Spectral aspect subconvex bounds for some L-functions

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Dedicated to Papa and Mummy

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Contents

Notations & Abbreviations

$e(z)$	$e^{2\pi iz}$
$a \sim b$	$a \leq b < 2b$
$f(x) \sim g(x)$	$f(x)/g(x) \to 1$, as $x \to \infty$
$\alpha \ll A$	$\|\alpha\| \leq cA$, c some absolute postive constant
$\alpha \ll_f A$	$\|\alpha\| \leq c(f)A$, $c(f)$ some postive constant depending on f
$\alpha = O_f(A)$	$\|\alpha\| \leq c(f)A$
ϵ	an arbitrarily small constant
$\mathcal{C}_c^\infty(X)$	set of compactly supported smooth functions with support in X

Introduction

Automorphic L-functions play a significant role in the modern number theory. The most classic examples of such functions are the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \Re(s) > 1, \tag{0.0.1}$$

and the Dirichlet L-function

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \quad \Re(s) > 1, \tag{0.0.2}$$

where χ is a Dirichlet character modulo M and p runs through all primes. The above functions are related to the study of prime numbers. In fact non-vanishing of $\zeta(1+it)$, $t \in \mathbb{R}$, gives us the prime number theorem and the non-vanishing of $L(1, \chi)$ yields the Dirichlet theorem for primes in an arithmetic progression (see [23]). The Riemann zeta function $\zeta(s)$ was considerably studied by B. Riemann in his famous memoir in 1860. He proved that $\zeta(s)$ has a meromorphic continuation on the whole complex plane $\mathbb{C}$ with a simple pole at $s = 1$ and satisfies a functional equation of the form (see [23])

$$\xi(s) = \frac{1}{2}\pi^{-s/2} s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s) = \xi(1-s). \tag{0.0.3}$$

Thus $\zeta(s)$ is well understood in the half plane $\Re s > 1$ (due to the absolute convergence) and $\Re s < 0$ (due to the functional equation). In the memoir, Riemann proposed a conjucture, famously known as the Riemann Hypothesis, that all the 'non-trivial' zeros of $\zeta(s)$ lie on the 'critical' line $\Re(s) = 1/2$. It has numerous remarkable applications in number theory, one of them being the Lindelöf Hypothesis which asserts that

$$\zeta(1/2 + it) \ll_\epsilon (1 + |t|)^\epsilon,$$

for any $\epsilon > 0$. A similar phenomenon occurs for the Dirichlet L-function $L(s, \chi)$ or more generally for any degree, $d \geq 1$, automorphic L-function

$$L(s, F) = \sum_{n=1}^{\infty} \frac{\lambda_F(n)}{n^s} = \prod_p \prod_{j=1}^{d} \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1}, \quad \Re(s) > 1,$$

where $\lambda_F(n)$'s are the normalised $(\lambda_F(1) = 1)$ Fourier coefficients of an automorphic form F and $\alpha_j(p)$, $1 \leq j \leq d$, are the local parameters of $L(s, F)$ at p. Like $\zeta(s)$, $L(s, F)$ has a meromorphic continuation (with at most poles at $s = 0$ and $s = 1$) and satisfies a functional equation of the form (see [42])

$$\Lambda(s, F) = q(F)^{s/2} \pi^{-ds/2} \prod_{j=1}^{d} \Gamma\left(\frac{s + \kappa_j}{2}\right) L(s, F) = \varepsilon(F) \Lambda(1 - s, \bar{F}), \qquad (0.0.4)$$

where $q(F)$ is the conductor, $\varepsilon(F)$ is the root number of $L(s, F)$, κ_j's $(\in \mathbb{C})$ are the Langlands parameters of $L(s, F)$ at infinity and $\bar{F}$ is the dual form of F.

It is one of the most sought-after problems to understand the size of $L(s, F)$ on the critical line $\Re(s) = 1/2$. It turns out that the size of $L(1/2 + it, F)$ can be expressed in terms of the analytic conductor $\mathfrak{q}(t, F)$, which is defined as (see [[42], Chapter 5])

$$\mathfrak{q}(t, F) = q(F) \prod_{j=1}^{d} (3 + |1/2 + it + \kappa_j|). \qquad (0.0.5)$$

Using the functional equation and the Phragmén-Lindelöf convexity principle, it follows that

$$L(1/2 + it, F) \ll_\epsilon \mathfrak{q}(t, F)^{1/4+\epsilon}. \qquad (0.0.6)$$

The above bound is known as the convexity bound. It is conjectured that

$$L(1/2+it,F) \ll_\epsilon \mathfrak{q}(t,F)^\epsilon,$$

with any $\epsilon > 0$. This is known as the generalized Lindelöf Hypothesis which follows from the Grand Riemann Hypothesis (GRH). In many applications (which we discuss shortly), the crucial step is to improve on the convexity bound for the relevent L-functions, in the sense of reducing the exponent $1/4$ by a positive number, however small that number may be. Such bounds are known as subconvexity bounds. Mostly, in applications, one seeks subconvexity estimates with respect to some subfamily (i.e. only one of the parameters t, $q(F)$ or (κ_j) varies). If t varies (F is fixed), we call it t-aspect, if $q(F)$ varies, we call it level aspect and if (κ_j) varies, we call it spectral aspect. Getting subconvexity bounds is a challanging task and has remained open for most L-functions. Moreover, for higher degree L-functions, the subconvexity problem becomes even more difficult due to the increase in the complexity of the L-functions and the lack of sufficient tools. The purpose of this thesis to obtain subconvex bounds for some specific degree six L-functions in the spectral aspect. Now we briefly recall some important instances of subconvexity bounds and their applications.

0.1 Degree one L-functions

For degree one L-functions, for $\zeta(s)$, the first t-aspect subconvexity bound was proved by Hardy-Littlewood (written down by Landau [52] in 1924) and Weyl [90] using 'Weyl differencing trick' in 1921 independently, who proved the following result:

$$\zeta(1/2+it) \ll_\epsilon (1+|t|)^{1/4-1/12+\epsilon},$$

for any $\epsilon > 0$. Since then it has been improved by several people and the latest bound $(13/84+\epsilon)$ is due to Bourgain [16]. A beautiful application of the t-aspect subconvexity for $\zeta(s)$ was given by Ingham in 1937 (see [[19], Chapter V]), who proved that

$$\zeta(1/2+it) \ll_\epsilon (1+|t|)^{c+\epsilon} \Rightarrow \pi(x+x^\theta)-\pi(x) \sim \frac{x^\theta}{\log x},$$

for any $\theta > (1+4c)/(2+4c)$. Thus, corresponding to the current bound, which is $13/84$, we get

$$\pi(x+x^{34/55+\epsilon})-\pi(x) \sim \frac{x^{34/55+\epsilon}}{\log x},$$

which also implies that $p_{n+1}-p_n < p_n^{34/55+\epsilon}$, for sufficiently large n, where p_n denotes the nth prime number (see [38] and [5] also for more details on gaps between primes). Another interesting application was given by Conrey, Ghosh and Gonek [21] who showed that if $\zeta(1/2+it) \ll_\epsilon (1+|t|)^{c+\epsilon}$ then the number of simple zeros of $\zeta(s)$ inside the rectangle

$$\{s=\sigma+it : 0<\sigma<1, 0<t<T\}$$

are at least $T^{\theta-\epsilon}$ for any $\epsilon>0$ and T large enough, where

$$\theta = \max\{1/(1+6c), (\sqrt{1+16c+16c^2}-1-4c)/4c\}.$$

Note that we expect all the zeros to be simple.

In the level aspect, i.e., when the modulus M of χ tends to ∞, the first subconvexity bound for $L(s,\chi)$ is due to Burgess [17], who proved that

$$L(1/2,\chi) \ll_\epsilon M^{1/4-1/16+\epsilon}. \tag{0.1.1}$$

He used a variant of the Weyl differencing trick but in a purely arithmetic context and with significant differences. In a breakthrough work, around four decades later, Conrey and Iwaniec [22] improved the Burgess bound to the Weyl strength bound

$$L(1/2,\chi) \ll_\epsilon M^{1/4-1/12+\epsilon}$$

when χ is a quadratic character. They used the moment method along with the non-negativity of the L-value $L(1/2, f \times \chi)$, where f is a holomorphic or Maass cusp form for $SL(2,\mathbb{Z})$. This result was recently extended to any Dirichlet characters by Petrow and Young [79], [80] following the ideas of [22]. In the depth aspect (i.e., when χ is a Dirichlet character modulo M^r, M prime and $r \to \infty$) , the latest bound is due to Milićević [66] who proved the following sub-Weyl bound

$$L(1/2, \chi) \ll (M^r)^{1/6-\delta},$$

for some $\delta > 0$. He developed the theory of p-adic exponent pairs to achieve the above result.

0.2 Degree two L-functions

For degree two L-functions, the first t-aspect subconvexity bounds were proved by Good [31] and Meurman [68] for holomorphic and Maass cusp forms for $\mathrm{SL}(2,\mathbb{Z})$ respectively. They used spectral theory of the hyperbolic Laplacian to achieve the following Weyl strength bound

$$L(1/2 + it, f) \ll_{\epsilon,f} [(1 + |t|)^2]^{1/4-1/12+\epsilon}, \tag{0.2.1}$$

for any $\epsilon > 0$. where f is a holomorphic or Maass cusp form for $\mathrm{SL}(2,\mathbb{Z})$. In [43], Jutila gave simplified proof of the above bound using summation formulae. Using the delta symbol approach developed by Munshi [71], Aggarwal and Singh [4] and Acharya, Maiti, Singh and the author [1] also obtained the t-aspect subconvex bound of the above strength.

In the level aspect, subconvexity problem for $\mathrm{GL}(2)$ L-functions were settled by Duke, Friedlander and Iwaniec in a series of articles [[24]-[28]] (see also Blomer-Harcos-Michel [8]). They developed the amplification method to achieve these results. A striking application of these results is the uniform distribution of certain lattice points in $\mathbb{Z}^3$ on a sphere centered at the origin with increasing radius, without imposing Linnik's condition and in a quantitative sense (see [63]). Indeed, consider the set

$$V_n = \{m/|m| \in \mathbb{S}^2; m \in \mathbb{Z}^3, |m|^2 = m_1^2 + m_2^2 + m_3^2 = n\}.$$

Then V_n's are uniformly distributed on $\mathbb{S}^2$ as $n \to \infty$ through integers n such that n square-free and $n \not\equiv 0,\ 4,\ 7 \bmod 8$. Cogdell, Piatetsky-Shapiro and Sarnak (see [20]), obtained subconvexity for some twists of a holomorphic Hilbert modular form over a totally real number field and resolved remaining cases of Hilbert's eleventh problem, which asks which integers are integrally represented by a given quadratic form over a number field. This result was later improved (getting a Burgess-type exponent) by Blomer and Harcos [12] and hence they obtained a better error term in Hilbert's eleventh problem. In the twist aspect, Munshi [75] applied the $\mathrm{GL}(2)$ delta symbol approach (see [74]) and obtained the following Burgess type bound

$$L(1/2, f \times \chi) \ll_{f,\epsilon} M^{1/2-1/8+\epsilon},$$

where χ is a Dirichlet character modulo M, a prime (the same exponent (in a more general setting) was previously known by Blomer-Harcos [11] using a different approach). In this same paper, he also recovered Burgess's bound (0.1.1) by taking f an Eisenstein series (Burgess bound for $\mathrm{GL}(2)$ was previously known due to Bykovskii [18] using the moment method approach). Following the $\mathrm{GL}(1)$ delta symbol approach (see [71]), Munshi and Singh [76] obtained the following Weyl strength bound

$$L(1/2, f \times \chi) \ll_{f,\epsilon} (M^r)^{1/2-1/6+\epsilon},$$

where χ is a primitive character of modulus M^r, M prime and $r \equiv 0 \bmod 3$.

In the spectral aspect, Iwaniec [40] proved subconvex bounds for Hecke Maass cusp forms for $\mathrm{SL}(2, \mathbf{Z})$ using the amplification method. In a groundbreaking work, Michel and Venkatesh [65] proved subconvexity bounds for any $\mathrm{GL}(2)$ L-functions over number fields uniformly in all parameters. Their approach uses tools of representation theory of adele groups and the proof is based on the realization of L-functions as periods.

0.3 Degree three L-functions

Let π be a Hecke-Maass cusp form of type $(\nu_1,\, \nu_2)$ for $\mathrm{SL}(3,\, \mathbb{Z})$. Let the normalised Fourier coefficinets of π be given by $\lambda_\pi(m_1, m_2)$ (so that $\lambda_\pi(1,1) = 1$). The Langlands parameters $(\alpha_1, \alpha_2, \alpha_3)$ associated with π are defined as (see [30])

$$\alpha_1 = -\nu_1 - 2\nu_2 + 1,\;\; \alpha_2 = -\nu_1 + \nu_2,\;\; \text{and}\; \alpha_3 = 2\nu_1 + \nu_2 - 1. \tag{0.3.1}$$

The dual form $\tilde{\pi}$ has Langlands parameters $(-\alpha_3, -\alpha_2, -\alpha_1)$. The L-series associated with π is given by

$$L(s, \pi) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(1,n)}{n^s}, \quad \Re(s) > 1. \tag{0.3.2}$$

$L(s, \pi)$ satisfies a functional equation of the form

$$\gamma(s,\pi) L(s,\pi) = \gamma(s, \tilde{\pi}) L(1-s, \tilde{\pi}),$$

where

$$\gamma(s,\pi) = \prod_{j=1}^{3} \pi^{-s/2} \Gamma\left(\frac{s-\alpha_j}{2}\right).$$

The first subconvexity result for a $\mathrm{GL}(3)$ L-function was proved by X. Li [58]. For a self-dual ($\tilde{\pi} = \pi$) Hecke-Maass cusp form for $\mathrm{SL}(3,\, \mathbb{Z})$, she proved the following t-aspect subconvex bound

$$L(1/2+it, \pi) \ll_{\pi,\epsilon} [(1+|t|)^3]^{1/4-1/48+\epsilon}. \tag{0.3.3}$$

Her method was motivated by the Conrey-Iwaniec [22] moment method. This result relied on non-negativity of the central values $L(1/2, \pi \times f)$, where f is holomorphic/ Maass cusp form for $\mathrm{SL}(2, \mathbb{Z})$, for which self-duality of π was necessary (see [54]). Thus her approach does not work for generic $\mathrm{GL}(3)$ forms.

In the level aspect, Blomer [9] proved a subconvex bound for quadratic twists of self-dual $\mathrm{GL}(3)$ L-functions using the moment method approach. Indeed Blomer used the Kuznetsov trace formula for $\mathrm{GL}(2)$ and the Voronoi formula for $\mathrm{GL}(3)$ to prove the following bound

$$L(1/2, \pi \times \chi) \ll_{\pi,\epsilon} M^{5/8+\epsilon},$$

where M is a large squarefree odd integer and χ is a primitive quadratic character modulo M. In this result also, non-negativity of central values of centain L-functions was playing a crucial role. Hence a new approach was needed to tackle the subconvexity problem for generic π and non-quadratic χ.

In a series of papers ([69]-[73]), Munshi developed a new and ingenious approach, popularly known as the delta method, to resolve the subconvexity problem for any degree three L-functions that allowed him to generalize Li's and Blomer's results. In [72], Munshi used the Petersson trace formula as a delta method to prove the following level aspect subconvex bound

$$L(1/2, \pi \times \chi) \ll_{\pi,\epsilon} M^{3/4-1/1612+\epsilon}, \tag{0.3.4}$$

where π is a Hecke- Maass cusp form for $\mathrm{SL}(3, \mathbb{Z})$ and χ is a Dirichlet character modulo M. While proving it he had to assume the Ramanujan conjecture and the Ramanujan-Selberg conjecture. In [74], he removed these conditions and improved the above exponent to $3/4 - 1/308 + \epsilon$ by introducing 'mass transfer trick'. By analysing Munshi's method [72] more closely Holowinsky and Nelson [35] simplified his proof by developing 'a key identity' and obtained a better exponent $3/4-1/36+\epsilon$. Their idea was extended by Lin [59] to the t-aspect who obtained the following uniform bound

$$L(1/2 + it, \pi \times \chi) \ll_{\pi,\epsilon} ((1 + |t|)M)^{3/4-1/36+\epsilon}.$$

In [71], Munshi used Kloosterman's version of the circle method to prove the following t-aspect subconvex bound

$$L(1/2 + it, \pi) \ll_{\pi,\epsilon} (1 + |t|)^{3/4-1/16+\epsilon}, \tag{0.3.5}$$

for any $\mathrm{SL}(3, \mathbb{Z})$ Hecke-Maass cusp form π. Thus he generalised Li's result (0.3.3) for all $\mathrm{GL}(3)$ L-functions. The conductor lowering trick was the new and crucial input in this paper. In [3], Aggarwal simplified Munshi's proof of (0.3.5) and obtained a better exponent $3/4 - 3/40 + \epsilon$. In the depth aspect (twists of π by a character χ of prime power modulus p^n, p odd prime and $n \to \infty$), Sun and Zhao [86] followed Munshi's t-aspect [71] approach and obtained the following subconvex bound

$$L(1/2, \pi \times \chi) \ll_{\pi,\epsilon} p^{3/4}(p^n)^{3/4-3/40+\epsilon}.$$

In the spectral aspect, the first subconvex estimate was given by Blomer and Buttcane [10] for a $\mathrm{GL}(3)$ form in a breakthrough work. They considered the case when the Langlands parameters (α_i)'s of π are in generic positions, i.e., away from the Weyl chamber walls and away from self-dual forms. Let

$$\alpha_i \asymp T, \quad |\alpha_i - \alpha_j| \asymp T, \quad 1 \leq i \neq j \leq 3,$$

($A \asymp B$ means $c_1|A| \leq |B| \leq c_2|A|$ for some positive constants c_1 and c_2). Then they proved

$$L(1/2, \pi) \ll T^{3/4-1/120000}. \tag{0.3.6}$$

They implemented an amplified fourth moment averaged over $\mathrm{GL}(3)$ forms with Langlands parameters in an ball of radius $O(T^\epsilon)$ about π. Application of the $\mathrm{GL}(3)$ Kuznetsov formula was the main and crucial input in this case. They did not address the case when the Langlands parameters are in non-generic positions, i.e., are close to the Weyl chambers, as, in this case, 'the spectral measure' drops, so that the average over $\mathrm{GL}(3)$ forms becomes less powerful. In a recent preprint [49], the author along with Mallesham and Singh proved subconvexity for $\mathrm{GL}(3)$ L-functions whenever the Langlands parameters are in non-generic positions. In fact, we prove a more general result by considering the Rankin-Selberg L-functions associated to $\mathrm{GL}(3)$ and $\mathrm{GL}(2)$ forms. Using similar ideas we also consider the generic position case. Indeed we get a result of type (0.3.6) (with a better exponent 3/4-1/16) under some assumption.

0.4 Rankin-Selberg L-functions

Let f and g be holomorphic Hecke cusp forms of weights k_f and k_g respectively or Hecke-Maass cusp forms of Laplace eigenvalues $1/4+\nu_f^2$ and $1/4+\nu_g^2$ respectively for $SL(2,\mathbb{Z})$. Let $\lambda_f(n)$ and $\lambda_g(n)$ be the normalised Fourier coefficients of f and g respectively. The Rankin-Selberg L-series associated to f and g is given by

$$L(s,f\otimes g)=\zeta(2s)\sum_{n=1}^{\infty}\frac{\lambda_f(n)\lambda_g(n)}{n^s},\quad \Re(s)>1. \tag{0.4.1}$$

The above series has a analytic continuation to all of $\mathbb{C}$, except for $g=\overline{f}$, in which case, there is a simple pole at $s=1$. The Rankin-Selberg L-function is known to be automorphic (see [81]). In fact, it is an example of a degree 4 automorphic L-function.

In the weight aspect (when f is a holomorphic cusp form of large weight k_f and g is a holomorphic cusp form of fixed weight k_g), Sarnak [83] proved the following subconvex bound

$$L(1/2,f\otimes g)\ll_{g,\epsilon} k_f^{1-7/165+\epsilon}. \tag{0.4.2}$$

In the same article, he gave a beautiful application of this result in equidistribution. Corresponding to f we associate a density μ_f (normalized to be a probability measure on $SL(2,\mathbb{Z})\backslash\mathbb{H}$) given by

$$\mu_f:=y^k|f(z)|^2\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

Quantum unique ergodicity (QUE) conjecture says that, as $k_f\to\infty$ or $\nu_f\to\infty$, we have

$$\mu_f\overset{w}{\to}\frac{1}{\mathrm{Vol}(SL(2,\mathbb{Z})\backslash\mathbb{H})}\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \tag{0.4.3}$$

(in the sense of integration against continuous functions of compact support on $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$). From Watson's thesis [89] it follows that a subconvex estimate (as k_f varies) for the special value $L(1/2,f\otimes f\otimes g)$ of this degree 8 L-function will resolve the QUE with a power saving error term. Moreover it follows easily that

$$L(s, f \otimes f \otimes g) = L(s, \mathrm{sym}^2 f \otimes g) L(s, g). \tag{0.4.4}$$

Thus the QUE follows from the subconvexity of $L(1/2, \mathrm{sym}^2 f \otimes g)$, which is a degree six Rankin-Selbeg L-function associated to a $\mathrm{GL}(3)$ form $\mathrm{sym}^2 f$ and a $\mathrm{GL}(2)$ form g. Furthermore if f is a CM form then $L(s, \mathrm{sym}^2 f \otimes g)$ factors as $L(s, h \otimes g) L(s, g \times \chi)$, where h is a cusp form for $\mathrm{SL}(2, \mathbb{Z})$ and χ is a Dirichlet character. Thus Sarnak [83] deduces QUE for CM forms using (0.4.2). Using similar ideas, Lau, Liu and Ye [55] further improved the the exponent in (0.4.2) to $2/3 + \epsilon$.

In the t-aspect, Michel and Venketesh [65] proved subconvexity bounds using representation theoritic approach. Recently, using a similar approach, Blomer, Jana and Nelson [13] obtained a Weyl strength subconvex bound

$$L(1/2 + it, f \otimes g) \ll_{f,g,\epsilon} (1 + |t|)^{2/3+\epsilon}.$$

Using the delta symbol approach, Acharya, Sharma and Singh [2] also obtained a subconvex bound with exponent $9/10 + \epsilon$ (weaker than the Weyl bound).

In the level aspect, Kowalski, Michel and Vanderkam [48], Michel [62], Harcos and Michel [32], and Michel and Venkatesh [64] settled the subconvexity problem in the case when the level of f varies and the level of g is fixed. In [62], Michel gives a striking application to the equidistribution of Gross-Heegner points on Shimura curves associated to definite quaternion algebras over $\mathbb{Q}$, while in [32], Harcos and Michel provide the equidistribution of incomplete Galois orbits of Heegner points on Shimura curves associated with indefinite quaternion algebras over $\mathbb{Q}$. In the case, when the levels P_f and P_g, say, of both the forms f and g vary, subconvexity is known, in a certain range, due to Holowinsky and Munshi [33]. They used the amplified second moment method to achieve the following result

$$L(1/2, f \otimes g) \ll_{f,g,\epsilon} (P_f P_g)^{1/2 - \delta(\eta) + \epsilon},$$

for some $\delta(\eta) > 0$ and $(P_f, P_g) = 1$ with $P_f \sim P_g^{\eta}$, $0 < \eta < 2/21$. Ye [91] extended this for all η assuming that the form with the smaller level is holomorphic. Ye's result was recently generalised to all forms (holomorphic and Maass) by Raju [85], getting a much better exponent, using the delta method approach.

0.5 $\mathbf{GL(3) \times GL(2)}$ L-functions

Let π be a Hecke–Maass cusp form of type (ν_1, ν_2) for $\mathrm{SL}(3, \mathbb{Z})$ with the normalised Fourier coefficients $\lambda_\pi(n_1, n_2)$ and with the Langlands parameters $(\alpha_i)_{1\leq i\leq 3}$ and f be a holomorphic/Maass Hecke cusp form for $\mathrm{SL}(2, \mathbb{Z})$ with the normalised Fourier coefficients $\lambda_f(n)$. Let $1/4 + \nu_f^2$, $\nu_f > 0$, be the Laplace eigenvalue of the Hecke Maass cusp f or k_f be the weight of the holomorphic Hecke cusp form f. The associated Rankin–Selberg L-series is given by (see Section 1.3)

$$L(s, \pi \times f) = \sum_{n,r\geq 1}\sum \frac{\lambda_\pi(n, r)\, \lambda_f(n)}{(nr^2)^s}, \quad \Re(s) > 1. \tag{0.5.1}$$

This series is known to be automorphic (see [47]) and in fact, it is a example of a degree 6 automorphic L function. The above series extends to an entire function and satisfies a functional equation of the following form (see [15] for details)

$$\gamma(s, \pi \times f)L(s, \pi \times f) = \varepsilon_{\pi\times f}\, \gamma(1 - s, \bar{\pi} \times \bar{f})L(1 - s, \bar{\pi} \times \bar{f}),$$

with a gamma factor of degree 6

$$\gamma(s, \pi \times f) = \pi^{3s} \prod_{j=1}^{2}\prod_{i=1}^{3} \Gamma\left(\frac{s + \beta_j - \alpha_i}{2}\right),$$

where $\beta_1 = (k_f - 1)/2$, $\beta_2 = (k_f + 1)/2$ if f is a holomorphic form and $\beta_1 = \epsilon_f - i\nu_f$, $\beta_2 = \epsilon_f + i\nu_f$ if f is a Maass form. Here $\epsilon_f = 0$ if f is an even Maass form and $\epsilon_f = -1$ if f is an odd form. Subconvexity for this class of Rankin–Selberg L-functions is a holy grail for eminent number theorists. Indeed, subconvexity estimates for $L(1/2, \pi \times f)$ in the $\mathrm{GL}(3)$ level and in the $\mathrm{GL}(3)$ spectral aspect (see (0.4.4)) resolves the quantum unique ergodicity (QUE) conjecture (see (0.4.3)) in the quantitative sense. One can also recover subconvexity for some lower degree

L-functions from the subconvexity of this class of Rankin–Selberg L-functions (by taking π or f an Eisenstein series).

The first subconvex bound for such L-functions in the $\mathrm{GL}(2)$ spectral aspect is due to X. Li [58]. She considered π to be a self-dual $\mathrm{GL}(3)$ form and obtained the following result

$$L(1/2, \pi \times f) \ll_{\pi,\epsilon} (1 + |\nu_f|)^{11/8+\epsilon}, \tag{0.5.2}$$

She adapted Conrey-Iwaniec [22] moment method to prove the above result. In fact, she analysed the first moment average of $L(s, \pi \times u_j)$ over an orthonormal basis $u_j, j \geq 1,$ of even Hecke-Maass cusp forms for $\mathrm{SL}(2, \mathbb{Z})$. Non-negativity of the central values $L(1/2, \pi \times f)$, which is a deep result due to Lapid [54], played a key role in her proof, for which self-duality of π was necessary. Thus her approach does not work for generic $\mathrm{GL}(3)$ forms. The main purpose of this thesis is to generalise Li's spectral aspect result (0.5.2) to all $\mathrm{SL}(3, \mathbb{Z})$ forms (see Theorem 0.7.1). We discuss it in Chapter 2.

Following Li's approach, Blomer [9] considered $L(s, \pi \times f \times \chi)$, where π is a self-dual form and χ is a primitive quadratic character modulo M and obtained the following twist aspect subconvex bound

$$L(1/2, \pi \times f \times \chi) \ll_{\pi,f,\epsilon} M^{5/4+\epsilon}.$$

Non-negativity of the central values $L(1/2, \pi \times f \times \chi)$ was the key input in this result also. In the $\mathrm{GL}(2)$ level aspect, using the amplified first moment approach, Khan [45] proved the following bound

$$L(1/2, \pi \times f) \ll_{\pi} P_f^{3/4-1/2001},$$

under the assumption

$$\sum_{n<L} \lambda_f(n)^2 \gg_{\epsilon} L^{1-\epsilon},$$

for $L > P_f^{1/4+1/2001}$, where P_f, a prime number, is the level of a $\mathrm{GL}(2)$ form f

and π is a self dual $\mathrm{GL}(3)$ form. He also uses the deep result of Lapid [54] on non-negativity of the central values $L(1/2, \pi \times f)$. Thus new ideas were needed to resolve the subconvexity problem for this class of Rankin-Selberg L-functions for generic π and non-quadratic χ.

In a recent pioneering work, Munshi [77] considered the t-aspect subconvexity problem for $\mathrm{GL}(3) \times \mathrm{GL}(2)$ L-functions and he obtained the following bound

$$L\left(1/2 + it, \pi \times f\right) \ll_{\epsilon, f, \pi} (1 + |t|)^{3/2 - 1/51 + \epsilon}, \tag{0.5.3}$$

where π is any (not necessarily self-dual) Masss cusp form for $\mathrm{SL}(3, \mathbb{Z})$, and f is a Holomorphic/Maass cusp form for $\mathrm{SL}(2, \mathbb{Z})$. Munshi applied the delta method approach, which he developed in a series of articles [69]-[73], along with the conductor lowering trick which he introduced in [71]. More specifically, he used the delta method of Duke, Friedlander and Iwaniec (DFI) (see [[42], Chapter 20]) to separate the oscillatory factors. The key input in this paper was his observation that the character sum, emerging after the summation formulae, essentially boils down to an additive character, which is very specific to Rankin–Selberg convolutions of the type $\mathrm{GL(n+1)} \times \mathrm{GL(n)}$. The exponent $3/2 - 1/51$ was recently improved to $3/2 - 3/20$ by Lin and Sun [61] following a similar approach.

Following Munshi's template, Sharma [84] considered the same problem in the twist aspect. Indeed, he proved the following bound

$$L(1/2, \pi \times f \times \chi) \ll_{\pi, f, \epsilon} M^{3/2 - 1/16 + \epsilon},$$

for any $\mathrm{SL}(3, \mathbb{Z})$ form π and $\mathrm{SL}(2, \mathbb{Z})$ form f. Here χ is any (non-necessarily quadratic) Dirichlet character modulo M, a prime. Thus he generalised Blomer's result [9] to all $\mathrm{GL}(3)$ forms and all Dirichlet characters. Moreover, he also improved Holowinsky and Nelson's exponent [35], $3/4 - 1/36$, to $3/4 - 1/32$ by taking f an Eisenstein series. The key input in this paper was the implementation of the mass transfer trick, which was introduced by Munshi in [74]. Following Sharma's approach, Lin, Michel and Sawin [60] generalised his result by replacing χ by a 'generic trace function'. In the depth aspect, the author along with Mallesham and Singh

[50] adapted Munshi's t-aspect template [77] in the p-adic setting and obtained the following subconvex bound

$$L(1/2, \pi \times f \times \chi) \ll_{\pi,f,\epsilon} (M^r)^{3/2-3/20+\epsilon},$$

where χ is a primitive Dirichlet character modulo M^r, M a prime and $r \geq 2$. In the hybrid level aspect, the author, Munshi and Singh [51] applied the DFI delta method approach and obtained the following subconvex bound

$$L(1/2, \pi \times f) \ll_{\epsilon} (P_\pi^2 P_f^3)^{1/4-\delta}, \quad \delta > 0,$$

in the range $P_f^{1/2} < P_\pi < P_f^{3/2}$, where P_π is a prime which is the level of a $\mathrm{GL}(3)$ form π and P_f is a prime such that $(P_f, P_\pi) = 1$ and it is the level of a $\mathrm{GL}(2)$ form f. A crucial observation in this paper was that the conductor lowering trick was not necessary and thus the proof was less technical and cleaner (in comparision with other articles of Munshi). We note that if one could extend the above result to $P_f = 1$, it would resolve the QUE in the level aspect.

Motivated by Munshi's t-aspect result [77], we considered the spectral aspect case. In a recent priprint, the author, Mallesham and Singh [49] proved subconvexity for $L(1/2, \pi \times f)$ in the $\mathrm{GL}(3)$ spectral aspect (in non-generic and generic positions (under some assumptions)). We take up the $\mathrm{GL}(2)$ spectral aspect case in this thesis. We state it in Section 0.7 and give the proof in Chapter 2.

0.6 Other higher degree L-functions

For other higher degree L-functions subconvexity problem is open in the level aspect, expect for the triple product L-functions on $\mathrm{GL}(2)$, in which case it was proved by Venkatesh [88]. More precisely, he proved the following subconvex bound

$$L(1/2, f \times g \times h) \ll_{g,h} P_f^{1-\delta},$$

for some $\delta > 0$, where f, g and h are some $\mathrm{GL}(2)$ form with P_f being the level of f. He used representation theoretic approach to achieve this result. In the same year, Bernstein and Reznikov [7] considered the same L-function in the spectral aspect and proved the following result

$$L(1/2, f \times g \times h) \ll_{g,h,\epsilon} (1 + |\nu_f|)^{2-1/3+\epsilon},$$

where ν_f is the spectral parameter of the $\mathrm{GL}(2)$ form f. Their approach was also representation theoritic. Recently, Blomer, Jana and Nelson [13] improved their exponent $(5/3)$ to the Weyl strength subconvex exponent $4/3$ by combining representation theory, local harmonic analysis, and analytic number theory altogether.

A few months ago, Nelson [78] announced an extraordinary result, in which he resolved the subconvexity problem for any $\mathrm{GL(n)}$ L-functions in the t-aspect and the spectral aspect (having 'uniform parameter growth'). His method uses a lot of tools from representation theory and is motivated by the fundamental work of Michel and Venkatesh [65]. We note that Nelson's result also covers the result proved in this thesis. However our method is quite different, less technical and uses only tools from analytic number theory. Moreover, needless to say, our exponents are better then his expontents in the corresponding setup.

0.7 Statement of results

Let π be a Hecke-Maass cusp form of type (ν_1, ν_2) for $\mathrm{SL}(3, \mathbb{Z})$ with the Langlands parameters $(\alpha_1, \alpha_2, \alpha_3)$ and f be a holomorphic cusp form with weight k_f or a Hecke-Maass cusp form corresponding to the Laplacian eigenvalue $1/4 + \nu_f^2$, $\nu_f \geq 1$, for $\mathrm{SL}(2, \mathbb{Z})$. Let $L(1/2, \pi \times f)$ be the central value of the Rankin-Selberg L-series (0.5.1) at $1/2$.

0.7.1 $\mathrm{GL}(2)$ spectral aspect

As the main result, we prove the following theorem in this thesis, which gives the $\mathrm{GL}(2)$ spectral aspect subconvexity bound for the Rankin-Selberg L-functions associated to π and f. We will follow Remark 2.0.3 for the notations.

Theorem 0.7.1. *Let π and f be as above. Let $\nu_f \asymp k_f$. Then we have*

$$L\left(1/2, \pi \times f\right) \ll_{\pi,\epsilon} k_f^{3/2-1/51+\epsilon}, \tag{0.7.1}$$

for any $\epsilon > 0$.

The above theorem generalises Li's $\mathrm{GL}(2)$ spectral aspect result [58] to all $\mathrm{GL}(3)$ forms.

0.8 Discussion on the proof

The method of proof of the above result is motivatived by Munshi's t-aspect result [77], which we discuss briefly.

0.8.1 Munshi's approach

We recall from (0.5.3) that Munshi proves the following result

$$L\left(1/2 + it, \pi \times f\right) \ll_{\epsilon,f,\pi} \left(1 + |t|\right)^{3/2-1/51+\epsilon},$$

for any $\epsilon > 0$. Upon using the functional equation, the problem boils down to getting some cancellations in the following sum

$$\sum_{n \sim N} \lambda_\pi(n,1)\lambda_f(n)n^{-it}, \quad N \asymp t^3.$$

He initiates the proof by applying the DFI delta method (to separate $\lambda_\pi(n,1)$ and $\lambda_f(n)n^{-it}$) along with the conductor lowering trick (to reduce the modulus in the

DFI). Thus he ends up into

$$\int_K^{2K} \sum_{q\sim Q} \sideset{}{^\star}\sum_{a \bmod q} \sum_{n\sim N} \lambda_\pi(n,1) n^{iv} e\left(\frac{an}{q}\right) \sum_{m\sim N} \lambda_f(m)\, m^{-i(t+v)} e\left(\frac{-am}{q}\right) \mathrm{d}v,$$

where K is a parameter $K < t$ which he chooses optimally later and $Q = t^{3/2}/\sqrt{K}$. Here the situation seems to be worse a priori, as we have lost N in the above sum. However he gains structurely and he manages to gain it back later.

In the second step he applies summation formulae to the sum over n and m, and he saves $\sqrt{NK}/t$ in the m-sum and $N^{1/4}/K^{3/4}$ in the n-sum. Then he analyses the v-integral in which he gets square-root cancellations, in other words, he saves $\sqrt{K}$. The analysis of the a-sum also gives square-root cancellations and he saves $\sqrt{q}$ from it. Hence in total he saves N/t so far and he is left with the following sum

$$\sum_{q\sim Q} \sum_{n\sim K^{3/2}N^{1/2}} \lambda_\pi(1,n) \sum_{m\sim t^2/K} \lambda_f(m)\mathfrak{C}\,\mathfrak{I},$$

in which he needs to save t and a bit more, say, t^η. Here $\mathfrak{I}$ is an integral transform which oscillates like n^{iK} with respect to n, and the character sum $\mathfrak{C}$ is given by

$$\mathfrak{C} = \sideset{}{^*}\sum_{a \bmod q} S\left(\bar{a}, n; q\right)\, e\left(\frac{\bar{a}m}{q}\right) \approx qe\left(-\frac{\bar{m}n}{q}\right).$$

Next he applies Cauchy to break the involution and arrives at

$$\sum_{n\sim K^{3/2}N^{1/2}} \left| \sum_{q\sim Q} \sum_{m\sim t^2/K} \lambda_f(m) e\left(-\frac{\bar{m}n}{q}\right) \mathfrak{I} \right|^2,$$

in which t^2 (plus extra) is needed to be saved. In the end game strategy, he applies the Poisson summation formula to the sum over n. In the zero frequency he saves t^2Q/K which is more then t^2 provided $K < t$. In the non-zero frequencies, he saves $K^{3/2}N^{1/2}/(\sqrt{Q^2K}) \times Q$, which is good enough if $K > t^{1/2}$. Thus he succeeds by chosing K optimally between $\sqrt{t}$ and t. Notice that there is an extra Q in the saving of the non-zero frequencies. It is a crucial factor which he obtains due to the additive (with respect to n) character $e\left(-\bar{m}n/q\right)$, which comes due to the $\mathrm{GL}(3) \times \mathrm{GL}(2)$

structure. This is the key input in this paper.

0.8.2 Our approach

On applying the functional equation, our problem boils down to getting cancellations in

$$S(N) = \sum_{n \sim N} \lambda_\pi(n,1)\lambda_f(n),$$

where $N \asymp k_f^3$ in Theorem 0.7.1. To prove Theorem 0.7.1, following Munshi, we apply DFI delta method to separate $\lambda_\pi(n,1)$ and $\lambda_f(n)$ along with the conductor lower trick. Then applications of summation formulae followed by Cauchy and Poisson gives us the result. We also get the structural advantage of the $\mathrm{GL}(3) \times \mathrm{GL}(2)$ type and hence we are able to save more (than the usual) in the Poisson. The main technical input of this theorem is to get square-root cancellations in the integral transforms. Indeed, after summation formulae, the integral transform (for f holomorphic) looks like

$$\mathrm{I} = \int U(y) e(\mathfrak{a} y^{1/3}) J_{k_f - 1}(\mathfrak{b}\sqrt{y})\, \mathrm{d}y,$$

where $\mathfrak{a} \asymp \mathfrak{t}$, for some $\mathfrak{t} < k_f$ to be chosen later, $\mathfrak{b} \asymp k_f$ and U is a smooth bump function supported on $[1/2, 5/2]$ (here the symbol $A \asymp B$ means $c_1 k_f^{-\epsilon}|A| \leq |B| \leq c_2 k_f^{\epsilon}|A|$ for some positive constants c_1 and c_2). Here the argument $\mathfrak{b} y^{1/2}$ of the Bessel function is in 'transitional range' ($\mathfrak{b} \asymp k_f$), in which case, a 'nice' asymptotic expansion (uniform in k_f) is not known. We get desired cancellations ($\mathrm{I} \ll 1/k_f$) using the integral representation

$$J_{k-1}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e\left(\frac{(k-1)\tau - x \sin \tau}{2\pi} \right) \mathrm{d}\tau,$$

followed by a chain of stationary phase analysis. We discuss full details of the proof in Chapter 2.

Chapter 1

Preliminary lemmas

We will use some known lemmas in the proof of our results. We will record them in this chapter.

1.1 Automorphic form for $\mathrm{GL}(2)$

In this section, we recall some basic facts about automorphic forms for $\mathrm{SL}(2,\mathbb{Z})$ (for details see [41] and [42]).

1.1.1 Holomorphic cusp forms

Let f be a holomorphic Hecke cusp of weight k_f for the full modular group $\mathrm{SL}(2,\mathbb{Z})$. The Fourier expansion of f at ∞ is given by

$$f(z)=\sum_{n=1}^{\infty}\lambda_f(n)\,n^{(k-1)/2}\,e(nz),\quad z\in\mathbb{H},$$

where we assume that f is arithmetically normalized so that $\lambda_f(1)=1$. We have the following well-known estimate due to Deligne:

$$|\lambda_f(n)|\leq d(n),\quad n\geq 1, \tag{1.1.1}$$

where $d(n)$ is the divisor function. The Hecke L-function associated with the form f is given by

$$L(s,f)=\sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^s}\ =\prod_p\left(1-\lambda_f(p)p^{-s}+p^{-2s}\right)^{-1},\quad \Re s>1.$$

Hecke proved that $L(s,f)$ admits an analytic continuation to the whole complex plane and satisfies the functional equation

$$\Lambda(s,f)=\varepsilon(f)\ \Lambda(1-s,\overline{f}),$$

where $\varepsilon(f)$ is a root number and $\overline{f}$ is the dual form of f, and the completed L-function is given by

$$\Lambda(s,f)=\pi^{-s}\,\Gamma\left(\frac{s+(k_f+1)/2}{2}\right)\Gamma\left(\frac{s+(k_f-1)/2}{2}\right)L(s,f).$$

We now state the Voronoi summation formula for the form f.

Lemma 1.1.1. *Let $\lambda_f(n)$ be as above and g be a smooth, compactly supported function on $(0,\infty)$. Let a, $q\in\mathbb{Z}$ with $(a,q)=1$. Then we have*

$$\sum_{n=1}^{\infty}\lambda_f(n)\,e\left(\frac{an}{q}\right)g(n)=\frac{2\pi i_f^k}{q}\sum_{n=1}^{\infty}\lambda_f(n)\,e\left(-\frac{dn}{q}\right)\,h(n),$$

where $ad\equiv 1(\bmod q)$ and

$$h(y)=\int_0^{\infty}g(x)\,J_{k_f-1}\left(\frac{4\pi\sqrt{xy}}{q}\right)\,dx.$$

Proof. See [[29], P. 792]. See [[48], Theorem A.4] also for general level. □

1.1.2 Maass cusp forms

Let f be a Hecke-Maass cusp form for $\mathrm{SL}(2,\mathbb{Z})$ with Laplace eigenvalue $1/4+\nu_f^2$. The Fourier series expansion of f at ∞ is given by

$$f(z)=\sqrt{y}\sum_{n\neq 0}\lambda_f(n)K_{i\nu_f}(2\pi|n|y)e(nx),$$

where $K_{i\nu_f}(y)$ is the Bessel function of the third kind and f is normalized so that $\lambda_f(1)=1$. The Ramanujan-Petersson conjecture predicts that

$$|\lambda_f(n)|\leq d(n).$$

The work of H. Kim and P. Sarnak [46] tells us that $|\lambda_f(n)|\ll n^{7/64+\epsilon}$. The L-function associated with the form f is given by

$$L(s,f)=\sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^s},\quad \Re s>1.$$

It extends to an entire function and satisfies a functional equation of the form

$$\Lambda(s,f)=\varepsilon(f)\Lambda(1-s,\overline{f}),$$

where $\varepsilon(f)$ is the root number with $|\varepsilon(f)|=1$ and the completed L-function $\Lambda(s,f)$ is given by

$$\Lambda(s,f)=\pi^{-s}\Gamma\left(\frac{s+\epsilon+i\nu}{2}\right)\Gamma\left(\frac{s+\epsilon-i\nu}{2}\right)L(s,f).$$

Here $\epsilon=0$ if f is even and $\epsilon=1$ if f is odd. We now recall Rankin-Selberg bound for the Fourier coefficients in the following lemma.

Lemma 1.1.2. *Let $\lambda_f(n)$ be the normalised Fourier coefficients of a holomorphic/-Maass cusp form. Then we have*

$$\sum_{1\leq n\leq x}|\lambda_f(n)|^2\ll_\epsilon C(f)^\epsilon x^{1+\epsilon}, \tag{1.1.2}$$

where $C(f) = k_f^2$ *if* f *is holomorphic and* $C(f) = 1 + \nu_f^2$ *if* f *is a Maass form.*

Proof. Iwaniec [[40], Lemma 1]. $\square$

Next we state the Voronoi formula for Maass cusp forms.

Lemma 1.1.3. *Let* $\lambda_f(n)$ *be as above and* g *be a smooth, compactly supported function on* $(0, \infty)$. *Let* a, $q \in \mathbb{Z}$ *with* $(a, q) = 1$. *Then we have*

$$\sum_{n=1}^{\infty} \lambda_f(n)\, e\left(\frac{an}{q}\right) h(n) = q \sum_{\pm} \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n} e\left(\mp \frac{dn}{q}\right) H^{\pm}\left(\frac{n}{q^2}\right),$$

where $ad \equiv 1 \bmod q$ *and*

$$H^{\pm}(y) = \frac{\epsilon_f^{(1\mp 1)/2}}{4\pi^2 i} \int_{\sigma} (\pi^2 x)^{-s} (C^+(-s) \pm C^-(-s)) \tilde{g}(-s)\, \mathrm{d}s,$$

with

$$C^+(s) = \frac{\Gamma\left(\frac{1+s+i\nu_f}{2}\right)\Gamma\left(\frac{1+s-i\nu_f}{2}\right)}{\Gamma\left(\frac{-s+i\nu_f}{2}\right)\Gamma\left(\frac{-s-i\nu_f}{2}\right)}, \quad C^-(s) = \frac{\Gamma\left(\frac{2+s+i\nu_f}{2}\right)\Gamma\left(\frac{2+s-i\nu_f}{2}\right)}{\Gamma\left(\frac{1-s+i\nu_f}{2}\right)\Gamma\left(\frac{1-s-i\nu_f}{2}\right)}.$$

Here $\epsilon_f = \pm 1$ *depending on* f *even or odd.*

Proof. See [[48], Theorem A.4]. $\square$

1.2 Automorphic forms for $\mathrm{GL}(3)$

In this section, we will recall some results about the Maass cusp forms for $\mathrm{SL}(3, \mathbb{Z})$. This section, except for notation, is taken from [58]. Let π be a Hecke-Maass cusp form of type (ν_1, ν_2) for $SL(3, \mathbb{Z})$. Let $\lambda_\pi(n, r)$ denote the normalised Fourier coefficients of π. Let

$$\alpha_1 = -\nu_1 - 2\nu_2 + 1,\ \alpha_2 = -\nu_1 + \nu_2 \text{ and } \alpha_3 = 2\nu_1 + \nu_2 - 1$$

be the Langlands parameters for π (see [30]). The dual form $\tilde{\pi}$ has Langlands parameters $(-\alpha_3, -\alpha_2, -\alpha_1)$. The L-series associated with π is given by

$$L(s,\pi) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(1,n)}{n^s}, \quad \Re(s) > 1. \tag{1.2.1}$$

$L(s,\pi)$ satisfies a functional equation of the form

$$\gamma(s,\pi)L(s,\pi) = \gamma(s,\tilde{\pi})L(1-s,\tilde{\pi}),$$

where

$$\gamma(s,\pi) = \prod_{j=1}^{3} \pi^{-s/2}\Gamma\left(\frac{s-\alpha_j}{2}\right).$$

Let g be a compactly supported smooth function on $(0,\infty)$ and

$$\widetilde{g}(s) = \int_0^\infty g(x)x^{s-1}\mathrm{d}x$$

be its Mellin transform. For $\ell = 0$ and 1, we define

$$\gamma_\ell(s) := \frac{\pi^{-3s-\frac{3}{2}}}{2} \prod_{i=1}^{3} \frac{\Gamma\left(\frac{1+s+\alpha_i+\ell}{2}\right)}{\Gamma\left(\frac{-s-\alpha_i+\ell}{2}\right)}. \tag{1.2.2}$$

Set $\gamma_\pm(s) = \gamma_0(s) \mp \gamma_1(s)$ and let

$$G_\pm(y) = \frac{1}{2\pi i}\int_{(\sigma)} y^{-s}\,\gamma_\pm(s)\,\widetilde{g}(-s)\,\mathrm{d}s, \tag{1.2.3}$$

where $\sigma > -1 + \max\{-\Re(\alpha_1), -\Re(\alpha_2), -\Re(\alpha_3)\}$. With the aid of the above terminology, we now state the $\mathrm{GL}(3)$ Voronoi summation formula in the following lemma:

Lemma 1.2.1. *Let $g(x)$ and $\lambda_\pi(n,r)$ be as above. Let a, $\bar{a} \in \mathbb{Z}$, $q \in \mathbb{N}$ with $(a,q) = 1$, and $a\bar{a} \equiv 1 (\mathrm{mod}\ q)$. Then we have*

$$\begin{aligned}&\sum_{n=1}^{\infty} \lambda_\pi(n,r)e\left(\frac{an}{q}\right)g(n)\\ &= q\sum_{\pm}\sum_{n_1|qr}\sum_{n_2=1}^{\infty} \frac{\lambda_\pi(n_1,n_2)}{n_1n_2} S\left(r\bar{a}, \pm n_2; qr/n_1\right) G_\pm\left(\frac{n_1^2 n_2}{q^3 r}\right),\end{aligned}$$

where $S(a,b;q)$ is the Kloosterman sum which is defined as follows:

$$S(a,b;q)=\sum_{x \bmod \mathrm{q}}^{\star} e\left(\frac{ax+b\bar{x}}{q}\right).$$

Proof. See [67]. □

We also need to extract the oscillations of the integral transform $G_\pm$. To this end, we have the following lemma:

Lemma 1.2.2. *Let $G_\pm(x)$ be as above, and $g(x) \in C_c^\infty(X,2X)$. Then for any fixed form π, any integer $K \geq 1$ and $xX \gg 1$, we have the following expression for $G_\pm(x)$:*

$$x\int_0^\infty g(y)\sum_{j=1}^{K}\frac{c_j(\pm)e\left(3(xy)^{1/3}\right)+d_j(\pm)e\left(-3(xy)^{1/3}\right)}{(xy)^{j/3}}\,\mathrm{d}y+O_K\left((xX)^{\frac{-K+5}{3}}\right),$$

where $c_j(\pm)$ and $d_j(\pm)$ are some constants depending only on α_i's, for $i=1,\,2,\,3$.

Proof. See [[57], Lemma 6.1]. □

The following lemma, which gives the Ramanujan conjecture on average, is also well-known.

Lemma 1.2.3. *We have*

$$\sum\sum_{n_1^2 n_2 \leq x} |\lambda_\pi(n_1,n_2)|^2 \ll_\epsilon C(\pi)^\epsilon\, x^{1+\epsilon},$$

where $C(\pi)=\prod_{i=1}^3(1+|\alpha_j|)$ is the analytic conductor of π.

Proof. See [56]. □

1.3 GL(3) × GL(2) L-functions

The Rankin-Selberg L-series associated to π and f is given by

$$L(s,\pi\times f)=\sum\sum_{n,r\geq 1}\frac{\lambda_\pi(n,r)\,\lambda_f(n)}{(nr^2)^s},\quad \Re(s)>1. \tag{1.3.1}$$

The above series extends to an entire function and satisfies a functional equation of the following form

$$\gamma(s,\pi\times f)L(s,\pi\times f)=\varepsilon_{\pi\times f}\,\gamma(1-s,\bar\pi\times\bar f)L(1-s,\bar\pi\times\bar f),$$

where $\varepsilon_{\pi\times f}$ is the root number having modulus 1 and $\gamma(s,\pi\times f)$ is a gamma factor of degree 6

$$\gamma(s,\pi\times f)=\pi^{3s}\prod_{j=1}^{2}\prod_{i=1}^{3}\Gamma\left(\frac{s+\beta_j-\alpha_i}{2}\right),$$

where $\beta_1=(k_f-1)/2$, $\beta_2=(k_f+1)/2$ if f is a holomorphic form and $\beta_1=\epsilon_f-i\nu_f$, $\beta_2=\epsilon_f+i\nu_f$ if f is a Maass form. Here $\epsilon_f=0$ if f is even and $\epsilon_f=1$ if f is odd.

Later, we will be estimating $L(1/2,\pi\times f)$. The following lemma expresses it in terms of a weighted Dirichlet series.

Lemma 1.3.1. *Let $\beta_j-\alpha_i\asymp \mathfrak{t}$, $\mathfrak{t}\gg 1$, for all i and j. Then $L(1/2,\pi\times f)$ has the following expression*

$$\sum\sum_{n,r\geq 1}\frac{\lambda_\pi(n,r)\,\lambda_f(n)}{(nr^2)^{1/2}}V\left(\frac{nr^2}{\mathfrak{t}^3}\right)+\varepsilon_{\pi\times f}\sum\sum_{n,r\geq 1}\frac{\lambda_{\bar\pi}(n,r)\,\lambda_{\bar f}(n)}{(nr^2)^{1/2}}\overline{V}\left(\frac{nr^2}{\mathfrak{t}^3}\right), \tag{1.3.2}$$

where $V(x)$ is a smooth function satisfying

$$x^jV^{(j)}(x)\ll_A(1+|x|)^{-A},$$

for any positive integer A and $j\geq 0$.

Proof. See [[42], page 100]. □

Thus the above lemma limits both sums in (1.3.2) effectively to the terms with $nr^2 \ll \mathfrak{t}^{3+\epsilon}$, as $V(nr^2/\mathfrak{t})$ is negligibly small, ($\mathfrak{t}^{-A}$ for any $A > 1$) for $nr^2 \gg \mathfrak{t}^{3+\epsilon}$. We now estimate $L(1/2, \pi \times f)$ in terms of an exponential sum.

Lemma 1.3.2. *Let π and f be as above, with π fixed. Then, for large $\nu_f \asymp k_f$ and any θ such that $0 < \theta < 3/2$, we have*

$$L\left(1/2, \pi \times f\right) \ll_{\pi,\epsilon} k_f^{\epsilon} \sup_{r \leq k_f^{\theta}} \sup_{k_f^{3-\theta}/r^2 \leq N \leq k_f^{3+\epsilon}/r^2} \frac{|S_r(N)|}{N^{1/2}} + k_f^{(3-\theta)/2+\epsilon}, \tag{1.3.3}$$

where $S_r(N)$ is a sum of the form

$$S_r(N) := \sum_{n=1}^{\infty} \lambda_{\pi}(n,r) \lambda_f(n) V\left(\frac{n}{N}\right), \tag{1.3.4}$$

for some smooth function V supported in $[1,2]$ and satisfying $V^{(j)}(x) \ll_j k_f^{\epsilon}$ for any integer $j \geq 0$.

Proof. By Lemma 1.3.1, we see that

$$L(1/2, \pi \times f) = \sum\sum_{n,r \geq 1} \frac{\lambda_{\pi}(n,r)\, \lambda_f(n)}{(nr^2)^{1/2}} V\left(\frac{nr^2}{\mathfrak{t}^3}\right) + \varepsilon_{\pi \times f}\, \text{dual sum},$$

where $V(x)$ is a smooth function which is negligibly small if $x \gg \mathfrak{t}^{\epsilon}$. We proceed with the first sum as the calculations for the dual sum are the same. Note that $\mathfrak{t} \asymp k_f$, as π is fixed. Hence we see that

$$\begin{aligned} L\left(1/2, \pi \times f\right) &\ll \left| \sum\sum_{nr^2 \ll k_f^{3+\epsilon}} \frac{\lambda_{\pi}(n,r)\, \lambda_f(n)}{(nr^2)^{1/2}} V\left(\frac{nr^2}{k_f^3}\right) \right| \\ &\leq \left| \sum_{r \leq k_f^{(3+\epsilon)/2}} \frac{1}{r} \sum_{n \leq k_f^{3+\epsilon}/r^2} \frac{\lambda_{\pi}(n,r)\, \lambda_f(n)}{n^{1/2}} V\left(\frac{nr^2}{k_f^3}\right) \right|. \end{aligned}$$

We split the above sum as follows:

$$\sum_{r\leq k_f^{(3+\epsilon)/2}}\ \sum_{n\leq k_f^{3+\epsilon}/r^2} = \sum_{r\leq k_f^{\theta}}\ \sum_{k_f^{3-\theta}/r^2\leq n\leq k_f^{3+\epsilon}/r^2} + \sum_{r\leq k_f^{\theta}}\ \sum_{n< k_f^{3-\theta}/r^2} + \sum_{k_f^{\theta}<r\leq k_f^{(3+\epsilon)/2}}\ \sum_{n\leq k_f^{3+\epsilon}/r^2},$$

where $\theta > 0$ is a constant which will be chosen later optimally. Using the Ramanujan bound on average

$$\sum\sum_{n_1^2 n_2\leq x} |\lambda_\pi(n_1,n_2)|^2 \ll x^{1+\epsilon}, \quad \sum_{1\leq n\leq x} |\lambda_f(n)|^2 \ll_\epsilon x^{1+\epsilon},$$

we see that the last two sums are bounded by $k_f^{(3-\theta)/2+\epsilon}$. Hence we arrive at

$$L\left(1/2,\pi\times f\right) \ll \left| \sum_{r\leq k_f^{\theta}} \frac{1}{r} \sum_{k_f^{3-\theta}/r^2\leq n\leq k_f^{3+\epsilon}/r^2} \frac{\lambda_\pi(n,r)\,\lambda_f(n)}{n^{1/2}} V\left(\frac{nr^2}{k_f^3}\right)\right| + k_f^{(3-\theta)/2+\epsilon}.$$

Using a smooth dyadic partition of unity U, we see that the inner most sum is at most

$$\sup_{k_f^{3-\theta}/r^2\leq N\leq k_f^{3+\epsilon}/r^2} \sum_{n=1}^{\infty} \frac{\lambda_\pi(n,r)\,\lambda_f(n)}{n^{1/2}} U\left(\frac{n}{N}\right) V\left(\frac{nr^2}{k_f^3}\right),$$

which can be written as

$$\sup_{k_f^{3-\theta}/r^2\leq N\leq k_f^{3+\epsilon}/r^2} \frac{|S_r(N)|}{N^{1/2}},$$

where

$$S_r(N) := \sum_{n=1}^{\infty} \lambda_\pi(n,r)\,\lambda_f(n) V_{r,N}\left(\frac{n}{N}\right),$$

with $V_{r,N}(x) = x^{-1/2}U(x)V(Nr^2x/k_f^3)$. Note that $V_{r,N}(x)$ is supported on $[1,2]$ and satisfies $V_{r,N}^{(j)}(x) \ll_j k_f^{\epsilon}$ (the bound independent of r and N) for any integer $j\geq 0$. Henceforth we ignore the dependence on r and N and assume that $V_{r,N}$ is the same function for all r and N and call it $V(x)$ (abusing notation). Finally, taking supremum over r, we get the lemma. □

1.4 DFI delta method

Let $\delta : \mathbb{Z} \to \{0, 1\}$ be defined by

$$\delta(n) = \begin{cases} 1 & \text{if } n = 0; \\ 0 & \text{otherwise.} \end{cases}$$

The above function can be used to separate the oscillations involved in a sum, say, $\sum_{n\sim X} a(n)\, b(n)$. Furthermore, we seek a nice Fourier expansion of $\delta(n)$. We mention here an expansion for $\delta(n)$ which is due to Duke, Friedlander and Iwaniec (see [[42], Chapter 20]). Let $L \geq 1$ be a large real number. For $n \in [-2L, 2L]$, we have

$$\delta(n) = \frac{1}{Q} \sum_{1\leq q\leq Q} \frac{1}{q} \sideset{}{^\star}\sum_{a \bmod q} e\left(\frac{na}{q}\right) \int_{\mathbb{R}} g(q,x)\, e\left(\frac{nx}{qQ}\right) \mathrm{d}x, \tag{1.4.1}$$

where $Q = 2L^{1/2}$. The $\star$ on the sum indicates that the sum over a is restricted by the condition $(a, q) = 1$. The function g is the only part in the above formula which is not explicitly given. Nevertheless, we only need the following properties of g in our analysis (for the proof, see [[36], Lemma 15] and [[2], Lemma 2.1]) . For any $B > 1$, we have

1. $g(q,x) = 1 + h(q,x), \quad$ with $\quad h(q,x) = O_B\left(\frac{Q}{q}\left(\frac{q}{Q} + |x|\right)^B\right).$
2. $|x|^j \frac{\partial^j}{\partial x^j} g(q,x) \ll_j \log Q \min\left\{\frac{Q}{q}, \frac{1}{|x|}\right\}, \; j \geq 1.$
3. $g(q,x) \ll_B |x|^{-B}.$
4. $\int_{\mathbb{R}} |g(q,x)| + |g(q,x)|^2 \,\mathrm{d}x \ll_\epsilon Q^\epsilon. \qquad (1.4.2)$

Using the third property we observe that the effective range of the x-integral in (1.4.1) is $[-Q^\epsilon, Q^\epsilon]$. We record the above observations in the following lemma.

Lemma 1.4.1. *Let δ be as above. Let $L \geq 1$ be a large parameter. Then, for $n \in [-2L, 2L]$, we have*

$$\delta(n) = \frac{1}{Q} \sum_{1 \leq q \leq Q} \frac{1}{q} \sideset{}{^\star}\sum_{a \bmod q} e\left(\frac{na}{q}\right) \int_{\mathbb{R}} W(x/Q^{\epsilon}) g(q,x)\, e\left(\frac{nx}{qQ}\right) \mathrm{d}x + O(L^{-2020}),$$

where $Q = 2L^{1/2}$, g is a function satisfying (1.4.2) and $W(x)$ is a non-negative smooth bump function supported in $[-2, 2]$, with $W(x) = 1$ for $x \in [-1, 1]$ and $W^{(j)}(x) \ll_j 1$, for $j \geq 0$.

Proof. See [[42], Chapter 20], [[36], Lemma 15] and [[2], Lemma 2.1]. □

1.5 Bessel function

In this section, we will recall some well-known expansions of the Bessel functions of the first kind. For $k \geq 2$ an integer, let J_{k-1} be the Bessel function of the first kind and of order $k-1$, which is defined as

$$J_{k-1}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e\left(\frac{(k-1)\tau - x\sin\tau}{2\pi}\right) \mathrm{d}\tau, \tag{1.5.1}$$

for any $x \in \mathbb{R}$. The following lemma gives an expression for the Bessel function when the order k is fixed.

Lemma 1.5.1. *For fixed k, we have*

$$J_{k-1}(2\pi x) = e(x) W_{k-1}(x) + e(-x)\overline{W}_{k-1}(x),$$

where W_{k-1} is a smooth function satisfying

$$x^j {W_{k-1}}^{(j)}(x) \ll_{j,k} \frac{1}{\sqrt{x}},$$

for $x \gg 1$ and $j \geq 0$.

Proof. See [[34], Sec. 4.5]. □

In the analysis of the integral transforms in Chapter 2, we require a uniform asymptotic expansion of $J_{k-1}(x)$ for large values of k and x. The following lemma provides one such asymptotic expansion.

Lemma 1.5.2. *Let $x \geq (k-1)^{1+\epsilon/2}$ be a positive real number. Then, as $k \to \infty$, we have*

$$J_{k-1}(x) = \left(\frac{2}{\pi(k-1)w}\right)^{1/2} \left[\cos\left(Z(w)\right) \sum_{j=0}^{\infty} \frac{P_j\left(\frac{1}{w-\tan^{-1} w}\right)}{(k-1)^j}\right] + \left(\frac{2}{\pi(k-1)w}\right)^{1/2} \left[\sin\left(Z(w)\right) \sum_{j=1}^{\infty} \frac{P_j\left(\frac{1}{w-\tan^{-1} w}\right)}{(k-1)^j}\right], \tag{1.5.2}$$

where $Z(w) = (k-1)(w - \tan^{-1} w) - \pi/4$ and

$$w = \left(\frac{x^2}{(k-1)^2} - 1\right)^{1/2},$$

and P_j is a polynomial of the degree j with coefficients which are bounded functions of $k-1$ and $\log(x/(k-1))$ with $P_0 \equiv 1$.

Proof. Let $x = (k-1)\sec\beta$, with $0 < \beta < \pi/2$. Thus, as $x \geq (k-1)^{1+\epsilon/2}$, we have $\sec\beta \geq (k-1)^{\epsilon/2}$ and

$$\xi := (k-1)(\tan\beta - \beta) \geq (k-1)(\sqrt{(k-1)^{\epsilon} - 1} - \pi/2).$$

Thus, on using formula (63) on page 58 of [53], we get

$$J_{k-1}((k-1)\sec\beta) = \left(\frac{2}{\pi(k-1)\tan\beta}\right)^{1/2} \left[\cos f_1(\beta) \sum_{j=0}^{\infty} \frac{P_j\left(\frac{1}{\tan\beta-\beta}\right)}{(k-1)^j}\right] + \left(\frac{2}{\pi(k-1)\tan\beta}\right)^{1/2} \left[\sin f_1(\beta) \sum_{j=1}^{\infty} \frac{P_j\left(\frac{1}{\tan\beta-\beta}\right)}{(k-1)^j}\right],$$

where $f_1(\beta) = (k-1)(\tan\beta - \beta) - \pi/4$, and P_j represents a polynomial of the degree j with coefficients which are bounded functions of $k-1$ and $\log\sec\beta$ with $P_0 \equiv 1$. Now substituting $(k-1)\sec\beta = x$ and $\tan\beta = w$, we get the lemma. □

The expansion (1.5.2) can be truncated at any stage to get

Corollary 1.5.3. *Under the assumptions of Lemma 1.5.2, we have*

$$J_{k-1}(x) = \sum_{\pm} \epsilon(\pm) \sum_{j=0}^{2019} \frac{e\left(\pm \frac{(k-1)(w-\tan^{-1} w)}{2\pi}\right) P_j\left(\frac{1}{w-\tan^{-1} w}\right)}{\sqrt{\pi} w^{1/2} (k-1)^{j+1/2}} + O\left(\frac{1}{k^{2020}}\right),$$

where $\epsilon(\pm) \in \{1, -1\}$.

Proof. The statement follows directly from Lemma 1.5.2. □

For $0 < x \leq (k-1)^{1-\epsilon/2}$, we have the following lemma.

Lemma 1.5.4. *Let* $x = (k-1)z$ *with* $0 < z \ll (k-1)^{-\epsilon/2}$. *Then for* $k > k_0(\epsilon)$, *where* $k_0(\epsilon)$ *is a large absolute constant (depending on* ϵ*), we have*

$$J_{k-1}(x) \ll \exp\{-(k-1)/6\}.$$

Proof. By Lemma 4.2 of [82], we have

$$|J_{k-1}((k-1)z)| \leq A_1 (k-1)^{-1/2} (1-z^2)^{-1/4} \exp\left\{-\frac{1}{3}(k-1)(1-z^2)^{3/2}\right\},$$

for $0 < z \leq \sqrt{1 - \frac{1}{(k-1)^{2/3}}}$, $k \geq 16$ and some absolute constant A_1. Note that, by assumption, we have $z \leq (k-1)^{-\epsilon/2}$. Thus, $1 - z^2 \geq 1/2^{2/3}$ for $k > k_0(\epsilon)$, and we get

$$|J_{k-1}((k-1)z)| \leq A_1 2^{1/6} \exp\left\{-\frac{1}{6}(k-1)\right\}.$$

Hence the lemma follows. □

1.6 Gamma function

In this section we recall standard properties of the gamma function and Stirling's asymptotic formula for the gamma function $\Gamma(s)$. For any $z : |\arg z| \leq \pi - \epsilon$, we have

$$\Gamma(z) = \sqrt{\frac{2\pi}{z}} \left(\frac{z}{e}\right)^z \left\{\sum_{j=0}^{J-1} \frac{a_j}{z^j} + O_{\epsilon,J}\left(\frac{1}{|z|^J}\right)\right\}, \quad J \geq 1, \tag{1.6.1}$$

where a_j's are some absolute constants depending upon ϵ only (see [42]). Furthermore for $z = \sigma + i\tau$ with fixed $\sigma \in \mathbb{R}$, $|\tau| \geq 10$ and any $M > 0$, we have

$$\Gamma(\sigma + i\tau) = e^{-\frac{\pi}{2}|\tau|}\, |\tau|^{\sigma - \frac{1}{2}} \exp\left(i\tau \,\log \frac{|\tau|}{e}\right) g_{\sigma,M}(\tau) + O_{\sigma,M}(|\tau|^{-M}), \tag{1.6.2}$$

where $\tau^j g_{\sigma,M}^{(j)}(\tau) \ll_{j,\sigma,M} 1$ for $j \geq 0$.

We also record Legendre's duplication formula

$$\frac{\Gamma(z)}{\Gamma(1/2 - z)} = \frac{\cos(\pi z)\, \Gamma(2z)}{\sqrt{\pi}\, 2^{2z-1}}. \tag{1.6.3}$$

1.7 Stationary phase analysis

We also require to estimate the exponential integral of the form

$$I = \int_a^b g(x) e(f(x)) dx, \tag{1.7.1}$$

where f and g are real valued smooth functions on the interval $[a, b]$. We recall the following lemma on exponential integrals.

Lemma 1.7.1. *Let I, f and g be as above. Let $V(g)$ denotes the total variation of $g(x)$ on $[a, b]$ plus the maximum modulus of $g(x)$ on $[a, b]$. Then, if f' is monotone and $|f'(x)| \geq \mu_1 > 0$ for $x \in [a, b]$, we have $I \ll V(g)/\mu_1$. For $r > 1$, let $|f^{(r)}(x)| \geq \mu_r > 0$. Then we have $I \ll_r V(g)/\mu_r^{1/r}$. Moreover, let $f'(x) \geq B$ and $f^{(j)}(x) \ll B^{1+\epsilon}$ for $j \geq 2$ together with $Supp(g) \subset (a, b)$ and $g^{(j)}(x) \ll_{a,b,j} 1$. Then we have*

$$I \ll_{a,b,j,\epsilon} B^{-j+\epsilon}.$$

Proof. See [[71], Subsection 2.2] and [[39], Lemma 5.1.2, Lemma 5.1.4]. □

The above lemma can be used for $r = 1$ whenever the phase function f does not have any stationary point. We will also apply it for $r = 2$ and 3. In case there is a single stationary point, we use the following stationary phase expansion.

Lemma 1.7.2. *Let I, f and g be as above. Let $0 < \delta < 1/10$, X, Y, U, $Q > 0$, $Z := Q + X + Y + b - a + 1$, and assume that*

$$Y \geq Z^{3\delta},\, b - a \geq U \geq \frac{QZ^{\frac{\delta}{2}}}{\sqrt{Y}}.$$

Further, assume that g satisfies

$$g^{(j)}(x) \ll_j \frac{X}{U^j}, \quad \text{for } j = 0,\, 1,\, 2, \ldots$$

Suppose that there exists a unique $x_0 \in [a, b]$ such that $f'(x_0) = 0$, and the function f satisfies

$$f''(x) \gg \frac{Y}{Q^2}, \quad f^{(j)}(x) \ll_j \frac{Y}{Q^j}, \quad \text{for } \ j = 1,\, 2,\, 3, \ldots.$$

Then we have

$$I = \frac{e(f(x_0))}{\sqrt{f''(x_0)}} \sum_{n=0}^{\lfloor 3\delta^{-1}A \rfloor} p_n(x_0) + O_{A,\delta}\left(Z^{-A}\right),$$
$$p_n(x_0) = \frac{e^{\pi i/4}}{n!}\left(\frac{i}{2f''(x_0)}\right)^n G^{(2n)}(x_0),$$

where $A > 0$ is arbitrary, and

$$G(x) = g(x)e(F(x)), \quad \text{with} \quad F(x) = f(x) - f(x_0) - \frac{1}{2}f''(x_0)(x - x_0)^2.$$

Furthermore, each p_n is a rational function in $f', f'', \ldots$, satisfying

$$\frac{d^j}{dx_0^j} p_n(x_0) \ll_{j,n} X\left(\frac{1}{U^j} + \frac{1}{Q^j}\right)\left(\left(\frac{U^2Y}{Q^2}\right)^{-n} + Y^{-\frac{n}{3}}\right).$$

Proof. See [[14], Lemma 8.1]. □

Chapter 2

$\mathrm{GL}(3)\times\mathrm{GL}(2)$ L-functions: $\mathrm{GL}(2)$ spectral aspect

Let π be a Hecke-Maass cusp form of type (ν_1,ν_2) for $\mathrm{SL}(3,\mathbb{Z})$ with the Langlands parameters $(\alpha_1,\alpha_2,\alpha_3)$ and f be a holomorphic cusp form with weight k_f or a Hecke-Maass cusp form corresponding to the Laplacian eigenvalue $1/4+\nu_f^2$, $\nu_f\geq 1$, for $\mathrm{SL}(2,\mathbb{Z})$. The Rankin-Selberg L-series associated to π and f is given by

$$L(s,\pi\times f)=\sum\sum_{n,r\geq 1}\frac{\lambda_\pi(n,r)\,\lambda_f(n)}{(nr^2)^s},\quad \Re(s)\gg 1, \tag{2.0.1}$$

where $\lambda_\pi(n,r)$ and $\lambda_f(n)$ are the normalised Fourier coefficients of π and f respectively. The above series extends to an entire function and satisfies a functional equation of the following form

$$\gamma(s,\pi\times f)L(s,\pi\times f)=i^{k_f}\gamma(1-s,\bar{\pi}\times\bar{f})L(1-s,\bar{\pi}\times\bar{f}),$$

with a gamma factor of degree 6

$$\gamma(s,\pi\times f)=\pi^{3s}\prod_{j=1}^{2}\prod_{i=1}^{3}\Gamma\left(\frac{s+\beta_j-\alpha_i}{2}\right),$$

where $\beta_1=(k_f-1)/2$, $\beta_2=(k_f+1)/2$ if f is a holomorphic form and $\beta_1=\epsilon_f-i\nu_f$, $\beta_2=\epsilon_f+i\nu_f$ if f is a Maass form. Here $\epsilon_f=0$ if f is even and $\epsilon_f=1$ if f is odd. In this chapter, we will analyse $L(s, \pi \times f)$ at the central point $1/2$. More specifically, we are interested in estimating $L(1/2, \pi \times f)$ with respect to the $\mathrm{GL}(2)$ form f, keeping π fixed. Let $\nu_f \asymp k_f$. Then using the Phragmén-Lindelöf principle, we see that

$$L(1/2, \pi \times f) \ll_{\pi,\epsilon} (k_f^6)^{1/4+\epsilon}.$$

Note that k_f^6 is the analytic conductor in this case (see (0.0.5)). It is an interesting problem to improve the exponent $1/4$. Indeed, the first improvement in this case was made by Li [58] for the self-dual forms π. In this chapter, we will prove Theorem 0.7.1, which will generalise Li [58]'s result to all π (not necessarily self-dual). Recall from Theorem 0.7.1 that we need to prove the following

$$L\left(1/2, \pi \times f\right) \ll_{\pi,\epsilon} k_f^{3/2-1/51+\epsilon}, \tag{2.0.2}$$

for any $\epsilon > 0$. Using Lemma 1.3.2, we see that

$$L\left(1/2, \pi \times f\right) \ll_{\pi,\epsilon} k_f^{\epsilon} \sup_{r \leq k_f^{\theta}} \sup_{k_f^{3-\theta} \leq Nr^2 \leq k_f^{3+\epsilon}} \frac{|S_r(N)|}{N^{1/2}} + k_f^{(3-\theta)/2+\epsilon}, \tag{2.0.3}$$

where $S_r(N)$ is a sum of the form

$$S_r(N) := \sum_{n=1}^{\infty} \lambda_\pi(n,r)\lambda_f(n) V\left(\frac{n}{N}\right), \tag{2.0.4}$$

for some smooth function V supported in $[1,2]$, satisfying $V^{(j)}(x) \ll_j k_f^{\epsilon}$ for any integer $j \geq 0$ and $\int_{\mathbb{R}} V = 1$. We prove the following proposition.

Proposition 2.0.1. *For $0 < \eta < 1$, we have*

$$\frac{S_r(N)}{N^{1/2} k_f^{3/2+\epsilon}} \ll k_f^{-1/2+2\eta} + r^{1/2} k_f^{-\eta/2} + r^{1/2} \frac{k_f^{3/2-\eta/2}}{N^{1/2}} + k_f^{-1/6+3\eta/4}, \tag{2.0.5}$$

where the implied constant depends on η.

Theorem 0.7.1 follows from Proposition 2.0.1. Indeed, using $k_f^{3-\theta} \ll Nr^2 \ll k_f^{3+\epsilon}$ and $r \ll k_f^{\theta}$ in (2.0.5), we get

$$\frac{S_r(N)}{N^{1/2}k_f^{3/2+\epsilon}} \ll k_f^{-1/2+2\eta} + k_f^{\theta/2-\eta/2} + k_f^{2\theta-\eta/2} + k_f^{-1/6+3\eta/4}.$$

Hence to get subconvexity, we need all of the above exponents to be negative. So the first and the third term gives $1/4 > \eta > 4\theta$, and consequently the third and the fourth terms dominate the rest. Thus the above bound reduces to

$$\frac{S_r(N)}{N^{1/2}k_f^{3/2+\epsilon}} \ll k_f^{2\theta-\eta/2} + k_f^{-1/6+3\eta/4}.$$

The optimal choice for η is given by $\eta = 8\theta/5 + 2/15$. On plugging this in (2.0.3), we get

$$L(1/2, \pi \times f) \ll k_f^{3/2+6\theta/5-1/15+\epsilon} + k_f^{3/2-\theta/2+\epsilon},$$

and with the optimal choice $\theta = 2/51$, we get Theorem 0.7.1. In the rest of the chapter we prove Proposition 2.0.1.

Remark 2.0.2. *We will carry out the whole analysis for the holomorphic cusp form f, as the analysis for the Maass forms is similar. Indeed, in this case, we use the Rankin-Selberg bound (1.1.2) in place of Delinge's bound. We refer to [37] to see complete details.*

Remark 2.0.3 (Notation)**.** *In this chapter, the notation $\alpha \ll A$ will mean that for any ϵ, there is a constant c such that $|\alpha| \leq ck_f^{\epsilon}A$. $\alpha \asymp A$ means $c_1k_f^{-\epsilon}A \leq \alpha \leq c_2k_f^{\epsilon}A$. For notational convenience, we will ignore f from k_f and write k in place of k_f. By negligibly small we mean $O(k^{-A})$ for any $A > 1$. In particular we take $A = 2020$.*

2.1 The delta method and outline of the proof

Let's consider

$$S_r(N)=\sum_{n=1}^{\infty}\lambda_\pi(n,r)\lambda_f(n)V\left(\frac{n}{N}\right). \tag{2.1.1}$$

A trivial estimation (applying the Cauchy inequality followed by the Rankin-Selberg bound) of the above sum gives us $S_r(N)\ll N$. Thus, to prove Proposition 2.0.1, we need to show some cancellations in $S_r(N)$.

2.1.1 An application of the delta method

As a first step, following Munshi [77], we separate the oscillatory terms $\lambda_\pi(n,r)$ and $\lambda_f(n)$ involved in (2.1.1). We use the delta method of Duke, Friedlander and Iwaniec as a device to separate these terms. We also apply the conductor lowering trick introduced by Munshi in [71]. For this purpose, we introduce an extra t-integral. In fact, we rewrite $S_r(N)$ as

$$\begin{aligned}S_r(N)=&\frac{1}{\mathcal{T}}\int_{\mathbb{R}}V\left(\frac{t}{\mathcal{T}}\right)\sum\sum_{\substack{n,m=1\\ n=m}}^{\infty}\lambda_\pi(n,r)\lambda_f(m)\left(\frac{n}{m}\right)^{it}V\left(\frac{n}{N}\right)U\left(\frac{m}{N}\right)dt,\\ =&\frac{1}{\mathcal{T}}\int_{\mathbb{R}}V\left(\frac{t}{\mathcal{T}}\right)\sum\sum_{n,m=1}^{\infty}\delta(n-m)\lambda_\pi(n,r)\lambda_f(m)\left(\frac{n}{m}\right)^{it}V\left(\frac{n}{N}\right)U\left(\frac{m}{N}\right)dt,\end{aligned} \tag{2.1.2}$$

where $k^\epsilon<\mathcal{T}:=k^{1-\eta}<k^{1-\epsilon}$ is a parameter which will be chosen later optimally, and U is a smooth function supported in $[1/2,5/2]$ with $U(x)=1$ for $x\in[1,2]$, and $U^{(j)}(x)\ll_j 1$ for any integer $j\geq 0$. Recall that we have $\int_{\mathbb{R}}V=1$. Consider the t-integral

$$\frac{1}{\mathcal{T}}\int_{\mathbb{R}}V\left(\frac{t}{\mathcal{T}}\right)\left(\frac{m}{n}\right)^{it}\mathrm{d}t.$$

On applying integration by parts repeatedly, we observe that the above integral is negligibly small unless $|n-m|\ll k^\epsilon N/\mathcal{T}$. Thus the t-integral reduces the size of the equation $n = m$. Thus, on applying Lemma 1.4.1 to (2.1.2) with $L = k^\epsilon N/\mathcal{T}$, and $Q = k^\epsilon\sqrt{N/\mathcal{T}}$, we see that $S_r(N)$ is transformed into

$$\begin{aligned} S_r(N) =& \frac{1}{Q\mathcal{T}} \int_{\mathbb{R}} W(x/Q^\epsilon) \int_{\mathbb{R}} V\left(\frac{t}{\mathcal{T}}\right) \sum_{1\le q\le Q} \frac{g(q,x)}{q} \sideset{}{^\star}\sum_{a \bmod q} \\ & \times \sum_{n=1}^{\infty} \lambda_\pi(n,r) e\left(\frac{an}{q}\right) e\left(\frac{nx}{qQ}\right) n^{it} V\left(\frac{n}{N}\right) \\ & \times \sum_{m=1}^{\infty} \lambda_f(m) m^{-it} e\left(\frac{-am}{q}\right) e\left(\frac{-mx}{qQ}\right) U\left(\frac{m}{N}\right) \mathrm{d}t\, \mathrm{d}x + O(k^{-2020}). \end{aligned}$$

Next we break the q-sum into dyadic segments $q \sim C$, with $1 \ll C \ll Q$ and write

$$S_r(N) = \sum_{\substack{1\ll C\ll Q \\ \text{dyadic}}} S_r(N,C) + O(k^{-2020}), \tag{2.1.3}$$

where

$$\begin{aligned} S_r(N,C) =& \frac{1}{Q\mathcal{T}} \int_{\mathbb{R}} W(x/Q^\epsilon) \int_{\mathbb{R}} V\left(\frac{t}{\mathcal{T}}\right) \sum_{q\sim C} \frac{g(q,x)}{q} \sideset{}{^\star}\sum_{a \bmod q} \\ & \times \sum_{n=1}^{\infty} \lambda_\pi(n,r) e\left(\frac{an}{q}\right) e\left(\frac{nx}{qQ}\right) n^{it} V\left(\frac{n}{N}\right) \\ & \times \sum_{m=1}^{\infty} \lambda_f(m) m^{-it} e\left(\frac{-am}{q}\right) e\left(\frac{-mx}{qQ}\right) U\left(\frac{m}{N}\right) \mathrm{d}t\, \mathrm{d}x. \end{aligned} \tag{2.1.4}$$

2.1.2 Sketch of the proof

In this subsection, we will discuss rough ideas to get non-trivial cancellations in $S_r(N)$ given in (2.1.4). For simplicity, we consider the generic case, i.e., $N = k^3$, $r = 1$ and $q \sim Q = \sqrt{N/\mathcal{T}} = k^{3/2}/\mathcal{T}^{1/2}$. Thus $S_r(N)$ is roughly given by

$$\frac{1}{Q\mathcal{T}} \int_{\mathcal{T}}^{2\mathcal{T}} \sum_{q\sim Q} \frac{1}{q} \sideset{}{^\star}\sum_{a \bmod q} \sum_{n\sim N} \lambda_\pi(n,1) n^{it} e\left(\frac{an}{q}\right) \sum_{m\sim N} \lambda_f(m)\, m^{-it} e\left(\frac{-am}{q}\right) \mathrm{d}t.$$

Note that we have ignored the x-integral, as it does not contribute in the generic case, and we have also supressed all the weight functions. On estimating the above sum trivially, we get $S_r(N) \ll N^{2+\epsilon}$. Hence, to get non-trivial cancellations, we need to save N plus a little more, say, k^δ. In other words, we need to show

$$S_r(N) \ll N^2/(Nk^\delta),$$

for some $\delta > 0$. In the next step, we dualize the sum over n and m (we give full details in Section 2.2). Consider the sum over n

$$\mathrm{S}_3 = \sum_{n \sim N} \lambda_\pi(n,1) n^{it} e\left(\frac{an}{q}\right).$$

On applying the $\mathrm{GL}(3)$ Voronoi summation formula to the above sum, we arrive at (see Lemma 2.2.1)

$$\mathrm{S}_3 \approx \frac{N^{2/3}}{q} \sum_{n_2 \sim Q^3 \mathcal{T}^3/N} \frac{\lambda_\pi(1,n_2)}{n_2^{1/3}} S(\bar{a}, \pm n_2; q)\, \mathrm{I}_3(...),$$

where $\mathrm{I}_3(...)$ is an integral transform in which we need to get square root cancellations, i.e., need to show $\mathrm{I}_3(...) \ll 1/\sqrt{\mathcal{T}}$ (obtaining this bound is necessary, otherwise we won't get sufficient savings at the last). Next we apply the $\mathrm{GL}(2)$ Voronoi formula to the sum over m and we get (see Lemma 2.2.3 for details)

$$\sum_{m \sim N} \lambda_f(m)\, m^{-it} e\left(\frac{-am}{q}\right) \approx \frac{N}{q} \sum_{m \sim Q^2 k^2/N} \lambda_f(m) e\left(\frac{\bar{a}m}{q}\right) \mathrm{I}_2(...),$$

where $\mathrm{I}_2(...)$ is an integral transform in which we need to get full cancellations, i.e., need to show $\mathrm{I}_2(...) \ll 1/k$ (proving this bound is a necessary step to obtain our main result). Next we analyse the sum over a which is given by

$$\mathfrak{C} = \sideset{}{^*}\sum_{a \bmod q} S\left(\bar{a}, n_2; q\right)\, e\left(\frac{\bar{a}m}{q}\right) \approx q e\left(-\frac{\bar{m}n_2}{q}\right).$$

We observe that the above sum becomes an additive character with respect to n_2 (which saves us extra q when we apply the poisson after Cauchy). Thus, we arrive at the following expression of $S_r(N)$:

$$\frac{1}{Q\mathcal{T}} \frac{N}{Q^2\mathcal{T}} \frac{N}{Q} \sum_{q \sim Q} \sum_{n_2 \sim \mathcal{T}^{3/2} N^{1/2}} \lambda_\pi(1,n_2) \sum_{m \sim k^2/\mathcal{T}} \lambda_f(m) e\left(-\frac{\bar{m}n_2}{q}\right) \mathfrak{J},$$

where $\mathfrak{J}$ is an integral transform involving the t-integral, $\mathrm{I}_2(...)$ and $\mathrm{I}_3(...)$. We analyse it in Section 2.4. We observe that

$$\mathfrak{J} \ll \mathcal{T} \frac{1}{\sqrt{\mathcal{T}}} \frac{1}{\sqrt{\mathcal{T}}} \frac{1}{k}.$$

Note that a saving of $\sqrt{\mathcal{T}}$ comes from the t-integral, another saving of $\sqrt{\mathcal{T}}$ comes from the $\mathrm{GL}(3)$-integral and the saving of k comes from the $\mathrm{GL}(2)$ integral. The factor $\mathcal{T}$ reflects the length of the t-integral. Thus, on plugging it in place of $\mathfrak{J}$, we see that

$$\begin{aligned} S_r(N) &\ll \sum_{q \sim Q} \sum_{n_2 \sim \mathcal{T}^{3/2} N^{1/2}} |\lambda_\pi(1, n_2)| \left| \sum_{m \sim k^2/\mathcal{T}} \lambda_f(m) e\left(-\frac{\bar{m} n_2}{q}\right) \mathfrak{J} \right| \\ &\ll Q \mathcal{T}^{3/2} N^{1/2} \frac{k^2}{\mathcal{T}} \frac{1}{k} \ll Nk. \end{aligned}$$

Thus we now need to save $k^{1+\delta}$. Next we apply Cauchy's inequality to the sum over n_2 to get rid of the $\mathrm{GL}(3)$ coefficients. Thus we arrive at (see Subsection 2.3.1)

$$(\mathcal{T}^{3/2} N^{1/2})^{1/2} \left(\sum_{n_2 \sim \mathcal{T}^{3/2} N^{1/2}} \left| \sum_{q \sim Q} \sum_{m \sim k^2/\mathcal{T}} \lambda_f(m)\, e\left(-\frac{\bar{m} n}{q}\right) \mathfrak{J} \right|^2 \right)^{1/2}.$$

The end game strategy is to apply the Poisson to the sum over n_2 (we carry out the details in Subsection 2.3.2). Opening the absolute value square followed by the Poisson, we observe that we save the whole length, i.e., $k^2 Q/\mathcal{T}$ in the zero-frequency ($n_2 = 0$ case) which suffices if $k^2 Q/\mathcal{T} > k^2$ which implies that $\mathcal{T} < k$. On the other hand, in the non-zero frequencies ($n_2 \neq 0$ case), we save

$$\frac{\mathcal{T}^{3/2} N^{1/2}}{(Q^2 \mathcal{T})^{1/2}}.$$

Here the factor $Q^2 \mathcal{T}$ in the denominator reflects the size of the conductor, which is given by

$$\text{conductor} = \text{arithmetic conductor} \times \text{analytic conductor}.$$

Note that the arithmetic conductor is of size Q^2 and the analytic conductor is of size $\mathcal{T}$ (because $\mathfrak{J}$ oscillates like $n_2^{i\mathcal{T}}$ with respect to n_2). We also save Q due to the presence of the additive character $e(-\bar{m}n/q)$. Thus the total savings in the non-zero frequencies turns out to be

$$\frac{\mathcal{T}^{3/2}N^{1/2}}{(Q^2\mathcal{T})^{1/2}} \times Q = \mathcal{T}N^{1/2},$$

which suffices if $\mathcal{T}N^{1/2} > k^2$ which boils down to $\mathcal{T} > k^{1/2}$. Hence, to succeed, we must ensure that $k^{1/2} < \mathcal{T} < k$, which is done by chosing $\mathcal{T}$ optimally.

2.2 Applications of Voronoi formulae

In this section, we will analyse the sum over n and m in (2.1.4) using the Voronoi summation formulae.

2.2.1 The $\mathrm{GL}(3)$ Voronoi

Let's consider the sum over n

$$\mathrm{S}_3 := \sum_{n=1}^{\infty} \lambda_\pi(n,r) e\left(\frac{an}{q}\right) e\left(\frac{nx}{qQ}\right) n^{it} V\left(\frac{n}{N}\right) \tag{2.2.1}$$

in (2.1.4). We analyze it using the $\mathrm{GL}(3)$ Voronoi summation formula (see Lemma 1.2.1). In the present set-up, we have $g(n) = e\left(nx/qQ\right) n^{it} V\left(n/N\right)$ and $X = N$. Thus, on applying Lemma 1.2.1 to the above sum, we get

$$\mathrm{S}_3 = q \sum_{\pm} \sum_{n_1|qr} \sum_{n_2=1}^{\infty} \frac{\lambda_\pi(n_1,n_2)}{n_1 n_2} S\left(r\bar{a}, \pm n_2; qr/n_1\right) G_{\pm}\left(n_2^\star\right), \tag{2.2.2}$$

where $n_2^\star := n_1^2 n_2/(q^3 r)$ and $G_\pm(n_2^\star)$ is the integral transform defined in (1.2.3). Next we extract the oscillations of the integral transform $G_\pm(n_2^\star)$ using Lemma (1.2.2), which gives the following expression for $G_{\pm}(n_2^\star)$ in the range $n_2^\star N \gg k^\epsilon$

$$n_2^\star \int_0^\infty g(z) \sum_{j=1}^{K_0} \frac{c_j(\pm)e(3(n_2^\star z)^{1/3}) + d_j(\pm)e(-3(n_2^\star z)^{1/3})}{(n_2^\star z)^{j/3}} \, \mathrm{d}z + O_\epsilon(k^{-2020}), \tag{2.2.3}$$

where $K_0 = [\frac{6060}{\epsilon} + 5] + 1$ with $[.]$ denoting the greatest integer function. From now on we will continue our analysis with the terms corresponding to $j = 1$, as other terms can be treated in a similar way and in fact, give us better estimates (see Remark 2.2.2). Thus on plugging the above expression corresponding to the term $j = 1$ into (2.2.2), we arrive at

$$\frac{N^{2/3+it}}{qr^{2/3}} \sum_{\pm} \sum_{n_1|qr} n_1^{1/3} \sum_{n_2=1}^{\infty} \frac{\lambda_\pi(n_1, n_2)}{n_2^{1/3}} S(r\bar{a}, \pm n_2; qr/n_1) \mathrm{I}_3(n_1^2 n_2, q, x),$$

where

$$\mathrm{I}_3(n_1^2 n_2, q, x) := \int_0^\infty z^{-1/3} V(z) z^{it} e\left(\frac{Nxz}{qQ} \pm \frac{3(Nn_1^2 n_2 z)^{1/3}}{qr^{1/3}} \right) \mathrm{d}z. \tag{2.2.4}$$

On applying the change of variable $z \mapsto z^3$ followed by the integration by parts (differentiating $3z^2 V(z^3) z^{i3t} e(Nxz^3/qQ)$ and integrating $e(\pm 3(Nn_1^2 n_2)^{1/3} z/qr^{1/3})$) j-times to the above integral, we observe that

$$\mathrm{I}_3(n_1^2 n_2, q, x) \ll_j \left(\mathcal{T} + \frac{N|x|}{qQ} \right)^j \left(\frac{qr^{1/3}}{(Nn_1^2 n_2)^{1/3}} \right)^j,$$

for any integer $j \geq 0$, and it is negligibly small if

$$n_1^2 n_2 \geq \; k^\epsilon \max\left\{ \frac{C^3 \mathcal{T}^3 r}{N}, \mathcal{T}^{3/2} N^{1/2} r \right\} =: N_0. \tag{2.2.5}$$

Recall that $q \sim C$. Now it remains to analyse $G_{\pm}(n_2^\star)$ for $n_2^\star N \ll k^\epsilon$, which is given as

$$\begin{aligned} G_{\pm}(n_2^\star) &= \frac{1}{2\pi i}\int_{(\sigma)} (n_2^\star)^{-s}\,\gamma_{\pm}(s)\tilde{g}(-s)\,\mathrm{d}s \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} (n_2^\star)^{-\sigma-i\tau}\,\gamma_{\pm}(\sigma+i\tau)\,\tilde{g}(-\sigma-i\tau)\,\mathrm{d}\tau. \end{aligned} \tag{2.2.6}$$

We will consider this case in Subsection 2.6.3. We conclude this subsection by summarising the above discussion in the following lemma.

Lemma 2.2.1. *Let* S_3 *be as in* (2.2.1)*. Then, for* $n_2^\star N = n_1^2 n_2 N/(q^3 r) \gg k^\epsilon$*, we have*

$$\begin{aligned} \mathrm{S}_3 = \frac{N^{2/3+it}}{qr^{2/3}} \sum_{\pm}\sum_{n_1|qr} n_1^{1/3} \sum_{n_2\le N_0/n_1^2} \frac{\lambda_\pi(n_1,n_2)}{n_2^{1/3}}\, S(r\bar{a},\pm n_2; qr/n_1)\,\mathrm{I}_3(n_1^2 n_2,\, q,\, x), \\ + \text{other lower order terms} + O(k^{-2020}), \end{aligned} \tag{2.2.7}$$

where $\mathrm{I}_3(n_1^2 n_2,\, q,\, x)$ *is the integral transform defined in* (2.2.4) *and* N_0 *is the range of* $n_1^2 n_2$ *defined in* (2.2.5)*. For the non-generic case* $n_2^\star N \ll k^\epsilon$*, we have*

$$\mathrm{S}_3 = q\sum_{\pm}\sum_{n_1|qr}\sum_{n_2=1}^{\infty} \frac{\lambda_\pi(n_1,n_2)}{n_1 n_2} S\left(r\bar{a},\pm n_2; qr/n_1\right) G_{\pm}\left(n_2^\star\right), \tag{2.2.8}$$

where $G_{\pm}(n_2^\star)$ *is the integral transform* (2.2.6)*.*

Remark 2.2.2. *The other lower order terms come from the expression of* (2.2.3) *corresponding to* $j = 2, 3\ldots$*. On comparing it with the main term* $(j = 1)$*, we see that the integral transforms in these terms are exactly same as that of the main term. However, there is an extra factor* $(n_2^\star N)^{(j-1)/3}$ *in the denominator (after the change of variable* $z \to Nz$*). For instance, for* $j = 2$*, it is* $(n_2^\star N)^{1/3}$ *which is of size* $\mathcal{T}$ *in the generic case. Thus the final bound turns out to be* $1/\mathcal{T}$ *times the final bound of the main term. We save even more for other* j*. From now on, we will proceed with the main term of* (2.2.7)*.*

2.2.2 $\mathrm{GL}(2)$ Voronoi

We now consider the sum over m in (2.1.4), which is given as

$$\mathrm{S}_2 := \sum_{m=1}^{\infty} \lambda_f(m) m^{-it} e\left(\frac{-am}{q}\right) e\left(\frac{-mx}{qQ}\right) U\left(\frac{m}{N}\right). \tag{2.2.9}$$

On applying the $\mathrm{GL}(2)$ Voronoi summation formula (see Lemma 1.1.1) to the above sum with $g(m) = m^{-it} e(-mx/(qQ)) U(m/N)$, we get

$$\begin{aligned}\mathrm{S}_2 &= \frac{2\pi i^k}{q} \sum_{m=1}^{\infty} \lambda_f(m) e\left(\frac{\bar{a}m}{q}\right) \int_0^{\infty} y^{-it} e\left(\frac{-xy}{qQ}\right) U\left(\frac{y}{N}\right) J_{k-1}\left(\frac{4\pi\sqrt{my}}{q}\right) \mathrm{d}y \\ &= \frac{2\pi i^k N^{1-it}}{q} \sum_{m=1}^{\infty} \lambda_f(m) e\left(\frac{\bar{a}m}{q}\right) \int_0^{\infty} U(y) y^{-it} e\left(\frac{-Nxy}{qQ}\right) J_{k-1}\left(\frac{4\pi\sqrt{mNy}}{q}\right) \mathrm{d}y,\end{aligned}$$

Next we analyse the integral transform

$$\mathrm{I}_2(m, q, x) := \int_0^{\infty} U(y) y^{-it} e\left(\frac{-Nxy}{qQ}\right) J_{k-1}\left(\frac{4\pi\sqrt{mNy}}{q}\right) \mathrm{d}y. \tag{2.2.10}$$

to determine the range of m. We claim that $\mathrm{I}_2(m, q, x)$ is negligibly small unless

$$M := \frac{C^2 (k-1)^2 k^{-\epsilon}}{N} \leq m \leq k^{\epsilon} \max\left(\frac{(k-1)^2 C^2}{N}, \mathcal{T}\right) =: M_0. \tag{2.2.11}$$

In fact, in the range $m < M$, we have

$$4\pi\sqrt{mNy}/q \leq 4\pi\sqrt{mNy}/C < 4\pi\sqrt{5/2}(k-1)^{1-\epsilon/2} \ll (k-1)^{1-\epsilon/2}.$$

Thus, by Lemma 1.5.4, the integral transform $\mathrm{I}_2(m, q, x)$ is negligibly small.

Now we consider the range $m > M_0$. We note that $4\pi\sqrt{mNy}/q > (k-1)^{1+\epsilon/2}$. Thus, we can apply Langer's expansion (see Lemma 1.5.2) for the Bessel function J_{k-1}. On applying Corollary 1.5.3 with $x = 4\pi\sqrt{mNy}/q$, we see that $\mathrm{I}_2(m, q, x)$, up to a negligible error term, is given by

$$\sum_{j=0}^{2019} \frac{1}{(k-1)^{j+1/2}} \int_0^{\infty} U_j(y) y^{-it} e\left(\frac{-Nxy}{qQ}\right) e\left(\pm\frac{(k-1)(w - \tan^{-1} w)}{2\pi}\right) \mathrm{d}y,$$

where $U_j(y) = U(y) P_j\left((w - \tan^{-1} w)^{-1}\right) w^{-1/2}$ with

$$w = \left(\frac{x^2}{(k-1)^2} - 1\right)^{1/2} = \left(\frac{16\pi^2 m N y}{q^2(k-1)^2} - 1\right)^{1/2},$$

and P_j is a polynomial of the degree j with coefficients which are bounded functions of k. Note that $w > ((k-1)^{\epsilon} - 1)^{1/2}$. Thus

$$w - \tan^{-1} w = w - \frac{\pi}{2} + \tan^{-1}\frac{1}{w} \asymp w \asymp \frac{\sqrt{mN}}{C(k-1)},$$

and $U_j^{(\ell)}(y) \ll_\ell k^{\epsilon\ell}$ for any integer $\ell \geq 0$. Next we apply integration by parts ℓ-times to the above y-integral. To this end, let $g(y) = U_j(y) y^{-it} e\left(\frac{-Nxy}{qQ}\right)$ and $h(y) = \pm i(k-1)(w - \tan^{-1} w)$. We now write the y-integral as follows:

$$\int_0^\infty g(y)\, e^{h(y)}\, \mathrm{d}y = \int_0^\infty \frac{g(y)}{h'(y)} h'(y)\, e^{h(y)}\, \mathrm{d}y = \int_0^\infty \frac{g(y)}{h'(y)}\, \mathrm{d}(e^{h(y)}).$$

By integration by parts, we see that

$$\int_0^\infty g(y)\, e^{h(y)}\, \mathrm{d}y = -\int_0^\infty e^{h(y)}\, \mathrm{d}\left(\frac{g(y)}{h'(y)}\right) = -\int_0^\infty g_1(y)\, e^{h(y)}\, \mathrm{d}y,$$

where $g_1(y) = \frac{\mathrm{d}}{\mathrm{d}y}\frac{g(y)}{h'(y)}$. Note that

$$g_1(y) = \frac{h'(y)g'(y) - g(y)h''(y)}{(h'(y))^2} \ll \frac{|g'(y)|}{|h'(y)|} \ll \frac{(\mathcal{T} + N|x|/(qQ))}{(k-1)\sqrt{mN}/(q(k-1))},$$

as $h''(y) \asymp h'(y) \asymp (k-1)\frac{\sqrt{mN}}{q(k-1)}$ and $g(y) \ll k^{\epsilon}$. We iterate the same procedure with the new y-integral $\int_0^\infty g_1(y)\, e^{h(y)}\, \mathrm{d}y$ and we thus get

$$\begin{aligned}\mathrm{I}_2(m, q, x) &\ll_\ell \left(\mathcal{T} + \frac{N|x|}{qQ}\right)^{\ell} \left(\frac{1}{(k-1)\sqrt{mN}/(q(k-1))}\right)^{\ell} \\ &\ll \left(\frac{\mathcal{T}C}{\sqrt{M_0 N}} + \frac{N|x|}{Q\sqrt{M_0 N}}\right)^{\ell} \ll \left(\frac{k^{\epsilon}\mathcal{T}}{k} + \frac{1}{k^{\epsilon}}\right)^{\ell} \ll \frac{1}{k^{\epsilon\ell}}.\end{aligned}$$

Upon taking ℓ sufficiently large, we get the claim. We end this subsection by summarizing the above arguments in the following lemma.

Lemma 2.2.3. *Let S_2 be the sum over m as given in (2.2.9). Then we have*

$$\mathrm{S}_2 = \frac{2\pi i^k N^{1-it}}{q} \sum_{M \le m \le M_0} \lambda_f(m) e\left(\frac{\bar{a}m}{q}\right) \mathrm{I}_2(m,\, q,\, x) + O(k^{-2020}), \tag{2.2.12}$$

where

$$\mathrm{I}_2(m,\, q,\, x) = \int_0^\infty U(y) y^{-it} e\left(\frac{-Nxy}{qQ}\right) J_{k-1}\left(\frac{4\pi\sqrt{mNy}}{q}\right) \mathrm{d}y,$$

and M and M_0 are the ranges of m defined in (2.2.11).

2.3 Cauchy and Poisson

After the applications of the Voronoi formulae and applying Lemma 2.2.1 and Lemma 2.2.3 to (2.1.4), we see that $S_r(N)$ in (2.1.3) given by

$$\begin{aligned}
&\frac{N^{5/3}}{Q\mathcal{T}r^{2/3}} \sum_{\substack{1 \ll C \ll Q \\ \text{dyadic}}} \sum_{q \sim C} \frac{1}{q^3} \sideset{}{^\star}\sum_{a \bmod q} \sum_{\pm} \sum_{n_1 | qr} n_1^{1/3} \\
&\times \sum_{n_2 \le N_0/n_1^2} \frac{\lambda_\pi(n_1, n_2)}{n_2^{1/3}} S(r\bar{a}, \pm n_2; qr/n_1) \\
&\times \sum_{M \le m \le M_0} \lambda_f(m) e\left(\frac{\bar{a}m}{q}\right) \mathrm{J}_\pm(m,\, n_1^2 n_2,\, q) + O(k^{-2020}),
\end{aligned} \tag{2.3.1}$$

where

$$\mathrm{J}_\pm(m,\, n_1^2 n_2,\, q) = \int_{\mathbb{R}} \int_{\mathbb{R}} W(x/Q^\epsilon)\, g(q,\, x)\; \mathrm{I}_2(m,\, q,\, x)\, \mathrm{I}_3(n_1^2 n_2,\, q,\, x)\, V\left(\frac{t}{\mathcal{T}}\right) \mathrm{d}t\, \mathrm{d}x. \tag{2.3.2}$$

We recall that we are in the situation where the n_2 variable satisfies $n_1^2 n_2 N/q^3 r \gg k^\epsilon$. We will keep this fact in mind from now on. In this section, we will analyse (2.3.1) using the Cauchy inequality and the Poisson summation formula.

2.3.1 Cauchy's inequality

On writing $q = q_1 q_2$ with $q_1|(n_1 r)^\infty$, $(n_1 r,\, q_2) = 1$, we see that the expression in (2.3.1) is dominated by

$$\sup_{C\ll Q} \frac{N^{5/3}\log Q}{Q\mathcal{T} r^{2/3} C^3} \sum_{\pm} \sum_{\frac{n_1}{(n_1,r)}\ll C} n_1^{1/3} \sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1 r)^\infty} \sum_{n_2\le N_0/n_1^2} \frac{|\lambda_\pi(n_1,n_2)|}{n_2^{1/3}} \times \Big| \sum_{q_2\sim C/q_1} \sum_{M\le m\le M_0} \lambda_f(m)\mathcal{C}_\pm(q,n_2,m)\mathrm{J}_\pm(m,n_1^2 n_2,q)\Big|, \tag{2.3.3}$$

where the character sum $\mathcal{C}_\pm(q,n_2,m) = \mathcal{C}_\pm(\dots)$ is defined as

$$\mathcal{C}_\pm(\dots) := \sideset{}{^\star}\sum_{a \bmod q} S(r\bar{a}, \pm n_2; qr/n_1) e\left(\frac{\bar{a}m}{q}\right) = \sum_{d|q} d\mu\left(\frac{q}{d}\right) \sideset{}{^\star}\sum_{\substack{\alpha \bmod qr/n_1 \\ n_1\alpha\equiv -m \bmod d}} e\left(\pm\frac{\bar{\alpha}n_2}{qr/n_1}\right).$$

We note that q is q_1q_2 and the q_2 variable in the sum over q_2 in (2.3.3) satisfies $(q_2, n_1 r) = 1$. From now on we will keep these facts in mind and use them whenever required.

Next we analyse the expression inside $|\;|$. We first split the sum over m into dyadic blocks $m \sim M_1$, $M \ll M_1 \ll M_0$ and then apply the Cauchy inequality to the sum over n_2 in (2.3.3) to arrive at

$$S_r(N) \ll \sup_{\substack{M\ll M_1\ll M_0 \\ C\ll Q}} \frac{N^{5/3}(QM_0)^\epsilon}{Q\mathcal{T} r^{2/3} C^3} \sum_{\pm} \sum_{\frac{n_1}{(n_1,r)}\ll C} n_1^{1/3}\Theta^{1/2} \sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1 r)^\infty} \sqrt{\Omega_\pm}, \tag{2.3.4}$$

where

$$\Theta = \sum_{n_2\ll N_0/n_1^2} \frac{|\lambda_\pi(n_1,n_2)|^2}{n_2^{2/3}}, \tag{2.3.5}$$

and

$$\Omega_\pm = \sum_{n_2\ll N_0/n_1^2} \Big| \sum_{q_2\sim C/q_1} \sum_{m\sim M_1} \lambda_f(m)\mathcal{C}_\pm(q,n_2,m)\mathrm{J}_\pm(m,n_1^2 n_2,q)\Big|^2. \tag{2.3.6}$$

Recall that

$$\frac{(k-1)^2C^2}{N}k^{-\epsilon}=M\ll M_1\ll M_0=k^{\epsilon}\max\left(\frac{(k-1)^2C^2}{N},\,\mathcal{T}\right),$$
$$N_0=k^{\epsilon}\max\left\{\frac{(C\mathcal{T})^3\,r}{N},\,\mathcal{T}^{3/2}N^{1/2}r\right\}. \tag{2.3.7}$$

2.3.2 The Poisson summation formula

Next we apply the Poisson summation formula to the sum over n_2 with the modulus $\mathfrak{q}:=q_1q_2q_2'r/n_1$ in (2.3.6). To this end, we first split the sum over n_2 into dyadic blocks $n_2\sim\tilde{N}/n_1^2$, $\tilde{N}\ll N_0$. Then opening the absolute value square in (2.3.6), we arrive at

$$\Omega_{\pm}=\sum\sum_{q_2,\,q_2'\sim C/q_1}\ \sum\sum_{m,\,m'\sim M_1}\lambda_f(m)\lambda_f(m')\,\Delta_{\pm},$$

where

$$\Delta_{\pm}=\sum_{\tilde{N}}\sum_{n_2\in\mathbb{Z}}\phi\left(\frac{n_1^2n_2}{\tilde{N}}\right)\mathcal{C}_{\pm}(q,n_2,m)\overline{\mathcal{C}_{\pm}(q',n_2,m')}\mathrm{J}_{\pm}(m,n_1^2n_2,q)\overline{\mathrm{J}_{\pm}(m',n_1^2n_2,q')},$$

$q'=q_1q_2'$ and $\phi(w)$ is a non-negative smooth function supported on $[2/3,\,3]$ with $\phi(w)=1$ for $w\in[1,\,2]$ and $\phi^{(j)}(w)\ll_j 1$. Now applying the change of variable

$$n_2\to n_2\mathfrak{q}+\beta,\quad \beta \bmod \mathfrak{q},$$

we get the following expression for $\Delta_{\pm}$:

$$\Delta_{\pm}=\sum_{\tilde{N}}\sum_{\beta \bmod \mathfrak{q}}\mathcal{C}_{\pm}(q,\beta,m)\overline{\mathcal{C}_{\pm}(q',\beta,m')}$$
$$\times\sum_{n_2\in\mathbb{Z}}\phi\left(\frac{n_2\mathfrak{q}+\beta}{\tilde{N}/n_1^2}\right)\mathrm{J}_{\pm}(m,n_1^2(n_2\mathfrak{q}+\beta),q)\,\overline{\mathrm{J}_{\pm}(m',n_1^2(n_2\mathfrak{q}+\beta),q')}.$$

On applying the Poisson summation formula to the sum over n_2 , we arrive at

$$\Omega_{\pm} = \sum_{\tilde{N}} \frac{\tilde{N}}{n_1^2} \sum\sum_{q_2,\, q_2' \sim C/q_1} \sum\sum_{m,\, m' \sim M_1} \lambda_f(m)\lambda_f(m') \sum_{n_2 \in \mathbb{Z}} \mathfrak{C}_{\pm}\mathcal{J}_{\pm}, \tag{2.3.8}$$

where

$$\begin{aligned}\mathfrak{C}_{\pm} &= \frac{1}{\mathfrak{q}} \sum_{\beta \bmod \mathfrak{q}} \mathcal{C}_{\pm}(q,\beta,m)\, \overline{\mathcal{C}_{\pm}(q',\beta,m')}\, e\left(\frac{n_2\beta}{\mathfrak{q}}\right) \\ &= \sum\sum_{\substack{d|q \\ d'|q'}} dd'\mu\left(\frac{q}{d}\right)\mu\left(\frac{q'}{d'}\right) \mathop{\sum\nolimits^{\star}\sum\nolimits^{\star}}_{\substack{\alpha \bmod qr/n_1 \quad \alpha' \bmod q'r/n_1 \\ n_1\alpha \equiv -m \bmod d \quad n_1\alpha' \equiv -m' \bmod d' \\ \pm\bar{\alpha}q_2' \mp \bar{\alpha}' q_2 \equiv -n_2 \bmod \mathfrak{q}}} 1,\end{aligned} \tag{2.3.9}$$

and

$$\mathcal{J}_{\pm} = \int_{\mathbb{R}} \phi(w)\, \mathrm{J}_{\pm}(m,\tilde{N}w,q)\, \overline{\mathrm{J}_{\pm}(m',\tilde{N}w,q')}\, e\left(-\frac{n_2\tilde{N}w}{q_1q_2q_2'rn_1}\right) \mathrm{d}w. \tag{2.3.10}$$

Now on applying the Ramanujan bound (1.1.1) for the Fourier coefficients $\lambda_f(m)$ and $\lambda_f(m')$, we get

$$\Omega_{\pm} \ll k^{\epsilon} \sup_{\tilde{N} \ll N_0} \frac{\tilde{N}}{n_1^2} \sum\sum_{q_2,\, q_2' \sim C/q_1} \sum\sum_{m,\, m' \sim M_1} \sum_{n_2 \in \mathbb{Z}} |\mathfrak{C}_{\pm}||\mathcal{J}_{\pm}|, \tag{2.3.11}$$

as there are at most $\log N_0 (\ll k^{\epsilon})$ many $\tilde{N}$'s.

2.4 Estimates for the integral transform

In this section we will analyse the integral transform

$$\mathcal{J}_{\pm} = \int_{\mathbb{R}} \phi(w)\, \mathrm{J}_{\pm}(m,\tilde{N}w,q)\, \overline{\mathrm{J}_{\pm}(m',\tilde{N}w,q')}\, e\left(-\frac{n_2\tilde{N}w}{q_1q_2q_2'rn_1}\right) \mathrm{d}w, \tag{2.4.1}$$

where

$$\begin{aligned}
\mathrm{J}_{\pm}(...) &= \int_{\mathbb{R}} \int_{\mathbb{R}} W(x/Q^{\epsilon})\, g(q,\, x)\; \mathrm{I}_2(m,\, q,\, x)\, \mathrm{I}_3(\tilde{N}w,\, q,\, x)\, V\left(\frac{t}{\mathcal{T}}\right)\, \mathrm{d}t\, \mathrm{d}x \\
&= \int_{\mathbb{R}} W(x/Q^{\epsilon}) g(q,x) \int_{\mathbb{R}} V\left(\frac{t}{\mathcal{T}}\right) \int_0^{\infty} U(y) y^{-it} \int_0^{\infty} V(z) z^{it-1/3} \\
&\quad \times e\left(\frac{Nx(z-y)}{qQ} \pm \frac{3(N\tilde{N}wz)^{1/3}}{qr^{1/3}}\right) J_{k-1}\left(\frac{4\pi\sqrt{mNy}}{q}\right)\, \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}t\, \mathrm{d}x,
\end{aligned} \tag{2.4.2}$$

and $\mathrm{J}_{\pm}(m', \tilde{N}w, q')$ is similarly defined. Recall that $N\tilde{N}/q^3 r \gg k^{\epsilon}$. We first analyse $\mathrm{J}_{\pm}(m,\, \tilde{N}w,\, q)$.

Lemma 2.4.1. *Let $\mathrm{J}_{\pm}(...) = \mathrm{J}_{\pm}(m,\, \tilde{N}w,\, q)$ be as above. Let I_{z-y} and L_{z-y} denote the x and the t-integral respectively in $\mathrm{J}_{\pm}(...)$. Then either I_{z-y} or L_{z-y} is negligibly small unless $|u| = |z-y| \ll k^{\epsilon} C/(Q\mathcal{T})$.*

Proof. We consider two cases.

Case 1. $q \sim C \ll Q^{1-\epsilon}$.

Consider the integral over x in (2.4.2) which is given by

$$\begin{aligned}
\mathrm{I}_{z-y} &= \int_{\mathbb{R}} W(x/Q^{\epsilon}) g(q,x) e\left(\frac{Nx(z-y)}{qQ}\right)\, \mathrm{d}x \\
&= Q^{\epsilon} \int_{\mathbb{R}} W(x) g(q, xQ^{\epsilon}) e\left(\frac{NxQ^{\epsilon}(z-y)}{qQ}\right)\, \mathrm{d}x.
\end{aligned}$$

We now split the above integral as

$$\int_{\mathbb{R}} ...\, \mathrm{d}x = \int_{-Q^{-2\epsilon}}^{Q^{-2\epsilon}} ...\, \mathrm{d}x + \int_D ...\, \mathrm{d}x,$$

where $D = [-2,\, 2] \backslash [-Q^{-2\epsilon},\, Q^{-2\epsilon}]$. Note that, for $x \in [-Q^{-2\epsilon},\, Q^{-2\epsilon}]$, we have

$$g(q, xQ^{\epsilon}) = 1 + h(q, xQ^{\epsilon}) = 1 + O\left(\frac{Q}{q}\left(\frac{q}{Q} + |x|Q^{\epsilon}\right)^B\right) \ll 1 + O(Q^{-2020}).$$

Thus, in this range, we can replace $g(q, xQ^{\epsilon})$ by 1 at the cost of a negligible error term. Then by repeated integration by parts we see that the integral is negligibly small unless

$$|z-y| \ll k^{\epsilon} C/(Q\mathcal{T}). \tag{2.4.3}$$

Now we consider the complementary range, i.e., $x \in D$. Note that, using the second property (see (1.4.2)) of $q(q,x)$, we have

$$x^j \frac{\partial^j}{\partial x^j} g(q, xQ^{\epsilon}) \ll \log Q \min\left\{\frac{Q}{q}, \frac{1}{|x|Q^{\epsilon}}\right\} \ll Q^{3\epsilon}.$$

Thus, on using integration by parts repeatedly, we see that the integral is negligibly small unless (2.4.3) holds true.

Case 2. $q \sim C \gg Q^{1-\epsilon}$.

In this case, we consider the t-integral in (2.4.2) which is given by

$$\mathrm{L}_{z-y} = \int_{\mathbb{R}} V\left(\frac{t}{\mathcal{T}}\right)\left(\frac{z}{y}\right)^{it} \mathrm{d}t.$$

Now applying the change of variable $t \to t\mathcal{T}$ followed by integration by parts repeatedly, we conclude that the t-integral is negligibly small unless

$$|z-y| \ll k^{\epsilon}/\mathcal{T} \ll k^{2\epsilon} C/(Q\mathcal{T}).$$

Thus, combining Case 1 and Case 2, we get the lemma. □

Lemma 2.4.2. *Let* $\mathrm{J}_{\pm}(...) = \mathrm{J}_{\pm}(m,\, \tilde{N}w,\, q)$ *be as in* (2.4.2). *Then we have*

$$\mathrm{J}_{\pm}(m,\, \tilde{N}w,\, q) = \int_{\mathbb{R}} V\left(\frac{t}{\mathcal{T}}\right) \int_{u \ll \frac{k^{\epsilon} C}{Q\mathcal{T}}} \mathrm{I}_u\, \mathrm{I}_{\pm}(m, \tilde{N}w, q)\, \mathrm{d}u\, \mathrm{d}t + O(k^{-2020}), \tag{2.4.4}$$

where I_u *and* $\mathrm{I}_{\pm}(m, \tilde{N}w, q)$ *are the integrals defined in* (2.4.6) *and* (2.4.7) *respectively, with the weight function* $U_{u,t}$ *satisfying* $U_{u,t}^{(j)}(y) \ll_j k^{\epsilon j}$ *for* $j \geq 0$.

Proof. Using Lemma 2.4.1, we write $z - y = u$ with $u \ll k^{\epsilon} C/(Q\mathcal{T})$ in (2.4.2). Thus we see that

$$\mathrm{J}_{\pm}(m,\, \tilde{N}w,\, q) = \int_{\mathbb{R}} V\left(\frac{t}{\mathcal{T}}\right) \int_{u \ll \frac{k^{\epsilon} C}{Q\mathcal{T}}} \mathrm{I}_u\, \mathrm{I}_{\pm}(m, \tilde{N}w, q)\, \mathrm{d}u\, \mathrm{d}t + O(k^{-2020}), \tag{2.4.5}$$

where

$$\mathrm{I}_u = \int_{\mathbb{R}} W(x/Q^\epsilon) g(q,x) e\left(\frac{Nxu}{qQ}\right) \mathrm{d}x, \tag{2.4.6}$$

and

$$\mathrm{I}_\pm(m, \tilde{N}w, q) = \int_0^\infty U_{u,t}(y) e\left(\pm \frac{3(N\tilde{N}w(y+u))^{1/3}}{qr^{1/3}}\right) J_{k-1}\left(\frac{4\pi\sqrt{mNy}}{q}\right) \mathrm{d}y, \tag{2.4.7}$$

with $U_{u,t}(y) = U(y)V(y+u)(1+u/y)^{it-1/3}$. Note that, for $y \in \mathrm{Supp}(U)$, we have

$$\frac{\partial^j}{\partial y^j}\left(1+\frac{u}{y}\right)^{it} = \frac{\partial^j}{\partial x^j} \exp\left(it \log\left(1+\frac{u}{y}\right)\right) \ll_j k^{\epsilon j}, \quad j \geq 0.$$

Thus $U_{u,t}^{(j)}(y) \ll_j k^{\epsilon j}$ for $j \geq 0$. Hence the lemma follows. □

The analysis for $\mathrm{J}_\pm(m', \tilde{N}w, q')$ is exactly same. We record it in the following lemma.

Lemma 2.4.3. *Let $\mathrm{J}_\pm(m', \tilde{N}w, q')$ be the integral transform defined by replacing q by q' and m by m' in (2.4.2). Let $\mathrm{I}_{z'-y'}$ and $\mathrm{L}_{z'-y'}$ denote the x' and t'-integral respectively in $\mathrm{J}_\pm(m', \tilde{N}w, q')$. Then either $\mathrm{I}_{z'-y'}$ or $\mathrm{L}_{z'-y'}$ is negligibly small unless $|u'| = |z'-y'| \ll k^\epsilon C/(Q\mathcal{T})$, and in which case, we have*

$$\mathrm{J}_\pm(m', \tilde{N}w, q') = \int_{\mathbb{R}} V\left(\frac{t'}{\mathcal{T}}\right) \int_{u' \ll \frac{k^\epsilon C}{Q\mathcal{T}}} \mathrm{I}_{u'}\, \mathrm{I}_\pm(m', \tilde{N}w, q')\, \mathrm{d}u'\, \mathrm{d}t' + O(k^{-2020}), \tag{2.4.8}$$

where $\mathrm{I}_{u'}$ and $\mathrm{I}_\pm(m', \tilde{N}w, q')$ are the integrals corresponding to (2.4.6) and (2.4.7) respectively.

On plugging the expressions from (2.4.4) and (2.4.8) into (2.4.1), we see that

$$\mathcal{J}_\pm = \int_{\mathbb{R}} \int_{\mathbb{R}} V\left(\frac{t}{\mathcal{T}}\right) V\left(\frac{t'}{\mathcal{T}}\right) \int_{u \ll \frac{k^\epsilon C}{Q\mathcal{T}}} \int_{u' \ll \frac{k^\epsilon C}{Q\mathcal{T}}} \mathrm{I}_u\, \bar{\mathrm{I}}_{u'}\, \mathfrak{J}_\pm\, \mathrm{d}u'\, \mathrm{d}u\, \mathrm{d}t'\, \mathrm{d}t + O(k^{-2020}), \tag{2.4.9}$$

where

$$\mathfrak{I}_{\pm} := \int_{\mathbb{R}} \phi(w)\, \mathrm{I}_{\pm}(m, \tilde{N}w, q)\, \overline{\mathrm{I}_{\pm}(m', \tilde{N}w, q')}\, e\left(-\frac{n_2 \tilde{N} w}{q_2 q_2' q_1 r n_1}\right) \mathrm{d}w, \tag{2.4.10}$$

which we will analyse now. We have the following proposition:

Proposition 2.4.4. *Let $\mathfrak{I}_{\pm}$ be the integral transform defined as above. Then $\mathfrak{I}_{\pm}$ is negligibly small unless*

$$n_2 \ll k^{\epsilon} \frac{C N^{1/3} r^{2/3} n_1}{q_1 \tilde{N}^{2/3}} := N_2, \tag{2.4.11}$$

in which case we have

$$\mathfrak{I}_{\pm} \ll \frac{k^{\epsilon} C^2}{M_1 N}. \tag{2.4.12}$$

Furthermore, if $q \sim C \gg k^{1+\epsilon}$ and $n_2 \neq 0$, then we have

$$\mathfrak{I}_{\pm} \ll \frac{C r^{1/3} k^{2/3}}{k^2 (N\tilde{N})^{1/3}}. \tag{2.4.13}$$

Before proving the propositon, we will analyze $\mathrm{I}_{\pm}(m, \tilde{N}w, q)$ and $\mathrm{I}_{\pm}(m', \tilde{N}w, q')$. We have the following lemma.

Lemma 2.4.5. *Let $\mathrm{I}_{\pm}(m, \tilde{N}w, q)$ be the integral transform defined in (2.4.7). Let $\mathfrak{b} = 4\pi\sqrt{mN}/q$ and $\mathfrak{a} = \mathfrak{a}(q, r) := 3(N\tilde{N})^{1/3}/(qr^{1/3})$. Then $\mathrm{I}_{\pm}(m, \tilde{N}w, q)$ is negligibly small unless $\mathfrak{a} \leq k^{\epsilon}\mathfrak{b}$. In the case, $\mathfrak{a} \leq k^{-\epsilon}\mathfrak{b}$, we have*

$$\mathrm{I}_{\pm}(m, \tilde{N}w, q) \ll k^{\epsilon}/\mathfrak{b}.$$

Furthermore, if $q \sim C \gg k^{1+\epsilon}$, then $\mathfrak{b} \asymp k$ and we have

$$\mathrm{I}_{\pm}(m, \tilde{N}w, q) = \frac{e\left(F(\tau_0)\right)}{\sqrt{F''(\tau_0)}} \frac{c_3 \mathfrak{a}^{9/2} w^{3/2}}{\mathfrak{b}^5 \tau_0^5 \sqrt{1-\tau_0^2}} U_{u,t}^2\left(\left(\frac{4\pi\mathfrak{a} w^{1/3}}{3\mathfrak{b}\tau_0}\right)^6\right) + O\left(k^{-2020}\right), \tag{2.4.14}$$

where τ_0 is the stationary point of the phase function

$$F(\tau) = \frac{(k-1)\sin^{-1}\tau}{2\pi} + \frac{16\pi^2\mathfrak{a}^3 w}{27\mathfrak{b}^2\tau^2},$$

which is given by (2.4.27), $c_3 = c_2\, e(1/8) = 3\sqrt{2}(4\pi/3)^5 e(1/4)$ and $U^2_{u,t}$ is a smooth bump function satisfying $U^{2(j)}_{u,t} \ll_j k^{\epsilon j}$. In the remaining case, i.e., $k^{-\epsilon}\mathfrak{b} \le \mathfrak{a} \le k^{\epsilon}\mathfrak{b}$, $\mathrm{I}_{\pm}(m, \tilde{N}w, q)$ is given by

$$\frac{c_2\mathfrak{a}^{9/2}w^{3/2}}{\mathfrak{b}^5}\int_{b_1/2}^{1} \frac{1}{\tau^5\sqrt{1-\tau^2}} U^1_{u,t}\left(\left(\frac{4\pi\mathfrak{a}w^{1/3}}{3\mathfrak{b}\tau}\right)^6\right) e\left(F(\tau)\right) \mathrm{d}\tau + O\left(k^{-2020}\right),$$

where $b_1 := 4\pi(2/3)^{1/3}\mathfrak{a}/(3(2.5)^{1/6}\mathfrak{b})$. Here $U^1_{u,t}$ is a smooth bump function satisfying $U^{1(j)}_{t,u} \ll_j k^{\epsilon j}$.

Remark 2.4.6. *From the statement of Lemma 2.2.1, we infer that $\mathfrak{a} \gg k^{\epsilon}$. We will keep on using this fact form now on (without mentioning it explicitly).*

Proof. Let's recall from (2.4.7) that

$$\begin{aligned}\mathrm{I}_{\pm}(m, \tilde{N}w, q) &= \int_0^{\infty} U_{u,t}(y) e\left(\pm\frac{3(N\tilde{N}w(y+u))^{1/3}}{qr^{1/3}}\right) J_{k-1}\left(\frac{4\pi\sqrt{mNy}}{q}\right) \mathrm{d}y \\ &= \int_{1/2}^{5/2} U_{u,t}(y)\, e\left(\pm\mathfrak{a}w^{1/3}(y+u)^{1/3}\right) J_{k-1}\left(\mathfrak{b}\sqrt{y}\right) \mathrm{d}y. \qquad (2.4.15)\end{aligned}$$

Consider the term $e(\pm\mathfrak{a}w^{1/3}(y+u)^{1/3})$. It can be written as

$$e(\pm\mathfrak{a}w^{1/3}(y+u)^{1/3}) = e(\pm\mathfrak{a}w^{1/3}y^{1/3})\, e(\pm\mathfrak{a}w^{1/3}y^{1/3}((1+u/y)^{1/3}-1)).$$

Note that

$$\frac{\partial^j}{\partial y^j} e(\pm\mathfrak{a}w^{1/3}y^{1/3}((1+u/y)^{1/3}-1)) \ll_j k^{\epsilon j}, \quad j \ge 0.$$

This is obvious for $j=0$. We will verify it for $j=1$ (for other j, it follows similarly). Let $f(y,w) := \pm\mathfrak{a}w^{1/3}y^{1/3}((1+u/y)^{1/3}-1)$. Thus for $j=1$ we have

$$\frac{\partial}{\partial y} e(f(y,w)) = e(f(y,w))(\pm\mathfrak{a})w^{1/3}\left(\frac{(1+u/y)^{1/3}-1}{3y^{2/3}} - \frac{u}{3y^{5/3}(1+u/y)^{2/3}}\right).$$

Thus, using $y,\, w \asymp 1$ and $(1+u/y)^{1/3}-1 \ll u$, we see that

$$\frac{\partial}{\partial y} e(f(y,w)) \ll \mathfrak{a} u \ll \frac{(N\tilde{N})^{1/3}}{Cr^{1/3}} \frac{Ck^{\epsilon}}{Q\mathcal{T}} \ll \frac{(NN_0)^{1/3}}{Qr^{1/3}} \frac{Qk^{\epsilon}}{Q\mathcal{T}} \ll k^{\epsilon},$$

where we used (2.3.7) for the expression of N_0. Hence we can insert $e(f(y,w))$ into the weight function $U_{u,t}(y)$. Thus we arrive at the following expression:

$$\mathrm{I}_{\pm} := \mathrm{I}_{\pm}(m, \tilde{N}w, q) = \int_{1/2}^{5/2} U^0_{u,t}(y)\, e\left(\pm \mathfrak{a} w^{1/3} y^{1/3}\right) J_{k-1}\left(\mathfrak{b}\sqrt{y}\right) \mathrm{d}y, \tag{2.4.16}$$

where $U^0_{u,t}(y) = U_{u,t}(y) e(f(y,w))$ is the new weight function

$$\text{supported in } [1/2, 5/2] \text{ and satisfying } U^{0(j)}_{u,t}(y) \ll_j k^{\epsilon j},\; j \geq 0. \tag{2.4.17}$$

To analyze (2.4.16) further, we use an integral representation of the Bessel function J_{k-1}. Thus, on applying (1.5.1) to the Bessel function J_{k-1}, we see that

$$\mathrm{I}_{\pm} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(k-1)\tau} \int_{1/2}^{5/2} U^0_{u,t}(y) e\left(\pm \mathfrak{a} w^{1/3} y^{1/3} - \mathfrak{b}\sqrt{y} \sin\tau/2\pi\right) \mathrm{d}y\, \mathrm{d}\tau.$$

We now split the τ-integral as follows:

$$\int_{-\pi}^{\pi} \ldots \mathrm{d}\tau = \int_{0}^{\pi/2} \ldots \mathrm{d}\tau + \int_{\pi/2}^{\pi} \ldots \mathrm{d}\tau + \int_{-\pi/2}^{0} \ldots \mathrm{d}\tau + \int_{-\pi}^{-\pi/2} \ldots \mathrm{d}\tau.$$

Let $\mathrm{I}^{(i)}_{\pm}$ denote the i-th integral in the right hand side of the above expression for $i = 1,\, 2,\, 3$ and 4. Let's first consider $\mathrm{I}^{(1)}_{\pm}$ which is defined as follows:

$$\mathrm{I}^{(1)}_{\pm} = \frac{1}{2\pi} \int_{0}^{\pi/2} e^{i(k-1)\tau} \int_{1/2}^{5/2} U^0_{u,t}(y) e\left(\pm \mathfrak{a} w^{1/3} y^{1/3} - \mathfrak{b}\sqrt{y} \sin\tau/2\pi\right) \mathrm{d}y\, \mathrm{d}\tau. \tag{2.4.18}$$

Next we apply stationary phase analysis to the y-integral. By the change of variable $y \to y^3$, we arrive at the following expression of the y-integral

$$\int_{\sqrt[3]{1/2}}^{\sqrt[3]{5/2}} 3y^2 U^0_{u,t}(y^3) e\left(\pm \mathfrak{a} w^{1/3} y - \mathfrak{b} y^{3/2} \sin\tau/2\pi\right) \mathrm{d}y.$$

Note that if we have negative sign with $\mathfrak{a}$, then the above integral is negligibly small by Lemma 1.7.1. Thus, we proceed with the y-integral of $\mathrm{I}_{+}^{(1)}$, which is given by

$$\int_{\sqrt[3]{1/2}}^{\sqrt[3]{5/2}} 3y^2 U_{u,t}^0(y^3) e\left(\mathfrak{a}w^{1/3}y - \mathfrak{b}y^{3/2}\sin\tau/2\pi\right) \mathrm{d}y.$$

Here the phase function is given by $f_1(y) = \mathfrak{a}w^{1/3}y - \mathfrak{b}y^{3/2}\sin\tau/2\pi$. On computing the first order derivative, we see that the stationary point occurs at $y_0 = \left(\frac{4\pi\mathfrak{a}w^{1/3}}{3\mathfrak{b}\sin\tau}\right)^2$. Note that

$$\sqrt[3]{1/2} \leq y_0 \leq \sqrt[3]{5/2} \iff \frac{4\pi}{3}\frac{\mathfrak{a}w^{1/3}}{\mathfrak{b}(2.5)^{1/6}} \leq \sin\tau \leq \frac{4\pi}{3}\frac{\mathfrak{a}w^{1/3}}{\mathfrak{b}(0.5)^{1/6}}.$$

Let $b_1 := \frac{4\pi}{3}\frac{\mathfrak{a}(2/3)^{1/3}}{\mathfrak{b}(2.5)^{1/6}}$ and $b_2 := \frac{4\pi}{3}\frac{3^{1/3}\mathfrak{a}}{\mathfrak{b}(0.5)^{1/6}}$. We consider three cases.

Case 1. $\mathfrak{a} \geq k^{\epsilon}\mathfrak{b}$.

In this case we have $b_1 \geq 2$. Thus there is no stationary point in the range $[(1/2)^{1/3}, (5/2)^{1/3}]$. Moreover,

$$f_1'(y) = \mathfrak{a}w^{1/3} - 3\mathfrak{b}\sqrt{y}\sin\tau/(4\pi) \gg \mathfrak{b}, \;\; f_1^{(j)}(y) \ll \mathfrak{b}, j \geq 2.$$

Hence, by Lemma (1.7.1), the integral is negligibly small. This proves the first part of the lemma.

Case 2. $\mathfrak{a} \leq k^{-\epsilon}\mathfrak{b}$.

In this case we have $0 < b_1/2 < 2b_2 \ll k^{-\epsilon} < 1$. we now split the τ-integral in (2.4.18) as follows:

$$\int_0^{\pi/2} \ldots \mathrm{d}\tau = \int_0^{\sin^{-1}(b_1/2)} \ldots \mathrm{d}\tau + \int_{\sin^{-1}(b_1/2)}^{\sin^{-1} 2b_2} \ldots \mathrm{d}\tau + \int_{\sin^{-1} 2b_2}^{\pi/2} \ldots \mathrm{d}\tau.$$

Note that the first and the third integrals of the right side of the above expression are negligibly small due to the absence of the stationary point. Hence it boils down to analyse the second integral which is given by

$$\int_{\sin^{-1}(b_1/2)}^{\sin^{-1}2b_2} e^{i(k-1)\tau} \int_{\sqrt[3]{1/2}}^{\sqrt[3]{5/2}} 3y^2 U_{u,t}^0(y^3) e\left(\mathfrak{a}w^{1/3}y - \mathfrak{b}y^{3/2}\sin\tau/2\pi\right) \mathrm{d}y\, \mathrm{d}\tau. \quad (2.4.19)$$

On applying the stationary phase analysis (Lemma 1.7.2) to the y-integral with the parameters $X = Q = 1$, $U = k^{-\epsilon}$, $\delta = 1/11$ and $Y = \mathfrak{a}$, we see that, it is given by

$$\frac{e(f_1(y_0))}{\sqrt{|f_1''(y_0)|}} \sum_{n=0}^{\lfloor 3\delta^{-1}A \rfloor} p_n(y_0) + O(\mathfrak{a}^{-A}),$$

for any arbitrary large $A > 0$, where $p_0(y_0) = c_1 y_0^2 U_{u,t}^0(y_0^3)$, $c_1 = 3e(1/8)$, $y_0 = \left(\frac{4\pi\mathfrak{a}w^{1/3}}{3\mathfrak{b}\sin\tau}\right)^2$, $\frac{\mathrm{d}^j}{\mathrm{d}y_0^j} p_n(y_0) \ll_{n,j} \frac{k^{\epsilon j}}{\mathfrak{a}^{n/3}} \ll k^{\epsilon j}$ and $f_1(y) = \mathfrak{a}w^{1/3}y - \mathfrak{b}y^{3/2}\sin\tau/2\pi$. Hence, on plugging the values of y_0, $f_1(y_0)$, $f_1''(y_0)$ and choosing A large enough, we, upto a negligible small error term, get the following expression for the y-integral:

$$\frac{c_2\mathfrak{a}^{9/2}w^{3/2}}{\mathfrak{b}^5\sin^5\tau} U_{u,t}^1\left(\left(\frac{4\pi\mathfrak{a}w^{1/3}}{3\mathfrak{b}\sin\tau}\right)^6\right) e\left(\frac{16\pi^2\mathfrak{a}^3 w}{27\mathfrak{b}^2\sin^2\tau}\right), \quad (2.4.20)$$

where $c_2 = c_1\sqrt{2}(4\pi/3)^5$ and $U_{u,t}^1$ is the new weight function. On plugging the above expression in place of the y-integral into (2.4.19), we arrive at

$$\frac{c_2\mathfrak{a}^{9/2}w^{3/2}}{\mathfrak{b}^5} \int_{\sin^{-1}(b_1/2)}^{\sin^{-1}2b_2} \frac{1}{\sin^5\tau} U_{u,t}^1\left(\left(\frac{4\pi\mathfrak{a}w^{1/3}}{3\mathfrak{b}\sin\tau}\right)^6\right) e\left(\frac{(k-1)\tau}{2\pi} + \frac{16\pi^2\mathfrak{a}^3 w}{27\mathfrak{b}^2\sin^2\tau}\right) \mathrm{d}\tau.$$

On applying the change of variable $\sin\tau \to \tau$, we arrive at

$$\frac{c_2\mathfrak{a}^{9/2}w^{3/2}}{\mathfrak{b}^5} \int_{b_1/2}^{2b_2} \frac{1}{\tau^5\sqrt{1-\tau^2}} U_{u,t}^1\left(\left(\frac{4\pi\mathfrak{a}w^{1/3}}{3\mathfrak{b}\tau}\right)^6\right) e\left(F(\tau)\right) \mathrm{d}\tau, \quad (2.4.21)$$

where

$$F(\tau) = \frac{(k-1)\sin^{-1}\tau}{2\pi} + \frac{16\pi^2\mathfrak{a}^3 w}{27\mathfrak{b}^2\tau^2}.$$

Next we apply the second derivative bound to the above integral. On computing the first and the second order derivative of $F(\tau)$, we see that

$$F'(\tau) = \frac{(k-1)}{2\pi\sqrt{1-\tau^2}} - \frac{32\pi^2\mathfrak{a}^3 w}{27\mathfrak{b}^2\tau^3},$$
$$F''(\tau) = \frac{(k-1)\tau}{2\pi(1-\tau^2)^{3/2}} + \frac{32\pi^2\mathfrak{a}^3 w}{9\mathfrak{b}^2\tau^4} \gg \frac{\mathfrak{a}^3}{\mathfrak{b}^2\tau^4} \gg \frac{\mathfrak{b}^2}{\mathfrak{a}}. \tag{2.4.22}$$

Thus on applying Lemma 1.7.1 to (2.4.21), we see that (2.4.21) is bounded by

$$\frac{\mathsf{Var}\, g + \max|g|}{\min\sqrt{F''(\tau)}} \ll \frac{k^\epsilon \mathfrak{a}^{9/2}}{\mathfrak{b}^5(\mathfrak{a}/\mathfrak{b})^5\sqrt{\mathfrak{b}^2/\mathfrak{a}}} = \frac{k^\epsilon}{\mathfrak{b}},$$

where $\mathsf{Var}\, g$ denotes the total variation of the weight function

$$g(\tau) = \frac{c_2\mathfrak{a}^{9/2}w^{3/2}U^1_{u,t}\left(\left(4\pi\mathfrak{a}w^{1/3}/3\mathfrak{b}\tau\right)^6\right)}{\mathfrak{b}^5\tau^5\sqrt{1-\tau^2}}.$$

Hence $\mathrm{I}_\pm^{(1)} \ll k^\epsilon/\mathfrak{b}$. On analyzing other $\mathrm{I}_\pm^{(i)}$'s in a similar fashion, we get

$$\mathrm{I}_\pm = \mathrm{I}_\pm(m, N_0 w, q) \ll k^\epsilon/\mathfrak{b}.$$

Now we proceed to prove (2.4.14). We will give details for $\mathrm{I}_\pm^{(1)}$ only, as the analysis for other $\mathrm{I}_\pm^{(i)}$'s is similar. Let $q \sim C \gg k^{1+\epsilon}$. Note that this condition assures that $\mathfrak{b} \asymp k$, as, by (2.3.7), we have

$$k^{-\epsilon}(k-1)^2C^2/N \ll M_1 \ll k^\epsilon \max\left((k-1)^2C^2/N,\, \mathcal{T}x^2\right) \ll k^\epsilon(k-1)^2C^2/N. \tag{2.4.23}$$

We also have

$$\mathfrak{a} = \frac{3(N\tilde{N})^{1/3}}{qr^{1/3}} \ll \frac{(NN_0)^{1/3}}{qr^{1/3}} \ll \frac{k^\epsilon\sqrt{\mathcal{T}N}}{C} \ll \frac{\sqrt{\mathcal{T}N}}{k^{1+\epsilon}} \ll (k\mathcal{T})^{1/2}, \tag{2.4.24}$$

The second inequality follows from (2.3.7). Indeed we have $N_0 \ll k^{\epsilon}\mathcal{T}^{3/2}\sqrt{N}r$. Thus

$$\mathfrak{a} \ll (k\mathcal{T})^{1/2} = k^{1-\eta/2} < k \asymp \mathfrak{b}, \tag{2.4.25}$$

as $\mathcal{T} = k^{1-\eta} < k$. We now apply the stationary phase analysis to (2.4.21). The stationary point of the phase function $F(\tau)$ occurs at τ_0, where τ_0 satisfies

$$\frac{(k-1)}{2\pi\sqrt{1-\tau_0^2}} = \frac{32\pi^2\mathfrak{a}^3 w}{27\mathfrak{b}^2\tau_0^3} \iff \frac{\tau_0^3}{\sqrt{1-\tau_0^2}} = \left(\frac{4\pi}{3}\right)^3 \frac{\mathfrak{a}^3 w}{\mathfrak{b}^2(k-1)}.$$

Simplifying it further, we see that τ_0 satisfies the polynomial equation

$$\tau^6 - \mathfrak{c}^2(1-\tau^2) = 0,$$

where $\mathfrak{c} = \mathfrak{c}(w) := \left(\frac{4\pi}{3}\right)^3 \frac{\mathfrak{a}^3 w}{\mathfrak{b}^2(k-1)}$. Upon letting $\tau_0^2 = \tau_1$, the above polynomial reduces to the cubic polynomial equation $\tau_1^3 - \mathfrak{c}^2(1-\tau_1) = 0$, which can be solved using the Cardano's method. In fact, as the discriminant of the cubic is negative, it has only one real root which can be found as follows: Let $\theta_1 + \theta_2$ be the real root. Upon substituting it into the cubic, we get

$$\theta_1^3 + \theta_2^3 + (3\theta_1\theta_2 + \mathfrak{c}^2)(\theta_1 + \theta_2) - \mathfrak{c}^2 = 0,$$

which leads to the following system of equations:

$$3\theta_1\theta_2 + \mathfrak{c}^2 = 0, \quad \theta_1^3 + \theta_2^3 - \mathfrak{c}^2 = 0,$$

from which θ_1 and θ_2 are found using the quadratic equation formula. Thus the real root is given by

$$\sqrt[3]{\frac{\mathfrak{c}^2}{2} + \sqrt{\frac{\mathfrak{c}^4}{4} + \frac{\mathfrak{c}^6}{27}}} + \sqrt[3]{\frac{\mathfrak{c}^2}{2} - \sqrt{\frac{\mathfrak{c}^4}{4} + \frac{\mathfrak{c}^6}{27}}}.$$

Hence we get

$$\tau_0=\tau_0(w)=\left(\sqrt[3]{\frac{\mathfrak{c}^2}{2}+\sqrt{\frac{\mathfrak{c}^4}{4}+\frac{\mathfrak{c}^6}{27}}}+\sqrt[3]{\frac{\mathfrak{c}^2}{2}-\sqrt{\frac{\mathfrak{c}^4}{4}+\frac{\mathfrak{c}^6}{27}}}\right)^{1/2}$$

$$=\sqrt[6]{\frac{\mathfrak{c}^2}{2}+\sqrt{\frac{\mathfrak{c}^4}{4}+\frac{\mathfrak{c}^6}{27}}}\left(1-\frac{3}{\mathfrak{c}^2}\left(\sqrt{\frac{\mathfrak{c}^4}{4}+\frac{\mathfrak{c}^6}{27}}-\frac{\mathfrak{c}^2}{2}\right)^{2/3}\right)^{1/2}$$

$$=\sqrt[6]{\frac{\mathfrak{c}^2}{2}+\sqrt{\frac{\mathfrak{c}^4}{4}+\frac{\mathfrak{c}^6}{27}}}\left(1-\frac{3}{\mathfrak{c}^2}\left(\sqrt{\frac{\mathfrak{c}^4}{4}+\frac{\mathfrak{c}^6}{27}}-\frac{\mathfrak{c}^2}{2}\right)^{2/3}\right)^{1/2}. \tag{2.4.26}$$

Now expanding the above expression using the binomial theorem, we see that

$$\tau_0=\tau_0(w)=c_1\mathfrak{h}(w)+c_3(\mathfrak{h}(w))^3+c_3(\mathfrak{h}(w))^5...+c_{2n-1}(\mathfrak{h}(w))^{2n-1}+..., \tag{2.4.27}$$

where c_i's, $i=1,\,3,\,5,\,\cdots$, are some non-zero explicit absolute constants and

$$\mathfrak{h}(w)=\frac{\mathfrak{a}w^{1/3}}{\mathfrak{b}^{2/3}(k-1)^{1/3}}.$$

Note that the series in (2.4.27) is convergent and each binomial expansion in (2.4.26) is justified as $\mathfrak{c}\ll\mathfrak{a}^3/(\mathfrak{b}^2(k-1))\ll k^{-3\eta/2}$. Next we analyse the higher order derivatives of the phase function $F(\tau)$. On using (2.4.22) and computing other higher order derivatives of $F(\tau)$, we get

$$F''(\tau)\asymp\mathfrak{b}^2/\mathfrak{a}=\mathfrak{a}(\mathfrak{a}/\mathfrak{b})^{-2},\quad F'(\tau)\ll\mathfrak{a}(\mathfrak{a}/\mathfrak{b})^{-1},$$
$$F^{(j)}(\tau)=\frac{(k-1)}{2\pi}\frac{\mathrm{d}^{j-2}}{\mathrm{d}\tau^{j-2}}\frac{\tau}{(1-\tau^2)^{3/2}}+\frac{32\pi^2\mathfrak{a}^3w}{9\mathfrak{b}^2}\frac{\mathrm{d}^{j-2}(\tau^{-4})}{\mathrm{d}\tau^{j-2}}\ll\mathfrak{a}(\mathfrak{a}/\mathfrak{b})^{-j},\ j\geq 3,$$

where we used the fact $\mathfrak{a}\ll\mathfrak{b}\asymp k$ and

$$\frac{\mathrm{d}^{j-2}}{\mathrm{d}\tau^{j-2}}\frac{\tau}{(1-\tau^2)^{3/2}}\ll_{j,\epsilon}1,\quad b_1/2\leq\ \tau\leq 2b_2<1.$$

On computing the derivatives of the weight function

$$g(\tau)=\frac{c_2\mathfrak{a}^{9/2}w^{3/2}U^1_{u,t}\left(\left(4\pi\mathfrak{a}w^{1/3}/3\mathfrak{b}\tau\right)^6\right)}{\mathfrak{b}^5\tau^5\sqrt{1-\tau^2}},$$

we see that

$$g^{(i)}(\tau)\ll\mathfrak{a}^{-1/2}\left(\mathfrak{a}/\mathfrak{b}\right)^{-i},\quad i=0,1,2,...$$

Thus, on applying Lemma 1.7.2 with $X=\mathfrak{a}^{-1/2}$, $Q=U=\mathfrak{a}/\mathfrak{b}$ and $Y=\mathfrak{a}$ to the τ-integral in (2.4.21), we get (2.4.14).

Case 3. $k^{-\epsilon}\mathfrak{b}\leq\mathfrak{a}\leq k^{\epsilon}\mathfrak{b}$.

In this case we can assume that $b_1/2<1$, otherwise, we get back to the starting point of the discussion in Case 1. Consider $\mathrm{I}^{(1)}_{\pm}$ which is given by

$$\mathrm{I}^{(1)}_{\pm}=\frac{1}{2\pi}\int_0^{\pi/2}e^{i(k-1)\tau}\int_{1/2}^{5/2}U^0_{u,t}(y)e\left(\pm\mathfrak{a}w^{1/3}y^{1/3}-\mathfrak{b}\sqrt{y}\sin\tau/2\pi\right)\mathrm{d}y\,\mathrm{d}\tau. \tag{2.4.28}$$

We split the τ-integral as fellows:

$$\int_0^{\pi/2}...\,\mathrm{d}\tau=\int_0^{\sin^{-1}(b_1/2)}...\,\mathrm{d}\tau+\int_{\sin^{-1}(b_1/2)}^{\pi/2}...\,\mathrm{d}\tau.$$

The first integral on the right side is negligibly small due to the absence of the stationary point. Consider the second integral which is given by

$$\int_{\sin^{-1}(b_1/2)}^{\pi/2}e^{i(k-1)\tau}\int_{\sqrt[3]{1/2}}^{\sqrt[3]{5/2}}3y^2U^0_{u,t}(y^3)e\left(\mathfrak{a}w^{1/3}y-\mathfrak{b}y^{3/2}\sin\tau/2\pi\right)\mathrm{d}y\,\mathrm{d}\tau. \tag{2.4.29}$$

On analyzing the y-integral like Case 2, we get the lemma. □

Next we consider $I_{\pm}(m',\tilde{N}w,q')$ which is defined by replacing m by m' and q by q' in (2.4.7). On analysing $\mathrm{I}_{\pm}(m',\tilde{N}w,q')$ like $\mathrm{I}_{\pm}(m,\tilde{N}w,q)$, we get the following lemma

Lemma 2.4.7. *Let* $\mathrm{I}_{\pm}(m',\tilde{N}w,q')$ *be as above. Let* $\mathfrak{b}'=4\pi\sqrt{m'N}/q'$ *and* $\mathfrak{a}'=\mathfrak{a}'(q',r):=3(N\tilde{N})^{1/3}/(q'r^{1/3})$. *Then* $\mathrm{I}_{\pm}(m',\tilde{N}w,q')$ *is negligibly small unless* $\mathfrak{a}'\leq$

$k^{\epsilon}\mathfrak{b}'$. *In the case,* $\mathfrak{a}' \leq k^{-\epsilon}\mathfrak{b}'$, *we have*

$$\mathrm{I}_{\pm}(m', \tilde{N}w, q') \ll k^{\epsilon}/\mathfrak{b}'.$$

Furthermore, if $q' \sim C \gg k^{1+\epsilon}$, *then we have*

$$\mathrm{I}_{\pm}(m', \tilde{N}w, q') = \frac{e\left(f_2(\tau_0')\right)}{\sqrt{f_2''(\tau_0')}} \frac{c_3 \mathfrak{a}'^{9/2} w^{3/2}}{\mathfrak{b}'^5 \tau_0'^5 \sqrt{1-\tau_0'^2}} U^2_{u',t'}\left(\left(\frac{4\pi\mathfrak{a}' w^{1/3}}{3\mathfrak{b}'\tau_0'}\right)^6\right) + O\left(k^{-2020}\right), \tag{2.4.30}$$

where τ_0' *is the stationary point of the phase function*

$$f_2(\tau') = \frac{(k-1)\sin^{-1}\tau'}{2\pi} + \frac{16\pi^2\mathfrak{a}'^3 w}{27\mathfrak{b}'^2\tau'^2},$$

given by replacing $\mathfrak{h}(w)$ *by* $\mathfrak{h}'(w) = \mathfrak{a}' w^{1/3}/(\mathfrak{b}'^{2/3}(k-1)^{1/3})$ *in* (2.4.27) *and* $c_3 = c_2 e(1/8) = 3\sqrt{2}(4\pi/3)^5 e(1/4)$. *In the remaining case, i.e.,* $k^{-\epsilon}\mathfrak{b}' \leq \mathfrak{a}' \leq k^{\epsilon}\mathfrak{b}'$, $\mathrm{I}_{\pm}(m', \tilde{N}w, q')$ *is given by*

$$\frac{c_2 \mathfrak{a}'^{9/2} w^{3/2}}{\mathfrak{b}'^5} \int_{b_1'/2}^{1} \frac{1}{\tau'^5\sqrt{1-\tau'^2}} U^1_{u',t'}\left(\left(\frac{4\pi\mathfrak{a}' w^{1/3}}{3\mathfrak{b}'\tau'}\right)^6\right) e\left(f_2(\tau')\right) \mathrm{d}\tau',$$

where $b_1' := 4\pi(2/3)^{1/3}\mathfrak{a}'/(3(2.5)^{1/6}\mathfrak{b}')$.

Proof of Proposition 2.4.4. Consider the integral transform $\mathfrak{J}_{\pm}$

$$\mathfrak{J}_{\pm} := \int_{\mathbb{R}} \phi(w)\, \mathrm{I}_{\pm}(m, \tilde{N}w, q)\, \overline{\mathrm{I}_{\pm}(m', \tilde{N}w, q')}\, e\left(-\frac{n_2\tilde{N}w}{q_2 q_2' q_1 r n_1}\right) \mathrm{d}w, \tag{2.4.31}$$

where

$$\mathrm{I}_{\pm}(m, \tilde{N}w, q) = \int_{1/2}^{5/2} U_{u,t}(y)\, e\left(\pm\mathfrak{a} w^{1/3} y^{1/3}\right) J_{k-1}\left(\mathfrak{b}\sqrt{y}\right) \mathrm{d}y.$$

Note that

$$\frac{\partial^j}{\partial w^j}\mathrm{I}_{\pm}(m, \tilde{N}w, q) \ll \mathfrak{a}^j, \quad j \geq 0.$$

Similarly, it follows that

$$\frac{\partial^j}{\partial w^j}\mathrm{I}_{\pm}(m', \tilde{N}w, q') \ll \mathfrak{a}'^j, \quad j \geq 0.$$

Hence, on applying integration by parts j-times to the w-integral in (2.4.31), we see that

$$\mathfrak{J}_{\pm} \ll (k^{\epsilon}+\mathfrak{a}+\mathfrak{a}')^j \left(\frac{q_2 q_2' q_1 r n_1}{n_2 \tilde{N}}\right)^j \ll \left(\frac{(N\tilde{N})^{1/3}}{Cr^{1/3}}\right)^j \left(\frac{C^2 r n_1}{q_1 n_2 \tilde{N}}\right)^j = \left(\frac{N^{1/3} C r^{2/3} n_1}{q_1 n_2 \tilde{N}^{2/3}}\right)^j.$$

Thus, $\mathfrak{J}_{\pm}$ is negligibly small if

$$\frac{N^{1/3} C r^{2/3} n_1}{q_1 n_2 \tilde{N}^{2/3}} \ll \frac{1}{k^{\epsilon}} \iff n_2 \gg k^{\epsilon} \frac{C N^{1/3} r^{2/3} n_1}{q_1 \tilde{N}^{2/3}}.$$

Next we prove the bound $\mathfrak{J}_{\pm} \ll k^{\epsilon} C^2/(M_1 N)$.

Case 1. $\mathfrak{a} \not\asymp \mathfrak{b}$, i.e., $\mathfrak{a}' \asymp \mathfrak{a} \ll k^{-\epsilon}\mathfrak{b} \asymp k^{-\epsilon}\mathfrak{b}'$ or $\mathfrak{a}' \asymp \mathfrak{a} \gg k^{\epsilon}\mathfrak{b} \asymp k^{\epsilon}\mathfrak{b}'$.

In the case $\mathfrak{a} \gg k^{\epsilon}\mathfrak{b}$, on applying Lemma 2.4.5 to $\mathrm{I}_{\pm}(m, \tilde{N}w, q)$, we see that $\mathfrak{J}_{\pm}$ is negligibly small. In the other case, i.e., $\mathfrak{a}' \asymp \mathfrak{a} \ll k^{-\epsilon}\mathfrak{b} \asymp k^{-\epsilon}\mathfrak{b}'$, on applying Lemma 2.4.5 and Lemma 2.4.7 to (2.4.31), we get

$$\mathfrak{J}_{\pm} \leq \int_{\mathbb{R}} \phi(w)\, |\mathrm{I}_{\pm}(m, \tilde{N}w, q)|\, |\overline{\mathrm{I}_{\pm}(m', \tilde{N}w, q')}|\, \mathrm{d}w \ll \frac{k^{\epsilon}}{\mathfrak{b}\mathfrak{b}'} \ll \frac{k^{\epsilon} C^2}{M_1 N}, \tag{2.4.32}$$

Case 2. $\mathfrak{a} \asymp \mathfrak{b}$, i.e., $k^{-\epsilon}\mathfrak{b} \ll \mathfrak{a} \ll k^{\epsilon}\mathfrak{b}$.

On applying the last part of Lemma 2.4.5 and Lemma 2.4.7 to (2.4.31), we see that

$$\begin{aligned} \mathfrak{J}_{\pm} \ll \frac{(\mathfrak{a}\mathfrak{a}')^{9/2}}{(\mathfrak{b}\mathfrak{b}')^5} & \int_{b_1/2}^{1} \int_{b_1'/2}^{1} \frac{1}{\tau^5\sqrt{1-\tau^2}} \frac{1}{\tau'^5\sqrt{1-\tau'^2}} \\ & \times \left| \int_{2/3}^{3} g_3(\tau, \tau', w) e\left(w f_3(\tau, \tau')\right) \mathrm{d}w \right| \mathrm{d}\tau\, \mathrm{d}\tau', \end{aligned} \tag{2.4.33}$$

where

$$f_3(\tau, \tau') = \frac{16\pi^2 \mathfrak{a}^3}{27\mathfrak{b}^2\tau^2} - \frac{16\pi^2 \mathfrak{a}'^3}{27\mathfrak{b}'^2\tau'^2} - \frac{n_2 \tilde{N}}{q_2 q_2' q_1 r n_1}$$

and

$$g_3(\tau, \tau', w) = \phi(w) w^3 U^1_{u,t}\left(\left(\frac{4\pi\mathfrak{a}w^{1/3}}{3\mathfrak{b}\tau}\right)^6\right) \bar{U^1}_{u',t'}\left(\left(\frac{4\pi\mathfrak{a}'w^{1/3}}{3\mathfrak{b}'\tau'}\right)^6\right).$$

On applying the change of variable $\tau \to 1/\sqrt{\tau}$, $\tau' \to 1/\sqrt{\tau'}$, we arrive at

$$\begin{aligned}\mathfrak{I}_{\pm} \ll \frac{(\mathfrak{a}\mathfrak{a}')^{9/2}}{(\mathfrak{b}\mathfrak{b}')^5} \int_1^{4/b_1^2} \int_1^{4/b_1'^2} \frac{\tau^{3/2}}{2\sqrt{\tau-1}} \frac{\tau'^{3/2}}{2\sqrt{\tau'-1}} \\ \times \left| \int_{2/3}^{3} g_3(1/\sqrt{\tau}, 1/\sqrt{\tau'}, w) e\left(\frac{16\pi^2\mathfrak{a}^3 w}{27\mathfrak{b}^2} f_4(\tau,\tau')\right) \mathrm{d}w \right| \mathrm{d}\tau\, \mathrm{d}\tau',\end{aligned} \tag{2.4.34}$$

where

$$f_4(\tau,\tau') = \tau - \frac{\mathfrak{a}'^3\mathfrak{b}^2}{\mathfrak{a}^3\mathfrak{b}'^2}\tau' - \frac{27 n_2 \tilde{N} \mathfrak{b}^2}{16\pi^2 q_2 q_2' q_1 r n_1 \mathfrak{a}^3}.$$

Now using the change of variable

$$\frac{\mathfrak{a}'^3\mathfrak{b}^2}{\mathfrak{a}^3\mathfrak{b}'^2}\tau' + \frac{27 n_2 \tilde{N} \mathfrak{b}^2}{16\pi^2 q_2 q_2' q_1 r n_1 \mathfrak{a}^3} \to \tau'$$

we arrive at the following expression of the w-integral in (2.4.34):

$$\int_{2/3}^{3} g_3(\ldots, w) e\left(w \frac{16\pi^2\mathfrak{a}^3}{27\mathfrak{b}^2}(\tau-\tau')\right) \mathrm{d}w.$$

On applying integration by parts repeatedly, we see that the above integral is negligibly small unless

$$|\tau - \tau'| \ll k^{\epsilon}\mathfrak{b}^2/\mathfrak{a}^3.$$

Now writing $\tau - \tau' = \tau_2$, with $\tau_2 \ll k^{\epsilon}\mathfrak{b}^2/\mathfrak{a}^3$, and estimating all the integrals in (2.4.34) trivially, we get

$$\mathfrak{I}_{\pm} \ll \frac{(\mathfrak{a}\mathfrak{a}')^{9/2}}{(\mathfrak{b}\mathfrak{b}')^5} \frac{k^{\epsilon}\mathfrak{b}^2}{\mathfrak{a}^3} \ll \frac{1}{(\mathfrak{b}\mathfrak{b}')^{1/2}} \frac{k^{\epsilon}}{\mathfrak{b}} \ll \frac{k^{\epsilon}C^2}{M_1 N},$$

where we used the fact $\mathfrak{a}' \asymp \mathfrak{a} \asymp \mathfrak{b} \asymp \mathfrak{b}'$. Hence we get (2.4.12). Now we proceed to prove the last part. Let $q \sim C \gg k^{1+\epsilon}$. Note that we also have $q' \sim C \gg k^{1+\epsilon}$. Note that in this situation we have $\mathfrak{a} \ll k^{-\epsilon}\mathfrak{b}$, $\mathfrak{a}' \ll k^{-\epsilon}\mathfrak{b}'$ and $\mathfrak{b} \asymp \mathfrak{b}' \asymp k$ (see (2.4.23) and (2.4.25)). On substituting the main term of $\mathrm{I}_{\pm}(m, \tilde{N}w, q)$ from (2.4.14) and the main term of $\mathrm{I}_{\pm}(m', \tilde{N}w, q')$ from (2.4.30) into (2.4.31), we arrive at the following expression:

$$\frac{c_3^2(\mathfrak{a}\mathfrak{a}')^{9/2}}{(\mathfrak{b}\mathfrak{b}')^5}\int_{\mathbb{R}}\phi_1(w)e\left(f_5(w)\right)\mathrm{d}w, \tag{2.4.35}$$

where

$$\begin{aligned}\phi_1(w) =&\frac{1}{\sqrt{f''(\tau_0)}}\frac{1}{\tau_0^5\sqrt{1-\tau_0^2}}\frac{1}{\sqrt{f_2''(\tau_0')}}\frac{1}{\tau_0'^5\sqrt{1-\tau_0'^2}}\\ &\times U_{u,t}^2\left(\left(\frac{4\pi\mathfrak{a}w^{1/3}}{3\mathfrak{b}\tau_0}\right)^6\right)\bar{U^2}_{u',t'}\left(\left(\frac{4\pi\mathfrak{a}'w^{1/3}}{3\mathfrak{b}'\tau_0'}\right)^6\right),\end{aligned} \tag{2.4.36}$$

and

$$f_5(w) = \frac{(k-1)(\sin^{-1}\tau_0 - \sin^{-1}\tau_0')}{2\pi} + \frac{16\pi^2}{27}\left(\frac{\mathfrak{a}^3 w}{\mathfrak{b}^2\tau_0^2} - \frac{\mathfrak{a}'^3 w}{\mathfrak{b}'^2\tau_0'^2}\right) - \frac{\tilde{N}n_2 w}{q_2q_2'q_1rn_1},$$

in which we apply the third derivative bound. Recall from (2.4.27) that

$$\tau_0 = \tau_0(w) = c_1\mathfrak{h}(w) + c_3(\mathfrak{h}(w))^3 + c_3(\mathfrak{h}(w))^5 ... + c_{2n-1}(\mathfrak{h}(w))^{2n-1} + ..., \tag{2.4.37}$$

with

$$\mathfrak{h}(w) = \frac{\mathfrak{a}w^{1/3}}{\mathfrak{b}^{2/3}(k-1)^{1/3}}, \quad \mathfrak{b} = \frac{4\pi\sqrt{mN}}{q} \quad \text{and} \quad \mathfrak{a} = \frac{3(N\tilde{N})^{1/3}}{qr^{1/3}}.$$

and τ_0' is similarly defined. On applying the change of variable $w \to w^3$ in (2.4.35), we see that the phase function is given by

$$\begin{aligned}&\frac{(k-1)(\sin^{-1}\tau_0(w^3) - \sin^{-1}\tau_0'(w^3))}{2\pi}\\ &+\frac{16\pi^2}{27}\left(\frac{\mathfrak{a}^3w^3}{\mathfrak{b}^2\tau_0^2(w^3)} - \frac{\mathfrak{a}'^3w^3}{\mathfrak{b}'^2\tau_0'^2(w^3)}\right) - \frac{\tilde{N}n_2w^3}{q_2q_2'q_1rn_1}.\end{aligned}$$

On applying the Taylor series expansion of $\sin^{-1}\tau_0(w^3)$, we see that

$$\begin{aligned}\sin^{-1}\tau_0(w^3) &= \tau_0(w^3) + (\tau_0(w^3))^3/6 + \cdots\\ &= d_1\mathfrak{h}(w^3) + d_3(\mathfrak{h}(w^3))^3 + \cdots\\ &= d_1\frac{\mathfrak{a}w}{\mathfrak{b}^{2/3}(k-1)^{1/3}} + d_3\frac{\mathfrak{a}^3w^3}{\mathfrak{b}^2(k-1)} + \cdots,\end{aligned}$$

where d_1, d_3 $\cdots$ are some absolute constants. Thus,

$$\frac{\mathrm{d}^3}{\mathrm{d}w^3}\sin^{-1}\tau_0(w^3) \ll \frac{\mathfrak{a}^3}{\mathfrak{b}^2(k-1)}.$$

Similarly,

$$\frac{\mathrm{d}^3}{\mathrm{d}w^3}\sin^{-1}\tau_0'(w^3) \ll \frac{\mathfrak{a}'^3}{\mathfrak{b}'^2(k-1)}.$$

Next we consider $\mathfrak{a}^3 w^3/(\mathfrak{b}^2\tau_0^2(w^3))$. On applying the Taylor series expansion, we get

$$\begin{aligned}
\frac{\mathfrak{a}^3 w^3}{\mathfrak{b}^2\tau_0^2(w^3)} = \frac{(k-1)(\mathfrak{h}(w^3))^3}{\tau_0^2(w^3)} &= \frac{(k-1)\mathfrak{h}(w^3)}{c_1^2}\left(1+\frac{c_3(\mathfrak{h}(w^3))^3}{c_1\mathfrak{h}(w^3)}+\cdots\right)^{-2}\\
&= \frac{(k-1)\mathfrak{h}(w^3)}{c_1^2}\left(1-\frac{2c_3(\mathfrak{h}(w^3))^3}{c_1\mathfrak{h}(w^3)}-\cdots\right)\\
&= \frac{(k-1)}{c_1^2}\left(\mathfrak{h}(w^3)-\frac{2c_3(\mathfrak{h}(w^3))^3}{c_1}-\cdots\right)
\end{aligned}$$

Thus,

$$\frac{\mathrm{d}^3}{\mathrm{d}w^3}\frac{\mathfrak{a}^3 w^3}{\mathfrak{b}^2\tau_0^2(w^3)} \ll \frac{\mathfrak{a}^3}{\mathfrak{b}^2}.$$

A similar analysis also gives us

$$\frac{\mathrm{d}^3}{\mathrm{d}w^3}\frac{\mathfrak{a}'^3 w^3}{\mathfrak{b}'^2\tau_0'^2(w^3)} \ll \frac{\mathfrak{a}'^3}{\mathfrak{b}'^2}.$$

Hence, upon combining the above estimates, we conclude that

$$\frac{\mathrm{d}^3 f_5(w^3)}{\mathrm{d}w^3} = O\left(\frac{\mathfrak{a}^3}{\mathfrak{b}^2}+\frac{\mathfrak{a}'^3}{\mathfrak{b}'^2}\right) - \frac{6\tilde{N}n_2}{q_2q_2'q_1rn_1}.$$

Since $n_2 \neq 0$, we note that

$$\frac{\mathfrak{a}^3}{\mathfrak{b}^2}+\frac{\mathfrak{a}'^3}{\mathfrak{b}'^2} \ll \frac{N\tilde{N}}{C^3rk^2} \ll \frac{(k^3/r^2)\tilde{N}}{C^2rk^{3+\epsilon}} \ll \frac{\tilde{N}}{k^\epsilon C^2 r(n_1,r)} \ll \frac{\tilde{N}}{k^\epsilon (C^2/q_1)rn_1} \ll \frac{k^{-\epsilon}6\tilde{N}|n_2|}{q_2q_2'q_1rn_1}.$$

In the first inequality, we used the fact $\mathfrak{a} \asymp \mathfrak{a}'$, $\mathfrak{b} \asymp \mathfrak{b}' \asymp k$. For the second inequality, we used $Nr^2 \ll k^{3+\epsilon}$ and $C \gg k^{1+\epsilon}$, while for the second last inequality, $(n_1, r) \geq n_1/q_1$, is being used. Hence, we see that

$$\left|\frac{\mathrm{d}^3 f_5(w^3)}{\mathrm{d}w^3}\right| = \left|O\left(\frac{\mathfrak{a}^3}{\mathfrak{b}^2}+\frac{\mathfrak{a}'^3}{\mathfrak{b}'^2}\right) - \frac{6\tilde{N}n_2}{q_2q_2'q_1rn_1}\right| \gg \frac{\mathfrak{a}^3}{\mathfrak{b}^2}+\frac{\mathfrak{a}'^3}{\mathfrak{b}'^2} \asymp \frac{N\tilde{N}}{C^3rk^2}.$$

On computing the variation of $\phi_1(w)$, given in (2.4.36), we see that

$$\operatorname{Var}\phi_1 \ll \frac{1}{\sqrt{\mathfrak{b}^2/\mathfrak{a}}}\frac{1}{(\mathfrak{a}/k)^5}\frac{1}{\sqrt{\mathfrak{b}'^2/\mathfrak{a}'}}\frac{1}{(\mathfrak{a}'/k)^5} \ll \frac{1}{\mathfrak{b}^2/\mathfrak{a}}\frac{1}{(\mathfrak{a}/k)^{10}}, \tag{2.4.38}$$

where we used $F''(\tau_0) \asymp \mathfrak{b}^2/\mathfrak{a}$, $f_2''(\tau_0') \asymp \mathfrak{b}'^2/\mathfrak{a}'$, $\tau_0 \asymp \mathfrak{a}/(\mathfrak{b}^{2/3}(k-1)^{1/3}) \asymp \mathfrak{a}/k$ and $\tau_0' \asymp \mathfrak{a}'/k$. Hence, on applying the third derivative bound (see Lemma 1.7.1) to (2.4.35), we see that (2.4.35) is bounded by

$$\frac{c_3^2(\mathfrak{a}\mathfrak{a}')^{9/2}}{(\mathfrak{b}\mathfrak{b}')^5}\frac{\operatorname{Var}\phi_1 + \max|\phi_1|}{\min|f_5(w^3)|^{1/3}} \ll \frac{\mathfrak{a}^9}{\mathfrak{b}^{10}}\frac{1}{\mathfrak{b}^2/\mathfrak{a}}\frac{1}{(\mathfrak{a}/k)^{10}}\frac{(C^3rk^2)^{1/3}}{(N\tilde{N})^{1/3}} \asymp \frac{Cr^{1/3}k^{2/3}}{k^2(N\tilde{N})^{1/3}}.$$

Hence we get Proposition 2.4.4. □

We conclude this section by giving a final estimation of the main integral $\mathcal{J}_\pm$ defined in (2.4.1) in the following corollary.

Corollary 2.4.8. *Let $\mathcal{J}_\pm$ be the integral transform defined as in (2.4.1). Then we have*

$$\mathcal{J}_\pm \ll \frac{k^\epsilon C^4}{Q^2 M_1 N}. \tag{2.4.39}$$

Furthermore, if $C \gg k^{1+\epsilon}$ and $n_2 \neq 0$, then we have

$$\mathcal{J}_\pm \ll \frac{k^\epsilon C^2}{Q^2}\frac{Cr^{1/3}k^{2/3}}{k^2(N\tilde{N})^{1/3}}. \tag{2.4.40}$$

Proof. Let's recall from (2.4.9) that

$$\mathcal{J}_\pm = \int_{\mathbb{R}}\int_{\mathbb{R}} V\left(\frac{t}{T}\right)V\left(\frac{t'}{T}\right)\int_{u \ll \frac{k^\epsilon C}{QT}}\int_{u' \ll \frac{k^\epsilon C}{QT}} \mathrm{I}_u\,\bar{\mathrm{I}}_{u'}\,\mathfrak{J}_\pm\,\mathrm{d}u'\,\mathrm{d}u\,\mathrm{d}t'\,\mathrm{d}t + O(k^{-2020}),$$

where

$$\mathrm{I}_u = \int_{\mathbb{R}} W(x/Q^\epsilon)g(q,x)e\left(\frac{Nxu}{qQ}\right)\mathrm{d}x,$$

and $\mathrm{I}_{u'}$ is similarly defined. On applying the bound $\mathfrak{J}_{\pm} \ll k^{\epsilon}C^2/(M_1N)$ from Proposition 2.4.4, we see that

$$|\mathcal{J}_{\pm}| \ll \frac{k^{\epsilon}C^2}{M_1N}\int_{\mathbb{R}}\int_{\mathbb{R}} V\left(\frac{t}{\mathcal{T}}\right) V\left(\frac{t'}{\mathcal{T}}\right)\int_{u\ll\frac{k^{\epsilon}C}{Q\mathcal{T}}}\int_{u'\ll\frac{k^{\epsilon}C}{Q\mathcal{T}}} |\mathrm{I}_u|\,|\bar{\mathrm{I}}_{u'}|\,\mathrm{d}u'\,\mathrm{d}u\,\mathrm{d}t'\,\mathrm{d}t. \tag{2.4.41}$$

Consider the u-integral

$$\int_{u\ll\frac{k^{\epsilon}C}{Q\mathcal{T}}} |\mathrm{I}_u|\,\mathrm{d}u \ll \int_{u\ll\frac{k^{\epsilon}C}{Q\mathcal{T}}}\int_{\mathbb{R}} W(x/Q^{\epsilon})|g(q,x)|\mathrm{d}x\,\mathrm{d}u \ll \frac{k^{\epsilon}C}{Q\mathcal{T}}Q^{\epsilon},$$

where we used Property 4 (see (1.4.2)) of $g(q,x)$. The same bound holds for the u'-integral as well. Thus, on plugging these bounds into (2.4.41) and estimating the t and t'-integral trivially, we get (2.4.39). On analysing the u, u', t and t'-integrals as above and applying the bound (2.4.13) from Proposition 2.4.4, we get the second part of the corollary. □

2.5 Analysis of the zero frequency: $n_2 = 0$

With all the ingredients in the hands, we now give final estimates for $S_r(N)$, given in (2.3.4), in the present and coming sections. The zero frequency case, i.e., $n_2 = 0$, needs to be analysed diffenently. Let $\Omega_{\pm}^{0}$ denote the contribution of the zero frequency to $\Omega_{\pm}$, given in (2.3.8), and let $S_r^0(N)$ denote the contribution of $\Omega_{\pm}^{0}$ to $S_r(N)$. We have the following lemma:

Lemma 2.5.1. *Let $\Omega_{\pm}^{0}$ and $S_r^0(N)$ be defined as above. Then we have*

$$\Omega_{\pm}^{0} \ll \frac{k^{\epsilon}N_0C^6r}{q_1n_1^2Q^2N}(C+M_1),$$

and

$$S_r^0(N) \ll k^{\epsilon}r^{1/2}N^{1/2}k^{3/2-\eta/2},$$

where $\mathcal{T} = k^{1-\eta}$.

Proof. Let's recall from (2.3.11) that

$$\Omega_{\pm}^{0} \ll k^{\epsilon} \sup_{\tilde{N} \ll N_0} \frac{\tilde{N}}{n_1^2} \sum\sum_{q_2, q_2' \sim C/q_1} \sum\sum_{m, m' \sim M_1} |\mathfrak{C}_{\pm}| \, |\mathcal{J}_{\pm}|. \tag{2.5.1}$$

Consider the congruence condition

$$\pm \bar{\alpha} q_2' \mp \bar{\alpha}' q_2 \equiv n_2 \bmod q_1 q_2 q_2' r / n_1$$

given in the expression (2.3.9) of $\mathfrak{C}_{\pm}$. For $n_2 = 0$, it follows that $q_2 = q_2'$ and $\alpha = \alpha'$. Hence, we get

$$\begin{aligned}
\mathfrak{C}_{\pm} &= \sum\sum_{d, d' | q} d d' \mu\left(\frac{q}{d}\right) \mu\left(\frac{q}{d'}\right) \sideset{}{^\star}\sum_{\substack{\alpha \bmod qr/n_1 \\ n_1 \alpha \equiv -m \bmod d \\ n_1 \alpha \equiv -m' \bmod d'}} 1 \\
&\ll \sum\sum_{\substack{d, d' | q \\ (d,d') | (m-m')}} d d' \frac{qr}{n_1 [d/(n_1, d), \, d'/(n_1, d')]} \ll \sum\sum_{\substack{d, d' | q \\ (d,d') | (m-m')}} d d' \frac{qr}{[d, d']}.
\end{aligned}$$

On plugging the above expression and the bound $\mathcal{J}_{\pm} \ll k^{\epsilon} C^4/(Q^2 M_1 N)$ from Corollary 2.4.8 into (2.5.1), we get

$$\begin{aligned}
\Omega_{\pm}^{0} &\ll k^{\epsilon} \sup_{\tilde{N} \ll N_0} \frac{\tilde{N}}{n_1^2} \sum_{q_2 \sim C/q_1} qr \sum\sum_{d, d' | q} (d, \, d') \sum\sum_{\substack{m, m' \sim M_1 \\ (d, \, d') | (m-m')}} \frac{k^{\epsilon} C^4}{Q^2 M_1 N} \\
&\ll \frac{k^{\epsilon} N_0 C^4}{n_1^2 Q^2 M_1 N} \sum_{q_2 \sim C/q_1} qr \sum\sum_{d, d' | q} (M_1 (d, d') + M_1^2) \\
&\ll \frac{k^{\epsilon} N_0 C^4}{n_1^2 Q^2 M_1 N} \sum_{q_2 \sim C/q_1} qr (M_1 q + M_1^2) \ll \frac{k^{\epsilon} N_0 C^6 r}{q_1 n_1^2 Q^2 N} (C + M_1).
\end{aligned}$$

Hence we have the first part of the lemma. On substituting the above bound in place of $\Omega_{\pm}$ in (2.3.4), we see that $S_r^0(N)$ is dominated by

$$\sup_{\substack{M_1 \ll M_0 \\ C \ll Q}} \frac{N^{5/3+\epsilon}}{Q \mathcal{T} r^{2/3} C^3} \sum_{\frac{n_1}{(n_1, r)} \ll C} n_1^{1/3} \Theta^{1/2} \sum_{\frac{n_1}{(n_1, r)} | q_1 | (n_1 r)^{\infty}} \frac{C^3 (N_0 r)^{1/2}}{n_1 q_1^{1/2} Q \sqrt{N}} \left(\sqrt{M_1} + \sqrt{C}\right).$$

Recall that Θ is defined in (2.3.5) as

$$\Theta = \sum_{n_2 \ll N_0/n_1^2} \frac{|\lambda_\pi(n_1, n_2)|^2}{n_2^{2/3}}.$$

Executing the q_1-sum trivially and replacing the range for n_1 by the longer range $n_1 \ll Cr$, we get

$$S_r^0(N) \ll k^\epsilon \sup_{\substack{M_1 \ll M_0 \\ C \ll Q}} \frac{N^{2/3}(N_0 r)^{1/2}}{r^{2/3}\sqrt{N}} \sum_{n_1 \ll Cr} \frac{(n_1, r)^{1/2}}{n_1^{7/6}} \Theta^{1/2} \left(\sqrt{M_1} + \sqrt{C}\right).$$

Next we evaluate the n_1-sum, using Cauchy's inequality and the Rankin-Selberg bound (see Lemma 1.2.3), as follows:

$$\sum_{n_1 \ll Cr} \frac{(n_1, r)^{1/2}}{n_1^{7/6}} \Theta^{1/2} \ll \left[\sum_{n_1 \ll Cr} \frac{(n_1, r)}{n_1}\right]^{1/2} \left[\sum\sum_{n_1^2 n_2 \le N_0} \frac{|\lambda_\pi(n_1, n_2)|^2}{(n_1^2 n_2)^{2/3}}\right]^{1/2} \ll_{\pi,\epsilon} N_0^{1/6+\epsilon}. \tag{2.5.2}$$

Thus we arrive at

$$S_r^0(N) \ll k^\epsilon \frac{N^{2/3} N_0^{2/3}}{r^{1/6}\sqrt{N}} \left(\sqrt{M_0} + \sqrt{Q}\right). \tag{2.5.3}$$

Note that

$$Q = k^\epsilon \sqrt{N/\mathcal{T}} \ll k^{3/2+\epsilon}/\sqrt{\mathcal{T}} \ll k^{2+\epsilon}/\mathcal{T} \ll k^{2+\epsilon} Q^2/N.$$

The second inequality follows because of $\mathcal{T} < k$. We also have

$$M_0 = k^\epsilon \max\left((k-1)^2 C^2/N, \, \mathcal{T}\right) \ll k^{2+\epsilon} Q^2/N$$

and

$$N_0 = k^\epsilon \max\left\{(C\mathcal{T})^3 r/N, \, \mathcal{T}^{3/2} N^{1/2} r\right\} \ll k^\epsilon (Q\mathcal{T})^3 r/N \ll k^\epsilon \mathcal{T}^{3/2}\sqrt{N} r.$$

Finally, upon using the above bounds in (2.5.3), we get

$$S_r^0(N) \ll \frac{k^\epsilon r^{2/3}\mathcal{T}N}{r^{1/6}\sqrt{N}}\frac{kQ}{\sqrt{N}} \ll k^\epsilon r^{1/2}N^{1/2}k^{3/2-\eta/2}.$$

Hence the lemma follows. □

2.6 Analysis of the non-zero frequencies: $n_2 \neq 0$

It now remains to estimate $S_r(N)$ corresponding to the non-zero frequencies, i.e., $n_2 \neq 0$. We will consider two cases, small q 's and large q's. To start with, we analyse the character sum $\mathfrak{C}_\pm$ given in (2.3.9). We have the following lemma.

Lemma 2.6.1. *Let $\mathfrak{C}_\pm$ be as in (2.3.9). Then, for $n_2 \neq 0$, we have*

$$\mathfrak{C}_\pm \ll \frac{q_1^2\, r(m,n_1)}{n_1} \sum\sum_{\substack{d_2|(q_2, n_1q_2' \mp mn_2)\\ d_2'|(q_2', n_1q_2 \pm m'n_2)}} d_2d_2'\,.$$

Proof. Let's recall from (2.3.9) that

$$\mathfrak{C}_\pm = \sum\sum_{\substack{d|q\\ d'|q'}} dd'\mu\left(\frac{q}{d}\right)\mu\left(\frac{q'}{d'}\right) \sideset{}{^\star}\sum_{\substack{\alpha \bmod qr/n_1\\ n_1\alpha\equiv -m \bmod d}} \sideset{}{^\star}\sum_{\substack{\alpha' \bmod q'r/n_1\\ n_1\alpha'\equiv -m' \bmod d'\\ \pm\bar{\alpha}q_2' \mp \bar{\alpha}'q_2 \equiv -n_2 \bmod q_1q_2q_2'r/n_1}} 1.$$

Using the Chinese Remainder theorem, we observe that $\mathfrak{C}_\pm$ can be dominated by a product of two sums $\mathfrak{C}_\pm \ll \mathfrak{C}_\pm^{(1)}\mathfrak{C}_\pm^{(2)}$, where

$$\mathfrak{C}_\pm^{(1)} = \sum\sum_{d_1, d_1'|q_1} d_1d_1' \sideset{}{^\star}\sum_{\substack{\beta \bmod \frac{q_1r}{n_1}\\ n_1\beta \equiv -m \bmod d_1}} \sideset{}{^\star}\sum_{\substack{\beta' \bmod \frac{q_1r}{n_1}\\ n_1\beta' \equiv -m' \bmod d_1'\\ \pm\overline{\beta}q_2' \mp \overline{\beta'}q_2 + n_2 \equiv 0 \bmod q_1r/n_1}} 1$$

and

$$\mathfrak{C}_\pm^{(2)} = \sum\sum_{\substack{d_2|q_2\\ d_2'|q_2'}} d_2d_2' \sideset{}{^\star}\sum_{\substack{\beta \bmod q_2\\ n_1\beta \equiv -m \bmod d_2}} \sideset{}{^\star}\sum_{\substack{\beta' \bmod q_2'\\ n_1\beta' \equiv -m' \bmod d_2'\\ \pm\overline{\beta}q_2' \mp \overline{\beta'}q_2 + n_2 \equiv 0 \bmod q_2q_2'}} 1.$$

On analysing the second sum $\mathfrak{C}_{\pm}^{(2)}$, we get $\beta \equiv -m\bar{n}_1 \bmod d_2$ and $\beta' \equiv -m'\bar{n}_1 \bmod d_2'$, as $(n_1, q_2 q_2') = 1$. Then using the congruence modulo $q_2 q_2'$, we conclude that

$$\mathfrak{C}_{\pm}^{(2)} \ll \sum_{\substack{d_2 | (q_2, n_1 q_2' \mp m n_2) \\ d_2' | (q_2', n_1 q_2 \pm m' n_2)}} \sum d_2 d_2'.$$

In the first sum $\mathfrak{C}_{\pm}^{(1)}$, the congruence condition determines β' uniquely in terms of β, and hence

$$\mathfrak{C}_{\pm}^{(1)} \ll \sum_{d_1, d_1' | q_1} \sum d_1 d_1' \sideset{}{^\star}\sum_{\substack{\beta \bmod q_1 r/n_1 \\ n_1 \beta \equiv -m \bmod d_1}} 1 \ll \frac{r\, q_1^2\, (m, n_1)}{n_1}.$$

Hence we have the lemma. □

2.6.1 $S_r(N)$ for small q

In this subsection, we will estimate $S_r(N)$ for small values of q. Let $\Omega_{\pm}^{\neq 0}$ denote the part of $\Omega_{\pm}$ (defined in (2.3.8)) which is complement to $\Omega_{\pm}^{0}$ (contribution of $n_2 \neq 0$) and let $S_r^{\neq 0}(N)$ denote the part of $S_r(N)$ corresponding to $\Omega_{\pm}^{\neq 0}$. We have the following lemma.

Lemma 2.6.2. *Let $\Omega_{\pm}^{\neq 0}$ and $S_r^{\neq 0}(N)$ be as above. Then, for $C \ll k^{1+\epsilon}$, we have*

$$\Omega_{\pm}^{\neq 0} \ll \frac{k^{\epsilon} r^2 C^7 (\mathcal{T} N)^{1/2}}{n_1^2 q_1 Q^2 M_1 N} \left(\frac{C^2 n_1}{q_1^2} + \frac{C M_1 n_1}{q_1} + M_1^2 \right). \tag{2.6.1}$$

Furthermore, let $S_{r,small}^{\neq 0}(N)$ denote the contribution of $C \ll k^{1+\epsilon}$ to $S_r^{\neq 0}(N)$. Then we have

$$S_{r,small}^{\neq 0}(N) \ll r^{1/2} k^{3-\eta/2}, \tag{2.6.2}$$

where $\mathcal{T} = k^{1-\eta}$.

Proof. We start by analysing $\Omega_{\pm}^{\neq 0}$ which is defined using (2.3.11). On applying Lemma 2.6.1 for $\mathfrak{C}_{\pm}$ in (2.3.11), we get the following expression for $\Omega_{\pm}^{\neq 0}$:

$$\Omega_{\pm}^{\neq 0} \ll \frac{k^{\epsilon} q_1^2 r}{n_1^3} \sup_{\tilde{N} \ll N_0} \tilde{N} \sum\sum_{q_2,\, q_2' \sim \frac{C}{q_1}} \sum\sum_{\substack{d_2 | q_2 \\ d_2' | q_2'}} d_2 d_2' \sum\sum_{\substack{m,\, m' \sim M_1 \\ n_1 q_2' \mp m n_2 \equiv 0 \bmod d_2 \\ n_1 q_2 \pm m' n_2 \equiv 0 \bmod d_2'}} \sum_{n_2 \in \mathbb{Z} - \{0\}} (m, n_1) |\mathcal{J}_{\pm}|.$$

On using the bound $\mathcal{J}_{\pm} \ll k^{\epsilon} C^4/(Q^2 M_1 N)$ from Corollary 2.4.8, and Proposition 2.4.4 for the range of n_2, we get, upto a negligible error term, the following expression for $\Omega_{\pm}^{\neq 0}$

$$\frac{k^{\epsilon} q_1^2 r C^4}{n_1^3 Q^2 M_1 N} \sup_{\tilde{N} \ll N_0} \tilde{N} \sum\sum_{q_2,\, q_2' \sim \frac{C}{q_1}} \sum\sum_{\substack{d_2 | q_2 \\ d_2' | q_2'}} d_2 d_2' \sum\sum_{\substack{m,\, m' \sim M_1 \\ n_1 q_2' \mp m n_2 \equiv 0 \bmod d_2 \\ n_1 q_2 \pm m' n_2 \equiv 0 \bmod d_2'}} \sum_{1 \leq |n_2| \ll N_2} (m, n_1).$$

Further writing $q_2 d_2$ in place of q_2 and $q_2' d_2'$ in place of q_2', we arrive at

$$\Omega_{\pm}^{\neq 0} \ll \frac{k^{\epsilon} q_1^2 r C^4}{n_1^3 Q^2 M_1 N} \sup_{\tilde{N} \ll N_0} \tilde{N} \sum\sum_{d_2, d_2' \ll C/q_1} d_2 d_2' \sum\sum_{\substack{q_2 \sim \frac{C}{d_2 q_1} \\ q_2' \sim \frac{C}{d_2' q_1}}} \sum\sum_{\substack{m,\, m' \sim M_1 \\ n_1 q_2' d_2' \mp m n_2 \equiv 0 \bmod d_2 \\ n_1 q_2 d_2 \pm m' n_2 \equiv 0 \bmod d_2'}} \sum_{1 \leq |n_2| \ll N_2} (m, n_1). \tag{2.6.3}$$

Next we count the number of m and m' in the above expression as follows:

$$\begin{aligned}
\sum_{\substack{m \sim M_1 \\ n_1 q_2' d_2' \mp m n_2 \equiv 0 \bmod d_2}} (m, n_1) &\leq \sum_{\ell | n_1} \ell \sum_{\substack{m \sim M_1/\ell \\ n_1 q_2' d_2' \bar{\ell} \mp m n_2 \equiv 0 \bmod d_2}} 1 \\
&\leq \sum_{\ell | n_1} \ell \left((d_2, n_2) + \frac{M_1}{\ell d_2/(d_2, n_2)} \right) \\
&\ll n_1^{\epsilon} (d_2,\, n_2) \left(n_1 + \frac{M_1}{d_2} \right).
\end{aligned}$$

In the above estimate we have used the fact $(d_2, n_1) = 1$, and hence ℓ can be inverted modulo d_2. We now count the number of m in a similar fashion as follows:

$$\sum_{\substack{m' \sim M_1 \\ n_1 q_2 d_2 \pm m' n_2 \equiv 0 \bmod d_2'}} 1 \ll \left((d_2',\, n_1 q_2 d_2,\, n_2) + \frac{M_1}{d_2'/(d_2',\, n_1 q_2 d_2,\, n_2)} \right)$$
$$= (d_2',\, n_1 q_2 d_2) \left(1 + \frac{M_1}{d_2'}\right).$$

Thus the number of m and m' in (2.6.3) is dominated by

$$k^{\epsilon} (d_2', n_1 q_2 d_2)\, (d_2, n_2) \left(n_1 + \frac{M_1}{d_2}\right) \left(1 + \frac{M_1}{d_2'}\right).$$

On substituting the above bound in (2.6.3), we arrive at

$$\frac{k^{\epsilon} q_1^2 r C^4}{n_1^3 Q^2 M_1 N} \sup_{\tilde{N} \ll N_0} \tilde{N} \sum\sum_{d_2, d_2' \ll C/q_1} d_2 d_2' \sum\sum_{\substack{q_2 \sim \frac{C}{d_2 q_1} \\ q_2' \sim \frac{C}{d_2' q_1}}}$$
$$\times \sum_{1 \leq |n_2| \ll N_2} (d_2', n_1 q_2 d_2)(d_2, n_2) \left(n_1 + \frac{M_1}{d_2}\right) \left(1 + \frac{M_1}{d_2'}\right).$$

Now summing over n_2, and q_2', we get the following expression:

$$\frac{k^{\epsilon} q_1 r C^5}{n_1^3 Q^2 M_1 N} \sup_{\tilde{N} \ll N_0} \tilde{N} N_2 \sum\sum_{d_2, d_2' \ll C/q_1} d_2 \sum_{q_2 \sim \frac{C}{d_2 q_1}} (d_2', n_1 q_2 d_2) \left(n_1 + \frac{M_1}{d_2}\right) \left(1 + \frac{M_1}{d_2'}\right).$$

Next we sum over d_2' to get

$$\Omega_{\pm}^{\neq 0} \ll \frac{k^{\epsilon} q_1 r C^5}{n_1^3 Q^2 M_1 N} \sup_{\tilde{N} \ll N_0} \tilde{N} N_2 \sum_{d_2 \ll C/q_1} d_2 \sum_{q_2 \sim \frac{C}{d_2 q_1}} \left(n_1 + \frac{M_1}{d_2}\right) \left(\frac{C}{q_1} + M_1\right).$$

Finally executing the remaining sums, we get

$$\Omega_{\pm}^{\neq 0} \ll \frac{k^{\epsilon} r C^6}{n_1^3 Q^2 M_1 N} \sup_{\tilde{N} \ll N_0} \tilde{N} N_2 \left(\frac{C n_1}{q_1} + M_1\right) \left(\frac{C}{q_1} + M_1\right). \tag{2.6.4}$$

Note that, using the expression $N_2 = k^\epsilon C N^{1/3} r^{2/3} n_1/(q_1 \tilde{N}^{2/3})$ from (2.4.11) and $N_0 \ll k^\epsilon \mathcal{T}^{3/2}\sqrt{N}r$ from (2.3.7), we have

$$\sup_{\tilde{N} \ll N_0} \tilde{N} N_2 \ll k^\epsilon \frac{C r^{2/3} n_1}{q_1} (N\tilde{N})^{1/3} \ll k^\epsilon \frac{C r^{2/3} n_1}{q_1} (N N_0)^{1/3} \ll \frac{k^\epsilon r n_1}{q_1} (\mathcal{T} N)^{1/2} C. \tag{2.6.5}$$

Hence, on plugging the above expression into (2.6.4), we arrive at

$$\begin{aligned} \Omega_{\pm}^{\neq 0} &\ll \frac{k^\epsilon r^2 C^7 (\mathcal{T}N)^{1/2}}{n_1^2 q_1 Q^2 M_1 N} \left(\frac{C n_1}{q_1} + M_1 \right) \left(\frac{C}{q_1} + M_1 \right) \\ &\ll \frac{k^\epsilon r^2 C^7 (\mathcal{T}N)^{1/2}}{n_1^2 q_1 Q^2 M_1 N} \left(\frac{C^2 n_1}{q_1^2} + \frac{C n_1 M_1}{q_1} + M_1^2 \right). \end{aligned} \tag{2.6.6}$$

Hence we get the first part of the lemma. Let's consider the third term on the right side of the above expression. On substituting it in place of $\Omega_\pm$ in $S_r(N)$ in (2.3.4), we arrive at

$$\begin{aligned} &\sup_{\substack{M \ll M_1 \ll M_0 \\ C \ll k^{1+\epsilon}}} \frac{N^{5/3+\epsilon}}{Q\mathcal{T} r^{2/3} C^3} \sum_{\pm} \sum_{\frac{n_1}{(n_1,r)} \ll C} n_1^{1/3} \Theta^{1/2} \sum_{\frac{n_1}{(n_1,r)} | q_1 | (n_1 r)^\infty} \left(\frac{r^2 C^7 (\mathcal{T}N)^{1/2} M_1}{n_1^2 q_1 Q^2 N} \right)^{1/2} \\ &\ll \sup_{\substack{M \ll M_1 \ll M_0 \\ C \ll k^{1+\epsilon}}} \frac{N^{5/3+\epsilon}}{Q\mathcal{T} r^{2/3} C^3} \frac{r (\mathcal{T}N)^{1/4} C^{7/2} M_1^{1/2}}{Q\sqrt{N}} \sum_{n_1 \ll Cr} n_1^{-2/3} \Theta^{1/2} \sum_{\frac{n_1}{(n_1,r)} | q_1 | (n_1 r)^\infty} \frac{1}{q_1^{1/2}} \\ &\ll \sup_{\substack{M \ll M_1 \ll M_0 \\ C \ll k^{1+\epsilon}}} \frac{N^{5/3+\epsilon}}{Q\mathcal{T} r^{2/3}} \frac{r (\mathcal{T}N)^{1/4} C^{1/2} M_1^{1/2}}{Q\sqrt{N}} \sum_{n_1 \ll Cr} \frac{\sqrt{(n_1,r)}}{n_1^{7/6}} \Theta^{1/2} \\ &\ll k^\epsilon r^{1/2} k^{3-\eta/2}, \end{aligned}$$

where in the second last inequality, we used

$$\sum_{n_1 \ll Cr} \frac{\sqrt{(n_1,r)}}{n_1^{7/6}} \Theta^{1/2} \ll_{\pi,\epsilon} N_0^{1/6+\epsilon}$$

from (2.5.2), $C \ll k^{1+\epsilon}$, $N_0 \ll k^\epsilon r \sqrt{N} \mathcal{T}^{3/2}$ and $M_0 \ll k^{4+\epsilon}/N$ as $C \ll k^{1+\epsilon}$. We now consider the second term in the right hand side of (2.6.6). We see that its contribution to $S_r(N)$ in (2.3.4) is given by

$$\begin{aligned}
&\sup_{\substack{M \ll M_1 \ll M_0 \\ C \ll k^{1+\epsilon}}} \frac{N^{5/3+\epsilon}}{Q\mathcal{T}r^{2/3}C^3} \sum_{\pm} \sum_{\frac{n_1}{(n_1,r)} \ll C} n_1^{1/3}\Theta^{1/2} \sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^\infty} \left(\frac{r^2C^7(\mathcal{T}N)^{1/2}C}{n_1q_1^2Q^2N}\right)^{1/2} \\
&\ll \sup_{\substack{M \ll M_1 \ll M_0 \\ C \ll k^{1+\epsilon}}} \frac{N^{5/3+\epsilon}}{Q\mathcal{T}r^{2/3}C^3} \frac{r(\mathcal{T}N)^{1/4}C^{7/2}C^{1/2}}{Q\sqrt{N}} \sum_{n_1 \ll Cr} n_1^{-1/6}\Theta^{1/2} \sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^\infty} \frac{1}{q_1} \\
&\ll \sup_{\substack{M \ll M_1 \ll M_0 \\ C \ll k^{1+\epsilon}}} \frac{N^{5/3+\epsilon}}{Q\mathcal{T}r^{2/3}} \frac{r(\mathcal{T}N)^{1/4}C}{Q\sqrt{N}} \sum_{n_1 \ll Cr} \frac{(n_1,r)}{n_1^{7/6}}\Theta^{1/2} \\
&\ll k^{3-\eta/2}.
\end{aligned}$$

In the second last inequality, we used the bound

$$\begin{aligned}
\sum_{n_1 \ll Cr} \frac{(n_1,r)}{n_1^{7/6}}\sqrt{\Theta} &\ll \left[\sum_{n_1 \ll Cr} \frac{(n_1,r)^2}{n_1}\right]^{1/2} \left[\sum\sum_{n_1^2n_2 \leq N_0} \frac{|\lambda_\pi(n_1,n_2)|^2}{(n_1^2n_2)^{2/3}}\right]^{1/2} \\
&\ll_{\pi,\epsilon} r^{1/2}N_0^{1/6+\epsilon}.
\end{aligned}$$

Lastly we consider the the first term in (2.6.6). We observe that

$$\frac{C^2n_1}{q_1^2M_1} \ll \frac{n_1}{q_1^2}\frac{Nk^\epsilon}{k^2} \ll \frac{n_1k^{1+\epsilon}}{q_1},$$

where we used $M_1 \gg C^2k^2/(Nk^\epsilon)$ (see (2.3.7)). Note that while analysing the second term we had the factor n_1C/q_1, in which we bound C by $k^{1+\epsilon}$ later. Now on estimating the q_1 and n_1 sum as done in the analysis of the second term we see that the contribution of the first term is dominated by the contribution of the second term. Finally on combining all the estimates we get the lemma. □

2.6.2 Estimates for generic q

Now we tackle the case when $C \gg k^{1+\epsilon}$ and $n_2 \neq 0$. Let $S^{\neq 0}_{r,generic}(N)$ denote the part of $S^{\neq 0}_r(N)$ which is complement to $S^{\neq 0}_{r,small}(N)$ (i.e., the contribution of $C \gg k^{1+\epsilon}$) and $n_2 \neq 0$ to $S_r(N)$. We have the following lemma.

Lemma 2.6.3. *Let $S^{\neq 0}_{r,generic}(N)$ be as above. Then we have*

$$S^{\neq 0}_{r,generic}(N) \ll N^{1/2}k^{3/2-1/6+3\eta/4}. \tag{2.6.7}$$

Proof. Let's recall (see (2.6.4)) from the analysis of $\Omega^{\neq 0}_{\pm}$ in the proof of Lemma 2.6.2 that

$$\Omega^{\neq 0}_{\pm} \ll \frac{k^{\epsilon} r C^6}{n_1^3 Q^2 M_1 N} \sup_{\tilde{N}\ll N_0} \tilde{N} N_2 \left(\frac{Cn_1}{q_1}+M_1\right)\left(\frac{C}{q_1}+M_1\right). \tag{2.6.8}$$

To get this, we had used the bound $\mathcal{J}_{\pm} \ll k^{\epsilon}C^4/(Q^2M_1N)$. For $C \gg k^{1+\epsilon}$, we have a better bound for $\mathcal{J}_{\pm}$ (see Corollary 2.4.8). In fact, we have

$$\mathcal{J}_{\pm} \ll \frac{k^{\epsilon}C^2}{Q^2}\frac{Cr^{1/3}k^{2/3}}{k^2(N\tilde{N})^{1/3}} \asymp \frac{k^{\epsilon}C^4}{Q^2M_1N}\frac{Cr^{1/3}k^{2/3}}{(N\tilde{N})^{1/3}}, \tag{2.6.9}$$

where we used $\sqrt{M_1N}/C \asymp k$ for $C \gg k^{1+\epsilon}$. Thus, on using the above bound, we see that

$$\Omega^{\neq 0}_{\pm} \ll \frac{k^{\epsilon} r C^6}{n_1^3 Q^2 M_1 N} \times Cr^{1/3}k^{2/3} \times \sup_{\tilde{N}\ll N_0} \frac{\tilde{N} N_2}{(N\tilde{N})^{1/3}} \left(\frac{Cn_1}{q_1}+M_1\right)\left(\frac{C}{q_1}+M_1\right). \tag{2.6.10}$$

Recall from (2.6.5) that

$$\sup_{\tilde{N}\ll N_0} \frac{\tilde{N} N_2}{(N\tilde{N})^{1/3}} \ll k^{\epsilon}\frac{Cr^{2/3}n_1}{q_1}, \tag{2.6.11}$$

and

$$\sup_{\tilde{N}\ll N_0} \frac{\tilde{N} N_2}{(N\tilde{N})^{1/3}} = \frac{N_0N_2}{(NN_0)^{1/3}}.$$

Thus, we see that

$$\Omega^{\neq 0}_{\pm} \ll \frac{k^{\epsilon} r C^6}{n_1^3 Q^2 M_1 N} \times Cr^{1/3}k^{2/3} \times \frac{N_0N_2}{(NN_0)^{1/3}} \left(\frac{C^2n_1}{q_1^2}+\frac{Cn_1M_1}{q_1}+M_1^2\right). \tag{2.6.12}$$

On comparing it with (2.6.6), we observe that we have an extra factor

$$\frac{Cr^{1/3}k^{2/3}}{r^{1/3}(N\mathcal{T})^{1/2}} \ll \frac{Qk^{2/3}}{(N\mathcal{T})^{1/2}} = k^{\eta-1/3+\epsilon}$$

in this case. Hence we get

$$\Omega_{\pm}^{\neq 0} \ll \frac{k^{\epsilon}r^2C^7(\mathcal{T}N)^{1/2}}{n_1^2q_1Q^2M_1N} \times k^{\eta-1/3}\left(\frac{C^2n_1}{q_1^2} + \frac{Cn_1M_1}{q_1} + M_1^2\right).$$

Note that

$$\frac{C^2n_1}{q_1^2M_1} \ll \frac{n_1}{q_1^2}\frac{Nk^{\epsilon}}{k^2} \ll \frac{n_1Nk^{\epsilon}}{q_1k^2},$$

where we used $M_1 \gg C^2k^2/(Nk^{\epsilon})$ (see (2.3.7)). Thus, we see that

$$\begin{aligned}
\frac{C^2n_1}{q_1^2M_1} + \frac{Cn_1}{q_1} + M_1 &\ll \frac{n_1Nk^{\epsilon}}{q_1k^2} + \frac{Qn_1}{q_1} + M_0 \\
&\ll \frac{n_1Nk^{\epsilon}}{q_1k^2} + \frac{n_1k^{\epsilon}}{q_1}\sqrt{\frac{N}{\mathcal{T}}} + \frac{Q^2k^{2+\epsilon}}{N} \\
&\ll \frac{(n_1,r)Nk^{\epsilon}}{k^2} + (n_1,r)k^{\epsilon}\sqrt{\frac{N}{\mathcal{T}}} + \frac{k^{2+\epsilon}}{\mathcal{T}} \\
&\ll \frac{(n_1,r)k^{1+\epsilon}}{r^2} + \frac{(n_1,r)k^{3/2+\epsilon}}{r\mathcal{T}^{1/2}} + \frac{k^{2+\epsilon}}{\mathcal{T}} \ll \frac{k^{2+\epsilon}}{\mathcal{T}},
\end{aligned}$$

where we used $M_0 \ll Q^2k^{2+\epsilon}/N$, $Nr^2 \ll k^{3+\epsilon}$, $Q = k^{\epsilon}\sqrt{N/\mathcal{T}}$, $\mathcal{T} \ll k$ and $n_1/q_1 \leq (n_1,r)$. Thus we see that

$$\Omega_{\pm}^{\neq 0} \ll \frac{k^{\epsilon}r^2C^7(\mathcal{T}N)^{1/2}}{n_1^2q_1Q^2N} \times k^{\eta-1/3} \times \frac{k^{2+\epsilon}}{\mathcal{T}},.$$

On substituting it into $S_r(N)$, we arrive at

$$\begin{aligned}
&\sup_{C\ll Q}\frac{N^{5/3+\epsilon}}{Q\mathcal{T}r^{2/3}C^3}\sum_{\pm}\sum_{\frac{n_1}{(n_1,r)}\ll C} n_1^{1/3}\Theta^{1/2}\sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^{\infty}}\left(\frac{r^2C^7(\mathcal{T}N)^{1/2}}{n_1^2q_1Q^2N}\right)^{1/2} \times \frac{k^{5/6+\eta/2}}{\sqrt{\mathcal{T}}} \\
&\ll \sup_{C\ll Q}\frac{N^{5/3+\epsilon}}{Q\mathcal{T}r^{2/3}C^3}\frac{r(\mathcal{T}N)^{1/4}C^{7/2}}{Q\sqrt{N}}\sum_{n_1\ll Cr} n_1^{-2/3}\Theta^{1/2}\sum_{\frac{n_1}{(n_1,r)}|q_1|(n_1r)^{\infty}}\frac{1}{q_1^{1/2}} \times \frac{k^{5/6+\eta/2}}{\sqrt{\mathcal{T}}} \\
&\ll \sup_{C\ll Q}\frac{N^{5/3+\epsilon}}{Q\mathcal{T}r^{2/3}}\frac{r(\mathcal{T}N)^{1/4}C^{1/2}}{Q\sqrt{N}}\sum_{n_1\ll Cr}\frac{\sqrt{(n_1,r)}}{n_1^{7/6}}\Theta^{1/2} \times \frac{k^{5/6+\eta/2}}{\sqrt{\mathcal{T}}} \\
&\ll N^{1/2}k^{3/2-1/6+3\eta/4}.
\end{aligned}$$

Hence the lemma follows. □

2.6.3 Estimates for the error term

In this subsection, we give estimates for $S_r(N)$ corresponding to the non-generic case $n_2^\star N \ll k^\epsilon$ (see Lemma 2.2.1). Recall from (2.2.8) that, if $n_2^\star N = n_1^2 n_2 N/(q^3 r) \ll k^\epsilon$, then we have

$$\mathrm{S}_3 = q \sum_{\pm} \sum_{n_1 | qr} \sum_{n_2=1}^{\infty} \frac{\lambda_\pi(n_1, n_2)}{n_1 n_2} S\left(r\bar{a}, \pm n_2; qr/n_1\right) G_{\pm}\left(n_2^\star\right), \tag{2.6.13}$$

where $G_{\pm}(n_2^\star)$ is a integral transform given in (2.2.6). On plugging (2.6.13) and (2.2.12) in place of S_3 and S_2 respectively into (2.1.4) we arrive at

$$\begin{aligned} S_r(N) =& \frac{2\pi i^k N^{1-it}}{Q\mathcal{T}} \sum_{1\leq q\leq Q} \frac{1}{q} \sum_{\pm} \sum_{n_1|qr} \sum_{n_2 \ll \frac{q^3 r k^\epsilon}{n_1^2 N}} \frac{\lambda_\pi(n_1,n_2)}{n_1 n_2} \\ &\times \sum_{M\leq m\leq M_0} \lambda_f(m)\, \mathcal{C}_{\pm}(\ldots)\, \mathrm{I}_4(q, m, n_1^2 n_2) + O(k^{-2020}), \end{aligned} \tag{2.6.14}$$

where

$$\begin{aligned} \mathcal{C}_{\pm}(\ldots) :=& \sideset{}{^\star}\sum_{a \bmod q} S(r\bar{a}, \pm n_2; qr/n_1) e\left(\frac{\bar{a}m}{q}\right) \\ =& \sum_{d|q} d\mu\left(\frac{q}{d}\right) \sideset{}{^\star}\sum_{\substack{\alpha \bmod qr/n_1 \\ n_1\alpha \equiv -m \bmod d}} e\left(\pm\frac{\bar{\alpha}n_2}{qr/n_1}\right) \ll (n_1, q)\left(q + \frac{qr}{n_1}\right), \end{aligned} \tag{2.6.15}$$

and

$$\mathrm{I}_4(q, m, n_1^2 n_2) = \int_{\mathbb{R}} W(x/Q^\epsilon) \int_{\mathbb{R}} V\left(\frac{t}{\mathcal{T}}\right) g(q, x)\, \mathrm{I}_2(m,\, q,\, x)\, G_{\pm}\left(n_2^\star\right) \mathrm{d}t\, \mathrm{d}x,$$

with

$$\mathrm{I}_2(m,\, q,\, x) = \int_0^\infty U(y) y^{-it} e\left(\frac{-Nxy}{qQ}\right) J_{k-1}\left(\frac{4\pi\sqrt{mNy}}{q}\right) \mathrm{d}y,$$

and

$$\begin{aligned} G_{\pm}(n_2^\star) &= \frac{1}{2\pi i}\int_{(\sigma)} (n_2^\star)^{-s}\,\gamma_{\pm}(s)\tilde{g}(-s)\,\mathrm{d}s \\ &= \frac{N^{it}}{2\pi}\int_{-\infty}^{\infty}\frac{\gamma_{\pm}(\sigma+i\tau)}{(n_2^\star N)^{\sigma+i\tau}}\int_0^{\infty} e\left(\frac{z_1 N x}{qQ}\right) V\left(z_1\right) z_1^{-\sigma-i\tau+it}\,\frac{\mathrm{d}\,z_1}{z_1}\,\mathrm{d}\tau, \end{aligned} \tag{2.6.16}$$

where $\sigma > -1+\max\{-\Re(\alpha_1), -\Re(\alpha_2), -\Re(\alpha_3)\}$. On analysing the x and t-integral like in Lemma 2.4.1, we get the following restriction

$$|z_1 - y| \ll k^\epsilon q/Q\mathcal{T}.$$

Thus, on replacing z_1 by $y+u$ with $u \ll k^\epsilon q/Q\mathcal{T}$, we essentially arrive at

$$\mathrm{I}_4(q,m,n_1^2 n_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\gamma_{\pm}(\sigma+i\tau)}{(n_2^\star N)^{\sigma+i\tau}}\int_{\mathbb{R}} V\left(\frac{t}{\mathcal{T}}\right) N^{it}\int_{u\ll\frac{k^\epsilon q}{Q\mathcal{T}}} \mathrm{I}_u\,\mathrm{I}_5(...)\,\mathrm{d}u\,\mathrm{d}t\mathrm{d}\tau,$$

where

$$\mathrm{I}_u = \int_{\mathbb{R}} W(x/Q^\epsilon) g(q,x) e\left(\frac{Nxu}{qQ}\right)\,\mathrm{d}x,$$

and

$$\mathrm{I}_5(...) = \mathrm{I}_5(m,\,q,\,u,\tau) = \int_0^{\infty} U_{t,u,\tau}(y) y^{-i\tau} J_{k-1}\left(\frac{4\pi\sqrt{mNy}}{q}\right)\,\mathrm{d}y,$$

with $U_{t,u,\tau}(y) = U(y)y^{-\sigma}(1+u/y)^{-\sigma-i\tau+it}$. The analysis of the above integral is similar to $\mathrm{I}_{\pm}$ which is analysed in Lemma 2.4.5. Indeed, by the z_1-integral in (2.6.16), we observe that $\tau \asymp N|x|/qQ$. Now going back to $\mathrm{I}_{\pm}$, we note that $N|x|/qQ \asymp (N\tilde{N})^{1/3}/qr^{1/3}$ (see (2.2.4)). Thus $\tau \asymp (N\tilde{N})^{1/3}/qr^{1/3}$. Hence, in terms of the oscillations of the phase functions, both the integrals, $\mathrm{I}_5(...)$ and $\mathrm{I}_{\pm}$ are same. Thus using the stationary phase analysis as done in Lemma 2.4.5, we see that

$$\mathrm{I}_5(m,\,q,\,u,\tau) \ll \frac{k^\epsilon q^{1/2}}{(mN)^{1/4}}.$$

We now move the contour σ in (2.6.16) to the left up to $\sigma = -5/2$ passing through the poles given by

$$\frac{1+\sigma+i\tau+\alpha_i+\ell}{2} = 0 \iff \sigma+i\tau = -1-\alpha_i-\ell.$$

Thus, on treating the u and t-integral trivially, we get

$$\begin{aligned}\mathrm{I}_4(q,m,n_1^2n_2) \ll (n_2^\star N)^{5/2}\frac{k^\epsilon q^{3/2}}{Q(mN)^{1/4}}\int_{-\infty}^{\infty}|\gamma_\pm(-5/2+i\tau)|\mathrm{d}\tau \\ + \frac{k^\epsilon q^{3/2}}{Q(mN)^{1/4}}\sum_{\ell=0,1}\sum_{i=1}^{3}(n_2^\star N)^{1+\ell+\Re\alpha_i}.\end{aligned}$$

Now using the Stirling bound

$$|\gamma_\pm(-5/2+i\tau)| \ll (1+|\tau|)^{3(-5/2+1/2)} = (1+|\tau|)^{-6},$$

we arrive at

$$\mathrm{I}_4(q,m,n_1^2n_2) \ll \frac{k^\epsilon q^{3/2}}{Q(mN)^{1/4}}\left((n_2^\star N)^{5/2} + \sum_{\ell=0,1}\sum_{i=1}^{3}(n_2^\star N)^{1+\ell+\Re\alpha_i}\right).$$

Note that $(n_2^\star N)^{5/2} = (n_2^\star N)^{1/2+2} \ll k^\epsilon (n_2^\star N)^{1/2}$, and

$$\sum_{i=1}^{3}(n_2^\star N)^{1+\ell+\Re\alpha_i} = \sum_{i=1}^{3}(n_2^\star N)^{1/2+\beta_i} \ll k^\epsilon (n_2^\star N)^{1/2},$$

as $1+\ell+\Re\alpha_i = 1/2+\beta_i$ for some $\beta_i > 0$. Thus we get

$$\mathrm{I}_4(q,m,n_1^2n_2) \ll \frac{k^\epsilon q^{3/2}}{Q(mN)^{1/4}}(n_2^\star N)^{1/2} = \frac{k^\epsilon (n_1^2n_2)^{1/2}N^{1/4}}{Qm^{1/4}r^{1/2}}. \tag{2.6.17}$$

Thus, on plugging the above bound, the bound (2.6.15) for $\mathcal{C}_\pm(\ldots)$ and $|\lambda_f(m)| \ll m^\epsilon$ into (2.6.14), we arrive at

$$S_r(N) \ll \sum_{1\le q\le Q}\frac{N^{5/4}M_0^{3/4}}{Q^2\mathcal{T}r^{1/2}}\sum_{n_1|qr}\sum_{n_2\ll\frac{q^3rk^\epsilon}{n_1^2N}}\frac{|\lambda_\pi(n_1,n_2)|}{n_1n_2}n_1n_2^{1/2}\left(1+\frac{r}{n_1}\right). \tag{2.6.18}$$

We estimate the sum over n_1 and n_2 as follows:

$$\begin{aligned}
&\sum_{n_1|qr} \sum_{n_2 \ll \frac{q^3 r k^\epsilon}{n_1^2 N}} \frac{|\lambda_\pi(n_1,n_2)|}{n_1 n_2} (n_1^2 n_2)^{1/2} \left(1 + \frac{r}{n_1}\right) \\
&\ll \sum_{n_1|qr} \sum_{n_2 \ll \frac{q^3 r k^\epsilon}{n_1^2 N}} |\lambda_\pi(n_1,n_2)| \frac{r}{\sqrt{n_2}} \\
&\ll \left(\sum\sum_{n_1^2 n_2 \ll k^\epsilon q^3 r/N} |\lambda_\pi(n_1,n_2)|^2 \right)^{1/2} \left(\sum_{n_1|qr} \sum_{n_2 \ll \frac{q^3 r k^\epsilon}{n_1^2 N}} \frac{r^2}{n_2} \right)^{1/2} \\
&\ll \frac{k^\epsilon q^{3/2} r^{3/2}}{\sqrt{N}}.
\end{aligned}$$

Hence we get

$$S_r(N) \ll k^\epsilon \sum_{1 \leq q \leq Q} \frac{N^{5/4} M_0^{3/4}}{Q^2 \mathcal{T} r^{1/2}} \frac{q^{3/2} r^{3/2}}{\sqrt{N}} \ll \frac{k^{3/2+\epsilon} N r}{\mathcal{T}^2} \ll \sqrt{N} k^{1+2\eta+\epsilon}, \tag{2.6.19}$$

where we used $M_0 \ll k^{2+\epsilon}/\mathcal{T}$ and $Nr^2 \ll k^{3+\epsilon}$.

Finally pulling together the bounds from Lemma 2.5.1, Lemma 2.6.2, Lemma 2.6.3 and the bound for the error term (2.6.19) we get Proposition 2.0.1.

Bibliography

[1] R. Acharya; Sumit Kumar; G. Maiti; S. K. Singh: *Subconvexity for $GL(2)$ L-functions in t-aspect*, Acta Arithmetica **194** (2020), no. 2, 111–133.

[2] R. Acharya; P. Sharma; S. K. Singh: *t-aspect subconvexity for $GL(2) \times GL(2)$ L-functions*, Journal of Number Theory, **240** (2022), 296-324.

[3] K. Aggarwal: *A new subconvex bound for $GL(3)$ L-functions in the t-aspect*, Int. J. Number Theory, **17**(5), 1111–1138, 2021.

[4] K. Aggarwal; S. K. Singh: *Subconvexity bound for $GL(2)$ L-functions: t-aspect*. Mathematika, 67(1): 71–99, 2021.

[5] R. C. Baker; G. Harman; J. Pintz: *The difference between consecutive primes. II*, Proc. London Math. Soc. (3) **83** (2001), no. 3, 532–562.

[6] J. Bernstein; A. Reznikov: *Periods, subconvexity of L-functions and representation theory*, J. Differential Geom. **13** (2005), 129-141.

[7] J. Bernstein; A. Reznikov: *Subconvexity bounds for triple L-functions and representation theory*, Ann. of Math. (2) **172** (2010), no. 3, 1679–1718.

[8] V. Blomer; G. Harcos; P. Michel: *Bounds for modular L-functions in the level aspect*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 5, 697–740.

[9] V. Blomer: *Subconvexity for twisted L-functions on $\mathrm{GL}(3)$*. Amer. J. Math., **135** (5), 1385–1421, 2012.

[10] V. Blomer; J. Buttcane: *On the subconvexity problem for L-functions on $GL(3)$*, Ann. Sci. Éc. Norm. Supér. (4) **53** (2020), no. 6, 1441–1500.

[11] V. Blomer; G. Harcos: *Hybrid bounds for twisted L-functions*, J. Reine Angew. Math. **621** (2008), 53–79; Addendum, ibid. **694** (2014), 241-244.

[12] V. Blomer; G. Harcos: *Twisted L-functions over number fields and Hilbert's eleventh problem*, Geom. Funct. Anal.(GAFA), **20** (2010), no. 1, 1–52.

[13] V. Blomer; S. Jana; P. Nelson: *The Weyl bound for triple product L-functions*, arXiv:2101.12106.

[14] V. Blomer; R. Khan; M. Young: *Distribution of Maass holomorphic cusp forms*, Duke Math. J. **162** (2013), 2609-2644.

[15] A. Booker; M. Krishnamurthy; M. Lee: *New integral representations for Rankin-Selberg L-functions*, arXiv:1804.07721.

[16] J. Bourgain: *Decoupling, exponential sums and the Riemann zeta function*, J. Amer. Math. Soc. **30** (2017), no. 1, 205–224.

[17] D. Burgess: *On character sums and L-series. II*, Proc. London. Soc. **13** (1963), 524-536.

[18] V. A. Bykovskii: *A trace formula for the scalar product of Hecke series and its applications*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **226** (1996), no. Anal. Teor. Chisel i Teor. Funktsii. 13, 14–36, 235–236.

[19] K. Chandrasekharan: *Arithmetical functions, Die Grundlehren der mathematischen Wissenschaften* , Band 167 Springer-Verlag, New York-Berlin 1970.

[20] J. Cogdell: *On sums of three squares*, Les XXIIèmes Journées Arithmetiques (Lille, 2001). J. Théor. Nombres Bordeaux **15** (2003), no. 1, 33–44.

[21] J. B. Conrey; A. Ghosh; S. M. Gonek: *Simple zeros of the zeta function of a quadratic number field. I*, Invent. Math. **86** (1986), no. 3, 563–576.

[22] B. Conrey; H. Iwaniec: *The cubic moment of central values of automorphic L-functions*, Ann. of Math. **151** (2000), 1175-1216.

[23] H. Davenport: *Multiplicative number theory, Third edition*, Revised and with a preface by Hugh L. Montgomery, Graduate Texts in Mathematics, **76**, 2000.

[24] W. Duke; J. Friedlander; H. Iwaniec: *Bounds for automorphic L-functions*, Invent. Math. **112** (1993), 1-8.

[25] W. Duke; J. Friedlander; H. Iwaniec: *Bounds for automorphic L-functions. II*, Invent. Math. **115** (1994), 219-239.

[26] W. Duke; J. Friedlander; H. Iwaniec: *Erratum: Bounds for automorphic L-functions. II*, Invent. Math. **140** (2000), 227–242.

[27] W. Duke; J. B. Friedlander; H. Iwaniec: *Bounds for automorphic L-functions. III*. Invent. Math. **143** (2001), no. 2, 221–248.

[28] W. Duke; J. B. Friedlander; H. Iwaniec: *The subconvexity problem for Artin L-functions*. Invent. Math. **149** (2002), no. 3, 489–577.

[29] W. Duke; H. Iwaniec: *Bilinear forms in the Fourier coefficients of half-integral weight cusp forms and sums over primes*. Math. Ann. **286**, 783–802 (1990).

[30] D. Goldfeld: *Automorphic forms and L-functions for the group $GL(n, \mathbb{R})$*, Cambridge Stud. Adv. Math. , **99**, Cambridge Univ. Press, Cambridge, 2006.

[31] A. Good: *The square mean of Dirichlet series associated with cusp forms*, Mathematika, **29** (2), 1982, 278-295.

[32] G. Harcos; P. Michel: *The subconvexity problem for Rankin-Selberg L-function and equidistribution of Heegner points. II*, Invent. Math. **163** (2006), 581-655.

[33] R. Holowinsky; R. Munshi: *Level aspect subconvexity for Rankin-Selberg L-functions*, Automorphic representations and L-functions, 311–334, Tata Inst. Fundam. Res. Stud. Math., **22**, Tata Inst. Fund. Res., Mumbai, 2013.

[34] R. Holowinsky; R. Munshi; Z. Qi: *Hybrid subconvexity bounds for $L(1/2, sym^2 f \otimes g)$*, Math. Z. **283** (2016), no. 1-2, 555-579.

[35] R. Holowinsky; P. D. Nelson: *Subconvex bounds on $gl3$ via degeneration to frequency zero*, Mathematische Annalen, **372**(1-2), 299–319, 2018.

[36] B. Huang: *On the Rankin–Selberg problem*, Math. Ann. **381**, 1217–1251 (2021).

[37] B. Huang: *Uniform bounds for* $GL(3)\times GL(2)$ L *-functions*, arXiv:2104.13025.

[38] M. N. Huxley: *On the difference between consecutive primes*, Invent. Math., **15** (1972), 164–170.

[39] M. N. Huxley: *Area, lattice points, and exponential sums*, London Mathematical Society Monographs. New Series, **13**, The Clarendon Press, Oxford University Press, New York, 1996. Oxford Science Publications.

[40] H. Iwaniec: *The spectral growth of automorphic* L*-functions*, J. reine angew. Math. **428** (1992), 139-159.

[41] E. Iwaniec: *Topics in Classical Automorphic Forms*, Graduate text in mathematics **17**, American Mathematical Society, Providence, RI, 1997.

[42] H. Iwaniec; E. Kowalski: *Analytic Number Theory*, American Mathematical Society Colloquium Publication **53**, American Mathematical Society, Providence, RI, 2004.

[43] M. Jutila: *Lectures on a Method in the Theory of Exponential Sums*, Tata Inst. Fund. Res. Lectures on Math. and Phys.,**80**, Springer, Berlin, 1987.

[44] M. Jutila: *Mean values of Dirichlet series via Laplace transforms*, in Analytic Number Theory (Kyoto, 1996), pp. 169-207. Cambridge Univ. Press, Cambridge, 1997.

[45] R. Khan: *On the subconvexity problem for* $\mathrm{GL}(3)\times\mathrm{GL}(2)$ L*-functions*, Forum Math. **27** (2015), no. 2, 897–913.

[46] H. Kim; P. Sarnak: *Refined estimates towards the Ramanujan and Selberg conjectures*, J. American Math. Soc. **16**, (2003), 175-181.

[47] H. H. Kim; F. Shahidi: *Functorial products for* $GL_2\times GL_3$ *and functorial symmetric cube for* $GL2$, C. R. Acad. Sci. Paris Ser. I Math. **331** (2000), no. 8, 599–604.

[48] E. Kowalski; P. Michel; J. Vanderkam: *Rankin-Selberg* L*-function in the level aspect*, Duke Math. J. **114** (2002), no. 1, 123–191.

[49] S. Kumar; K. Mallesham.; S. K. Singh: *Sub-convexity bound for $GL(3) \times GL(2)$ L-functions: $GL(3)$-spectral aspect*, https://arxiv.org/abs/2006.07819, 2020.

[50] S. Kumar; K. Mallesham; S. K. Singh: *Sub-convexity bound for $GL(3) \times GL(2)$ L-functions: the depth aspect*, Math. Z. (2022), https://doi.org/10.1007/s00209-022-02993-x.

[51] S. Kumar; R. Munshi; S. K. Singh: *Sub-convexity bound for $GL(3) \times GL(2)$ L-functions: Hybrid level aspect*, 2021, arXiv:2103.03552.

[52] E. Landau: *Über die ζ-Funktion und die L-Funktionen*, Math. Z. **20** , 105–125, 1924.

[53] R. E. Langer: *On the asymptotic solutions of ordinary differential equations, with an application to the Bessel functions of large order*, Transactions of the American Mathematical Society, Volume **33**, (1931), no. 1, 23-64.

[54] E. M. Lapid: *On the nonnegativity of Rankin-Selberg L-functions at the center of symmetry*, Int. Math. Res. Not. (2003), no. 2, 65–75.

[55] Y. Lau; J. Liu; Y. Ye: *A new bound $k^{2/3+\epsilon}$ for Rankin-Selberg L-functions for Hecke congruence subgroups*, IMRP Int. Math. Res. Pap. (2006), Art. ID 35090, 78. MR 2235495.

[56] Xiannan Li: *Upper Bounds on L-Functions at the Edge of the Critical Strip*, International Mathematics Research Notices, Volume 2010, Issue 4, 2010, Pages 727–755, https://doi.org/10.1093/imrn/rnp148.

[57] Xiaoqing Li: *The central value of the Rankin-Selberg L-functions*. Geom. funct. anal. **18** (2009), 1660-1695.

[58] Xiaoqing Li: *Bounds for $GL(3) \times GL(2)$ L-functions and $GL(3)$ L-functions*, Annals of Math. **173** (2011), 301-336.

[59] Y. Lin: *Bounds for twists of $GL(3)$ L-functions*, J. Eur. Math. Soc. (JEMS) **23**(6), 1899-1924, 2021.

[60] Y. Lin; P. Michel; W. Sawin: *Algebraic twists of* $GL_3 \times GL_2$ L*-functions* (2019). arxiv:1912.09473.

[61] Y. Lin; Q. Sun: *Analytic twists of* $GL_3 \times GL_2$ *automorphic forms*, Int. Math. Res. Not. IMRN 2021, no. 19, 15143–15208.

[62] P. Michel: *The subconvexity problem for Rankin-Selberg L-function and equidistribution of Heegner points*, Ann. of Math. **160** (2004), 185-236.

[63] P. Michel: Analytic number theory and families of automorphic L-function, IAS, Park City 2006.

[64] P. Michel; A. Venkatesh: *Heegner points and nonvanishing of Rankin-Selberg* L*-functions*. Analytic number theory, 169–183, Clay Math. Proc., **7**, Amer. Math. Soc., Providence, RI, 2007.

[65] P. Michel; A. Venkatesh: *The subconvexity problem for* $GL(2)$. Publ. Math. IHES **111** (2010), 171–280.

[66] D. Milićević: *Sub-Weyl subconvexity for Dirichlet* L*-functions to prime power moduli*, Compos. Math. **152** (2016), no. 4, 825-875.

[67] S. D. Miller; W. Schmid: *Automorphic distributions, L-functions, and Voronoi summation for* $GL(3)$, Ann. of Math. **164** (2006), 423-488.

[68] T. Meurman: *On the order of the Maass* L*-function on the critical line*, in Number Theory, Vol. I (Budapest, 1987), pp. 325-354. Colloq. Math. Soc. János Bolyai, 51. North-Holland, Amsterdam, 1990.

[69] R. Munshi: *Bounds for twisted symmetric square* L*-functions*, J. Reine Angew. Math., **682** 65–88, 2013.

[70] R. Munshi: *The circle method and bounds for* L*-functions-I*, Math. Ann., **358**(1- 2), 389–401, 2014.

[71] R. Munshi: *The circle method and bounds for* L*-functions-III:* t*-aspect subconvexity for* $GL(3)$ L*-functions*, Journal of American Mathematical Society, Volume **28**(4), 913-938, 2015.

[72] R. Munshi: *The circle method and bounds for L-functions-IV: subconvexity for twist of $GL(3)$ L-functions*, Ann. Math(2) **182**(2), 617-672, 2015.

[73] R. Munshi: *The circle method and bounds for L-functions- II: Subconvexity for twists of $GL(3)$ L-functions*, Amer. J. Math., **137**(3), 791–812, 2015.

[74] R. Munshi: *Twists of $GL(3)$ L-functions*, In: Müller W., Shin S.W., Templier N. (eds) Relative Trace Formulas. Simons Symposia. Springer, Cham (2021), https://doi.org/10.1007/978-3-030-68506-5 11.

[75] R. Munshi: *A note on Burgess bound*, Geometry, algebra, number theory, and their information technology applications, 273–289, Springer Proc. Math. Stat., 251, Springer, Cham, 2018.

[76] R. Munshi; S. K. Singh: *Weyl bound for p-power twist of $GL(2)$ L-functions*, Algebra Number Theory **13** (2019), no. 6, 1395–1413.

[77] R. Munshi: *Subconvexity for $GL(3) \times GL(2)$ L-functions in t-aspect*, J. Eur. Math. Soc, DOI: 10.4171/JEMS/1131.

[78] P. D. Nelson: *Bounds for standard L-functions*, 2021, arXiv:2109.15230.

[79] I. Petrow; M.P. Young: *The Weyl bound for Dirichlet L-functions of cube-free conductor*, Annals of Math., **192** (2020), no. 2, 437–486. 11M06 (11F66).

[80] I. Petrow; M.P. Young: *The fourth moment of Dirichlet L-functions along a coset and the Weyl bound*, arXiv: 1908.10346.

[81] D. Ramakrishnan: *Modularity of the Rankin-Selberg L-series, and multiplicity one for $SL(2)$*, Ann. of Math. (2) **152** (2000), no. 1, 45–111.

[82] R. Rankin: *The vanishing of Poincaré series*, Proc. Edin. Math. Soc. **23** (1980), no. 2, 151-161.

[83] P. Sarnak: *Estimates for Rankin-Selberg L-functions and quantum unique ergodicity*, J. Funct. Anal. **184** (2001), 419-453.

[84] P. Sharma: *Subconvexity for $GL(3) \times GL(2)$ twists in level aspect*, arxiv 2019.

[85] C. Raju: *Sub-convexity problem for Rankin-Selberg L-functions*, arXiv:1807.11092.

[86] Q. Sun; R. Zhao: *Bounds for GL_3 L-functions in depth aspect*, Forum Math., **31**(2), 303–318, 2019.

[87] E. C. Titchmarsh: *The theory of the Riemann Zeta-function (revised by D. R. Heath-Brown)*, Clarendon Press, Oxford (1986).

[88] A. Venkatesh: *Sparse equidistribution problems, period bounds and subconvexity*, Ann. of Math. (2) **172** (2010), no. 2, 989–1094.

[89] T. Watson: *Central Value of Rankin Triple L-Function for Unramified Maass Cusp Forms*, thesis, Princeton, 2001.

[90] H. Weyl: *Zur Abschätzung von $\zeta(1+it)$*, Math. Z. **10** (1921), 88-101.

[91] Zhilin Ye: *The second moment of Rankin-Selberg L-function and hybrid subconvexity bound*, arXiv preprint arXiv:1404.2336 (2014).

List of Publications

1. Sumit Kumar:

   Subconvexity bound for $\mathrm{GL}(3) \times \mathrm{GL}(2)$ L-functions in $\mathrm{GL}(2)$ spectral aspect.

   Preprint.