, we have $< b_1, ..., b_n >_e = A_e$. Therefore some power of e is in the ideal $< b_1, ..., b_n >$. Therefore for any $p \in$ Spec(A), $I \subset p$ if and only if $< b_1, ..., b_n > \subset p$.

We claim that $I =< b_1, ..., b_n >$.

Note that it is enough to prove our claim locally. Let $p \in$ Spec(A). If p does not contain I then we have, $I_p =< b_1, ..., b_n > A_p = A_p$, hence without loss of generality we may assume that $I \subset p$. Note that in the ring A_p we have $I_p =< b_1, ..., b_n > A_p + I_p^2$. Again using Lemma 2.1.3 there exists $s \in I_p^2$ such that $I_p =< b_1, ..., b_n, s > A_p$, with $s(1-s) \in< b_1, ..., b_n > A_p$. Since A_p is a local ring and $s \in I_p^2 \subset pA_p$ implies that $1 - s \in A_p^*$. Therefore $s \in< b_1, ..., b_n > A_p$ and this gives us $I_p =< b_1, ..., b_n > A_p$. $\square$

Remark 2.1.6 Note that in the last part of the above proof we actually show that in a local ring A_p, any set of generators of J/J^2, where $J \subset A_p$ is an ideal, can be lifted to a set of generators of J.

Let us recall the following series of definitions:

Definition 2.1.7

1. Let A be a ring . A row vector $(a_0, ..., a_n) \in A^{n+1}$ of length $n+1$ is said to be a unimodular row of length $n+1$, if there exists $(b_0, ..., b_n) \in A^{n+1}$ such that $\sum_{i=0}^{n} a_i b_i = 1$. We will denote $\mathrm{Um}_{n+1}(A)$ as the set of all unimodular row vectors of length $n+1$ over the ring A.

2. A projective $A-$module Q is said to be a stably free module if there exists integer $n \geq 0$ such that $Q \oplus A^n$ is a free $A-$module.

3. Let A be a ring and P be a projective $A-$module. An element $p \in P$ is called *unimodular* if there is an A-linear map $\phi : P \twoheadrightarrow A$ such that $\phi(p) = 1$. Let $\mathrm{Um}(P)$ denote the set of all unimodular elements of P.

An interesting example of a unimodular row comes from the rows (or columns) of invertible matrices. If a unimodular row v comes from the rows (or columns) of an invertible matrix, we will call that v is completable. The following lemma states that if the length of the unimodular row is large enough ($\geq \dim(A) + 2$) then it is always completable. We will give a proof of this fact using Theorem 2.1.4.

Lemma 2.1.8. *Let A be a commutative Noetherian ring of dimension d and $v \in Um_n(A)$, where $n \geq d+2$. Then v is completable.*

Proof Let $v = (v_1, ..., v_n)$. Using Theorem 2.1.4 we can find $\lambda_i \in A$, for $i = 1, ..., n-1$ such that $\mathrm{ht}(< u_1, ..., u_{n-1} >_{v_n}) \geq n-1 > \dim(A)$, where $u_i = v_i + \lambda_i v_n$. Furthermore since $\mathrm{ht}(< v_1, ..., v_n >) = \infty \geq n$ we have $\mathrm{ht}(< u_1, ..., u_{n-1} >) \geq n-1 > \dim(A)$. Therefore we get $< u_1, ..., u_{n-1} > A = A$. Note that by the choice of u_i's there exists $\epsilon_1 \in \mathrm{GL}_n(A)$ such that $(v_1, ..., v_{n-1}, v_n)\epsilon_1 = (u_1, ..., u_{n-1}, v_n)$. Since $1 \in < u_1, ..., u_{n-1} >$, there exists $\epsilon_2 \in \mathrm{GL}_n(A)$ such that $(u_1, ..., u_{n-1}, v_n)\epsilon_2 = (0, ..., 1)$. Thus we get $(v_1, ..., v_n) = (0, ..., 1)\alpha$, where $\alpha = (\epsilon_1\epsilon_2)^{-1}$.This completes the proof. □

Remark 2.1.9 Note that this matrix α is in fact an elementary matrix (for definition see 12.1.6).

Theorem 2.1.4 gives us that in a ring A, for any projective $A-$module P of rank n we can always find an ideal $I \subset A$ such that $\mathrm{ht}(I) \geq n$ and a surjection $\alpha : P/IP \twoheadrightarrow I/I^2$. We are mostly interested in the case when I is locally complete intersection ideal of height n and P/IP is a free A/I-module of rank n. In the above set-up we are mainly trying to investigate the following questions:

(i) Is $\mu(I) = n$?

(ii) Can one get a surjective lift $\phi : A^n \twoheadrightarrow I$ of α ?

(iii) Suppose that P has a unimodular element. Then is I a complete intersection ideal ?

We shall end this section with a result, which is an accumulation of results of various authors. A detailed proof can be found in ([21], Theorem 2.5 and Corollary 2.7) so we opted to skip the proof.

Theorem 2.1.10. *Let R be an affine algebra of dimension $d \geq 2$ over $\overline{\mathbb{F}}_p$. Then,*

1. *Every locally complete intersection ideal of height d is complete intersection.*

2. *Any projective R-module P of rank d with trivial determinant has a unimodular element.*

2.2 An improvement of a result by Mandal and Murthy

The purpose of this section is to give a detailed proof of a slightly an improved version of a result by S. Mandal and M. P. Murthy ([42], Theorem 3.2). The next lemma allow us to reduce the proof of Theorem 2.2.5 for the reduced rings only.

Lemma 2.2.1. *Let A be a Noetherian ring, P be a projective A-module, $I \subset A$ and $K \subset I^2$ be two ideals. Moreover assume that $\tilde{\phi} : P \twoheadrightarrow I/K$ is a surjective map. Suppose that there exists a surjective map $\phi : P \twoheadrightarrow I/I \cap \mathfrak{n}$, which satisfies $\phi \otimes A/K \cap \mathfrak{n} = \tilde{\phi} \otimes A/\mathfrak{n}$, where $\mathfrak{n}$ is the nil-radical of A. Then there exists a surjective map $\Phi : P \twoheadrightarrow I$, such that $\Phi \otimes A/K = \tilde{\phi}$.*

Proof Let 'bar' denote going modulo $\mathfrak{n}$. Since $\overline{I}/\overline{K}$ can be identified with $I/(K + I \cap \mathfrak{n})$ and by the hypothesis we have $\phi \otimes \overline{A}/\overline{K} = \tilde{\phi} \otimes \overline{A}$, we consider the following fiber product diagram:

$$\begin{tikzcd} I/(K \cap \mathfrak{n}) \arrow[r] \arrow[d] & I/K \arrow[d] \\ \overline{I} = I/(I \cap \mathfrak{n}) \arrow[r] & I/(K + I \cap \mathfrak{n}) = \overline{I}/\overline{K} \end{tikzcd}$$

The maps $\tilde{\phi}$ and ϕ shall patch to give a surjective map $\tilde{\Phi} : P \twoheadrightarrow I/(K \cap \mathfrak{n})$ such that $\tilde{\Phi} \otimes A/K = \tilde{\phi}$. Since P is projective, we get a lift (might not be surjective) $\Phi : P \to I$ of $\tilde{\Phi}$. Then we have $\Phi(P) + K \cap \mathfrak{n} = I$ and $\Phi \otimes A/K = \tilde{\Phi} \otimes A/K = \tilde{\phi}$. By Lemma 2.1.3 there exists $e \in K \cap \mathfrak{n}$, such that $e(1-e) \in \Phi(P)$. Since $e \in \mathfrak{n}$, $1 - e$ is a unit and thus we get $e \in \Phi(P)$, i.e. $I = \Phi(P)$ and this completes the proof. $\square$

Before going to the next theorem we need to recall the following definitions first.

Definition 2.2.2

1. Let A be a commutative Noetherian ring. We say that the projective stable range of A (notation: $psr(A)$) is n if n is the least positive integer such that for any projective $A-$module P of rank n and $(p,a) \in \mathsf{Um}(P \oplus A)$, there exists $q \in P$ such that $p + aq \in \mathsf{Um}(P)$.

2. Let A be a ring. Let P be a projective $A-$module such that either P or P^* has a unimodular element. We choose $\phi \in P^*$ and $p \in P$ such that $\phi(p) = 0$. We define an endomorphism ϕ_p as the composite $\phi_p : P \to A \to P$, where $A \to P$ is the map sending $1 \to p$. Then by a transvection we mean an automorphism of P, of the form $1 + \phi_p$, where either $\phi \in \mathsf{Um}(P^*)$ or $p \in \mathsf{Um}(P)$. By $\mathsf{E}(P)$ we denote the subgroup of $\mathrm{Aut}\,(P)$ generated by all transvections.

The next result can be found in ([46], Theorem 3.7). Here we just restate their result with a slight improvement in the dimension two case. For the proof we just mimic their arguments.

Lemma 2.2.3. *Let R be an affine algebra of dimension $d \geq 2$ over $\overline{\mathbb{F}}_p$ and $\mathfrak{a} \subset R$ be an ideal. Suppose that P is a projective R-module of rank d having a unimodular element and $p \in P$ is such that $\overline{p} \in$ Um$(P/\mathfrak{a}P)$, where 'bar' denotes going modulo $\mathfrak{a}$. Then there exists $q \in$ Um(P) such that $p \equiv q$ (modulo $\mathfrak{a}$).*

Proof Since transvections have lift, it is enough to have $psr(R) \leq d$. We elaborate. Since $\overline{p} \in \mathsf{Um}(P/\mathfrak{a}P)$, there exists $a \in \mathfrak{a}$ such that $(p, a) \in \mathsf{Um}(P \oplus R)$. Now if $psr(P) \leq d$, then there exists $y \in P$ such that $p + ay \in Um(P)$. We can then take $q = p + ay$.

For $d \geq 3$, by ([46], Theorem 3.7) we have $psr(A) \leq d$. So the only remaining case is $d = 2$. But for $d = 2$, the same proof of ([46], theorem 3.7) goes through as well, as in the proof of ([46], theorem 3.7) it was enough to show that any projective module of rank 2 has a unimodular element, which follows from theorem 2.1.10. $\square$

We shall end this section with the following improvement of ([42], Theorem 3.2). We essentially follow their proof, with some small adjustments to suit our requirements. Before that we shall recall the following definition.

Definition 2.2.4 (Order ideal) Let A be a ring and P be a projective module. Let $m \in P$. Then the *order ideal* of m is defined as:

$$\mathcal{O}(m, P) = \mathcal{O}(m) = \{f(m) : f \in \mathsf{Hom}_A(P, A)\}$$

Theorem 2.2.5. *Let R be an affine algebra of dimension $d \geq 2$ over $\overline{\mathbb{F}}_p$ and $I \subset R$ be an ideal. Let P be a projective R-module of rank $\geq d$ such that there is a surjection $\overline{f} : P \twoheadrightarrow I/K$, where $K \subseteq I^2$ is an ideal. Then $\overline{f}$ lifts to a surjection $f : P \twoheadrightarrow I$.*

Proof Note that if $\text{rank}(P) > d$ then the proof follows by mimicking the arguments given in the proof of Lemma 2.1.5. Therefore, with out loss of generality we may assume that $\text{rank}(P) = d$. By Theorem 2.2.1 it is enough to take R to be reduced. Applying Swan's Bertini theorem [60] we can find a lift (not necessarily surjective) $f' : P \longrightarrow I$ of $\overline{f}$, such that $f'(P) = IJ$ where J is a product of distinct smooth maximal ideals of height d and J is co-maximal with K. Using Theorem 2.1.10, we get $p_d \in P$ such that $P \cong P' \oplus Rp_d$ for some R-module P' of rank $d-1$. Since J is a finite product of distinct smooth maximal ideals, by the Chinese Remainder Theorem we get $\mu(J/J^2) = d$. Now note that P/JP is a free $R/J-$module of rank d and $f' \otimes R/J$ is surjective. Thus there are $p_1, \cdots, p_{d-1} \in P'$ so that $f'(p_1), \cdots, f'(p_d)$ form a basis of J/J^2.

Consider the order ideal

$$N = \mathcal{O}(p_1 \wedge \cdots \wedge p_d) = \{\phi(p_1 \wedge \cdots \wedge p_d) \,|\, \phi \in (\wedge^d P)^*\}.$$

As P/JP is (R/J)-free having basis as the images of $p_1, \cdots, p_d$ in P/JP, the image of $p_1 \wedge \cdots \wedge p_d$ in R/J is a unit. Therefore $J + N = R$ and hence $J + NK = R$.

Now we split the proof into two separate cases.

Case 1. Let $d \geq 3$. As $J + NK = R$, there exists $h \in J$ such that $1 - h \in NK$. Note that we can always choose h to be a non-zero divisor. (If h is a zero-divisor, we can find $c \in R$ such that $h + c(1-h)$ is a non-zero divisor. Write $h' = h + c(1-h)$. We note that $1 - h' = 1 - h + hc - c = (1-h)(1-c) \in NK$. We can work with h' instead of h).

Let 'bar' denote reduced modulo $< h >$. By Theorem 2.1.10, $\overline{J}$ is a complete intersection of height $d-1$. Further, $\overline{P}$ is $\overline{R}$-free (as $\overline{N} = \overline{R}$) with basis $\overline{p_1}, \cdots, \overline{p_d}$ and we have $\overline{f'}(\overline{P}) = \overline{J}$.

By [57] (for details see [42], Lemma 3.1) there exists $(\overline{\lambda_1}, \cdots, \overline{\lambda_d}) \in \text{Um}_d(\overline{R})$ such that $\sum \overline{\lambda_i}\, \overline{f'}(\overline{p_i}) = 0$. Let $x = \sum \lambda_i p_i$, then $\overline{x} = \sum \overline{\lambda_i}\, \overline{p_i} \in \text{Um}(\overline{P})$. Using lemma 2.2.3, we can find $p' \in \text{Um}(P)$ such that $\overline{p'} \equiv \overline{x} \equiv \sum \overline{\lambda_i}\, \overline{p_i}$ and hence $\overline{f'}(\overline{p'}) \equiv 0$. Thus $f'(p') = ah$ for some $a \in R$.

Let $\mathfrak{m}$ be any maximal ideal containing J. As $f' \otimes R/J : P/JP \twoheadrightarrow J/J^2$, the image of $f'(p') = ah$ will be part of a basis of $\mathfrak{m}/\mathfrak{m}^2$ and in particular, the image of ah in $\mathfrak{m}/\mathfrak{m}^2$ is nonzero. Since $h \in J \subset \mathfrak{m}$, it follows that $a \notin \mathfrak{m}$. Hence we get $J + (a) = R$ and $a = a(1-h) + ah \in I$. Since $I/f'(P) = I/IJ \cong (I+J)/J = R/J$, it follows that $I = f'(P)+ < a >$.

Define $f : P \longrightarrow I$ by $f|_{P''} := f'|_{P''}$ and $f(p') := a$. Then f is surjective and f lifts $\overline{f}$ as $ah - a \in K$. This completes the proof in this case.

Case 2. Let $d = 2$. Since J is a complete intersection of height 2, we get $h_1, h_2 \in J$ such that $J = (h_1, h_2)$. Let 'bar' denote going modulo NK. Then we have $(\overline{h_1}, \overline{h_2}) \in \mathrm{Um}_2(\overline{R})$. Using a similar argument as in Lemma 2.2.3 we can find $(h_1', h_2') \in \mathrm{Um}_2(R)$ such that $h_i \equiv h_i'$ (modulo NK). Let $\sigma \in SL_2(R)$ be such that $(h_1', h_2')\sigma = (0, 1)$. By replacing (h_1, h_2) by $(h_1, h_2)\sigma$ we may assume that $h_2 \equiv 1$ modulo NK.

Let 'tilde' denote going modulo $< h_2 >$. Then, as before, $\widetilde{P}$ is $\widetilde{R}$-free with basis $\widetilde{p_1}, \widetilde{p_2}$ and $\widetilde{J} = (\widetilde{h_1}) = (\widetilde{f'(p_1)}, \widetilde{f'(p_2)})$. By [57] (or see Lemma 3.1, [42]) choose $(\widetilde{\lambda_1}, \widetilde{\lambda_2}) \in \mathrm{Um}_2(\widetilde{R})$ with $\widetilde{\lambda_1}\widetilde{f'(p_1)} + \widetilde{\lambda_2}\widetilde{f'(p_2)} = 0$. If $x = \lambda_1 p_1 + \lambda_2 p_2$ then $\widetilde{x} \in \mathrm{Um}(\widetilde{P})$. As in Case 1, this can be lifted to a unimodular element p' of P. The rest of the proof is exactly the same as in Case 1. $\square$

The next lemma is known as "moving lemma". We restate it to suit our requirements. The proof is given in ([33], Corollary 2.14). To reestablish this version one just needs to use Theorem 2.2.5 in the appropriate place. For the sake of completeness we sketch the proof.

Lemma 2.2.6. *(Moving Lemma) Let A be a commutative Noetherian ring and $I \subset A$ be an ideal such that $\mu(I/I^2) = n$. Let $I =< a_1, ..., a_n > +I^2$. Then there exists an ideal $J \subset A$, either of height n or $J = A$, with the property that $I \cap J =< b_1, ..., b_n >$, with $a_i - b_i \in I^2$ and $I + J = A$. Moreover if A is an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $n + 1$ and ht$(I) \geq 1$ then J can be chosen to be co-maximal with any ideal of height ≥ 1.*

Proof Since the first part of the proof is exactly the same as of ([33], Corollary 2.14) we begin with the assumption that A is an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $n + 1$. For the sake of completeness of the proof, we will point out the exact place where we use this assumption.

Let K be an ideal of height ≥ 1 and 'bar' denotes going modulo $K \cap I^2$. Let $B = A/K \cap I^2$, then $\dim(B) \leq n$. In the ring B we get $\overline{I} =< \overline{a}_1, ..., \overline{a}_n > +\overline{I}^2$. Using Theorem 2.2.5 there exists $b_i \in A$ such that $\overline{I} =< \overline{b}_1, ..., \overline{b}_n >$, with $a_i - b_i \in I^2 \cap K$. Thus we get $I =< b_1, ..., b_n > +I^2 \cap K$. This is the only place one needs to use the assumption that A is an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $n+1$. To establish the first part of the theorem one can follow the remaining part of the proof.

Using Lemma 2.1.3 there exists $e \in I^2 \cap K$ such that $I =< b_1, ..., b_n, e >$ and $e(1 - e) \in < b_1, ..., b_n >$. By Theorem 2.1.4 replacing b_i with $b_i + \lambda_i e$, for suitably chosen λ_i we may assume that ht$(< b_1, ..., b_n >_e) \geq n$. Define $J =< b_1, ..., b_n, 1 - e >$. Note that if $J = A$ then the proof ends here, hence without loss of generality we may assume that $J \subset A$ is a proper ideal.

Since $e \in J^2 \cap K$ and $e(1-e) \in < b_1, ..., b_n >$ we have $J + I^2 \cap K = A$ and $I \cap J =< b_1, ..., b_n >$ respectively. Therefore only remaining is to show that ht$(J) = n$.

Since $1 - e \in J$ and ht$(< b_1, ..., b_n >_e) \geq n$, any prime ideal of A containing J must contain $< b_1, ..., b_n >$ and will not contain e. Hence we get ht$(J) \geq n$. Note that since $e(1-e) \in < b_1, ..., b_n >$ implies that $1 - e \in J^2$, thus we have $J =< b_1, ..., b_n > + J^2$. By the Remark 2.1.6 we get $\mu(J_p) \leq n$ for any $p \in$ Spec(A). Using Generalized Krull's Principal Ideal Theorem we get ht$(I) = n$. This completes the proof. □

2.3 Some miscellaneous results

We begin this section with the following interesting lemma which can be found in ([38] Lemma 1.1, Chapter III)

Lemma 2.3.1. *Let A be a commutative ring, and $I \subset A[T]$ be an ideal containing a monic polynomial. Let $J \subset A$ be an ideal such that $I + J[T] = A[T]$. Then $I \cap A + J = A$*

Proof Let $S = A[T]/I \supset A/A \cap I$, and let $\overline{J}$ be the image of J in $A/A \cap I$. The hypothesis means that $\overline{J}S = S$. Since S is integral over $A/A \cap I$, the "Going Up" Theorem for integral extensions. implies that $\overline{J} = A/A \cap I$, that is $(A \cap I) + J = A$. □

Let us recall the following definitions:

Definition 2.3.2

1. For any commutative ring R with 1, the stable range (denoted by sr(R)) of R is the smallest natural number r, with the property that for any $(u_1, ..., u_{r+1}) \in$ Um$_{r+1}(R)$, there exists $\lambda_i \in R$, $i = 1, ..., r$ such that $(u_1 + \lambda_1 u_{r+1}, ..., u_r + \lambda_r u_{r+1}) \in$ Um$_r(R)$.

2. For any commutative ring R with 1, stable dimension (denoted by $sdim(R)$) of R is defined by $sdim(R) := sr(R) - 1$.

The following result can be found in ([73], Corollary 17.3).

Lemma 2.3.3. *Let $F \to A$ be a finitely generated algebra over a field F which is algebraic over a finite field. Then $sr(A) \leq \max\{2, \dim(A)\}$.*

The next result is an accumulation of results from different authors. For a proof one can see ([21], Corollary 2.4).

Theorem 2.3.4. *([21], Corollary 2.4) Let R be an affine algebra over $\overline{\mathbb{F}}_p$, and let $I \subset R$ be an ideal such that $\dim(R/I) \leq 1$. Then, we have the following assertions:*

(1) *The canonical map $SL_n(R) \twoheadrightarrow SL_n(R/I)$ is surjective for $n \geq 3$.*

(2) *If $\dim(R) = 3$, then the canonical map $SL_2(R) \twoheadrightarrow SL_2(R/I)$ is surjective.*

The following proposition is an easy consequence of the above theorem. One can find a proof in ([21], Theorem 4.1) hence we skip the proof.

Proposition 2.3.5. *Let R be an affine algebra of dimension $d \geq 3$ over $\overline{\mathbb{F}}_p$. Let $I \subset R$ be an ideal of height $d-1$ such that $\mu(I/I^2) = d-1$. Let a surjection $\alpha : (R/I)^{d-1} \twoheadrightarrow I/I^2$ be such that it has a surjective lift $\theta : R^{d-1} \twoheadrightarrow I$. Then the same is true for any $\alpha\sigma : (R/I)^{d-1} \twoheadrightarrow I/I^2$, where $\sigma \in SL_{d-1}(R/I)$.*

The following lemma is an interesting consequence of Theorem 2.3.4. One can find a proof of the same in ([14], Lemma 5.3). For the sake of completeness we give a proof.

Lemma 2.3.6. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d+1 \geq 3$ and $I \subset R$ be an ideal such that $ht(I) = \mu(I/I^2) = d$. Let $\overline{f} \in R/I$ be a unit. Moreover assume that $I =< f_1, ..., f_d > +I^2$ has a lift to a set of generators of I. Let $(g_1, ..., g_d) = (f_1, ..., f_d)\alpha$, where $\alpha \in GL_d(R/I)$, be such that $\det(\alpha) = \overline{f}^2$. Then $I =< g_1, ..., g_d > +I^2$ also has a lift to a set of generators of I.*

Proof Let $I =< h_1, ..., h_d >$ where $h_i - f_i \in I^2$. Note that after an elementary transformation we may always assume ht$(< h_1, ..., h_i >) = i$, for $i = 1, ..., d$. Let $B = R/ < h_3, ..., h_d >$ and 'bar' denote going modulo $< h_3, ..., h_d >$. Then $\dim(B) \leq 3$.

Let $f \in R$ be a lift of $\overline{f}$. Since f is a unit modulo I, we have $g \in R$ such that $fg - 1 \in I$. Note that $(\overline{g}^2, \overline{h}_2, -\overline{h}_1) \in \mathrm{Um}_3(B)$. By a result Swan-Towber [68] the unimodular row $(\overline{g}^2, \overline{h}_2, -\overline{h}_1)$ is completable to an invertible matrix in $\mathrm{SL}_3(B)$. Using ([14], 5.2) we get $\tau \in M_2(B)$ such that $(\overline{h}_1, \overline{h}_2)\tau = (\overline{h}'_1, \overline{h}'_2)$, where $\overline{I} =< \overline{h}'_1, \overline{h}'_2 >$ and $\det(\tau) - f^2 \in I$.

Thus in the ring R we get, $I =< h'_1, h'_2, h_3, ..., h_d >$. Define $\theta = \tau \perp I_{d-2} \in \mathrm{GL}_d(R/I)$. Then note that $(h_1, h_2, h_3, ..., h_d)\theta = (h'_1, h'_2, h_3, ..., h_d)$ and $\det(\theta) - f^2 \in I$. Since $\det(\theta) - \det(\alpha) \in I$, there exists $\epsilon' \in \mathrm{SL}_d(R/I)$ such that $\theta\epsilon' = \alpha$. Since $\dim(R/I) = 1$, by ([20], 2.3) the natural map $\mathrm{SL}_d(R) \twoheadrightarrow \mathrm{SL}_d(R/I)$ is surjective. Therefore we can lift ϵ' and get $\epsilon \in \mathrm{SL}_d(R)$ such that they are equal modulo I. Let $(G_1, ..., G_d) = (h'_1, h'_2, h_3, ..., h_d)\epsilon$. Then note that $I =< G_1, ..., G_d >$. It only remains to show $G_i - g_i \in I^2$.

Consider any $d-$tuple $[(a_1, ..., a_d)]$ as a map $(R/I)^d \to I/I^2$ sending $e_i \to \overline{a}_i$. Then we have $[(G_1, ..., G_d)] = [(h'_1, h'_2, h_3, ..., h_d)\epsilon] = [(h_1, h_2, h_3, ..., h_d)\theta\epsilon] = [(h_1, ..., h_d)\theta\epsilon'] = [(h_1, ..., h_d)\alpha] = [(f_1, ..., f_d)\alpha] = [(g_1, ..., g_d)]$. This completes the proof. $\square$

The following lemma can be found in [14]. This lemma is used in Chapter 3. We shall give a detailed proof.

Lemma 2.3.7. *Let A be a Noetherian ring and $J \subset A$ be an ideal of height n. Let $f(\neq 0) \in A$ such that J_f is a proper ideal of A_f. Assume that $J_f = (a_1, ..., a_n)$, where $a_i \in J$. Then, there exists $\sigma \in SL_n(A_f)$ such that $(a_1, ..., a_n)\sigma = (b_1, ..., b_n)$, where $b_i \in J \subset A$ and $ht(< b_1, ..., b_n > A) = n$.*

Proof Let $\mathfrak{I} = \{\sigma \in \mathrm{SL}_n(A_f) : (a_1, ..., a_n)\sigma = (b_1, ..., b_n), b_i \in J \subset A\}$. Then note that $I_n \in \mathfrak{I}$. For any $\sigma \in \mathfrak{I}$, we define $N(\sigma) = \mathrm{ht}(< b_1, ..., b_n > A)$, where $(a_1, ..., a_n)\sigma = (b_1, ..., b_n)$. Let $\sigma \in \mathfrak{I}$. If $N(\sigma) = n$, then we are done. So, let us assume that $N(\sigma) < n$. It is enough to produce another $\sigma' \in \mathfrak{I}$ such that $N(\sigma) < N(\sigma')$. In the remaining part of the proof we will prove this in the following steps:

Step-1 $N(\sigma) = \mathrm{ht}(< b_1, ..., b_{n-1} >)$.

Proof Using Theorem 2.1.4 we get $c_i = b_i + \lambda_i b_n$, where $\lambda_i \in A$, for $i = 1, ..., n-1$ such that $\mathrm{ht}(< c_1, ..., c_{n-1} >_{b_n}) \geq n-1$. Note that there exists $\epsilon \in E_n(A)$ such that $(a_1, ..., a_n)\sigma\epsilon = (c_1, ..., c_{n-1}, b_n)$. Thus $\sigma\epsilon \in \mathfrak{I}$. Also note that since $\epsilon \in E_n(A)$ we have $N(\sigma) = N(\sigma\epsilon)$. Thus replacing σ with $\sigma\epsilon$ we may assume that $\mathrm{ht}(< b_1, ..., b_{n-1} >_{b_n}) \geq n-1$. Let p be a minimal prime ideal of A containing $< b_1, ..., b_{n-1} >$ such that $\mathrm{ht}(p) = \mathrm{ht}(< b_1, ..., b_{n-1} >)$.

If $b_n \notin p$ then $\mathrm{ht}(p) \geq n-1$. And by our assumption $\mathrm{ht}(p) = \mathrm{ht}(< b_1, ..., b_{n-1} >) \leq \mathrm{ht}(< b_1, ..., b_n >) = N(\sigma) \leq n-1$. Thus in this case we have

$$n - 1 = \mathrm{ht}(p) = \mathrm{ht}(< b_1, ..., b_{n-1} >) \leq N(\sigma) \leq n - 1.$$

That is $N(\sigma) = \mathrm{ht} < b_1, ..., b_{n-1} >$.

If $b_n \in p$, then note that $N(\sigma) = \mathrm{ht}(< b_1, ..., b_n >) = \mathrm{ht}(< b_1, ..., b_{n-1} >)$. Hence our claim is achieved.

Step-2 For any minimal prime ideal p containing $< b_1, ..., b_{n-1} >$, if $b_n \in p$, then p must contain f.

Proof First of all note that such a minimal prime ideal p always exists. If for all minimal prime ideal $p \supset < b_1, ..., b_{n-1} >$ misses b_n then note that $N(\sigma) = \mathrm{ht}(< b_1, ..., b_n >) > \mathrm{ht}(< b_1, ..., b_{n-1} >)$ which is not possible as shown in Step-1.

Proof by contradiction. Suppose p be as mentioned above and $f \notin p$. Since $f \notin p$, we have $N(\sigma) = \text{ht}(< b_1, ..., b_{n-1} >) \leq \text{ht}(p) = \text{ht}(p_f)$. Now note that since p is minimal containing $< b_1, ..., b_{n-1} >$ and $f \notin p$ implies that p_f is minimal over $< b_1, ..., b_{n-1} >_f$ and thus we get $\text{ht}(p) = \text{ht}(p_f) \leq n-1$. But also note that $< b_1, ..., b_n > \subset p$ and $f \notin p$ implies that $n = \text{ht}(< b_1, ..., b_n >_f) \leq \text{ht}(p_f)$ which is a contradiction. Therefore, $f \in p$.

Step-3 There exists $\sigma_1 \in \mathfrak{I}$ such that $(a_1, ..., a_n)\sigma_1 = (b_1, ..., b_{n-1}, b_n + x^r)$, for some suitably chosen $x \in A$.

Proof Since $I_f =< b_1, ..., b_n >_f$ is a proper ideal of A_f so is $< b_1, ..., b_{n-1} >_f$. Using Step-2 we can assure there exists a minimal prime ideal q containing $< b_1, ..., b_{n-1} >$ which do not contain b_n . Thus $\bigcap_q q \neq \phi$, where the intersection runs over all minimal prime ideals of $< b_1, ..., b_{n-1} >$ which do not contain b_n. Let K_2 be the set consisting of all minimal prime ideals of $< b_1, ..., b_{n-1} >$ which do not contains b_n and K_1 is the complement of K_2 inside the set of all minimal prime ideals of $< b_1, ..., b_{n-1} >$. Also note that using prime avoidance lemma we have $\bigcap_{p \in K_2} p \not\subset \bigcup_{p \in K_1} p$. We choose $x \in \bigcap_{p \in K_2} p - \bigcup_{p \in K_1} p$. Note that $xf \in \sqrt{< b_1, ..., b_{n-1} >}$. Let $r \in \mathbb{N}$ be such that $(xf)^r \in < b_1, ..., b_{n-1} >$. There exists $\alpha \in E_n(A_f)$ such that $(b_1, ..., b_n)\alpha = (b_1, ..., b_{n-1}, b_n + (xf)^r)$. Let $\alpha_1 = \text{diag}\,(1, ..., 1, f^r) \in M_n(A)$. Then note that $\alpha_1 \in \text{GL}_n(A_f)$. Let $\theta = \alpha_1 \alpha_f \alpha_1^{-1} \in \text{SL}_n(A_f)$ and $\sigma_1 = \sigma\theta$. Then note that $\sigma_1 \in \mathfrak{I}$. Now we observe:

$(a_1, ..., a_n)\sigma_1 = (b_1, ..., b_n)\theta = (b_1, ..., b_n)\alpha_1\alpha_f\alpha_1^{-1} = (b_1, ..., b_{n-1}, b_n f^r)\alpha_f\alpha_1^{-1}$
$= (b_1, ..., b_{n-1}, b_n f^r + (xf)^r)\alpha_1^{-1} = (b_1, ..., b_{n-1}, b_n + x^r)$.

Step-4 $N(\sigma_1) > N(\sigma)$.

Proof It is enough to show that no minimal prime ideal of $< b_1, ..., b_{n-1} >$ contains $b_n + x^r$. If so, then $N(\sigma) = \text{ht}(< b_1, ..., b_{n-1} >) < \text{ht}(< b_1, ..., b_{n-1}, b_n + x^r >) = N(\sigma_1)$ and we will be done.

Suppose $p \supset < b_1, ..., b_{n-1} >$ be a minimal prime ideal. If $p \in K_1$ then since $b_n \in p$ and $x \notin p$ implies that $b_n + x^r \notin p$. And if $p \in K_2$ then since $b_n \notin p$ and $x \in p$ implies that $b_n + x^r \notin p$. This completes the proof. □

Le A be a ring. Recall that the ring $A(T)$ is obtained from $A[T]$ by inverting all monic polynomials. The next result is due to G. Horrocks [30].

Theorem 2.3.8. *Let A be a local ring and P be a projective $A[T]$-module. Assume that the projective $A(T)$-module $P \otimes A(T)$ is free. Then P is a free $A[T]$-module.*

The following is a global version of the previous theorem. It was proved by D. Quillen [52]. This is known as Affine Horrocks' Theorem.

Theorem 2.3.9. *[52] Let A be a commutative Noetherian ring and P be a projective $A[T]$-module. Assume that the projective $A(T)$-module $P \otimes A(T)$ is free. Then P is a free $A[T]$-module.*

The next theorem is a partial answer to an open question asked by M. P. Murthy [47]. One can find a proof in ([20], Theorem 3.2).

Theorem 2.3.10. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ and $n \geq 2$ be an integer. Let $I \subset R[T]$ be an ideal containing a monic polynomial such that $\mu(I/I^2) = n \geq dim(R[T]/I) + 1$. Then I is generated by n elements. Moreover, any set of n generators of I/I^2 can be lifted to a set of n generators of I in the following cases:*

1. *$n = 2 = \mu(I/I^2) = ht(I) = \dim(R)$;*

2. *$n \geq 3$.*

The following lemma is known as Quillen's Splitting Lemma [52].

Lemma 2.3.11. *Let A be a ring and $s, t \in A$ be such that $As + At = A$. Let $\sigma(T) \in GL_n(A_{st}[T])$ be such that $\sigma(0) = Id$. Then $\sigma(T) = (\psi_2(T))t(\psi_1(T))s$, where $\psi_1(T) \in GL_n(A_t[T])$ such that $\psi_1(0) = Id$ and $\psi_1(T) = Id$ modulo $< s >$ and $\psi_2(T) \in GL_n(A_s[T])$ such that $\psi_2(0) = Id$ and $\psi_2(T) = Id$ modulo $< t >$.*

Chapter 3

On a question of Nori

3.1 Main theorem

In this section we focus on a question asked by M. V. Nori (Theorem 3.1.2) and its subsequent developments. We improve the bound imposed by S. Mandal [41], for non extended ideals of affine algebras over $\overline{\mathbb{F}}_p$, which contains a monic polynomial. Before going to our main theorem we shall state the following lemmas. The proof of the following lemma is standard so we choose to sketch the proof for the sake of completeness.

Lemma 3.1.1. *Let A be an commutative Noetherian ring and $I \subset A[T]$ be any ideal containing a monic polynomial and $J = I \cap A$. Suppose that P is a projective A-module of rank $n \geq 2$ and $\overline{\phi} : P[T] \twoheadrightarrow I/(I^2T)$ is a surjection. Moreover assume that there exists $j \in J^2$ and a surjective map $\phi'(T) : P_{1+j}[T] \twoheadrightarrow I_{1+j}$, which lifts $\overline{\phi} \otimes A_{1+j}[T]$. Then there exists a surjective map $\phi(T) : P[T] \twoheadrightarrow I$, which lifts $\overline{\phi}$.*

Proof Choose any lift (may not be surjective) $\alpha(T) : P[T] \to I$ of $\overline{\phi}$. As $\alpha(T)$ lifts $\overline{\phi}$ we have a surjection $\alpha(0) : P \twoheadrightarrow I(0)$. Thus over the ring $A_{j(1+j)}[T]$ we have

$$0 \longrightarrow (K_1)_j \longrightarrow P_{j(1+j)}[T] \xrightarrow{\phi'(T)_j} I_{j(1+j)} \longrightarrow 0$$

$$0 \longrightarrow (K_2[T])_{j(1+j)} \longrightarrow P_{j(1+j)}[T] \xrightarrow{\alpha(0)\otimes A_{j(1+j)}[T]} I_{j(1+j)} \longrightarrow 0$$

where $K_1 = \ker(\phi'(T))$ and $K_2 = \ker(\alpha(0))$. Now note that going modulo T we have, $(\phi'(0))_j = (\alpha(0))_{j(1+j)}$, as they both match with any lift of $\overline{\phi}_{j(1+j)}$ modulo $(I^2TA_{j(1+j)}[T])$. Since I_{1+j} contains a monic polynomial, by Theorem 2.3.8 K_1 is locally extended from A_{j+1}

and by Theorem 2.3.9 K_1 is globally extended. And K_2 being extended from A_j follows from the fact the map itself is extended. Hence using ([51], Lemma 2) we can find an automorphism τ of $P_{j(1+j)}[T]$ such that $\tau(0) = Id$ and $(\alpha(0) \otimes A_{j(1+j)}[T])\tau = \phi'(T)_j$. Then applying Lemma 2.3.11 we get $\tau(T) = (\tau_1(T))_{1+j}(\tau_2(T))_j$, where $\tau_1(T) \in \mathrm{Aut}\,(P_j[T])$ and $\tau_2(T) \in \mathrm{Aut}\,(P_{1+j}[T])$. Then a standard patching argument completes the proof. $\square$

Theorem 3.1.2. *Let A be an affine algebra over $\overline{\mathbb{F}}_p$ and $I \subset A[T]$ be an ideal containing a monic polynomial. Assume that $I =< f_1, ..., f_n > +(I^2T)$, where $n \geq \max\{(dim\frac{A[T]}{I}+1), 2\}$. Then there exists $F_i \in I$, $i = 1, ..., n$, such that, $I =< F_1, ..., F_n >$, with $F_i - f_i \in (I^2T)$, for all $i = 1, ..., n$.*

Proof We shall divide the proof into the following two cases:

Case-1($n \geq 3$) Let $f \in I$ be a monic polynomial. Without loss of generality we may assume that f_1 to be monic, by replacing f_1 with $f_1+T^kf^2$, for some suitably chosen $k > 0$. By Lemma 2.1.3 there exists $e \in I^2T$, such that $I =< f_1, ..., f_n, e >$, with $e(1-e) \in< f_1, ..., f_n >$.

Let $J = I \cap A$. By Lemma 3.1.1 it is enough to find $j \in J^2$ and $F_i \in I_{1+j}$ such that $I_{1+j} =< F_1, ..., F_n >$, with $f_i - F_i \in (I^2T)_{1+j}$.

Let $B = \frac{A[T]}{<J^2[T],f_1>}$. Since f_1 is monic, $\dim(B) = \dim(\frac{A}{J^2}) = \dim(\frac{A}{J}) = \dim(\frac{A[T]}{I}) \leq n-1$. Let 'bar' denote modulo $< J^2[T], f_1 >$.

In the ring B, we have $\overline{I} =< \overline{f}_2, ..., \overline{f}_n > +\overline{I^2T}$. By Theorem 2.2.5, we get $\overline{h}_i \in \overline{I}$ such that $\overline{I} =< \overline{h}_2, ..., \overline{h}_n >$, with $\overline{f}_i - \overline{h}_i \in \overline{I^2T}$, for all $i = 2, ..., n$. Hence we get, $I =< f_1, h_2, ..., h_n > +J^2[T]$, where, $f_i - h_i \in< I^2T, J^2[T], f_1 >$ for all $i = 2, ..., n$. Note that by an elementary transformation we may further assume (we are not changing the notations h_i's here) that $h_i - f_i \in I^2T + J^2[T]$, for all $i = 2, ..., n$. Define $F_i(T) = h_i(T) - h_i(0) + f_i(0)$. Then $F_i - f_i = (h_i(T) - f_i(T)) - (h_i(0) - f_i(0)) \in I^2T$ and $F_i \equiv h_i$ modulo $J^2[T]$, for all $i = 2, ..., n$. Thus we get $I =< f_1, F_2, ..., F_n > +J^2[T]$, with $F_i - f_i \in I^2T$, for all $i = 2, ..., n$.

Again applying Lemma 2.1.3 we can find $s \in J^2A[T]$ with $s(1-s) \in< f_1, F_2, ..., F_n >$. Let $I' =< f_1, F_2, ..., F_n, 1-s >$. Then we have $I \cap I' =< f_1, F_2, ..., F_n >$, and $I' + J^2[T] = A[T]$. Since I' contains a monic polynomial (namely f_1), then by Lemma 2.3.1 we can find $j \in J^2$, such that, $1 + j \in I' \cap A$. We get $I_{1+j} =< f_1, F_2, ..., F_n >_{1+j}$ with $f_i - F_i \in (I^2T)_{1+j}$.

Case-2($n = 2$) By [41] we may assume that $\dim(A[T]/I) + 1 = 2$. Let $J = I \cap A$. By Lemma 3.1.1 it is enough to find $j \in J^2$ and $h_i \in I_{1+j}$ such that $I_{1+j} =< h_1, h_2 >$, with $f_i - h_i \in (I^2T)_{1+j}$.

Since I contains a monic polynomial, we have $\dim(A[T]/I) = \dim(A/J) = \dim(A/J^2) = 1$. Let $C = \frac{A}{J^2}$, then in the ring $C[T]$ we have

$$\overline{I} =< \overline{f}_1, \overline{f}_2 > +\overline{I^2T}.$$

Using Theorem 2.2.5 we can find $g_i \in I$, such that $I =< g_1, g_2 > +J^2[T]$, where $g_i - f_i \in I^2T + J^2[T]$. Let $h_i(T) = g_i(T) - g_i(0) + f_i(0)$. Then $h_i - g_i \in J^2[T]$ implies that $I =< h_1, h_2 > +J^2[T]$ and $h_i - f_i \in I^2T$. Now since I contains a monic polynomial and J is a proper ideal, the ideal $< h_1, h_2 >$ contains a monic. By Lemma 2.1.3 there exists $s \in J^2A[T]$ with $s(1-s) \in < h_1, h_2 >$. Let $I' =< h_1, h_2, 1-s >$. Then I' contains a monic polynomial with $I \cap I' =< h_1, h_2 >$ and $I' + J^2[T] = A[T]$. Then by Lemma 2.3.1, there exists $j \in J^2$, such that $1 + j \in I' \cap A$. Thus we get $I_{1+j} =< h_1, h_2 >_{1+j} [T]$, with $f_i - h_i \in (I^2T)_{1+j}$ and this completes the proof. $\square$

3.2 A projective version of the main theorem

Here we shall prove a projective version of the above theorem.

Theorem 3.2.1. *Let A be an affine algebra over $\overline{\mathbb{F}}_p$ and $I \subset A[T]$ be any proper ideal containing a monic polynomial. Suppose that P is a projective A-module of rank n, where $n \geq \max\{(\dim A[T]/I + 1), 2\}$. Then any surjective map $\overline{\phi} : P[T] \twoheadrightarrow I/I^2T$ lifts to a surjective map $\phi(T) : P[T] \twoheadrightarrow I$.*

Proof Let $J = I \cap A$. Since I contains a monic polynomial $\dim(A[T]/I) = \dim(A/J) \leq n-1$. Since rank$(P/JP) = n > \dim(A/J)$ then by a result of J. P. Serre [59] P/JP has a free direct summand of rank one. Then by Nakayam's Lemma we can find $s \in J$ such that $P_{1+s} \cong Q \oplus B$, where $B = A_{1+s}$ is an affine algebra over $\overline{\mathbb{F}}_p$ and Q is a projective B-module of rank $n-1$. Also note that $B[T]/IB[T] \cong (A[T]/I)_{1+s}$ gives us the fact $\dim(B[T]/IB[T]) \leq \dim((A[T]/I))$. Let $\alpha(T) : P[T] \to I$ be any lift of $\overline{\phi}$, then we have a surjective map $\alpha(T) \otimes B[T]/ITB[T] = \overline{\phi} \otimes B[T] : (Q \oplus B)[T] \twoheadrightarrow IB[T]/I^2TB[T]$. Let $\delta(T)$ be any lift of $\overline{\phi} \otimes B[T]$ and $f_0 \in I$ be a monic polynomial. Then replacing $\delta(T)(0,1)$ by $\delta(T)(0,1) + T^k f_0^2$, for some suitably chosen $k > 1$, we may assume that $\delta(T)(0,1) = f$ is a monic polynomial in $IB[T]$.

Case-1($n \geq 3$) Define $C = B[T]/(J^2B[T], f)$. Then C is an affine algebra over $\overline{\mathbb{F}}_p$ and since (J^2B, f) contains a monic polynomial, namely, f, we have $\dim(C) = \dim(B/J^2B) = \dim(B/JB) = \dim(A/J) = \dim(A[T]/I) \leq n-1$. $Q[T] \otimes C$ is a projective C-module of rank $n-1$. Also note that $(\delta(T)\otimes C)|_{Q[T]\otimes C} = (\overline{\phi}\otimes C)|_{Q[T]\otimes C} : Q[T]\otimes C \twoheadrightarrow IC/I^2TC$ is a surjective map, with rank$(Q[T]\otimes C) = n-1 \geq \dim(C)$. Then by Lemma 2.2.5 there exists a surjective map $\overline{\psi} : Q[T]\otimes C \twoheadrightarrow IC$, which lifts $\overline{\phi}\otimes C$. Let $\psi : Q[T]\otimes B[T] \to IB[T]$ be a lift of $\overline{\psi}$. In the ring $B[T]$ we have $Im(\psi) + J^2B[T] + (f) = IB[T]$. Then by Lemma 2.1.3 there exists $e \in J^2B[T]$ with $e(1-e) \in (Im(\psi), f)$, such that $(Im(\psi), f, e) = I$. Define $I' = (Im(\psi), f, 1-e)$, then $IB[T]\cap I' = (Im(\psi), f)$, $I' + J^2B[T] = B[T]$, and I' contains a monic polynomial. Using Lemma 2.3.1 we can find $t \in J^2B$, such that $1+t \in I'\cap B$. Thus we get $IB_{1+t}[T] = (Im(\psi), f)B_{1+t}[T]$. Now note that since $t \in J^2B$, $t = \frac{b}{(1+s)^{k'}}$, for some $k' \geq 0$. Then by further localizing at $(1+s)^{k'}$ one can show that $(B_{1+t})_{(1+s)^{k'}} = [(A_{1+s})_{1+t}]_{(1+s)^{k'}} \cong A_{1+j}$, for some $j \in J$. Define $\omega'(T) : P_{1+j}[T] \to I_{1+j}$, by $\omega'(T)|_{Q_{1+j}[T]} = \psi_{(1+s)^k}$ and $\omega'(T)(0,1) = f$. Then $\omega'(T)$ is a surjective map. Also note that $\omega'(T) = (\delta(T))_{1+j}$ modulo $(J^2B_{1+j}[T], f, I^2TB_{1+j}[T])$. Since $f \in Im(\omega'_{1+s})$ is a monic polynomial, we can find a transvection ζ of $(Q_{1+j}[T]\oplus B_{1+j}[T])$ such that $\omega'(T)|_{Q_{1+j}[T]}\zeta = (\delta(T))_{1+j}$ modulo $(J^2B_{1+j}[T], f, I^2TB_{1+j}[T])$. So we can replace $\omega'(T)$ by $\omega'(T)\zeta$ (without changing the notations) and may assume that $\omega'(T)\zeta = (\delta(T))_{1+j} = (\alpha(T))_{1+j}$ modulo $(J^2B_{1+j}[T], I^2TB_{1+j}[T])$. Define $\omega : P_{1+j}[T] \to I_{1+j}$, by $\omega(T) = \omega'(T) - (\omega'(0) - \alpha_{1+j}(0))$. Then we have the following:

(1) $\omega(T) = \omega'(T)$ modulo J^2B.

(2) $\omega(0) = \alpha_{1+j}(0)$.

We get $\omega(T) = \alpha_{1+j}(T) = \overline{\phi}_{1=+j}$ modulo $I^2TB[T]$.

Again we have, $\alpha(0) : P \to I(0)$ a surjective map. Therefore, $(\gamma(T) =)\alpha(0)\otimes A_j[T] : P_j[T] \to I(0)A_j[T](= A_j[T])$ is also a surjective map and $\gamma(T) = \overline{\phi} \mod (I^2TA_j[T])$.

Thus in the ring $A_{j(1+j)}[T]$ we have two surjective maps $(\omega(T))_j : P_{j(1+j)}[T] \twoheadrightarrow I_{j(1+j)}$ and $(\gamma(T))_{1+j} : P_{j(1+j)}[T] \to I_{j(1+j)}$. Let $K_1 = \text{Ker}(\omega(T))$ and $K_2 = \text{Ker}(\gamma(T))$. Then we have the following two exact sequences

$$0 \longrightarrow (K_1)_j \longrightarrow P_{j(1+j)}[T] \xrightarrow{\omega_j} I_{j(1+j)} \longrightarrow 0$$

$$0 \longrightarrow (K_2)_{1+j} \longrightarrow P_{j(1+j)}[T] \xrightarrow{\gamma_{1+j}} I_{j(1+j)} \longrightarrow 0.$$

Note that going modulo T we have, $(\omega(0))_j = (\gamma(0))_{1+j}$, as they both matches with any lift of $\overline{\phi}_{j(1+j)}$ modulo $(I^2TA_{j(1+j)}[T])$. Since I_{1+j} contains a monic polynomial then by Theorem 2.3.8 K_1 is locally extended from A_{j+1} and by Theorem 2.3.9 K_1 is globally extended. Also K_2 is extended from A_j follows from the fact the map γ itself is extended. Using ([51], Lemma 2) we can find an automorphism τ of $P_{j(1+j)}[T]$ such that $\tau(0) = Id$ and $\gamma_{1+j}\tau = \omega_j$. Then applying Lemma 2.3.11 we get $\tau(T) = (\tau_1(T))_{1+j}(\tau_2(T))_j$, where $\tau_1(T) \in \text{Aut}\,(P_j[T])$ and $\tau_2(T) \in \text{Aut}\,(P_{1+j}[T])$. Then a standard patching argument completes the proof in this case.

Case-2($n = 2$) By [41] we can assume $\dim(A[T]/I) + 1 = 2$. By Lemma 3.1.1 it is enough to find $j \in J^2$ and a surjection $\omega : P_{1+j}[T] \twoheadrightarrow I_{1+j}$ which lifts $\overline{\phi}(T)_{1+j}$.

Since I contains a monic polynomial, we have $\dim(A[T]/I) = \dim(A/J) = \dim(A/J^2) = 1$. Let $C = \frac{A}{J^2}$, and 'bar' denotes going modulo J^2, then in the ring $C[T]$ we have

$$\overline{\phi}(T) : \overline{P}[T] \twoheadrightarrow \overline{I}/\overline{I^2 T}.$$

Using Theorem 2.2.5 we can find a lift $\overline{\psi}(T) : P[T] \twoheadrightarrow I/J^2[T]$ of $\overline{\phi}(T)$. Let us define $\omega(T) = \overline{\psi}(T) - \overline{\psi}(0) + \overline{\phi}(0) : P[T] \twoheadrightarrow I/J^2[T]$. Thus we get $I = \omega(P[T]) + J^2[T]$. Now since I contains a monic polynomial and J is a proper ideal in A, the ideal $\omega(P[T])$ must contains a monic polynomial. By Lemma 2.1.3 there exists $s \in J^2A[T]$ with $s(1-s) \in \omega(P[T])$. Let $I' =< \omega(P[T]), 1 - s >$. Then I' contains a monic polynomial, $I \cap I' = \omega(P[T])$ and $I' + J^2[T] = A[T]$. Then by Lemma 2.3.1, there exists $j \in J^2$, such that $1 + j \in I' \cap A$. Thus we get $\omega(P[T])_{1+j} : P_{1+j}[T] \twoheadrightarrow I_{1+j}$, is a surjective lift of $\overline{\phi}(T)_{1+j}$ and this completes the proof. □

Corollary 3.2.2. *Let A be an affine algebra over $\overline{\mathbb{F}}_p$ and $I \subset A[T]$ be any proper ideal containing a monic polynomial. Let P be a projective A-module of rank n, where $n \geq \max\{(\dim A[T]/I + 1), 2\}$ and $\lambda : P \twoheadrightarrow I(0)$ be a surjection. Suppose that there exists a surjective map $\overline{\phi} : P[T]/IP[T] \twoheadrightarrow I/I^2$ such that $\overline{\phi(0)} = \lambda(0) \otimes A/I(0)$. Then there exists a surjective map $\phi : P[T] \twoheadrightarrow I$ such that ϕ lifts $\overline{\phi}$ and $\phi(0) = \lambda$.*

Proof Follows from the Remark 3.9 of [12] and using Theorem 3.2.1. □

3.3 A relative version

In [43] S. Mandal and R. Sridharan proved a relative version of Mandal's theorem quoted before. Their result has been crucial to the development of the Euler class theory. We now improve the bound of their result when the base ring is an affine algebras over $\overline{\mathbb{F}}_p$.

Theorem 3.3.1. *Let A be an affine algebra over $\overline{\mathbb{F}}_p$. $I = I_1 \cap I_2 \subset A[T]$ be an ideal such that:*

1. *I_1 contains a monic polynomial.*

2. *$I_1 =< f_1, ..., f_n > + I_1^2$, where $n \geq \max\{(\dim(A[T]/I_1) + 1), 2\}$.*

3. *$I_2 = I_2(0)A[T]$, is extended from A.*

Suppose that there exists $a_i \in I(0)$ with $a_i - f_i(0) \in I_1(0)^2$ for $i = 1, ..., n$, such that

$$I(0) =< a_1, ..., a_n > .$$

Then there exists $h_i \in I$ with $h_i(0) = a_i$, for $i = 1, ..., n$ such that

$$I =< h_1(T), ..., h_n(T) >$$

.

Proof Let $J_1 = I_1 \cap A$. Since I_1 contains a monic polynomial and I_2 is extended from A, by Lemma 2.3.1 we can find $s \in J_1$ and $t \in I_2(0)$ such that $s + t = 1$. Note that in the ring $A_t[T]$, we have $I_1A_t[T] =< f_1, ..., f_n > +I^2$, and $I_1(0)A_t = I(0)A_t =< a_1, ..., a_n > A_t$, with $f_i(0) - a_i \in I_1(0)^2A_t$. Thus by Corollary 3.2.2 there exists $g_i \in I_1A_t[T]$, for $i = 1, ..., n$ such that $I_1A_t[T] =< g_1, ..., g_n >$, with $g_i(0) = a_i$. Now consider the following two exact sequences

$$0 \longrightarrow (K_1)_s \longrightarrow A^n_{st}[T] \xrightarrow{(g_1,...,g_n)} IA_{st}[T](= I_1A_st[T] = A_st[T]) \longrightarrow 0$$

$$0 \longrightarrow (K_2)_{st} \longrightarrow A^n_{st}[T] \xrightarrow{(a_1,...,a_n)\otimes A_{st}[T]} IA_{st}[T](= I(0)_{st}A[T] = A_{st}[T]) \longrightarrow 0.$$

where K_1 is kernel of the map from $A^n_t[T] \to I_1A_t[T]$ induced by $(g_1, ..., g_n)$, which is extended from A_t by Theorem 2.3.9. And K_2 is the kernel of the map from $A^n[T] \to I(0)A[T]$ induced by $(a_1, ..., a_n) \otimes A[T]$, which is also extended from A, as the map itself is extended. Then using ([51], Lemma 2 and Proposition 2, or see [7], Lemma 3.4) we can find $\alpha(T) \in SL_n(A_{st}[T])$ such that $\alpha(0) = I_d$. Thus by Lemma 2.3.11 $\alpha(T) = (\alpha_1(T))_s(\alpha_2(T))_t$, where $\alpha_1(T) \in SL_n(A_s[T])$ and $\alpha_2(T) \in SL_n(A_t[T])$. Then a standard patching argument completes the proof. □

Theorem 3.3.2. *Let A be an affine algebra over $\overline{\mathbb{F}}_p$. $I = I_1 \cap I_2 \subset A[T]$, is an ideal and P is a projective A-module such that,*

(i) *I_1 contains a monic polynomial.*

(ii) *$I_2 = I_2(0)A[T]$ is an extended ideal.*

(iii) $I_1 + I_2 = A[T]$.

(iv) *rank$(P) = n \geq \max\{(\dim(A[T]/I_1) + 1), 2\}$.*

Suppose that there exists surjections $\rho : P \twoheadrightarrow I(0)$ and $\overline{\delta} : P[T]/I_1P[T] \twoheadrightarrow I_1/I_1^2$ such that $\overline{\delta} = \rho \otimes A/I_1(0)$. Then there exists a surjection $\eta : P[T] \twoheadrightarrow I$ such that $\eta(0) = \rho$.

Proof Using Corollary 3.2.2 and following the same argument used as in Theorem 3.3.1 the proof follows. $\square$

3.4 The case of dimension two

In this section we shall show that if $\dim(A) = \text{ht}(I) = \mu(I/I^2) = 2$, then the question asked by M. V. Nori does have an affirmative answer over the base field $\overline{\mathbb{F}}_p$ (Theorem 3.4.6). In other words one can not construct such an example as of ([12], Example 3.15) over $\overline{\mathbb{F}}_p$. We shall prove a local version of the dimension two case. Before that we shall need the following lemma.

Lemma 3.4.1. *Let A be an affine domain of dimension $d \geq 1$ over $\overline{\mathbb{F}}_p$ and $\mathfrak{m}_1, \cdots, \mathfrak{m}_r$ be maximal ideals of A. Let $S = A \setminus (\mathfrak{m}_1 \cup \cdots \cup \mathfrak{m}_r)$ and $I \subset S^{-1}A[T]$ be an ideal of height d. Then the natural map $SL_2(S^{-1}A[T]) \longrightarrow SL_2(S^{-1}A[T]/I)$ is surjective.*

Proof There is an ideal $J \subset A[T]$ such that $S^{-1}J = I$. Let $K = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_r$. There are two possibilities: $J + K[T] = A[T]$ or $J + K[T] \subsetneqq A[T]$. In the first case, we have $\dim(S^{-1}A[T]/I) = 0$ and we are done.

Now we consider the second case when $J + K[T]$ is a proper ideal. In this case, it is easy to see that $\dim(A[T]/J) = \dim(S^{-1}A[T]/I) = 1$. Note that $S^{-1}A[T]/I$ is the direct limit of affine $\overline{\mathbb{F}}_p$-algebras of dimension one. It now follows from Theorem 2.3.4 that $\text{SL}_2(S^{-1}A[T]) \longrightarrow \text{SL}_2(S^{-1}A[T]/I)$ is surjective. $\square$

Theorem 3.4.2. *Let A be an affine domain of dimension 2 over $\overline{\mathbb{F}}_p$ and $\mathfrak{m}_1, \cdots, \mathfrak{m}_r$ be some smooth maximal ideals of A. Let $S = A \setminus (\mathfrak{m}_1 \cup \cdots \cup \mathfrak{m}_r)$ Consider the ring $R = S^{-1}A$. Let $I \subset R[T]$ be an ideal such that: (1) $I + \mathfrak{J}[T] = R[T]$, where $\mathfrak{J}$ is the Jacobson radical of R; (2) $\mu(I/(I^2T)) = \text{ht}(I) = 2$. Let $I = (f_1, f_2) + (I^2T)$ be given. Then there exist $F_1, F_2 \in I$ such that $I = (F_1, F_2)$ and $F_i - f_i \in (I^2T)$ for $i = 1, 2$.*

Proof By Lemma 2.1.3 there exists $e \in (I^2T)$ such that $I = (f_1, f_2, e)$ where $e(1-e) \in (I^2T)$. Then, $I_e = R[T]_e = (1, 0)$ and $I_{1-e} = (f_1, f_2)_{1-e}$. The unimodular row $(f_1, f_2)_{e(1-e)}$ can be completed to a matrix in $\text{SL}_2(R[T]_{e(1-e)})$ and by a standard patching argument we obtain a surjection $P \twoheadrightarrow I$ where P is a projective $R[T]$-module of rank two with trivial determinant. As R is smooth and semilocal, it follows that P is free and therefore, $I = (g_1, g_2)$. There exist is a matrix $\overline{\sigma} \in GL_2(R[T]/I)$ such that $(\overline{f}_1, \overline{f}_2) = (\overline{g}_1, \overline{g}_2)\overline{\sigma}$. Let $\det(\overline{\sigma}) = \overline{u}$ and let $\overline{uv} = \overline{1}$

in $R[T]/I$. The unimodular row $(v, g_2, -g_1) \in \mathrm{Um}_3(R[T])$ is completable (as R is smooth semilocal). Therefore, using ([14], Lemma 5.2) we can find $h_1, h_2 \in I$ such that $I = (h_1, h_2)$ and $(\overline{f}_1, \overline{f}_2) = (\overline{h}_1, \overline{h}_2)\overline{\theta}$ for some $\overline{\theta} \in \mathrm{SL}_2(R[T]/I)$. Applying the lemma above, we can find a lift $\theta \in \mathrm{SL}_2(R[T])$ of $\overline{\theta}$. Let $(h_1, h_2)\theta = (H_1, H_2)$.

From the above paragraph, we have: $I = (H_1, H_2)$, where $H_i - f_i \in I^2$. We still have to lift the generators of $I(0)$, namely, $f_1(0), f_2(0)$. Since $I + \mathfrak{J}R[T] = R[T]$, we have $I(0) = R$ and the rows $(f_1(0), f_2(0))$, $(H_1(0), H_2(0))$ are both unimodular. As R is semilocal there is a matrix $\alpha \in E_2(R)$ such that $(f_1(0), f_2(0)) = (H_1(0), H_2(0))\alpha$. Let $\alpha = \prod E_{ij}(a_{ij})$, $a_{ij} \in R$. As $I(0) = R$, there exists $\lambda_{ij} \in I$ such that $\lambda_{ij}(0) = a_{ij}$. Let $\Delta = \prod E_{ij}(\lambda_{ij}) \in E_2(R[T])$. Taking $(F_1, F_2) = (H_1, H_2)\Delta$ we are done. $\square$

We shall recall some results which will be used to prove the main theorem in this section.

Lemma 3.4.3. *([12], Lemma 3.5) Let A be a regular domain containing a field k, $I \subset A[T]$ an ideal, $J = I \cap A$ and $B = A_{1+J}$. Let P be a finitely generated projective A-module and $\phi : P[T] \twoheadrightarrow I/I^2T$ be a surjective map. Suppose that there exists a surjection $\theta : P_{1+J} \twoheadrightarrow I_{1+J}$ such that θ is a lift of $\phi \otimes B$. Then there exists a surjection $\Phi : P[T] \twoheadrightarrow I$ such that Φ is a lift of ϕ.*

Theorem 3.4.4. *[35] Let R be an affine algebra of dimension one over $\overline{\mathbb{F}}_p$. Then, $SK_1(R)$ is trivial.*

Proposition 3.4.5. *([67], 9.10) Let A be a ring and I be an ideal. Let $\gamma \in Sp_{2t}(A/I)$, $t \geq 1$. If the class of γ is trivial in $K_1Sp(A/I)$ and if $2t \geq sr(A) - 1$, then γ has a lift $\alpha \in Sp_{2t}(A)$.*

Now we are ready to prove our main result in this section.

Theorem 3.4.6. *Let R be an affine domain of dimension two over $\overline{\mathbb{F}}_p$. Let $I \subset R[T]$ be an ideal such that $\mu(I/(I^2T)) = ht(I) = 2$ and $R/(I \cap R)$ is smooth. Let $I = (f_1, f_2) + (I^2T)$ be given. Then there exist $F_1, F_2 \in I$ such that $I = (F_1, F_2)$ and $F_i - f_i \in (I^2T)$ for $i = 1, 2$.*

Proof Let $J = I \cap R$. Let 'tilde' denote reduction modulo (J^2T). We have $\tilde{I} = (\tilde{f}_1, \tilde{f}_2) + (\tilde{I^2T})$. As $\dim(R[T]/(J^2T)) \leq 2$, by Theorem 2.2.5 there exist $g_1, g_2 \in I$ such that $\tilde{I} = (\tilde{g}_1, \tilde{g}_2)$ such that $\tilde{g}_i - \tilde{f}_i \in (\tilde{I^2T})$. Therefore, $I = (g_1, g_2) + (J^2T)$ such that $g_i - f_i \in (I^2T)$. Using Lemma 2.1.3 there exist $e \in J^2T$ such that $I =< g_1, g_2, e >$ and $e(1-e) \in < g_1, g_2 >$. Moreover by Theorem 2.1.4 replacing g_i by $g_i + \lambda_i e$ (and retaining the same notations) we may assume that ht$(< g_1, g_2 >_e) \geq 2$. Let $I' =< g_1, g_2, 1-e >$. Then we have $I' + (J^2T) = R[T]$, ht$(I') \geq 2$ and $I \cap I' = (g_1, g_2)$.

If $I' = R[T]$, then we are done. Therefore, we assume that I' is proper and $\text{ht}(I') = 2$. We have $I' = (g_1, g_2) + I'^2$. Note that $I'(0) = R$. Applying ([12], Remark 3.9) we can lift g_1, g_2 so that $I' = (h_1, h_2) + (I'^2T)$ where $h_i - g_i \in I'^2$ for $i = 1, 2$.

Let $J' = I' \cap R$. Let $B = R_{1+J}$ and $C = B_{1+J'} = R_{1+J+J'}$. Note that since R/J is smooth, the ideal of singular locus of R/J (which is extended from R) is co-maximal with J, hence the ring B is smooth. This implies that C is smooth, being further localization of a smooth ring. It has been proved in ([12], Theorem 3.8, Step-1) that the ring C is semilocal.

We have $I'C[T] = (h_1, h_2) + (I'^2T)$. Since $I'C[T] + (J^2T)C[T] = C[T]$ and J is contained in the Jacobson radical of C, we can apply Theorem 3.4.2 and ensure that h_1, h_2 can be lifted to a set of generators of $I'C[T]$. Now, we can apply Lemma 3.4.3 and obtain:

$$I'B[T] = (k_1, k_2) \text{ such that } k_i - h_i \in (I'^2T)B[T].$$

Note that, in view of Lemma 3.4.3, to prove the theorem it will be enough to show that $IB[T] = (\alpha_1, \alpha_2)$ such that $\alpha_i - g_i \in (I^2T)B[T]$. The remaining part of the proof is dedicated to show this only.

We have $I'B[T] + (J^2T)B[T] = B[T]$. Let us write $D = B[T]/J^2B[T]$ and 'bar' denote modulo $J^2B[T]$. Now, $(\overline{k_1}, \overline{k_2}) \in \text{Um}_2(D)$. As $(\overline{k_1}, \overline{k_2})$ is a unimodular row of length two, there is a matrix $\sigma \in \text{SL}_2(D)$ such that $(\overline{k_1}, \overline{k_2})\sigma = (\overline{1}, \overline{0})$.

Claim: σ can be lifted to a matrix $\tau \in SL_2(B[T])$.

Proof of the claim. Since $B[T] = R_{1+J}[T]$, we observe that B is the direct limit of affine $\overline{\mathbb{F}}_p$-algebras of dimension two and therefore, applying ([73], Corollary 17.3) we obtain: $\text{sr}(B[T]) \leq max\{2, \dim(B[T])\} = 3$.

Let us now consider $SK_1(D)$ and $K_1Sp(D)$. We have

$$D_{\text{red}} = B[T]/\sqrt{J}[T] = B[T]/J[T] = (R/J)[T],$$

since J is reduced. Since R/J is smooth, we have $SK_1((R/J)[T]) = SK_1((R/J)$ and $K_1Sp((R/J)[T]) = K_1Sp(R/J)$. Since $\dim(R/J) = 1$, by Theorem 3.4.4 we have $SK_1(R/J) = 0$ and applying ([73], Lemma 16.2) we further obtain that $K_1Sp(R/J) = SK_1(R/J)$ and hence it is trivial as well.

As for any ring C, we know that $SK_1(C) = SK_1(C_{\text{red}})$, we conclude that $SK_1(D) = 0$. On the other hand, we know that for a ring C, the natural map $K_1Sp(C) \longrightarrow K_1Sp(C_{\text{red}})$ is injective. Therefore, from the above computation we see that $K_1Sp(D) = 0$.

We can now apply Swan's result (Proposition 3.4.5), with $t = 1$. Since $Sp_2(B[T])$ is the same as $\mathsf{SL}_2(B[T])$, we are done.

Let $(k_1, k_2)\tau = (k_1', k_2')$. Then $I' = (k_1', k_2')$ and $k_1' \equiv 1$ modulo $J^2B[T]$ and $k_2' \equiv 0$ modulo $J^2B[T]$. Write $(\beta_1, \beta_2) = (k_1', k_2')\epsilon$, where $\epsilon = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, (here $\beta_1 = k_1'$ and $\beta_2 = k_1' + k_2'$). Then $\beta_i \equiv 1$ modulo $J^2B[T]$ for $i = 1, 2$.

We now write $B[T] = A$ and introduce a new variable X and consider the following ideals in $A[X]$:

$$K' = (\beta_1, X + \beta_2), \ \ K'' = IA[X], \ \ K = K' \cap K''$$

Let us write the ideal $\beta_1 A$ as $\mathfrak{n}$. We have $K_{1+\mathfrak{n}} = K'_{1+\mathfrak{n}} = (\beta_1, X + \beta_2)$. Recall that we have $I \cap I' = (g_1, g_2)$, implying that $(I \cap I')A = (g_1, g_2)A$. Let $(g_1, g_2)\tau = (g_1', g_2')$ and write $(l_1, l_2) = (g_1', g_2')\epsilon$. Then also we have $(I \cap I')A = (l_1, l_2)A$ and all the relations are retained.

Now $(l_1, l_2)A_{1+\mathfrak{n}} = I'A_{1+\mathfrak{n}} = K'_{1+\mathfrak{n}}(X = 0) = K_{1+\mathfrak{n}}(X = 0)$.

We also have $K_{1+\mathfrak{n}}(X = 0) = (\beta_1, \beta_2)$. Then note that $(l_1 - \beta_1, l_2 - \beta_2) = (g_1 - k_1, g_2 - k_2)\tau\epsilon \in I'^2A_{1+\mathfrak{n}} \times I'^2A_{1+\mathfrak{n}}$ as $g_i - k_i = (g_i - h_i) + (h_i - k_i) \in (I'^2B[T])$. Since $K'_{1+\eta}(X = 0) =< \beta_1, \beta_2 >=< l_1, l_2 >$, there exists $\alpha \in \mathsf{GL}_2(A_{1+\mathfrak{n}})$ such that $(\beta_1, \beta_2)\alpha = (l_1, l_2)$. Let $(\beta_1, X + \beta_2)\alpha = (G_1(X), G_2(X))$, then $G_i(0) = l_i$ for $i = 1, 2$.

Recall that $\mathfrak{n} = (\beta_1)$ is comaximal with $J^2B[T]$. We choose some $s \in \mathfrak{n}$ such that $1 + s \in J^2B[T]$ and $K'A_{1+sA} = (G_1(X), G_2(X))$ with $G_i(0) = l_i$ for $i = 1, 2$.

Let $\phi : A_{1+sA}[X]^2 \twoheadrightarrow K_{1+sA}$ be the surjection corresponding to $K_{1+sA} = (G_1(X), G_2(X))$. And we have a surjection $\psi : A_s[X]^2 \twoheadrightarrow K_s$ induced by the Following: $K_s = K''_s = I_s = (l_1, l_2)$.

The surjections $\phi_s : A_{s(1+sA)}[X]^2 \twoheadrightarrow K_{s(1+sA)}$ and $\psi_{1+sA} : A_{s(1+sA)}[X]^2 \twoheadrightarrow K_{s(1+sA)}$ agree when $X = 0$. As both the kernels are free, by a standard patching argument we obtain $K = (H_1(X), H_2(X))$ such that $H_i(0) = l_i$ for $i = 1, 2$.

Now, $I = K(1 - \beta_2) = (H_1(1 - \beta_2), H_2(1 - \beta_2))$. We write $H_i(1 - \beta_2) = \alpha_i$. As the constant term of H_i is l_i and $\beta_2 \equiv 1$ modulo $J^2B[T]$, it follows that $IA = (\alpha_1, \alpha_2)$ with $\alpha_i - l_i$ modulo $J^2B[T]$.

Let us now revert back to the original notations (recall: $A = B[T] = R_{1+J}[T]$). We have thus far been able to establish the following:

$$I_{1+J} = (\alpha_1, \alpha_2) \text{ such that } \alpha_i \equiv l_i \text{ modulo } J^2_{1+J}[T]$$

Also recall that we started with $I = (g_1, g_2) + (J^2T)$ and then applied some automorphisms on (g_1, g_2) to get (l_1, l_2). So we have $I = (l_1, l_2) + (J^2T)$. However α_1, α_2 are lifts modulo $J^2[T]$. We need to find γ_1, γ_2 so that $I_{1+J} = (\gamma_1, \gamma_2)$ with $\gamma_i - l_i \in J^2[T]_{1+J}$ and $\gamma_i(0) = l_i(0)$ for $i = 1, 2$. Once we have done this we can apply inverses of the said automorphisms on (γ_1, γ_2) to solve the problem. The remaining part of the proof is dedicated to find such γ_i.

We have $I(0)_{1+J} = (l_1(0), l_2(0)) = (\alpha_1(0), \alpha_2(0))$ such that $\alpha_i(0) - l_i(0) \in (J^2)_{1+J}$.

Note that $J \subset I(0)$ and $J^2 \subset JI(0)$. Therefore, we can write $\alpha_1(0) - l_1(0) = c\alpha_1(0) + d\alpha_2(0)$, where $c, d \in JB$. Similarly, $\alpha_2(0) - l_2(0) = e\alpha_1(0) + f\alpha_2(0)$, where $e, f \in JB$. Putting it in another way,

$$(\alpha_1(0), \alpha_2(0))\delta = (l_1(0), l_2(0))$$

where $\delta = \begin{pmatrix} 1-c & -e \\ -d & 1-f \end{pmatrix}$. Note that the determinant of the above matrix is 1 modulo J. Since J is contained in the Jacobson radical of R_{1+J}, it is an invertible matrix in R_{1+J}.

Let $(\gamma_1, \gamma_2) = (\alpha_1, \alpha_2)\delta$. Then note that:

1. $I_{1+J} = (\gamma_1, \gamma_2)$, as $\delta \in \mathsf{GL}_2(R_{1+J})$;

2. $\gamma_i - l_i \in I^2_{1+J}$, as $\gamma_1 - l_1 = (\gamma_1 - \alpha_1) - (\alpha_1 - l_1) = -c\alpha_1 - d\alpha_2 + (\alpha_1 - l_1) \in I^2_{1+J}$, and $\gamma_2 - l_2 = (\gamma_2 - \alpha_2) - (\alpha_2 - l_2) = -e\alpha_1 - f\alpha_2 + (\alpha_2 - l_2) \in I^2_{1+J}$;

3. $\gamma_i(0) = l_i(0)$ for $i = 1, 2$.

This completes the proof. □

3.5 Question of Nori: Precise obstruction

As mentioned earlier, in the appendix of a paper by S. Mandal [41], M. V. Nori asked the following question, which is motivated by certain results in topology. For the convenience of understanding, we state the "free" version of the question below.

Question 3.5.1 Let R be a smooth affine domain of dimension d over a field k and $I \subset R[T]$ be an ideal of height n such that $\mu(I/I^2T) = n$, where $2n \geq d + 3$. Assume that $I = (f_1, \cdots, f_n) + (I^2T)$ is given. Then, do there exist $F_i \in I$ $(i = 1, \cdots, n)$, such that $I = (F_1, \cdots, F_n)$ where $F_i - f_i \in (I^2T)$ for $i = 1, \cdots, n$?

If I contains a monic polynomial, then S. Mandal [41] proved that the answer is in the affirmative where he needs the ring R to be just Noetherian. Ideals containing monic polynomials are of a different league and let us leave them out of our discussion. So, from now on we assume that I does not contain monic in the above question. Nori's question has been answered comprehensively. First, Mandal-Varma [44] proved it to be true when R is local. Bhatwadekar-Sridharan [12] gave an affirmative answer when $n = d \geq 3$ and k is infinite perfect. Later, Bhatwadekar-Keshari settled it in the affirmative [10] for $2n \geq d+3$ with the same assumption on k.

On the other hand, Bhatwadekar-Mohan Kumar-Srinivas gave an example in [12, Example 6.4] to show that Nori's question will have a negative answer if R is not smooth (even when R is local). They constructed an example of a normal affine $\mathbb{C}$-domain R of dimension 3 which has an isolated singularity at the origin, and an ideal $I \subset R[T]$ of height 3 such that a given set of generators of $I/(I^2T)$ cannot be lifted to a set of generators of I.

The results and the example stated above had profound impact on the development of the theory in understanding the behaviour of projective modules and local complete intersection ideals in past twenty years. Among recent instances, the Bhatwadekar-Sridharan solution played a crucial role in computing the group of isomorphism classes of oriented stably free R-modules of rank d where R is a smooth affine domain of dimenson d over $\mathbb{R}$ ([23], see also [24]). Further, Asok-Fasel [1] used it successfully to establish the isomorphism between the d-th Euler class group and the d-th Chow-Witt group (also the isomorphism between the weak Euler class group and the Chow group) — thus establishing a long standing conjecture.

In this context, we delve deep into this phenomenon and pose the following rephrased question.

Question 3.5.2 Let R be an affine domain of dimension d over a field k and $I \subset R[T]$ be an ideal of height n such that $\mu(I/I^2T) = n$, where $2n \geq d+3$. Assume that $I = (f_1, \cdots, f_n) + (I^2T)$ is given. Then, what is the precise obstruction for I to have a set of generators $F_1, \cdots, F_n$ such that $F_i - f_i \in (I^2T)$ for $i = 1, \cdots, n$?

Obviously we have left out the case when I contains a monic polynomial. We prove that the obstruction lies in the fact as to whether $I \cap R$ is contained in only smooth maximal ideals or not. More precisely, we prove the following result. We have decided to give the details for the case $n = d$. We shall comment on the other versions in the sequel.

Theorem 3.5.3. *Let R be a an affine domain of dimension d over an infinite perfect field k and $I \subset R[T]$ be an ideal of height n such that $\mu(I/I^2T) = n$, where $2n \geq d+3$. Assume that $I = (f_1, \cdots, f_n) + (I^2T)$ is given. Assume further that $I \cap R$ is contained only in smooth maximal ideals of R. Then, there exist $F_i \in I$ ($i = 1, \cdots, n$), such that $I = (F_1, \cdots, F_n)$ where $F_i - f_i \in (I^2T)$ for $i = 1, \cdots, n$.*

To prove the above result, the most crucial proposition is the following improvement of [12, Theorem 3.8]. Note that we do not assume P to be extended.

Proposition 3.5.4. *Let A be a domain containing a field. Let $I \subset A[T]$ be an ideal such that $J := I \cap A$ is contained only in smooth maximal ideals. Let P be a projective $A[T]$-module such that there is a surjection*

$$\overline{\varphi} : P \twoheadrightarrow I/(I^2T).$$

Assume that there is a surjection

$$\theta : P_{1+J} \twoheadrightarrow I_{1+J},$$

such that θ is a lift of $\overline{\varphi} \otimes A_{1+J}$. Then, there is a surjection $\Phi : P \twoheadrightarrow I$ which lifts $\overline{\varphi}$.

Proof From the map θ, clearing denominators we can find $s_1 \in J$ such that $\theta : P_{1+s_1} \twoheadrightarrow I_{1+s_1}$ is surjective (we are using the same notation θ). We can also find $s_2 \in J$ such that A_{1+s_2} is smooth. We now take $(1+s) := (1+s_1)(1+s_2)$ and consider $\theta : P_{1+s} \twoheadrightarrow I_{1+s}$. Note that, as A_{1+s} is a regular ring containing a field, by a result of Lindel [39], the module P_{1+s} is extended from A_{1+s}.

The map $\overline{\varphi}$ induces a surjection, say, $\overline{\varphi}(0) : P/TP \twoheadrightarrow I(0)$. As $s \in J$, we have $I(0)_s[T] = I_s = A_s[T]$. Therefore, we have $\overline{\varphi}(0)_s : (P/TP)_s \twoheadrightarrow I_s$. Then we have a surjection $\alpha : P_s \twoheadrightarrow I_s$ (composing $\overline{\varphi}(0)_s$ with the canonical map $P_s \twoheadrightarrow (P/TP)_s$).

We now proceed to patch the two maps $\theta : P_{1+s} \twoheadrightarrow I_{1+s}$ and $\alpha : P_s \twoheadrightarrow I_s$. We move to $A_{s(1+s)}[T]$. As $P_{s(1+s)}$ is extended from $A_{s(1+s)}$, we have a projective $A_{s(1+s)}$-module, say, P' such that $P'[T] = P_{s(1+s)}$. We finally have:

$$\theta_s : P'[T] \twoheadrightarrow I_{s(1+s)} \, (= A_{s(1+s)}[T]), \text{ and}$$

$$\alpha_{1+s} : P'[T] \twoheadrightarrow I_{s(1+s)} \, (= A_{s(1+s)}[T]),$$

where θ_s and α_{1+s} are equal modulo (T). Since $A_{s(1+s)}$ is a regular ring containing a field, we also note that the kernels of θ_s and α_{1+s} are both extended from $A_{s(1+s)}$. Therefore, by [14, Lemma 2.9] there is an isomorphism $\sigma : P'[T] \xrightarrow{\sim} P'[T]$ such that $\sigma(0) = id$ and $\alpha_{1+s}\sigma = \theta_s$. We can now patch $\theta : P_{1+s} \twoheadrightarrow I_{1+s}$ and $\alpha : P_s \twoheadrightarrow I_s$ using Plumstead's patching technique (see [51]) to obtain a surjection $\Phi : P \twoheadrightarrow I$. It is then easy to check that Φ lifts $\overline{\varphi}$. $\square$

We now present the following *"projective"* version of Theorem 3.5.3 mentioned above. This is an improvement of the result of Bhatwadekar-Sridharan [12, Theorem 3.8]. The proof is essentially contained in [12]. We just give a sketch and for the details we refer to their paper.

Theorem 3.5.5. *Let R be an affine domain of dimension $d \geq 3$ over an infinite perfect field k and $I \subset R[T]$ be an ideal of height d such that $J := I \cap R$ is contained only in smooth maximal ideals. Let P be a projective $R[T]$-module of rank d such that there is a surjection*

$$\overline{\varphi} : P \twoheadrightarrow I/(I^2T).$$

Then, there is a surjection $\Phi : P \twoheadrightarrow I$ which lifts $\overline{\varphi}$.

Proof Following the proof of [12, Lemma 3.6] we obtain a lift $\varphi \in \text{Hom}_{R[T]}(P, I)$ of $\overline{\varphi}$ such that if $\varphi(P) = I''$, then:

1. $I'' + (J^2T) = I$;
2. $I'' = I \cap I'$, where $I' \subset R[T]$ is an ideal of height $\geq d$;
3. $I' + (J^2T) = R[T]$.

If $I' = R[T]$, then we are done. Therefore, we assume that I' is a proper ideal of height d.

Let $J' = I' \cap R$. Then it is proved in [12, Proof of Theorem 3.8] that $\dim(R/(J + J')) = 0$. As a consequence, $R_{1+J+J'}$ is semilocal. If we write $B = R_{1+J}$, then note that $B_{1+J'B} = R_{1+J+J'}$. Therefore $R_{1+J+J'}$ is smooth (as B is so).

Since B is a smooth k-algebra, there is a projective B-module P' such that $P' \otimes B[T] = P \otimes B[T]$. In simpler notation, we write this as $P'[T]$. From (2) above, we see that $\varphi \otimes R[T]/I'$ induces a surjection from P to I'/I'^2 which in turn induces a surjection $\varphi' : P'[T] \twoheadrightarrow I'B[T]/I'^2B[T]$. From (3) we deduce that $I'(0) = R$ and hence $I'(0)B = B$. As JB is contained in the Jacobson radical of B, it is easy to see that P' has a unimodular element. This implies that there is a surjection $\alpha : P' \twoheadrightarrow I'(0)B = B$.

Therefore, applying [12, Remark 3.9] we can lift φ' to a surjection

$$\overline{\lambda} : P'[T] \twoheadrightarrow I'B[T]/(I'^2T)B[T].$$

As $B_{1+J'B}$ is a smooth semilocal k-algebra, it follows from [12, Theorem 2.13] that $\overline{\lambda}_{1+J'B}$ has a surjective lift from $P'_{1+J'B}[T]$ to $I'B_{1+J'B}[T]$. Applying Proposition 3.5.4 we obtain a surjection

$$\lambda : P'[T] \twoheadrightarrow I'B[T],$$

which lifts $\overline{\lambda}$.

In view of Proposition 3.5.4 above, it is now enough to show that $\overline{\varphi} \otimes B[T] : P \otimes B[T] \twoheadrightarrow IB[T]/(I^2T)B[T]$ can be lifted to a surjection $\theta : P \otimes B[T] \twoheadrightarrow IB[T]$. This is exactly what has been proved in Steps 3 and 4 of [12, Theorem 3.8]. $\square$

Remark 3.5.6 The above theorem shows that the condition that $I \cap R$ is contained only smooth maximal ideals is sufficient to find surjective lifts, as mentioned in the questions above. The necessity of this condition follows from the local version of the example of Bhatwadekar-Mohan Kumar-Srinivas [12, Example 6.4].

Remark 3.5.7 One may wonder whether instead of the smoothness condition on $I \cap R$, we can impose it on I itself. Unfortunately, that would not work. In the the example of Bhatwadekar-Mohan Kumar-Srinivas mentioned above, $R[T]/I$ is smooth and the lifting fails.

We now proceed to show some interesting applications of Proposition 3.5.4 and Theorem 3.5.5. For a commutative Noetherian $\mathbb{Q}$-algebra R of dimension $d \geq 3$, the d-th Euler class group $E^d(R[T])$ was defined in [19]. It was further proved that the canonical map $\phi : E^d(R) \longrightarrow E^d(R[T])$ is injective. The morphism ϕ is an isomorphism if R is smooth but it may not be surjective if R is not smooth (see [19] for the details). In this context, we may ask, precisely which Euler cycles $(I, \omega_I) \in E^d(R[T])$ have a preimage in $E^d(R)$? We answer this question in the following form.

Theorem 3.5.8. *Let R be an affine domain of dimension $d \geq 3$ over a field k of characteristic zero. Let $(I, \omega_I) \in E^d(R[T])$ be such that $I \cap R$ is contained only in smooth maximal ideals. Then (I, ω_I) is in the image of the canonical morphism $\phi : E^d(R) \longrightarrow E^d(R[T])$.*

Proof As k is infinite, applying [12, Lemma 3.3] we can find $\sigma \in k$ such that either $I(\sigma) = R$ or ht$(I(\sigma)) = d$. Changing T by $T - \sigma$, we may assume that either $I(0) = R$ or ht$(I(0)) = d$. If $I(0) = R$, then ω_I can be lifted to a surjection $\alpha : R[T]^d \twoheadrightarrow I/(I^2T)$. Then α lifts to a surjection from $R[T]^d$ to I and consequently, $(I, \omega_I) = 0$. Therefore, we assume that $I(0)$ is proper of height d. Then (I, ω_I) induces $(I(0), \omega_{I(0)}) \in E^d(R)$. If $(I(0), \omega_{I(0)}) = 0$ in $E^d(R)$, then also ω_I can be lifted to a surjection $\alpha : R[T]^d \twoheadrightarrow I/(I^2T)$ and we will be done by Theorem 3.5.5 (taking P to be free). So let $(I(0), \omega_{I(0)}) \neq 0$ in $E^d(R)$.

Using the moving lemma [14, Corollary 2.14] together with Swan's Bertini Theorem [13, Theorem 2.11], we can find a reduced ideal $K \subset R$ of height d which is comaximal with $I \cap R$ and a local orientation ω_K such that $(I(0), \omega_{I(0)}) + (K, \omega_K) = 0$ in $E^d(R)$.

As K is reduced of height d and $\mu(K/K^2) = d$, we observe that K is product of a finite number of distinct smooth maximal ideals of R.

Let $L = I \cap K[T]$. The local orientations ω_I and ω_K will induce ω_L and we have

$$(L, \omega_L) = (I, \omega_I) + (K[T], \omega_{K[T]}) = 0 \text{ in } E^d(R[T]).$$

As $(L(0), \omega_{L(0)}) = (I(0), \omega_{I(0)}) + (K, \omega_K) = 0$, it follows that ω_L can be lifted to a surjection $\lambda : R[T]^d \twoheadrightarrow L/(L^2T)$.

Now $L \cap R = (I \cap R) \cap K$. Since K is reduced and $I \cap R$ is contained only in smooth maximal ideals of R, it follows that $L \cap R$ is contained only in smooth maximal ideals of R. Therefore, by Theorem 3.5.5 λ can be lifted to a surjection $\alpha : R[T]^d \twoheadrightarrow L$. As a consequence, $(L, \omega_L) = 0$ in $E^d(R[T])$, and we have

$$(I, \omega_I) = -(K[T], \omega_{K[T]}) \in \phi(E^d(R)).$$

□

Remark 3.5.9 With notations as above, let $(I, \omega_I) \in E^d(R[T])$ be such that: I is a non-extended ideal of $R[T]$ and I does not contain a monic polynomial. If we further drop the condition that $I \cap R$ is contained only in smooth maximal ideals, then there is an example [18, Remark 3.4] which shows that (I, ω_I) may not be in the image of ϕ.

Using Proposition 3.5.4 and following the arguments given in Theorem 3.5.5, one can easily prove the following results.

Theorem 3.5.10. *Let R be a domain of dimension d containing a field k (no restriction on k). Let $I \subset R[T]$ be an ideal such that ht$(I) = n = \mu(I/I^2T)$, where $2n \geq d+3$. Let $I = (f_1, \cdots, f_n) + (I^2T)$ be given. Assume that there exist $F_1, \cdots, F_n \in IR(T)$ such that $IR(T) = (F_1, \cdots, F_n)$ where $F_i - f_i \in I^2R(T)$. Assume further that $I \cap R$ is contained only in smooth maximal ideals of R. Then there are $g_1, \cdots, g_n \in I$ such that $I = (g_1, \cdots, g_n)$ with $g_i - f_i \in (I^2T)$.*

And also the result mentioned at the beginning of this section:

Theorem 3.5.11. *Let R be a an affine domain of dimension d over an infinite perfect field k and $I \subset R[T]$ be an ideal of height n such that $\mu(I/I^2T) = n$, where $2n \geq d+3$. Assume that $I = (f_1, \cdots, f_n) + (I^2T)$ is given. Assume further that $I \cap R$ is contained only in smooth maximal ideals of R. Then, there exist $F_i \in I$ $(i = 1, \cdots, n)$, such that $I = (F_1, \cdots, F_n)$ where $F_i - f_i \in (I^2T)$ for $i = 1, \cdots, n$.*

Remark 3.5.12 The couple of theorems given above were proved in [10] (Proposition 4.9 and Theorem 4.13) assuming R to be smooth. We remark that [10, Proposition 4.9] is crucially used to prove [10, Theorem 4.13].

Chapter 4

An obstruction group on affine algebras over $\overline{\mathbb{F}}_p$

4.1 Some addition and subtraction principles

Proposition 4.1.1. *(Addition Principle) Let A be an affine algebra over $\overline{\mathbb{F}}_p$ of $\dim(A) = d$. Suppose that $I_1, I_2 \subset A$ be two co-maximal ideals of height n, where $2n \geq d+2$, such that $I_1 =< a_1, ..., a_n >$ and $I_2 =< b_1, ..., b_n >$. Let $I = I_1 \cap I_2$. Then there exists $c_i \in I$ such that $I =< c_1, ..., c_n >$, with $c_i - a_i \in I_1^2$ and $c_i - b_i \in I_2^2$.*

Proof Without loss of generality we may assume that $d > n$, as the case $d < n$ follows from Lemma 2.1.5 and $d = n$ follows from [14]. Note that we can always perform elementary transformations on $(a_1, ..., a_n)$ and $(b_1, ..., b_n)$. Let $B = A/ < b_1, ..., b_n >$ and 'bar' denotes going modulo $< b_1, ..., b_n >$. Note that $(\overline{a}_1, ..., \overline{a}_n) \in \mathrm{Um}_n(B)$. Since $n \geq d - n + 2 \geq \dim(B) + 2$, we shall have $(\overline{a}_1, ..., \overline{a}_n) \sim_E (\overline{1}, ..., \overline{0})$. Adding suitable multiples of a_n to a_i's Theorem 2.1.4 we may further assume ht $< a_1, ..., a_{n-1} >= n-1$. Since $\overline{a}_n = 0$, we may still have $(\overline{a}_1, ..., \overline{a}_n) \sim_E (\overline{1}, ..., \overline{0})$. Thus we get $< a_1, ..., a_{n-1} > + I_2 = A$.

Let $C = A/ < a_1, ..., a_{n-1} >$, and 'tilde' denotes going modulo $< a_1, ..., a_{n-1} >$. Since $< a_1, ..., a_{n-1} > + I_2 = A$, we have $(\widetilde{b}_1, .., \widetilde{b}_n) \in \mathrm{Um}_n(C)$. Also note that $n \geq (d-n+1)+1 \geq \dim(C) + 1$ therefore by Lemma 2.3.3 we get $(\widetilde{b}_1, .., \widetilde{b}_n) \sim_E (\widetilde{1}, ..., \widetilde{0})$. Thus we may further assume that $(\widetilde{b}_1, .., \widetilde{b}_n) = (\widetilde{1}, ..., \widetilde{0})$. Again as before without altering the assumption $\widetilde{b}_n = \widetilde{0}$ we may also assume $ht < b_1, ..., b_{n-1} >= n-1$ and thus $(\widetilde{b}_1, .., \widetilde{b}_n)$. Hence we get

(i) $< a_1, .., a_{n-1} > + < b_1, .., b_{n-1} >= A$,

(ii) ht $< a_1, .., a_{n-1} >= ht < b_1, .., b_{n-1} >= n-1$.

Now define $J_1 =< a_1, ..., a_{n-1}, a_n + T >\subset A[T]$, $J_2 =< b_1, ..., b_{n-1}, b_n + T >\subset A[T]$ and $J = J_1 \cap J_2$ be an ideal of $A[T]$ containing a monic polynomial. Since $J_1 + J_2 = A[T]$, by the Chinese Remainder Theorem $J/J^2 = J_1/J_1^2 \oplus J_2/J_2^2$. Thus we can find $g_i \in J$, for $i = 1, ..., n$ such that $J =< g_1, ..., g_n > +J^2$, with $g_i - a_i \in J_1^2$, $g_i - b_i \in J_2^2$, for $i = 1, ..., n-1$, $g_n - a_n - T \in J_1^2$ and $g_n - b_n - T \in J_2^2$. Also $A[T]/J \cong A[T]/J_1 \oplus A[T]/J_2$, gives us the fact that $\dim(A[T]/J) = \max\{\dim(A[T]/J_1), \dim(A[T]/J_2)\} = \max\{\dim(A/ < a_1, ..., a_{n-1} >), \dim(A/ < b_1, ..., b_{n-1} >)\} = d - n + 1 \leq n - 1$.

Thus by Theorem 2.2.5 we can find $h_i \in J$, $i = 1, ..., n$ be such that $J =< h_1, ..., h_n >$, with $h_i - g_i \in J^2$. Let $h_i(0) = c_i$, for $i = 1, ..., n$, then $I_1 \cap I_2 =< c_1, ..., c_n >$, with $c_i - a_i \in I_1^2$ and $c_i - b_i \in I_2^2$, for $i = 1, ..., n$. □

Proposition 4.1.2. *(Subtraction Principle) Let A be an affine algebra over $\overline{\mathbb{F}}_p$ of $\dim(A) = d$. Suppose that $I_1, I_2 \subset A$ be two comaximal ideals of height n, where $2n \geq d + 2$, such that $I_1 =< a_1, ..., a_n >$ and $I = I_1 \cap I_2 =< c_1, ..., c_n >$ with $c_i - a_i \in I_1^2$, for all $i = 1..., n$. Then there exists $b_i \in I$ such that $I_1 =< b_1, ..., b_n >$, with $c_i - b_i \in I_2^2$.*

Proof Note that as before we may always assume $d \geq n + 1$, as the case $n = d$ follows from ([14], Theorem 3.3). Without loss of generality we, can perform elementary transformation on $(a_1, ..., a_n)$. As, we can perform the same elementary transformation on $(c_1, ..., c_n)$ and ensure the condition $c_i - a_i \in I_1^2$ remains unaltered.

Let $B = A/I_2^2$, then $\dim(B) = \dim(A/I_2^2) \leq d - n$. Let 'bar' denotes going modulo I_2^2. Since we have $n \geq \dim(B) + 2$, we can assume $(\overline{a}_1, ..., \overline{a}_n) = (\overline{1}, ..., \overline{0})$, after some suitable elementary transformation. Again adding some suitable multiples of a_n to $a_1, ..., a_{n-1}$ we may further assume that Theorem 2.1.4 ht$(< a_1, ..., a_{n-1} >) = n - 1$, without altering the assumption $(\overline{a}_1, ..., \overline{a}_n) = (\overline{1}, ..., \overline{0})$. Replacing a_n by $a_1 + a_n$, we can also assume $\overline{a}_n = \overline{1}$.

Define $J_1 =< a_1, ..., a_{n-1}, a_n + T >$, $J_2 = I_2A[T]$ and $J = J_1 \cap J_2$. Then we get

(i) J_1 contains a monic polynomial.

(ii) $J_1 =< a_1, ..., a_{n-1}, a_n + T > +J_1^2$, with $\dim(A[T]/J_1) + 1 = \dim(A/ < a_1, ..., a_{n-1} >) + 1 \leq d - n + 2 \leq n$. (note that the assumption $d > n$ ensure the fact $n \geq 3$)

(iii) J_2 is an extended ideal.

(iv) $J(0) = I_1 \cap I_2 = I =< c_1, ..., c_n >$ with $c_i - a_i \in I_1^2 = J_1(0)^2$, for all $i = 1..., n$.

Thus applying Theorem 3.3.2, we can obtain $J =< h_1, ..., h_n >$, such that evaluating $h_i(T)$ at $T = 0$ match with c_i, for all $i = 1..., n$. We define b_i to be the evaluation of $h_i(T)$ at $T = 1 - a_n$, that is, $b_i := h_i(1 - a_n)$ for all $i = 1, ..., n$. Then we get $I_2 = J_2(1 - a_n) =$

$J_1(1-a_n)\cap J_2(1-a_n) = J(1-a_n) =< b_1, ..., b_n >$, with $c_i - b_i = h_i(0) - b_i = h_i(1-a_n) - b_i = 0$ modulo I_2^2 and this completes the proof. □

4.2 The Euler class group of affine algebras over $\overline{\mathbb{F}}_p$

Let A be an affine algebra over $\overline{\mathbb{F}}_p$, with $\dim(A) = d$. From now onwards we shall assume $d \geq 3$ and n is an integer satisfying $2n \geq d + 2$.

Let J be an ideal of height n such that J/J^2 is generated by n elements. Let α and β be two surjections from $(A/J)^n$ to J/J^2. We say α and β are related if there exists an elementary automorphism σ of $(A/J)^n$ such that $\alpha\sigma = \beta$. This defines an equivalence relations on the set of surjections from $(A/J)^n$ to J/J^2. Let $[\alpha]$ denote the equivalence class of α. If $\overline{a}_1, ..., \overline{a}_n$ generate J/J^2, we obtain a surjection $\alpha : (A/J)^n \twoheadrightarrow J/J^2$, sending $\overline{e}_i$ to $\overline{a}_i$. We say $[\alpha]$ is given by the set of generators $\overline{a}_1, ..., \overline{a}_n$ of J/J^2.

Definition 4.2.1 Let G be the free abelian group on the set B of pairs (J, ω_J), where:
(i) $J \subset A$ is an ideal of height n,
(ii) Spec(A/J) is connected,
(iii) J/J^2 is generated by n elements, and
(iv) $\omega_J : (A/J)^n \twoheadrightarrow J/J^2$ is an equivalence class of surjections $\alpha : (A/J)^n \twoheadrightarrow J/J^2$.

Let $J \subset A$ be a proper ideal. Get $J_i \subset A$ such that $J = J_1 \cap J_2 \cap ... \cap J_r$, where J_i's are proper, pairwise co-maximal and Spec(A/J_i) is connected. It was proved in ([16], lemma 4.1) that such a decomposition is unique. We shall say that J_i are the connected components of J.

Let $J \subset A$ be an ideal of height n, J/J^2 is generated by n elements and $J = \cap J_i$ be the decomposition of J into its connected components. Then note that for every i, ht$(J_i) = n$ and by Chinese remainder theorem J_i/J_i^2 is generated by n elements. Let $\omega_J : (A/J)^n \twoheadrightarrow J/J^2$ be a surjection. Then in a natural way ω_J gives rise to surjections $\omega_{J_i} : (A/J_i)^n \twoheadrightarrow J/J_i{}^2$. We associate to the pair (J, ω_J), the element $\sum(J_i, \omega_{J_i})$ of G.

Let H be the subgroup of G generated by the set S of pairs (J, ω_J), where $\omega_J : (A/J)^n \to \to J/J^2$ has a surjective lift $\theta : A^n \twoheadrightarrow J$. Then we define the quotient group G/H as the n-th Euler Class group of A denoted as $E^n(A)$.

Theorem 4.2.2. *Let A be an affine algebra over $\overline{\mathbb{F}}_p$ of $\dim(A) = d$ and n be an integer satisfying $2n \geq d + 2$. Let $I \subset A$ be an ideal of height n be such that I/I^2 is generated by n elements and $\omega_I : (A/I)^n \twoheadrightarrow I/I^2$ be an equivalent class of surjections. Suppose that* *the image of (I, ω_I) is zero in the Euler Class group $E^n(A)$ of A. Then I is generated by n elements and ω_I can be lifted to a surjection $\theta : A^n \twoheadrightarrow I$.*

To prove the Theorem 4.2.2, we shall need the following Lemma. The proof of the following Lemma can be found in ([48], Lemma 4.1).

Lemma 4.2.3. *Let F be a free abelian group with basis $\{e_i\}_{i \in T}$ and $\sim$ be an equivalence relation on $\{e_i\}_{i \in T}$. Define, $x \in F$ to be reduced if $x = e_1 + ... + e_r$, $(i \neq j)$. Define $supp(x) = \{e_i\}_{i=1}^r$. Define, $x \in F$ to be nicely reduced if it is reduced and e_i and e_j are not equivalent for $i \neq j$. Let $S \subset F$ be such that :*

1) Every element of S is nicely reduced.

2) Suppose $x, y, x + y$ are nicely reduced. If any two are in S, then so is the third.

3) $x \in F$, $x \notin S$. Let $K \subset T$ and $|K| < \infty$. Then there exists $y \in F$ such that ,

i) y is nicely reduced.

ii) $x + y \in S$.

iii) $y + e_k$ is nicely reduced for all $k \in K$.

Let $H =< S >$. Then $x \in H$ is nicely reduced implies that $x \in S$.

Proof of Theorem 4.2.2: We take F to be the free abelian group generated by the set B, as defined in 4.2.1. Define a relation '$\sim$' on B as $(J, \omega_J) \sim (I, \omega_I)$ if $I = J$. Then it is an equivalence relation.

Let $S \subset G$ be as in 4.2.1. In view of the above Lemma, it is enough to show that the three conditions in Lemma 4.2.3 are satisfied. Condition (i) is clear, almost from the definition. The addition and subtraction principles (4.1.1 and 4.1.2) will yield condition (ii). Finally, applying the moving Lemma ([16], Corollary 2.4), it is clear that (iii) is also satisfied. $\square$

4.3 The Euler class of stably free modules

Let A be a smooth affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 3$. In this section we shall assign a *'1-stably free module'* P of rank n to its *'n-th Euler class'* and show that P has an unimodular element if and only if its Euler class is trivial. Let $v = (v_0, ..., v_n) \in \mathrm{Um}_{n+1}(A)$, where $2n \geq d + 2$. Note that the case $d = n$ is done in [14]. So without loss of generality we may further assume $d \geq n + 1$. Let $P = A^{n+1}/ < v >$, where $< v >$ means the principal ideal of A^{n+1} generated by the element v. Thus we have $P \oplus A \cong A^{n+1}$. Let p_i denotes the image of e_i in P, for $i = 0, ..., n$ and denote $p = (p_0, ..., p_n)$. Then $P = \sum_{i=0}^n Ap_i$ and $\sum_{i=0}^n v_i p_i = 0$. We shall define a map $\mathrm{Um}_{n+1}(A) \to E^n(A)$, for all $2n \geq d + 2$ and shall assign P with an element of the group $E^n(A)$ in the following way:

Let $\lambda : P \twoheadrightarrow J$ be a surjection, where $J \subset A$ be an ideal of height n. Since P is a stably free A-module of rank $\geq d-n+2 \geq \dim(A/J)+2$, by Lemma 2.1.8 P/JP is a free A/J-module. Note that $\lambda : P \twoheadrightarrow J$ shall induce a surjection $\overline{\lambda} : P/JP \twoheadrightarrow J/J^2$ and since P/JP is free, J/J^2 is generated by n elements.

Let 'bar' denote going modulo J. Since $\dim(A/J) \leq d-n$ we have $n+1 \geq d-n+1 \geq \dim(A/J)+1 \geq sr(A/J)+1$, thus by Lemma 2.3.3 $\overline{v} \sim_E \overline{e}_1$. Hence there exists $\epsilon \in E_{n+1}(A)$ with $\overline{\epsilon e_1^T} = \overline{v}^T$, and thus we have $\overline{p\epsilon} = (\overline{0}, \overline{u}_1, ..., \overline{u}_n)$. Hence $\{\overline{u}_1, ..., \overline{u}_n\}$ forms a basis of the free A/J-module P/JP. Let ω_J be the surjection induced by λ, sending each $\overline{u}_i \to \overline{\lambda(u_i)}$, for $i = 1, ..., n$.

We define $e(P, v, p) := (J, \omega_J) \in E^n(A)$.

To prove this is well defined one needs to check the followings:

Theorem 4.3.1. *With the same notations as earlier suppose that there exists another $\lambda' : P \rightarrow \rightarrow J'$ be another surjection, where $J' \subset A$ be an ideal of height n and we get $\omega_{J'}$ in the same way as discuss earlier, then $(J, \omega_J) = (J', \omega_{J'})$ in $E^n(A)$.*

Proof By ([16], Lemma 5.1) there exists an ideal $I \subset A[T]$ of height n and a surjection $\lambda(T) : P[T] \twoheadrightarrow I$, such that $\lambda(0) = \lambda$ and $\lambda(1) = \lambda'$. Let $N = (I \cap A)^2$. Then $n-1 \leq \text{ht}(N) \leq n$ and by our assumption $\dim(A/N) \geq 2$. Thus by Lemma 2.3.3 we have $sr(A/N) \leq \dim(A/N) \leq d-n+1 < d-n+3 \leq n+1$. Hence there exists $\alpha \in E_{n+1}(A)$, with $\overline{\alpha e_1^t} = \overline{v}^t$, and thus we have $\overline{p\alpha} = (\overline{0}, \overline{w}_1, ..., \overline{w}_n)$. Thus$\{\overline{w}_1, ..., \overline{w}_n\}$ forms a basis of the free A/N-module P/NP and also a basis of the free $A[T]/I$-module $P[T]/IP[T]$. Therefore, as earlier we obtain $\{\overline{\lambda}(T)(\overline{w}_1), ..., \overline{\lambda}(T)(\overline{w}_n)\}$ as a set of generators of I/I^2, and setting $T = 0$ and 1 we obtain generators of J/J^2 and J'/J'^2 receptively and hence the surjections $\omega_J : (A/J)^n \twoheadrightarrow J/J^2$ and $\omega'_J : (A/J')^n \twoheadrightarrow J'/J'^2$.

Then by ([16], Proposition 5.2) there exists an ideal $K \subset A$ of height n comaximal with J and J', and a surjection $\omega_K : (A/K)^n \twoheadrightarrow K/K^2$, such that $(J, \omega_J) + (K, \omega_K) = (J', \omega'_J) + (K, \omega_K)$ in $E^n(A)$ and hence $(J, \omega_J) = (J', \omega'_J)$. □

Theorem 4.3.2. *Suppose that there exists $\epsilon' \in E_{n+1}(A)$,with $\overline{\epsilon}'\overline{e}_1^T = \overline{v}^T$ such that, $\overline{p\epsilon'} = (\overline{0}, \overline{u}'_1, ..., \overline{u}'_n)$ and ω'_J be a surjection induced by λ, sending each $\overline{u}'_i \to \overline{\lambda(u'_i)}$, for $i = 1, ..., n$, then $(J, \omega_J) = (J, \omega'_J)$.*

Theorem 4.3.2 follows from the following lemma.

Lemma 4.3.3. *Suppose that there exists $\epsilon' \in E_{n+1}(A)$, with $\overline{\epsilon}'\overline{e}_1^T = \overline{v}^T$, such that $\overline{p\epsilon'} = (\overline{0}, \overline{u}'_1, ..., \overline{u}'_n)$. Then there exists $\theta \in E_n(A/J)$, such that $(\overline{u}_1, ..., \overline{u}_n)\theta = (\overline{u}'_1, ..., \overline{u}'_n)$.*

Proof Since $\overline{\epsilon e_1^T} = \overline{v}^T = \overline{\epsilon}'\overline{e}_1^T$, we have $\overline{\epsilon}^{-1}\epsilon'\overline{e}_1^T = \overline{e}_1^T$ and $\epsilon^{-1}\epsilon' \in E_{n+1}(A)$. Therefore there exists $\theta \in SL_n(A/J) \cap E_{n+1}(A/J)$, such that $(\overline{u}_1, ..., \overline{u}_n)\theta = (\overline{u}_1', ..., \overline{u}_n')$. Since $n \geq d-n+2 > \dim(A/J) = sr(A/J)$, by ([72], Theorem 3.2) $\theta \in E_n(A/J)$. □

Hence the assignment of (J, ω_J) associated to (P, v, p) is well defined and we shall denote it by $e(v) \in E^n(A)$. From the definition of $E^n(A)$ it follows that for any two unimodular rows $u, v \in \mathrm{Um}_{n+1}(A)$, if $u \sim_E v$, then $e(v) = e(u)$. Thus we obtain a set-theocratic map

$$e : \mathrm{Um}_{n+1}(A)/E_{n+1}(A) \to E^n(A).$$

Theorem 4.3.4. *Suppose that v and P be as defined before. Then P has a unimodular element if and only if $e(v) = 0$ in $E^n(A)$.*

Proof First assume that P has a unimodular element, say $\omega \in \mathrm{Um}(P)$. Thus we have $P \cong Q \oplus A\omega$. Let $\lambda : P \twoheadrightarrow J$ be a surjection, where $J \subset A$ be an ideal of height n. By Theorem 2.1.4 without loss of generality we may assume that, $\mathrm{ht}(\lambda(Q)) = n-1$. Let $N = \lambda(Q)$ and 'bar' denote going modulo N. Since $n+1 \geq d-n+3 = \dim(A/N)+2$, there exists $\epsilon \in E_{n+1}(A)$ with $\overline{\epsilon e_1^t} = \overline{v}^t$, and thus we have $\overline{p\epsilon} = (\overline{0}, \overline{w}_1, ..., \overline{w}_n)$. Hence $\{\overline{w}_1, ..., \overline{w}_n\}$ shall form a basis of the free A/N-module P/NP.

Let $\overline{\omega} = \sum_{i=1}^n \overline{a}_i\overline{w}_i$. Since $\overline{\omega}$ is a unimodular element of the free A/N-module P/NP, we have $(\overline{a}_1, ..., \overline{a}_n) \in \mathrm{Um}_n(A/N)$. Since $\dim(A/N) = d-n+1 \geq 2$ we have, $n \geq \dim(A/N) + 1 \geq sr(A/N)+1$, and thus $(\overline{a}_1, ..., \overline{a}_n) \sim_E \overline{e}_1$. Hence $(\overline{w}_1, ..., \overline{w}_n)$ can be taken by an elementary automorphism to a basis $\{\overline{\omega}, \overline{t}_2, ..., \overline{t}_n\}$, where $\overline{t}_i \in Q/NQ$, for $i = 2, ..., n$. Let $\overline{\lambda}(t_i) = \overline{b}_i$, for $i = 2, ..., n$. Thus we got $J =< b_2, ..., b_n, c > +J^2$ and $N =< b_2, ..., b_n > +N^2$, where $c = \lambda(\omega)$. Then by Lemma 2.1.3 there exists $e \in N^2$, such that $e(1-e) \in< b_2, ..., b_n >$. Let $N =< b_2, ..., b_n, e >$ and $J =< N, c >=< b_2, ..., b_n, e+(1-e)c >$. Therefore we get a lift of $J =< b_2, ..., b_n, c > +J^2$, which implies that $e(v) = 0$ in $E^n(A)$.

Conversely assume that $e(v) = 0$ in $E^n(A)$. Let $\lambda : P \twoheadrightarrow J$, be a surjection, where $J \subset A$ of height n. Since $n+1 \geq \dim(A/J)+2$, we can find $\alpha \in E_{n+1}(A)$ such that $\overline{v}^t = \overline{\alpha e_1^t}$, where 'bar' denotes going modulo J. Thus we have $\overline{p\alpha} = (\overline{0}, \overline{u}_1, ..., \overline{u}_n)$ and a basis $\{\overline{u}_1, ..., \overline{u}_n\}$ of the free A/J-module P/JP. Hence we get $J =< \lambda(u_1), ..., \lambda(u_n) > +J^2$. Since $e(v) = 0$, we can have $J =< b_1, ..., b_n >$, with $\lambda(u_i) - b_i \in J^2$, for $i = 1, ..., n$. Moreover using Theorem 2.1.4 we can find $c_1, ..., c_{n-1} \in A$, such that $\mathrm{ht}(< b_1 + c_1b_n, ..., b_n + c_nb_n >) = n-1$. Let $d_i = b_i + c_ib_n$, for $i = 1, ..., c_{n-1}$ and $d_n = b_n$.

Let $I =< d_1, ..., d_{n-1}, d_n + T >$ and 'prime' denotes going modulo I. Then we have $P[T] = (A[T])^{n+1}/ < v >$. We see that $\dim(A[T]/I) = \dim(A/ < d_1, ..., d_{n-1} >) =$

$d-n+1 \leq n-1 < n+1$, and hence we can find $\Gamma(T) \in E_{n+1}(A[T])$, such that $\Gamma'(T){e'_1}^t = {v'}^t$. Thus we get $p'\Gamma'(T) = (0', u'_1(T), ..., u'_n(T))$, and $\{u'_1(T), ..., u'_n(T)\}$ shall form a basis of the free $A[T]/I$-module $P[T]/IP[T]$. Evaluating at $T = 0$, we get $\overline{p}\overline{\Gamma}(0) = (\overline{0}, \overline{u}_1(0), ..., \overline{u}_n(0))$, with $\Gamma(0) \in E_{n+1}(A)$.

Thus by Lemma 4.3.3 we can find $\theta_1 \in E_n(A/J)$ such that, $(\overline{u}_1, ..., \overline{u}_n) = (\overline{u}_1(0), ..., \overline{u}_n(0))\theta_1$. Therefore there exists $\theta_2 \in E_{n+1}(A/J)$ such that,

$$(\overline{u_1 + c_1 u_n}, ..., \overline{u_{n-1} + c_{n-1} u_n}, \overline{u_n}) = (\overline{u_1(0)}, ..., \overline{u_n(0)})\theta_2$$

Note that $\overline{\lambda(u_i + c_i u_n)} = \overline{d_i}$, for all $i = 1, ..., n-1$ and $\overline{\lambda(u_n)} = \overline{d_n}$. Also note that $A/J \cong (A[T]/I)/ < t >$, where $t = T'$. Thus the map $A[T]/I \twoheadrightarrow A/J$ is surjective and so thus $E_n(A[T]/I) \twoheadrightarrow E_n(A/J)$. Hence we can find $\tau(T) \in E_n(A[T])$, which is a lift of θ_2. Let

$$(u_1(T)', ..., u_n(T)')\tau(T)' = (w_1(T)', ..., w_n(T)')$$

Since $\{u_1(T)', ..., u_n(T)'\}$ is a basis of the free $A[T]/I$-module $P[T]/IP[T]$, $\{w_1(T)', ..., w_n(T)'\}$ is also a basis. Define a surjection $\theta : P[T]/IP[T] \twoheadrightarrow I/I^2$, sending $w_i(T)' \to d'_i$ for $i = 1, ..., n-1$ and $w_n(T)' \to d_n + T$. Then since $\tau(0)' = \theta_2$, it follows that $\theta(0) = \overline{\lambda}$. Since we have $d \geq n+2$ this gives us $\dim(A[T]/I) + 1 \leq d - n + 2 \leq n$, by Corollary 3.2.2 there exists a surjective lift Θ of $\theta : P[T] \twoheadrightarrow I$ which matches at $T = 0$. Since $T + d_n \in I$, setting $T = 1 - d_n$ we obtain a surjection from $\gamma = \Theta(1 - d_n) : P \twoheadrightarrow A$, which completes the proof. $\square$

Lemma 4.3.5. *Let n be even. Let $J =< a_1, ..., a_n >$ be an ideal of height n and u be a unit modulo J. Let $\omega_J : (A/J)^n \twoheadrightarrow J/J^2$ be given by the set of generators $\overline{ua_1}, \overline{a_2}, ..., \overline{a_n}$ of J/J^2. Let $v \in A$ be such that $1 - uv \in J$. Then $e(v, a_1, ..., a_n) = (J, \omega_J)$.*

Proof Let $Q = A^{n+1}/ < (v, a_1, ..., a_n) >$ and q_i denotes the image of e_i in P, for $i = 0, ..., n$. Let $\mu : Q \twoheadrightarrow J$ be a surjection sending $q_0 \to 0$ and $q_i \to a_{i+1}$ if i is odd, and $q_i \to -a_{i-1}$ if i is even. Thus modulo J we get $(v, a_1, ..., a_n) = (v, 0, ..., 0)$. By Whitehead's Lemma, the diagonal matrix given by $\operatorname{diag}(\overline{v}, \overline{u}, \overline{1}, \overline{1} ..., \overline{1}) \in E_n(A/J)$. Hence $e(v, a_1, ..., a_n) = (J, \omega_J)$, where ω_J is given by the set of generators $\overline{ua_2}, -\overline{a}_1, \overline{a}_4, -\overline{a}_3, ..., \overline{a}_n, -\overline{a}_{n-1}$ of J/J^2. Applying Whitehead's lemma again, we see that ω_J is given by the set of generators $\overline{ua_1}, \overline{a}_2, ..., \overline{a}_n$ of J/J^2. $\square$

The next theorem can be found in ([31], Theorem 4.1).

Theorem 4.3.6. *Let R be a commutative ring, $n \geq 3$ and $sdim(A) \leq 2n-4$. Then the universal weak Mennicke symbol wms : $Um_n(A)/E_n(A) \to WMS_n(R)$ is bijective. Thus $Um_n(A)/E_n(A)$ has a group structure.*

Since A is an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 3$ we have $sdim(A) \leq d-1$. Thus by Theorem 4.3.6, for all n satisfying $2n \geq d+1$, we have a group structure on $\mathrm{Um}_{n+1}(A)/E_{n+1}(A)$.

Theorem 4.3.7. *For all n satisfying $2n \geq d+2$, the map e : $Um_{n+1}(A)/E_{n+1}(A) \to E^n(A)$ is a group homomorphism.*

Proof If n is odd, since every one-stably free module of odd rank has a unimodular element, using Theorem 4.3.4 it follows that e is the zero map, hence nothing to prove. So we shall assume n is even. Without loss of generality we may further assume A is a domain. Since $2(n+1) \geq d+4 \geq d+3$, by ([70], Lemma 3.2) it is enough to prove that if $(x, a_1, ..., a_n)$ and $(1-x, a_1, ..., a_n)$ are unimodular then we have

$$e(x, a_1, ..., a_n) + e(1-x, a_1, ..., a_n) = e(x(1-x), a_1, ..., a_n).$$

Let $y = 1-x$. We may assume that $xy \neq 0$. Let 'bar' denotes going modulo xy. Then adding a suitable multiple of $\overline{a}_1$ to $\overline{a}_i$ for $i = 2, ..., n$ we may assume that ht $< \overline{a}_2, ..., \overline{a}_n >= n-1$, and hence ht$(< x, a_2, ..., a_n >) = n$. Thus we may assume ht$(< y, a_2, ..., a_n >) = n =$ ht$(< x, a_2, ..., a_n >)$.

Let $b_1 \in A$ be such that $1 + a_1 b_1 \in < xy, a_2, ..., a_n >$. Now since $(x, a_1, a_2 ..., a_n) \sim_E (-a_1, x, a_2 ..., a_n)$, we have $e(x, a_1, a_2 ..., a_n) = e(-a_1, x, a_2 ..., a_n)$.

Since n is even by Lemma 4.3.5, $e(-a_1, x, a_2 ..., a_n) = e(x, a_1, a_2 ..., a_n) = (J_1, \omega_{J_1})$, where $J_1 =< x, a_2, ..., a_n >$ and ω_{J_1} is given by the set of generators $\overline{b_1 x}, \overline{a}_2, ..., \overline{a}_n$ of J_1/J_1^2. Similarly we get $e(y, a_1, a_2 ..., a_n) = (J_2, \omega_{J_2})$, where $J_2 = (y, a_2, ..., a_n)$ and ω_{J_2} is given by the set of generators $\overline{b_1 y}, \overline{a}_2, ..., \overline{a}_n$ of J_2/J_2^2 and $e(xy, a_1, a_2 ..., a_n) = (J_3, \omega_{J_3})$, where $J_3 = (xy, a_2, ..., a_n)$ and ω_{J_3} is given by the set of generators $\overline{b_1 xy}, \overline{a}_2, ..., \overline{a}_n$ of J_3/J_3^2. Since $J_3 = J_1 \cap J_2$ and $x + y = 1$ we have $(J_3, \omega_{J_3}) = (J_1, \omega_{J_1}) + (J_2, \omega_{J_2})$ and this completes the proof. $\square$

Chapter 5

On a question of Roitman

5.1 Over some finite $\mathbb{Z}$-algebras

By a finite $\mathbb{Z}$-algebra we mean a finitely generated algebra over $\mathbb{Z}$. In this section we give an affirmative answer of a question asked by M. Roitman. The following lemma is crucial to our proof. The proof of the lemma is motivated from [14].

Lemma 5.1.1. *Let A be a finite $\mathbb{Z}$-algebra of dimension $d \geq 3$. Moreover assume that there exists an integer $n \geq 2$ be such that $n \in A^*$. Let P be a projective A-module of rank d with trivial determinant. Let $\chi : A \cong \wedge^d P$ be an isomorphism. Let $I \subset A$ be an ideal of height $\geq d-1$ and $J \subset A$ be an ideal of height $\geq d$ such that $I + J = A$. Suppose that there exist surjections $\alpha : P \twoheadrightarrow I \cap J$ and $\beta : A^d \twoheadrightarrow I$. Let 'bar' denotes going modulo I. Suppose that there exists an isomorphism $\delta : \overline{A}^d \cong \overline{P}$ with the following properties:*
(i) $\overline{\beta} = \overline{\alpha}\delta$;
(ii) $\wedge^d \delta = \overline{\chi}$.
Then there exists a surjection $\gamma : P \twoheadrightarrow J$ such that $\gamma \otimes A/J = \alpha \otimes R/J$.

Proof Let β correspond to $I =< a_1, ..., a_d >$. Going modulo J^2 we may assume (i) $< a_1, ..., a_{d-1} > +J^2 = A$, (ii) $a_d \in J^2$ and replacing a_i by $a_i + \lambda_i a_d$ (without changing its notations), for some $\lambda_i \in A$, $i = 1, ..., d-1$ we may assume (iii) ht$(< a_1, ..., a_{d-1} >) = d-1$. Now replacing a_d by $a_d + \lambda$, for some $\lambda \in< a_1, ..., a_{d-1} >$, with $\lambda - 1 \in J^2$, we may assume $a_d - 1 \in J^2$.

Consider the following ideals in $A[T]$: $K' =< a_1, ..., a_{d-1}, a_d + T >$, $K'' = J[T]$ and $K = K' \cap K''$. Then it is enough to show there exists a surjection $\theta : P[T] \twoheadrightarrow K$ such that $\theta(0) = \alpha$. As if we can do, then specializing at $1 - a_d$ we get $\gamma := \theta(1 - a_d) : P \twoheadrightarrow J$. Since

$a-1 \in J^2$, we have $\gamma \otimes A/J = \theta(0) \otimes A/J = \alpha \otimes A/J$. In the rest of the proof we will find such an θ.

$\dim(A[T]/K') = \dim(A/ < a_1, ..., a_{d-1} >) \leq 1$, hence the module $P[T]/K'P[T]$ is free of rank d. We choose an isomorphism $\kappa(T) : (A[T]/K')^d \cong P[T]/K'P[T]$ such that $\wedge^d \kappa(T) = \chi \otimes A[T]/K'$. Therefore, $\wedge^d \kappa(0) = \wedge^d \delta$. Thus $\kappa(0)$ and δ differs by an element $\alpha \in \mathrm{SL}_d(A/I)$. By ([73], Theorem 16.4), $SK_1(A/I) = 0$, thus for all $d \geq 3$, $\mathrm{SL}_d(A/I) = E_d(A/I)$, hence we can lift $\alpha \in E_d(A)$ and use this to alter $\kappa(T)$ so that $\kappa(0) = \delta$.

Sending the canonical basis vectors to $a_1, ..., a_{d-1}, a_d + T$ respectively we have a surjection from $(A[T])^d \twoheadrightarrow K'$, which induces a surjection $\epsilon(T) : (A[T]K')^d \twoheadrightarrow K'/K'^2$.

Let $\phi(T) := \epsilon(T)\kappa(T)^{-1} : P[T]/K'P[T] \twoheadrightarrow K'/K'^2$. Note that $\phi(0) = \epsilon(0)\kappa(0)^{-1} = \alpha \otimes A/I$. Since $d \geq \dim(A[T]/K') + 2 = 3$, using ([43], Theorem 2.3) we get $\theta(T) : P[T] \twoheadrightarrow K$ such that $\theta(0) = \alpha$. $\square$

Remark 5.1.2 The only place at which the hypothesis $d \geq 3$ is used in the proof, is to establish the fact that there exists a natural surjection $\mathrm{SL}_d(A) \twoheadrightarrow \mathrm{SL}_d(A/I)$. But this fact can be obtained using the same arguments given in ([20], Corollary 2.3) and hence the above result can be proved for $d \geq 2$.

Corollary 5.1.3. *Let A be a finite $\mathbb{Z}$-algebra of dimension $d \geq 3$. Moreover assume that there exists an integer $n \geq 2$ be such that $n \in A^*$. Let P be a projective A-module with trivial determinant of rank d. Let $\chi : A \cong \wedge^d P$ be an isomorphism. Let $I \subset A$ be an ideal of height $\geq d-1$ such that there exists surjections $\alpha : P \twoheadrightarrow I$ and $\beta : A^d \twoheadrightarrow I$. Let 'bar' denotes going modulo I. Suppose that there exists an isomorphism $\delta : \overline{A}^d \cong \overline{P}$ with the following properties:*
(i) $\overline{\beta} = \overline{\alpha}\delta$;
(ii) $\wedge^d \delta = \overline{\chi}$.
Then P has a unimodular element.

Proof The proof follows from Theorem 5.1.1, taking $J = A$. $\square$

Theorem 5.1.4. *Let A be a finite $\mathbb{Z}$-algebra of dimension $d \geq 1$. Moreover assume that there exists an integer $n \geq 2$ such that $n \in A^*$. Let P be a projective $A[T]-$module with trivial determinant of rank d and $J \subset A[T]$ be an ideal of height d containing a monic polynomial. Suppose that there exists a surjection $\alpha : P \twoheadrightarrow J$ then P has a unimodular element.*

Proof For $d = 1$ the theorem follows from the Theorem 2.3.9 and for $d = 2$ the proof is done in ([8], Proposition 3.3). Therefore we may assume that $d \geq 3$.

Fix $\chi : A[T] \cong \wedge^d P$. Let 'bar' denote going modulo J. Since P has trivial determinant and $\dim(A[T]/J) \leq 1$, P/IP is a free $A[T]$-module of rank d. Let $\delta : (A[T]/J)^d \cong P/JP$ be such that $\wedge^d \delta = \chi \otimes A[T]/J$. Let $\omega = (\alpha \otimes A[T]/J)\delta : (A[T]/J)^d \twoheadrightarrow J/J^2$. Since J contains a monic polynomial and $\dim(A[T]/J) + 2 \leq 3 \leq d$, by (N. M. Kumar) there exists $\beta : (A[T])^d \twoheadrightarrow J$ such that $\beta \otimes A[T]/J = \omega$.

Since $\wedge^d P$ is extended from the ring A in a view of ([15], Theorem 2.3) it is enough to show that P/TP has a unimodular element. Let us define some notations:

$P/TP =: P(0)$, $J \otimes A[T]/ < T >=: J(0)$,

$\alpha \otimes A[T]/ < T >=: \alpha(0) : P(0) \twoheadrightarrow J(0)$,

$\beta \otimes A[T]/ < T >=: \beta(0) : A^d \twoheadrightarrow J(0)$,

$\omega \otimes A/J(0) = \omega(0) : (A/J(0))^d \twoheadrightarrow J(0)/J(0)^2$

$\delta \otimes A[T]/ < T >=: \delta(0) : (A/J(0))^d \xrightarrow{\sim} P(0)/J(0)P(0)$ and

$\chi \otimes A[T]/ < T >=: \chi(0) : A \xrightarrow{\sim} \wedge^d P(0)$.

Since $J \cap A \subset J(0)$, ht$(J(0)) \geq d - 1$.

Now note that we have the followings:

(i) $\beta(0) \otimes A/J(0) = (\alpha(0) \otimes A/J(0))\delta(0)(= \omega(0))$ and;

(ii) $\wedge^d \delta(0) = \chi(0) \otimes A/J(0)$.

Therefore using Lemma 5.1.3, $P(0)$ has a unimodular element. □

5.2 Over affine $\overline{\mathbb{F}}_p$-algebras

Recall that 1-stably free modules are those which are given by a unimodular row.

Theorem 5.2.1. *Let R be an affine $\overline{\mathbb{F}}_p-$algebra of dimension $d \geq 2$. Let P be a 1-stably free $R[T]$-module of rank $d-1$. Assume that $P \otimes R(T)$ has a unimodular element. Then P also has a unimodular element.*

Proof If $d = 2$, then the result follows trivially. If $d = 3$, then since P is of rank two with trivial determinant, we observe that $P \otimes R(T)$ is actually free. Then, by the Affine Horrocks Theorem, P is free. Therefore, we assume that $d \geq 4$. As we shall apply the Euler class theory developed in Section 4.3, we take $2(d-1) \geq (d+1) + 2$, implying that $d \geq 5$. Apparently, the only case that seems to be left out is $d = 4$ and rank$(P) = 3$. But it is easy to see that 1-stably free modules of odd rank always have a unimodular element. Therefore, all the cases will be covered by this theorem once we complete the following arguments (with $d \geq 5$).

Since R contains an infinite field, namely, $\overline{\mathbb{F}}_p$, we can follow the arguments of ([15], Lemma 3.1) and obtain an $R[T]$-linear surjection $\lambda : P \twoheadrightarrow I$, where $I \subset R[T]$ is an ideal of height $d-1$

and I contains a monic polynomial. Let P correspond to the unimodular row $v \in \mathrm{Um}_d(R[T])$. Using λ we can compute the Euler class of P (or v), as in Section 4.3 and obtain

$$e(v) = (I, \omega_I) \in E^{d-1}(R[T]).$$

As I contains a monic polynomial, it follows from ([20], Theorem 3.2) that $\omega_I : (R[T]/I)^{d-1} \twoheadrightarrow I/I^2$ has a surjective lift $\theta : R[T]^{d-1} \twoheadrightarrow I$. In other words, $(I, \omega_I) = 0$ in $E^{d-1}(R[T])$. Consequently, $e(v) = 0$ and by Theorem 4.3.4, P has a unimodular element. $\square$

Chapter 6

Monic inversion principle

6.1 Main theorem

Here we shall give a direct proof of a Monic inversion principle, over the base field $\overline{\mathbb{F}}_p$ without any further restriction on the ring. Let R be an affine $\overline{\mathbb{F}}_p-$algebra. Note that with the condition $\gcd(p, \dim(R)) = 1$ or taking R to be smooth one can obtain a proof using ([19] or [24]). The idea of the proof is to produce an ideal I_2 (in our proof), which contains a monic polynomial. Since it has a monic polynomial, any set of generators of I_2/I_2^2 gets lift to a set of generators of I_2. Now we shall use the subtraction principle (Proposition 4.1.2) repeatedly to get a lift of the set of generators, we started with.

Theorem 6.1.1. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$ and $I \subset R[T]$ be an ideal such that ht$(I) = \mu(I/I^2) = d$. Assume that $I =< f_1, ..., f_d > +I^2$. Suppose that there exists $F_i \in IR(T)$ such that $IR(T) =< F_1, ..., F_d >$, with $F_i - f_i \in IR(T)^2$, $(i = 1, ..., d)$. Then there exists $g_i \in I$, such that $I =< g_1, ..., g_d >$, where $g_i - f_i \in I^2$.*

Proof

Case- 1 ($d = 2$) Note that if the ideal I contains a monic polynomial in T, a lift exists by Theorem 2.3.10 (without any further assumption), hence without loss of generality we may assume that I does not contain a monic. Let $I =< f_1, f_2 > +I^2$ induce the surjection $\omega_I : (R[T]/I)^2 \twoheadrightarrow I/I^2$, sending the canonical basis e_i to the image f_i in I/I^2. Then using a standard patching argument there exists a projective $R[T]-$module P with trivial determinant of rank 2 and a surjection $\alpha : P \twoheadrightarrow I$. Fix an isomorphism $\chi : R[T] \cong \wedge^2 P$. Let α and χ induce $I =< f_1', f_2' > +I^2$.

Let 'bar' denote going modulo I^2. By Theorem 2.3.4 there exists $\overline{\sigma} \in \mathsf{GL}_2(R[T]/I)$ with determinant $\overline{f}$ such that $(\overline{f}_1, \overline{f}_2) = (\overline{f}'_1, \overline{f}'_2)\overline{\sigma}$. Following ([14], Lemma 2.7 and Lemma 2.8) , there exists a projective $R[T]$-module P_1 of rank 2 having trivial determinant, a trivialization χ_1 of $\wedge^2 P_1$, and a surjection $\beta : P_1 \twoheadrightarrow I$ such that if the set of generators of I/I^2 induced by β and χ_1 (with respect to a fixed basis of $R[T]^2$ induced by χ_1 say $\{\eta_1, \eta_2\}$) is $\overline{h}_1, \overline{h}_2$, then $(\overline{h}_1, \overline{h}_2) = (\overline{f}'_1, \overline{f}'_2)\overline{\delta}$, where $\overline{\delta} \in \mathsf{GL}_2(R[T]/I)$ has determinant $\overline{f}$. Thus $(\overline{f}_1, \overline{f}_2)\overline{\gamma} = (\overline{h}_1, \overline{h}_2)$, where $\overline{\gamma} = \overline{\sigma}^{-1}\overline{\delta} \in \mathsf{SL}_2(R[T]/I)$. Since by Theorem 2.3.4 the natural map $\mathsf{SL}_d(R[T]) \twoheadrightarrow \mathsf{SL}_d(R[T]/I)$ is surjective, get $\gamma \in \mathsf{SL}_2(R[T])$ such that γ is a lift of $\overline{\gamma}$.

Now note that it is enough to show that the set of generators $I =< h_1, h_2 > +I^2$ has a lift to a set of generators of I. Suppose that $I =< a_1, a_2 >$ is such a lift. Then we define $(g_1, g_2) = (a_1, a_2)\gamma^{-1}$. Then note that $(g_1 - f_1, g_2 - f_2) = (a_1 - h_1, a_2 - h_2)\gamma^{-1} \in I^2 \times I^2$. Thus the remaining part of the proof in this case is dedicated to find a lift of $\{h_1, h_2\}$.

Since the set of generators (f_1, f_2) has a lift in the ring $R(T)$, so does the set of generators (h_1, h_2). Now recall that $R(T)$ is a two dimensional Noetherian ring and in the two dimensional set-up, the theory of Euler class group (defined in [14]), and the Euler class one does not need any additional assumption. Hence we can use the Euler class group theory in the ring $R(T)$ freely.

Note that in the ring $R(T)$ we have $e(P_1 \otimes R(T), \chi_1 \otimes R(T)) = (IR(T), \omega)$, where $\omega : (R(T)/IR(T))^2 \twoheadrightarrow IR(T)/IR(T)^2$ by sending $e_i \to h_i$. Since (h_1, h_2) has a lift $P_1 \otimes R(T)$ has a unimodular element. Thus there exists a monic polynomial $g \in R[T]$ such that P_{1g} has a unimodular element. As P_1 has trivial determinant, P_{1g} is free implying that P_1 is free by Theorem 2.3.9.

Thus there exists an isomorphism $\eta : (R[T])^2 \cong P_1$, such that $\wedge^2\eta = \chi_1$. Let $H_i = \beta\eta(\eta_i)$, then note that $I =< H_1, H_2 >$ and $\overline{H}_1 \wedge \overline{H}_2 = \overline{h}_1 \wedge \overline{h}_2$ in $\wedge^2(I/I^2)$. Hence we can find $\sigma \in \mathsf{GL}_2(R[T]/I)$ such that $(\overline{H}_1, \overline{H}_2)\sigma = (\overline{h}_1, \overline{h}_2)$. Moreover since $\overline{H}_1 \wedge \overline{H}_2 = \overline{h}_1 \wedge \overline{h}_2$, we can take $\sigma \in \mathsf{SL}_2(R[T]/I)$. Now by using Theorem 2.3.4 there exists $\tau \in \mathsf{SL}_2(R[T])$ such that $\overline{\tau} = \sigma$. Let $(g_1, g_2) = (H_1, H_2)\tau$. Then $(\overline{g}_1, \overline{g}_2) = (\overline{H}_1, \overline{H}_2)\overline{\tau} = (\overline{H}_1, \overline{H}_2)\sigma = (\overline{h}_1, \overline{h}_2)$, hence we are done with the proof in this case.

Case- 2 ($d \geq 3$) There is a monic polynomial $f \in R[T]$ such that $I_f =< F_1, ..., F_d >$, with $F_i - f_i \in I_f^2$. Let $B = R[T]/ < f > \cap I^2$ and 'bar' denote going modulo $< f > \cap I^2$. Note that $\dim(B) \leq d$, thus in the ring B we get $\overline{I} =< \overline{f}_1, ..., \overline{f}_d > + \overline{I}^2$. Using Theorem 2.2.5 we get $\overline{h}_i \in \overline{I}$ such that $\overline{I} =< \overline{h}_1, ..., \overline{h}_d >$, with $\overline{f}_i - \overline{h}_i \in \overline{I}^2$. That is, we get $I =< h_1, ..., h_d >$ $+I^2 \cap < f >$. Using Lemma 2.1.3 there exits $e \in I^2 \cap < f >$ be such that $I =< h_1, ..., h_d, e >$, and $e(1-e) \in < h_1, ..., h_d >$. By Theorem 2.1.4 replacing h_i by $h_i + e\lambda_i$ we may assume that ht$(< h_1, ..., h_d >)_e = d$ or $< h_1, ..., h_d >_e = R[T]_e$. Note that if $< h_1, ..., h_d >_e = R[T]_e$ then $I =< h_1, ..., h_d >$ with $f_i - h_i \in I^2\cap < f >$ and we are done in this case. Thus without loss of generality we may assume that ht$(< h_1, ..., h_d >_e) = d$.

Let $I_1 =< h_1, ..., h_d, 1 - e >$. Then note that $I \cap I_1 =< h_1, ..., h_d >$, $I_1+ < e >= I_1+ < f >= I_1 + I = R[T]$ and ht$(I_1) = d$. Since $I_1+ < f >= R[T]$ and ht$(I_1) = d$, we have ht$((I_1)_f) = d$ as well. Since $h_i - f_i \in I^2$, to prove the theorem it is enough to lift $I =< h_1, ..., h_d > +I^2$ to a set of generators of I. Since $I + I_1 = R[T]$, $I \cap I_1 =< h_1, ..., f_d >$ will induce $I_1 =< h_1, ..., h_d > +I_1^2$. In view of the subtraction principle (Proposition 4.1.2),to prove the theorem, it is enough to lift $I_1 =< h_1, ..., h_d > +I_1^2$ to a set of generators of I_1.

Note that in the ring $R[T]_f$,the set of generators $I_f =< h_1, ..., h_d > R[T]_f + I_f^2$ lifts to a set of generators of I_f. Hence again by the subtraction principle (Proposition 4.1.2) there exists $l_i \in (I_1)_f$, such that $(I_1)_f =< l_1, ..., l_d >$, with $h_i - l_i \in (I_1)_f^2$. Let $k \geq 1$ be an integer such that $f^{2k}l_i \in I_1$, for all i. Since f is unit modulo I_1, to find a lift of $I_1 =< h_1, ..., h_d > +I_1^2$ by Lemma 2.3.6 it is enough to lift $I_1 =< f^{2k}l_1, ..., f^{2k}l_d > +{I_1}^2$ to a set of generators of I_1. Therefore, we can replace l_i with $f^{2k}l_i$ and may assume that $l_i \in I_1$.

Using Lemma 2.3.7 we get $\epsilon \in \mathrm{SL}_d(R[T]_f)$ such that $(l_1, ..., l_d)\epsilon = (l'_1, ..., l'_d)$,where $l'_i \in I_1$ and ht$(< l'_1, ..., l'_d > R[T]) = d$.

Let $< l'_1, ..., l'_d > R[T] = \bigcap_{i=1}^r q_i \bigcap_{i=r+1}^n q_i$ be the reduced primary decomposition, where q_i's are p_i-primary ideals in $R[T]$, such that $f \notin p_i$ for $i \leq r$ and $f \in p_i$ for all $i > r$. Since $(I_1)_f =< l'_1, ..., l'_d >_f$ is a proper ideal of height d, we must have $r \geq 1$. Let $I_2 = \bigcap_{i=r+1}^n q_i$. Therefore, ht$(I_2) \geq d$ and some power of f is in I_2.

Now note that $I_1 = \bigcap_{i=1}^r q_i$. To prove this note that $< l'_1, ..., l'_d > R[T]_f = \bigcap_{i=1}^r q_{if}$ implies that $< l'_1, ..., l'_d > R[T] \subset \bigcap_{i=1}^r q_i$. Now suppose that there exists another p-primary ideal q (where $p \neq p_i$, $i = 1, ..., r$) in the reduced primary decomposition of the ideal $< l'_1, ..., l'_d > R[T]$. This gives us (note that $p_f \neq p_{if}$, for $i = 1, ..., r$, as $p \neq p_i$) $p_f = R[T]_f$. This implies that $f \in p$ which is not possible as $I_1+ < f >= R[T]$ implies $p+ < f >= R[T]$. Thus $I_1 = \bigcap_{i=1}^r q_i$.

Hence we get (1) I_2 contains a monic polynomial (some power of f), (2) $I_1 + I_2 = R[T]$, (as going modulo I_1 any power of f is a unit) and (3) $I_1 \cap I_2 =< l'_1, ..., l'_d > R[T]$.

Note that (3) gives us for any prime $p \supset I_2$, we must have $\mu((I_2)_p) \leq d$ and hence ht$(I_2) = d$. Since $I_1+I_2 = R[T]$, $I_1\cap I_2 =< l'_1, ..., l'_d > R[T]$ will induce $I_1 =< l'_1, ..., l'_d > +I_1^2$ and $I_2 =< l'_1, ..., l'_d > +I_2^2$. Since I_2 contains a monic and $d \geq 3$, by Theorem 2.3.10 $I_2 =< l'_1, ..., l'_d > +I_2^2$ can be lifted to a set of generators of I_2. Therefore applying the subtraction principle (Proposition 4.1.2) we can lift $I_1 =< l'_1, ..., l'_d > +I_1^2$ to a set of generators of I_1.

Since f is a monic polynomial (in particular a non zero divisor), $\det(\epsilon) = 1$ in $R[T]_f$ implies that $\det(\epsilon) = 1$ in the ring $R[T]$. As $I_1 + < f >= R[T]$, we have $R[T]/I_1 = (R[T]/I_1)_f$. Therefore $\epsilon \in \mathsf{SL}_d(R[T]/I_1)$. Since $I_1 =< l'_1, ..., l'_d > +I_1^2$ has a lift to a set of generators of I_1 and $(l_1, ..., l_d)\epsilon = (l'_1, ..., l'_d)$ by Proposition 2.3.5 we can lift $I_1 =< l_1, ..., l_d > +I_1^2$ to a set of generators of I_1. Therefore, we can lift $I_1 =< h_1, ..., h_d > +I_1^2$ to a set of generators of I_1. This completes the proof. □

Chapter 7

Splitting criterion via an obstruction class in an obstruction group

Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$ and P be a finitely generated projective $R[T]$-module of rank d, with a trivial determinant (via χ). The purpose of this section is to define an Euler cycle for the triplet (P, λ, χ), in the group $E^d(R[T])$ and show that the vanishing of this Euler cycle is sufficient for P to have a unimodular element, where λ is a generic section. First we will prove some addition and subtraction principles. For most of the proofs in this section we will frequently move to the ring $R(T)$, prove the results in $R(T)$ then using Theorem 6.1.1 we will come back to the ring $R[T]$. Some of the results below were proved for Noetherian ring containing $\mathbb{Q}$ in [19].

7.1 Some addition and subtraction principles

Proposition 7.1.1. *(Addition principle) Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$. Let $I, J \subset R[T]$ be two co-maximal ideals, each of height d. Suppose that $I = (f_1, ..., f_d)$ and $J = (g_1, ..., g_d)$. Then $I \cap J = (h_1, ..., h_d)$ where $h_i - f_i \in I^2$ and $h_i - g_i \in J^2$.*

Proof Since ht$(I) =$ ht(J) in the ring $R(T)$ both the ideals $IR(T)$ and $JR(T)$ are of height $\geq d$. Now note that if one of them is of height $> d$, then there is nothing to prove. So without loss of generality we may assume that each ideal is of height d.

Since $I + J = R[T]$, using the Chinese Remainder Theorem we have $I \cap J/(I \cap J)^2 \cong I/I^2 \oplus J/J^2$. Hence the given set of generators of I and J will induce a set of generators a_i's of $(I \cap J)/(I \cap J)^2$ such that $a_i - f_i \in I^2$ and $a_i - g_i \in J^2$. Thus to prove the theorem it is enough to find a lift of $I \cap J =< a_1, ..., a_d > +(I \cap J)^2$ to a set of generators of $I \cap J$.

In the ring $R(T)$, we have $\text{ht}(I) = \text{ht}(J) = \text{ht}(I \cap J) = \dim(R(T)) = d$. Hence using ([14], Theorem 3.2) we can find $H_i \in (I \cap J)R(T)$ such that $(I \cap J)R(T) =< H_1, ..., H_d > R(T)$, with $H_i - f_i \in IR(T)^2$ and $H_i - g_i \in JR(T)^2$. Now use Theorem 6.1.1 to conclude the proof. □

Proposition 7.1.2. *(Subtraction principle) Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$. Let $I, J \subset R[T]$ be two co-maximal ideals, each of height d. Suppose that $I = (f_1, ..., f_d)$ and $I \cap J = (h_1, ..., h_d)$ where $h_i - f_i \in I^2$. Then there exists $g_i \in J$ such that $J = (g_1, ..., g_d)$ with $h_i - g_i \in J^2$.*

Proof The proof uses the same arguments as in Proposition 7.1.1 with slight modification, so we will only sketch a proof. As before, without loss of generality we may assume $\text{ht}(IR(T)) = \text{ht}(JR(T)) = \text{ht}((I \cap J)R(T)) = \dim(R(T)) = d$. Since $I + J = R[T]$, we get $J =< h_1, ..., h_d > +J^2$. Again observe that to prove the theorem it is enough to find a lift of $J =< h_1, ..., h_d > +J^2$ to a set of generators of J. Now using ([14], Theorem 3.3) in the ring $R(T)$ we can find $G_i \in JR(T)$ such that $G_i - h_i \in JR(T)^2$. Then as before we can use Theorem 6.1.1 to complete the proof. □

7.2 An obstruction group

Proposition 7.2.1. *Let R be affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$ and $I \subset R[T]$ be an ideal of height d. Moreover suppose that α and β are two surjections from $(R[T]/I)^d \twoheadrightarrow I/I^2$ such that there exists $\sigma \in SL_d(R[T]/I)$ with the property that $\alpha\sigma = \beta$. Then if α can be lifted to a surjection $\theta : (R[T])^d \twoheadrightarrow I$ then so can β.*

Proof Since $\dim(R[T]/I) \leq 1$, using Theorem 2.3.4, we can find $\epsilon \in \text{SL}_d(R[T])$, which lifts σ. Since $\epsilon \in \text{SL}_d(R[T])$ and θ is a surjection, it follows that $\theta\epsilon : (R[T])^d \twoheadrightarrow I$ is also a surjection. Thus it is only remains to show that $(\theta\epsilon) \otimes (R[T]/I) = \beta$. But this follows from the fact that $\epsilon \otimes R[T]/I = \sigma$ and $\theta \otimes R[T]/I = \alpha$. □

Now we proceed to define the d-th Euler class group of $R[T]$ where R is an affine algebra of dimension $d \geq 2$.

Definition 7.2.2 Let $I \subset R[T]$ be an ideal of height d such that I/I^2 is generated by d elements. Let α and β be two surjections from $(R[T]/I)^d \twoheadrightarrow I/I^2$. We say α and β are related if there exists $\sigma \in \text{SL}_d((R[T]/I)^d$ be such that $\alpha\sigma = \beta$. This defines an equivalence relation on the set of surjections from $(R[T]/I)^d \twoheadrightarrow I/I^2$. Let $[\alpha]$ denote the equivalance class of α. If $f_1, ..., f_d$ generate I/I^2 , we obtain a surjection $\alpha : (R[T]/I)^d \twoheadrightarrow I/I^2$, sending e_i to f_i. We say $[\alpha]$ is given by the set of generators $f_1, ..., f_d$ of I/I^2.

Let G be the free Abelian group on the set B of pairs (I, ω_J), where:

(i) $I \subset R[T]$ is an ideal of height d,

(ii) Spec$(R[T]/I)$ is connected,

(iii) I/I^2 is generated by d elements, and

(iv) $\omega_I : (R[T]/I)^d \twoheadrightarrow I/I^2$ is an equivalence class of surjections $\alpha : (R[T]/I)^d \twoheadrightarrow I/I^2$.

Let $J \subset R[T]$ be a proper ideal. we get $J_i \subset R[T]$ such that $J = J_1 \cap J_2 \cap ... \cap J_r$, where J_i 's are proper, pairwise co-maximal and Spec$(R[T]/J_i)$ is connected. We shall say that J_i are the connected components of J.

Let $K \subset R[T]$ be an ideal of height d, K/K^2 is generated by d elements and $K = \cap K_i$ be the decomposition of K into its connected components. Then note that for every i, ht$(K_i) = d$ and by Chinese remainder theorem K_i/K_i^2 is generated by d elements. Let $\omega_K : (R[T]/I)^d \twoheadrightarrow K/K^2$ be a surjection. Then in a natural way ω_K gives rise to surjections $\omega_{K_i} : (R[T]/K_i)^d \twoheadrightarrow K_i/{K_i}^2$. We associate the pair (K, ω_K), to the element $\sum_{i=1}^r (K_i, \omega_{K_i})$ of G. We will call (K, ω_K) as a local orientation of K induced by ω_K.

Let H be the subgroup of G generated by the set S of pairs (J, ω_J), where $\omega_J : (R[T]/J)^d \to \to J/J^2$ has a surjective lift $\theta : (R[T]/J)^d \twoheadrightarrow J$. Then we define the quotient group G/H as the d-th Euler class group of $R[T]$ denoted as $E^d(R[T])$. A local orientation (J, ω_J) is said to be a global orientation if ω_J lifts to a set of generators of J.

Remark 7.2.3 Note that the decomposition of K into its connected components is unique by ([19], Lemma 4.5) as the proof of Lemma 4.5 does not require the assumption that the ring contains $\mathbb{Q}$.

Remark 7.2.4 The class of the pair (I, ω_I) defined above is well-defined by the Proposition 7.2.1.

Lemma 7.2.5. *Let R be an affine algebra of dimension $d \geq 2$, $I \subset R[T]$ be an ideal of height d such that I/I^2 is generated by d elements. Let $\omega_I : (R[T]/I)^d \twoheadrightarrow I/I^2$ be a local orientation of I. Suppose that the image of (I, ω_I) is zero in the Euler class group $E^d(R[T])$. Then, I is generated by d elements and ω_I can be lifted to a surjection $\theta : (R[T])^d \twoheadrightarrow I$.*

Proof The proof is done in Theorem 4.2.2 for $d \geq 3$ and in ([21], Theorem 4.7) for $d = 2$. $\square$

7.3 An obstruction class

Definition 7.3.1 (Euler cycle induced by a Projective module) Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$ and P be a projective $R[T]$-module of rank d having trivial determinant. Let $\chi : R[T] \cong \wedge^d P$ be an isomorphism. To the pair (P, χ), we associate an element $e(P, \chi)$ of $E^d(R[T])$ as follows:

Let $\lambda : P \twoheadrightarrow I$ be a surjection, where I is an ideal of $R[T]$ of height d. Let 'bar' denote going modulo I. We obtain an induced surjection $\lambda \otimes R[T]/I : P/IP \twoheadrightarrow I/I^2$. Note that, since P has trivial determinant and $\dim(R[T]/I) \leq 1$, P/IP is a free $R[T]/I$-module of rank d. We choose an isomorphism $\phi : (R[T]/I)^d \xrightarrow{\sim} P/IP$, such that $\wedge^d \phi = \chi \otimes R[T]/I$. Let ω_I be the surjection $(\lambda \otimes R[T]/I) \circ \phi : (R[T]/I)^d \twoheadrightarrow I/I^2$. We say that (I, ω_I) is an Euler cycle induced by the triplet (P, λ, χ).

Whenever the class of (I, ω_I) in $E^d(R[T])$ induced by the triplet (P, λ, χ) becomes independent of a certain choice of the pair (λ, I) we will define the Euler class $e(P, \chi)$ of the pair (P, χ) as the image of (I, ω_I) in $E^d(R[T])$.

Notation 7.3.2. Continuing with the above notations we might omit P sometimes and only say (I, ω_I) is induced by (λ, χ), if there are no confusions. By saying an Euler cycle induced by (P, χ) we mean to say an Euler cycle induced by the triplet (P, λ, χ), for some suitably chosen $\lambda \in P^*$.

Theorem 7.3.3. *Continuing with the notations as in the Definition 7.3.1, furthermore assume that $(d-1)! \in R^*$. Then the assignment sending the pair (P, χ) to the element $e(P, \chi)$, as described above, is well defined.*

Proof Let $\mu : P \twoheadrightarrow J$ be another surjection where $J \subset R[T]$ is an ideal of height d. Let (J, ω_J) be obtained from (μ, χ). Then we need to show $(I, \omega_I) = (J, \omega_J)$ in $E(R[T])$.

Applying Lemma 2.2.6, get an ideal $K \subset R[T]$ such that K is co-maximal with I, J and there exists a surjection $\nu : (R[T])^d \twoheadrightarrow I \cap K$ such that $\nu \otimes R[T]/I = \omega_I$. Since I and K are co-maximal ν induces a local orientation ω_K of K and we have $(I, \omega_I) + (K, \omega_K) = 0$ in $E(R[T])$.

Let $L = K \cap J$, then again as before since K and J are co-maximal ω_K and ω_J together with L will induce a local orientation ω_L of L and we have $(L, \omega_L) = (K, \omega_K) + (J, \omega_J)$ in $E(R[T])$. Thus to prove the theorem it is enough to show that $(L, \omega_L) = 0$, that is by Theorem 6.1.1 it is enough to show that $(L \otimes R(T), \omega_L \otimes R(T)) = 0$ in $E^d(R(T))$.

Since $(d-1)! \in R^*$, $e(P \otimes R(T), \chi \otimes R(T))$ is well-defined in $E^d(R(T))$ (see [14], Section 4), hence the result follows. □

In the next theorem we will show that the vanishing of any Euler cycle in the Euler class group induced by the triplet (P, λ, χ) is sufficient for the projective module to have an unimodular element.

Theorem 7.3.4. *Let R be an affine algebra of dimension $d \geq 2$ and P be a projective $R[T]$-module with trivial determinant of rank d. Moreover assume that there exists a surjection $\lambda : P \twoheadrightarrow I$ and (λ, χ) induces an Euler cycle (I, ω_I). Suppose that $(I, \omega_I) = 0$ in $E^d(R[T])$. Then P has a unimodular element.*

Proof Recall that (λ, χ) induces (I, ω_I) means there exists an isomorphism $\phi : (R[T]/I)^d \overset{\sim}{\to} P/IP$, such that $\wedge^d \phi = \chi \otimes R[T]/I$ and $\omega = (\lambda \otimes R[T]/I) \circ \phi$.

Note that in a view of ([15], Theorem 3.4) to show that P has a unimodular element it is enough to show that $P \otimes R(T)$ has a unimodular element.

Since (I, ω_I) vanishes in $E(R[T])$ by Lemma 7.2.5 there exists a surjection $\theta : (R[T])^d \twoheadrightarrow I$, such that $\theta \otimes R[T]/I = \omega$.

Thus in the ring $R(T)$ we have:

(i) $\lambda \otimes R(T) : P \otimes R(T) \twoheadrightarrow IR(T)$;

(ii) $\theta \otimes R(T) : (R(T))^d \twoheadrightarrow IR(T)$;

(iii) $\phi : (R[T]/I)^d \overset{\sim}{\to} P/IP$ such that, $\omega \otimes R(T)/IR(T) = (\lambda \otimes R(T)/IR(T)) \circ (\phi \otimes R(T)/IR(T))$ and $\wedge^d(\phi \otimes R(T)/IR(T)) = \chi \otimes R(T)/IR(T)$.

Hence using ([14], Corollary 3.4) $P \otimes R(T)$ has a unimodular element and this completes the proof. $\square$

The same proof will give us the following corollary and we therefore omit the proof.

Corollary 7.3.5. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ with $\dim(R) = d \geq 2$. Let P and Q be projective $R[T]$-modules of rank d and $d-1$ respectively, such that their determinants are free. Let $\chi : \wedge^d P \cong \wedge^d(Q \oplus R[T])$ be an isomorphism. Let $I \subset R[T]$ be an ideal of height d such that there exists surjections $\alpha : P \twoheadrightarrow I$ and $\beta : Q \oplus R[T] \twoheadrightarrow I$. Let 'bar' denotes going modulo I. Suppose that there exists an isomorphism $\delta : \overline{P} \cong \overline{Q \oplus R[T]}$ with the following properties:*

(i) $\overline{\beta}\delta = \overline{\alpha}$;

(ii) $\wedge^d \delta = \overline{\chi}$.

Then P has a unimodular element.

The next theorem is a stronger version of Theorem 7.3.4 with an extra assumption on the ring R. Together with Theorem 7.3.3 it says that the assignments of (P, λ, χ) to an element

$e(P,\chi) \in E^d(R[T])$, is precisely the obstruction class for P to have a unimodular element. In this case we will call the Euler class group $E^d(R[T])$ is the obstruction group to detect the existence of a unimodular element in a projective $R[T]$-module (with trivial determinant) of rank equal to the dimension of R.

Theorem 7.3.6. *Let R be an affine algebra of dimension $d \geq 2$ (with an extra assumption $(d-1)! \in R^*$) and P be a projective $R[T]$-module with trivial determinant (via χ) of rank d. Then P has a unimodular element if and only if $e(P,\chi) = 0$ in $E^d(R[T])$.*

Proof Note that only remaining part is to prove the if part. So we may begin with the assumption that P has a unimodular element. Since $(d-1)! \in R^*$, by Theorem 7.3.3, $e(P,\chi)$ is well defined. Thus enough to show any $(I,\omega_I) \in E^d(R[T])$ which represents the same class as of $e(P,\chi)$ vanishes.

In a view of Theorem 6.1.1, it is enough to show that $(IR(T), \omega_I \otimes R(T)/IR(T))$ vanishes. But note that since $(d-1)! \in R(T)^*$, we can use ([14], Corollary 4.4) in the ring $R(T)$. Thus since $P \otimes R(T)$ has a unimodular element $(IR(T), \omega_I \otimes R(T)/IR(T)) = 0$ in $E(R(T))$, and this completes a proof. $\square$

For the next result we will sketch a proof to avoid repeating same arguments used earlier in this paper.

Theorem 7.3.7. *Let R be a $d(\geq 2)$-dimensional smooth affine algebra over $\overline{\mathbb{F}}_p$ (for $d \geq 3$ with an additional assumption that R is a domain) and $I \subset R[T]$ be an ideal such that $\mu(I/I^2) = ht(I) = d$. Then any set of generators of $I =< f_1, ..., f_d > +I^2$ can be lifted to a set of generators of I.*

Proof

Case -1 ($d = 2$) Following the same proof of (Theorem 6.1.1, case 1) we get P_1. Since R is smooth and $\dim(R) \leq 2$, P_1 is extended from R and hence free by ([21], Corollary 2.9) (as P_1 has trivial determinant). Then again we can follow the last paragraph of the proof of (Theorem 6.1.1, case 1) to conclude the result.

Case -2 ($d \geq 3$) Let $I =< f_1, ..., f_d > +I^2$. Then to prove the theorem it is enough to find a lift of f_i's to a set of generators of I. Note that $I(0) =< f_1(0), ..., f_d(0) > +I(0)^2$. By Theorem 2.2.5 there exists $b_i \in I(0)$ such that $I(0) =< b_1, ..., b_d >$ with $f_i(0) - b_i \in I(0)^2$. This b_i's together f_i's will induce a set if generators $I =< g_1, ..., g_d > +I^2T$, where $g_i(0) = b_i$

and $f_i - g_i \in I^2$, for $i = 1, ..., d$ (see [12], Remark 3.9). Now use ([12], Corollary 3.8) to complete the proof. $\square$

Corollary 7.3.8. *Let R be a $d(\geq 2)$-dimensional smooth affine algebra over $\overline{\mathbb{F}}_p$ (for $d \geq 3$ with an additional assumption that R is a domain) and P be a projective $R[T]$-module of rank d with trivial determinant, then P has a unimodular element.*

Proof Since by Theorem 7.3.7 any local orientation of an ideal I (with the property $\text{ht}(I) = \mu(I/I^2)$) can be lifted to a global orientation of I. Thus by Lemma 7.2.5, $E^d(R[T]) = 0$, hence in particular, any Euler cycle induced by the pair (P, χ) vanishes thus by Theorem 7.3.4, P has a unimodular element. $\square$

Remark 7.3.9 Corollary 7.3.8 easily follows from [46] since P is extended from R (as R being smooth). This corollary is an Euler class theoretic treatment of the same.

Chapter 8

A splitting criterion via projective generation of complete intersection curves over algebraically closed field of characteristic $\neq 2$

8.1 A cancellation result in dimension two

In this section we will prove a cancellation result in dimension 2 over some C_1 field of characteristic $\neq 2$. Note that if the ring is smooth this result is due to A. A. Suslin ([64], Theorem 2.4). Here we drop the smoothness assumption of A. A. Suslin's proof using a clever observation by P. Raman which is crucial in our set-up. The remaining part of this section is devoted to rediscover some of its consequences as a splitting criterion of projective modules.

Theorem 8.1.1. *Let k be an algebraically closed field of characteristic $p \neq 2$. Let R be a two dimensional affine algebra over the C_1 field $k(T)$, which is essentially of finite type over k. Then stably free modules over R are free. In other words, Um$_3(R) = e_1$SL$_3(R)$.*

Proof Let $R = S^{-1}A$, where A is affine algebra over k and S be a multiplicatively closed subset of A. Let $\mathfrak{I}_A$ and $\mathfrak{I}_R$ be the ideal of singular locus of A and R respectively. Then note that since k is perfect ht$(\mathfrak{I}_A) \geq 1$ and thus ht$(S^{-1}\mathfrak{I}_A) \geq 1$. Now also note that $S^{-1}\mathfrak{I}_A \subset \mathfrak{I}_R$. To show this it is enough to show for any $a = \frac{t}{s} \in S^{-1}\mathfrak{I}_A$, R_a is smooth. Put $R_a = (S^{-1}A)_{\frac{t}{s}} = S^{-1}(R_t)$. Now R_t is smooth as $t \in \mathfrak{I}_A$ and hence so is R_a. Thus we can always assume the ideal of singular locus $\mathfrak{I}_R$ of R has a positive height.

Let $(a_1, a_2, a_3) \in \mathrm{Um}_3(R)$. Let $B = R/\mathfrak{J}_R$ and 'bar' denote going modulo $\mathfrak{J}_R$. Then in the B we have got $\mathrm{Um}_3(B) = e_1 E_3(B)$. Since $E_3(R) \twoheadrightarrow E_3(B)$ is a surjection, we can always replace (a_1, a_2, a_3) by $(a_1, a_2, a_3)\epsilon$ for some $\epsilon \in E_3(R)$ and may assume $a_3 - 1 \in \mathfrak{J}_R$ and $a_1, a_2 \in \mathfrak{J}_R$. Note that $< a_1, a_2, a_3 >= R \not\subset \bigcup p$, where the union runs over all minimal prime ideals of R. Hence using Prime Avoidance Lemma replacing a_3 by $a_3 + \lambda_1 a_1 + \lambda_2 a_2$ we may also assume $a_3 \not\subset \bigcup p$ that is a_3 is a non-zero divisor, keeping the fact $a_3 - 1 \in \mathfrak{J}_R$ unaltered.

Let $C = R/ < a_3 >$. Then note that since $a_3 - 1 \in \mathfrak{J}_R$, C is a smooth curve. Let 'tilde' denote going modulo a_3. Since $p \neq 2$, by ([64], Proposition 1.4) $SK_1(C)$ is a $2-$divisible group and by ([64], Proposition 1.7) $SK_1(C) \cong K_1 Sp(C)$. Thus following A. A. Suslin's proof of ([64], Theorem 2.4) we have $[\tilde{a_1}, \tilde{a_2}] = [\tilde{b_1}^2, \tilde{b_2}]$ in $SK_1(C)$. Therefore we get $\alpha \in \mathrm{SL}_2(C) \cap E(C)$ such that $(\tilde{a_1}, \tilde{a_2})\alpha = (\tilde{b_1}^2, \tilde{b_2})$. Since $[\alpha] = 0$ in $SK_1(C)$ implies that $[\alpha] = 0$ in $K_1 Sp(C)$, thus we have $\alpha \in \mathrm{SL}_2(C) \cap Ep(C)$. Hence using ([64], Lemma 2.1) we get $\beta \in \mathrm{SL}_2(R) \cap Ep(R)$ such that $\alpha \equiv \beta \mod (< a_3 >)$. Thus $(a_1, a_2, a_3)(\beta \perp 1) = (b_1^2, b_2, a_3) \mod (E_3(R))$. Using a result of Swan-Towber (b_1^2, b_2, a_3) is completable and so is (a_1, a_2, a_3). $\square$

8.2 A splitting criterion

Let R be a $d(\geq 2)$ dimensional affine algebra over an algebraically closed field k. Using the arguments given in Section 7.3, one can define the d-th weak Euler class group $E_0^d(R[T])$ as defined in [19].

Theorem 8.2.1. *Let R be a $d(\geq 2)$ dimensional affine algebra over an algebraically closed field k of $char(k) = p \neq 2$. Then $E^d(R(T)) \cong E_0^d(R(T))$.*

Proof It is enough to show that for any ideal $I \subset R(T)$ with $\mathrm{ht}(I) = \mu(I/I^2) = \mu(I) = d$, and for any local orientation ω_I of I, $(I, \omega_I) = 0$ in $E^d(R(T))$.

Let $I =< a_1, ..., a_d >$ and ω_I be induced by $I =< f_1, ..., f_d > + I^2$. Replacing $(a_1, ..., a_d)$ by $(a_1, ..., a_d)\epsilon$ we may always assume $\dim(R(T)/ < a_3, ..., a_d >) \leq 2$ for some suitably chosen $\epsilon \in E_d(R(T))$. Let $B = R(T)/ < a_3, ..., a_d >$, then note that B is at-most a two dimensional affine algebra over a C_1 field $k(T)$, where k is algebraically closed of characteristic $p \neq 2$. Also B is essentially of finite type over k. Let 'bar' denote going modulo $< a_3, ..., a_d >$. As an $R(T)/I$-module two sets of generators of I/I^2 must differ by some invertible matrix $\alpha \in \mathrm{GL}_d(R(T)/I)$. Let $\det(\alpha) = a \in (R(T)/I)^*$. Get $b \in R(T)$ be such $ab - 1 \in I$. Then note that $(\overline{b}, \overline{a}_2, -\overline{a}_1) \in \mathrm{Um}_3(B) = e_1 \mathrm{SL}_3(B)$ (by 8.1.1). Using ([14], 5.2) get $\tau \in M_2(B)$ such that $(\overline{a}_1, \overline{a}_2)\tau = (\overline{a}'_1, \overline{a}'_2)$, where $\overline{I} =< \overline{a}'_1, \overline{a}'_2 >$ and $\det(\tau) - a \in I$.

Thus in the ring $R(T)$ we get, $I =< a_1', a_2', a_3, ..., a_d >$. Define $\theta = \tau \perp I_{d-2} \in \mathsf{GL}_d(R(T)/I)$ then note that $(a_1, a_2, a_3, ..., a_d)\theta = (a_1', a_2', a_3, ..., a_d)$ and $\det(\theta) - a \in I$. Since $\det(\theta) - \det(\alpha) \in I$, there exists $\epsilon' \in \mathsf{SL}_d(R(T)/I) = E_d(R(T)/I)$ such that $\theta\epsilon' = \alpha$. Since $\dim(R(T)/I) = 0$, the natural map $E_d(R(T)) \twoheadrightarrow \mathsf{SL}_d(R(T)/I) = E_d(R(T)/I)$ is surjective. Therefore we can lift ϵ' and get $\epsilon \in E_d(R(T))$ such they are equal modulo I. Let $(F_1, ..., F_d) = (a_1', a_2', a_3, ..., a_d)\epsilon$. Then note that $I =< F_1, ..., F_d >$. It only remains to show $F_i - f_i \in I^2$.

Consider any $d-$tuple $[(a_1, ..., a_d)]$ as a map $(R[T]/I)^d \to I/I^2$ sending $e_i \to \overline{a}_i$. Then we have $[(F_1, ..., F_d)] = [(a_1', a_2', a_3, ..., a_d)\epsilon] = [(a_1, a_2, a_3, ..., a_d)\theta\epsilon] = [(a_1, ..., a_d)\theta\epsilon'] = [(a_1, ..., a_d)\alpha] = [(f_1, ..., f_d)]$. This completes the proof. $\square$

Theorem 8.2.2. *Let R be a $d(\geq 2)$ dimensional affine algebra over an algebraically closed field k of $char(k) = p \neq 2$. Let P be a projective $R[T]$-module of rank d with trivial determinant and $I \subset R[T]$ be an ideal of height d such that there is a surjection $\phi : P \twoheadrightarrow I$. If $\mu(I) = d$ then P has a unimodular element.*

Proof Note that if I contains a monic polynomial $f \in R[T]$, then P_f has a unimodular element via the map $\phi \otimes A[T]_f$. Thus using ([15], Theorem 3.4) P has a unimodular element. Thus we may always assume I does not contain any monic polynomial.

Let $IR(T)$ be the extension of the ideal I in the ring $R(T)$. Therefore in the ring $R(T)$ we have $\mathsf{ht}(IR(T)) = \mu(IR(T)) = \mu(IR(T)/I^2R(T)) = d$. Thus using Theorem 8.2.1, for any local orientation ω_I of $IR(T)$, we must have $(IR(T), \omega_I) = 0$. Let the surjection ϕ induce a local orientation ω of I. Since $(IR(T), \omega \otimes R(T)) = 0$ in $E^d(R(T))$, the projective $R(T)-$module $P \otimes R(T)$ has a unimodular element. We obtain a monic polynomial $f \in R[T]$, such that P_f has a unimodular element and therefore $\mathsf{Um}(P) \neq \phi$ by ([15],Theorem 3.4). $\square$

Remark 8.2.3 Note that for $char(k) = 0$, Theorem 8.1.1 holds without any restrictions (see [55], Proposition 3.1). Thus the same proofs of Theorem 8.2.1 and Theorem 8.2.2 will go through as well in the characteristic 0 setup, which drops the domain assumption in ([15], Theorem 4.5).

Corollary 8.2.4. *Let R be a $d(\geq 2)$dimensional affine algebra over algebraically closed field k of characteristic $\neq 2$ and P be a stably free $R[T]$-module of rank d. Then P has a unimodular element.*

Proof Note that in view of Theorem 8.2.2 and Remark 8.2.3 it is enough to show that P maps surjectively onto an ideal $I \subset R[T]$, such that $\mathsf{ht}(I) = \mu(I) = d$. Since P is stably free

$R[T]$-module of rank d, by [51], $R[T] \oplus P \cong (R[T])^{d+1}$. We have the following short exact sequence

$$0 \longrightarrow (R[T])^d \longrightarrow R[T] \oplus P \xrightarrow{(a,-\alpha)} R[T] \longrightarrow 0 \ .$$

Using Theorem 2.1.4, we may replace α with $\alpha + a\beta$, for some $\beta \in P^*$, and assume $\text{ht}(\alpha(P)) = d$ or $\alpha(P) = R[T]$. Note that by this replacement of α, the kernel remain unaltered. If $\alpha(P) = R[T]$, then this proves the theorem. So let us assume that $\alpha(P) = I \subset R[T]$ be a proper ideal of height d. Then by ([14], Lemma 2.8(i)) we get $\mu(I) = d$. □

8.3 A necessary and sufficient condition for splitting of a projective module on polynomial algebras over $\overline{\mathbb{F}}_p$

This section is a sequel of the previous section. Here we have shown that we can strengthen the results of the previous section whenever the base field is $\overline{\mathbb{F}}_p$, with the assumption $p \neq 2$. We begin with the following remark.

Remark 8.3.1 Note that following the arguments given in ([19], Section 6) and using Theorem 6.1.1 one can define the $d-$th weak Euler class group $E_0^d(R[T])$ without the assumption that ring contains the field of rationals.

Theorem 8.3.2. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$. Further assume $p \neq 2$. Then $E^d(R[T]) \cong E_0^d(R[T])$.*

Proof Let $I \subset R[T]$ be an ideal such that $\mu(I) = \text{ht}(I) = d$. Note that since the canonical map $E(R[T]) \to E_0(R[T])$ is surjective it is enough to show any local orientation $I =< f_1, ..., f_d > +I^2$ has a lift.

Note that for I containing a monic polynomial it follows from Theorem 2.3.10 so without loss of generality we may assume I does not contain a monic polynomial, i.e. $\text{ht}(I) = \text{ht}(IR(T))$. By Theorem 8.2.1 we have $E^d(R(T)) \cong E_0^d(R(T))$, thus there exists $F_i \in IR(T)$ such that $IR(T) =< F_1, ..., F_d >$, with $f_i - F_i \in IR(T)^2$. Now use Theorem 6.1.1, to get a lift of f_i's. □

Theorem 8.3.3. *Let R be a $d(\geq 2)$ dimensional affine algebra over $\overline{\mathbb{F}}_p$ with $(d-1)! \in R^*$. Let P be a projective $R[T]$-module with trivial determinant of rank d and $I \subset R[T]$ be an ideal of height d such that there is a surjection $\phi : P \twoheadrightarrow I$. Then P has a unimodular element if and only if $\mu(I) = d$.*

Proof Note that if $\mu(I) = d$, then by Theorem 8.2.2 P has a unimodular element (even for $d = 2$ and without P having trivial determinant). We now assume that P has a unimodular element. Note that again as before we may always assume I does not contain any monic polynomial. Since P has a unimodular element so does $P \otimes R(T)$. Let ω be the local orientation of I induced by ϕ and an isomorphism $\chi : \wedge^d P \cong R[T]$. Note that in the ring $R(T)$ we have $e(PR(T), \chi \otimes R(T)) = (IR(T), \omega \otimes R(T))$ in the group $E^d(R(T))$. Since $P \otimes R(T)$ has a unimodular element we have $\mu(I \otimes R(T)) = d$, thus using Theorem 8.2.1 we have $(IR(T), \omega \otimes R(T)) = 0$. Now use Theorem 6.1.1 to complete the proof. $\square$

Chapter 9

On Laurent polynomial algebras

9.1 On two conjectures

The purpose of this section is to improve the bounds of a similar question asked by M. P. Murthy and M. V. Nori, on Laurent polynomial algebras over $\overline{\mathbb{F}}_p$. Before we jump into the main theorems we begin with the following definitions.

Definition 9.1.1

(i) A Laurent polynomial f in $A[X, X^{-1}]$ is said to be doubly monic if the coefficients of the highest and the lowest degree terms are units.

(ii) A polynomial f in $A[X]$ is said to be special monic if f is monic and the constant term is 1.

Theorem 9.1.2. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ and $I \subset R[T, T^{-1}]$ be an ideal containing a doubly monic Laurent polynomial. Moreover assume $I =< f_1, ..., f_n > +I^2$, with $n \geq \max\{(\dim R[T, T^{-1}]/I + 1), 2\}$. Then there exists $g_i \in I$, for $i = 1, ..., n$, such that $I = (g_1, ..., g_n)$, with $g_i - f_i \in I^2$.*

Proof Let $J = I \cap R[T]$. Define $h_i = T^k f_i \in R[T]$, for all $i = 1, ..., n$. Since the image of the matrix $\operatorname{diag}(\frac{1}{T^k}, \frac{1}{T^k}, ..., \frac{1}{T^k})$ is in $\mathrm{GL}_n(R[T, T^{-1}]/I^2)$ we have $I =< h_1, ..., h_n > +I^2$.

We claim that $J =< h_1, ..., h_n > +J^2$. Note that to prove the claim it is enough show for all prime ideals $p \subset R[T]$, we have $JR[T]_p =< h_1, ..., h_n > R[T]_p + J^2 R[T]_p$. Since I contains a doubly monic Laurent polynomial, this gives us the fact J contains a special monic polynomial say h. Then by replacing h_1 with $h_1 + h^{k'}(T - h_1(0) + 1)$, for some suitably chosen $k' > 1$ we may further assume h_1 is a special monic without changing the assumption $\frac{h_1}{T^k} - f_1 \in I^2$. If $T \in p$

then $h_1 \notin p$, so in this case we have $JR[T]_p = R[T]_p =< h_1, ..., h_n > R[T]_p + J^2R[T]_p$. And if $T \notin p$ then this gives us $JR[T]_p = IR[T, T^{-1}]_{pR[T,T^{-1}]} =< h_1, ..., h_n > R[T, T^{-1}]_{pR[T,T^{-1}]} + I^2R[T, T^{-1}]_{pR[T,T^{-1}]} =< h_1, ..., h_n > R[T]_p + J^2R[T]_p$. Hence this establishes our claim.

Since I contains a doubly monic Laurent polynomial, we have $\dim(R[T]/J) = \dim(R[T, T^{-1}]/I)$. Thus by Theorem 2.3.10 we can find $G_i \in J$, such that $J =< G_1, ..., G_n >$ with $G_i - h_i \in J^2$, for $i = 1, ..., n$. Now $I = JR[T, T^{-1}]$, gives us the fact $I =< G_1, ..., G_n >$. Define $g_i = \frac{G_i}{T^k}$, then note that $I =< g_1, ..., g_n >$, and $g_i - f_i = \frac{G_i - h_i}{T^k} \in J^2R[T, T^{-1}] = I^2$. This completes the proof. □

Theorem 9.1.3. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ and $I \subset R[T, T^{-1}]$ be an ideal containing a doubly monic Laurent polynomial. Moreover assume that $I =< f_1, ..., f_n > +(I^2(T - 1))$, with $n \geq \max\{(\dim R[T, T^{-1}]/I + 1), 2\}$. Then there exist $g_i \in I$, for $i = 1, ..., n$, such that $I = (g_1, ..., g_n)$, with $g_i - f_i \in I^2(T - 1)$.*

Proof Let $J = I \cap R[T]$. Since I is extended from $R[T]$, we have $J(1) = I(1) =< f_1(1), ..., f_n(1) >$. We can find a suitable k such that $h_i = T^k f_i \in R[T]$, for all $i = 1, ..., n$. Then note that we have $J(1) = I(1) =< h_1(1), ..., h_n(1) >$ and $I =< h_1, ..., h_n > +I^2(T-1)$.

We claim that $J =< h_1, ..., h_n > +J^2(T - 1)$. Note that it is enough show for all prime ideals $p \subset R[T]$, we have $JR[T]_p =< h_1, ..., h_n > R[T]_p + J^2(T-1)R[T]_p$. Since I contains a doubly monic Laurent polynomial, this gives us the fact J contains a special monic polynomial say h. Then by replacing h_1 with $h_1 + h^2(T-1)^{k'} + h_1h(T-1)$, for some suitably chosen $k' > 0$ we may further assume h_1 is special monic without changing the assumption $\frac{h_1}{T^k} - f_1 \in I^2(T-1)$. If $T \in p$ then $h_1 \notin p$, so in this case we have $JR[T]_p = R[T]_p =< h_1, ..., h_n > R[T]_p + J^2(T-1)R[T]_p$. And if $T \notin p$ then this gives us $JR[T]_p = IR[T, T^{-1}]_{pR[T,T^{-1}]} =< h_1, ..., h_n > R[T, T^{-1}]_{pR[T,T^{-1}]} + I^2(T - 1)R[T, T^{-1}]_{pR[T,T^{-1}]} =< h_1, ..., h_n > R[T]_p + J^2(T - 1)R[T]_p$. Hence this completes the proof of our claim.

Since I contains a doubly monic Laurent polynomial, we have $\dim(R[T]/J) = \dim(R[T, T^{-1}]/I)$. Therefore, by Theorem 3.2.2 we can find $G_i \in J$, such that $J =< G_1, ..., G_n > R[T, T^{-1}]$ with $G_i - h_i \in J^2(T - 1)$, for $i = 1, ..., n$. As $I = JR[T, T^{-1}]$, we have $I =< G_1, ..., G_n >$. Define $g_i = \frac{G_i}{T^k}$. Then we note that $I =< g_1, ..., g_n >$, and $g_i - f_i = \frac{G_i - h_i}{T^k} \in J^2(T - 1)R[T, T^{-1}] = I^2(T - 1)$. This completes the proof. □

9.2 An obstruction group and obstruction class

In this section we shall develop the theory of an obstruction group on the Laurent polynomial algebras, which can governed the splitting problem. The philosophy is to develop the theory parallelly with the theory we developed for the polynomial $\overline{\mathbb{F}}_p-$algebras, which was absent in the literature even when the ground field is algebraically closed.

Notation 9.2.1. Let R be a ring. We denote $\mathfrak{R} = S^{-1}R[T,T^{-1}]$, where $S \subset R[T,T^{-1}]$ is the multiplicatively closed set consisting all doubly monic Laurent polynomials.

Theorem 9.2.2. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$ and $I \subset R[T,T^{-1}]$ be an ideal such that $ht(I) = \mu(I/I^2) = d$. Moreover assume that $I =< f_1, ..., f_d > +I^2$. Suppose that there exists $F_i \in I\mathfrak{R}$ be such that $I\mathfrak{R} =< F_1, ..., F_d >$, with $F_i - f_i \in (I\mathfrak{R})^2$. Then there exists $g_i \in I$ be such that $I =< g_1, ..., g_d >$, where $g_i - f_i \in I^2$.*

Proof Following the proof of Theorem 6.1.1 one can establish the fact that it is enough to assume that I contains a doubly monic Laurent polynomial. Then using Theorem 9.1.2 the proof completes. $\square$

We are now stating results on Laurent polynomial rings without proofs. The proofs follow the same arguments as in the case of polynomial algebras (see Section 7.1), and are hence omitted.

Proposition 9.2.3. *(Addition principle) Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$. Let $I, J \subset R[T,T^{-1}]$ be two co-maximal ideals, each of height d. Suppose that $I = (f_1, ..., f_d)$ and $J = (g_1, ..., g_d)$. Then $I \cap J = (h_1, ..., h_d)$ where $h_i - f_i \in I^2$ and $h_i - g_i \in J^2$.*

Proposition 9.2.4. *(Subtraction principle) Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$. Let $I, J \subset R[T,T^{-1}]$ be two co-maximal ideals, each of height d. Suppose that $I = (f_1, ..., f_d)$ and $I \cap J = (h_1, ..., h_d)$ where $h_i - f_i \in I^2$. Then there exists $g_i \in J$ such that $J = (g_1, ..., g_d)$ with $h_i - g_i \in J^2$.*

Mimicking the same proof given in Proposition 7.2.1 one can establish the following proposition.

Proposition 9.2.5. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$ and $I \subset R[T,T^{-1}]$ be an ideal of height d. Moreover suppose that α and β are two surjections from $(R[T,T^{-1}]/I)^d \to$ $\to I/I^2$ such that there exists $\sigma \in SL_d(R[T,T^{-1}]/I)$ with the property that $\alpha\sigma = \beta$. If α can be lifted to a surjection $\theta : (R[T,T^{-1}])^d \twoheadrightarrow I$ then β can also be lifted to a surjection.*

Definition 9.2.6 Let $I \subset R[T,T^{-1}]$ be an ideal of height d such that I/I^2 is generated by d elements. Let α and β be two surjections from $(R[T,T^{-1}]/I)^d \twoheadrightarrow I/I^2$. We say that α and β are related if there exists $\sigma \in \mathrm{SL}_d(R[T,T^{-1}]/I)$ be such that $\alpha\sigma = \beta$. This defines an equivalence relation on the set of surjections from $(R[T,T^{-1}]/I)^d \twoheadrightarrow I/I^2$. Let $[\alpha]$ denote the equivalance class of α. If $f_1,...,f_d$ generate I/I^2 , we obtain a surjection $\alpha : (R[T,T^{-1}]/I)^d \twoheadrightarrow I/I^2$, sending e_i to f_i. We say $[\alpha]$ is given by the set of generators $f_1,...,f_d$ of I/I^2.

Let G be the free abelian group on the set B of pairs (I,ω_J), where:

(i) $I \subset R[T,T^{-1}]$ is an ideal of height d,

(ii) Spec$(R[T,T^{-1}]/I)$ is connected,

(iii) I/I^2 is generated by d elements, and

(iv) $\omega_I : (R[T,T^{-1}]/I)^d \twoheadrightarrow I/I^2$ is an equivalence class of surjections $\alpha : (R[T,T^{-1}]/I)^d \twoheadrightarrow I/I^2$.

Let $J \subset R[T,T^{-1}]$ be a proper ideal. Get $J_i \subset R[T,T^{-1}]$ such that $J = J_1 \cap J_2 \cap ... \cap J_r$, where J_i 's are proper, pairwise co-maximal and Spec$(R[T,T^{-1}]/J_i)$ is connected. We shall say that J_i are the connected components of J.

Let $K \subset R[T,T^{-1}]$ be an ideal of height d, K/K^2 is generated by d elements and $K = \cap K_i$ be the decomposition of K into its connected components. Then note that for every i, ht$(K_i) = d$ and by Chinese Remainder Theorem K_i/K_i^2 is generated by d elements. Let $\omega_K : (R[T,T^{-1}]/I)^d \twoheadrightarrow K/K^2$ be a surjection. Then in a natural way ω_K gives rise to surjections $\omega_{K_i} : (R[T,T^{-1}]/K_i)^d \twoheadrightarrow K_i/{K_i}^2$. We associate the pair (K,ω_K), to the element $\sum_{i=1}^r (K_i,\omega_{K_i})$ of G. We will call it (K,ω_K) a local orientation of K induced by ω_K.

Let H be the subgroup of G generated by the set S of pairs (J,ω_J), where $\omega_J : (R[T,T^{-1}]/J)^d \twoheadrightarrow J/J^2$ has a surjective lift $\theta : (R[T,T^{-1}]/J)^d \twoheadrightarrow J$. Then we define the quotient group G/H as the d-th Euler class group of $R[T,T^{-1}]$ denoted as $E^d(R[T,T^{-1}])$. A local orientation (J,ω_J) is said to be a global orientation if ω_J lifts to a set of generators of J. With a slight abuse of notation we might sometimes use $E(R[T,T^{-1}])$ instead of $E^d(R[T,T^{-1}])$ in this section.

Lemma 9.2.7. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$, $I \subset R[T,T^{-1}]$ be an ideal of height d such that I/I^2 is generated by d elements. Let $\omega_I : (R[T,T^{-1}]/I)^d \twoheadrightarrow I/I^2$ be a local orientation of I. Suppose that the image of (I,ω_I) is zero in the Euler class group $E(R[T,T^{-1}])$. Then, I is generated by d elements and ω_I can be lifted to a surjection $\theta : (R[T,T^{-1}])^d \twoheadrightarrow I$.*

Proof The proof is done in Theorem 4.2.2 for $d \geq 3$ and in ([21], Theorem 4.7) for $d = 2$. □

Definition 9.2.8 (Local orientation induced by a Projective module) Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$ and P be a projective $R[T,T^{-1}]$-module of rank d having trivial determinant. Let $\chi : R[T,T^{-1}] \cong \wedge^d P$ be an isomorphism. To the pair (P,χ), we associate an element $e(P,\chi)$ of $E(R[T,T^{-1}])$ as follows:

Let $\lambda : P \twoheadrightarrow I$ be a surjection, where I is an ideal of $R[T,T^{-1}]$ of height d. Let 'bar' denotes going modulo I. We obtain an induced surjection $\lambda \otimes R[T,T^{-1}]/I : P/IP \twoheadrightarrow I/I^2$. Note that, since P has trivial determinant and $\dim(R[T,T^{-1}]/I) \leq 1$, P/IP is a free $R[T,T^{-1}]/I$-module of rank d. We choose an isomorphism $\phi : (R[T,T^{-1}]/I)^d \xrightarrow{\sim} P/IP$, such that $\wedge^d \phi = \chi \otimes R[T,T^{-1}]/I$. Let ω_I be the surjection $(\lambda \otimes R[T,T^{-1}]/I) \circ \phi : (R[T,T^{-1}]/I)^d \twoheadrightarrow I/I^2$. We say that (I,ω_I) is a local orientation induced by P together with the pair (λ,χ).

Whenever the class of (I,ω_I) in $E(R[T,T^{-1}])$ induced by the pair (P,χ) becomes independent of a certain choice of an ideal I we will define the Euler class $e(P,\chi)$ of the pair (P,χ) as the image of (I,ω_I) in $E(R[T,T^{-1}])$.

With the above results and definitions in hand, one easily obtains the following series of results. The proofs can be mimicked from the case of polynomial algebra as described in the previous sections. We decided not to repeat the arguments.

Theorem 9.2.9. *Continuing with the notations as in the Definition 9.2.8, furthermore assume $(d-1)! \in R^*$, then the assignment sending the pair (P,χ) to the element e(P,χ), as described above, is well defined.*

Theorem 9.2.10. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$ and P be a projective $R[T,T^{-1}]$-module with trivial determinant of rank d. Moreover assume that there exists a surjection $\lambda : P \twoheadrightarrow I$ and (λ,χ) induces a local orientation (I,ω_I). Suppose that $(I,\omega_I) = 0$ in $E(R[T,T^{-1}])$ then P has a unimodular element.*

Corollary 9.2.11. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ with $\dim(R) = d \geq 2$. Let P and Q be projective $R[T,T^{-1}]$-modules of rank d and $d-1$ respectively, such that their determinants are free. Let $\chi : \wedge^d P \cong \wedge^d(Q \oplus R[T,T^{-1}])$ be an isomorphism. Let $I \subset R[T,T^{-1}]$ be an ideal of height d such that there exists surjections $\alpha : P \twoheadrightarrow I$ and $\beta : Q \oplus R[T,T^{-1}] \twoheadrightarrow I$. Let 'bar' denotes going modulo I. Suppose that there exists an isomorphism $\delta : \overline{P} \cong \overline{Q \oplus R[T,T^{-1}]}$ with the following properties:*

(i) $\overline{\beta}\delta = \overline{\alpha}$;

(ii) $\wedge^d \delta = \overline{\chi}$.

Then P has a unimodular element.

Theorem 9.2.12. *Let R be an affine algebra of dimension $d \geq 2$ (with an extra assumption $(d-1)! \in R^*$) and P be a projective $R[T,T^{-1}]$-module with trivial determinant (via χ) of rank d. Then P has a unimodular element if and only if $e(P,\chi) = 0$ in $E^d(R[T,T^{-1}])$.*

Theorem 9.2.13. *Let R be a $d(\geq 2)$ dimensional affine algebra over an algebraically closed field k of $char(k) = p \neq 2$. Recall that $\mathfrak{R}$ is as defined in the Notation 9.2.1. Then $E^d(\mathfrak{R}) \cong E_0^d(\mathfrak{R})$.*

Theorem 9.2.14. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$. Further assume $p \neq 2$. Then $E^d(R[T, T^{-1}]) \cong E_0^d(R[T, T^{-1}])$.*

Chapter 10

Segre class of an ideal

10.1 On polynomial algebras over $\overline{\mathbb{F}}_p$

The purpose of this section is to weaken the hypothesis on the height of the ideal of the Theorem 8.3.3. As an application we develop the idea of Segre class of an ideal in polynomial algebra over $\overline{\mathbb{F}}_p$.

Theorem 10.1.1. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$ and $I \subset R[T]$ be an ideal such that ht$(I) \geq 2$ and $\mu(I/I^2) = d$. Moreover assume that $I =< f_1, ..., f_d > +I^2$. Suppose that there exists $F_i \in IR(T)$ be such that $IR(T) =< F_1, ..., F_d >$, with $F_i - f_i \in IR(T)^2$. Then there exists $g_i \in I$ be such that $I =< g_1, ..., g_d >$, where $g_i - f_i \in I^2$.*

Proof Note that for $d = 2$, proof follows from Theorem 6.1.1 thus without loss of generality we may assume that $d \geq 3$. By Lemma 2.1.3 there exists $e \in I^2$ such that $e(1-e) \in < f_1, ..., f_d >$. Replacing f_i by $f_i + e\lambda_i$ (without changing it's notations) using Theorem 2.1.4 we may assume that ht$(< f_1, ..., f_d >_e) \geq d$. Let $J =< f_1, ..., f_d, 1 - e >$. Then we have $J + I^2 = R[T]$, ht$(J) \geq d$, $J =< f_1, ..., f_d > +J^2$ and $I \cap J =< f_1, ..., f_d >$. Note that if ht$(J) > d$ then we are done, thus enough to assume that ht$(J) = d$.

Now in the ring $R(T)$, we have $IR(T) =< F_1, ..., F_d >$, with $F_i - f_i \in IR(T)^2$. Note that ht$(IR(T)) \geq$ ht$(I) \geq 2$ and ht$(JR(T)) \geq$ ht$(J) = d \geq 3$, thus by ([22], Proposition 2.2), there exists $G_i \in JR(T)$ such that $G_i - f_i \in JR(T)^2$. Using Theorem 6.1.1 get $h_i \in J$ such that $J =< h_1, ..., h_d >$, with $h_i - f_i \in J^2$. Thus in the ring $A[T]$ we get

(i) ht$(I) \geq 2$ and ht$(J) = d \geq 3$;

(ii) $I^2 + J = R[T]$;

(iii) $I \cap J =< f_1, ..., f_d >$;

(iv) $J =< h_1, ..., h_d >$, with $f_i - h_i \in J^2$.

Let $B = R[T]/I^2$ and 'bar' denotes going modulo I^2. Since B is an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $\leq d-1$ we have $(\overline{h}_1, ..., \overline{h}_d) \in \text{Um}_d(B) = e_1 E_d(B)$, whenever $d \geq 3$. Thus we can replace $(h_1, ..., h_d)$ by $(h_1, ..., h_d)\epsilon$ (without changing it's notations), for some $\epsilon \in E_d(R[T])$ to assume $h_d - 1 \in I^2$ and $h_i \in I^2$ for all $i < d$. Again note that by ([60], Lemma 2) we can assume $\text{ht}(< h_1, ..., h_{d-1} >) \geq d-1$, without losing the assumption $h_d - 1 \in I^2$.

Let $A = R[T, X]$, $K_1 =< h_1, ..., h_{d-1}, h_d + X > A$, $K_2 = IA$ and $K_3 = K_1 \cap K_2$. Then note that it is enough to show $K_3 =< a_1(X), ..., a_d(X) >$ such that $a_i(0) = f_i$. As then specializing K_3 at $h_d - 1$ completes the proof.

To prove this note that $\dim(A/K_1) = \dim(R[T]/ < h_1, ..., h_{d-1} >) \leq 2$. Hence for $d \geq 3$ using Theorem 3.3.1 we can find a required set of generators of K_3 which matches with f_i's at $X = 0$. □

The next two corollaries are generalizations of Proposition 7.1.1 and Proposition 7.1.2 in which we relax the hypothesis on the height of the ideals. The proof essentially uses the same arguments as of Proposition 7.1.1 and Proposition 7.1.2. One just have to use Theorem 10.1.1 and ([22], Proposition 2.1, Proposition 2.2) instead of Theorem 6.1.1 and ([14], Theorem 3.2 and Theorem 3.3) in the appropriate places. For the sake of completeness we sketch the proofs.

Corollary 10.1.2. *(Addition principle) Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$. Let $I, J \subset R[T]$ be two co-maximal ideals, each of height ≥ 2. Suppose that $I = (f_1, ..., f_d)$ and $J = (g_1, ..., g_d)$. Then $I \cap J = (h_1, ..., h_d)$ where $h_i - f_i \in I^2$ and $h_i - g_i \in J^2$.*

Proof Since the heights of both the ideas I and J are ≥ 2, in the ring $R(T)$ both the ideals $IR(T)$ and $JR(T)$ are of heights ≥ 2. Since $I + J = R[T]$, using the Chinese Remainder Theorem we have $I \cap J/(I \cap J)^2 \cong I/I^2 \oplus J/J^2$. Hence the given set of generators of I and J will induce a set of generators a_i's of $(I \cap J)/(I \cap J)^2$ such that $a_i - f_i \in I^2$ and $a_i - g_i \in J^2$. Thus to prove the theorem it is enough to find a lift of $I \cap J =< a_1, ..., a_d > +(I \cap J)^2$ to a set of generators of $I \cap J$.

In the ring $R(T)$, we have $\text{ht}((I \cap J)R(T)) \geq 2$. Hence using ([22], Proposition 2.1) we can find $H_i \in (I \cap J)R(T)$ such that $(I \cap J)R(T) =< H_1, ..., H_d > R(T)$, with $H_i - f_i \in IR(T)^2$ and $H_i - g_i \in JR(T)^2$. Now use Theorem 10.1.1 to conclude the proof. □

Corollary 10.1.3. *(Subtraction principle) Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$. Let $I, J \subset R[T]$ be two co-maximal ideals, each of height ≥ 2. Suppose that $I = (f_1, ..., f_d)$ and $I \cap J = (h_1, ..., h_d)$ where $h_i - f_i \in I^2$. Then there exists $g_i \in J$ such that $J = (g_1, ..., g_d)$ with $h_i - g_i \in J^2$.*

Proof The proof uses the same arguments as in Proposition 10.1.2 with slight modification, so we will only sketch a proof. As before, we have ht$(IR(T))$, ht$(JR(T))$ and ht$((I \cap J)R(T)) \geq 2$. Since $I + J = R[T]$, we get $J =< h_1, ..., h_d > +J^2$. Again observe that to prove the theorem it is enough to find a lift of $J =< h_1, ..., h_d > +J^2$ to a set of generators of J. Now using ([22], Proposition 2.2) in the ring $R(T)$ we can find $G_i \in JR(T)$ such that $G_i - h_i \in JR(T)^2$. Then as before we use Theorem 10.1.1 to complete the proof. $\square$

Definition 10.1.4 (Segre class of an ideal) Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$. Let $I \subset R[T]$ be an ideal of height ≥ 2 such that $\mu(I/I^2) = d$. Let $\omega_I : (R[T]/I)^d \twoheadrightarrow I/I^2$ be a surjection which induces $I =< f_1, ..., f_d > +I^2$. Using Lemma 2.2.6 get $J \subset R[T]$ of either height d or $J = R[T]$ such that $I \cap J =< g_1, ..., g_d >$, with $f_i - g_i \in I^2$. Let $\omega_J : (R[T]/J)^d \twoheadrightarrow J/J^2$ be a map sending $e_i \to g_i$. Then we define $s(I, \omega_I) = -(J, \omega_J) \in E^d(R[T])$ whenever ht$(J) = d$ and $s(I, \omega_I) = 0$ whenever $J = R[T]$.

The next theorem says the above definition of Segre class is well-defined. The proof is exactly the same as of ([22], Proposition 3.2), just one needs to apply Lemma 2.2.6, Corollary 10.1.2 and Corollary 10.1.3 in the appropriate places.

Proposition 10.1.5. *The Segre class of (I, ω_I) as described above, is well defined.*

Proof Note that following the arguments of ([22], Proposition 3.2) we may assume all the ideals which are going to appear are proper ideals. With continuing the notation of the definition 10.1.4, suppose that $(J', \omega_{J'})$ be another pair with the properties (i) ht$(J') = d$, (ii) $I + J' = R[T]$ and (iii) $I \cap J' =< g_1, ..., g_d >$, with $f_i - g_i \in I^2$. Let $\omega_{J'}$ is induced by the set of generators $J' =< g_1, ..., g_d > +J'^2$. Thus we have to show $(J, \omega_J) = (J', \omega_{J'})$ in $E^d(R[T])$.

Using Lemma 2.2.6, get $K \subset R[T]$ of height d and a local orientation ω_K such that K is co-maximal with $I \cap J \cap J'$ and $(J, \omega_J) + (K, \omega_K) = 0$ in $E^d(R[T])$. Thus enough to prove that $(J', \omega_{J'}) + (K, \omega_K) = 0$ in $E^d(R[T])$.

Using Lemma 2.2.6, get $L \subset R[T]$ of height d such that L is co-maximal with $I \cap J \cap J' \cap K$ and $L \cap I$ is generated d many elements.

Now note that to prove $(J', \omega_{J'}) + (K, \omega_K) = 0$ we need to show $J' \cap K$ is generated by d many elements which is compatible with both $\omega_{J'}$ and ω_K. Note that by our choice we have $J \cap K + I \cap L = R[T]$. Using Corollary 10.1.2 we get $J \cap K \cap I \cap L$ is generated by the appropriate set of generators. Since $J \cap I$ is generated by d elements, by the subtraction principle (Corollary 10.1.3) it follows that $K \cap L$ is generated by d elements with appropriate set of generators.

Since $I \cap J'$ and $K \cap L$ are both generated by d elements and they are co-maximal, by the addition principle (Corollary 10.1.2) $J' \cap K \cap I \cap L$ is generated by d elements with appropriate set of generators. Since $I \cap L$ is generated by d many elements using subtraction principle (Corollary 10.1.3) $J' \cap K$ is generated by appropriate set of generators. Keeping track of the generators, it follows that $(J', \omega_{J'}) + (K, \omega_K) = 0$ in $E^d(R[T])$. This completes the proof. □

Theorem 10.1.6. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$. Let $I \subset R[T]$ be an ideal of height ≥ 2. Let $\omega_I : (R[T]/I)^d \twoheadrightarrow I/I^2$ be surjection. Suppose that $s(I, \omega_I) = 0$ then ω_I can be lifted to a surjection $(R[T])^d \twoheadrightarrow I$.*

Proof Let ω_I induces $I =< f_1, ..., f_d > +I^2$. By the definition of $s(I, \omega_I)$ there exists an ideal $J \subset R[T]$ of either height d or $J = R[T]$ such that $I \cap J =< g_1, ..., g_d >$, with $f_i - g_i \in I^2$. Let $\omega_J : (R[T]/J)^d \twoheadrightarrow J/J^2$ be a map sending $e_i \to g_i$. Now if $J = R[T]$, then the theorem follows thus with out loss of generality we may assume ht$(J) = d$. Thus $s(I, \omega_I) = 0$ gives us $(J, \omega_J) = 0$ in $E^d(R[T])$. By Lemma 7.2.5, we can get $a_i \in J$ such that $J =< a_1, ..., a_d >$, with $a_i - g_i \in J^2$. Now use Corollary 10.1.3, to get a surjective lift of ω_I. □

Remark 10.1.7 Note that the converse of the above theorem is also true. As if ω_I is a global orientation then using the subtraction principle (Corollary 10.1.3) the result follows.

The following theorem is on the additivity of the Segre classes.

Theorem 10.1.8. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$ and $I_1, I_2 \subset R[T]$ be tow co-maximal ideals of height ≥ 2. Suppose that there exists surjections $\omega_{I_i} : (R[T]/I_i)^d \twoheadrightarrow I_i/I_i^2$, for $i = 1, 2$. Then we have $s(I_1 \cap I_2, \omega_{I_1 \cap I_2}) = s(I_1, \omega_{I_1}) + s(I_2, \omega_{I_2})$.*

Proof Suppose that ω_{I_1} is induced by $I_1 =< f_1, ..., f_d > +I_1^2$ and ω_{I_2} is induced by $I_2 =< g_1, ..., g_d > +I_2^2$. Note that by the definition using Lemma 2.2.6 we can choose pairs (J_1, ω_{J_1}) and (J_2, ω_{J_2}) such that J_1 is co-maximal with $I_1 \cap I_2$ and J_2 is co-maximal with $I_1 \cap I_2 \cap J_1$ such that $s(I_i, \omega_{I_i}) = (J_i, \omega_{J_i})$, for $i = 1, 2$.

Now since $J_1 + J_2 = R[T]$, we have $(J_1 \cap J_2, \omega_{J_1 \cap J_2}) = (J_1, \omega_{J_1}) + (J_2, \omega_{J_2})$ in $E^d(R[T])$, hence this completes the proof. □

10.2 On Laurent polynomial algebras over $\overline{\mathbb{F}}_p$

In Chapter 9 we develop the machineries, which allow us to mimic the work done in the previous sections. Hence we omit the proofs in this section.

Theorem 10.2.1. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$ and $I \subset R[T,T^{-1}]$ be an ideal such that ht$(I) \geq 2$ and $\mu(I/I^2) = d$. Moreover assume that $I =< f_1, ..., f_d > +I^2$. Suppose that there exists $F_i \in I\mathfrak{R}$ be such that $I\mathfrak{R} =< F_1, ..., F_d >$, with $F_i - f_i \in I\mathfrak{R}^2$. Then there exists $g_i \in I$ be such that $I =< g_1, ..., g_d >$, where $g_i - f_i \in I^2$.*

Corollary 10.2.2. *(Addition principle) Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$. Let $I, J \subset R[T,T^{-1}]$ be two co-maximal ideals, each of height ≥ 2. Suppose that $I = (f_1, ..., f_d)$ and $J = (g_1, ..., g_d)$. Then $I \cap J = (h_1, ..., h_d)$ where $h_i - f_i \in I^2$ and $h_i - g_i \in J^2$.*

Corollary 10.2.3. *(Subtraction principle) Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$. Let $I, J \subset R[T,T^{-1}]$ be two co-maximal ideals, each of height ≥ 2. Suppose that $I = (f_1, ..., f_d)$ and $I \cap J = (h_1, ..., h_d)$ where $h_i - f_i \in I^2$. Then there exists $g_i \in J$ such that $J = (g_1, ..., g_d)$ with $h_i - g_i \in J^2$.*

Definition 10.2.4 (Segre class of an ideal) Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$. Let $I \subset R[T,T^{-1}]$ be an ideal of height ≥ 2 such that $\mu(I/I^2) = d$. Let $\omega_I : (R[T,T^{-1}]/I)^d \twoheadrightarrow I/I^2$ be a surjection which induces $I =< f_1, ..., f_d > +I^2$. Using Lemma 2.2.6 get $J \subset R[T,T^{-1}]$ of either height d or $J = R[T,T^{-1}]$ such that $I \cap J =< g_1, ..., g_d >$, with $f_i - g_i \in I^2$. Let $\omega_J : (R[T,T^{-1}]/J)^d \twoheadrightarrow J/J^2$ be a map sending $e_i \to g_i$. Then we define $s(I, \omega_I) = -(J, \omega_J) \in E^d(R[T,T^{-1}])$ whenever ht$(J) = d$ and $s(I, \omega_I) = 0$ whenever $J = R[T,T^{-1}]$.

Proposition 10.2.5. *The Segre class of (I, ω_I) as described above, is well defined.*

Theorem 10.2.6. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$. Let $I \subset R[T,T^{-1}]$ be an ideal of height ≥ 2. Let $\omega_I : (R[T,T^{-1}]/I)^d \twoheadrightarrow I/I^2$ be surjection. $s(I, \omega_I) = 0$ if and only if ω_I can be lifted to a surjection $(R[T,T^{-1}])^d \twoheadrightarrow I$.*

Theorem 10.2.7. *Let R be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$ and $I_1, I_2 \subset R[T,T^{-1}]$ be two co-maximal ideals of height ≥ 2. Suppose that there exists surjections $\omega_{I_i} : (R[T,T^{-1}]/I_i)^d \twoheadrightarrow I_i/I_i^2$, for $i = 1, 2$. Then we have $s(I_1 \cap I_2, \omega_{I_1 \cap I_2}) = s(I_1, \omega_{I_1}) + s(I_2, \omega_{I_2})$.*

Chapter 11

Equivalence of two conjectures

11.1 Equivalence of two conjectures

Recall the following questions, which was instigate in the introduction of the thesis:

Question 11.1.1 Let A be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension d and $I \subset A[T]$ be an ideal such that $\mu(I/I^2) = \text{ht}(I) = d$. Then is $\mu(I) = n$?

Question 11.1.2 Let A be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension d and $I \subset A[T]$ be an ideal such that $\mu(I/I^2) = \text{ht}(I) = d$. Further assume that $I =< f_1, ..., f_d > +I^2$. Then can we lift f_i's to a set of generators of I ?

Question 11.1.3 Let A be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension d and $I \subset A[T]$ be an ideal of height d. Further assume that $I =< f_1, ..., f_d > +I^2T$, then does there exists $g_i \in I$, such that $I =< g_1, ..., g_d >$, with $f_i - g_i \in I^2T$?

We shall show in Theorem 11.1.7 that all the above questions are equivalent whenever the ring is taken as an affine algebra over $\overline{\mathbb{F}}_p$ (with some additional hypothesis on p). We shall begin with the following lemma.

Lemma 11.1.4. *Suppose that R is a Noetherian ring of dimension $d \geq 2$, $(d-1)! \in R^*$, $I \subset R[T]$ is an ideal with ht$(I) = d$, and $I =< f_1, ..., f_d > +I^2T$. Furthermore assume that there exists $s \in I \cap R$ and $h_i \in IB[T]$ such that $I =< h_1, ..., h_d >$, with $f_i - h_i \in I^2TB[T]$, where $B = R_{1+s}$. Then there exists $g_i \in I$ such that $I =< g_1, ..., g_d >$, with $f_i - g_i \in I^2T$.*

Proof Let $C = R_{s(1+sR)}$, then $\dim(C) \leq d-1$. In the ring $C[T]$ we have $IC[T] =< h_1, ..., h_d > C[T] =< f_1(0), ..., f_d(0) > C[T] = C[T]$. Then by ([54], Corollary 2.5), there exists $\alpha(T) \in \mathrm{SL}_d(C[T])$, such that $(h_1, ..., h_d)\alpha(T) = (f_1(0), ..., f_d(0))(= (h_1(0), ..., h_d(0)))$. Furthermore replacing $\alpha(T)$ by $\alpha(T)\alpha(0)^{-1}$, we may assume $\alpha(0) = i_d$. Then a standard patching argument give rise to the desired set of generators of the ideal I. $\square$

Theorem 11.1.5. *Suppose that A is an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 3$, and $(d-1)! \in A^*$. Then an affirmative answer of the Question 11.1.2 will imply an affirmative answer of the Question 11.1.3.*

Proof Let $J = A \cap I$. By ([7], Lemma 3.6) there exists $I' \subset A[T]$ of $\mathrm{ht}(I') = d$ and $p_i \in I'$, such that

$(i)\, I' + (J^2T) = A[T]$.

$(ii)\, I \cap I' =< p_1, ..., p_d >$, where $f_i - p_i \in I^2T$.

Since $I + I' = A[T]$, tensoring $I \cap I' =< p_1, ..., p_d >$ by $A[T]/I'$ and using Chinese remainder theorem we get $I' =< p_1, ..., p_d > + I'^2$. We claim that I' has a lift to a set of generators of I'.

Note that if $\mathrm{ht}(I') > d$, then the only possibility is $I' = A[T]$, as if $\mathrm{ht}(I') = d + 1$, then by Suslin's Monic Polynomial Theorem I' contains a monic polynomial in T and hence by ([40], Corollary 1.5) we will get $\mu(I) = d$, and this will leads us to a contradiction! Now if $I' = A[T]$, this will imply $I =< p_1, ..., p_d >$, and then we are done with the theorem. So only nontrivial case remains $\mathrm{ht}(I') = d$.

Now an affirmative answer of the Question 11.1.2 will assure the existence of $a_i \in I'$ such that $I' =< a_1, ..., a_d >$, with $p_i - a_i \in I'^2$.

Let $C = A_{1+J}$, then in the ring $C[T]$, we have $IC[T] =< p_1, ..., p_d > C[T] + (I^2T)C[T]$, and $I'C[T] =< p_1, ..., p_d > C[T] + I'^2C[T]$, has a lift $I'C[T] =< a_1, ..., a_d > C[T]$. By Lemma 11.1.4 it is enough to find an $s \in J$ and $h_i \in IA_{1+s}[T]$, with $IA_{1+s}[T] =< h_1, ..., h_d >$, where $h_i - p_i \in (I^2T)A_{1+s}[T]$.

Claim There exists $\sigma \in E_d(C[T])$ such that $\sigma(a_1, ..., a_d) = (b_1, ..., b_d)$, satisfying

$(i)\; < b_1, ..., b_{d-1} > C[T] + (J^2T)C[T] = C[T]$

$(ii)\; \dim(C[T]/ < b_1, ..., b_{d-1} > C[T]) \leq 1$, and

$(iii)\, b_d - 1 \in (J^2T)C[T]$.

First we complete the proof with assuming the above claim and later we will prove the claim. Get $s \in J$ such that all the conditions of the above claim holds. Let $D = A_{1+s}$, then note that D is an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$. Thus we have $(I \cap I')C[T] =< p_1, ..., p_d > D[T]$, and $I'D[T] =< a_1, ..., a_d >$, with $p_i - a_i \in I'^2D[T]$. Let $\sigma(p_1, ..., p_d) = (g_1, ..., g_d)$.

Then note that $g_i - b_i \in I'^2 D[T]$.

Define $R = D[T, X]$ and $N =< b_1, ..., b_{d-1} > D[T]$. Also set $K_1 = (NR, X + b_d)$, $K_2 = IR$, $K_3 = K_1 \cap K_2$. Then note that

(i) K_1 contains a monic polynomial,

(ii) $K_2 = IR$, is an extended ideal,

(iii) $K_1 + K_2 = R$, and

(iv) $\dim(R/K_1) = \dim(D[T]/ < b_1, ..., b_{d-1} > D[T]) = 2 \leq d - 1$.

Also note that $K_3(0) = K_1(0) \cap K_2(0) = I'C[T] \cap IC[T] =< f_1, ..., f_d > C[T] =< g_1, ..., g_d > C[T]$ and $K_1 =< b_1(X), ..., b_{d-1}(X), b_d + X >$, will induced $K_1 =< b_1(X), ..., b_{d-1}(X), b_d + X > +K_1^2$, where $b_i(0) - g_i \in I'^2 C[T] = K_1(0)^2$, for $i = 1, ..., d-1$ and $b_d - g_d \in I'^2 C[T] = K_1(0)^2$.

Then by Theorem 3.3.1 there exists $H_i(X) \in K_3$, such that $K_3 =< H_1(X), ..., H_d(X) >$ with, $H_i(0) = p_i$. Set $h_i = H_i(1 - b_d)$. Thus we get $< h_1, ..., h_d >= K_3(1 - b_d) = IC[T]$, and since $b_d - 1 \in (J^2T)C[T] \subset (I^2T)C[T]$, we get $h_i - p_i \in (I^2T)C[T]$.

Proof of the claim. Let $B = C[T]/(J^2T)C[T]$ and 'bar' denotes going modulo $(J^2T)C[T]$. Then note that $(\overline{a}_1, ..., \overline{a}_d) \in \mathrm{Um}_d(B)$. Note that JB is contained in the Jacobson radical of B. Also note that to show $E_d(B)$ acts transitively on $\mathrm{Um}_d(B)$ it is enough to show going modulo JB, it acts transitively. This follows from the fact that $B/JB \cong (A/J)[T]$ and thus $\dim(B/JB) = \dim((A/J)[T]) \leq 2$ then apply (Theorem 2.6, [7]) to get that $E_d(B)$ acts transitively on $\mathrm{Um}_d(B)$.

Thus there exists an $\sigma \in E_d(C[T])$, such that $\sigma(a_1, ..., a_d) = (b_1, ..., b_d)$, where $(\overline{b}_1, ..., \overline{b}_{d-1}) \in \mathrm{Um}_{d-1}(B)$ and $b_d \in (J^2T)C[T]$. Moreover by Theorem 2.1.4, adding suitable multiples of b_d to b_i, $i = 1, ..., d-1$, we can further assume $\mathrm{ht}(b_1, ..., b_{d-1})_{b_d} \geq d - 1$. Now since $< b_1, ..., b_d > C[T] = I'C[T]$, and $\mathrm{ht}(I'C[T]) = d$, this implies that $\mathrm{ht}(b_1, ..., b_{d-1}) \geq d - 1$. Since $< b_1, ..., b_{d-1} > C[T] + (J^2T)C[T] = C[T]$, and $(J^2T)C[T] \subset Jac(C[T])$, by (Lemma 3.1, [7]) we have any maximal ideal of $C[T]$, containing $< b_1, ..., b_{d-1} > C[T]$ has height less than or equals to d. Thus we get $\dim(C[T]/ < b_1, ..., b_{d-1} > C[T]) \leq 1$.

Since $< b_1, ..., b_{d-1} > C[T] + (J^2T)C[T] = C[T]$, there exists λ_i, $i = 1, ..., d-1$, such that $\lambda_1 b_1 + ... + \lambda_{d-1} b_{d-1} - 1 \in (J^2T)C[T]$. Replacing b_d by $b_d + \lambda_1 b_1 + ... + \lambda_{d-1} b_{d-1}$, we may further assume $b_d - 1 \in (J^2T)C[T]$. Hence this proves the claim and the theorem. □

Theorem 11.1.6. *Suppose that A is an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$. Then an affirmative answer of the Question 11.1.3 will imply an affirmative answer of the Question 11.1.2 and in particular, of the Question 11.1.1.*

Proof Assume $I =< f_1, ..., f_d > +I^2$. Since ht$(I^2) = d \geq 1$ and ht$(< T >) = 1$, we have ht$(I^2T) \geq 1$. Let 'bar' denotes going modulo (I^2T). Let $B = A[T]/I^2T$, then B is an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 2$, and in the ring B we have $\overline{I} =< \overline{f}_1, ..., \overline{f}_d > +\overline{I}^2$. Then by Theorem 2.2.5 we can find $h_i \in I$, such that $\overline{I} =< \overline{h}_1, ..., \overline{h}_d >$, with $\overline{f}_i - \overline{h}_i \in \overline{I}^2$ i.e. $I =< h_1, ..., h_d > +I^2T$, with $h_i - f_i \in I^2$.

Now an affirmative answer of the Question 11.1.3 will ensure the existence $g_i \in I$, such that $I =< g_1, ..., g_d >$, with $g_i - h_i \in I^2T \subset I^2$. Thus we actually get $I =< g_1, ..., g_d >$, $g_i - f_i = (g_i - h_i) + (h_i - f_i) \in I^2$. This completes the proof. $\square$

Theorem 11.1.7. *Let A be an affine algebra over $\overline{\mathbb{F}}_p$ of dimension $d \geq 3$, $(d-1)! \in A^*$. Let $I \subset A[T]$, be an ideal such that ht$(I) = \mu(I/I^2) = d$. Then the followings are equivalent*

(i) $\mu(I) = d$.

(ii) *If $I =< f_1, ..., f_d > +I^2$, then it has a lift to a set of generators of I.*

(iii) *If $I =< f_1, ..., f_d > +I^2T$, then it has a lift to a set of generators of I.*

Proof (i) implies that (ii) follows from Theorem 8.3.2.

(ii) implies that (iii) follows from Theorem 11.1.5.

(iii) implies that (i) follows from Theorem 11.1.6. $\square$

Part II

On real affine algebras

Chapter 12

Preliminaries

Notations for part II:

Unless otherwise stated we fix the following notations for the rest of this part:

- A will stand for a commutative Noetherian ring with $1 \neq 0$.
- R will stand for a a real affine algebra of dimension d satisfying one of the following conditions:
  - there are no real maximal ideals;
  - the intersection of all real maximal ideals has height at least 1.

12.1 Stably free modules and unimodular rows

The purpose of this section is to recall some basic definitions related to stably free modules and unimodular rows and collect various result related to the freeness of a stably free module which will be used throughout this part.

Definition 12.1.1 An A-module P is said to be a stably free module of type n, if $P \oplus A^n \cong A^m$ for some $m \in \mathbb{N}$. In this case rank$(P) = m - n$. We will say P is stably free if it is stably free of type n for some $n \in \mathbb{N}$.

Let P be a stably free module of rank n and type 1. Then we have an isomorphism $P \oplus A \cong A^{n+1}$. This isomorphism will induce the following short exact sequence

$$0 \to P \to A^{n+1} \to A \to 0.$$

Recall that we can assign any surjective $A-$linear map $A^{n+1} \twoheadrightarrow A$ with a row vector $(a_0, ..., a_n) \in A^{n+1}$, having the property that there exists $(b_0, ..., b_n) \in A^{n+1}$ such that $\sum_{i=0}^{n} a_i b_i = 1$. Thus in the above process for any stably free module P of rank n and type 1 we can find a row vector $(a_0, ..., a_n)$ of length $n+1$ such that $P = \{(x_0, ..., x_n) \in A^{n+1} : \sum_{i=0}^{n} a_i x_i = 0\}$.

Definition 12.1.2 A row vector $(a_0, ..., a_n) \in A^{n+1}$ of length $n+1$ is said to be a unimodular row of length $n+1$, if there exists $(b_0, ..., b_n) \in A^{n+1}$ such that $\sum_{i=0}^{n} a_i b_i = 1$. We will denote by $\mathsf{Um}_{n+1}(A)$ the set of all unimodular row vectors of length $n+1$ over the ring A.

Example 12.1.3 (i) Consider the ring

$$A = \frac{\mathbb{R}[X_1, ..., X_n]}{< X_1^2 + ... + X_n^2 - 1 >}$$

. Let 'bar' denote going modulo the ideal $< X_1^2 + ... + X_n^2 - 1 >$. Then $(\overline{X}_1, ..., \overline{X}_n)$ is a unimodular row of length n.

(ii) Over any ring A, let $\alpha \in \mathsf{GL}_n(A)$ be an invertible $n \times n$ matrix. Let $v_i = e_i\alpha$, is the i-th row of α, where e_i is the row vector in A^n, consisting only non zero entry 1, at the i-th position. Then v_i is a unimodular row of length n.

Definition 12.1.4 A unimodular row $v \in \mathsf{Um}_n(A)$ of length n is said to be completable if there exists $\alpha \in \mathsf{GL}_n(A)$ such that $e_1\alpha = v$, that is, v is the, first row of an invertible $n \times n$ matrix α.

Remark 12.1.5 We would like to remark that, if v is completable, then we can always choose $\alpha \in \mathsf{SL}_{n+1}(A)$ by replacing the second row $e_2\alpha$ of α with $(\det(\alpha))^{-1}e_2\alpha$.

Definition 12.1.6 In a ring A, $\mathsf{E}_n(A)$ is the subgroup of $\mathsf{GL}_n(A)$ generated by the matrices $E_{ij}(\lambda) = I_n + e_{ij}(\lambda)$, where $i \neq j$. Recall that $e_{ij}(\lambda)$ is the matrix with only possible non zero entry is λ at the $(i,j)-$th position and I_n is the identity matrix.

Convention 12.1.7. An invertible matrix ϵ is said to be an elementary matrix if $\epsilon \in \mathsf{E}_n(A)$.

Definition 12.1.8 A unimodular row $v \in \mathsf{Um}_n(A)$ is said to be elementarily completable if there exists $\epsilon \in \mathsf{E}_n(A)$ such that $e_1\epsilon = v$.

Lemma 12.1.9. *Let A be a ring and $P = \{(x_0, ..., x_n) \in A^{n+1} : \sum_{i=0}^{n} a_i x_i = 0\}$ be a stably free module of rank n, where $(a_0, ..., a_n) \in$ Um$_{n+1}(A)$. Then P is free if and only if $(a_0, ..., a_n)$ is completable.*

Proof Let us assume that P to be free. Consider an isomorphism $\phi : P \xrightarrow{\sim} A^n$. Then note that we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r] \arrow[d, "\phi"] & A^{n+1} \arrow[r] \arrow[d, dashed, "\sigma'"] & A \arrow[r] \arrow[d, equal, "i"] & 0 \\ 0 \arrow[r] & A^n \arrow[r] & A^{n+1} \arrow[r] & A \arrow[r] & 0. \end{tikzcd}$$

Where the map $\sigma' : A^{n+1} \to A^{n+1}$ is defined in the following way:
Let $v = (v_0, ..., v_n) \in A^{n+1}$, then there exists a unique $p \in P$ and $a \in A$ such that $v = a(a_0, ..., a_n) + p$, where $p = (y_0, ..., y_n) \in A^{n+1}$, with $\sum_{i=0}^{n} a_i y_i = 0$.
We define $\sigma'(v) = i(a)e_1 + \phi(p)$, where $i : A \to A$ is the identity map.
Since ϕ and i are both isomorphisms by five lemma σ' is also an isomorphism, that is, $\sigma' \in \mathrm{GL}_{n+1}(A)$. Now note that $(a_0, ..., a_n) = 1.(a_0, ..., a_n) + 0$. Hence a local checking ensures that $e_1\sigma = (a_0, ..., a_n)$, where $\sigma = (\sigma')^{-1}$.
Conversely assume there exists $\sigma \in \mathrm{SL}_{n+1}(A)$ such that $e_1\sigma = (a_0, ..., a_n)$. We can consider $(a_0, ..., a_n)$ and e_1 as surjective maps $A^{n+1} \twoheadrightarrow A$. Then note that $P = \ker(a_0, ..., a_n) = \ker(e_1\sigma) \cong \ker(e_1) = A^n$. □

Notation 12.1.10. Let $v, w \in \mathrm{Um}_n(A)$. We shall say $v \sim w$ if there exists $\alpha \in \mathrm{SL}_n(A)$ such that $v\alpha = w$. Moreover, if the matrix $\alpha \in \mathrm{E}_n(A)$ then we shall say $v \sim_E w$.

Remark 12.1.11 One can check that '$\sim$' (respectively '$\sim_E$') induces an equivalence relation on the set of all unimodular rows of a fixed length. A unimodular row $v \in \mathrm{Um}_n(A)$ is completable (respectively elementarily completable) if and only if $v \sim e_1$ (respectively $v \sim_E e_1$).

Theorem 12.1.12. *Let $(a_1, ..., a_n) \in Um_n(A)$ be such that it contains a shorter length unimodular row. Then $(a_1, ..., a_n)$ is elementarily completable.*

Proof Without loss of generality we may assume that the row $(a_2, ..., a_n)$ is unimodular. Hence we can find $b_2, ..., b_n \in A$ such that $1 - a_1 = \sum_{i=2}^{n} a_i b_i$ i.e. $\sum_{i=2}^{n} a_i b_i + a_1 = 1$. Thus note that $(a_1, ..., a_n) \sim_E (1, a_2, ..., a_n) \sim_E (1, 0, ..., 0)$. Hence $(a_1, ..., a_n)$ is elementarily completable. □

The next lemma is an application of the prime avoidance lemma, for a proof one can see ([53], Lemma 2.1.9).

Lemma 12.1.13. *Let A be a ring. Let $I \subset A$ be an ideal generated by n elements $a_1, ..., a_n$ such that ht$(I) \geq n$, $n \geq 1$. Then there exists $\theta \in$ E$_n(A)$ such that*

$$(a_1, ..., a_n)\theta = (d_1, ..., d_n),$$

where $d_1, ..., d_n$ generate I and ht$(d_1, ..., d_i) \geq i$ for $1 \leq i \leq n$.

Corollary 12.1.14. *Let $\dim(A) = n$. Then for all $r \geq n+2$, we have Um$_r(A) = e_1$E$_r(A)$.*

Proof Let $v = (a_1, ..., a_r) \in$ Um$_r(A)$ be a unimodular row of length r, where $r \geq \dim(A)+2$. Then by Lemma 12.1.13, there exists $\theta \in$ E$_r(A)$ and $w = (b_1, ..., b_r) = (a_1, ..., a_r)\theta$, such that ht$(< b_1, ..., b_{r-1} >) \geq r-1 > n$, that is, $(b_1, ..., b_{r-1}) \in$ Um$_{r-1}(A)$. Since the unimodular row $(b_1, ..., b_r)$ contains a shorter length unimodular row, it is elementarily completable by Theorem 12.1.12. Thus we have $v \sim_E w \sim_E e_1$ and hence v is elementarily completable . $\square$

Remark 12.1.15 Note that the proof of the above Corollary tells us more than just elementary completion of a unimodular row. In fact the proof shows that in any unimodular row of length greater than the dimension of the ring plus two, by adding suitable multiples of an entry with the other entries we can get a new unimodular row consisting of a shorter length unimodular row.

The above Corollary 12.1.14 and Lemma 12.1.9 proves that any stably free A-module rank n and type 1 is free provided $n \geq \dim(A)+1$. Now consider any stably free A-module P of rank n and type r with $n \geq \dim(A)+1$. Then note that $P \oplus A^{r-1}$ is a stably free A-module rank $n+r-1$ and type 1 and hence free. Repeating this process r many times we can conclude that P is free. Next we state a result which is very crucial to prove a particular unimodular row is completable. For $r = 2$ the result is due to R. G. Swan and J. Towber [68], and for arbitrary r the result is due to A. A. Suslin ([62], Theorem 2).

Theorem 12.1.16. *Let A be a ring. Let $(a_0, ..., a_r) \in$ Um$_{r+1}(A)$ be a unimodular row, and $n_0, n_1, ..., n_r$ be positive integers. Suppose that $\prod_{i=0}^{r} n_i$ is divisible by $r!$. Then there exists a matrix $\alpha \in$ SL$_{r+1}(A)$ with $(a_0^{n_0}, a_1^{n_1}, ..., a_r^{n_r})$ as the first row.*

We end this section with a result due to A. A. Suslin ([64], Theorem 2.4) and an observation of P. Raman. For a proof one can see ([55], Proposition 3.1).

Theorem 12.1.17. *Let A be an affine algebra of dimension n over a field k. Assume that, for any prime $p < n$ one of the following conditions is satisfied:*
(a) $p \neq char(k)$, $c.d._p(k) < 1$;

(b) $p = char(k)$ and k is perfect.

Then $Um_{n+1}(A) = e_1 SL_{n+1}(A)$.

12.2 Application of Swan's Bertini theorem

In this section we review a version of (Theorem 12.2.6) Swan's Bertini theorem, for a proof one can see ([66], Theorem 1.3). The purpose of this section is to use Swan's Bertini theorem and some divisibility argument of symplectic K_1 groups to show that any stably free R-module of rank d is free whenever we take $\dim(R) = d$ (where R is as defined at the beginning of this chapter). This is an improvement of A. A. Suslin's result ([64]) over the ring R. We begin with the following definitions.

Definition 12.2.1 Let A be a ring. An A-module P is said to be a projective module if there exists another A-module Q such that $P \oplus Q$ is a free $A-$module.

Definition 12.2.2 Let A be a ring. The rank of a projective $A-$module P is the function rk : Spec$(R) \to \mathbb{N} \cup \{0\}$, defined by $p \to \dim(P \otimes Q(A/p))$, where $Q(A/p)$ is the field of fraction of A/p.

Convention 12.2.3. Unless otherwise stated through out the thesis by saying a projective module we always mean a finitely generated projective module with a constant rank function.

Definition 12.2.4 Let A be a ring and P be a projective A-module. An element $p \in P$ is said to be a unimodular element if the order ideal defined by $\mathcal{O}(p, P) = \mathcal{O}(p) = \{f(p) : f \in$ Hom $_A(P, A)\}$ contains 1.

Remark 12.2.5 Let A be a ring and P be a projective A-module. P has a unimodular element if and only if there a projective A-module Q such that $P \cong Q \oplus A$.

Theorem 12.2.6. *([66], Theorem 1.3) Let $V =$ Spec(A) be a smooth affine variety over an infinite field k. Let Q be a finitely generated projective A-module of rank r. Let $(q, a) \in Q \oplus A$ be a unimodular element. Then there exists a $y \in Q$ such that $I = o_Q(q + ay)$ has the following properties:*

- *(i) The subscheme $U =$ Spec(A/I) of V is smooth over k and $\dim(U) = \dim(V) - r$, unless $U = \phi$.*
- *(ii) If $\dim(U) \neq 0$ then U is a variety.*

Let A be a ring and $\alpha \in \mathsf{GL}_n(A)$. Then we can embed $\mathsf{GL}_n(A) \subset \mathsf{GL}_{n+1}(A)$ as $\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$ $\in \mathsf{GL}_{n+1}(A)$. The set $\mathsf{GL}(A)$ is the direct limit of $\mathsf{GL}_i(A)$. In an obvious way a group structure is defined on it, which coincides with the group structures on $\mathsf{GL}_n(A)$. We set $\mathsf{GL}(A) = \bigcup_{n\in\mathbb{N}} \mathsf{GL}_n(A)$, $\mathsf{SL}(A) = \bigcup_{n\in\mathbb{N}} \mathsf{SL}_n(A)$ and $\mathsf{E}(A) = \bigcup_{n\in\mathbb{N}} \mathsf{E}_n(A)$. By a result of J. P. Serre we know that $\mathsf{E}(A)$ is a normal subgroup of $\mathsf{GL}(A)$.

Definition 12.2.7 We define $K_1(A) = \mathsf{GL}(A)/\mathsf{E}(A)$ and $SK_1(A) = \mathsf{SL}(A)/\mathsf{E}(A)$.

Definition 12.2.8 A matrix in $M_r(A)$ is said an alternating matrix if it has the form $\nu - \nu^T$, for some ν in $M_r(A)$.

Let $\alpha \in M_r(A)$ and $\beta \in M_s(A)$, then we define, $\alpha \perp \beta := \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in M_{r+s}(A)$.

Define χ_r inductively as

$$\chi_1 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathsf{E}_2(A) \ \textit{and} \ \chi_{r+1} := \chi_r \perp \chi_1.$$

Then by the definition, χ_r is an alternating matrix and by induction it can be shown that $\chi_r \in \mathsf{E}_{2r}(A)$.

Definition 12.2.9 For any natural number r we define $Sp_{2r}(A) := \{\alpha \in SL_{2r}(A) : \alpha^T \chi_r \alpha = \chi_r\}$.

We define a bijection σ on $\mathbb{N}$, setting $\sigma(2i) = 2i - 1$ and $\sigma(2i - 1) = 2i$ for any natural number i.

Definition 12.2.10 For $1 \leq i \neq j \leq 2r$ and $a \in A$ we set

$$\begin{aligned} SE_{i,j}(a) &= I_{2r} + e_{i,j}(a), && \text{if } i = \sigma(j), \\ SE_{i,j}(a) &= I_{2r} + e_{i,j}(a) - (-1)^{i+j} e_{\sigma(j),\sigma(i)}(a), && \text{if } i \neq j \neq \sigma(i). \end{aligned}$$

We define $Ep_{2r}(A)$ to be the subgroup generated by $SE_{i,j}(\lambda)$, where $\lambda \in A$. Set $Sp(A) = \bigcup_{r\in\mathbb{N}} Sp_{2r}(A) \subset SL(A)$ and $Ep(A) = \bigcup_{r\in\mathbb{N}} Ep_{2r}(A) \subset E(A)$.

Theorem 12.2.11. *[72] For a ring A, $Ep(A)$ is a normal subgroup of $Sp(A)$.*

Definition 12.2.12 We define $K_1Sp(A) := Sp(A)/Ep(A)$.

The next two results are consequences of A. A. Suslin's result [64], which follows easily from ([49], Proposition 3 and 4).

Theorem 12.2.13. *Let C be a smooth real curve having no real maximal ideals then $SK_1(C)$ is a divisible group.*

Theorem 12.2.14. *Let C be a smooth real curve having no real maximal ideal then the natural homomorphism $K_1Sp(C) \to SK_1(C)$ is an isomorphism.*

The following lemma is also due to A. A. Suslin. For a proof one can see ([64], Corollary 2.3).

Lemma 12.2.15. *Let A be a commutative Noetherian ring and $(a_0, ..., a_n) \in Um_{n+1}(A)$ where $n \geq 2$ such that $\dim(A/ < a_2, ..., a_n >) \leq 1$ and $\dim(A/ < a_3, ..., a_n >) \leq 2$. Moreover assume that there exists $\alpha \in SL_2(A/ < a_2, ..., a_n >) \cap E_3(A/ < a_2, ..., a_n >)$ such that $(\overline{a}_0, \overline{a}_1)\alpha = (\bar{b}_0, \bar{b}_1)$. Then there exists $\gamma \in E_{n+1}(A)$ such that $(a_0, ..., a_n)\gamma = (b_0, b_1, a_3, ..., a_n)$.*

The next proposition is a slightly modified version of a well known fact to suit our needs. This proposition tells us whenever we are dealing with unimodular rows of length $d + 1$ more often than not, it is enough to assume smoothness. The proof is essentially based on a clever observation of P. Raman, that one may avoid singularities on the A. A. Suslin's proof of ([64], Theorem 2.4). We give a detailed proof.

Proposition 12.2.16. *Let A be an affine algebra over a perfect field k of dimension n and $v = (v_0, ..., v_n) \in Um_{n+1}(A)$. Assume that $\mathbb{S}$ is a collection of some maximal ideals of A such that $\mathscr{I} = \bigcap_{m \in \mathbb{S}} m$ has a positive height. Then there exists $\epsilon \in E_{n+1}(A)$ and $u = (u_0, ..., u_n) = v\epsilon$ such that for any $1 \leq i \leq n$, $A/ < u_0, ..., u_{i-1} >$ is a smooth affine algebra (domain if $i < n$) of dimension $n - i$ and mSpec$(A/ < u_0, ..., u_{i-1} >) \cap \mathbb{S} = \phi$.*

Proof Without loss of generality we may assume that A is reduced. Since k is perfect, the ideal $\mathscr{J}$, defining the singular locus of A has a positive height. Let $I = \mathscr{I}\mathscr{J}$, then by our hypothesis ht$(I) \geq 1$. Thus going modulo I, we can find $\omega \in E_{n+1}(A/I)$ such that $\omega v = e_1 \mod (I)$. Get a lift $\Omega \in E_{n+1}(A)$ of ω. Then we have $\Omega v = w = (w_0, ..., w_n)$, where $1 - w_0 \in I$ and $w_i \in I$ for all $i \geq 1$. Thus it is enough to prove the theorem for w. So without loss of generality we may begin with the assumption $1 - v_0 \in I$ and $v_i \in I$ for $i \geq 1$.
Also we observe that it is enough to take $i = 1$. As if we prove for $i = 1$, then we can repeat

the same steps on A/v_0A to get the result on $A/<v_0,...,v_{i-1}>$. Since the completion is elementary we can always come back to our initial ring A . Moreover since $1-v_0 \in I \subset \mathscr{I}$ we have $\mathbb{S} \cap m\mathrm{Spec}(A/v_0A) = \phi$.

By Theorem 12.2.6, we get $\lambda_j \in A$, for $j = 1, ..., n$, such that replacing v_0 by $v_0' = v_0 + \sum \lambda_j v_j$, gives us $\mathrm{Spec}(A/v_0'A)$ is a n-dimensional smooth variety outside the singularities of A. Note that we still have $v_0' - 1 \in I$. Let J be the ideal defining the singularities of $A/v_0'A$. Then to show $A/v_0'A$ is smooth it is enough to show $J = A/v_0'A$. Since $A/v_0'A$ is smooth outside the singularities A we have $\overline{I} \subset J$, where 'bar' denotes going modulo v_0'. But $v_0' - 1 \in I$ gives us the fact that $\overline{I} = A/v_0'A$. Thus we get $J = A/v_0'A$. So by taking $u = (v_0', v_1, ..., v_n)$ completes the proof. □

Theorem 12.2.17. *Let R be as defined at the beginning of this chapter. Any stably free $R-$modules of rank d is free, in particular, $\mathrm{Um}_{d+1}(R) = e_1 SL_{d+1}(R)$.*

Proof First we remark that if $d < 2$ then there is nothing to prove. Thus without loss of generality we may assume that $d \geq 2$. Note that if the closure of the set of $\mathbb{R}$-rational points in $\mathrm{Spec}(R)$, has dimension $\leq d-1$ then this is done in ([50], Theorem 3.2). So enough to take R as a real affine algebra having no real maximal ideal. Let $v = (v_0, ..., v_d) \in \mathrm{Um}_{d+1}(R)$. Using Lemma 12.2.16, we may begin with assuming $C = R/<v_0, ..., v_{d-2}>$ is a smooth curve. Let 'bar' denote going modulo $<v_0, ..., v_{d-2}>$. Since $SK_1(C)$ is divisible by Theorem 12.2.13, there exists $\sigma \in \mathrm{SL}_2(C) \cap E_3(C)$ such that $\sigma(\overline{v}_0, \overline{v}_1) = (\overline{a}^{d!}, \overline{b})$. By Theorem 12.2.14, we can further assume $\sigma \in \mathrm{SL}_2(C) \cap Ep(C)$. Then by Lemma 12.2.15 we can find $\epsilon \in E_{d+1}(R)$. such that $v\epsilon = (a^{d!}, b, v_3, ..., v_d) \in e_1\mathrm{SL}_{d+1}(R)$. □

Remark 12.2.18 Let $w = (w_0, ..., w_n) \in \mathrm{Um}_{n+1}(A)$. The factorial row $(w_0, ..., w_{n-1}, w_n^{n!}) \in e_1\mathrm{SL}_{n+1}(A)$. Note that from the proof of Theorem 12.2.17 it follows that any unimodular row of length $n+1$ can be transformed to a factorial row elementarily.

The next proposition is a slight variation (see [55], Proposition 3.3) of the above theorem in our set-up. This form will be required to improve the injective stability of SK_1 and K_1Sp in Chapter 14.

Proposition 12.2.19. *Let $v \in \mathrm{Um}_{d+1}(R)$ be such that $v \cong e_1$ modulo $<t>$ for some $t \neq 0$. Then v can be completed to a $\sigma \in SL_{d+1}(R)$ with $\sigma \equiv I_{d+1}$ modulo $<t>$.*

Proof Note that we can always find $w = (w_i)_i \in R^{d+1}$ such that $v = e_1 + tw$. Thus we get $(1+\lambda_1, \lambda_2, ..., \lambda_{d+1}) \in \mathrm{Um}_{d+1}(R)$ such that $1 + \lambda_1 + w_1 + t\sum_{i=1}^{d+1} \lambda_i w_i = 1$, that is, $\lambda_1 + w_1 = -t\sum_{i=1}^{d+1} \lambda_i w_i$.

Let $B = R[T]/ < T^2 - Tt >$ and 'bar' denote going modulo $< T^2 - Tt >$. Let $v(T) = e_1 + Tw$ and $u(T) = (1 + T\lambda_1, T\lambda_2, ..., T\lambda_{d+1})$. Then note that $v(T)u(T)^T = 1 + T(w_1 + \lambda_1) + T^2(\sum_{i=1}^{d+1} \lambda_i w_i) = 1 + T(T-t)(\sum_{i=1}^{d+1} \lambda_i w_i)$. That is, $\overline{v(T)u(T)^T} = \overline{1}$. Thus $\overline{v(T)} \in$ Um$_{d+1}(B)$. Hence by Theorem 12.2.17, there exists $\alpha(T) \in$ SL$_{d+1}(B)$, such that $\overline{v(T)} = \overline{e}_1\alpha(T)$. Let $\sigma = \alpha(0)^{-1}\alpha(t)$. Then note that we have $\sigma \in$ SL$_{d+1}(R)$ and $e_1\sigma = v$ where $\sigma \cong I_{d+1}$ modulo $< t >$. □

12.3 Cancellation of projective modules

The purpose of this section is to recall some definitions and results related to the cancellation problem of projective modules and to prove (Theorem 12.3.5) that any projective R-module of rank d is cancellative whenever we take $d = \dim(R)$. This is an improvement of a classical result of H. Bass [2] over the ring R.

Definition 12.3.1 For any ring A, a projective $A-$module P of rank n is said to be cancellative if $P \oplus A^r \cong P' \oplus A^r$ implies that $P \cong P'$, where P' is another projective $A-$module.

After Theorem 12.2.17 a natural question arises whether a projective $R-$module of rank d is cancellative whenever we take $\dim(R) = d$. This turns out to be affirmative in our case. Again the result must be well-known and an easy consequence of Theorem 12.2.17, but we did not find any suitable reference. We begin with a classical result of H. Bass, for a proof one can see [2].

Theorem 12.3.2. *Let A be a ring of dimension n and P be a projective A-module of rank $r \geq n+1$. Then P is cancellative.*

The next lemma is a projective version of Lemma 12.1.9. As the argument of the proof is same as of the proof of the Lemma 12.1.9, we opted to omit the proof.

Lemma 12.3.3. *Let A be a ring and P be a projective A-module. Then P is cancellative if and only if for any $(a,p) \in$ Um$(A \oplus P)$ there exists $\sigma \in \mathrm{Aut}\,(A \oplus P)$ such that $\sigma(a,p) = (1,0)$.*

Definition 12.3.4 Let A be a ring. Let P be a projective $A-$module such that either P or P^* has a unimodular element. We choose $\phi \in P^*$ and $p \in P$ such that $\phi(p) = 0$. We define an endomorphism ϕ_p as the composite $\phi_p : P \to A \to P$, where $A \to P$ is the map sending $1 \to p$. Then by a transvection we mean an automorphism of P, of the form $1 + \phi_p$, where either $\phi \in$ Um(P^*) or $p \in$ Um(P). By E(P) we denote the subgroup of $\mathrm{Aut}\,(P)$ generated by all transvections.

Theorem 12.3.5. *Let $\dim(R) = d \geq 2$ and P be a finitely generated projective $R-$module of rank d. Then P is cancellative.*

Proof Note that for the case whenever the intersection of all real maximal ideals of R has a positive height, it is done in ([50], Theorem 3.2). Therefore, we assume that R has no real maximal ideals. We will show that for any $(a, p) \in \mathsf{Um}(R \oplus P)$, there exists $\sigma \in \mathrm{Aut}\,(R \oplus P)$ such that $\sigma(a, p) = (1, 0)$. Furthermore we can assume R to be reduced. Let J be the ideal defining singular locus. Then ht$(J) \geq 1$. Moreover we can find a nonzero divisor $t \in J$ such that P_t is free. Let F be the free $A-$module of rank d. Let $s = t^l$ be such that $sP \subset F$.
Since s is a nonzero divisor, by ([5], Proposition 2.13) $\mathsf{Um}(R \oplus P) \twoheadrightarrow \mathsf{Um}(R/sR \oplus P/sP) = e_1\mathsf{E}(R/sR \oplus P/sP)$ is surjective. Therefore with out loss of generality we may assume $a - 1 \in sR$ and $p \in sP \subset F$. Hence we may take $p = (a_1, ..., a_d) \in F$.
Using Lemma 12.2.16, we may further assume $B = R/ < a, a_1, ..., a_{d-2} >$ is smooth of dimension 1, infact the proof of lemma assures that we would not lose the fact that $a - 1 \in < s >$.
Let 'bar' denote going modulo $< a, a_1, ..., a_{d-2} >$. Then note that $\overline{P}$ is free over B. Using Theorem 12.2.13, we have $SK_1(B)$ is divisible. Then in B we can find $\overline{\epsilon} \in \mathsf{SL}_2(B) \cap E_3(B)$ such that $\overline{\epsilon}(\overline{a}_{d-1}, \overline{a}_d) = (\overline{b}_{d-1}, \overline{b}_d^{d!})$. Furthermore since $SK_1(B) \cong K_1Sp(B)$ we can take $\overline{\epsilon} \in \mathsf{SL}_2(B) \cap E_p(B)$.
Using ([64], Corollary 2.3) get $\gamma \in E_d(R/aR) = E(P/aP)$ such that $\gamma(\tilde{a}_1, ..., \tilde{a}_d) = (\tilde{a}_1, ..., \tilde{a}_{d-2}, \tilde{b}_{d-1}, \tilde{b}_d^{d!})$, where 'tilde' denotes going modulo a. Since $aR + sR = R$ and $F \subset P$ gives us $\tilde{P} = \tilde{F}$ hence $\mathsf{Um}(\tilde{P}) = \mathsf{Um}_d(R/aR)$. Then by ([5], Proposition 2.12) we can lift $\gamma \in E_d(R/aR)$ to $\alpha \in \mathrm{Aut}\,(P)$ and hence $\alpha p = (a_1, ..., a_{d-2}, b_{d-1}, b_d^{d!}) + a\lambda q$, for some $\lambda \in F$ and $q \in P$. Take $\sigma = 1 \perp \alpha$, then $\sigma(a, p) = (a, \alpha p) \equiv (a, a_1, ..., a_{d-2}, b_{d-1}, b_d^{d!}) \mod E(R \oplus P)$. Note that by Suslin's factorial theorem $(a, a_1, ..., a_{d-2}, b_{d-1}, b_d^{d!}) \equiv (1, 0) \mod \mathrm{Aut}\,(R \oplus P)$. Hence this completes the proof. $\square$

12.4 Excision ring and relative cases

The purpose of this section is to state and prove Theorem 12.4.8, a relative version of Theorem 12.2.17. This form will be needed to improve the injective stability of SK_1 in chapter 14. We begin with the following definition.

Definition 12.4.1 Let A be a ring and I an ideal in A. The Excision ring $A \oplus I$, has coordinate-wise addition and multiplication given by: $(a, i).(b, j) = (ab, aj + bi + ij)$. The additive identity of this ring $A \oplus I$ is $(0, 0)$ and the multiplicative identity is $(1, 0)$.

For a proof of the next proposition one can see ([32], Proposition 3.1).

Proposition 12.4.2. *Let A be an affine algebra of dimension n over a field k and I an ideal in A. Then the excision ring $A \oplus I$ is also an affine algebra of dimension n over the field k.*

Notation 12.4.3. Let $\pi_2 : A \oplus I \twoheadrightarrow A$ be the surjection, which sends $(r, i) \to r + i$.

Lemma 12.4.4. *$R \oplus I$ is also satisfies one of the following conditions:*
(i) $R \oplus I$ has no real maximal ideal.
(ii) The intersection of all real maximal ideals of $R \oplus I$ has positive height.

Proof Note that any ideal (apart from $R \oplus 0$) of $R \oplus I$ is of the form $J \oplus I'$, where $J \subset R$ and $I' \subset I$ are ideals of R. In particular, the maximal ideals are of the form $m \oplus I$, where $m \subset R$ is a maximal ideal. Now if R has no real maximal ideals then the residue field of $R \oplus I$ remains $\mathbb{C}$ and hence $R \oplus I$ also does not have any real maximal ideals.

Now suppose that the intersection of all real maximal ideals of R has a positive height. Let $\mathscr{J}$ be the intersection of all real maximal ideals. Then ht$(\mathscr{J}) \geq 1$. Note that the intersection of all real maximal ideals of $R \oplus I$ is $\mathscr{J} \oplus I$, which also has a positive height. Hence this completes the proof. $\square$

Definition 12.4.5 Let A be a ring and $I \subset A$ be an ideal. We define $\mathrm{Um}_n(A, I) := \{v \in \mathrm{Um}_n(A) : v = e_1 \mod (I)\}$. Any element $v \in \mathrm{Um}_n(A, I)$ will be called a relative unimodular row of length n with respect to the ideal I.

Definition 12.4.6 For any ring A and for any ideal $I \subset A$ we define $\mathrm{SL}_n(A, I) := \{\alpha \in \mathrm{SL}_n(A) : \alpha \equiv I_n \mod (I)\}$.

Definition 12.4.7 For any ring A and for any ideal $I \subset A$ we define $\mathrm{E}(A, I)$ to be the smallest normal subgroup of $\mathrm{E}_n(A)$ containing the element $E_{21}(x)$, $x \in I$.

Now we are ready to state and prove Theorem 12.4.8.

Theorem 12.4.8. *Let $d \geq 2$ and $I \subset R$ be an ideal. Then $Um_{d+1}(R, I) = e_1 SL_{d+1}(R, I)$.*

Proof Let $v = (v_0, ..., v_d) \in \mathrm{Um}_{d+1}(R, I)$ then note that $\tilde{v} = ((1, v_0 - 1), (0, v_1)..., (0, v_d)) \in \mathrm{Um}_{d+1}(R \oplus I, 0 \oplus I)$. Then by Theorem 12.2.17 get $\alpha \in \mathrm{SL}_{d+1}(R \oplus I)$ such that $e_1\alpha = \tilde{v}$. Let 'bar' denote going modulo $0 \oplus I$. Then we have $\overline{e}_1\overline{\alpha} = \overline{e}_1$, where $\overline{\alpha} \in \mathrm{SL}_{d+1}(R) \subset \mathrm{SL}_{d+1}(R \oplus I)$. Replacing α by $\overline{\alpha}^{-1}\alpha$ we may assume that $e_1\alpha = \tilde{v}$, where $\alpha \in \mathrm{SL}_{d+1}(R \oplus I, 0 \oplus I)$. Then

$e_1\mathsf{SL}_{d+1}(\pi_2)(\alpha) = v$, where $\mathsf{SL}_{d+1}(\pi_2) : \mathsf{SL}_{d+1}(R \oplus I) \to \mathsf{SL}_{d+1}(R)$ induced by π_2. Now note that we have the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathsf{SL}_{d+1}(R \oplus I, 0 \oplus I) \arrow[r] \arrow[d, dashed] & \mathsf{SL}_{d+1}(R \oplus I) \arrow[r] \arrow[d, "\mathsf{SL}_{d+1}(\pi_2)"] & \mathsf{SL}_{d+1}(\frac{R \oplus I}{0 \oplus I}) \arrow[d, "\overline{\mathsf{SL}_{d+1}(\pi_2)}"] \\
1 \arrow[r] & \mathsf{SL}_{d+1}(R, I) \arrow[r] & \mathsf{SL}_{d+1}(R) \arrow[r] & \mathsf{SL}_{d+1}(\frac{R}{I})
\end{tikzcd}$$

Thus $\mathsf{SL}_{d+1}(\pi_2)$ and $\overline{\mathsf{SL}_{d+1}(\pi_2)}$ will induce $\Gamma : \mathsf{SL}_{d+1}(R \oplus I, 0 \oplus I) \to \mathsf{SL}_{d+1}(R, I)$ such that the diagram commutes. Hence we actually get $\mathsf{SL}_{d+1}(\pi_2)(\alpha) \in \mathsf{SL}_{d+1}(R, I)$. This completes the proof. $\square$

12.5 Mennicke and weak Mennicke symbols

In this section we will briefly recall the Mennicke symbols and weak Mennicke symbols. Let us begin with the following definitions.

Definition 12.5.1 Let A be a ring. A Mennicke symbol of length $n+1 \geq 3$, is a pair (ψ, G), where G is a group and $\psi : \mathsf{Um}_{n+1}(A) \to G$ is a map such that:
ms_1. $\psi((0, ..., 0, 1)) = 1$ and $\psi(v) = \psi(v\epsilon)$ for any $\epsilon \in E_{n+1}(A)$;
ms_2. $\psi((b_1, ..., b_n, x))\psi((b_1, ..., b_n, y)) = \psi((b_1, ..., b_n, xy))$ for any two unimodular rows $(b_1, ..., b_n, x)$ and $(b_1, ..., b_n, y)$.

Remark 12.5.2 Clearly, a universal Mennicke symbol $(ms, MS_{n+1}(A))$ exists. It is universal in the sense that for any Mennicke symbol (ϕ, G) of length $n+1 \geq 3$, the map $\phi : \mathsf{Um}_{n+1}(A) \to G$ factors through the map ms via a unique morphism $MS_{n+1}(A) \to G$.

Definition 12.5.3 Let A be a ring. A weak Mennicke symbol of length $n+1 \geq 3$ is a pair (ψ, G) where G is a group and $\phi : \mathsf{Um}_{n+1}(A) \to G$ is a map such that the two following properties are satisfied:
wms_1. $\phi(1, 0, ..., 0) = 1$ and $\psi(v) = \phi(v\epsilon)$ if $\epsilon \in E_{n+1}(A)$.
wms_2. If the row $(a, a_1, ..., a_n) \in \mathsf{Um}_{n+1}(A)$ be such that $(1-a, a_1, ..., a_n)$ is also a unimodular row on length $n+1$ then $\phi((a, a_1, ..., a_n))\phi((1-a, a_1, ..., a_n)) = \phi(a(1-a), a_1, ..., a_n)$.

Remark 12.5.4 Clearly, a universal weak Mennicke symbol $(wms, WMS_{n+1}(A))$ exists. It is universal in the sense that for any Mennicke symbol (ϕ, G) of length $n+1 \geq 3$, the map $\phi : \mathrm{Um}_{n+1}(A) \to G$ factors through the map wms via a unique morphism $WMS_{n+1}(A) \to G$.

For any commutative Noetherian ring A of dimension n, W. van der Kallen defined in [69], an abelian group structure on $\mathrm{Um}_{n+1}(A)/E_{n+1}(A)$. Moreover in the same paper it was shown that the group $\mathrm{Um}_{n+1}(A)/E_{n+1}(A)$ coincides with the weak Mennicke symbol $WMS_{n+1}(A)$. Thus we will stick to the notation $WMS_{n+1}(A)$ only.

Definition 12.5.5 Let A be a ring. The abelian group $WMS_{n+1}(A)$ is said to have a nice group structure if for any two $(a, a_1, ..., a_n)$ and $(b, a_1, ..., a_n) \in \mathrm{Um}_{n+1}(A)$,

$$[(a, a_1, ..., a_n)] \star [(b, a_1, ..., a_n)] = [(ab, a_1, ..., a_n)]$$

holds, where $[-]$ denotes the class in the elementary orbit space of unimodular rows of length $n+1$.

Now recall the following variation of the Mennicke-Newman Lemma. Following proof is essentially taken from ([70], Lemma 3.2).

Lemma 12.5.6. *Let A be a ring of dimension $n \geq 1$ and $u, v \in \mathrm{Um}_{n+1}(A)$. Then there exists $\epsilon, \delta \in \mathrm{E}_{n+1}(A)$ and $x, y, a_i \in A$, $i = 1, ..., n$ such that $u\epsilon = (x, a_1, ..., a_n)$, $v\delta = (y, a_1, ..., a_n)$ and $x + y = 1$.*

Proof Let $u = (u_0, ..., u_n)$ and $v = (v_0, ..., v_n)$. Then note that $(u_0v_0, u_1, ..., u_n, v_1, ..., v_n) \in \mathrm{Um}_{2n+1}(A)$. Since $2n+1 \geq n+2$, by adding suitable multiples of u_0v_0 to $u_1, ..., u_n, v_1, ..., v_n$ we can make $(u_1, ..., u_n, v_1, ..., v_n) \in \mathrm{Um}_{2n}(A)$ (without changing the notations of u_is and v_is). Thus adding suitable multiples of $u_1, ..., u_n$ with u_0 and $v_1, ..., v_n$ with v_0 we can make $u_0 + v_0 = 1$ (without changing the notations of u_0 and v_0). Now by replacing u_i by adding a suitable multiple of u_0 with u_i and v_i by adding a suitable multiple of v_0 with v_i for all $i \geq 1$, we can have $u_i = v_i$ for all $i \geq 1$. $\square$

Proposition 12.5.7. *Let A be a ring of dimension $n \geq 2$. Suppose that the universal weak Mennicke symbol group $WMS_{n+1}(A)$ has a nice group structure. Then $WMS_{n+1}(A) \cong MS_{n+1}(A)$.*

Proof Since $WMS_{n+1}(A)$ has a nice group structure it is a Mennicke symbol of length $n+1$. Thus there exists a unique group morphism $f : MS_{n+1}(A) \to WMS_{n+1}(A)$ such that the following diagram commutes

$$\begin{tikzcd} \mathrm{Um}_{n+1}(A) \arrow[r, "ms"] \arrow[dr, "wms"'] & MS_{n+1}(A) \arrow[d, "f"] \\ & WMS_{n+1}(A) \end{tikzcd}$$

Again since $MS_{n+1}(A)$ is a weak Mennicke symbol of length $n+1$, there exists unique group morphism $g : WMS_{n+1}(A) \to MS_{n+1}(A)$ such that the following diagram commutes

$$\begin{tikzcd} \mathrm{Um}_{n+1}(A) \arrow[r, "wms"] \arrow[dr, "ms"'] & WMS_{n+1}(A) \arrow[d, "g"] \\ & MS_{n+1}(A) \end{tikzcd}$$

Note that to show the required isomorphism it is enough to show that both $f \circ g$ and $g \circ f$ are identity maps. To show this consider the following commutative diagram

$$\begin{tikzcd} \mathrm{Um}_{n+1}(A) \arrow[r, "wms"] \arrow[dr, "wms"'] & WMS_{n+1}(A) \arrow[d] \\ & WMS_{n+1}(A) \end{tikzcd}$$

Note that the vertical map is unique. Also observe that both the maps $i_d : WMS_{n+1} \to WMS_{n+1}(A)$ and $f \circ g : WMS_{n+1} \to WMS_{n+1}(A)$ satisfies the above commutative diagram. Hence $f \circ g = i_d$. Following the similar argument one can established the fact that $g \circ f = i_d$. This completes the proof. $\square$

We end this section with result due to Bass-Kubota. For a proof one can see ([69], Theorem 2.12).

Theorem 12.5.8. *Let A be a ring of dimension 1 and $I \subset A$ be an ideal. Then the Mennicke symbol MS_2 induces an isomorphism $SK_1(A, I) \xrightarrow{\sim} MS_2(A, I)$.*

12.6 Euler and weak Euler class groups

The purpose of this section is to briefly recall some definitions of Euler and weak Euler class groups. Unless otherwise stated for this section we will always assume that A is a commutative Noetherian ring containing $\mathbb{Q}$ of dimension $n \geq 2$. We recall the following definitions from [14].

Definition 12.6.1 Let G be the free abelian group on the set of pairs (J, ω_J), where:

(i) J is an ideal of A of height n;

(ii) J is m-primary for some maximal ideal m of A;

(iii) $\omega_J : (A/J)^n \twoheadrightarrow J/J^2$ is a surjective map of A/J-modules, is called a 'local orientation' of J.

Given an ideal $I \subset A$ of height n, let $I = \bigcap_i \eta_i$ be the unique irredundant primary decomposition of I, where η_i's are m_i-primary ideals, and $m_i \in m\text{Spec}(A)$ be distinct of height n. Then by the Chinese remainder theorem any local orientation ω_I uniquely defines $\omega_{\eta_i} : (A/\eta_i)^d \twoheadrightarrow \eta_i/\eta_i^2$. We associate the pair (I, ω_I), to the element $\sum_i (\eta_i, \omega_{\eta_i})$.

Let H be the subgroup of G generated by the set of pairs (J, ω_J), such that there exists a surjective $A-$module morphism $\Omega_J : A^n \twoheadrightarrow J$ with the property $\Omega \otimes A/J = \omega_J$. Such an ω_J will be called a 'global orientation' of J and such Ω_J will be called a 'lift' of ω_J.

The quotient group $E(A) := G/H$ is called the Euler class group of A.

Definition 12.6.2 Let G be the free abelian group on the set of ideals $J \subset A$, where:

(i) ht$(J) = n$;

(ii) J is m-primary for some maximal ideal m of A;

(iii) $\omega_J : (A/J)^n \twoheadrightarrow J/J^2$ is a surjective map of A/J-modules, is called a 'local orientation' of J.

Given an ideal $I \subset A$ of height n, let $I = \bigcap_i \eta_i$ be the unique irredundant primary decomposition of I, where η_i's are m_i-primary ideals, and $m_i \in m\text{Spec}(A)$ be distinct of height n. Then by the Chinese remainder theorem any local orientation ω_I uniquely defines $\omega_{\eta_i} : (A/\eta_i)^n \twoheadrightarrow \eta_i/\eta_i^2$. We associate to the (I), the element $\sum_i (\eta_i)$.

Let H be the subgroup of G generated by (J), such that $\mu(J) = n$, where $\mu(\,)$ denotes the number of minimal generator of .

The quotient group $E_0(A) := G/H$ is called the weak Euler class group of A.

The next result is due to M. Boratynski and M. P. Murthy (see [48], Theorem 2.2), which will be required in Chapter 16. Before that we will need the following definition.

Definition 12.6.3 Let A be a commutative Noetherian ring and $I \subset A$ be an ideal such that $\mu(I/I^2) = n$. I is said to be projectively generated if there exists a finitely generated projective $A-$module P of rank n and a surjection $P \twoheadrightarrow I$.

Theorem 12.6.4. *Let A be a Noetherian ring and $I \subset A$ be a local complete intersection ideal with ht$(I) = \mu(I/I^2) = n$. Moreover assume $I =< a_1, ..., a_n > +I^2$ and $J = I^{(n-1)!}+ <$*

$a_2, ..., a_n >$. Then J is a surjective image of a finitely generated projective $A-$module P of rank n (with trivial determinant).

The next result is due to N. M. Kumar. For a proof of the following version one can see ([48], Corollary 1.6).

Lemma 12.6.5. *Assume that R is reduced. Let $J \subset R$ be a local complete intersection ideal of height d and $I \subset R$ be an ideal such that $I + J = R$. Moreover assume that J and IJ are projectively generated. Then the following holds:*
(i) If R does not have any real maximal ideals. Then I is projectively generated
$(ii)\,(a)$ If the intersection of all real maximal ideals of R has height at-least 1. Then I is projectively generated provided ht$(I) \geq 2$.
$(ii)\,(b)$ If the intersection of all real maximal ideals of R has height at least 2. Then I is projectively generated.

Proof If the intersection of all real maximal ideals of R has height at-least 1 then it is done in ([48], Corollary 1.6) and if R does not have any real maximal ideals or the intersection of all real maximal ideals of R has height at-least 2 then use ([48], Theorem 1.3) taking F as the empty set. $\square$

The next lemma is a corollary of ([48], Theorem 1.3).

Lemma 12.6.6. *Let A be a reduced Noetherian ring of dimension n and $J \subset R$ be a local complete intersection ideal of height n and $I \subset R$ be an ideal such that $I^2 + J = R$. Moreover assume that J and IJ are projectively generated and ht$(I) \geq 2$. Then I is projectively generated.*

Proof Note that with the given conditions all the hypothesis of ([48], Theorem 1.3) are satisfied and hence using ([48], Theorem 1.3) we can obtain the result. $\square$

12.7 Chow groups and its divisibility

In this section we shall recall Chow groups and the n-th Chern class of a projective A-module of rank n. Let A be a reduced affine algebra of dimension n over a field k. Then $F^nK_0(A)$ denotes the subgroup of $K_0(A)$ generated by the images of all the residue fields of all smooth maximal ideals of height n. For a finitely generated projective $A-$module P of rank n, we define the $n-$th Chern class of P to be $c_n(P) := \sum(-1)^i(\wedge^i P^*)$, where P^* is the dual of P. If A is smooth, $c_n(P)$ maps to the top Chern class of P in the Chow group $CH^n(\text{Spec}(A))$ via the

Chern class map $c_n : K_0(A) \to CH^n(\mathrm{Spec}(A))$, constructed by Grothendieck. For motivation one can see ([48], Introduction).

We end the section with a couple of results (to suit our requirements) of a recent work by A. Krishna (see [34]), which will be used in Chapter 15.

Theorem 12.7.1. *([34], Theorem 6.7) Let A be a reduced affine algebra of dimension n over an algebraically closed field k and $X =$ Spec(A). Then $CH^n(X)$ is uniquely divisible.*

Theorem 12.7.2. *([34], Corollary 7.6 and 7.7) Let A be a reduced affine algebra of dimension n over an algebraically closed field k. Then $E_0(A) \cong E(A) \cong CH^n(\mathit{Spec}(A)) \cong F^n K_0(A)$ canonically.*

Chapter 13

W. van der Kallen's group structure on the orbit spaces of unimodular rows

13.1 A nice group structure of $WMS_{d+1}(R)$

For any ring A of dimension $n \geq 2$, in [31] W. van der Kallen has shown that the group $\mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A)$ is the universal weak $(n+1)$-Mennicke symbol group, $WMS_{n+1}(A)$. Obviously whenever the group $\mathrm{Um}_{n+1}(A)/\mathrm{E}_{n+1}(A)$ has a nice group structure, it coincides with the universal $(n+1)$-Mennicke symbol group $MS_{n+1}(A)$. Before going to the main results recall that R is a real affine algebra of dimension d, satisfying one of the following conditions:

- there are no real maximal ideals;
- the intersection of all real maximal ideals has height at least 1.

Theorem 13.1.1. *Let $d \geq 2$. Then the abelian group $\mathrm{Um}_{d+1}(R)/\mathrm{E}_{d+1}(R)$ has a nice group structure. That is for any $(a, a_1, ..., a_d)$ and $(b, a_1, ..., a_d) \in \mathrm{Um}_{d+1}(R)$ we have*

$$[(a, a_1, ..., a_d)] \star [(b, a_1, ..., a_d)] = [(ab, a_1, ..., a_d)].$$

In particular, $WMS_{d+1}(R) \cong MS_{d+1}(R)$.

Proof Without loss of generality we may assume that R is reduced (see [27], Lemma 3.5). Moreover if the intersection of all real maximal ideals of R has a positive height, then by Lemma

12.2.16 taking $\mathbb{S}$ to be the collection of all real maximal ideals, we may further assume that for any $2 \leq i \leq d$, $R/ < a_i, a_{i+1}, ..., a_d >$ is a smooth real affine algebra of dimension $i-1$, having no real maximal deals. Then by the product formula [69] we get

$$[(a, a_1, ..., a_d)] \star [(b, a_1, ..., a_d)] = [(a(b+p) - 1, (b+p)a_1, a_2, ..., a_d)]$$

where p is chosen such that $ap - 1 \in < a_2, ..., a_d >$.

Let $B = R/ < a_2, a_3, ..., a_d >$ and 'bar' denote going modulo $< a_2, a_3, ..., a_d >$. Then by Theorem 12.5.8 we have $SK_1(B) = MS_2(B)$. Thus in $MS_2(B)$ we get

$$[(\overline{a}(\overline{b}+\overline{p}) - \overline{1}, (\overline{b}+\overline{p})\overline{a}_1)] = [(\overline{a}(\overline{b}+\overline{p}) - \overline{1}, \overline{a}_1)].$$

Therefore we can find $\sigma \in \mathrm{SL}_2(B) \cap E_3(B)$ such that $(\overline{a}(\overline{b}+\overline{p}) - \overline{1}, (\overline{b}+\overline{p})\overline{a}_1)\sigma = (\overline{a}(\overline{b}+\overline{p}) - \overline{1}, \overline{a}_1)$. Using Theorem 12.2.14 we get $\sigma \in \mathrm{SL}_2(B) \cap Ep(B)$. Then by ([64], Corollary 2.3) there exists $\epsilon \in E_{d+1}(R)$ such that $\epsilon(a(b+p) - 1, (b+p)a_1, a_2, ..., a_d) = (a(b+p) - 1, a_1, a_2, ..., a_d)$. In other words we have

$$[(a(b+p) - 1, (b+p)a_1, a_2, ..., a_d)] = [(a(b+p) - 1, a_1, a_2, ..., a_d)]$$

. Now the choice of p gives us

$$[(a(b+p) - 1, a_1, a_2, ..., a_d)] = [(ab, a_1, a_2, ..., a_d)].$$

This completes the proof. □

Theorem 13.1.2. *Let $d \geq 2$ and $I \subset R$ be an ideal. Then the abelian group $WMS_{d+1}(R, I) = \frac{Um_{d+1}(R,I)}{E_{d+1}(R,I)}$ has a nice group structure. That is*

$$[(a, a_1, ..., a_d)] \star [(b, a_1, ..., a_d)] = [(ab, a_1, ..., a_d)]$$

where $[-]$ denotes the class in the relative elementary orbit space of unimodular rows of length $d+1$.

Proof By Proposition 12.4.2 and Lemma 12.4.4, $R \oplus I$ is a real affine algebra of dimension d satisfies the hypothesis of Theorem 13.1.1. Thus $WMS_{d+1}(R \oplus I)$ has a nice group structure. Therefore, using ([29], Lemma 3.6) the group $WMS_{d+1}(R, I)$ has a nice group structure. □

13.2 $MS_{d+1}(R)$ is a divisible group

In this section we prove a corollary of Theorem 13.1.1 which is closely related to a result by J. Fasel ([26]. Theorem 2.2).

Corollary 13.2.1. *For $d \geq 2$, The group $WMS_{d+1}(R)$ is a divisible group.*

Proof Let $v = (v_0, ..., v_d) \in \mathrm{Um}_{d+1}(R)$ and $n \in \mathbb{N}$. Then by Proposition 12.2.16 (taking $\mathbb{S}$ to be the collection of all real maximal ideals, whenever the intersection of all real maximal ideals of R has a positive height) we may assume $R/ < v_2, ..., v_d >$ is a smooth curve having no real maximal ideal. Let 'bar' denote going modulo $< v_2, ..., v_d >$ and $C = R/ < v_2, ..., v_d >$. Then by Theorem 12.5.8, we have $SK_1(C) = MS_2(C)$, which is divisible. Thus we get $\epsilon \in E_2(R)$ such that $(\overline{v}_0, \overline{v}_1) = \overline{\epsilon}(\overline{u}_0^n, \overline{u}_1)$. Hence we can always find $\gamma \in E_{d+1}(R)$ such that $v = \gamma(u_0^n, u_1, v_2, ..., v_d)$, where γ is of the form

$$\begin{pmatrix} \epsilon & *_{2,d-1} \\ 0_{d-1,2} & I_{d-1} \end{pmatrix}.$$

Thus in the group $WMS_{d+1}(R)$, we get $[v] = [u_0^n, u_1, v_2, ..., v_d]$. By Theorem 13.1.1 the group $WMS_{d+1}(R)$ has a nice group structure. Thus we have $[v] = [u_0^n, u_1, v_2, ..., v_d] = [u_0, u_1, v_2, ..., v_d]^n$. This completes the proof. $\square$

Chapter 14

Improved stability for K_1 of classical groups

Recall that R is a $d-$dimensional real affine algebra satisfies one of the two properties mentioned in Chapter 12. More often, in this chapter we shall take R to be smooth and $I \subset R$ to be a principal ideal. In this set-up we have shown that the injective stability range of $SK_1(R, I)$ and $K_1Sp(R, I)$ decreases by one.

14.1 Improved stability of SK_1

Theorem 14.1.1. *Let $I =< a >\subset R$, be a principal ideal. Let $\sigma \in \mathrm{SL}_{d+1}(R, I)$ be a stably elementary matrix. Then σ is isotopic to identity. Moreover if R is nonsingular, then $E_{d+2}(R, I) \cap \mathrm{SL}_{d+1}(R, I) = E_{d+1}(R, I)$, for $d \geq 3$.*

Proof By stability result [71], there exists $\sigma \in E_{d+2}(R, I) \cap \mathrm{SL}_{d+1}(R, I)$. Get $\tau(T) \in E_{d+2}(R[T], < T >)$ such that $\tau(a) = 1 \perp \sigma$. Let $t = T^2 - Ta \in R[T]$ be a non zero element and $v = e_1\tau(T) \in \mathrm{Um}_{d+2}(R[T], < t >)$. Thus by Theorem 12.4.8 there exists $\chi(T) \in \mathrm{SL}_{d+2}(R[T], < t >)$, such that $v = e_1\chi(T)$.
Thus $e_1\tau(T)\chi(T)^{-1} = e_1$. Hence $\tau(T)\chi(T)^{-1}$ is of the form $(1 \perp \rho(T)) \prod_{i=1}^{d+2} E_{i,1}(\lambda_i)$, where $\lambda_i \in < t >$, $\rho(T) \in \mathrm{SL}_{d+1}(R[T], < T >)$ and $\rho(a) = \sigma$. Let $\rho'(T) = \rho(aT) \in \mathrm{SL}_{d+1}(R[T], I)$.
Now since $\chi(T) \cong I_{d+2}$ modulo $< t >$, we have $\chi(0) = \chi(a) = I_{d+2}$. Thus $\rho'(0) = I_{d+1}$ and $\rho'(1) = \sigma$ that is ρ' is an isotopy of σ.
Since R is nonsingular by ([74],Theorem 3.3) $\rho(T) \in E_{d+1}(R[T], < T >)$ thus $\sigma = \rho(a) \in E_{d+1}(R, I)$. $\square$

14.2 Improved stability of K_1Sp

For the rest of the section we will shift a bit towards the symplectic matrices and prove an analogous result of the Theorem 14.1.1 for the symplectic group $K_1Sp(R, I)$. We begin with the following series of definitions from ([73], Chapter 1, Section 3 and 4).

Definition 14.2.1

1. For any matrix $\alpha \in M_r(A)$ and $\beta \in M_s(A)$ we denote $\alpha \perp \beta$ by the matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in M_{r+s}(A)$.

2. We inductively define an alternating matrix $\chi_r \in E_{2r}(A)$ as follows: $\chi_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in E_2(A)$ and $\chi_{r+1} := \chi_r \perp \chi_1$.

3. $Sp_{2n}(A) = \{\alpha \in \mathsf{SL}_{2n} : \alpha^T \chi_n \alpha = \chi_n\}$.

4. Let $I \subset A$ be an ideal then $Sp_{2n}(A, I) = \{\alpha \in Sp_{2n}(A) : \alpha \equiv I_{2n} \mod (I)\}$.

5. Let σ be the permutation of the natural numbers given by $\sigma(2i) = 2i-1$ and $\sigma(2i-1) = 2i$ for $i = 1, 2, ..., n$.

6. We define for $\lambda \in A$, $1 \leq i \neq j \leq 2n$,

$$se_{ij}(\lambda) = \begin{matrix} I_{2n} + e_{ij}(\lambda) & \text{if } i = \sigma(j); \\ I_{2n} + e_{ij}(\lambda) - (-1)^{i+j} e_{\sigma(j)\sigma(i)}(\lambda) & \text{if } i \neq \sigma(j), i < j. \end{matrix}$$

7. The subgroup of $Sp_{2n}(A)$ generated by $se_{ij}(\lambda)$, where $\lambda \in A$ and $i, j \leq 2n$ is called the elementary symplectic group $ESp_{2n}(A)$.

8. The group $ESp_{2n}(A, I)$ is defined to be the smallest normal subgroup of $ESp_{2n}(A)$ containing $se_{ij}(\lambda)$, where $\lambda \in I$ and $i, j \leq 2n$.

Theorem 14.2.2. *Let $d \equiv 1 \mod (4)$ and $I \subset R$ be an ideal of R, then $\mathrm{Um}_{d+1}(R, I) = e_1 Sp_{d+1}(R, I)$.*

Proof Note that for $d = 1$, $Sp_2(R, I) = SL_2(R, I)$ so there is nothing to prove so we may assume $d \geq 5$. Let $v \in \mathrm{Um}_{d+1}(R, I)$. Then note that if $v = (1 - i_1, i_2, ..., i_d) \in \mathrm{Um}_{d+1}(R, I)$, then v is $ESp_{d+1}(R, I)$ equivalent to $(1 - i_1, i_1 i_2, i_1 i_3, ..., i_1 i_d) \in \mathrm{Um}_{d+1}(R, < i_1 >) \subset \mathrm{Um}_{d+1}(R, I)$. So without loss of generality we may assume that $I =< t >$ is a principal ideal.

Using ([3] Proposition 3.1 and Theorem 3.2) It is enough to show $\frac{\mathrm{Um}_{d+1}(R,<t>)}{E_{d+1}(R,<t>)}$ has a nice group structure. To show that by ([29], Lemma 3.6) it is enough to show $WMS_{d+1}(R\oplus <t>)$ has a nice group structure. By ([29] Proposition 4.1) $R\oplus <t>$ is also a real affine algebra of dimension d. By Lemma 12.4.4 $R\oplus <t>$ also satisfies one of the properties that either there are no real maximal ideals, or the intersection of all real maximal ideals has height at least 1. Hence the result follows from Theorem 13.1.1. □

The proof of the next result will follow verbatim that of ([74], Theorem 3.3) in the linear case. One may also see ([4], Theorem 3.8) for details.

Theorem 14.2.3. *Let A be a regular ring essentially of finite type over a field k. Then $Sp_{2r}(A[X],(X)) = ESp_{2r}(A[X],(X))$, for $r \geq 2$.*

Theorem 14.2.4. *Let R be nonsingular. Let $d \geq 4$ and $I =< a >\subset R$ be a principal ideal. Moreover assume that if d is even then $4|d$. Let $n = 2[\frac{d+1}{2}]$, where $[-]$ denotes the smallest integer less than or equals to $-$. Then $K_1Sp(R,I) = \frac{Sp_n(R,I)}{Ep_n(R,I)}$.*

Proof Enough to show that $Sp_n(R,I)\cap Ep_{n+2}(R,I) = Ep_n(R,I)$. The proof is divided into the following cases:

Case-1 (d is odd) For d to be odd note that $n = d+1$. Let $\sigma \in Sp_{d+1}(R,I)\cap Ep_{d+3}(R,I)$. Then by ([17], Corollary 6.4) get $\rho(T) \in Sp_{d+1}(R[T])$ such that $\rho(1) = \sigma$ and $\rho(0) = I_{d+1}$. Replacing $\rho(T)$ by $\rho(aT)$ we may further assume $\rho(a) = I_{d+1}$. Since R is nonsingular by ([28], Theorem 5.3) $\rho(T) \in Ep_{d+1}(R[T],<T>)$ and thus $\sigma = \rho(a) \in Ep_{d+1}(R,I)$.

Case-2 (4 divides d) Here note that $n = d$. Let $\sigma \in Sp_d(R,I) \cap Ep_{d+2}(R,I)$. Get $\rho(T) \in Ep_{d+2}(R[T])$ such that $\rho(0) = I_{d+2}$ and $\rho(1) = I_2 \perp \sigma$. Further replacing $\rho(T)$ by $\rho(aT)$ we may assume $\rho(a) = I_2 \perp \sigma$. Let $v(T) = e_1\rho(T) \in \mathrm{Um}_{d+2}(R[T])$, then note that $v(T) = e_1 \mod <T^2 - T>$ i.e. $v(T) \in \mathrm{Um}_{d+2}(R[T],<T^2 - aT>)$. Then by Lemma 14.2.2, get $\alpha(T) \in Sp_{d+2}(R[T],<T^2 - aT>)$, such that $v(T) = e_1\rho(T) = e_1\alpha(T)$. Then note that

$$\rho(T)\alpha(T)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & * \\ * & 0 & \eta(T) \end{pmatrix}$$

for some $\eta(T) \in Sp_d(R[T],<T>) = Ep_d(R[T],<T>)$ (since R is regular) by Theorem 14.2.3. Thus $\eta(T)$ is a symplectic homotopy of σ. Therefore $\sigma = \eta(a) \in Ep_d(R,I)$. □

Chapter 15

Divisibility of the Euler class group $E(R)$

In this chapter, in addition we assume that R is reduced. We show that there is a canonical isomorphism between the groups $E(R)$, $E_0(R)$ and $F^dK_0(R)$. In particular, we prove that the group $E(R)$ is uniquely divisible.

15.1 A natural map $\delta_R : E_0(R \otimes_{\mathbb{R}} \mathbb{C}) \to E_0(R)$

We begin with an easy consequence of ([14], Lemma 5.4) which is crucial to this section.

Theorem 15.1.1. *Let $d \geq 2$. Suppose that R is a $d-$dimensional real affine algebra having no real maximal ideals. Then the canonical surjective map $E(R) \twoheadrightarrow E_0(R)$ is an isomorphism. In particular for any ideal $I \subset R$ with ht$(I) = \mu(I/I^2) = d$, and for any two local orientations ω_I and ω_I' of I one must have $(I, \omega_I) = (I, \omega_I') = (I)$.*

Notation 15.1.2. We shall introduced the following notations:

- Let (I) be the class of the ideal I in $E(R)$ irrespective of any local orientation.
- Let $R_{\mathbb{C}}$ be the "complexification" of the real affine algebra R, that is $R_{\mathbb{C}} := \frac{R[T]}{<T^2+1>} \cong R \otimes_{\mathbb{R}} \mathbb{C}$. Then note that $R \hookrightarrow R_{\mathbb{C}}$ is an integral extension. For any ideal $I \subset R$, $I_{\mathbb{C}}$ will be denote as the extension of the ideal I in the ring $R_{\mathbb{C}}$.

Lemma 15.1.3. *Suppose that R is a real affine algebra having no real maximal ideals. Then any maximal ideal of $R_{\mathbb{C}}$ is extended from R. In other words, for any $M \in m$Spec$(R_{\mathbb{C}})$, let $m = M \cap R \in m$Spec(R). Then we have $M = m_{\mathbb{C}}$, where $m_{\mathbb{C}} = m[T] + < T^2 + 1 >$ (to be precise $m_{\mathbb{C}}$ is the image of $m[T] + < T^2 + 1 >$ in $\frac{R[T]}{<T^2+1>}$).*

Proof For any maximal ideal $\eta \subset R$ define $\eta_{\mathbb{C}} := \eta[T]+ < T^2+1 >$. Note that $R/\eta \hookrightarrow R[T]/ < \eta[T], T^2+1 >$ is an integral extension and since $R/\eta \cong \mathbb{C}$ we have $\mathbb{C} \cong R/\eta \cong R[T]/ < \eta[T], T^2+1 >$. Hence we have $R_{\mathbb{C}}/\eta_{\mathbb{C}} = R[T]/ < \eta[T], T^2+1 > \cong R/\eta \cong \mathbb{C}$, that is $\eta_{\mathbb{C}} \in m\text{Spec}(R_{\mathbb{C}})$. Since $R \hookrightarrow R_{\mathbb{C}}$ is integral, for any maximal ideal $M \subset R_{\mathbb{C}}$, $M \cap R$ is also a maximal ideal of R. Let $m = M \cap A$. Then $m_{\mathbb{C}} \subset M$ and we have shown that for any maximal ideal $m \subset R$, $m_{\mathbb{C}} \subset R_{\mathbb{C}}$ is also a maximal ideal. Hence $m_{\mathbb{C}} = M$. □

Lemma 15.1.4. *Suppose that R is a real affine algebra having no real maximal ideals. Let $\mathfrak{I} \subset R_{\mathbb{C}}$ be such that ht$(\mathfrak{I}) = d$. Then ht$(I) = d$ and $I_{\mathbb{C}} = \mathfrak{I}$, where $I = \mathfrak{I} \cap R$.*

Proof Since R is reduced $R_{\mathbb{C}}$ is also reduced. Then by Lemma 15.1.3 we can take $\mathfrak{I} = m^1_{\mathbb{C}} \cap ... \cap m^n_{\mathbb{C}}$, where each $m^i \in m\text{Spec}(R)$ of height d. Then $I = \mathfrak{I} \cap R = m^1 \cap ... \cap m^n$ and thus ht$(I) = d$. Now $I_{\mathbb{C}} = I+ < T^2+1 >= m^1 \cap ... \cap m^n + < T^2+1 >=< m^1, T^2+1 > \cap ... \cap < m^n, T^2+1 >= m^1_{\mathbb{C}} \cap ... \cap m^n_{\mathbb{C}} = \mathfrak{I}$. □

Definition 15.1.5 (A map $\delta_R : E_0(R_{\mathbb{C}}) \to E_0(R)$) Suppose that R is a real affine algebra having no real maximal ideals of dimension $d \geq 2$. By Lemma 15.1.4, for any $I_{\mathbb{C}} \subset R_{\mathbb{C}}$ of height d, $I := I_{\mathbb{C}} \cap R \subset R$ is an ideal of height d. Thus we can always choose a set of generators of $I_{\mathbb{C}}$ are coming from I. Hence any set of generators of $I_{\mathbb{C}}/I^2_{\mathbb{C}}$ are also coming from I/I^2. Thus note that for any $(I_{\mathbb{C}}) \in E_0(R_{\mathbb{C}})$, we can chose $a_i \in I$ such that $I_{\mathbb{C}} =< a_1, ..., a_d > R_{\mathbb{C}} + I^2_{\mathbb{C}}$. Hence we can define the natural map $\delta_R : E_0(R_{\mathbb{C}}) \to E_0(R)$, by $\delta_R((I_{\mathbb{C}})) = (I)$.

Remark 15.1.6 Note that by the definition, δ_R is well defined as for any ideal $I_{\mathbb{C}} \subset R_{\mathbb{C}}$ with ht$(I_{\mathbb{C}}) = \mu(I_{\mathbb{C}}/I^2_{\mathbb{C}}) = d$, there exists unique $I = I_{\mathbb{C}} \cap R \subset R$, with the property ht$(I) = \mu(I/I^2) = d$.

Theorem 15.1.7. *The natural map $\delta_R : E_0(R_{\mathbb{C}}) \to E_0(R)$ is an isomorphism.*

Proof Let $(I_{\mathbb{C}})$ and $(J_{\mathbb{C}}) \in E_0(R_{\mathbb{C}})$ and $(K_{\mathbb{C}}) = (I_{\mathbb{C}}) - (J_{\mathbb{C}})$ in $E_0(R_{\mathbb{C}})$. Then it is enough to show that $(I) - (J) = (K)$ in $E_0(K)$. By moving Lemma ([33], Corollary 2.14) get $K' \subset R$ of height d, comaximal with I and J such that $J \cap K'$ is complete intersection. Then $(K') = -(J)$ and thus $(I) - (J) = (I) + (K') = (I \cap K')$ in $E_0(R)$. Now the choice of K' also gives us the fact that $K'_{\mathbb{C}} \subset R_{\mathbb{C}}$ of height d, comaximal with $I_{\mathbb{C}}$ and $J_{\mathbb{C}}$ such that $J_{\mathbb{C}} \cap K'_{\mathbb{C}}$ is complete intersection. Then $(K'_{\mathbb{C}}) = -(J_{\mathbb{C}})$ and thus $(K_{\mathbb{C}}) = (I_{\mathbb{C}}) - (J_{\mathbb{C}}) = (I_{\mathbb{C}}) + (K'_{\mathbb{C}}) = (I_{\mathbb{C}} \cap K'_{\mathbb{C}}) = ((I \cap K')_{\mathbb{C}})$ in $E_0(R_{\mathbb{C}})$. Thus $(K) = \delta_R(K_{\mathbb{C}}) = \delta_R(((I \cap K')_{\mathbb{C}})) = (K' \cap I) = (I) - (J)$.

Let $(I_{\mathbb{C}}) \in E_0(R_{\mathbb{C}})$ be such that I is a complete intersection ideal in R then so is $I_{\mathbb{C}}$ in $E_0(R_{\mathbb{C}})$ and thus δ_R is injective.

Let $(I) \in E_0(R)$. Since R is reduced we can take $I = m^1...m^n$, where m^i are maximal ideals of height d and $(I) = (m^1) + ... + (m^n)$. Then $I_{\mathbb{C}} = m^1_{\mathbb{C}}...m^n_{\mathbb{C}}$ and $\delta_R(I_{\mathbb{C}}) = I$. □

15.2 $E(R)$ is uniquely divisible

Theorem 15.2.1. *$E(R)$ is uniquely divisible.*

Proof Note that if the intersection of all real maximal ideals of R has a positive height, then by ([48], Lemma 2.10 and Remark 2.13) $E(R)$ is uniquely divisible. Hence it is enough to assume that R has no real maximal ideals. By ([34], Corollary 7.6) $E_0(R_{\mathbb{C}})$ is uniquely divisible and therefore $E_0(R)(\cong E(R))$ is uniquely divisible by Theorem 15.1.7. □

Theorem 15.2.2. *If R has no real maximal ideals then there is a canonical isomorphism $E_0(R) \cong E(R) \cong F^dK_0(R)$.*

Proof By Theorem 15.1.7 we have $E_0(R) \cong E(R)$ canonically. Also note that the canonical map $\delta_R : F^dK_0(R_{\mathbb{C}}) \to F^dK_0(R)$ is an isomorphism as there is a natural one-to-one correspondence between smooth maximal ideals of $R_{\mathbb{C}}$ and R. Hence we have the following commutative diagram:

$$\begin{tikzcd}
F^dK_0(R_{\mathbb{C}}) \arrow[r, "\delta_R"] \arrow[d, "\simeq"] & F^dK_0(R) \arrow[d] \\
E_0(R_{\mathbb{C}}) \arrow[r, "\delta_R"] & E_0(R)
\end{tikzcd}$$

Then the induced canonical map $F^dK_0(R) \to E_0(R)$ is also an isomorphism. Thus we have $E_0(R) \cong E(R) \cong F^dK_0(R)$. □

Chapter 16

Projective generation of a curve in polynomial extension

In this chapter we prove that any local complete intersection ideal $I \subset R[T]$ with $\mu(I/I^2) = \text{ht}(I) = d$ is projectively generated.

16.1 Projective generation of a locally complete intersection ideal of top height

We begin with recalling an easy computation (see [6], Remark 3.2.).

Lemma 16.1.1. *Let $I, J \subset R$ be two ideals of height d. Suppose that there exists $a_i \in R$ such that $I =< a_1, ..., a_d > +I^2$ and $J =< a_1, ..., a_{d-1} > +I^{(d-1)!}$. Then $(J) = (d-1)!(J)$ in $E(R)$.*

The following proposition is crucial to our main results in this section. The proposition asserts that it is enough to prove Theorem 16.1.3 for reduced rings only. Therefore the divisibility of the Euler class group comes into play. The idea of the proof follows from ([9], Proposition 2.15).

Proposition 16.1.2. *Let $I \subset R$ be such thath ht$(I) = \mu(I/I^2) = d$. Let η be the nilradical of R. Moreover assume that $\overline{I}$ is projectively generated, where 'bar' denotes going modulo η, then so is I.*

Proof Let $R_{red} = R/\eta$. There exists a projective $R_{red}-$module P' of rank d and a surjection $P' \twoheadrightarrow \overline{I}$. Thus we can find $\chi' : R_{red} \cong \wedge^d(P')$ such that $e(P', \chi') = (\overline{I}, \omega') = (\overline{I})$ in $E(R_{red})$, for some local orientation ω'. Using ([76], Lemma 2.2) we can find a projective

$R-$module P of rank d such that $P \otimes A/\eta \cong P/\eta P \cong P'$. Since the canonical map from $R^* \twoheadrightarrow (R_{red})^*$ is surjective, we may assume that $P/\eta P = P'$ and χ' is induced by some isomorphism $\chi : R \cong \wedge^d(P)$.

We choose an isomorphism $\sigma : (P/IP) \cong (R/I)^d$ such that $\wedge^d \sigma = \chi \otimes R/I$. We obtain the surjection $\alpha : P/IP \twoheadrightarrow I/I^2$ which is the composite $P/IP \to (R/I)^d \to I/I^2$. Thus note that $e(P,\chi) = (I,\alpha) = (I)$ in $E(R)$. Since $(\overline{I}) = e(P',\chi') = e(P/\eta P, \chi \otimes R/\eta)$ there exists $\beta : P' \twoheadrightarrow \overline{I}$ which lifts $\alpha \otimes R/\eta$. Now consider the following patching diagram:

$$\begin{tikzcd}
 & P/(I\cap\eta)P \arrow[rr] \arrow[dl, dashed, two heads, "\phi"'] \arrow[dd, two heads] & & P' = P/\eta P \arrow[dl, two heads, "\beta"] \arrow[dd] \\
I/(I^2\cap\eta) \arrow[rr] \arrow[dd] & & I/(\eta\cap I) = \overline{I} \arrow[dd] & \\
 & P/IP \arrow[rr] \arrow[dl, two heads, "\alpha"] & & P/(\eta+I)P \arrow[dl] \\
I/I^2 \arrow[rr] & & I/(I\cap\eta+I^2) &
\end{tikzcd}$$

We get a surjection $\phi : P/(I\cap\eta)P \twoheadrightarrow I/(I^2\cap\eta)$, by patching the surjections α and β. Then by the definition of a projective module we can obtain a map $\theta : P \to I$, such that the following diagram commutes.

$$\begin{tikzcd}
 & I \arrow[d, two heads] \\
P \arrow[ur, dashed, "\theta"] \arrow[r, two heads, "\phi\circ\pi"] & I/(I^2\cap\eta)
\end{tikzcd}$$

Where $\pi : P \twoheadrightarrow P/(I\cap\eta)P$ is the surjective quotient map. Thus we have $\theta(P) + I^2 \cap \eta = I$. Now an easy local checking ensures that $I = \theta(P)$. $\square$

Now we are ready to prove our main results of the section.

Theorem 16.1.3. *Any local complete intersection ideal $I \subset R$ with ht$(I) = \mu(I/I^2) = d \geq 3$, is projectively generated.*

Proof By Proposition 16.1.2 it is enough to take R to be reduced. Thus $E(R)$ is divisible by Theorem 15.2.1. We get $J \subset R$ with ht$(J) = \mu(J/J^2) = d$ such that $(I) = (d-1)!(J)$ in $E(R)$. Let $J =< a_1, ..., a_d > +J^2$ and $I' =< a_1, ..., a_{d-1} > +J^{(d-1)!}$. Then by Lemma 16.1.1, $(I') = (d-1)!(J)$ in $E(R)$.

Then by ([48], Theorem 2.2) there exists a surjection $P \twoheadrightarrow I'$, where P is a finitely generated projective R-module of rank d with trivial determinant. Thus we have $(I') = (I)$.

Since there are canonical isomorphisms between $F^dK_0(R) \cong E_0(R) \cong E(R)$, we must have $(R/I) = (R/I')$ in $F^dK_0(R)$. As J' is a surjective image of P, in $K_0(R)$ we have $C_d(P^*) = (R/I')$. That is we have $C_d(P^*) = (R/I)$ where $C_d(-)$ denotes the d-th Chern class.

Consider a surjection $\overline{\alpha} : P/IP \twoheadrightarrow I/I^2$. Let $f : P \to I$ be any lift of $\overline{\alpha}$ (that is $f \otimes R/I = \overline{\alpha}$). Then we have $I = f(P) + I^2$. Then we can find $I'' \subset R$ comaximal with I of height d such that $I \cap I'' = f(P)$. Thus in $K_0(R)$ we have $(R/I') = (R/I) = C_d(P*) = (R/II'') = (R/I) + (R/I'')$. Thus we get $(R/I'') = 0$ in $F^dK_0(R)$ and hence $(I'') = 0$ in $E(R)$.

Let 'bar' denote going modulo I''. Get $\delta^{-1} : P/I''P \xrightarrow{\sim} (R/I'')^d$ such that $\wedge^d\delta = \overline{\chi}$, where $\chi : \wedge^d P \cong R$. Let $\overline{\beta} = \overline{f}\delta : (R/I'')^d \twoheadrightarrow I''/I''^2$. Since $(I'') = 0$ get $\beta : R^d \twoheadrightarrow I''$ such that $\beta \otimes R/I'' = \overline{\beta}$ and $f : P \twoheadrightarrow I \cap I''$. Then note that $(\beta \otimes A/I'')\delta^{-1} = \overline{\beta}\delta^{-1} = \overline{f}$. Hence by Subtraction principle ([14], Theorem 3.3) there exists $\theta : P \twoheadrightarrow I$ such that $\theta \otimes R/I = f \otimes R/I = \overline{\alpha}$. □

Corollary 16.1.4. *Let $I \subset R$ be a locally complete intersection ideal of height $d \geq 3$ and P be a projective R-module of rank d. Suppose that $\overline{f} : P/IP \twoheadrightarrow I/I^2$ be a surjective map. Then there exists a surjective lift $f : P \twoheadrightarrow I$ of $\overline{f}$ if and only if $C_d(P^*) = (R/I)$ in $K_0(R)$.*

Proof Again, as before we may begin with the assumption that R is reduced. Note that if $f : P \twoheadrightarrow I$ then we have $C_d(P^*) = (R/I)$ in $K_0(R)$. So we assume that $C_d(P^*) = (R/I)$. Let $\alpha : P \to I$ be any lift (might not be surjective) of $\overline{f}$ (that is $\alpha \otimes R/I = \overline{f}$). Then we have $I = \alpha(P) + I^2$. Therefore we can find $I'' \subset R$ comaximal with I of height d such that $I \cap I'' = \alpha(P)$. In the group $K_0(R)$ we have $(R/I) = C_d(P^*) = (R/II'') = (R/I) + (R/I'')$. Therefore we get $(R/I'') = 0$ in $F^dK_0(R)$. Hence $(I'', \omega) = 0$ in $E(R)$ for any local orientation of I''.

Let 'bar' denote going modulo I''. Get $\delta^{-1} : P/I''P \xrightarrow{\sim} (R/I'')^d$ such that $\wedge^d\delta = \overline{\chi}$, where $\chi : \wedge^d P \cong R$. Let $\overline{\beta} = \overline{f}\delta : (R/I'')^d \twoheadrightarrow I''/I''^2$. Since $(I'') = 0$ get $\beta : R^d \twoheadrightarrow I''$ such that $\beta \otimes R/I'' = \overline{\beta}$ and $\alpha : P \twoheadrightarrow I \cap I''$. Then note that $(\beta \otimes R/I'')\delta^{-1} = \overline{\beta}\delta^{-1} = \overline{f}$. Hence by Subtraction principle ([14], Theorem 3.3) there exists $f : P \twoheadrightarrow I$ such that $f \otimes R/I = \alpha \otimes R/I = \overline{f}$. □

The next corollary is just a restatement of the above result in terms of Euler class groups and therefore we skip the proof.

Corollary 16.1.5. *Let $I \subset R$ be a locally complete intersection ideal of height $d \geq 3$ and P be a projective R-module of rank d. Suppose that $\overline{f} : P/IP \twoheadrightarrow I/I^2$ be a surjective map. Then there exists a lift $f : P \twoheadrightarrow I$ of $\overline{f}$ if and only if $e(P) = (I)$ in $E_0(R)$.*

In fact now we can prove a stronger version of Theorem 16.1.3.

Corollary 16.1.6. *Let $I \subset R$ be an ideal which is not contained in any minimal primes, such that $\mu(I/I^2) = d$. Then I is projectively generated in the following cases:*
(i) Whenever R has no real maximal ideal.
$(ii)\,(a)$ Whenever the intersection of all real maximal ideals of R has height at least 1 and ht$(I) \geq 2$.
$(ii)\,(b)$ Whenever the intersection of all real maximal ideals of R has height at-least 2.

Proof Let $I =< a_1, ..., a_d > +I^2$. By Lemma 2.1.3 get $e \in I^2$ be such that $e(1-e) \in< a_1, ..., a_d >$. Moreover by Theorem 2.1.4 replacing a_i by $a_i + \lambda_i e$ we may assume ht$(< a_1, ..., a_d >_e) = d$. Let $J =< a_1, ..., a_d, 1-e >$. Then note that
(i) ht$(J) = d$,
$(ii)\, I + J = R$,
$(iii)\, J =< a_1, ..., a_d > +J^2$.
Then by Theorem 16.1.3 J is projectively generated and so is I by Lemma 12.6.5. □

16.2 Projective generation of a curve in a polynomial extension

Theorem 16.2.1. *Any local complete intersection ideal $I \subset R[T]$ with ht$(I) = \mu(I/I^2) = d \geq 3$, is projectively generated.*

Proof Since $\mathbb{Q} \subset R$ we may assume that there exists $\lambda \in R$ such that either $I(\lambda) = R$ or ht$(I(\lambda)) = d$. Furthermore taking the transformation $T \to T - \lambda$ we may assume that either $I(0) = R$ or ht$(I(0)) = d$.

Suppose that we have ht$(I(0)) = d$. Then note that $I(0)$ is a local complete intersection ideal of height d. By Theorem 16.1.3, there exists a projective R-module P with trivial determinant, such that $f : P \twoheadrightarrow I(0)$ is a surjection. Therefore we have $C_d(P^*) = (R/I(0))$. Since $P[T]/IP[T] \cong (R[T]/I)^d$ and $\mu(I/I^2) =$ ht$(I) = d$, we get a surjection $\overline{\omega} : P[T]/IP[T] \twoheadrightarrow I/I^2$. This will induce $\overline{\omega(0)} : P/I(0)P \twoheadrightarrow I(0)/I(0)^2$. By Corollary 16.1.4 we can lift $\overline{\omega(0)}$. Altering f by a lift of $\overline{\omega(0)}$, we may further assume that $f \otimes R/I(0) = \overline{\omega(0)}$. Using ([12], Remark 3.9) $\overline{\omega}$ can be lifted to $\omega(T) : P[T] \twoheadrightarrow I/I^2T$. By ([9], Theorem 3.6) there exists a projective $R[T]-$module Q of rank d with $Q/TQ \cong P$ such that $Q \twoheadrightarrow I$ is a surjection.

If $I(0) = R$ then by ([12], Remark 3.9) any local orientation ω of I can be lifted to a surjection $(R[T])^d \twoheadrightarrow I/I^2T$. Therefore we may again apply ([9], Theorem 3.6), as before, to complete the proof. □

Theorem 16.2.2. *The canonical map $\Gamma : E(R[T]) \to E(R(T))$ is injective.*

Proof The proof is essentially reducing the question in terms of injectivity of the canonical map $\Gamma' : E(R) \to E(R(T))$. Note that whenever R is a local ring the Jacobson radical is the maximal ideal of R which is of height d, hence Γ is injective by ([19], Proposition 5.8. (1)). Therefore, in view of ([19], Theorem 5.4) it is enough to prove that the injectivity of the canonical map $\Gamma' : E(R) \to E(R(T))$.

Let $(I, \omega) \in E(R)$ be such that $(I \otimes R(T), \omega') = 0$ in $E(R(T))$, where $\omega' = \omega \otimes R(T)$. Then by Theorem 16.1.3, there exists a projective R-module P with trivial determinant of rank d and $\chi : \wedge^d P \cong R$ such that $e(P, \chi) = (I) = (I, \omega)$ in $E(R)$. Since $(I \otimes R(T), \omega') = 0$, by ([14], Corollary 4.4) $P \otimes R(T)$ has a unimodular element. Therefore by ([15], Theorem 3.4) $P \otimes R[T]$ has a unimodular element. Using ([19], Corollary 4.11) we get $e(P \otimes R[T], \chi \otimes R[T]) = (I \otimes R[T], \omega \otimes R[T]) = 0$ in $E(R[T])$. Since the canonical map $E(R) \to E(R[T])$ is injective (by [19], Theorem 5.4) we get $e(P, \chi) = (I, \omega) = 0$. □

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List of publications

1. S. Banerjee. Projective and stably free modules over real affine algebras. Nov. 2021. (available at https://arxiv.org/abs/2111.14384).

2. S. Banerjee and M. K. Das. On a question of Nori and it's applications. Feb. 2022. (available at https://arxiv.org/abs/2202.11676).

3. S. Banerjee and M. K. Das. Splitting criteria of projective modules on polynomial extensions over various base rings. Jun. 2022. (available at https://arxiv.org/abs/2206.06819).