SIMULTANEOUS REDUCTION OF A PAIR OF QUADRATIC FORMS

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SUMMARY. Given two quadratic forms $Q_1 = x'Ax$ and $Q_2 = x'Bx$ one of which (say Q_2) is p.d., it is well known that both are simultaneously reducible to forms containing square terms only by a real non-singular transformation and also by contragredient transformations. In this paper, necessary and sufficient conditions are obtained for other cases such as (a) Q_1 arbitrary, Q_2 n.n.d., (b) Q_1 arbitrary, Q_2 non-singular, and (c) Q_1 and Q_2 both arbitrary.

For a pair of symmetric matrices A and B of the same order and B n.n.d., the paper introduces the concept of a proper eigen value and a proper eigen vector of A with respect to B. Such eigen vectors are shown to determine the required transformation for the simultaneous reduction of associated quadratic forms.

1. Introduction

Given two quadratic forms $Q_1 = x'Ax$ and $Q_2 = x'Bx$ one of which (say Q_2) is p.d., it is well known that both are simultaneously reducible to forms containing square terms only by a real non-singular transformation and also by contragredient transformations. In this paper, necessary and sufficient conditions are obtained for these results to hold true in respect of a pair of quadratic forms Q_1 and Q_2 none of which may be p.d.

The following notations will be used throughout the paper. For a matrix G of order $n \times k$ and rank r,

$R(G)$ represents the rank of G,

$\mathcal{M}(G)$ represents the linear space generated by the columns of G,

G^- represents a g-inverse as defined by Rao (1962, 1967), and

$G^\perp$ represents a matrix of order $n \times (n-r)$ of rank $(n-r)$ such that $G'G^\perp = 0$.

I_r represents a unit matrix of order r.

A matrix G is called semi-simple if it is similar to a diagonal matrix, i.e., there exists a matrix L such that $L^{-1}GL$ is diagonal. A quadratic form is said to be non-negative definite (n.n.d.) if it does not take negative values and positive definite (p.d.) if it takes only positive values.

The concept of proper eigen values and vectors of a symmetric matrix A with respect to a n.n.d. matrix B is developed in Section 4. It is shown that proper eigen vectors of A with respect to B determine the transformation for simultaneous reduction of the associated quadratic forms.

The results obtained in this paper on simultaneous reduction of quadratic forms Q_1, Q_2 are summarised as follows.

Q_1	Q_2	n.s. condition for reducibility: single transformation	n.s. condition for reducibility: contragredient transformations
arbitrary	n.n.d.	$R(AB^1) = R((B^1)'AB^1)$ (Theorem 3.2)	$R(AB) = R(BAB)$ (Theorem 3.4)
n.n.d.	n.n.d.	always possible (Theorem 3.3)	always possible (Theorem 3.5)
arbitrary	non-singular	AB^{-1} is semi-simple (Theorem 5.1)	AB is semi-simple (Theorem 5.3)
arbitrary	arbitrary	$R(AB^1) = R((B^1)'AB^1)$	$R(AB) = R(BAB)$
		$(A - AB^1[(B^1)'AB^1]^-(B^1)'A)B^-$ is semi-simple (Theorem 5.2)	AB is semi-simple (Theorem 5.4)

2. Some preliminary lemmas

We shall state and prove certain lemmas which are needed elsewhere in this paper, and which are also of some independent interest.

Lemma 2.1 : *If X and Y are matrices such that $R(AX) = R(YA) = R(YAX)$, then the matrix $AX(YAX)^- YA$ is invariant under the choice of the g-inverse. If $R(AX) = R(X'AX)$ and A is symmetric, so is $AX(X'AX)^-X'A$.*

Proof: Let $(YAX)_1^-$ and $(YAX)_2^-$ be two different choices of the g-inverse. Observe $AX(YAX)_1^- YAX(YAX)_2^- YA = AX(YAX)_1^- YA = AX(YAX)_2^- YA$ using Corollary 1a.3 of Mitra (1968) to note that $(YAX)_1^- Y$ and $X(YAX)_2^-$ are g-inverses of AX and YA respectively. Symmetry of $AX(X'AX)^- X'A$ follows from the fact that a symmetric matrix will always have at least one (possibly infinitely many) symmetric g-inverse.

Lemma 2.2 : *If $R(AX) = R(X'A) = R(X'AX)$ and P be a projection operator projecting arbitrary vectors onto $\mathcal{M}(X)$, then*

$$AX(X'AX)^- X'A = AP(PAP)^- PA.$$

Proof: Since $X'AX = X'X(X'X)^-X'AX(X'X)^-X'X$ (see Theorem 2b of Rao (1967))

$$R(X(X'X)^-X'AX(X'X)^-X') = R(X(X'X)^-X'AX) = R(X'AX).$$

Hence repeated application of Corollary 1a.3 of Mitra (1968) will show that $(X'X)^-X'[X(X'X)^-X'AX(X'X)^-X']^-X(X'X)^-$ is a g-inverse of $X'AX$. Lemma 2.2 follows from Lemma 2.1, once $X(X'X)^-X'$ is identified as the projection operator P (see Rao, 1967).

Lemma 2.3 : *Let B_r^- be a symmetric and reflexive g-inverse of a symmetric and n.n.d. matrix B. Then the following statements are true.*

(a) *B_r^- is n.n.d.*

(b) *If $B_r^- = YY'$ be a rank factorisation of B_r^-, then there exists a left inverse C of Y such that $C'C$ is a rank factorisation of B and conversely.*

(c) *$B_r^- = G'BG$ for some g-inverse G of B and conversely.*

Proof : (a) is trivial. For (b), consider a rank factorisation $B = D'D$ of B. Since $YY'D'DYY' = YY'$, $Y'D'DY = I_r$ where $r = R(B) = R(B_r^-)$. Hence $Y'D' = L$ where L is orthogonal. Then $C = LD$ is the required matrix for $Y'D'L' = I_r$ and $(D'L')\,(LD) = B$. The converse is easy. (c) and its converse follow from Theorem 2b of Mitra (1968).

Lemma 2.4 : *Let A be a symmetric matrix of order n, C of order $n\times r$ and X of order $r\times n$ such that $XC = I_r$. Then*

$$\mathcal{M}(AC) \subset \mathcal{M}(X') \Longleftrightarrow R(C'A) = R(C'AC).$$

Proof : The $\Longrightarrow$ part is trivial. To prove the $\Longleftarrow$ part we note that it suffices to show that the rank condition implies the existence of a $r\times n$ matrix Y such that $\mathcal{M}(AC) \subset \mathcal{M}(Y')$ and YC is non-singular for $X = (YC)^{-1}Y$ is a left inverse of C and $\mathcal{M}(X') = \mathcal{M}(Y')$.

If $R(C'A) = p$, choose p linearly independent rows of $C'A$. Let these form the rows of a matrix Y_1. Take $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ where Y_2 is of order $(r-p)\times n$ and is so chosen that YC is non-singular. For example let $\begin{pmatrix} Y_1 \\ I_n \end{pmatrix} C = W = \begin{pmatrix} W_1 \\ C \end{pmatrix}$ where clearly $R(W) = r$. Since $R(C'A) = R(C'AC) = p \Longrightarrow R(W_1) = p$, one can choose $r-p$ independent rows of C such that these rows together with the rows of W_1 determine a non-singular matrix of order r. The corresponding $r-p$ rows of I_n may be taken to constitute the rows of Y_2.

3. Simultaneous reduction of two quadratic forms one of which is n.n.d.

Let $X'AX$ and $X'BX$ be two quadratic forms where the matrices A and B are of order n and the rank of B is $r \leqslant n$. Let $B^\perp$ be a $n\times(n-r)$ matrix of rank $n-r$ such that $BB^\perp = 0$. First we consider the case where B is n.n.d.

Theorem 3.1 : *Let B be n.n.d. of order n and rank r. Then there exists a $r\times n$ matrix L of rank r such that $LBL' = I_r$, the identity matrix of order r, and LAL' is diagonal.*

Proof : Since B is n.n.d, there exists a $n\times r$ matrix C such that $B = CC'$. Let D of order $r\times n$ be a left inverse of C, and Λ be the diagonal matrix of eigen values of DAD'. There exists an orthogonal matrix M of order r such that $MDAD'M' = \Lambda$. If $L = MD$, then $LAL' = \Lambda$ and

$$LBL' = MDCC'DM' = MM' = I_r \qquad \dots \quad (3.1)$$

which proves the theorem.

Theorem 3.2 : *Let B be n.n.d. Then a necessary and sufficient condition that there exists a non-singular transformation T such that TAT' and TBT' are both diagonal is*

$$R(NA) = R(NAN') \qquad \dots \quad (3.2)$$

where N' is written for $B^{\perp}$ for simplicity of notation.

Proof : The necessity is easy to establish. To prove sufficiency, consider

$$S = \begin{pmatrix} L-GN \\ N \end{pmatrix} \qquad \dots \quad (3.3)$$

where L is as defined in Theorem 3.1, G is $r\times(n-r)$ matrix, and N is of order $(n-r)\times n$ and rank $n-r$. If the condition (3.2) is satisfied, then G can be chosen such that

$$SAS' = \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix} \qquad \dots \quad (3.4)$$

where E and F are symmetric matrices, while

$$SBS' = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Let M and Q be orthogonal matrices such that $MEM' = \Lambda_1$ and $QFQ' = \Lambda_2$ are diagonal, and choose

$$T = \begin{pmatrix} M & 0 \\ 0 & Q \end{pmatrix} S. \qquad \dots \quad (3.5)$$

Then it is easy to see that

$$TAT' = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}, \qquad TBT' = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

which proves the theorem.

Theorem 3.3 : *Let A and B be both n.n.d. matrices. Then, there always exists a non-singular transformation T such that TAT' and TBT' are both diagonal.*

Proof : It is easy to show that $\mathcal{M}(NA) = \mathcal{M}(NAN')$ when A and B are both n.n.d., so that the condition (3.2) is automatically satisfied.

Theorem 3.4 : *Let B be n.n.d. Then a n.s. condition for the existence of a non-singular matrix T suh that TAT' and $(T')^{-1}AT^{-1}$ are both diagonal (i.e., A and B are reducible by contragredient transformations) is*

$$R(BA) = R(BAB). \qquad \dots \quad (3.6)$$

Proof : Necessity is easy to establish. To prove sufficiency let $B = CC'$. It is easy to see that the condition $R(BA) = R(BAB) \Longrightarrow R(C'A) = R(C'AC)$ and hence by Lemma 2.4, a left inverse L of C exists such that $\mathcal{M}(AC) \subset \mathcal{M}(L')$. Consider the partitioned matrices

$$S = \begin{pmatrix} L \\ N \end{pmatrix}, \quad S^{-1} = (C \,:\, F)$$

so that $LF = 0$. Since $\mathcal{M}(AC) \subset \mathcal{M}(L')$, $LF = 0 \Longrightarrow C'AF = 0$. Then

$$SBS' = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

$$(S')^{-1}AS^{-1} = \begin{pmatrix} C'AC & 0 \\ 0 & F'AF \end{pmatrix}.$$

Let M and Q be orthogonal matrices such that $M'C'ACM$ and $Q'F'AFQ$ are diagonal. Then the required transformation is

$$T = \begin{pmatrix} M' & 0 \\ 0 & Q' \end{pmatrix} S.$$

Theorem 3.5 : *Let A and B be n.n.d. matrices. Then there always exists a non-singular transformation T such that both TBT' and $(T')^{-1}AT^{-1}$ are diagonal.*

Theorem 3.6 : *Whatever may be the choice of the transformation T in Theorem 3.4 satisfying the conditions*

$$TBT' = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \qquad (T')^{-1}AT^{-1} = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \quad \dots \quad (3.7)$$

the non-zero elements of Λ_1 are precisely the non-zero eigen values of the matrix BA.

Proof : The result is established by considering the product $TBT'(T')^{-1}AT^{-1} = TBAT^{-1}$.

The following theorem is well-known and is reproduced here for completeness of results on simultaneous reduction of quadratic forms.

Theorem 3.6 : *Let A and B be real symmetric matrices of the same order. Then there exists an orthogonal matrix L such that both $L'AL$ and $L'BL$ are diagonal, if and only if A commutes with B. In the latter case AB is symmetric and $L'ABL$ is also diagonal.*

4. Eigen values and vectors of a matrix with respect to n. n. d. matrix

Let A and B be two square symmetric matrices of which B is n.n.d. Let λ be a constant and w be a vector such that

$$Aw = \lambda Bw, \quad Bw \neq o. \quad \dots \quad (4.1)$$

Then λ is called a *proper* eigen value and w, a *proper* eigen vector of A with respect to B. It is seen that there may exist a vector w such that $Aw = Bw = o$, in which case the equation $Aw = \lambda Bw$ is satisfied for arbitrary λ. Such a vector w is called an improper eigen vector. The space of improper eigen vectors is precisely the intersection of the spaces $\mathcal{M}(A^{\perp})$ and $\mathcal{M}(B^{\perp})$. We shall now study the proper eigen values and vectors.

Theorem 4.1 : *Let $R(B) = r$ and $R(NAN') = R(NA)$ where N' is written for $B^{\perp}$. Then there are precisely r proper eigen values of A with respect to B, $\lambda_1, \lambda_2, \ldots, \lambda_r$ some of which may be repeated or null. Let $w_1, w_2, \ldots, w_r$ be the corresponding eigen vectors, each of which is standardised, i.e. $w_i' B w_i = 1$, $i = 1, 2, \ldots, r$, and W be the $n \times r$ matrix with the eigen vectors as columns. Then*

$$W'BW = I_r, \; W'AW = \Lambda_1, \quad W'AN' = 0 \qquad \ldots \quad (4.2)$$

where Λ_1 is a diagonal matrix of the eigen values $\lambda_1, \lambda_2, \ldots, \lambda_r$.

Proof : We make use of Theorem 3.2 which establishes, under the condition $R(NAN') = R(NA)$, the existence of a matrix T such that

$$TAT' = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} = U_1, \; TBT' = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} = U_2. \qquad \ldots \quad (4.3)$$

Let us consider the matrices U_1 and U_2. It is easy to see that the proper eigen values of U_1 with respect to U_2 are the diagonal values of Λ_1, which are r in number and the corresponding proper eigen vectors can be chosen as the columns of $\begin{pmatrix} I_r \\ 0 \end{pmatrix}$. Further, there are no improper eigen vectors if Λ_2 is of full rank. We choose

$$W = T' \begin{pmatrix} I_r \\ 0 \end{pmatrix} = (T_1' : T_2') \begin{pmatrix} I_r \\ 0 \end{pmatrix} = T_1'$$

so that $W'AW = \Lambda_1$, $W'BW = I_r$ and $W'AN' = 0$. Now,

$$U_1 \begin{pmatrix} I_r \\ 0 \end{pmatrix} = U_2 \begin{pmatrix} I_r \\ 0 \end{pmatrix} \Lambda_1$$

i.e.,

$$TAT' \begin{pmatrix} I_r \\ 0 \end{pmatrix} = TBT' \begin{pmatrix} I_r \\ 0 \end{pmatrix} \Lambda_1$$

$$TAW = TBW\Lambda_1 \Longrightarrow AW = BW\Lambda_1$$

which shows that Λ_1 is the matrix of proper eigen values of A with respect to B. Note that each column of BW is non-null, since $W'BW = I_r$, and the theorem is established.

Theorem 4.2 : *Let the condition $R(NAN') = R(NA)$ be satisfied. Then the non-null proper eigen values of A with respect to B are same as the non-null eigen values of $(A - AN'(NAN')^{-}NA)B^{-}$ and vice versa, for any choice of the g-inverses involved.*

Proof : Let $H = I - AN'(NAN')^{-}N$. Then $HAN' = 0 \Longrightarrow HA = DB$ for some matrix D. Also $DB = BD'$ since HA is symmetric by Lemma 2.1.

Let λ be a non-null eigen value of HAB^{-}. Then $HAB^{-}x = \lambda x \Longrightarrow NHAB^{-}x = \lambda Nx = \mathbf{o} \Longrightarrow x = Bz$ for some z. Choosing $w = H'B^{-}x$, we have

$$Bw = BH'B^{-}x = BB^{-}x = BB^{-}Bz = Bz = x$$

$$Aw = AH'B^{-}x = HAB^{-}x = \lambda x = \lambda Bw \neq \mathbf{o} \; ;$$

which shows that λ is a proper eigen value of A with repect to B.

To prove the converse, let λ be an eigen value of A with respect to B. Then $Aw = \lambda Bw$ for some vector w, and

$$\begin{aligned} HAB^{-}Bw = DBB^{-}Bw = DBw &= HAw \\ &= Aw - A'(NAN')^{-}N(\lambda Bw) \\ &= Aw = \lambda Bw \end{aligned}$$

which shows that λ is an eigen value of HAB^{-} with the corresponding eigen vector Bw.

Note that

$$(A - AN'(NAN')^{-}NA)B^{-} = (A - AP(PAP)^{-}PA)B^{-}$$

where P is the projection operator, $P = I - B(B^2)^{-}B$, so that we could have used P instead of N in the statement of Theorem 4.2.

Theorem 4.3 : *The necessary and sufficient condition that there are no improper eigen vectors is that the rank of NAN' is full.*

The result follows since there are no improper eigen values when Λ_2 in (4.3) has full rank.

Theorem 4.4 : *Let $R(NA) = R(NAN')$, W be the matrix of standardised proper eigen vectors of A with respect to B and Λ_1 be the corresponding diagonal matrix of proper eigen values. Then the transformation*

$$T = \begin{pmatrix} I_r & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} W' \\ N \end{pmatrix} \qquad \ldots \quad (4.4)$$

where M is the orthogonal matrix such that $MNAN'M'$ is diagonal provides the simultaneous reduction of A and B :

$$TAT' = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}, \quad TBT' = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \qquad \ldots \quad (4.5)$$

where Λ_2 is the matrix of eigen values of NAN'.

Theorem 4.4 which is simply a restatement of the results in Theorems 4.1 and 3.1, establishes the relationship between the transformation T and the proper eigen vectors of A with respect to B.

5. Simultaneous reduction of two arbitrary quadratic forms

In this section we establish the conditions under which two arbitrary quadratic forms can be simultaneously reduced by cogredient and contragredient transformations.

Theorem 5.1 :* *Let A and B be a pair of real symmetric matrices and B be non-singular. Then there exists a non-singular matrix T such that TAT' and TBT' are diagonal if and only if AB^{-1} is semi-simple.*

*Theorems 5.1 and 5.2 are due to C. G. Khatri. The authors wish to thank Khatri for allowing them to include his unpublished results in this paper. The proof of Theorem 5.1 is the same as that given by Khatri while the proof of Theorem 5.2 is materially different from and in some respects an improvement over the original proof of Khatri.

Proof: The 'only if' part is trivial. To prove the 'if' part we proceed as follows. Since AB^{-1} is semi-simple there exists a non-singular matrix L such that $L(AB^{-1})L^{-1}=D_1$ is diagonal. Write $LA=D_1LB$ and observe that $LAL'=D_1LBL'$, is symmetric. Therefore

$$D_1LBL'=(D_1\,LBL')'=LBL'D_1$$

since D_1 and LBL' are symmetric. The matrices LBL' and D_1 commute and by Thorem 3.6 there exists an orthogonal matrix M such that $MLBL'M'$ and MD_1M' are diagonal. Clearly $MLAL'M'=MLBL'M'MD_1M'$ is also diagonal.

Theorem 5.2 : *Let A and B be a pair of real symmetric matrices. Then a necessary and sufficient condition that there exists a non-singular transformation T such that TAT' and TBT' are both diagonal is*

$$R(NA)=R(NAN') \qquad \dots \quad (5.1)$$

$$(A-AN'(NAN')^-NA)B^- \qquad \dots \quad (5.2)$$

is semi-simple where N' is written for $B^{\perp}$.

Proof: Necessity is trivial. To prove sufficiency let us write $\Gamma=A-AN'(NAN')^-NA$. We note that since $\Gamma N'=0$, Γ is of the form JB. Hence if B^- be a g-inverse of B, $\Gamma B^-=JBB^-=JB(B^-)'BB^-=\Gamma B_r^-$ where B_r^- as in Lemma 2.3 represents a symmetric reflexive g-inverse of B. We may assume therefore that (5.2) holds true for some such g-inverse B_r^- of B. Since B is real and symmetric we can express B as

$$B=CDC' \qquad \dots \quad (5.3)$$

where $C'C=I_r$ and D is diagonal.

Let us write $B_r^-=Y\Lambda Y'$ where $Y'Y=I_r$ and Λ is diagonal. $BB_r^-B=B\Longrightarrow CDC'Y\Lambda Y'CDC'=CDC'\Longrightarrow C'Y\Lambda Y'CD=I_r\Longrightarrow C'Y\Lambda Y'C=D^{-1}\Longrightarrow\Lambda=(C'Y)^{-1}D^{-1}(Y'C)^{-1}\Longrightarrow B_r^-=Y(C'Y)^{-1}D^{-1}(Y'C)^{-1}Y'$. Writing $Z=Y(C'Y)^{-1}$

$$B_r^-=ZD^{-1}Z' \text{ with } C'Z=I_r. \qquad \dots \quad (5.4)$$

Now consider

$$S=\begin{pmatrix} Z'-GN \\ N \end{pmatrix}$$

where $G=Z'AN'(NAN')^-$ and observe that if (5.1) is satisfied

$$SAS'=\begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix}$$

where $E=Z'\Gamma Z$ and $F=NAN'$ are symmetric matrices while

$$SBS'=\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

since $Z'BZ=Z'CDC'Z=D$ and $NB=0$.

* See also Gantmacher (1959), Theorem 7, p. 53.

If ED^{-1} is semi-simple, then by Theorem 5.1 there exists a non-singular matrix M such that both MEM' and MDM' are diagonal. Let Q be an orthogonal matrix such that QFQ' is diagonal. Choose

$$T = \begin{pmatrix} M & 0 \\ 0 & Q \end{pmatrix} S$$

and check that TAT' and TBT' are diagonal. To complete the proof of Theorem 5.2 it remains therefore to show that (5.2) implies the semi-simplicity of ED^{-1}, which is true since if $\Gamma ZD^{-1}Z'$ is semi-simple,

$$\begin{pmatrix} Z' \\ N \end{pmatrix} \Gamma ZD^{-1}Z' \begin{pmatrix} Z' \\ N \end{pmatrix}^{-1} = \begin{pmatrix} Z' \\ 0 \end{pmatrix} \Gamma ZD^{-1}(I \quad 0) = \begin{pmatrix} Z'\Gamma ZD^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

is obviously semi-simple.

Theorem 5.3 : *Let A and B be a pair of real symmetric matrices and B be non-singular. Then there exists a non-singular matrix T such that TAT' and $(T')^{-1} BT^{-1}$ are diagonal if and only if AB is semi-simple.*

Proof : Theorem 5.3 follows as a simple consequence of Theorem 5.1.

Theorem 5.4 : *Let A and B be a pair of real symmetric matrices. Then a necessary and sufficient condition that there exists a non-singular transformation T such that TBT' and $(T')^{-1} AT^{-1}$ are both diagonal is that*

$$R(BA) = R(BAB) \qquad \dots \quad (5.5)$$

$$AB \textit{ is semi-simple.} \qquad \dots \quad (5.6)$$

Proof : Necessity is easy to establish. To prove sufficiency determine C as in (5.3) and note that

$$(5.5) \Longleftrightarrow R(C'A) = R(C'AC).$$

Hence proceeding as in the proof of Theorem 3.4 one can determine S such that

$$SBS' = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

$$(S')^{-1}AS^{-1} = \begin{pmatrix} C'AC & 0 \\ 0 & F'AF \end{pmatrix}.$$

Note that $(5.6) \Longrightarrow$

$$(S')^{-1}AS^{-1}SBS' = \begin{pmatrix} C'ACD & 0 \\ 0 & 0 \end{pmatrix}$$

is semi-simple. Hence $C'ACD$ is semi-simple. Since D is non-singular, by Theorem 5.3 there exists a non-singular matrix M such that MDM' and $(M')^{-1}\ C'ACM^{-1}$ are diagonal. Let Q be an orthogonal matrix such that $Q'F'AFQ$ is diagonal. Then the required transformation is

$$T = \begin{pmatrix} M & 0 \\ 0 & Q' \end{pmatrix} S$$

References

Gantmacher, F. R. (1959) : *Applications of the Theory of Matrices*, Translation by J. L. Brenner, Interscience Publishers, Inc., New York.

Mitra, S. K. (1968) : On a generalised inverse of a matrix and applications. *Sankhyā*, Series A, 30, 107-114.

Rao, C. R. (1962) : A note on a generalized inverse of a matrix with applications to problems in mathematical statistics. *J. Roy. Stat. Soc.*, B, 24, 152-158.

——— Calculus of generalized inverse of matrices, Part I : General theory. *Sankhyā*, Series A, 29, 317-342.

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