On Some Issues Of Stochastic Comparisons & Their Applications

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by

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under the guidance of

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Dedicated To

My Beloved Parents

Shri Ranjan Kumar Panja and Smt.Archana Panja

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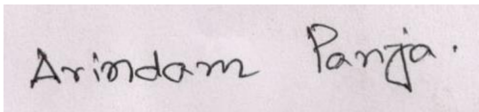

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Chapter 1

Introduction and a Brief Review of Literature

1.1 Introduction

One of the main objectives of statistics is the comparison of random quantities. These comparisons are mainly based on the comparison of some measures associated to these random quantities. For example, it is very common to compare two rvs in terms of their means, medians or variances. In some situations, comparisons based only on two single measures are not very informative. For instance, let us consider two rvs X and Y with respective cdfs $F(t) = 1 - e^{-t^2}$ and $G(t) = 1 - e^{-\frac{2}{\sqrt{\pi}}t}$ for all $t \geq 0$ respectively. Here we have $E[X] = E[Y] = \sqrt{\pi}/2$. If X and Y represent the random lifetimes of two devices, or the survival lifetimes of patients under two different treatments, then we would say that X has the same expected survival time than Y, if we just considered the mean values.However, if we took into account the probability of surviving at a fixed time $t \geq 0$, then $P[X \geq t] \leq P[Y \geq t]$ for all $t \in [0, 2/\sqrt{\pi}]$ and $P[X \geq t] \geq P[Y \geq t]$, for $t \in [2/\sqrt{\pi}, \infty)$.

Consequently here sfs provide more concrete information to compare these two rvs. The necessity of providing more detailed comparisons of two random quantities has motivated the development of the theory of stochastic orders, which has grown significantly during the last 50 years. Stochastic order refers to the comparisons of two random quantities in some stochastic sense. It is an important tool which has been used in many diverse areas of statistics, mathematics, economics, physics, biology and so on.

One of the most important area where stochastic orders studied extensively is reliability theory. Reliability is a popular concept that has been studied for decades as a commendable attribute of a living organism or a mechanical system. After the experience of second world war with complex military systems, the demand of reliability theory has grown up. An early application of the reliability theory could be found in the area of machine maintenance (cf. Barlow and Proschan [12]). Reliability theory describes different aspects of a system, namely, whether a system is good or bad, how long a system may survive, what the failure rate of a system is, etc. In reality, we deal with many different complex systems. In most of the cases, the basic structures of these systems match with those of the well known coherent systems (to be discussed later). Thus, the reliability study of a coherent system has received considerable attention from researchers/engineers. Indeed, one may be interested to know about the useful tools/theories developed so far to study the reliability of a system. Clearly, this is case-dependent. For example, if we want to know whether a system is good or bad, then the stochastic orders are useful tools for this; if we want to know how long a system will survive, then the mean residual life function is one which gives some idea about this. Thus, it is important to study different reliability tools/theories in order to study system reliability.

In reliability, stochastic orders are basically used to compare the lifetimes or the remaining lifetimes of two systems; in econometrics, these are used to compare different income inequalities from various random prospects; in biological sciences, these are used to know the effectiveness of a particular drug by comparing the lifetimes or residual lifetimes of the control group of living organisms. Like stochastic orders, stochastic ageing is also another important concept which has many applications in reliability theory. Different stochastic ageing properties describe how a system improves or deteriorates with age. Study of different closure properties of various ageing classes is one of the important problems in reliability theory. For example, if the components of a system have some ageing property, then it is important to study whether the corresponding system has also the same ageing property or not. Such a study is meaningful because this helps us to find out how the reliability of a system can be determined from knowledge of the reliabilities of its components.

Now-a-days, the systems which are used in industry, are very costly as well as complicated in nature. The failure of such systems may create a great monetary loss. Thus, the researchers want to find out new ways by which the system lifetime may be increased. One of the popular/useful ways to enhance the lifetime of a system is the allocation of redundant components into the system. Thus, the study of different allocation strategies is one of the frontier areas of research in reliability theory.

Another area where stochastic orders also applied extensively is actuarial science. One of the important problems in actuarial science is the study of comparing risks. Risk describe the potential loss of an individual or a company and defined by a rv X. Very often a decision makers has to choose an action given some uncertain alternative. For example an investor who wants to allocate his resources in different investment opportunities, an individual who has to decide weather he/she will buy a policy or not. Famous expected utility hypothesis states that for a rational decision maker there exist a utility function u such that Y will be more risky than X if and only if $E[u(X)] \leq E[u(Y)]$. However very often for every decision maker it is almost impossible to express the utility function explicitly. Therefore a natural question arises whether there any criteria available to compare two risk X and Y when there is only partial knowledge available about the corresponding utility function, say, u belongs to some prescribed class $\mathcal{F}$ of functions. Also there are situations where a group of decision makers with different utility functions, then a natural question can occur whether all the members of the group come to the same decision. Each of the above mention cases lead to the stochastic order relationship s.t.

$$X \preceq Y \text{ if } E[u(X)] \leq E[u(Y)] \text{ for all } u \in \mathcal{F}. \tag{1.1.1}$$

Clearly for different choices of u will lead to different stochastic orders.

1.2 Review of Literature

A detailed literature survey concerned with the problems studied in this thesis is given in this section. We divide this section into eleven subsections. The subsection 1.2.1 consists of notation, nomenclature, acronym and abbreviations. Some important measures, namely, failure rate, reverse hazard rate and mean residual life are discussed in subsection 1.2.3. In subsection 1.2.2, we give the definitions of different majorization. In subsection 1.2.3 Some important reliability measures are discussed. In subsection 1.2.4 different stochastic orders and their application have been discussed. Different stochastic ageing classes are discussed in subsection 1.2.5. In subsection 1.2.6 Copula theory is discussed. A brief discussion on the coherent system and its applications is given in subsection 1.2.7. The different kind of redundancies and their usefulness are discussed in subsection 1.2.9. In subsection 1.2.10, we discuss different type of claim amounts. In subsection 1.2.11 different semi-parametric models have been discussed.

1.2.1 Notation, Nomenclature, Acronym and Abbreviations

Below we give notation, nomenclature, acronyms, and abbreviations that will be used throughout the thesis.

Notation

X	underlying nonnegative rv.
$f_X(\cdot)$	probability density function of X.

$F_X(\cdot)$	cumulative cdf of X.
$\bar{F}_X(\cdot)$	survival (reliability) function of X.
$r_X(\cdot)$	hazard (failure) rate function of X.
$\tilde{r}_X(\cdot)$	reversed hazard (failure) rate function of X.
$\mu_X(\cdot)$	mean residual life function of X.
X_t	$(X - t \mid X > t)$.
$X_{(t)}$	$(t - X \mid X \leq t)$.
$\mathbf{X}$	An array of new components.
$\tau_{[n]}$	Lifetime of a coherent system having n number of components.
$\tau_{k:n}$	Lifetime of a k-out-of-n system.
$X \wedge Y$	$\min\{X, Y\}$.
$X \vee Y$	$\max\{X, Y\}$
$\phi_{\tau_{[n]}(\mathbf{x})}$	state of $\tau_{[n]}(\mathbf{X})$ at time t.
$\phi_{\tau_{k:n}(\mathbf{x})}$	state of $\tau_{k:n}(\mathbf{X})$ at time t.
$h_{[n]}(\cdot)$	Reliability function of $\tau_{[n]}$.
$h_{k:n}(\cdot)$	Reliability function of $\tau_{k:n}$.
$h_{[n]}(\mathbf{p})$	$h_{[n]}\,(p_1, p_2, \ldots, p_n)$, $0 < p_i < 1$, for all $i = 1, 2, \ldots, n$.
$h_{k:n}(\mathbf{p})$	$h_{k:n}\,(p_1, p_2, \ldots, p_n)$, $0 < p_i < 1$, for all $i = 1, 2, \ldots, n$.
$h_{[n]}(p)$	$h_{[n]}(\mathbf{p})$ whenever $p_i = p$, for all $i = 1, 2, \ldots, n$.
$h_{k:n}(p)$	$h_{k:n}(\mathbf{p})$ whenever $p_i = p$, for all $i = 1, 2, \ldots, n$.

Acronym and Abbreviations

cdf	cumulative distribution function.
sf	survival (reliability) function.
pdf	probability distribution function.
st	usual stochastic.
sp	stochastic precedence.
hr	hazard rate.
rhr	reversed hazard rate.
lr	likelihood ratio.
hr↑	up shifted hazard rate.
hr↓	down shifted hazard rate.
rhr↑	up shifted rhr.
lr↑	up shifted likelihood ratio.
lr↓	down shifted likelihood ratio.
disp	dispersive.
ILR	increasing likelihood ratio.
DLR	decreasing likelihood ratio.
IFR	increasing failure rate.
DFR	decreasing failure rate.
IFRA	increasing in failure rate average.
DFRA	decreasing in failure rate average.

NBU	new better than used.
DMRL	decreasing mean residual life.
IMRL	increasing mean residual life.

Nomenclature

$\bar{F}_X(\cdot)$	$1 - F_X(\cdot)$.
$k(t)$	first derivative of $k(t)$ with respect to t.
$k(t)$	second derivative of $k(t)$ with respect to t.
$k(t)$	third derivative of $k(t)$ with respect to t.
$a \overset{\text{sign}}{=} b$	a and b have the same sign.
$a \overset{\text{def.}}{=} b$	b is defined as a.
$X =_{st} Y$	X and Y have the same distribution.
$X =_{sp} Y$	$P(X > Y) = P(Y > X)$.
$F_X^{-1}(y)$	$\inf\{x : F_X(x) \geq y\}$.
$\mathbb{R}$	$\{x : -\infty < x < \infty\}$.
$\log(x)$	logarithm of x with base e.
iid	independent and identically distributed.
increasing	non–decreasing.
decreasing	nonincreasing.

We use the convention of $a/0$ to be equal to ∞ whenever $a > 0$.

1.2.2 Preliminaries

Given a vector $\boldsymbol{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, denote $x_{(1)} \leq x_{(2)} \leq ... \leq x_{(n)}$ as increasing arrangement of $x_1, x_2, \ldots, x_n$.

Definition 1.2.1. *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$ be any two vectors.*

(i) The vector $\boldsymbol{x}$ is said to majorize the vector $\boldsymbol{y}$, i.e., $\boldsymbol{x}$ is larger than $\boldsymbol{y}$ in majorization order (denoted as $\boldsymbol{x} \overset{m}{\succeq} \boldsymbol{y}$) if (cf. [96])

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}, \text{ for all } j = 1,\ 2,\ \ldots, n-1, \text{ and } \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}.$$

(ii) The vector $\boldsymbol{x}$ is said to weakly supermajorize the vector $\boldsymbol{y}$, denoted as $\boldsymbol{x} \overset{w}{\succeq} \boldsymbol{y}$ if (cf. [96])

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}, \text{ for all } j = 1, 2, \ldots, n.$$

(iii) *The vector $\boldsymbol{x}$ is said to weakly supermajorize the vector $\boldsymbol{y}$, denoted as $\boldsymbol{x} \succeq_w \boldsymbol{y}$ if (cf. [96])*

$$\sum_{i=j}^{n} x_{(i)} \geq \sum_{i=j}^{n} y_{(i)}, \text{ for all } j = 1, 2, \ldots, n-1.$$

(iv) *The vector $\boldsymbol{x}$ is said to be p-larger than the vector $\boldsymbol{y}$ (denoted as $\boldsymbol{x} \overset{p}{\succeq} \boldsymbol{y}$) if (cf. [25])*

$$\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)}, \text{ for all } j = 1, 2, \ldots, n.$$

It can be seen that

$$\mathrm{x} \overset{m}{\succeq} \mathrm{y} \Rightarrow \mathrm{x} \overset{w}{\succeq} \mathbf{y} \Rightarrow \mathrm{x} \overset{p}{\succeq} \mathbf{y}.\square$$

Definition 1.2.2. *(cf. [96]) Let A and B be two $m \times n$ matrices. Further let $\mathbf{a_1^R}, \ldots \mathbf{a_m^R}$ and $\mathbf{b}_1^R, \ldots \mathbf{b}_m^R$ are the rows of A and B respectively, so that each of these quantities is a row vectors of length n. Then A is said to*

(i) *row majorize B (denoted by $A >^{row} B$) if $\mathbf{a}_i^R \overset{m}{\succeq} \mathbf{b}_i^R$, $i = 1, \ldots, m$.*

(ii) *row weak majorize B (denoted by $A >^{w} B$) if $\mathbf{a}_i^R \overset{w}{\succeq} \mathbf{b}_i^R$, $i = 1, \ldots, m$.*

(iii) *chain majorize B (denoted by $A \gg B$) if there exists a finite number of $n \times n$ T-transform matrices, $T_1, T_2, \ldots, T_k$ such that $B = AT_1T_2 \ldots T_k$.*

Any T-transform matrix has the form

$$T = \lambda I + (1 - \lambda)Q$$

where $0 \leq \lambda \leq 1$ and Q is a permutation matrix that just interchanges two coordinates. It is to be noted that $A \gg B \Rightarrow A >^{row} B \Rightarrow A >^{w} B$.

Lemma 1.2.1. *[96] Let $I \subseteq \mathbb{R}$ be an open interval and let $\zeta : I^n \to \mathbb{R}$ be continuously differentiable. Necessary and sufficient conditions for ζ to be Schur-convex (resp. Schur-concave) on I^n are that ζ is symmetric on I^n, and for all $i \neq j$,*

$$(u_i - u_j)\left(\zeta_{(i)}(\boldsymbol{u}) - \zeta_{(j)}(\boldsymbol{u})\right) \geq (\text{resp.} \leq)\ 0 \text{ for all } \boldsymbol{u} = (u_1, u_2, ..., u_n) \in I^n,$$

where $\zeta_{(k)}(\boldsymbol{u}) = \partial\zeta(\boldsymbol{u})/\partial u_k$.

Lemma 1.2.2. *[96] Let $\mathcal{A} \subseteq \mathbb{R}^n$, and $\zeta : \mathcal{A} \to \mathbb{R}$ be a function. Then, for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}$,*

$$\boldsymbol{x} \overset{w}{\succeq} \boldsymbol{y} \implies \zeta(\boldsymbol{x}) \geq (\textit{resp.} \ \leq) \ \zeta(\boldsymbol{y})$$

if and only if ζ is both decreasing (resp. increasing) and Schur-convex (resp. Schur-concave) on $\mathcal{A}$.

Lemma 1.2.3. *[72] Let $\zeta : (0, \infty)^n \to \mathbb{R}$ be a function. Then,*

$$\boldsymbol{x} \overset{\mathrm{p}}{\succeq} \boldsymbol{y} \implies \zeta(\boldsymbol{x}) \geq (\textit{resp.} \leq) \ \zeta(\boldsymbol{y})$$

if and only if the following two conditions hold:

(i) $\zeta(e^{v_1}, \ldots, e^{v_n})$ is Schur-convex (resp. Schur-concave) in $(v_1, \ldots, v_n)$,
(ii) $\zeta(e^{v_1}, \ldots, e^{v_n})$ is decreasing (resp. increasing) in each v_i, for $i = 1, \ldots, n$,

where $v_i = \ln x_i$, for $i = 1, \ldots, n$.

In what follows, we introduce a notation. Let

$$\begin{aligned} \mathcal{U}_n &= \left\{ (\boldsymbol{x}, \boldsymbol{y}) = \begin{pmatrix} x_1 & \ldots & x_n \\ y_1 & \ldots & y_n \end{pmatrix} : x_i > 0, y_i > 0, \text{and} \right. \\ & \qquad \left. (x_i - x_j)(y_i - y_j) \geq 0, \forall\, i, j = 1, \ldots, n \right\}. \end{aligned}$$

$$\begin{aligned} \mathcal{S}_n &= \left\{ (\boldsymbol{x}, \boldsymbol{y}) = \begin{pmatrix} x_1 & \ldots & x_n \\ y_1 & \ldots & y_n \end{pmatrix} : x_i > 0, y_i > 0, \text{and} \right. \\ & \qquad \left. (x_i - x_j)(y_i - y_j) \leq 0, \forall\, i, j = 1, \ldots, n \right\}. \end{aligned}$$

Notation. Let us denote the following notations:

(i) $\mathcal{D} = \{(x_1, x_2, ..., x_n) : x_1 \geq x_2 \geq ... \geq x_n \geq 0\}$.
(ii) $\mathcal{D}_n^+ = \{(x_1, x_2, ..., x_n) : x_1 \geq x_2 \geq ... \geq x_n > 0\}$.
(iii) $\mathcal{E} = \{(x_1, x_2, ..., x_n) : 0 \leq x_1 \leq x_2 \leq ... \leq x_n\}$.
(iv) $\mathcal{E}_n^+ = \{(x_1, x_2, ..., x_n) : 0 < x_1 \leq x_2 \leq ... \leq x_n\}$.

Lemma 1.2.4. *[96] Let $\varphi : \mathcal{E} \to \mathbb{R}$ is continuously differentiable on the interior of $\mathcal{E}$. Then, for $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{E}$,*

$$\boldsymbol{x} \overset{m}{\succeq} \boldsymbol{y} \implies \varphi(\boldsymbol{x}) \geq (\textit{resp.} \ \leq) \ \varphi(\boldsymbol{y})$$

iff $\varphi_{(k)}(\boldsymbol{z})$ *is increasing (resp. decreasing) in* $k = 1, \ldots, n$*, where* $\varphi_{(k)} = \partial\varphi(\boldsymbol{z})/\partial z_k$ *denotes the partial derivative of* φ *with respect to its kth argument.*

Lemma 1.2.5. [96] *Let* $\varphi : S \to \mathbb{R}$ *be a function,* $S \subseteq \mathbb{R}^n$*. Then, for* $\boldsymbol{x}, \boldsymbol{y} \in S$*,*

$$\boldsymbol{x} \succeq_w \boldsymbol{y} \implies \varphi(\boldsymbol{x}) \geq (\textit{resp.} \ \leq) \ \varphi(\boldsymbol{y})$$

iff φ *is increasing (resp. decreasing) and Schur-convex (resp. Schur-concave) on* S*. Similarly,*

$$\boldsymbol{x} \overset{w}{\succeq} \boldsymbol{y} \implies \varphi(\boldsymbol{x}) \geq (\textit{resp.} \ \leq) \ \varphi(\boldsymbol{y})$$

iff φ *is decreasing (resp. increasing) and Schur-convex (resp. Schur-concave) on* S*.*

Lemma 1.2.6. *[57, 96] Let* $\varphi : \mathcal{D}(\mathcal{E}) \to \mathbb{R}$ *be a continuous function and continuously differentiable on the interior of* $\mathcal{D}\ (\mathcal{E})$*. Then*

$$\varphi(\boldsymbol{x}) \geq \varphi(\boldsymbol{y}) \textit{ whenever } \boldsymbol{x} \succeq_w \boldsymbol{y} \textit{ on } \mathcal{D}\ (\mathcal{E})$$

iff $\varphi_{(k)}(z)$ *is a non-negative decreasing (increasing) function in* k *for all* z *in the interior of* $\mathcal{D}\ (\mathcal{E})$*. Similarly,*

$$\varphi(\boldsymbol{x}) \geq \varphi(\boldsymbol{y}) \textit{ whenever } \boldsymbol{x} \overset{w}{\succeq} \boldsymbol{y} \textit{ on } \mathcal{D}\ (\mathcal{E})$$

iff $\varphi_{(k)}(z)$ *is a non-positive decreasing (increasing) function of* k *for all* z *in the interior of* $\mathcal{D}\ (\mathcal{E})$*.*

Lemma 1.2.7. *(cf. [96])*

(i) For all increasing convex function h, $x \preceq_w y \implies (h(x_1), h(x_2), ..., h(x_n)) \preceq_w (h(y_1), h(y_2), ..., h(y_n))$

(ii) For all increasing concave function h, $x \preceq^w y \implies (h(x_1), h(x_2), ..., h(x_n)) \preceq^w (h(y_1), h(y_2), ..., h(y_n))$

(iii) For all decreasing convex function h, $x \preceq^w y \implies (h(x_1), h(x_2), ..., h(x_n)) \preceq_w (h(y_1), h(y_2), ..., h(y_n))$

(iv) For all decreasing concave function h, $x \preceq_w y \implies (h(x_1), h(x_2), ..., h(x_n)) \preceq^w (h(y_1), h(y_2), ..., h(y_n))$

Lemma 1.2.8. *[43] For two n-dimensional Archimedean copulas* K_{φ_1} *and* K_{φ_2}*, if* $\phi_2 \circ \varphi_1$ *is superadditive, then* $K_{\varphi_1}(\boldsymbol{u}) \leq K_{\varphi_2}(\boldsymbol{u})$ *for all* $\boldsymbol{u} \in [0, 1]^n$*.*

Lemma 1.2.9. *(Chebyshevs inequality) If* $a_1, a_2, .., a_n; b_1, b_2, ..., b_n$ *be all real numbers,then*

(i) $\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right) \leq \left(\frac{1}{n}\sum_{i=1}^{n} a_i b_i\right)$ *if either* $a_1 \leq a_2 \leq ... \leq a_n$ *and* $b_1 \leq b_2 \leq ... \leq b_n$*, or* $a_1 \geq a_2 \geq ... \geq a_n$ *and* $b_1 \geq b_2 \geq ... \geq b_n$*;*

(ii) $\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right) \geq \left(\frac{1}{n}\sum_{i=1}^{n} a_i b_i\right)$ *if either* $a_1 \leq a_2 \leq ... \leq a_n$ *and* $b_1 \geq b_2 \geq ... \geq b_n$*, or* $a_1 \geq a_2 \geq ... \geq a_n$ *and* $b_1 \leq b_2 \leq ... \leq b_n$*.*

1.2.3 Some Important Measures in the Theory of Reliability

In this subsection we discuss some important measures which are very useful in reliability theory, namely, hazard rate, rhr, mean residual life and so on.

Definition 1.2.3. *Let* X *be a nonnegative rv. Then the sf of* X *is given by*

$$\bar{F}_X(x) = P(X > x),$$

- **Hazard rate (hr) function**

The lifetime of a system is completely characterized by its cdf. A realization of a lifetime is manifested by failure, death or some other end-event. So, the information on the probability of failure of an operating item in the next (usually sufficiently small) interval of time is really important in reliability analysis. The hr function is the one which measures this probability. The reliability analysts show their keen interest in studying this function. The hazard rate has variety of names in different fields, e.g. in extreme value theory, it is called the intensity function (cf. Gumbel [51]); in actuarial work, it is called the force of mortality. Sometimes it is also called age specific force of mortality and intensity of mortality (cf. Steffensen [126]). In statistics, its reciprocal for the normal distribution is called Mills ratio (cf. Barlow et al. [13]); in epidemiology, it is called the age specific failure rate, whereas in reliability theory we call it failure rate or hazard rate (cf. Barlow and Proschan [12]). Mathematically, hr function of a rv X is defined as

$$r_X(x) = \lim_{\Delta \to 0+} \frac{P(x < X \leq x + \Delta | X > x)}{\Delta},$$

If X is an absolutely continuous rv then $r_X(x)$ can be represented as

$$r_X(x) = \begin{cases} \frac{f_X(x)}{\bar{F}_X(x)}, & \text{if } \bar{F}_X(x) > 0 \\ \infty, & \text{if } \bar{F}_X(x) = 0. \end{cases}$$

Another important fact about the hazard rate function is that it uniquely determines the distribution by the relation

$$\bar{F}_X(x) = \exp\left\{-\int_0^x r_X(u)du\right\}.$$

The system reliability could be judged by the monotonic behavior of its hr function. For example, if the hr function is increasing, then the system is usually degrading in some suitable probabilistic sense. The different monotonic behavior of the hazard rate function such as increasing, decreasing, bathtub-shaped, reversed bathtub-shaped, roller-coaster, etc. have been extensively studied in the literature (cf.Lai and Xie [83]). To know more on the other properties of the hr function, we refer the reader to (Barlow and Proschan [12], Marshall and Olkin [95], and Finkelstein [46] among others).

- **Reversede hazard rate (rhr) function**

Like hr function, rhr function has also drawn attention due to its various useful properties in different areas of mathematics, statistics, economics and other related fields. It was first introduced by Mises [97] (as mentioned in Marshall and Olkin [95]), and was discussed briefly by Barlow *et al.* [13]. The rhr function has been shown to be used in modeling left-censored data by Andersen *et al.* [3], Sengupta and Nanda [121], and many other researchers. Andersen *et al.* [3] have stated that, in analyzing left-censored data, the rhr function plays the same role as the hr function plays in the analysis of right-censored data. In forensic science, it is used for estimating exact time of failure (death in case of human being) of a system. It can also be used in actuarial science, specially by the insurance companies, to decide on the premiums to be fixed for a new policy holder. For other applications of rhr function, one may refer to Sengupta and Nanda [120]. The rhr function of a rv X is defined as

$$\tilde{r}_X(x) = \lim_{\Delta\to 0+} \frac{P(x-\Delta < X \le x | X \le x)}{\Delta}.$$

If X is an absolutely continuous rv then $\tilde{r}_X(x)$ can be represented as

$$\tilde{r}_X(x) = \frac{f_X(x)}{F_X(x)}.$$

Like hr function, it also uniquely determines the distribution by the relation

$$F_X(x) = \exp\left\{-\int_x^\infty \tilde{r}_X(u)du\right\}.$$

Many different properties of the rhr function are studied by different researchers viz. Block *et al.* [19], Chandra and Roy [33], Sengupta and Nanda [121] and the references there in.

• Mean residual life (mrl) function

We have already discussed two important measures in the theory of reliability, namely, hr and rhr. Mean residual life is also another such important measure. It has huge applications in many diverse areas of statistics, physics, economics, biomedical science and related fields. In actuarial science, it is used to fix rates and benefits for life insurance; in biomedical science, it is used to analyze survivorship study. Demographers use it to study the human population. It is used to determine the optimal burn-in time which is an important screening method used in reliability theory. In replacement and repair strategies, although the shape of the hr function plays an important role, the mrl function is found to be more relevant than hr function. The hr is the instantaneous hr at any point of time whereas the mrl summarizes the entire residual life. Thus, the mrl has more intuitive appeal for modelling and analysis of failure data than the hazard rate. The mrl function of a rv X is defined as

$$\mu_X(x) = \begin{cases} E[X - x|X > x], & \text{for } x < x_0, \\ 0, & \text{otherwise,} \end{cases}$$

where $x_0 = \sup\{x : \bar{F}_X(x) > 0\}$. If X is an almost surely positive rv, then $\mu_X(0) = E(X)$. By the finiteness of $E(X)$ we have that $\mu_X(x) < \infty$, for all $x < \infty$. However, it is possible that $\lim_{x\to\infty} \mu_X(x) = \infty$. It is worth to mention here that

$$\mu_X(x) = \frac{1}{\bar{F}_X(x)} \int_x^\infty \bar{F}_X(u)du, \quad \text{when } x_0 = \infty.$$

Similar to the other measures discussed above, mrl function also uniquely determines the distribution by the relation

$$\bar{F}_X(x) = \frac{E(X)}{\mu_X(x)} \exp\left\{ - \int_0^x \frac{du}{\mu_X(u)} \right\} \quad \text{over } \{x : P(X > x) > 0\}.$$

In literature, many different properties of the mrl function are studied by different researchers, namely, Shaked and Shanthikumar [122], Marshall and Olkin [95], Finkelstein [46], Barlow and Proschan [12], Lai and Xie [83], and the references there in.

1.2.4 Stochastic Orders

Suppose we have two different systems, and we want to compare their reliabilities. Then the key question is − how to decide that one system is more reliable than the other? Stochastic orders are the effective solution to this problem. This is because once the cdfs of two lifetime rvs are known, stochastic orders use the complete information available regarding the underlying rvs through its distribution, whereas the other kind of comparison (say, in terms of means and/or variances) do not utilize the complete information as available with the distributions. So it is quite natural that stochastic orders will give better comparison than what is done in terms of means or variances. In literature many different types of stochastic orders have been defined. Each stochastic order has its individual importance. To study in details of these orders we refer the reader to Shaked and Shanthikumar [122], Kochar [78].

The concept of usual stochastic order was first introduced by Mann and Whitney [92]. This order compares the sfs of two distributions. The definition is given below.

Definition 1.2.4. *Let X and Y be two rvs. Then X is said to be smaller than Y in usual stochastic (st) order, denoted as $X \leq_{st} Y$, if*

$$\bar{F}_X(t) \leq \bar{F}_Y(t) \textit{ for all } t \in \ (-\infty, \infty).$$

This can equivalently be written as

$$F_X^{-1}(u) \leq F_Y^{-1}(u) \text{ for all } u \in (0,1).$$

There are many situations where more stronger concept than stochastic order is needed. For example as mentioned by Müller and Stoyan [102], suppose a person wants to buy a car and he/she has to choose between two types cars with different lifetimes X and X and Y. With the same price if $X \leq_{st} Y$ then he/she will choose the second one. But if someone wants to buy a used car which is two years old, with remaining lifetimes X' and Y',then he/she will decide according to the better remaining lifetimes i.e. $X' \leq_{st} Y'$ or $Y' \leq X'$. Here

$$P(X' > t) = P(X > t + 2|X > 2); \quad t \geq 0.$$

Now one may ask is the second type still better, i.e. $X' \leq_{st} Y'$? Unfortunately, this is not the case as shown in example 1.3.1 of Müller and Stoyan [102]. Consequently, more stronger assumption needed which ensures that usual stochastic order also holds for remaining lifetimes. i.e. when $[X|X > t] \leq_{st} [Y|Y > t] \quad \text{for all } t$?

i.e.

$$\begin{aligned} & P(X > s+t|X > t) \leq P(Y > s+t|Y > t) \quad \text{for all s} \geq 0 \text{ and all } t. \\ \Longrightarrow \quad & \frac{\bar{F}_Y(t)}{\bar{F}_X(t)} \leq \frac{\bar{F}_Y(s+t)}{\bar{F}_X(s+t)} \quad \text{for all } s \geq 0 \text{ and all } t. \end{aligned}$$

This is a motivation for hr order. Hazard rate order is the one which ensures the above fact. This order compares the hrs of two distributions. Below we give the definition of the hr order.

Definition 1.2.5. *Let X and Y be two absolutely continuous rvs with respective supports (l_X, u_X) and (l_Y, u_Y), where u_X and u_Y may be positive infinity, and l_X and l_Y may be negative infinity. Then X is said to be smaller than Y in hr order, denoted as $X \leq_{hr} Y$, if*

$$\frac{\bar{F}_Y(t)}{\bar{F}_X(t)} \text{ is increasing in } t \in (-\infty, max(u_X, u_Y)). \tag{1.2.1}$$

This can equivalently be written as

$$r_X(t) \geq r_Y(t), \text{ where defined.} \tag{1.2.2}$$

Further, we have that $X \leq_{hr} Y$ if, and only if the P-P plot is star-shaped w.r.t. $(1,1)$, i.e.

$$\frac{F_Y(F_X^{-1}(u)) - 1}{u-1} \text{ is increasing in } u. \tag{1.2.3}$$

rhr order is an another important stochastic order which is developed based on the concept of rhr function. It can obtained by replacing the sf by the cdf. This order was introduced by Keilson and Sumita [70]. The definition is given below.

Definition 1.2.6. *Let X and Y be two absolutely continuous rvs with respective supports (l_X, u_X) and (l_Y, u_Y), where u_X and u_Y may be positive infinity, and l_X and l_Y may be negative infinity. Then X is said to be smaller than Y in rhr order, denoted as $X \leq_{rhr} Y$, if*

$$\frac{F_Y(t)}{F_X(t)} \text{ is increasing in } t \in (\min(l_X, l_Y), \infty). \tag{1.2.4}$$

This can equivalently be written as

$$\tilde{r}_X(t) \leq \tilde{r}_Y(t). \tag{1.2.5}$$

Further, we have that $X \leq_{rhr} Y$ if, and only if the P-P plot is star-shaped w.r.t. $(0,0)$, i.e.

$$\frac{F_Y(F_X^{-1}(u))}{u} \text{ is increasing in } u. \tag{1.2.6}$$

It is to be noted that $X \leq_{hr} Y$ if and only if $[X|X > t] \leq_{st} [Y|Y > t]$ for all real t which actually compares lifetimes at the time t. Now if someone wants to know $[X|X \in A] \leq_{st} [Y|Y \in A]$ for all possible events A then this leads to the introduction of likelihood ratio order. Ross [117] introduced this useful stochastic order . This order is basically used as a sufficient condition for the above mentioned orders to hold.

Definition 1.2.7. *Let X and Y be two absolutely continuous rvs with respective supports (l_X, u_X) and (l_Y, u_Y), where u_X and u_Y may be positive infinity, and l_X and l_Y may be negative infinity. Then X is said to be smaller than Y in likelihood ratio (lr) order, denoted as $X \leq_{lr} Y$, if*

$$\frac{f_Y(t)}{f_X(t)} \textit{ is increasing in } t \in (l_X, u_X) \cup (l_Y, u_Y). \tag{1.2.7}$$

This is equivalent to the fact that

$$P(X \in B)P(Y \in A) \leq P(X \in A)P(Y \in B)$$

for all measurable sets A and B such that $A \leq B$, where $A \leq B$ means that for all $x \in A$ and $y \in B$, we have $x \leq y$ (cf. Müller [102]). Further, we have that $X \leq_{hr} Y$ if, and only if the P-P plot is convex i.e.

$$F_Y(F_X^{-1}(u)) \text{ is convex in } u. \tag{1.2.8}$$

Stochastic orders define till now compares size of the corresponding rvs. In many situation variability of the rv is of important. Suppose two rvs X and Y with same mean describes the return of two risky investments. Then to avoid risk any decision maker will choose the the one having less variability.

Also variability orders are useful for detecting the heterogeneity of a random sample with limited information. Suppose from the observe lifetimes of a "black box" parallel system someone want to determine whether the types of composing components are the same based on the available data (Kochar and Xu [77]). Variability ordering of heterogeneous and homogeneous systems will lead to the answer of this questions. Thus the variability ordering are of interest in the field of reliability and risk.

The two useful variability orders, namely, dispersive order and star order are discussed below.

Definition 1.2.8. *A rv X is said to be smaller than another rv Y in*

(a) *dispersive order, denoted as $X \leq_{disp} Y$, if*

$$F_X^{-1}(b) - F_X^{-1}(a) \leq F_Y^{-1}(b) - F_Y^{-1}(a) \text{ for all } 0 < a \leq b < 1.$$

(b) *star order (denoted by $X \leq_\star Y$) if $F_Y^{-1}(F_X(x))/x$ is increasing in $x \in \mathbb{R}_+$.*

The dispersive order is used to compare spread among the probability distributions. This order is sometimes called *tail order* (Jeon et al. [67], Kochar [75], Shaked and Shanthikumar [122]). Star order have been introduced in the literature to compare the skewness of probability distributions. The star order is also called more IFRA (increasing failure rate in average) order. If one rv is smaller than another in terms of star order, then this can be interpreted as the former rv ages faster than the later in the sense of the star ordering. For more discussion and applications see Barlow and Proschan [12] and Kochar [75].
Many applications of convolution operation are found in different areas of mathematics and engineering. It is of interest to know whether different stochastic orders are preserved under convolution. It is well known that the lr order is closed under convolution of independent rvs if the rvs under consideration have log-concave density functions. Shanthikumar and Yao [123] have introduced shifted lr order which is preserved under convolution without log-concavity condition. Later, Lillo et al. [90], and Di and Longobardi [39] have defined some other shifted stochastic orders. These orders are frequently used to study different stochastic inequalities. Many properties of these orders are studied by different authors, viz. Nakai [104], Belzunce et al. [16], Lillo et al. [90], Hu and Zhu [66] and the references there in. Below we give the formal definitions of shifted stochastic orders (Lillo et al. [90], Di and Longobardi [39], and Shanthikumar and Yao [123]).

Definition 1.2.9. *Let X and Y be two rvs with respective supports (l_X, u_X) and (l_Y, u_Y), where u_X and u_Y may be positive infinity, and l_X and l_Y may be negative infinity. Then X is said to be smaller than Y in*

1. *up shifted likelihood ratio (lr $\uparrow$) order, denoted as $X \leq_{lr\uparrow} Y$, if $X - x \leq_{lr} Y$, for all $x \geq 0$. This can equivalently be written as*

$$\frac{f_Y(t)}{f_X(t+x)} \text{ is increasing in } t \in (l_X - x, u_X - x) \cup (l_Y, u_Y),$$

for all $x \geq 0$*;([90, 122]).*

2. *down shifted likelihood ratio (lr* $\downarrow$*) order, denoted as* $X \leq_{lr\downarrow} Y$*, if* $X \leq_{lr} [Y - x|Y > x]$*, for all* $x \geq 0$*, or equivalently, if*

$$\frac{f_Y(t+x)}{f_X(t)} \text{ is increasing in } t \geq 0,$$

for all $x \geq 0$*;([90, 122]).*

3. *up shifted hazard rate (hr* $\uparrow$*) order, denoted as* $X \leq_{hr\uparrow} Y$*, if* $X - x \leq_{hr} Y$*, for all* $x \geq 0$*, which can equivalently be written as*

$$\frac{\bar{F}_Y(t)}{\bar{F}_X(t+x)} \text{ is increasing in } t \in (-\infty, u_Y),$$

for all $x \geq 0$ *([90]).*

4. *down shifted hazard rate (hr* $\downarrow$*) order, denoted as* $X \leq_{hr\downarrow} Y$*, if* $X \leq_{hr} [Y - x|Y > x]$*, for all* $x \geq 0$*, or equivalently, if*

$$\frac{\bar{F}_Y(t+x)}{\bar{F}_X(t)} \text{ is increasing in } t \geq 0,$$

for all $x \geq 0$ *([90]).*

5. *up shifted rhr (rhr* $\uparrow$*) order, denoted as* $X \leq_{rhr\uparrow} Y$*, if* $X - x \leq_{rhr} Y$*, for all* $x \geq 0$*, or equivalently, if*

$$\frac{F_Y(t)}{F_X(t+x)} \text{ is increasing in } t \in (l_X, \infty),$$

for all $x \geq 0$ *([39]).*

6. *down shifted rhr (rhr* $\downarrow$*) order, denoted as* $X \leq_{rhr\uparrow} Y$*, if* $X \leq_{rhr} [Y - x|Y > x]$*, for all* $x \geq 0$*, or equivalently, if*

$$\frac{F_Y(t+x)}{F_X(t)} \text{ is increasing in } t \in (l_X, \infty),$$

for all $x \geq 0$ *([39]).*

7. *up shifted mean residual life (mrl* $\uparrow$*) order, denoted as* $X \leq_{mrl\uparrow} Y$*, if* $X - x \leq_{mrl} Y$*, for all* $x \geq 0$*, or equivalently, if*

$$\frac{\int_{t+x}^{\infty} \bar{F}_Y(u)du}{\int_t^{\infty} \bar{F}_X(u)du} \text{ is increasing in } t \in (l_X, \infty),$$

for all $x \geq 0$ (Nanda et al. [106]).

8. *down shifted mean residual life (mrl $\downarrow$) order, denoted as $X \leq_{mrl\downarrow} Y$, if $X \leq_{mrl} Y - x$, for all $x \geq 0$, or equivalently, if*

$$\frac{\int_t^\infty \bar{F}_Y(u)du}{\int_{t+x}^\infty \bar{F}_X(u)du} \text{ is increasing in } t \in (l_X, \infty),$$

for all $x \geq 0$ (Nanda et al. [106]).

9. *up shifted mean inactivity time (mit $\uparrow$) order, denoted as $X \leq_{mit\uparrow} Y$, if $X - x \leq_{mit} Y$, for all $x \geq 0$, or equivalently, if*

$$\frac{\int_0^{t+x} F_Y(u)du}{\int_0^t F_X(u)du} \text{ is decreasing in } t \in (l_X, \infty),$$

for all $x \geq 0$ (Nanda et al. [105], Kayid et al. [69]).

10. *down shifted mean inactivity time (mit $\downarrow$) order, denoted as $X \leq_{mit\downarrow} Y$, if $X \leq_{mit} Y - x$, for all $x \geq 0$, or equivalently, if*

$$\frac{\int_0^x F_Y(u)du}{\int_0^{t+x} F_X(u)du} \text{ is decreasing in } t \in (l_X, \infty),$$

for all $x \geq 0$ (Nanda et al. [105], Kayid et al. [69]). □

The following diagram depicts relationship among the stochastic orders (cf. Shaked and Shanthikumar [122] and Lillo *et al.* [90]).

$$\begin{array}{ccccccc}
X \leq_{hr\uparrow} Y & \rightarrow & X \leq_{hr} Y & & & & X \leq_{icx} Y \\
\uparrow & & \uparrow & \searrow & & \nearrow & \\
X \leq_{lr\uparrow} Y & \rightarrow & X \leq_{lr} Y & & X \leq_{st} Y & & \\
\downarrow & & \downarrow & \nearrow & & \searrow & \\
X \leq_{rhr\uparrow} Y & \rightarrow & X \leq_{rhr} Y & & & & X \leq_{icv} Y
\end{array}$$

$$\begin{array}{ccccccc}
X \leq_{hr\downarrow} Y & \rightarrow & X \leq_{hr} Y & & & & X \leq_{icx} Y \\
\uparrow & & \uparrow & \searrow & & \nearrow & \\
X \leq_{lr\downarrow} Y & \rightarrow & X \leq_{lr} Y & & X \leq_{st} Y & & \\
 & & \downarrow & \nearrow & & \searrow & \\
 & & X \leq_{rhr} Y & & & & X \leq_{icv} Y
\end{array}$$

The diagram shows that the up (resp. down) shifted likelihood ratio order is the most strongest order whereas the increasing convex order and the increasing concave order are the weakest ones, and other orders lie between these orders. It is to be mentioned here that, in general, there is no relation between up and down shifted orders.

Clearly shifted stochastic orders are stronger than their respective usual versions of stochastic orders. Also, these shifted orders can be considered as generalization of their usual counterparts in some aspects. For instance, unlike lr ordering, shifted lr ordering preserves the order under convolution (Lillo et al. [90]). If $X \leq_{lr\uparrow} Y$, then $\kappa_X(t_1) \leq \kappa_Y(t_2)$ for $t_1 \geq t_2 \geq 0$, where $\kappa_X \equiv f/f$ and $\kappa_Y \equiv g/g$ (Lillo et al. [90]). Note that if $X \leq_{lr} Y$, then $\kappa_X(t) \leq \kappa_Y(t)$ for all $t \geq 0$. It is shown in Di and Longobardi [39] and Lillo et al. [90] that $X \leq_{hr\uparrow} Y \iff r_X(t_1) \geq r_Y(t_2)$ for $t_1 \geq t_2 \geq 0$. Note that $X \leq_{hr} Y$ implies $r_X(t) \geq r_Y(t)$ for all $t \geq 0$. Similarly, $X \leq_{rh\uparrow} Y \iff \tilde{r}_X(t_1) \leq \tilde{r}_Y(t_2)$ for $t_1 \geq t_2 \geq 0$ (Di and Longobardi [39]). Note that $X \leq_{rh} Y$ implies $\tilde{r}_X(t) \leq \tilde{r}_Y(t)$ for all $t \geq 0$. Also if $X \leq_{rh\uparrow} Y$, then $\bar{F}(t_1) \leq \bar{G}(t_2)$ for $t_1 \geq t_2 \geq 0$. If $X \leq_{mrl\uparrow} Y$, then $m_X(t_1) \leq m_Y(t_2)$ for $t_1 \geq t_2 \geq 0$, where $m_X(t) = \int_t^\infty \bar{F}(u)du \Big/ \bar{F}(t)$ is the mean residual life (mrl) of X (Nanda et al. [106]). If $X \leq_{mit\uparrow} Y$, $mit_X(t_1) \geq mit_Y(t_2)$ for $t_1 \geq t_2 \geq 0$, where $mit_X(t) = \int_0^t F(u)du \Big/ F(t)$ is known as mit (or reversed mean residual life) of X. Similar results are also shown for down shifted orders, e.g., if $X \leq_{hr\downarrow} Y$, then $r_X(t_1) \geq r_Y(t_2)$ for $t_2 \geq t_1 \geq 0$ (Lillo et al. [90]). Thus these shifted stochastic orders give us the flexibility that even at different points of time for the two variables, we can compare their hr, rhr, sf, mrl etc. One such specific instance is that we can compare the reliability of an used device and a new device using the shifted stochastic orders. For more discussion on those shifted orders including their applications and preservation properties, we refer to Aboukalam and Kayid [1], Naqvi et al. [108], Kayid et al. [69] and references therein.

1.2.5 Stochastic Ageings

Ageing describes how a unit ages with time. There are three types of ageing notions, namely, no ageing, positive ageing and negative ageing. By no ageing we mean that the age of a device has no effect on the distribution of the residual lifetime of the component. A component whose lifetime follows exponential distribution, has no ageing property. Positive ageing occurs when residual lifetime tends to decrease, in some probabilistic sense, with increasing age of a unit. Many different kinds of positive ageing classes viz. ILR, IFR, IFRA, DMRL etc. have been defined and discussed in Bryson and Siddiqui [27], Barlow and Proschan [12], Launer [84],Deshpande et al. [38],Loh [91], Klefsjö [74] and others. On the other hand, by negative ageing we mean that the residual lifetime of a component tends to increase, in some probabilistic sense, with increasing age of a component. The negative ageing is sometimes called anti-ageing or beneficial ageing. Besides the positive ageing classes, different negative ageing classes, namely, DLR, DFR, DFRA,IMRL etc. are also found in the literature. A more extensive discussion on this topic could be found in Lai and

Xie [83]. The following well known definitions of ageing classes may be obtained in Barlow and Proschan [12], Franco et al. [47], and Lai and Xie [83].

Definition 1.2.10. *Let X be an absolutely continuous rv. Then X is said to have*

1. *increasing likelihood ratio (ILR) (resp. decreasing likelihood ratio (DLR)) if*

$$f_X(t+x)/f_X(t) \text{ is decreasing (resp. increasing) in } t, \text{ for all } x \geq 0.$$

2. *increasing failure rate (IFR) (resp. decreasing failure rate (DFR)) if*

$$r_X(t) \text{ is increasing (resp. decreasing) in } t \geq 0.$$

3. *increasing failure rate in average (IFRA) (resp. decreasing failure rate in average (DFRA)) if*

$$\frac{1}{t}\int_0^t r_X(u)du \text{ is increasing (resp. decreasing) in } t > 0.$$

The interrelations among different ageing classes of a nonnegative rv, are given in the following flowcharts (Franco et al. [47], and Lai and Xie [83]).

$$\begin{array}{ccccc} \text{ILR} & \rightarrow & \text{IFR} & \rightarrow & \text{IFRA} \\ & & \downarrow & & \\ & & \text{DMRL} & & \end{array}$$

$$\begin{array}{ccccc} \text{DLR} & \rightarrow & \text{DFR} & \rightarrow & \text{DFRA} \\ & & \downarrow & & \\ & & \text{IMRL} & & \end{array}$$

1.2.6 Copula

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ have joint cdf F and joint sf $\bar{F}$. The marginal cdf and sf of X_i are F_i and $\bar{F}_i$, respectively, $i = 1, 2, \ldots, n$. If there exist C, $\bar{C} : [0,1]^n \mapsto [0,1]$ such that $F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$ and $\bar{F}(x_1, \ldots, x_n) = \bar{C}(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n))$ for all x_i, $i \in I_n$, then C and $\bar{C}$ are called the copula and survival copula respectively.

If $\varphi : [0, +\infty) \mapsto [0,1]$ with $\varphi(0) = 1$ and $\lim_{t\to+\infty} \varphi(t) = 0$, then $C(u_1, \ldots, u_n) = \varphi(\varphi^{-1}(u_1) + \ldots + \varphi^{-1}(u_n)) = \varphi(\sum_{i=1}^n \phi(u_i))$ for all $u_i \in (0,1]$, $i \in I_n$ is called an

Archimedean copula with generator φ provided $(-1)^k\varphi^{(k)}(t) \geq 0$, $k = 0, 1, \ldots, n-2$ and $(-1)^{n-2}\varphi^{(n-2)}(t)$ is decreasing and convex for all $t \geq 0$. Here $\phi = \varphi^{-1}$ is the right continuous inverse of φ so that $\phi(u) = \varphi^{-1}(u) = \sup\{t \in \mathbb{R} : \varphi(t) > u\}$. Archimedean copulas is a very important and widely used class of copulas, mainly because they are relatively easy to construct and study (we need only to find and study the generators) and a great variety of important families of copulas belong to this class (Nelsen [112]). For instance, Clayton family, Gumbel family, Frank family, Ali-Mikhail-Haq family, Gumbel-Hougaard family and Gumbel-Barnett family are in the families of Archimedean copulas (Nelsen [112]).

Lemma 1.2.10 (Navarro et al. [110]). *Let $T = \phi(X_1, .., X_n)$ be the lifetime of a coherent system based on possibly dependent components with lifetimes $X_1, .., X_n$, having a common reliability function $\bar{F}(t) = Pr(X_i > t)$. Then, the system sf can be written as*

$$\bar{F}_T(t) = h(\bar{F}(t)), \tag{1.2.9}$$

where h only depends on ϕ and on the survival copula of $X_1, .., X_n$.

Example 1.2.1. *Consider a system consisting of 3 components such that the system function if component 1 function and at least one of the components 2 and 3 function. Let X_1, X_2 and X_3 be the lifetimes of the components.The system lifetime is $T = \min(X_1, \max(X_2, X_3))$. Hence the minimal path sets are $\{1, 2\}, \{1, 3\}$ The sf of T can be written as*

$$\begin{aligned}
\bar{F}_T(t) &= Pr\left(\left\{X_{\{1,2\}} > t\right\} \cup \left\{X_{\{1,3\}} > t\right\}\right) \\
&= Pr\left(X_{\{1,2\}} > t\right) + Pr\left(X_{\{1,3\}} > t\right) - Pr\left(X_{\{1,2,3\}} > t\right) \\
&= \bar{F}(t, t, 0) + \bar{F}(t, 0, t) - \bar{F}(t, t, t) \\
&= K(\bar{F}(t), \bar{F}(t), 1) + K(\bar{F}(t), 1, \bar{F}(t)) - K(\bar{F}(t), \bar{F}(t), \bar{F}(t)) \\
&= h(\bar{F}(t)),
\end{aligned}$$

where

$$h(u) = K(u, u, 1) + K(u, 1, u) - K(u, u, u),$$

If K is exchangeable, $K(u, u, 1) = K(u, 1, u)$ and then $h(u) = 2K(u, u, 1) - K(u, u, u)$.

1.2.7 Coherent System

We frequently use the term system, although we do not briefly mention − what a system is. A system could be a mechanical system or it could be a living organism, for example, Radio, Car, TV, Airplane, etc. Roughly, we could think of a system that is formed by a collection of components, and they are connected in some fashion to create the whole. The basic principle of a system is that the failure or the survival of a system completely depends on the failure or the survival of its components. The functioning state of a system can be characterized in two ways − whether it is fully functioning or partially functioning. However, we consider only those systems which are either fully functioning or completely failed at any given point in time. This notion of two-state system was originally proposed and studied by Birnbaum et al. [18]. To identify the two states of a system, we assign a binary variable which takes value unity if the system is functioning, and zero if the system has failed.

Suppose $T(\mathbf{X})$ denote the lifetime of a system $\mathbf{X} = (X_1, X_2, \ldots, X_n)$. Further, let $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \in \{0, 1\}^n$ be the state vector of $\mathbf{X}$, where $x_i(t) = 1$ if the ith component is working at time t, and $x_i(t) = 0$ if it is not working at time t. Without any loss of generality, we write $\mathbf{x}$ in place of $\mathbf{x}(t)$, for mathematical simplicity, when there is no ambiguity. Suppose that the components are in some specific states. Then the question is − how to determine the state of the system? The mapping called *structure function*, denoted by $\xi_{T(\mathbf{x})}$, is the inter-link between the states of the components and that of the system, and is defined as

$$\xi_{T(\mathbf{x})} = \begin{cases} 1, & \text{if the system is functioning at time t} \\ 0, & \text{if the system has failed at time t.} \end{cases}$$

The reliability function of $T(\mathbf{X})$, denoted by $h_T(\cdot)$, is defined as the probability that it is working at time t. Thus,

$$h_T(t) = P(T(\mathbf{X}) > t) \quad = \quad P(\xi_{T(\mathbf{x})} = 1).$$

If the components are independent then the system reliability can be written as a function of component reliabilities, and hence

$$P(T(\mathbf{X}) > t) = h_T\left(\bar{F}_{X_1}(t), \bar{F}_{X_2}(t), \ldots, \bar{F}_{X_n}(t)\right).$$

Design engineers always like to design those type of systems which satisfy two basic requirements. Firstly, each of its components should have some importance to run the system. Secondly, if we replace a failed component by good one, then the system life must increase. On the basis of these two fundamental considerations, design engineers defined a system, called *coherent system*. Before discussing coherent system we give some basic definitions.

Definition 1.2.11. *The ith component of a system having structure function $\xi(\cdot)$ is said to be irrelevant if $\xi(\cdot)$ is constant in x_i, i.e., $\xi(1_i, \mathbf{x}) = \xi(0_i, \mathbf{x})$ for all $(\cdot_i, \mathbf{x})$, where $(\cdot_i, \mathbf{x}) = (x_1, x_2, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n)$.*

Definition 1.2.12. *The ith component of a system having structure function $\xi(\cdot)$ is said to be relevant if it is not irrelevant.*

Definition 1.2.13. *A structure function $\xi(\cdot)$ is said to be monotonically increasing if $\xi(\mathbf{x}) \leq \xi(\mathbf{y})$ whenever $\mathbf{x} \leq \mathbf{y}$, where the latter vector inequality is understood to be applied component-wise.*

Now we are in a position to define a coherent system.

Definition 1.2.14. *([12]) A system is said to be coherent if each of its components is relevant and its structure function is monotonically increasing.*

Let $\tau_{[n]}(\mathbf{X})$ be the lifetime of a coherent system formed by n independent components having lifetimes $\mathbf{X} = (X_1, X_2, \ldots, X_n)$. Then its structure function $\xi_{\tau_{[n]}(\mathbf{x})}$ is defined as

$$\xi_{\tau_{[n]}(\mathbf{x})} = \begin{cases} 1, & \text{if the coherent system having lifetime } \tau_{[\mathrm{n}]}(\mathbf{X}) \text{ is functioning} \\ 0, & \text{if the coherent system having lifetime } \tau_{[\mathrm{n}]}(\mathbf{X}) \text{ has failed,} \end{cases}$$

and its reliability function is given by

$$\begin{aligned} P(\tau_{[n]}(\mathbf{X}) > t) &= h_{[n]}(\bar{F}_{X_1}(t), \bar{F}_{X_2}(t), \ldots, \bar{F}_{X_n}(t)) \\ &= h_{[n]}(p_1, p_2, \ldots, p_n) \\ &= h_{[n]}(\mathbf{p}), \end{aligned}$$

where $p_i = \bar{F}_{X_i}(t),\ i = 1, 2, \ldots, n$. This $h_{[n]}(\mathbf{p})$ is sometimes called *distorted function* (Navarro and Spizzichino [111]),Navarro [109]); the corresponding distribution may be called *distorted distribution*. We write $h_{[n]}(p)$ in place of $h_{[n]}(\mathbf{p})$ whenever components are identically distributed.

Another well known system is k-out-of-n system which is a special type of coherent system. Many examples of k-out-of-n system are found in reality. An airplane which is capable of functioning if, and only if, at least two of its three engines function is an example of a 2-out-of-3 system.

Definition 1.2.15. *A system of n components is said to be a k-out-of-n system if and only if k of the n components function.*

Let $\tau_{k:n}(\mathbf{X})$ be the lifetime of a k-out-of-n system formed by n independent components $\mathbf{X} = (X_1, X_2, \ldots, X_n)$. Then its structure function is given by

$$\xi_{\tau_{k:n}(\mathbf{x})} = \begin{cases} 1, & \text{if } \sum_{i=1}^{n} x_i \geq k \\ 0, & \text{if } \sum_{i=1}^{n} x_i < k, \end{cases}$$

and its reliability function is given by

$$\begin{aligned} P(\tau_{k:n}(\mathbf{X}) > t) &= h_{k:n}(\bar{F}_{X_1}(t), \bar{F}_{X_2}(t), \ldots, \bar{F}_{X_n}(t)) \\ &= \sum_{j=0}^{n-k} \sum_{\{J:|J|=j\}} \left(\prod_{i \in J} (1 - \bar{F}_{X_i}(t)) \right) \left(\prod_{i \notin J} \bar{F}_{X_i}(t) \right), \end{aligned}$$

where J is any subset of $\{1, 2, .., n\}$ with at least k elements. $|J|$ is the cardinality of the set J. If $X_1, X_2, \ldots, X_n$ are iid rvs then $h_{k:n}(\cdot)$ can be written as

$$\begin{aligned} h_{k:n}(p) &= \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i} \\ &= \frac{n!}{(n-k)!(k-1)!} \int_0^p u^{k-1} (1-u)^{n-k} du, \quad \text{for } p \in (0, 1). \end{aligned}$$

The special cases of a k-out-of-n system are n-out-of-n system, known as series system and 1-out-of-n system, called parallel system. These systems are well studied in the literature by different researchers (cf. Barlow and Proschan [12], and Samaniego [118]).

Let $\tau_{n:n}(\mathbf{X})$ be the lifetime of a series system formed by n independent components $\mathbf{X} = (X_1, X_2, \ldots, X_n)$. Then its structure function is given by

$$\xi_{\tau_{n:n}(\mathbf{x})} = \min\{x_1, x_2, \ldots, x_n\} = \prod_{i=1}^{n} x_i,$$

and its reliability function is given by

$$\begin{aligned} P(\tau_{n:n}(\mathbf{X}) > t) &= h_{n:n}(\bar{F}_{X_1}(t), \bar{F}_{X_2}(t), \ldots, \bar{F}_{X_n}(t)) \\ &= \prod_{i=1}^{n} \bar{F}_{X_i}(t). \end{aligned}$$

Let $\tau_{1:n}(\mathbf{X})$ be the lifetime of a parallel (1-out-of-n) system formed by n independent components $\mathbf{X} = (X_1, X_2, \ldots, X_n)$. Then its structure function is given by

$$\begin{aligned}\xi_{\tau_{1:n}(\mathbf{x})} &= \max\{x_1, x_2, \ldots, x_n\} \\ &= 1 - \prod_{i=1}^{n}(1 - x_i),\end{aligned}$$

and its reliability function is given by

$$\begin{aligned}P(\tau_{1:n}(\mathbf{X}) > t) &= h_{1:n}(\bar{F}_{X_1}(t), \bar{F}_{X_2}(t), \ldots, \bar{F}_{X_n}(t)) \\ &= 1 - \prod_{i=1}^{n}(1 - \bar{F}_{X_i}(t)).\end{aligned}$$

Definition 1.2.16. *([76],[9]) A multiple-outlier model is a set of independent rvs $X_1, ..$ X_n of which $X_i \stackrel{st}{=} X, i = 1, ..., n_1$ and $X_i \stackrel{st}{=} Y, i = n_1 + 1, ..., n$ where $1 \leq n_1 < n$ and$X_i \stackrel{st}{=} X$ means that cdf of X_i is same as that of X. In other words, the set of independent rvs $X_1, .., X_n$ is said to constitute a multiple-outlier model if two sets of random variables $(X_1, .., X_{n_1})$ and $(X_{n_1+1}, .., X_{n_1+n_2})$ (where $n_1 + n_2 = n$), are homogeneous among themselves and heterogeneous between themselves.*

1.2.8 Order Statistics

Let $\{X_1, X_2, \ldots, X_n\}$ be a collection of rvs. If we arrange these rvs in an increasing order of magnitude, then there exists a unique order arrangement within $X_1, X_2, \ldots, X_n$. Suppose that $X_{1:n}$ denotes the smallest of $X_1, X_2, \ldots, X_n$; $X_{2:n}$ denotes the second smallest;... and $X_{n:n}$ denotes the largest. Then $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$, and these are collectively called the *order statistics* corresponding to the rvs $X_1, X_2, \ldots, X_n$. The kth smallest, $1 \leq k \leq n$, $X_{k:n}$ is called the kth order statistic. If $X_i, i = 1, 2, \ldots, n$ represents the lifetime of the ith component, then the reliability function of a k-out-of-n system formed by the components having lifetimes $X_1, X_2, \ldots, X_n$ is the same as that of the $(n - k + 1)$th order statistic $X_{n-k+1:n}$. Thus, to study a k-out-of-n system it is enough to study $X_{n-k+1:n}$, and vice versa. Different order statistics have different applications, for example, $X_{n:n}$ is of interest to study of floods and other extreme meteorological phenomena; $X_{1:n}$ is used in the survival analysis to measure the minimal survival time of a system; $X_{n:n} - X_{1:n}$ is a measure of dispersion, etc. In the literature, order statistics have been extensively studied in the case when the observations are independent and identically distributed. Due to the complicated expressions of the distributions in the non-identical case, only limited results are found in the literature. One may refer to David and Nagaraja [37],Balakrishnan and Rao [7], Balakrishnan and Rao [8] for results on the independent and non-identically distributed rvs. To know more on this topic, one may refer to Arnold et al. [4].

Let $X_1, X_2, \ldots, X_n$ be n independent rvs. Then the cumulative cdf of the kth order statistic is given by

$$F_{X_{k:n}}(t) = 1 - \sum_{j=0}^{k-1} \sum_{\{J:|J|=j\}} \left(\prod_{i \in J} F_{X_i}(t) \right) \left(\prod_{i \notin J} \bar{F}_{X_i}(t) \right).$$

If $X_1, X_2, \ldots, X_n$ are iid, then the above becomes

$$\begin{aligned} F_{X_{k:n}}(t) &= \sum_{i=k}^{n} \binom{n}{k} F_{X_1}^{k}(t) \bar{F}_{X_1}^{n-k}(t) \\ &= \frac{n!}{(n-k)!(k-1)!} \int_{0}^{F_{X_1}(t)} u^{k-1}(1-u)^{n-k} du, \end{aligned}$$

and the corresponding pdf is given by

$$f_{X_{k:n}}(t) = \frac{n!}{(n-k)!(k-1)!} F_{X_1}^{k-1}(t) \bar{F}_{X_1}^{n-k}(t) f_{X_1}(t).$$

1.2.9 Standby Component

It is an eternal truth that every system must collapse after certain time. For this reason, reliability engineers show their keen interest to find out different ways by which reliability of a system could be increased. Allocation of standby (also known as redundant or spare) component(s) into the system is an effective way to enhance the lifetime of a system. We may like to provide for each vital component of the system as many standby components as possible. Standby components are mostly of three types − hot (or active) standby, cold standby and warm standby. In hot standby, the original component and the redundant component work together under the same operational environment. In cold standby, the redundant component has zero hr when it is in inactive state. It starts to function under the usual environment (in which the system is running) only when the original component fails. On the other hand, warm standby describes an intermediate scenario. In warm standby, the redundant component undergoes two operational environments. Initially, it functions in a milder environment (in which a redundant component has non-zero failure rate which is less than its actual failure rate), thereafter it switches over to the usual environment after the original component fails. It might happen that the redundant component fails before switching over to the usual environment. Warm standby is sometimes called *general standby* because it contains both the hot standby and the cold standby as extreme cases. All the three types of redundancies are well studied in the literature cf. Barlow and Proschan [12], Brito et al. [26], Boland et al. [22], Boland et al. [23], Boland and El-Neweihi [21], She and Pecht [124], Sinha and Misra [125],Misra et al. [99], Misra et al. [100], Misra et al. [101]), Misra and Misra [98], Li and Hu [88], Valdés and Zequeira [128]) and the references there in.

Airplane engine is one of the examples of hot standby. Most of the small airplanes have three engines, and the airplane functions if at least two engines function. Here the third engine may be considered as the active redundancy of the airplane. Many examples of the cold standby are found in reality, for example, mobile batteries. Most of the mobiles (which are made in China) have two batteries − one battery is kept aside with the understanding that it will be used when the original battery stops to function. Here the battery kept aside is a cold standby of the system. As an application of warm standby, one may think of a situation where no non-zero lead time (the time between failure of a component and reinstate of a standby into the system) is allowed. To be specific, in an operation table if there is a power failure, non-zero lead time to get the lamp back to work cannot be allowed. For this, a switching and censoring device is used so that the shadowless lamp on the table does not get interrupted during operation in progress. As an another example, one may think of the players who are waiting outside the field while play is going on, with the understanding that if there is a need, a player from the pool may have to start playing. In this case the players in the pool cannot sit idle, rather they will keep on warming them up to get themselves activated as and when necessary.

1.2.10 Claim amounts

An insurance premium is the amount of money an individual or business pays for an insurance policy. Insurance premiums are paid for policies that cover healthcare, auto, home, and life insurance. Once earned, the premium is income for the insurance company. It also represents a liability, as the insurer must provide coverage for claims being made against the policy. Failure to pay the premium on the individual or the business may result in the cancellation of the policy. In this regard, the smallest and largest claim amounts can have an important role in insurance analysis since they provide useful information for determining suitable annual premium. Assume that $I_{p_1}, \ldots, I_{p_n}$ are independent Bernoulli rvs, independent of rvs X_is, with $E(I_{p_i}) = p_i$, $i = 1, \ldots, n$. where X_i represent total random claim that can be made during insurance policy. Associated with each X_i there is a Bernoulli rv I_{p_i} define as follows: $I_{p_i} = 1$ if i-th policy holder make claim and $I_{p_i} = 0$ if claim do not made by the policy holder. Then $X_i I_{p_i}$ represent the claim amount of a portfolio of risk. In actuarial science the vector $(X_1 I_{p_1}, X_2 I_{p_2}, \ldots, X_n I_{p_n})$ is called portfolio of risk. Now define

$X^*_{n:n} = \max(X^*_1, \ldots, X^*_n)$. In actuarial science, it represents the largest claim amount in a portfolio of risks (Barmalzan et al. [15], Balakrishnan et al. [10], Zhang et al. [136]). Again $\sum_{i=1}^n X_i I_{p_i}$ represents the aggregate claim amount for this portfolio of risks.

1.2.11 Different semi-parametric models

The Coxs PHR model is a popular and widely used semi-parametric model (Finkelstein [46]). A rv X is said to follow the PHR model, written as $X \sim PHR(\bar{F}, \lambda)$, if its sf can be expressed as

$$\bar{F}_X(t) = \bar{F}^\lambda(t), \tag{1.2.10}$$

where $\lambda(>0)$ is a constant and $\bar{F}(\cdot)$ is the baseline sf. From (1.2.10) it follows that $r_X(t) = \lambda r(t)$, where $r(\cdot)$ denotes the corresponding baseline hr function. Then the sf of X can be written as

$$\bar{F}_X(t) = e^{-\lambda R(t)}; \tag{1.2.11}$$

where $R(t) = \int_0^t r(u)du$ is the baseline cumulative hazard rate.

A rv X is said to follow the PRH model Gupta and Gupta [54], written as $X \sim PRH(\bar{F}, \lambda)$, if its cdf can be expressed as

$$F_X(t) = F^\lambda(t), \tag{1.2.12}$$

where $\lambda(>0)$ is a constant and $F(\cdot)$ is the baseline cdf. From (1.2.12) it follows that the rhr function of X is given by $\tilde{r}_X(t) = \lambda\tilde{r}(t)$, where $\tilde{r}(\cdot)$ denotes the corresponding baseline rhr function.

A r.v. X is said to follow the accelerated life (AL) model (also known as the scale model), if its cdf can be expressed as

$$F_X(t) = F(at), \tag{1.2.13}$$

where $a(>0)$ is a scale parameter and F is baseline distribution function. It is an well-known semi-parametric model and widely used in various applications (Finkelstein [46], Hazra et al. [61], Kochar and Torrado [79]).

Proportional odds (PO) model introduced by Bennett [17] is another very important model in reliability theory and survival analysis. Let X be rv with cdf $F_X(\cdot)$ and sf $\bar{F}_X(\cdot)$ and hr function $r_X(\cdot)$. The odds functions of X is defined by $\tau_X(t) = \bar{F}_X(t)/F_X(t)$. If the rv X represents lifetime of a component, then the odds function $\tau_X(t)$ represents the odds of surviving beyond time t. The rv X is said to satisfy PO model if $\tau_X(t) = \alpha\tau(t)$ for all admissible t, where τ is a baseline odds, i.e. odds function of the baseline variable, and α

is a proportionality constant known as proportional odds ratio. Then the sfs of X can be represented as

$$\bar{F}_X(t) = \frac{\alpha \bar{F}(t)}{1 - \bar{\alpha}\bar{F}(t)}, \tag{1.2.14}$$

where $\bar{\alpha} = 1 - \alpha$, and $\bar{F}(\cdot)$ is the corresponding baseline sf. That means X is following the PO model with baseline sf $\bar{F}(\cdot)$ denoted as $X \sim PO(\bar{F}, \alpha)$. For more interpretation and applications of the of PO model we refer to Collett [34], Kirmani and Gupta [73], Kundu et al. [81], Ross [117] and the references therein. Also, the model (1.2.14), with $0 < \alpha < \infty$, provides us a method of generating more flexible new family of distribution by introducing the parameter α to an existing family of distributions (Marshall and Olkin [94]). The family of distributions so obtained is known as Marshall-Olkin family of distributions (Cordeiro et al. [35], Marshall and Olkin [94]).

Let X be a rv with cdf F and sf $\bar{F} = 1 - F$, and Λ be a continuous rv with cdf H,pdf h and hr function r_F. A rv X^* is said to follow multiplicative frailty model with baseline distribution F and frailty rv Λ if its sf is given by

$$\bar{F}^*(t) = \int_0^\infty \bar{F}^\lambda(t) dH(\lambda.) \tag{1.2.15}$$

Here the frailty rv Λ serves as an unobserved random factor that modifies multiplicatively the underlying hr function r_F of an individual such that the individual is supposed to have hr $r_{F^*}(t)$ at age t, so that given $\Lambda = \lambda$, the conditional hr function of X^* will be $r_{F^*}(t|\lambda) = \lambda\, r_F(t)$, $t \geq 0$.

In analogy to the frailty model, to account for unobserved/unexplained heterogeneity in the rhrs of the experimental units, the resilience model (reversed frailty models) is introduced. A rv X^* is said to follow resilience model with baseline cdf F and resilience rv Λ if its cdf is given by

$$F^*(t) = \int_0^\infty F^\lambda(t) dH(\lambda). \tag{1.2.16}$$

1.3 A Brief Discussion on the Main Results of the Thesis

From the discussion given in the previous section, we have seen that stochastic orders and ageing properties have large applications in reliability theory, operations research, economics, actuarial science, biological science, forensic science, queuing theory, inventory, and related fields. In reliability theory, these are used to study different system reliabilities. Keeping the importance in mind, the present thesis is devoted to study different stochastic orderings and ageing properties, and their various applications in system reliability.

The thesis consists of seven chapters of which Chapter 1 is introductory. Here we discuss basic definitions, notations and a detailed survey of the literature related to the problems which are considered in the thesis. A brief discussion on the main results considered in Chapters 2-7 are presented below.

Chapter 2 : Stochastic comparisons of lifetimes of series and parallel systems

In this chapter, stochastic comparisons of series and parallel systems are discussed. First, we consider two series systems consisting of heterogeneous and dependent components with lifetimes following proportional odds models. Joint distribution of $X_1, X_2, .., X_n$ is modelled by Archimedean copula. Let $X_1, X_2, .., X_n$ be the lifetimes of the components of a series system. It is assumed that $X_i \sim PO(\bar{F}(x), \alpha_i), i = 1, ..., n$. Then the sf of the lifetime of the series system can be written as

$$\bar{F}_{X_{1:n}}(x) = P(X_{1:n} > x) = \varphi\left(\sum_{i=1}^{n} \xi\left(\bar{F}_{\alpha_i}(x)\right)\right) = S_1(\bar{F}(x), \boldsymbol{\alpha}, \varphi) \text{ (say)}, \tag{1.3.1}$$

where α_i is the odds of survival of X_i and φ is the generator of the Archimedean copula with $\xi(u) = \varphi^{-1}(u)$, $u \in (0, 1]$.

We establish sufficient conditions for usual stochastic ordering for the lifetimes of two heterogeneous and dependent series systems. With proper conditions on baseline rv and generator of the Archimedean copula it is established that one system dominates the other systems in the usual stochastic order under p-majorization of the odds vectors of the two systems.

Next, we consider the hr ordering of the lifetimes of two series systems. The hr function corresponding to system (1.3.1), can be written as

$$r_{X_{1:n}}(x) \quad = \quad r(x)\frac{\varphi\left(\sum_{i=1}^{n} \phi\left(\bar{F}_{\alpha_i}(x)\right)\right)}{\varphi\left(\sum_{i=1}^{n} \phi\left(\bar{F}_{\alpha_i}(x)\right)\right)} \sum_{i=1}^{n} \phi\left(\bar{F}_{\alpha_i}(x)\right) \frac{\bar{F}_{\alpha_i}(x)}{1 - \bar{\alpha}_i \bar{F}(x)}. \tag{1.3.2}$$

We established that with proper conditions on baseline rvs and the generator of Archimedean copula, hr ordering of two series systems holds under weak super majorization of their corresponding odds vectors.

Similarly, in another section, we consider stochastic comparison of lifetimes of parallel systems consisting of heterogeneous and dependent components with lifetimes following proportional odds models. The cdf of the lifetime of an n component parallel system, where the lifetime of ith component follows $PO(\bar{F}(x), \alpha_i), i = 1, .., n$, can be written as

$$F_{X_{n:n}}(x) = P(X_{n:n} \leq x) = P(X_i < x, i \in I_n) = \varphi\left(\sum_{i=1}^{n} \phi\left(F_{\alpha_i}(x)\right)\right). \tag{1.3.3}$$

We establish the usual stochastic ordering for the lifetimes of two parallel systems with heterogeneous and dependent components. With proper conditions on baseline rv and generator of the Archimedean copula one parallel system dominates the other the other system in usual stochastic order under weak super majorization of odds vectors of two systems. Next we consider the rhr function corresponding to system (3.3.1), which can be written as

$$\tilde{r}_{X_{n:n}}(x) = \frac{\tilde{r}(x)}{\bar{F}(x)} \frac{\varphi\left(\sum_{i=1}^{n} \xi_i\right)}{\varphi\left(\sum_{i=1}^{n} \xi_i\right)} \sum_{i=1}^{n} \frac{\varphi\left(\xi_i\right)}{\varphi\left(\xi_i\right)} \left(1 - \varphi\left(\xi_i\right)\right), \tag{1.3.4}$$

where $\xi_i = \phi\left(F_{\alpha_i}(x)\right)$.

With proper conditions on baseline distribution and the generator of Archimedean copula, rhr ordering of two parallel systems holds under weak super majorization of their corresponding odds vectors. Established results extend the results of Li and Li [86] from PHR, PRH and AL models to PO model.

Finally, we proposed two potential areas where the established results will be useful, one in comparing two series systems under random shock, and another in comparing the two smallest claim amounts in a portfolio of risk.

Chapter 3 : Dispersive and star ordering of sample extremes

In this chapter, we compare the lifetimes of series and parallel systems using two well-known variability orders, dispersive and star orderings. In many situations, it is important to investigate whether there is a significant effect on system lifetime as the heterogeneity among component lifetimes increases. We consider one coherent system with d.n.d. (dependent and non-identically distributed) components and another coherent system with d.i.d components. In both systems, component lifetimes follow PO models. The joint distribution of the components lifetimes is modelled by the Archimedean copula.

First we compare the lifetimes of these two series systems in terms of dispersive order. With proper conditions on generator of the Archimedean copula and odds vectors it is established that lifetime of a heterogeneous series system is dominated by the lifetime of a homogeneous series system in terms of dispersive order when the baseline distribution is decreasing failure rate (DFR). We also established sufficient conditions for star ordering of these two series systems by properly conditioning the generator of Archimedean copula and odds vectors. It is observed that dispersive ordering holds for series systems if $xr(x)$ is decreasing, where $r(x)$ is the hr function of baseline rv.

Similarly, we consider the comparison of parallel systems in terms of dispersive order. Here also, we consider two parallel systems with dependent components, one heterogeneous and another homogeneous. In both systems, component lifetimes follow PO models. Under certain conditions on generator of Archimedean copula and odds vectors, it is established that lifetime of heterogeneous parallel systems is dominated by homogeneous parallel systems in terms of dispersive order. It is observed that dispersive ordering holds when the baseline distribution has increasing rhr property. We also established sufficient conditions for star ordering of these two parallel systems under certain conditions on the generator of the Archimedean copula and odds vectors. It is observed that star ordering holds when $x\tilde{r}(x)$ is increasing, where $\tilde{r}(x)$ is the rhr function of the baseline rv. It is worth to mention that Fang et al. [43] established results for dispersive and star ordering of sample extremes when baseline follows PHR or PRH models. Li et al. [85] establish same results for AL random variables. Results in this chapter further extend these results from PHR, PRH and AL models to PO model. Throughout this chapter we provide counterexamples to validate the sufficient conditions. Finally some numerical examples are considered.

Chapter 4 : Stochastic comparisons of finite mixture models

In this chapter, we consider stochastic comparisons for finite mixture models. In many areas of reliability theory, survival analysis, and risk theory, finite mixture models play a significant role. Therefore, stochastic comparisons of finite mixture models pose significant problems to address. There are few research works in this direction. Let $X = (X_1, \ldots, X_n)$ be a vector of n rvs, where X_i denoting the lifetime of an item in the ith subpopulation with the cdf, sf and pdf as $F_i(\cdot)$, $\bar{F}_i(\cdot)$ and $f_i(\cdot)$, respectively. Then the cdf, sf and pdf of mixture of an item randomly drawn from these subpopulations are given by

$$F_{\mathbf{p}}(t) = \sum_{i=1}^{n} p_i F_i(t),\ \bar{F}_{\mathbf{p}}(t) = \sum_{i=1}^{n} p_i \bar{F}_i(t) \text{ and } f_{\mathbf{p}}(t) = \sum_{i=1}^{n} p_i f_i(t), \tag{1.3.5}$$

respectively, where p_i (> 0) is the mixing proportion (weighting factor) with $\sum_{i=1}^{n} p_i = 1$.

Recently, Hazra and Finkelstein [59] have derived some stochastic comparison results for two finite mixtures where corresponding rvs follow PHR, PRH or accelerated lifetime model, using the concept of multivariate chain majorization order.

We compare stochastically two finite mixture models where corresponding rv follow PO model in terms of weak mejorization of odds ratio and mixing portion. Next we consider multiple-outlier finite mixtures of $n = n_1 + n_2$ components where n_1 components are drawn from a particular homogeneous subpopulations and rest n_2 components are drawn from another homogeneous subpopulation. We provide sufficient conditions on the odds ratio vectors and the mixing proportion vectors under which the lifetime of two finite mixtures models can be compared with respect to the star order.

Next we generalize some of the results of two-component mixture models in Hazra and Finkelstein [59] to $n(> 2)$ component mixture model in case of multiple-outlier model and the results are obtained under weaker condition, namely row majorization order. When component lifetimes $X_1, .., X_n$ follow the PHR model with $X_i \sim PHR(\bar{F}, \lambda_i), i = 1, .., n,$

we established sufficient conditions on the hazard ratio vectors of the components and the mixing proportion vectors under which the lifetimes of two finite mixture models can be compared with respect to the hr order. Next we provide sufficient conditions on the baseline distribution, hazard ratio vectors and the mixing proportion vectors under which the lifetimes of two finite mixture models can be compared with respect to the star order. Similarly when components lifetimes $X_1, .., X_n$ follow the PRH model with $X_i \sim PRH(F, \lambda_i)$, we established sufficient conditions for rhr order and star order. Finally, the results are illustrated with numerical examples.

Chapter 5 : Some stochastic comparisons results on continuous mixture model

This chapter considered various shifted stochastic orderings of finite mixture models. Suppose $\{F_\alpha\}$ be a set of probability distributions, where the index α is governed by the distribution G. Then cdf F of continuous mixture of F_α is given by [12]

$$F(x) = \int_{-\infty}^{\infty} F_\alpha(x) dG(\alpha) \tag{1.3.6}$$

Two important continuous mixture model, frailty and resilience model are considered in this chapter. It may be noted that component may be subject to different levels of operating environments (e.g. voltage, stress, temperature) which is not fixed but changes over time. Component lifetimes and reliability depend on these random environmental variations. These environmental factors are unobservable and at the same time they are not ignorable either. In such cases a natural question arises whether the effect of these unobserved factors throughout lifetime will be dominating or not.

We study the effect of unobserved/unexplained heterogeneity in the hrs of the experimental units through frailty model. Similarly we consider the effect of unobserved or unexplained heterogeneity in the rhrs of the experimental units through resilience model. We established sufficient conditions for both up and down shifted likelihood ratio orders of frailty rv with its baseline rvs based on the condition that the baseline is ILR or DLR. Next we established up and down hr orders are established when baseline distribution belongs to IFR or DFR ageing classes. Up and down shifted mean residual life orders are also established when baseline distribution belongs to IMRL or DMRL ageing classes.

Similarly, we established sufficient conditions for both up and down shifted likelihood ratio orders of the resilience rv with its baseline rv. These orders are obtained when the baseline rv belongs to either ILR or DLR ageing class. Next we established up and down hazard rate orders when the baseline belongs to IRFR or DRFR ageing classes. It is also established up and down shifted mean inactivity time order when baseline belongs to IMIT ageing classes. Nanda and Das [107] established various shifted ordering for Marshall-Olkin extended distribution. In this chapter further investigation of various shifted ordering from

Marshall-Olkin extended distribution to frailty and resilience models have done.
Finally, two real data sets are analyzed for illustration purpose. One data set consists of Survival times in leukaemia (Hand et al. [58]) where the survival times of 43 patients suffering from chronic granulocytic leukaemia measured in days from the time of diagnosis. Another data set consists of the fatigue-life failures of ball-bearings (Hand et al. [58]).

Chapter 6 : Stochastic comparisons of coherent systems with active redundancy

This chapter considers various stochastic orderings for coherent systems under active redundancy allocation. Let $X = \{X_1, \ldots, X_n\}$ be a set of rvs denoting the lifetimes of the original components of a coherent system with common cdf F_0. The sf of the coherent system with dependent and identically distributed (d.i.d) components can be written as $\bar{F}_X(t) = h(\bar{F}_0(t))$, where h is the domination or distorted function. Suppose that for each of the n original components, m redundant components with lifetimes $Y_1, \ldots, Y_m$ having the cdfs $F_1, \ldots, F_m$, respectively, are allocated in parallel.

Kelkinnama [71] considered that the lifetime distributions of the original and redundant components follow the PRH or PRH model. In this chapter, we provide sufficient conditions to optimal selection of redundant components in a coherent system of dependent and identically distributed components in the sense of some stochastic orders based on the underlying distribution of the components lifetime. We considered that both original and redundant component lifetimes follow two important semi-parametric models, namely accelerated-life and proportional-odds models.

Let S_c denote the lifetime of this coherent system with active redundancy at the component level. The sf of S_c can be written as

$$\bar{F}_{S_c}(t) = h_\theta\left(1 - \prod_{i=0}^{m}(1 - \bar{F}_i(t))\right). \tag{1.3.7}$$

where θ_i is the parameter associated with the dependency structure (copula).

Suppose S_c and S_c^* are the lifetimes of two coherent systems with active redundancy at component levels. We establish sufficient conditions under which the lifetime of S_c and S_c^* can be compared in terms of usual stochastic, hr and rhr orders.

Next we compare the lifetimes of coherent systems with redundancy at the system level. Let S_0 be the lifetime of a coherent system consisting of d.i.d components, each having a common sf $\bar{F}_0$. Then the sf of S_0 is represented by

$$\bar{F}_{S_0}(t) = h_{\theta_0}(F_0(t))$$

where θ_0 is the parameter associated with dependency structure (copula). Let S_i be the lifetime of ith redundant system consisting of d.i.d components, each having a common sf

$\bar{F}_i, i-1, 2.., m$. The sf of S_i is represented by

$$\bar{F}_{S_i}(t) = h_{\theta_i}(F_i(t)), \quad i = 1, 2, .., m.$$

θ_i is the parameter associated with the dependency structure (copula). Active redundancy at the system level is allocated by connecting these m redundant systems in parallel with the original system S_0. Then the sf of this system with lifetime, say S_s can be written as

$$\bar{F}_{S_s}(t) = 1 - \prod_{i=0}^{m}(1 - h(\bar{F}_i(t))). \tag{1.3.8}$$

Let S_s and S_s^* be the lifetimes of two such coherent system with active redundancy at system levels. Sufficient conditions are established for comparing the lifetime of S_c and S_c^* under usual stochastic, hr and rhr order. Finally, we consider a real-world data set to illustrate the results Hand et al. [58]. The data set consists of the tensile strengths (in kg) of some cables, where each cable is composed of 12 wires.

Chapter 7 : Ordering properties of largest and aggregate claim amounts

In this chapter, we studied different types of stochastic ordering for the largest and aggregate claim amounts. The problem of comparison of number of claims and aggregate claim amounts with respect to some well-known stochastic orders are of interest from both theoretical and practical view points.

Let $X_1, X_2, .., X_n$ be the random claim amounts that can be made by a policy in an insurance period. Consider $I_{p_1}, \ldots, I_{p_n}$ are independent Bernoulli rvs representing the occurrence of these claims with $\mathbb{E}(I_{p_i}) = p_i$, $i = 1, \ldots, n$, where $I_{p_i} = 1$ if the i-th policyholder makes the random claim X_i and $I_{p_i} = 0$ if the policyholder does not make a claim. Then $X_{n:n}^* = \max(X_1 I_{p_1}, \ldots, X_n I_{p_n})$, represents the largest claim amount in a portfolio. It is assumed that odds function of each X_i is proportional to that of a baseline rv with proportionality constant (odds ratio) $\alpha_i > 0$, that is $X_i \sim PO(\bar{F}, \alpha_i)$, $i = 1, \ldots, n$, where $\bar{F}$ denotes the sf of the baseline rv.

We establish usual stochastic order for largest claim amount of two sets of portfolios. Usual stochastic ordering holds under the the weak super majorization of odds of claim vector $\alpha = (\alpha_1, .., \alpha_n)$ when both the portfolios are having common occurrence probability vector $p = (p_1, .., p_n)$. Similarly we establish stochastic ordering under the weak sub majorization of occurrence probability vector $p = (p_1, .., p_n)$ when both the portfolio having common odds of claim vector. It is also established that the usual stochastic ordering results for the largest claim amount holds when the occurrence probabilities are dependent. Our results further extend the results of Barmalzan and Najafabadi [14], Balakrishnan et al. [10] to different choice of semi-parametric models (Ex. PO, PRH).

Further we establish the rhr order of largest claim amounts for two sets of heterogeneous portfolios, assuming that the odds ratios are the same but the probabilities of occurrence of claims are different. We also considered stochastic comparisons on the largest claim amounts in case of multiple-outlier claims model with respect to the star order.

Next we consider the aggregate claim amount represented by $\sum_{i=1}^{n} I_i X_{\alpha_i}$. We established stochastic ordering of aggregate claim amount of two sets of portfolio under the majorization of odds of claim vector $\alpha = (\alpha_1, .., \alpha_n)$ when both the portfolio of risks having common occurrence probability vector. Established esults extend the results of Torrado and Navarro [127], Zhang et al. [140, 136] Numerical examples are provided to illustrate the results.

Chapter 8 : Future research directions

In this chapter, we discuss some possibilities of future research in the section Future Research Direction followed by a list of relevant references.

Chapter 2

Stochastic comparisons of series & parallel systems [1]

2.1 Introduction

There have been a number of works on stochastic comparisons of system lifetimes where component lifetimes follow different family of distributions (Khaledi and Kochar [72], Barmalzan et al. [15], Ding and Zhang [41], Fang et al. [43], Gupta et al. [53], Hazra et al. [61], Li and Fang [87], Navarro and Spizzichino [111]). However, most of the works have considered mutual independence among the concerned rvs. Recently, Fang et al. [43], Li and Fang [87] and Li and Li [86] have considered stochastic comparison of system lifetimes with dependent and heterogeneous component lifetimes following the proportional hazard rate (PHR) model.

Navarro and Spizzichino [111] have derived usual stochastic ordering for lifetimes of series and parallel systems having component lifetimes sharing a common copula, with the idea of mean reliability function associated with the common copula. Li and Fang [87] investigated stochastic order between two samples of dependent rvs following PHR model and having Archimedean survival copula. Fang et al. [43] derived some stochastic ordering results for minimum as well as for maximum of samples equipped with Archimedean survival copulas and following PHR model and proportional reversed hazard rate (PRH) model, respectively. Li and Li [86] investigated hr order on minimums of sample following PHR model, and reversed hr order on maximums of sample following PRH model, where both

[1] One paper based on this chapter has appeared under:

1. Stochastic comparisons of lifetimes of series and parallel systems with dependent and heterogeneous components. *Operations Research Letters*, **49**(**2**), 176-183, 2021.

the samples coupled with Archimedean survival copula.

However, there is no work on stochastic comparison of system lifetimes with dependent and heterogeneous component lifetimes following PO model.

In this chapter, the stochastic comparisons of lifetimes of series and parallel systems with dependent and heterogeneous components having lifetimes following the PO model. The joint distribution of component lifetimes is modelled by Archemedian survival copula. It is shown that the usual stochastic ordering and hr ordering hold for series systems under certain conditions, whereas for parallel system stochastic ordering and reversed hr ordering hold.

The organization of the chapter is as follows. In Section 2.2, we investigate stochastic comparisons between series systems of dependent and heterogeneous components having lifetimes following the PO model and dependency is modelled by Archimedean survival copulas. Section 2.3 investigates the same in case of parallel systems. Section 2.4 presents some potential applications of the proposed results.

2.2 Comparison of series systems

Here, the comparison of lifetimes of two series systems with heterogeneous and dependent components is considered. It is assumed that the lifetime vector $X = (X_1, X_2, ..., X_n)$ is a set of dependent random variables coupled with Archimedean survival copula with generator φ and following the PO model with baseline sf $\bar{F}$, denoted as $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi)$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{R}_+^n$ is the proportional odds ratio vector.

Lemma 2.2.1. *For any $x \in [0, 1]$, $S_1(x, \boldsymbol{\alpha}, \varphi)$ (1.3.1) is increasing in α_i, $i \in I_n$. Furthermore S_1 is schur-concave with respect to $\boldsymbol{\alpha}$.*

Proof: For $s \in I_n$,

$$\frac{\partial S_1}{\partial \alpha_s} = \varphi' \left(\sum_{i=1}^{n} \phi \left(\frac{\alpha_i x}{1 - \bar{\alpha}_i x} \right) \right) \phi' \left(\frac{\alpha_s x}{1 - \bar{\alpha}_s x} \right) \frac{x(1-x)}{(1 - \bar{\alpha}_s x)^2}.$$

Since both $\varphi(u)$ and $\phi(u)$ are decreasing for all $u \geq 0$, $\frac{\partial S_1}{\partial \alpha_s} \geq 0$. As a result $S_1(x, \boldsymbol{\alpha}, \varphi)$ is increasing in α_i, $i \in I_n$ for any $x \in [0, 1]$.

For $s \neq t$,

$$\begin{aligned}
&(\alpha_s - \alpha_t)\left(\frac{\partial S_1}{\partial \alpha_s} - \frac{\partial S_1}{\partial \alpha_t}\right) \\
= \quad &(\alpha_s - \alpha_t)\varphi'\left(\sum_{i=1}^{n} \phi\left(\frac{\alpha_i x}{1-\bar{\alpha}_i x}\right)\right)\left[\phi'\left(\frac{\alpha_s x}{1-\bar{\alpha}_s x}\right)\frac{x(1-x)}{(1-\bar{\alpha}_s x)^2} - \phi'\left(\frac{\alpha_t x}{1-\bar{\alpha}_t x}\right)\frac{x(1-x)}{(1-\bar{\alpha}_t x)^2}\right] \\
\overset{sign}{=} \quad &(\alpha_s - \alpha_t)\left(-\varphi'\left(\sum_{i=1}^{n} \phi(u_i)\right)\right)\left[\left(-\phi'(u_s)\right)\frac{1}{(1-\bar{\alpha}_s x)^2} - \left(-\phi'(u_t)\right)\frac{1}{(1-\bar{\alpha}_t x)^2}\right], \quad (2.2.1)
\end{aligned}$$

where $u_i = \frac{\alpha_i x}{1-\bar{\alpha}_i x}$ and $\overset{\text{`}sign'}{=}$ means equal in sign. Since both φ and ϕ are decreasing, and ϕ' is increasing, it follows from (2.2.1) that $(\alpha_s - \alpha_t)\left(\frac{\partial S_1}{\partial \alpha_s} - \frac{\partial S_1}{\partial \alpha_t}\right) \leq 0$. So from Lemma 1.2.1, S_1 is schur-concave in $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$.

Suppose there are two series systems formed out of n statistically dependent and heterogeneous components where the component lifetimes follow the PO model. The joint distribution of lifetimes of components is represented by Archimedean copula. Consider two such series systems with lifetime vectors $X = (X_1, X_2, ..., X_n)$ and $Y = (Y_1, Y_2, ..., Y_n)$ having respective proportionality odds ratio vectors $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, ..., \beta_n)$, where $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n_+$.
The following theorem compares the lifetimes of these series systems in the sense of usual stochastic order.

Theorem 2.2.1. *Suppose the lifetime vectors $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi_1)$ and $Y \sim PO(\bar{F}, \boldsymbol{\beta}, \varphi_2)$. If φ_1 or φ_2 is log-convex and $\phi_2 \circ \varphi_1$ is superadditive, then*

$$\boldsymbol{\alpha} \overset{p}{\succeq} \boldsymbol{\beta} \text{ implies } X_{1:n} \leq_{st} Y_{1:n}.$$

Proof: Write $v_i = \ln \alpha_i$, $i = 1, 2, ..., n$. Then as per (1.3.1),

$$\bar{F}_{X_{1:n}}(x) = \varphi_1\left(\sum_{i=1}^{n} \phi_1\left(\frac{e^{v_i}\bar{F}(x)}{1-(1-e^{v_i})\bar{F}(x)}\right)\right) = S_1(\bar{F}(x), (e^{v_1}, e^{v_2}, ..., e^{v_n}), \varphi_1).$$

Here $S_1(\bar{F}(x), (e^{v_1}, e^{v_2}, ..., e^{v_n}), \varphi_1)$ is symmetric with respect to $(v_1, v_2, ..., v_n) \in \mathbb{R}^n$. Now, for $s \in I_n$,

$$\frac{\partial S_1}{\partial v_s} = \varphi_1'\left(\sum_{i=1}^{n} \phi_1\left(\frac{e^{v_i}x}{1-(1-e^{v_i})x}\right)\right)\phi_1'\left(\frac{e^{v_s}x}{1-(1-e^{v_s})x}\right)\frac{x(1-x)e^{v_s}}{(1-(1-e^{v_s})x)^2},$$

so that $S_1(x,(e^{v_1},e^{v_2},...,e^{v_n}),\varphi_1)$ is increasing in each v_i, $i=1,2,...,n$ for any $x\in[0,1]$. Now, for $s\neq t$,

$$\begin{aligned}
& (v_s-v_t)\left(\frac{\partial S_1}{\partial v_s}-\frac{\partial S_1}{\partial v_t}\right)\\
= \; & (v_s-v_t)\left(-\varphi_1^{'}\left(\sum_{i=1}^{n}\phi_1\left(\frac{e^{v_i}x}{1-(1-e^{v_i})x}\right)\right)\right)\times\\
& \left[\left(-\phi_1^{'}\left(\frac{e^{v_s}x}{1-(1-e^{v_s})x}\right)\right)\frac{xe^{v_s}}{(1-(1-e^{v_s})x)^2}\right.\\
- \; & \left.\left(-\phi_1^{'}\left(\frac{e^{v_t}x}{1-(1-e^{v_t})x}\right)\right)\frac{xe^{v_t}}{(1-(1-e^{v_t})x)^2}\right]\\
\stackrel{sign}{=} \; & (v_s-v_t)\left[\left(-\frac{\varphi_1(\phi_1(u_s))}{\varphi_1^{'}(\phi_1(u_s))}\right)\frac{1}{1-(1-e^{v_s})x}-\left(-\frac{\varphi_1(\phi_1(u_t))}{\varphi_1^{'}(\phi_1(u_t))}\right)\frac{1}{1-(1-e^{v_t})x}\right],
\end{aligned} \tag{2.2.2}$$

where $u_s=\frac{e^{v_s}x}{1-(1-e^{v_s})x}$.

If φ_1 is log-convex, from (2.2.2) it follows that $(v_s-v_t)\left(\frac{\partial S_1}{\partial v_s}-\frac{\partial S_1}{\partial v_t}\right)\leq 0$. Hence from Lemma 1.2.1, $S_1(x,(e^{v_1},e^{v_2},...,e^{v_n}),\varphi_1)$ is schur-concave in $(v_1,v_2,...,v_n)$ if φ_1 is log-convex. Then from Lemma 1.2.3, it follows

$$\boldsymbol{\alpha}\stackrel{p}{\succeq}\boldsymbol{\beta} \text{ implies } S_1(\bar{F}(x),\boldsymbol{\alpha},\varphi_1)\leq S_1(\bar{F}(x),\boldsymbol{\beta},\varphi_1). \tag{2.2.3}$$

Since $\phi_2\circ\varphi_1$ is superadditive, from Lemma 1.2.8, we have

$$S_1(\bar{F}(x),\boldsymbol{\beta},\varphi_1)\leq S_1(\bar{F}(x),\boldsymbol{\beta},\varphi_2). \tag{2.2.4}$$

Thus combining (2.2.3) and (2.2.4) it follows that $S_1(\bar{F}(x),\boldsymbol{\alpha},\varphi_1)\leq S_1(\bar{F}(x),\boldsymbol{\beta},\varphi_2)$, that is $X_{1:n}\leq_{st}Y_{1:n}$. Now suppose φ_2 is log-convex, then

$$S_1(\bar{F}(x),\boldsymbol{\alpha},\varphi_2)\leq S_1(\bar{F}(x),\boldsymbol{\beta},\varphi_2). \tag{2.2.5}$$

Since $\phi_2\circ\varphi_1$ is superadditive, it follows

$$S_1(\bar{F}(x),\boldsymbol{\alpha},\varphi_1)\leq S_1(\bar{F}(x),\boldsymbol{\alpha},\varphi_2). \tag{2.2.6}$$

So combining 2.2.5 and 2.2.6 it follows that $X_{1:n}\leq_{st}Y_{1:n}$.

Corollary 2.2.1. *Suppose the lifetime vectors* $X\sim PO(\bar{F},\boldsymbol{\alpha},\varphi)$ *and* $Y\sim PO(\bar{F},\boldsymbol{\beta},\varphi)$. *If*

φ is log-convex, then

$$\boldsymbol{\alpha} \overset{p}{\succeq} \boldsymbol{\beta} \text{ implies } X_{1:n} \leq_{st} Y_{1:n}. \quad \square$$

The following counterexample shows that one may not get the the ordering result in Theorem 2.2.1 if the sufficient conditions on the generator functions are dropped.

Counterexample 2.2.1. Consider two series systems, each comprising of three dependent and heterogeneous components with respective survival functions $\bar{F}_{X_{1:3}}(x) = S_1(\bar{F}(x), \boldsymbol{\alpha}, \varphi_1)$ and $\bar{F}_{Y_{1:3}}(x) = S_1(\bar{F}(x), \boldsymbol{\beta}, \varphi_2)$ with $\bar{F}(x) = e^{-(x)^{1.5}}$, $x \geq 0$, $\boldsymbol{\alpha} = (2, 3, 5.5)$, $\boldsymbol{\beta} = (2.5, 3.5, 3.8)$ so that $\boldsymbol{\alpha} \overset{p}{\succeq} \boldsymbol{\beta}$. First we take $\varphi_1(x) = (2/(1 + e^x))^{1/\theta}$, $\theta = 0.9$ and $\varphi_2(x) = e^{1-(1+x)^{1/\eta}}$ with $\eta = 0.3$ so that $\phi_2 \circ \varphi_1$ is not super additive, and φ_1 and φ_2 are not log-convex. $\bar{F}_{X_{1:3}}(x)$ and $\bar{F}_{Y_{1:3}}(x)$ are depicted in the Figure 2.1 for some finite range of x. From this figure it is clear that the stochastic ordering result in Theorem 2.2.1 is not attained.

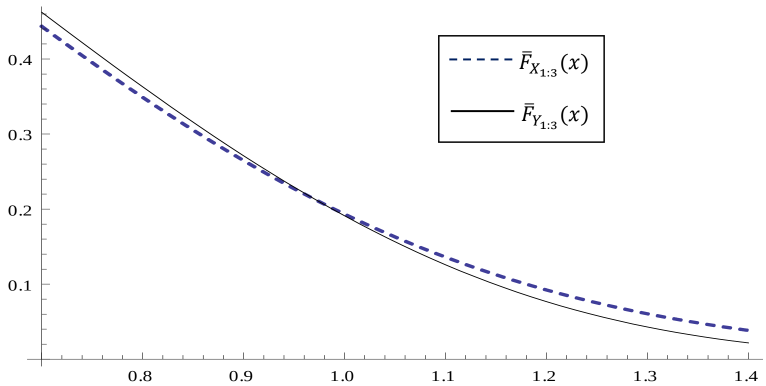

Figure 2.1: Plots of $\bar{F}_{X_{1:3}}(x)$ and $\bar{F}_{Y_{1:3}}(x)$ when $\phi_2 \circ \varphi_1$ is not super additive, and φ_1 and φ_2 are not log-convex.

Since p-larger order is weaker than weakly supermajorization order, the following theorem shows that one can get the ordering result in Theorem 2.2.1 under weakly supermajorization order with fewer condition.

Theorem 2.2.2. *Suppose the lifetime vectors $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi_1)$ and $Y \sim PO(\bar{F}, \boldsymbol{\beta}, \varphi_2)$. If $\phi_2 \circ \varphi_1$ is superadditive, then*

$$\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta} \text{ implies } X_{1:n} \leq_{st} Y_{1:n}.$$

Proof: From Lemma 2.2.1 and Lemma 1.2.2, it follows that

$$\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta} \text{ implies } S_1(\bar{F}(x), \boldsymbol{\alpha}, \varphi_1) \leq S_1(\bar{F}(x), \boldsymbol{\beta}, \varphi_1).$$

Since $\phi_2 \circ \varphi_1$ is superadditive, so from Lemma 1.2.8, it follows that, $S_1(\bar{F}(x), \boldsymbol{\beta}, \varphi_1) \leq S_1(\bar{F}(x), \boldsymbol{\beta}, \varphi_2)$. Combining the above results it follows $S_1(\bar{F}(x), \boldsymbol{\alpha}, \varphi_1) \leq S_1(\bar{F}(x), \boldsymbol{\beta}, \varphi_2)$. That is $X_{1:n} \leq_{st} Y_{1:n}$.

Remark 2.2.1. It is to be noted that super-additive assumption of $\phi_2 \circ \varphi_1$ is satisfied by many members of Archimedean survival copulas. For example, Archimedean survival copula with generators (i) $\varphi_1(t) = e^{1-(1+t)^{\frac{1}{\theta}}}$ and $\varphi_2(t) = \frac{\theta}{\log(e^\theta + t)}$, where $0 < \theta \leq 1$, (ii) $\varphi_1(t) = \frac{\theta}{\log(e^\theta + t)}$ and $\varphi_2(t) = \log(e+t)^{-1/\theta}$, where $\theta > 1$ and (iii) $\varphi_1(t) = e^{1-(1+t)^{\frac{1}{\theta_1}}}$ and $\varphi_2(t) = e^{1-(1+t)^{\frac{1}{\theta_2}}}$, where $\theta_2 \geq \theta_1 \geq 1$, satisfy super-additivity.

Corollary 2.2.2. *Suppose the lifetime vectors $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi)$ and $Y \sim PO(\bar{F}, \boldsymbol{\beta}, \varphi)$. Then*

$$\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta} \textit{ implies } X_{1:n} \leq_{st} Y_{1:n}.$$

Lemma 2.2.2. *$I_1(\boldsymbol{u}) = \frac{\varphi'(\sum_{i=1}^n u_i)}{\varphi(\sum_{i=1}^n u_i)} \sum_{i=1}^n \frac{\varphi(u_i)}{\varphi'(u_i)} (1 - \varphi(u_i))$ is increasing in u_s, $s \in I_n$ and Schur-convex with respect to $\boldsymbol{u} = (u_1, ..., u_n)$ if φ is log-concave and $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing and concave.*

Proof: Here $I_1(\mathbf{u})$ is symmetric in $\mathbf{u}$. For $s \in I_n$,

$$\begin{aligned}\frac{\partial I_1(\mathbf{u})}{\partial u_s} &= \frac{\partial}{\partial u_s}\left(\frac{\varphi'\left(\sum_{i=1}^n u_i\right)}{\varphi\left(\sum_{i=1}^n u_i\right)}\right)\sum_{i=1}^n \frac{\varphi(u_i)}{\varphi'(u_i)}(1-\varphi(u_i)) \\ &\quad + \frac{\varphi'\left(\sum_{i=1}^n u_i\right)}{\varphi\left(\sum_{i=1}^n u_i\right)}\frac{\partial}{\partial u_s}\left(\frac{\varphi(u_s)}{\varphi'(u_s)}(1-\varphi(u_s))\right).\end{aligned}$$

Since φ is log-concave, $\frac{\partial}{\partial u_s}\left(\frac{\varphi'(\sum_{i=1}^n u_i)}{\varphi(\sum_{i=1}^n u_i)}\right) \leq 0$ As $\varphi(1-\varphi)/\varphi'$ is decreasing implies

$$\frac{\partial}{\partial u_s}\left(\frac{\varphi(u_s)}{\varphi'(u_s)}(1-\varphi(u_s))\right) \leq 0.$$

Then using the fact that φ is deceasing, it implies $\frac{\partial I_1(\mathbf{u})}{\partial u_s} \geq 0$. So $I_1(\mathbf{u})$ is increasing in u_s

for any $s \in I_n$. For $s, t \in I_n$ with $s \neq t$,

$$\frac{\partial}{\partial u_s}\left(\frac{\varphi'\left(\sum_{i=1}^n u_i\right)}{\varphi\left(\sum_{i=1}^n u_i\right)}\right) = \frac{\partial}{\partial u_t}\left(\frac{\varphi'\left(\sum_{i=1}^n u_i\right)}{\varphi\left(\sum_{i=1}^n u_i\right)}\right).$$

Then

$$\begin{aligned}
& (u_s - u_t)\left(\frac{\partial I_1(\mathbf{u})}{\partial u_s} - \frac{\partial I_1(\mathbf{u})}{\partial u_t}\right) \\
= \; & (u_s - u_t)\frac{\varphi'\left(\sum_{i=1}^n u_i\right)}{\varphi\left(\sum_{i=1}^n u_i\right)}\left[\frac{\partial}{\partial u_s}\left(\frac{\varphi(u_s)}{\varphi'(u_s)}(1-\varphi(u_s))\right) - \frac{\partial}{\partial u_t}\left(\frac{\varphi(u_t)}{\varphi'(u_t)}(1-\varphi(u_t))\right)\right] \geq 0,
\end{aligned}$$

where the inequality follows from the fact that $\frac{\varphi(1-\varphi)}{\varphi'}$ is concave. So from lemma 1.2.1, $I_1(\mathbf{u})$ is Schur-convex with respect to $\mathbf{u}$. Next theorem established hr ordering of two series systems formed out of n statistically dependent and heterogeneous components having lifetimes following PO model.

Theorem 2.2.3. *Suppose the lifetime vectors $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi)$ and $Y \sim PO(\bar{F}, \boldsymbol{\beta}, \varphi)$. If φ is log-concave and $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing and concave (or convex), then*

$$\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta} \text{ implies } X_{1:n} \leq_{hr} Y_{1:n}.$$

Proof: From (1.3.2), it follows

$$\begin{aligned}
r_{X_{1:n}}(x) &= r(x)\frac{\varphi'\left(\sum_{i=1}^n \phi\left(\bar{F}_{\alpha_i}(x)\right)\right)}{\varphi\left(\sum_{i=1}^n \phi\left(\bar{F}_{\alpha_i}(x)\right)\right)}\sum_{i=1}^n \phi'\left(\bar{F}_{\alpha_i}(x)\right)\frac{\bar{F}_{\alpha_i}(x)}{1-\bar{\alpha}_i\bar{F}(x)} \\
&= \frac{r(x)}{F(x)}\frac{\varphi'\left(\sum_{i=1}^n \phi\left(\bar{F}_{\alpha_i}(x)\right)\right)}{\varphi\left(\sum_{i=1}^n \phi\left(\bar{F}_{\alpha_i}(x)\right)\right)}\sum_{i=1}^n \frac{\bar{F}_{\alpha_i}(x)}{\varphi'\left(\phi\left(\bar{F}_{\alpha_i}(x)\right)\right)}\frac{F(x)}{1-\bar{\alpha}_i\bar{F}(x)} \\
&= \frac{r(x)}{F(x)} I_1\left(\phi\left(\bar{F}_{\alpha_1}(x)\right), \ldots, \phi\left(\bar{F}_{\alpha_n}(x)\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
I_1\left(\phi\left(\bar{F}_{\alpha_1}(x)\right), \ldots, \phi\left(\bar{F}_{\alpha_n}(x)\right)\right) &= \frac{\varphi'\left(\sum_{i=1}^n \phi\left(\bar{F}_{\alpha_i}(x)\right)\right)}{\varphi\left(\sum_{i=1}^n \phi\left(\bar{F}_{\alpha_i}(x)\right)\right)} \\
&\quad \times \sum_{i=1}^n \frac{\varphi\left(\phi\left(\bar{F}_{\alpha_i}(x)\right)\right)}{\varphi'\left(\phi\left(\bar{F}_{\alpha_i}(x)\right)\right)}\left(1-\varphi\left(\phi\left(\bar{F}_{\alpha_i}(x)\right)\right)\right).
\end{aligned}$$

It is easy to check that $\phi\left(\bar{F}_{\alpha_i}(x)\right)$ is decreasing and convex in α_i. From Theorem A.2 (Chapter 5) of Marshall et al. Marshall et al. [96], $\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta}$ implies $\left(\phi\left(\bar{F}_{\alpha_1}(x)\right), \ldots, \phi\left(\bar{F}_{\alpha_n}(x)\right)\right) \succeq_w \left(\phi\left(\bar{F}_{\beta_1}(x)\right), \ldots, \phi\left(\bar{F}_{\beta_n}(x)\right)\right)$. From Lemma 2.2.2, $I_1(\mathbf{u})$ is increasing in u_i for $i \in I_n$ and Schur-convex with respect to $\mathbf{u}$ whenever φ is log-concave and $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing and concave. Then from Theorem A.8 (Chapter 3) of Marshall et al. Marshall et al. [96], it follows that

$$I_1\left(\phi\left(\bar{F}_{\alpha_1}(x)\right), \ldots, \phi\left(\bar{F}_{\alpha_n}(x)\right)\right) \geq I_1\left(\phi\left(\bar{F}_{\beta_1}(x)\right), \ldots, \phi\left(\bar{F}_{\beta_n}(x)\right)\right)$$

which implies $r_{X_{1:n}}(x) \geq r_{Y_{1:n}}(x)$, that is $X_{1:n} \leq_{hr} Y_{1:n}$.
Next the theorem will be proved when $\frac{\varphi(1-\varphi)}{\varphi'}$ is convex. Let $z_i = \phi\left(\bar{F}_{\alpha_i}(x)\right)$. Then the hr function is given by

$$r_{X_{1:n}}(x) = \frac{r(x)}{F(x)} \frac{\varphi'\left(\sum_{i=1}^n z_i\right)}{\varphi\left(\sum_{i=1}^n z_i\right)} \sum_{i=1}^n \frac{\varphi\left(z_i\right)}{\varphi'\left(z_i\right)}\left(1-\varphi\left(z_i\right)\right).$$

Now, for $s \in I_n$,

$$\begin{aligned} \frac{r_{X_{1:n}}(x)}{\partial \alpha_s} &= \frac{r(x)}{F(x)}\left[\frac{\partial}{\partial z_s}\left(\frac{\varphi'\left(\sum_{i=1}^n z_i\right)}{\varphi\left(\sum_{i=1}^n z_i\right)}\right) \frac{\partial z_s}{\partial \alpha_s} \sum_{i=1}^n \frac{\varphi\left(z_i\right)\left(1-\varphi\left(z_i\right)\right)}{\varphi'\left(z_i\right)} + \right. \\ & \left. \frac{\varphi'\left(\sum_{i=1}^n z_i\right)}{\varphi\left(\sum_{i=1}^n z_i\right)} \frac{\partial}{\partial z_s}\left(\frac{\varphi\left(z_s\right)\left(1-\varphi\left(z_s\right)\right)}{\varphi'\left(z_s\right)}\right) \frac{\partial z_s}{\partial \alpha_s}\right]. \end{aligned}$$

Note that z_s is decreasing in α_s and $\frac{\partial z_s}{\partial \alpha_s}$ is increasing in α_s. Since φ is log-concave and $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing, it follows $\frac{r_{X_{1:n}}(x)}{\partial \alpha_s} \leq 0$. Again

$$\frac{\partial}{\partial z_s}\left(\frac{\varphi'\left(\sum_{i=1}^n z_i\right)}{\varphi\left(\sum_{i=1}^n z_i\right)}\right) = \frac{\partial}{\partial z_t}\left(\frac{\varphi'\left(\sum_{i=1}^n z_i\right)}{\varphi\left(\sum_{i=1}^n z_i\right)}\right), \text{ for } s \neq t.$$

For $s \neq t$,

$$\begin{aligned} & \left(\alpha_s-\alpha_t\right)\left(\frac{r_{X_{1:n}}}{\partial \alpha_s}-\frac{r_{X_{1:n}}}{\partial \alpha_t}\right) \\ \overset{sign}{=} & \left(\alpha_s-\alpha_t\right)\left(\frac{\partial z_s}{\partial \alpha_s}-\frac{\partial z_t}{\partial \alpha_t}\right)+\left(\alpha_s-\alpha_t\right) \frac{\varphi'\left(\sum_{i=1}^n z_i\right)}{\varphi\left(\sum_{i=1}^n z_i\right)} \times \\ & \left[\left(-\frac{\partial}{\partial z_s}\left(\frac{\varphi\left(z_s\right)\left(1-\varphi\left(z_s\right)\right)}{\varphi'\left(z_s\right)}\right)\right)\left(-\frac{\partial z_s}{\partial \alpha_s}\right)\right. \\ & \left.-\left(-\frac{\partial}{\partial z_t}\left(\frac{\varphi\left(z_t\right)\left(1-\varphi\left(z_t\right)\right)}{\varphi'\left(z_t\right)}\right)\right)\left(-\frac{\partial z_t}{\partial \alpha_t}\right)\right] \leq 0, \end{aligned}$$

whenever $\frac{\varphi(1-\varphi)}{\varphi'}$ is convex in addition to the log-concave φ and decreasing $\frac{\varphi(1-\varphi)}{\varphi'}$. Thus

$r_{X_{1:n}}(x)$ is decreasing in α_i, $i \in I_n$ and Schur-convex in $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$. Then from Lemma 1.2.2, it follows

$$\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta} \text{ implies } r_{X_{1:n}}(x) \geq r_{X_{1:n}}(x).$$

Hence the theorem follows.

Corollary 2.2.3. *Suppose the lifetime vectors $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi)$ and $Y \sim PO(\bar{F}, \alpha\mathbf{1}, \varphi)$. Then, $X_{1:n} \leq_{hr} Y_{1:n}$ if $\alpha \geq \frac{1}{n}\sum_{i=1}^{n} \alpha_i$, φ is log-concave and $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing and concave (or convex). This follows from the Theorem 6.1.2 and using the fact that $(\alpha_1, \alpha_2, \ldots, \alpha_n) \overset{w}{\succeq} (\underbrace{\alpha, \alpha, \ldots, \alpha}_{n \text{ terms}})$, for $\alpha \geqslant \frac{1}{n}\sum_{i=1}^{n} \alpha_i$.*

Remark 2.2.2. It is to be noted that Archimedean copulas with generators $\varphi(t) = 2/(1+e^t)$ and $\varphi(t) = (-1+\theta)/(-e^t+\theta)$ for $-1 \leq \theta \leq 0$ are some examples of survival copula such that φ is log-concave, and $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing and convex.

The following counterexample shows that one may not get the the ordering result in Theorem 6.1.2 if the sufficient conditions on the generator functions are dropped.

Counterexample 2.2.2. Consider two series systems, each comprising of three dependent and heterogeneous components with respective hr functions $r_{X_{1:3}}(x)$ and $r_{Y_{1:3}}(x)$, with common baseline sf $\bar{F}(x) = e^{-(0.5x)^2}$, $x \geq 0$, $\boldsymbol{\alpha} = (0.2, 0.4, 0.6)$, $\boldsymbol{\beta} = (0.35, 0.55, 0.95)$ so that $\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta}$. First we take the common generator $\varphi(x) = \log(e+x)^{-1/a}$, $a = 0.1$, which is not log-concave but $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing and convex. Next, consider $\varphi(x) = (2/(1+e^x))^{1/a}$, $a = 0.2$, which is log-concave but $\frac{\varphi(1-\varphi)}{\varphi'}$ is neither decreasing nor convex. For both the cases $r_{X_{1:3}}(x)$ and $r_{Y_{1:3}}(x)$ are depicted in Figure 2.2(a) and 2.2(b) respectively for some finite range of x. From both the figures, it is observed that the hr ordering result in Theorem 6.1.2 is not attained.

2.3 Comparisons of parallel systems

Here, the comparisons of the lifetimes of two parallel systems consisting of dependent and heterogeneous components having lifetimes following the PO model, with respect to some stochastic orders, is considered.

Lemma 2.3.1. *For any $x \in [0,1]$, $S_2(x, \boldsymbol{\alpha}, \varphi)$(3.3.1) is decreasing in α_i, $i \in I_n$. Furthermore S_2 is Schur-convex with respect to $\boldsymbol{\alpha}$ whenever φ is log-concave.*

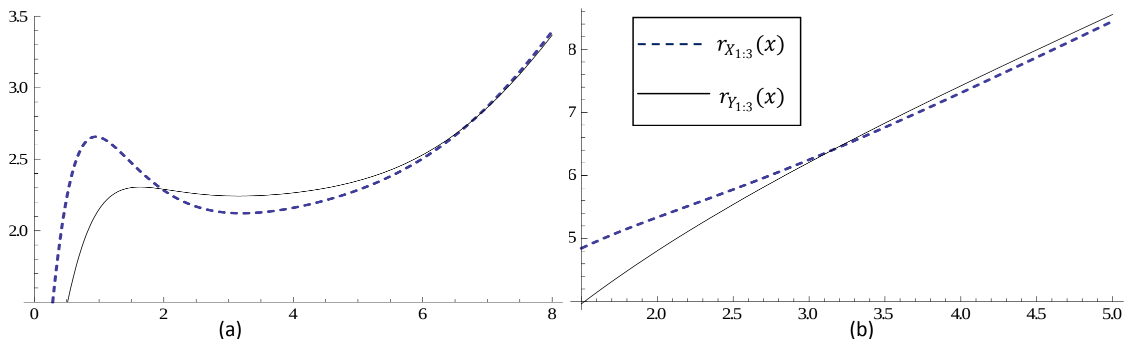

Figure 2.2: Plots of $r_{X_{1:3}}(x)$ and $r_{Y_{1:3}}(x)$ for (a) $\varphi(x)$ is not log-concave, (b) $\frac{\varphi(1-\varphi)}{\varphi'}$ is neither decreasing nor convex.

Proof: For $s \in I_n$,

$$\frac{\partial S_2}{\partial \alpha_s} = -\varphi'\left(\sum_{i=1}^{n}\phi\left(\frac{x}{1-\bar{\alpha}_i(1-x)}\right)\right)\phi'\left(\frac{x}{1-\bar{\alpha}_s(1-x)}\right)\frac{x(1-x)}{(1-\bar{\alpha}_s(1-x))^2}.$$

Since both $\varphi(u)$ and $\phi(u)$ are decreasing for all $u \geq 0$, $\frac{\partial S_2}{\partial \alpha_s} \leq 0$. So $S_2(x, \boldsymbol{\alpha}, \varphi)$ is decreasing in α_i for any $x \in [0, 1]$.
Let $v_i = \frac{x}{1-\bar{\alpha}_i(1-x)}$ then for $s \neq t$,

$$\begin{aligned}
& (\alpha_s - \alpha_t)\left(\frac{\partial S_2}{\partial \alpha_s} - \frac{\partial S_2}{\partial \alpha_t}\right) \\
= \quad & -(\alpha_s - \alpha_t)\varphi'\left(\sum_{i=1}^{n}\phi\left(v_i\right)\right)\left[\phi'\left(v_s\right)\frac{x(1-x)}{(1-\bar{\alpha}_s x)^2} - \phi'\left(v_t\right)\frac{x(1-x)}{(1-\bar{\alpha}_t x)^2}\right] \\
\overset{sign}{=} \quad & (\alpha_s - \alpha_t)\left[-\left(-\frac{\varphi(\phi(v_s))}{\varphi'(\phi(v_s))}\right)\frac{1}{1-\bar{\alpha}_s x} + \left(-\frac{\varphi(\phi(v_t))}{\varphi'(\phi(v_t))}\right)\frac{1}{1-\bar{\alpha}_t x}\right] \\
\geq \quad & 0,
\end{aligned}$$

where the last inequality is derived using the fact that φ is log-concave. So from Lemma 1.2.1, S_2 is Schur-convex in $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$.

Suppose there are two parallel systems with lifetime vectors $X = (X_1, X_2, ..., X_n)$ and $Y = (Y_1, Y_2, ..., Y_n)$, formed out of n dependent and heterogeneous components where the component lifetimes follow PO model. The following theorem compares the lifetimes of two such parallel systems in the sense of usual stochastic order.

Theorem 2.3.1. *Suppose the lifetime vectors $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi_1)$ and $Y \sim PO(\bar{F}, \boldsymbol{\beta}, \varphi_2)$.*

If φ_1 or φ_2 is log-concave and $\phi_1 \circ \varphi_2$ is superadditive, then

$$\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta} \textit{ implies } X_{n:n} \leq_{st} Y_{n:n}.$$

Proof: If φ_1 is log-concave, then from Lemma 2.3.1 and Lemma 1.2.2, it follows

$$\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta} \text{ implies } S_2(F(x), \boldsymbol{\alpha}, \varphi_1) \geq S_2(F(x), \boldsymbol{\beta}, \varphi_1). \tag{2.3.1}$$

Since $\phi_1 \circ \varphi_2$ is superadditive, so from Lemma 1.2.8 (by replacing φ_1 by φ_2 and vice versa), it follows

$$S_2(F(x), \boldsymbol{\beta}, \varphi_1) \geq S_2(F(x), \boldsymbol{\beta}, \varphi_2). \tag{2.3.2}$$

Combining (2.3.1) and (2.3.2), we get $S_2(F(x), \boldsymbol{\alpha}, \varphi_1) \geq S_2(F(x), \boldsymbol{\beta}, \varphi_2)$. That is $X_{n:n} \leq_{st} Y_{n:n}$. Now suppose φ_2 is log-concave, then

$$\begin{aligned} S_2(F(x), \boldsymbol{\alpha}, \varphi_1) &\geq S_2(F(x), \boldsymbol{\alpha}, \varphi_2) \\ &\geq S_2(F(x), \boldsymbol{\beta}, \varphi_2), \end{aligned}$$

where the first inequality follows from the fact that $\phi_1 \circ \varphi_2$ is superadditive, whereas the second inequality follows from the fact that $\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta}$. This proves the result.

Corollary 2.3.1. *Suppose the lifetime vectors $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi)$ and $Y \sim PO(\bar{F}, \boldsymbol{\beta}, \varphi)$. If φ is log-concave, then*

$$\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta} \textit{ implies } X_{n:n} \leq_{st} Y_{n:n}. \square$$

The following counterexample shows that one may not get the the ordering result in Theorem 2.3.1 if the sufficient conditions on the generator functions are dropped.

Counterexample 2.3.1. Consider two parallel systems, each comprising of three dependent and heterogeneous components with respective distribution functions $F_{X_{3:3}}(x) = S_2(F(x), \boldsymbol{\alpha}, \varphi_1)$ and $F_{Y_{3:3}}(x) = S_2(F(x), \boldsymbol{\beta}, \varphi_2)$, where $F(x) = 1 - e^{-x^{0.5}}$, $x \geq 0$, $\boldsymbol{\alpha} = (0.9, 1.45, 2.15)$, $\boldsymbol{\beta} = (1.2, 1.95, 2.65)$ so that $\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta}$. First consider $\varphi_1(x) = \theta_1/\log(x + e^{\theta_1})$ and $\varphi_2(x) = e^{1-(1+x)^{1/\theta_2}}$ with $\theta_1 = 0.9$ and $\theta_2 = 8$ so that neither φ_1 nor φ_2 is log-concave but $\phi_1 \circ \varphi_2$ is super additive. Next consider $\varphi_1(x) = e^{(1-e^x)/\theta_1}$ and $\varphi_2(x) = (2/(e^x + 1))^{1/\theta_2}$ with $\theta_1 = 0.9$ and $\theta_2 = 0.2$ so that φ_1 is log-concave but $\phi_1 \circ \varphi_2$ is not super additive. For both the cases $F_{X_{3:3}}(x)$ and $F_{Y_{3:3}}(x)$ are depicted in Figure 2.3(a) and 2.3(b) respectively for some finite range of x. From both the figures it is observe that the stochastic ordering result in Theorem 2.3.1 is not attained.

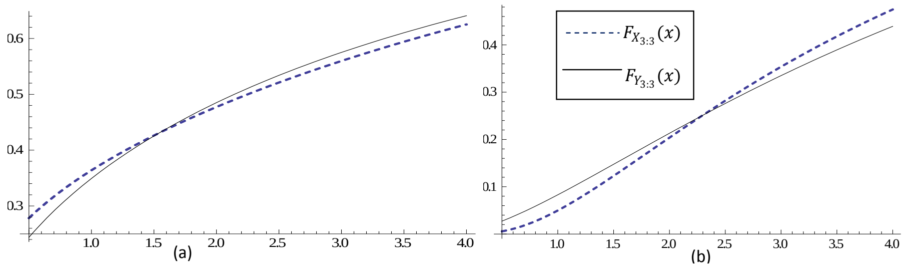

Figure 2.3: Plots of $F_{X_{3:3}}(x)$ and $F_{Y_{3:3}}(x)$ for (a) neither φ_1 nor φ_2 is log-concave, (b) $\phi_1 \circ \varphi_2$ is not super additive.

Next theorem established the reversed hr order of lifetimes of two parallel systems of dependent and heterogeneous components.

Theorem 2.3.2. *Suppose the lifetime vectors $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi)$ and $Y \sim PO(\bar{F}, \boldsymbol{\beta}, \varphi)$. If φ is log-concave and $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing and convex, then*

$$\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta} \text{ implies } X_{n:n} \leq_{rhr} Y_{n:n}.$$

Proof: From (1.3.4), the rhr function of $X_{n:n}$ is given by

$$\begin{aligned}
\tilde{r}_{X_{n:n}}(x) &= \frac{\tilde{r}(x)}{\bar{F}(x)} \frac{\varphi'\left(\sum_{i=1}^{n} \phi\left(F_{\alpha_i}(x)\right)\right)}{\varphi\left(\sum_{i=1}^{n} \phi\left(F_{\alpha_i}(x)\right)\right)} \sum_{i=1}^{n} \frac{F_{\alpha_i}(x)}{\varphi'\left(\phi\left(F_{\alpha_i}(x)\right)\right)} \bar{F}_{\alpha_i}(x) \\
&= \frac{\tilde{r}(x)}{\bar{F}(x)} \frac{\varphi'\left(\sum_{i=1}^{n} \phi\left(F_{\alpha_i}(x)\right)\right)}{\varphi\left(\sum_{i=1}^{n} \phi\left(F_{\alpha_i}(x)\right)\right)} \sum_{i=1}^{n} \frac{\varphi\left(\phi\left(F_{\alpha_i}(x)\right)\right)}{\varphi'\left(\phi\left(F_{\alpha_i}(x)\right)\right)} \left(1 - \varphi\left(\phi\left(F_{\alpha_i}(x)\right)\right)\right) \\
&= \frac{\tilde{r}(x)}{\bar{F}(x)} \frac{\varphi'\left(\sum_{i=1}^{n} \xi_i\right)}{\varphi\left(\sum_{i=1}^{n} \xi_i\right)} \sum_{i=1}^{n} \frac{\varphi\left(\xi_i\right)}{\varphi'\left(\xi_i\right)} \left(1 - \varphi\left(\xi_i\right)\right),
\end{aligned}$$

where $\xi_i = \phi\left(F_{\alpha_i}(x)\right)$. Now, for $s \in I_n$,

$$\begin{aligned}
\frac{\tilde{r}_{X_{n:n}}(x)}{\partial \alpha_s} &= \frac{\tilde{r}(x)}{\bar{F}(x)} \left[\frac{\partial}{\partial \xi_s} \left(\frac{\varphi'\left(\sum_{i=1}^{n} \xi_i\right)}{\varphi\left(\sum_{i=1}^{n} \xi_i\right)} \right) \frac{\partial \xi_s}{\partial \alpha_s} \sum_{i=1}^{n} \frac{\varphi\left(\xi_i\right)\left(1 - \varphi\left(\xi_i\right)\right)}{\varphi'\left(\xi_i\right)} + \right. \\
&\quad \left. \frac{\varphi'\left(\sum_{i=1}^{n} \xi_i\right)}{\varphi\left(\sum_{i=1}^{n} \xi_i\right)} \frac{\partial}{\partial \xi_s} \left(\frac{\varphi\left(\xi_s\right)\left(1 - \varphi\left(\xi_s\right)\right)}{\varphi'\left(\xi_s\right)} \right) \frac{\partial \xi_s}{\partial \alpha_s} \right].
\end{aligned}$$

Note that ξ_s is increasing in α_s and $\frac{\partial \xi_s}{\partial \alpha_s}$ is decreasing in α_s. Since φ is log-concave and $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing, it follows $\frac{\tilde{r}_{X_{n:n}}(x)}{\partial \alpha_s} \geq 0$. Again

$$\frac{\partial}{\partial \xi_s}\left(\frac{\varphi'\left(\sum_{i=1}^{n}\xi_i\right)}{\varphi\left(\sum_{i=1}^{n}\xi_i\right)}\right)=\frac{\partial}{\partial \xi_t}\left(\frac{\varphi'\left(\sum_{i=1}^{n}\xi_i\right)}{\varphi\left(\sum_{i=1}^{n}\xi_i\right)}\right),\text{ for } s\neq t.$$

For $s \neq t$,

$$\begin{aligned}
& (\alpha_s-\alpha_t)\left(\frac{\tilde{r}_{X_{n:n}}}{\partial\alpha_s}-\frac{\tilde{r}_{X_{n:n}}}{\partial\alpha_t}\right)\\
\stackrel{sign}{=}\; & (\alpha_s-\alpha_t)\left(\frac{\partial\xi_s}{\partial\alpha_s}-\frac{\partial\xi_t}{\partial\alpha_t}\right)+(\alpha_s-\alpha_t)\left(-\frac{\varphi'\left(\sum_{i=1}^{n}\xi_i\right)}{\varphi\left(\sum_{i=1}^{n}\xi_i\right)}\right)\times\\
& \left[\left(-\frac{\partial}{\partial\xi_s}\left(\frac{\varphi(\xi_s)(1-\varphi(\xi_s))}{\varphi'(\xi_s)}\right)\right)\frac{\partial\xi_s}{\partial\alpha_s}-\left(-\frac{\partial}{\partial\xi_t}\left(\frac{\varphi(\xi_t)(1-\varphi(\xi_t))}{\varphi'(\xi_t)}\right)\right)\frac{\partial\xi_t}{\partial\alpha_t}\right]\\
\leq\; & 0,
\end{aligned}$$

as $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing and convex. Thus $\tilde{r}_{X_{n:n}}(x)$ is increasing in α_i, $i \in I_n$ and schur-concave in $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$. Then from Lemma 1.2.2, it follows

$$\boldsymbol{\alpha} \stackrel{w}{\succeq} \boldsymbol{\beta} \text{ implies } \tilde{r}_{X_{n:n}}(x) \leq \tilde{r}_{Y_{n:n}}(x).$$

Hence the theorem follows. The following counterexample shows that one may not get the the ordering result in Theorem 2.3.2 if the sufficient conditions on the generator functions are dropped.

Counterexample 2.3.2. Consider two parallel systems, each comprising of four dependent and heterogeneous components with respective reversed hr functions $\tilde{r}_{X_{4:4}}(x)$ and $\tilde{r}_{Y_{4:4}}(x)$, with common baseline sf $\bar{F}(x) = e^{-x^3}$, $x \geq 0$, $\boldsymbol{\alpha} = (0.2, 0.6, 1.5, 3.5)$, $\boldsymbol{\beta} = (0.8, 0.9, 4.5, 5.5)$ so that $\boldsymbol{\alpha} \stackrel{w}{\succeq} \boldsymbol{\beta}$. First consider the common generator $\varphi(x) = (1/(ax+1))^{1/a}$, $a = 0.2$, which is not log-concave but $\frac{\varphi(1-\varphi)}{\varphi'}$ is decreasing and convex. Next consider $\varphi(x) = (2/(1+e^x))^{1/a}$, $a = 0.2$, which is log-concave but $\frac{\varphi(1-\varphi)}{\varphi'}$ is neither decreasing nor convex. For both the cases $\tilde{r}_{X_{4:4}}(x)$ and $\tilde{r}_{Y_{4:4}}(x)$ are depicted in Figure 2.4(a) and 2.4(b) respectively for some finite range of x. From both the figures it is observe that the reversed hr ordering result in Theorem 2.3.2 is not attained.

2.4 Applications

This section highlight some potential applications of the established results. Consider a series (or a parallel) system of n components having dependent lifetimes. It is quite practical that the odds functions of all the components (i.e. the odds of surviving beyond a specified time t) may not be the same for various possible reasons, like the components

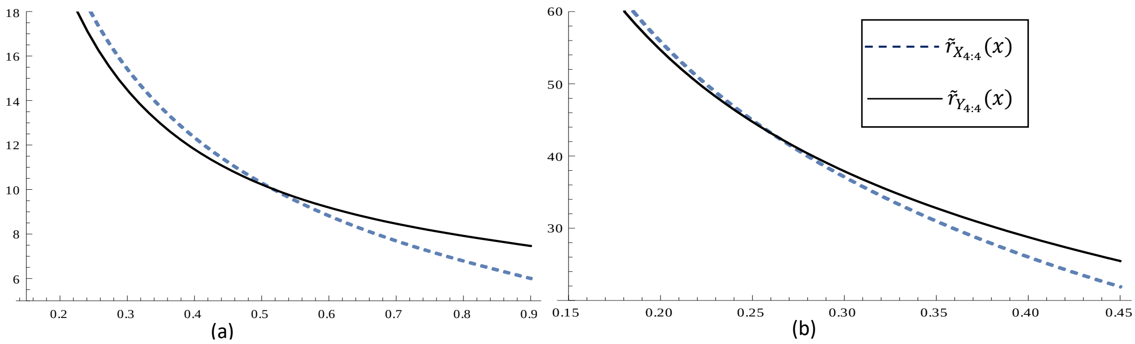

Figure 2.4: Plots of $\tilde{r}_{X_{4:4}}(x)$ and $\tilde{r}_{Y_{4:4}}(x)$ for (a) $\varphi(x)$ is not log-concave, (b) $\frac{\varphi(1-\varphi)}{\varphi'}$ is neither decreasing nor convex.

are manufactured by different manufacturing units or they are subjected to different levels of stress. So let the odds function of ith component is proportional to a baseline odds function with proportionality constant (odds ratio) α_i, $i = 1, 2, \ldots, n$. Now consider another series (or parallel) system of n dependent components having different odds ratios β_i, $i = 1, 2, \ldots, n$. Even if a same system operates in two different levels of environments/stress (e.g., voltage, temperature, compression and tension), then reliability characteristics (e.g., odds function) of a component of the system generally will not be the same in the two different environments. So it is a subject of interest to compare lifetimes of two such systems, i.e. under what conditions one system will be more reliable than other. Theorems 2.2.1 and 7.2.1 (resp. Theorem 2.3.1) give the conditions on the corresponding odds ratio vectors and the generators of the survival copulas under which a series (resp. parallel) system will have stochastically longer lifetime than that of the other. Similarly Theorem 6.1.2 (resp. Theorem 2.3.2) gives the conditions under which failure rate of a series (resp. parallel) system will be smaller than that of the other. Next it is shown that using proposed results one can compare the lifetime of two series systems whose components are subjected to random shock instantaneously Fang and Balakrishnan [42]. Suppose rv X_i denotes the lifetime of i-th component of the series system. Define Bernoulli rv I_{p_i} associated with X_i, where $I_{p_i} = 1$ if shock does not occur and 0 if shock occurs with $p_i = P(I_{p_i} = 1)$, $i = 1, \ldots, n$. Assume that $I_{p_1}, \ldots, I_{p_n}$ are independent rvs, and also they are independent of $X_1, \ldots, X_n$. Let $X_i^* = X_i I_{p_i}$, $i = 1, \ldots, n$, and denote $X_{1:n}^* = \min(X_1^*, \ldots, X_n^*)$. Similarly assume that $I_{q_1}, \ldots, I_{q_n}$ are independent Bernoulli rvs, and also they are independent of Y_i's with $q_i = P(I_{q_i} = 1)$, $i = 1, \ldots, n$. Denote $Y_{1:n}^* = \min(Y_1^*, \ldots, Y_n^*)$, where $Y_i^* = Y_i I_{q_i}$, $i = 1, \ldots, n$. Here $X_{1:n}^*$ represents the lifetime of a series system whose components are subjected to random shock instantaneously. Similarly $Y_{1:n}^*$ represents the the lifetime of another such series system. Now, if $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi_1)$ and $Y \sim PO(\bar{F}, \boldsymbol{\beta}, \varphi_2)$, then with the help of the Theorems 2.2.1, 7.2.1, 6.1.2, and the associated corollaries 2.2.1, 2.2.2, we can establish following stochastic comparisons between such smallest order statistics from the fact that $P(X^*_{1:n} > x) = P(X_1 > x, \ldots, X_n > x)P(I_{p_i} = 1, i \in I_n) = P(X_{1:n} > x) \prod_i^n p_i$.

Theorem 2.4.1. *Suppose the lifetime vectors* $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi_1)$ *and* $Y \sim PO(\bar{F}, \boldsymbol{\beta}, \varphi_2)$. *If* φ_1 *or* φ_2 *is log-convex,* $\phi_2 \circ \varphi_1$ *is superadditive and* $\prod_i^n p_i \leq \prod_i^n q_i$, *then*

$$\boldsymbol{\alpha} \overset{p}{\succeq} \boldsymbol{\beta} \text{ implies } X^*_{1:n} \leq_{st} Y^*_{1:n}.$$

Corollary 2.4.1. *Suppose the lifetime vectors* $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi)$ *and* $Y \sim PO(\bar{F}, \boldsymbol{\beta}, \varphi)$. *If* φ *is log-convex and* $\prod_i^n p_i \leq \prod_i^n q_i$, *Then*

$$\boldsymbol{\alpha} \overset{p}{\succeq} \boldsymbol{\beta} \text{ implies } X^*_{1:n} \leq_{st} Y^*_{1:n}.$$

Theorem 2.4.2. *Suppose the lifetime vectors* $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi_1)$ *and* $Y \sim PO(\bar{F}, \boldsymbol{\beta}, \varphi_2)$. *If* $\phi_2 \circ \varphi_1$ *is superadditive and* $\prod_i^n p_i \leq \prod_i^n q_i$, *then*

$$\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta} \text{ implies } X^*_{1:n} \leq_{st} Y^*_{1:n}.$$

Corollary 2.4.2. *Suppose the lifetime vectors* $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi)$ *and* $Y \sim PO(\bar{F}, \boldsymbol{\beta}, \varphi)$. *If* $\prod_i^n p_i \leq \prod_i^n q_i$, *then*

$$\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta} \text{ implies } X^*_{1:n} \leq_{st} Y^*_{1:n}.$$

Theorem 2.4.3. *Suppose the lifetime vectors* $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi)$ *and* $Y \sim PO(\bar{F}, \boldsymbol{\beta}, \varphi)$. *If* φ *is log-concave and* $\frac{\varphi(1-\varphi)}{\varphi'}$ *is decreasing and concave, then*

$$\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta} \text{ implies } X^*_{1:n} \leq_{hr} Y^*_{1:n}.$$

We will end this section by mentioning an other potential application. In actuarial science, $X^*_{1:n}$ corresponds to the smallest claim amount in a portfolio of risks Barmalzan et al. [15], Li and Li [86], Zhang et al. [136], where X_i's represent sizes of random claims of multiple risks covered by a policy that can be made in an insurance period and the corresponding I_{p_i}'s indicate the occurrence of these claims. That means $I_{p_i} = 1$ whenever the ith policy makes random claim X_i and $I_{p_i} = 0$ whenever there is no claim. Similarly suppose $Y^*_{1:n}$ represents the smallest claim amount in an another portfolio of risks. The above theorems can be used in stochastic comparisons between the smallest claim amounts of two different portfolio of risks.

Chapter 3

Dispersive & star ordering of sample extremes [1]

3.1 Introduction

Stochastic ordering has been widely used to compare the magnitude and variability of extreme order statistics. However, despite the importance and wide applications of the variability orders (e.g. dispersive order and star order), there are less research works in this direction as compared to the magnitude orders (e.g., stochastic order,hr order, rhr order, and lr order).

Skewed distributions often serve as reasonable models for system lifetimes, auction theory, insurance claim amounts, financial returns etc. and thus it is of interest to compare skewness of probability distributions (Wu et al. [130]). Recently, there have been a number of works on dispersive and star ordering of extreme order statistics of random samples from different family of distributions (Ding et al. [40], Fang et al. [43, 44], Kochar and Xu [77, 76], Li and Fang [87], Nadeb et al. [103], Zhang et al. [138, 137]). There are some research works on sample spacings also, like Xu and Li [132] established dispersive and star ordering for sample spacing from heterogeneous exponential distributions. An extensive literature review has been done in Balakrishnan and Zhao [11] on the stochastic comparison

[1] One paper based on this chapter have appeared as under:

1. Dispersive and star ordering of sample extremes from dependent random variables following the proportional odds model. *Communications in Statistics - Theory and Methods*, 2022, DOI: 10.1080/03610926.2022.2037643.

of order statistics corresponding to independent and heterogeneous rvs.

In case of dependent samples Zhao et al. [141] discussed stochastic comparisons of extreme order statistics from heterogeneous interdependent Weibull samples having a common Archimedean copula. They derived the results for usual stochastic order, rhr order and lr order when shape parameters are common but scale parameters are different and also when scale parameters are common but shape parameters are different. Li and Fang [87] derived the dispersive order between maximums of two PHR samples having a common Archimedean copula. For samples following scale model, Li and Li [86] obtained the dispersive and the star orders between minimums of one heterogeneous and one homogeneous samples sharing a common Archimedean copula. Fang et al. [43] investigated the dispersive order and the star order of extreme order statistics for the samples following PHR model with Archimedean survival copulas. Fang et al. [44] obtained the dispersive order between minimums of two scale proportional hazards samples with a common Archimedean survival copula. With resilience-scaled components, Zhang et al. [137] derived the dispersive and the star order between parallel systems, one consisting dependent heterogeneous components and another consisting homogeneous components sharing a common Archimedean survival copula.

In case of PO model, some authors, e.g. Kundu and Nanda [82], Kundu et al. [81], Panja et al. [114], Li and Li [89] have investigated stochastic comparison of this family of distributions and sample extreme in the sense of magnitude orders. To the best of our knowledge, there is no related study on the variability of extreme order statistics arising from independent or dependent rvs following the PO model. Motivated by this, in chapter, is devoted to the dispersive and the star ordering for comparing the minimums and the maximums of dependent samples following the PO model.
The organization of the rest of the chapter is as follows. In section 3.2, consider the comparisons of minimum order statistics from dependent samples following the PO model in terms of the dispersive order and the star order. Section 3.3 investigates the comparison of maximum order statistics in terms of dispersive and star orders. In section 3.4, some examples are provided for illustrative purpose.

3.2 Ordering for sample minimums

This section consider the dispersive ordering of minimums of dependent rvs. We compare stochastically the minimums of two dependent samples, one formed from heterogeneous rvs and another from homogeneous rvs. The following theorem consider the comparison of minimums of two samples, one from n dependent heterogeneous rvs following the PO model and another from n dependent homogeneous rvs following the PO model, in terms of dispersive order. The result holds for the decreasing failure rate (DFR) baseline distribution F. The distribution function F is said to be DFR if the corresponding hr $r(\cdot)$ is decreasing and increasing failure rate (IFR) distribution if $r(\cdot)$ is increasing.

Theorem 3.2.1. *Suppose $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi)$ and $Y \sim PO(\bar{F}, \alpha\boldsymbol{1}, \varphi)$. Then $X_{1:n} \leq_{disp} Y_{1:n}$ if the baseline distribution F is DFR, φ is log-convex, $\frac{\varphi}{\varphi'}$ is concave and $\alpha \geq \frac{1}{n}\sum_{i=1}^{n} \alpha_i$, for $0 \leq \alpha \leq 1$.*

Proof: The cdfs of $X_{1:n}$ and $Y_{1:n}$ are $F_1(x) = 1 - \varphi\left(\sum_{i=1}^{n} \phi\left(\bar{F}_{X_i}(x)\right)\right)$ and $G_1(x) = 1 - \varphi\left(n\phi\left(\bar{F}_{Y_1}(x)\right)\right)$, respectively, where $\bar{F}_{X_i}(x) = \frac{\alpha_i \bar{F}(x)}{1-\bar{\alpha}_i \bar{F}(x)}$ and $\bar{F}_{Y_1}(x) = \frac{\alpha \bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}$, $x \in \mathbb{R}$. The respective pdfs of $X_{1:n}$ and $Y_{1:n}$ are given by

$$f_1(x) = \varphi'\left(\sum_{i=1}^{n} \phi\left(\bar{F}_{X_i}(x)\right)\right) \sum_{i=1}^{n} \frac{\varphi\left(\phi\left(\bar{F}_{X_i}(x)\right)\right)}{\varphi'\left(\phi\left(\bar{F}_{X_i}(x)\right)\right)} \frac{r(x)}{1-\bar{\alpha}_i \bar{F}(x)}, \tag{3.2.1}$$

and

$$g_1(x) = n\varphi'\left(n\phi\left(\bar{F}_{Y_1}(x)\right)\right) \cdot \frac{r(x)}{1-\bar{\alpha}\bar{F}(x)} \cdot \frac{\varphi\left(\phi\left(\bar{F}_{Y_1}(x)\right)\right)}{\varphi'\left(\phi\left(\bar{F}_{Y_1}(x)\right)\right)},$$

Therefore

$$G_1^{-1}(x) = \bar{F}^{-1}\left(\frac{\varphi\left(\frac{1}{n}\phi(1-x)\right)}{\alpha + \bar{\alpha}\varphi\left(\frac{1}{n}\phi(1-x)\right)}\right).$$

So

$$G_1^{-1}(F_1(x)) = \bar{F}^{-1}\left(\frac{\varphi\left(\frac{1}{n}\sum_{i=1}^{n} \phi\left(\bar{F}_{X_i}(x)\right)\right)}{\alpha + \bar{\alpha}\varphi\left(\frac{1}{n}\sum_{i=1}^{n} \phi\left(\bar{F}_{X_i}(x)\right)\right)}\right) = \bar{F}^{-1}(\gamma(x)), \tag{3.2.2}$$

where $\gamma(x) = \frac{\varphi\left(\frac{1}{n}\sum_{i=1}^{n} \phi\left(\bar{F}_{X_i}(x)\right)\right)}{\alpha+\bar{\alpha}\varphi\left(\frac{1}{n}\sum_{i=1}^{n} \phi\left(\bar{F}_{X_i}(x)\right)\right)}$.

Now,

$$\begin{aligned} g_1(G_1^{-1}(F_1(x))) &= n\varphi'\left(n\phi\left(\frac{\alpha\gamma(x)}{1-\bar{\alpha}\gamma(x)}\right)\right) \cdot \frac{r\left(\bar{F}^{-1}(\gamma(x))\right)}{1-\bar{\alpha}\gamma(x)} \cdot \frac{\varphi\left(\phi\left(\frac{\alpha\gamma(x)}{1-\bar{\alpha}\gamma(x)}\right)\right)}{\varphi'\left(\phi\left(\frac{\alpha\gamma(x)}{1-\bar{\alpha}\gamma(x)}\right)\right)} \\ &= n\varphi'\left(\sum_{i=1}^{n} \phi\left(\bar{F}_{X_i}(x)\right)\right) \cdot \left(\alpha + \bar{\alpha}\varphi\left(\frac{1}{n}\sum_{i=1}^{n} \phi\left(\bar{F}_{X_i}(x)\right)\right)\right) \\ &\quad \times \frac{\varphi\left(\frac{1}{n}\sum_{i=1}^{n} \phi\left(\bar{F}_{X_i}(x)\right)\right)}{\varphi'\left(\frac{1}{n}\sum_{i=1}^{n} \phi\left(\bar{F}_{X_i}(x)\right)\right)} \cdot \frac{r\left(\bar{F}^{-1}(\gamma(x))\right)}{\alpha}. \end{aligned} \tag{3.2.3}$$

Note that $\bar{F}_{X_i}(x)$ is increasing and concave in α_i and $1/(1-\bar{\alpha}_i\bar{F}(x))$ is decreasing and convex in α_i. Also $\phi\left(\bar{F}_{X_i}(x)\right)$ is decreasing and convex in α_i if φ is log-convex. Now denote $\frac{1}{n}\sum_{i=1}^n \alpha_i = \boldsymbol{\alpha}^{avg}$ and $\eta(\alpha_i) = \phi\left(\bar{F}_{X_i}(x)\right)$. Then for $\alpha \geq \frac{1}{n}\sum_{i=1}^n \alpha_i = \boldsymbol{\alpha}^{avg}$, from the convexity and decreasing property of $\eta(\alpha_i) = \phi\left(\bar{F}_{X_i}(x)\right)$ with respect to α_i, implies $\frac{1}{n}\sum_{i=1}^n \eta(\alpha_i) \geq \eta(\boldsymbol{\alpha}^{avg}) \geq \eta(\alpha)$, which gives

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^n \phi\left(\bar{F}_{X_i}(x)\right) &\geq \phi\left(\bar{F}_{Y_1}(x)\right) \qquad (3.2.4)\\
\implies \frac{\alpha}{\bar{\alpha}} + \varphi\left(\frac{1}{n}\sum_{i=1}^n \phi\left(\bar{F}_{X_i}(x)\right)\right) &\leq \frac{\alpha}{\bar{\alpha}} + \bar{F}_{Y_1}(x)\\
\implies 1 - \frac{\frac{\alpha}{\bar{\alpha}}}{\frac{\alpha}{\bar{\alpha}} + \varphi\left(\frac{1}{n}\sum_{i=1}^n \phi\left(\bar{F}_{X_i}(x)\right)\right)} &\leq 1 - \frac{\frac{\alpha}{\bar{\alpha}}}{\frac{\alpha}{\bar{\alpha}} + \bar{F}_{Y_1}(x)}\\
\implies \frac{\varphi\left(\frac{1}{n}\sum_{i=1}^n \phi\left(\bar{F}_{X_i}(x)\right)\right)}{\alpha + \bar{\alpha}\varphi\left(\frac{1}{n}\sum_{i=1}^n \phi\left(\bar{F}_{X_i}(x)\right)\right)} &\leq \frac{\bar{F}_{Y_1}(x)}{\alpha + \bar{\alpha}\bar{F}_{Y_1}(x)}.
\end{aligned}$$

This implies $\gamma(x) \leq \bar{F}(x)$. As a result it implies $\bar{F}^{-1}(\gamma(x)) \geq x$. Now if $r(\cdot)$ is decreasing then

$$r(\bar{F}^{-1}(\gamma(x))) \leq r(x). \qquad (3.2.5)$$

Now (3.2.4), gives

$$\alpha + \bar{\alpha}\varphi\left(\frac{1}{n}\sum_{i=1}^n \phi\left(\bar{F}_{X_i}(x)\right)\right) \leq \frac{\alpha}{1-\bar{\alpha}\bar{F}(x)} \leq \alpha\frac{1}{n}\sum_{i=1}^n \frac{1}{1-\bar{\alpha}_i\bar{F}(x)},$$

since $\frac{1}{1-\bar{\alpha}_i\bar{F}(x)}$ is decreasing and convex in α_i. If $\frac{\varphi}{\varphi'}$ is concave, then

$$-\frac{\varphi\left(\frac{1}{n}\sum_{i=1}^n \phi\left(\bar{F}_{X_i}(x)\right)\right)}{\varphi'\left(\frac{1}{n}\sum_{i=1}^n \phi\left(\bar{F}_{X_i}(x)\right)\right)} \leq -\frac{1}{n}\sum_{i=1}^n \frac{\varphi\left(\phi\left(\bar{F}_{X_i}(x)\right)\right)}{\varphi'\left(\phi\left(\bar{F}_{X_i}(x)\right)\right)}.$$

Thus

$$\begin{aligned}
&\left(\alpha + \bar{\alpha}\varphi\left(\frac{1}{n}\sum_{i=1}^n \phi\left(\bar{F}_{X_i}(x)\right)\right)\right)\left(-\frac{\varphi\left(\frac{1}{n}\sum_{i=1}^n \phi\left(\bar{F}_{X_i}(x)\right)\right)}{\varphi'\left(\frac{1}{n}\sum_{i=1}^n \phi\left(\bar{F}_{X_i}(x)\right)\right)}\right)\\
&\leq \frac{\alpha}{n}\sum_{i=1}^n\left(-\frac{\varphi\left(\phi\left(\bar{F}_{X_i}(x)\right)\right)}{\varphi'\left(\phi\left(\bar{F}_{X_i}(x)\right)\right)}\right)\frac{1}{n}\sum_{i=1}^n \frac{1}{1-\bar{\alpha}_i\bar{F}(x)}. \qquad (3.2.6)
\end{aligned}$$

If φ is log-convex, then $-\frac{\varphi(x)}{\varphi'(x)}$ is increasing in x, so that $-\frac{\varphi(\phi(\bar{F}_{X_i}(x)))}{\varphi'(\phi(\bar{F}_{X_i}(x)))}$ is decreasing in α_i. So by Chebyshev's inequality the following inequality follows

$$\frac{1}{n}\sum_{i=1}^{n}\left(-\frac{\varphi\left(\phi\left(\bar{F}_{X_i}(x)\right)\right)}{\varphi'\left(\phi\left(\bar{F}_{X_i}(x)\right)\right)}\right)\cdot\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1-\bar{\alpha}_i\bar{F}(x)}\leq\frac{1}{n}\sum_{i=1}^{n}\left(-\frac{\varphi\left(\phi\left(\bar{F}_{X_i}(x)\right)\right)}{\varphi'\left(\phi\left(\bar{F}_{X_i}(x)\right)\right)}\right)\frac{1}{1-\bar{\alpha}_i\bar{F}(x)}. \tag{3.2.7}$$

From (3.2.5), (3.2.6), (3.2.7) and the fact that the common factor $\varphi'\left(\sum_{i=1}^{n}\phi\left(\bar{F}_{X_i}(x)\right)\right)$ in (3.2.1) and (3.2.3) is negative, implies $g_1(G_1^{-1}(F_1(x)))\leq f_1(x)$ for all $x\in\mathbb{R}$. Hence the theorem follows.

Remark 3.2.1. *It is to be noted that following generators of the Archimedean copula satisfy the conditions of the above theorem*

(i) $\varphi_1(x)=(\theta_1 x+1)^{-1/\theta_1}, \theta_1\in(1,\infty)$.
(ii) $\varphi_2(x)=\frac{\theta_2}{log(x+e^{\theta_2})}, \theta_2\in(0,\infty)$.

It may be of interest to know whether as in case of Theorem 3.2.1 it is possible to establish dispersive ordering for $\alpha\geq 1$ when the baseline distribution is IFR or DFR. The following counterexample shows that with these conditions, Theorem 3.2.1 cannot establish dispersive ordering even in case of samples from independent rvs.

Counterexample 3.2.1. *Consider the minimums of two samples, one having three independent and heterogeneous rvs, and another having three independent and homogeneous rvs with respective cdfs* $F_1(x)=1-\prod_{i=1}^{3}\left(\frac{\alpha_i\bar{F}(x)}{1-\bar{\alpha}_i\bar{F}(x)}\right)$ *and* $G_1(x)=1-\left(\frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}\right)^3$*, where* $\alpha_1=7$*,* $\alpha_2=25$*,* $\alpha_3=100$*,* $\alpha=(\alpha_1+\alpha_2+\alpha_3)/3=44$*, and* $\bar{F}(x)=e^{-(9x)^{0.9}}$*, so that the baseline distribution is DFR. Therefore*

$$g_1(G_1^{-1}(F_1(x)))=\frac{1}{\alpha}\,3\left(\prod_{i=1}^{3}\bar{F}_{X_i}(x)\right)\left(\alpha+\bar{\alpha}\left(\prod_{i=1}^{3}\bar{F}_{X_i}(x)\right)^{1/3}\right)r(\bar{F}^{-1}(\gamma(x))),$$

where $\gamma(x)=\frac{\left(\prod_{i=1}^{3}\bar{F}_{X_i}(x)\right)^{1/3}}{\alpha+\bar{\alpha}\left(\prod_{i=1}^{3}\bar{F}_{X_i}(x)\right)^{1/3}}$,
and

$$f_1(x)=\left(\prod_{i=1}^{3}\bar{F}_{X_i}(x)\right)r(x)\left(\sum_{i=1}^{3}\frac{1}{1-\bar{\alpha}_i\bar{F}(x)}\right).$$

$g_1(G_1^{-1}(F_1(x)))-f_1(x)$ plotted by substituting $x = t/(1-t)$, implies $x \in [0,\infty)$, and $t \in [0,1)$. The plot is shown in Figure 3.1(a) it is observed from the plot that $g_1(G_1^{-1}(F_1(x))) \nleq f_1(x)$ and also $g_1(G_1^{-1}(F_1(x))) \ngeq f_1(x)$.

Next consider $\alpha_1 = 0.78$, $\alpha_2 = 0.97$, $\alpha_3 = 67$, $\alpha = (\alpha_1 + \alpha_2 + \alpha_3)/3 = 22.9167$, and $\bar{F}(x) = e^{-x^3}$, so that the baseline distribution is IFR. Figure 3.1(b) illustrates the plot of $g_1(G_1^{-1}(F_1(x))) - f_1(x)$ by substituting $x = t/(1-t)$, so that for $x \in [0,\infty)$ implies $t \in [0,1)$. From Figure 3.1(b) it is observed that $g_1(G_1^{-1}(F_1(x))) - f_1(x) \nleq 0$ and also $g_1(G_1^{-1}(F_1(x))) - f_1(x) \ngeq 0$.

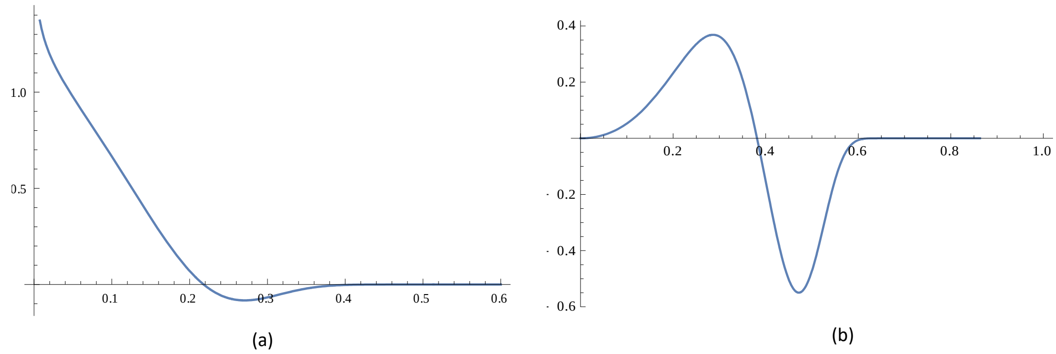

Figure 3.1: Plot of $g_1(G_1^{-1}(F_1(x))) - f_1(x)$ for $x = t/(1-t)$, $t \in [0,1]$ when baseline distribution is (a) DFR and (b) IFR.

The following theorem compare the minimums of two samples, both from n dependent homogeneous rvs following the PO model and with different Archimedean copulas.

Theorem 3.2.2. *Suppose $X \sim PO(\bar{F}, \alpha\boldsymbol{1}, \varphi_1)$ and $Y \sim PO(\bar{F}, \alpha\boldsymbol{1}, \varphi_2)$. Then $X_{1:n} \leq_{disp} Y_{1:n}$ if the baseline distribution is DFR, $\varphi_2(\phi_2(w)/n)/\varphi_1(\phi_1(w)/n)$ is increasing in w and $0 \leq \alpha \leq 1$.*

Proof: The cdfs of $X_{1:n}$ and $Y_{1:n}$ are given by $G_1(x) = 1 - \varphi_1\left(n\phi_1\left(\bar{F}_{X_1}(x)\right)\right)$, and $G_2(x) = 1 - \varphi_2\left(n\phi_2\left(\bar{F}_{X_1}(x)\right)\right)$, respectively, where $\bar{F}_{X_1}(x) = \frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}$, $x \in \mathbb{R}$. The respective pds are given by

$$g_1(x) = n\varphi_1'\left(n\phi_1\left(\bar{F}_{X_1}(x)\right)\right)\frac{\varphi_1\left(\phi_1\left(\bar{F}_{X_1}(x)\right)\right)}{\varphi_1'\left(\phi_1\left(\bar{F}_{X_1}(x)\right)\right)}\cdot\frac{r(x)}{1-\bar{\alpha}\bar{F}(x)}, \tag{3.2.8}$$

and

$$g_2(x) = n\varphi_2'\left(n\phi_2\left(\bar{F}_{X_1}(x)\right)\right)\frac{\varphi_2\left(\phi_2\left(\bar{F}_{X_1}(x)\right)\right)}{\varphi_2'\left(\phi_2\left(\bar{F}_{X_1}(x)\right)\right)}\cdot\frac{r(x)}{1-\bar{\alpha}\bar{F}(x)}.$$

Therefore

$$G_2^{-1}(G_1(x)) = \bar{F}^{-1}\left(\frac{\varphi_2\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(\bar{F}_{X_1}(x)\right)\right)\right)\right)}{\alpha + \bar{\alpha}\varphi_2\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(\bar{F}_{X_1}(x)\right)\right)\right)\right)}\right) = \bar{F}^{-1}(\eta(x)),$$

where $\eta(x) = \frac{\varphi_2\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(\bar{F}_{X_1}(x)\right)\right)\right)\right)}{\alpha+\bar{\alpha}\varphi_2\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(\bar{F}_{X_1}(x)\right)\right)\right)\right)}$.

$$\begin{aligned} g_2(G_2^{-1}(G_1(x))) &= n\varphi_2'\left(\phi_2\left(\varphi_1\left(n\phi_1\left(\bar{F}_{X_1}(x)\right)\right)\right)\right) \\ &\times \frac{\varphi_2\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(\bar{F}_{X_1}(x)\right)\right)\right)\right)}{\varphi_2'\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(\bar{F}_{X_1}(x)\right)\right)\right)\right)} \frac{r\left(\bar{F}^{-1}(\eta(x))\right)}{\alpha} \\ &\times \left(\alpha + \bar{\alpha}\varphi_2\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(\bar{F}_{X_1}(x)\right)\right)\right)\right)\right). \end{aligned} \tag{3.2.9}$$

From Lemma 3.9 of Fang et al. [43], for increasing $\varphi_2(\phi_2(w)/n)/\varphi_1(\phi_1(w)/n)$ implies $\varphi_2\left(n\phi_2\left(\bar{F}_{X_1}(x)\right)\right) \geq \varphi_1\left(n\phi_1\left(\bar{F}_{X_1}(x)\right)\right)$, which implies

$$\bar{F}_{X_1}(x) \geq \varphi_2\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(\bar{F}_{X_1}(x)\right)\right)\right)\right).$$

Again this gives $\bar{F}(x) \geq \eta(x)$ which implies $\bar{F}^{-1}(\eta(x)) \geq x$. Thus if $r(\cdot)$ is decreasing then

$$r(\bar{F}^{-1}(\eta(x))) \leq r(x). \tag{3.2.10}$$

Also for $\bar{\alpha} \geq 0$, $\varphi_2\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(\bar{F}_{X_1}(x)\right)\right)\right)\right) \leq \bar{F}_{X_1}(x)$ implies

$$\alpha + \bar{\alpha}\varphi_2\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(\bar{F}_{X_1}(x)\right)\right)\right)\right) \leq \frac{\alpha}{1-\bar{\alpha}\bar{F}(x)}. \tag{3.2.11}$$

Again from Lemma 3.9 of Fang et al. [43] by substituting $w = \varphi_1\left(n\phi_1\left(\bar{F}_{X_1}(x)\right)\right)$ in increasing $\frac{\varphi_1(\phi_1(w)/n)}{\varphi_2(\phi_2(w)/n)}$ implies

$$\begin{aligned} &\frac{\varphi_2'\left(\phi_2\left(\varphi_1\left(n\phi_1\left(\bar{F}_{X_1}(x)\right)\right)\right)\right)\varphi_2\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(\bar{F}_{X_1}(x)\right)\right)\right)\right)}{\varphi_2'\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(\bar{F}_{X_1}(x)\right)\right)\right)\right)} \\ &\qquad \leq \frac{\varphi_1'\left(n\phi_1\left(\bar{F}_{X_1}(x)\right)\right)\varphi_1\left(\phi_1\left(\bar{F}_{X_1}(x)\right)\right)}{\varphi_1'\left(\phi_1\left(\bar{F}_{X_1}(x)\right)\right)}. \end{aligned} \tag{3.2.12}$$

Now using (3.2.10), (3.2.11) and (3.2.12), from (3.2.9) and (3.2.8) implies
$g_2(G_2^{-1}(G_1(x))) \leq g_1(x)$ for all $x \in \mathbb{R}$. This completes the proof.

Remark 3.2.2. It is to be noted that Archimedean copula with generators as specified in (i) and (ii) below satisfy the condition that $\varphi_2(\phi_2(w)/n)/\varphi_1(\phi_1(w)/n)$ is increasing in w for all $n \in Z$.

(i) $\varphi_1(x) = \left(1 + x^{1/\theta_1}\right)^{-\theta_1}$ and $\varphi_2(x) = \frac{1}{(x^{1/\theta_2}+1)}$ where $\theta_1 \in (1, \infty), \theta_2 \in (1, \infty)$ and $1 < \theta_1 < \theta_2 < \infty$, then for any $n \in Z$,

$$\frac{d}{dx}(\varphi_2(\phi_2(x)/n)/\varphi_1(\phi_1(x)/n)) = \left[\frac{\left(1 + \left(\frac{(x^{-1/\theta_1}-1)^{\theta_1}}{n}\right)^{1/\theta_1}\right)^{\theta_1}}{x^2 \left(1 + \left(\frac{((1/x)-1)^{\theta_2}}{n}\right)^{1/\theta_2}\right)^2} \times \right.$$
$$\left. \left(-\frac{x\left(\frac{((1/x)-1)^{\theta_2}}{n}\right)^{1/\theta_2}}{x-1} \right) + \frac{x\left(1 + \left(\frac{((1/x)-1)^{\theta_2}}{n}\right)^{1/\theta_2}\right)\left(\frac{(x^{-1/\theta_1}-1)^{\theta_1}}{n}\right)}{(x^{1/\theta_1-1})\left(1 + \left(\frac{(x^{-1/\theta_1}-1)^{\theta_1}}{n}\right)^{1/\theta_1}\right)} \right]$$
$$\overset{sgn}{=} \left(-\frac{x\left(\frac{((1/x)-1)^{\theta_2}}{n}\right)^{1/\theta_2}}{x-1} \right) + \frac{x\left(1 + \left(\frac{((1/x)-1)^{\theta_2}}{n}\right)^{1/\theta_2}\right)\left(\frac{(x^{-1/\theta_1}-1)^{\theta_1}}{n}\right)}{(x^{1/\theta_1-1})\left(1 + \left(\frac{(x^{-1/\theta_1}-1)^{\theta_1}}{n}\right)^{1/\theta_1}\right)}.$$

Since $1 < \theta_1 < \theta_2 < \infty$, from the last expression it can be easily concluded that the derivative is non-negative for all $x \in [0, 1]$.

(ii) $\varphi_1(x) = (\theta_1 x + 1)^{-1/\theta_1}$ and $\varphi_2(x) = [\theta_2 x + 1]^{-1/\theta_2}$, for $0 < \theta_1 < \theta_2 < \infty$, then for any $n \in Z$,

$$\frac{d}{dx}(\varphi_2(\phi_2(x)/n)/\varphi_1(\phi_1(x)/n)) = \frac{(n-1)\left(\frac{x^{-\theta_1}+n-1}{n}\right)^{\frac{1}{\theta_1}}\left(\frac{x^{-\theta_2}+n-1}{n}\right)^{\frac{1}{\theta_2}}\left(x^{\theta_1} - x^{\theta_2}\right)}{x(x^{\theta_1}+n-1)(x^{\theta_2}+n-1)}.$$

From the above expression it is very clear that for $0 < \theta_1 < \theta_2 < \infty$ it is non-negative for all $x \in [0, 1]$.

The following corollary follows from Theorems 3.2.1 and 3.2.2. This corollary compare the minimums of two samples, one from from n dependent heterogeneous rvs following the PO model and another from n dependent homogeneous rvs following the PO model and with different Archimedean copulas.

Corollary 3.2.1. *Suppose $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi_1)$ and $Y \sim PO(\bar{F}, \alpha\boldsymbol{1}, \varphi_2)$. Then for $\alpha \geq \frac{1}{n}\sum_{i=1}^{n}\alpha_i$, $X_{1:n} \leq_{disp} Y_{1:n}$ if the baseline distribution is DFR, φ_1 is log-convex, $\frac{\varphi_1}{\varphi_1'}$ is concave, $\varphi_2(\phi_2(w)/n)/\varphi_1(\phi_1(w)/n)$ is increasing in w, and $0 \leq \alpha \leq 1$.*

Proof: Let $Z \sim PO(\bar{F}, \alpha\mathbf{1}, \varphi_1)$. Then Theorem 3.2.1, gives $X_{1:n} \leq_{disp} Z_{1:n}$. Again

Theorem 3.2.2, implies $Z_{1:n} \leq_{disp} Y_{1:n}$. This yields $X_{1:n} \leq_{disp} Y_{1:n}$.

Remark 3.2.3. It is to be noted that for any $n \in Z$ Archimedean copula with generators $\varphi_1(x) = \left(1 + x^{1/\theta_1}\right)^{-\theta_1}$ and $\varphi_2(x) = \frac{1}{\left(x^{1/\theta_2}+1\right)}$ where $\theta_1 \in (1, \infty), \theta_2 \in (1, \infty)$ and $1 < \theta_1 < \theta_2 < \infty$ satisfy the condition that $\varphi_2(\phi_2(w)/n)/\varphi_1(\phi_1(w)/n)$ is increasing in w for all $n \in Z$.

The following theorem compare the minimums of two samples, one from n dependent heterogeneous rvs following the PO model and another from n dependent homogeneous rvs following the PO model, in terms of star order.

Theorem 3.2.3. *Suppose $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi)$ and $Y \sim PO(\bar{F}, \alpha\boldsymbol{1}, \varphi)$. Then for $\alpha \geq \frac{1}{n}\sum_{i=1}^{n} \alpha_i$ we have $X_{1:n} \leq_{\star} Y_{1:n}$ if $xr(x)$ is decreasing, φ is log-convex, $\frac{\varphi}{\varphi'}$ is concave and $0 \leq \alpha \leq 1$.*

Proof: Using equations (3.2.1), (3.2.2) and (3.2.3), implies

$$
\begin{aligned}
& x^2 \frac{d}{dx}\left(\frac{G_1^{-1}(F_1(x))}{x}\right) \\
= \; & x\frac{d}{dx}\left(G_1^{-1}(F_1(x))\right) - G_1^{-1}(F_1(x)) \\
= \; & x\frac{f_1(x)}{g_1\left(G_1^{-1}(F_1(x))\right)} - G_1^{-1}(F_1(x)) \\
= \; & \frac{\alpha x r(x) \frac{1}{n}\sum_{i=1}^{n} \frac{\varphi\left(\phi\left(\bar{F}_{X_i}(x)\right)\right)}{\varphi'\left(\phi\left(\bar{F}_{X_i}(x)\right)\right)} \frac{1}{1-\bar{\alpha}_i\bar{F}(x)}}{r\left(\bar{F}^{-1}(\gamma(x))\right)\left(\alpha + \bar{\alpha}\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(\bar{F}_{X_i}(x)\right)\right)\right)\frac{\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(\bar{F}_{X_i}(x)\right)\right)}{\varphi'\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(\bar{F}_{X_i}(x)\right)\right)}} - F^{-1}(\gamma(x))
\end{aligned}
\tag{3.2.13}
$$

In Theorem 3.2.1, for $0 \leq \alpha \leq 1$ it is already proved that $\bar{F}^{-1}(\gamma(x)) \geq x$.
Now, if $xr(x)$ is decreasing in x, then $xr(x) \geq \bar{F}^{-1}(\gamma(x))r\left(\bar{F}^{-1}(\gamma(x))\right)$, that is

$$
\frac{xr(x)}{r(\bar{F}^{-1}(\gamma(x)))} \geq \bar{F}^{-1}(\gamma(x)). \tag{3.2.14}
$$

According to the equations (3.2.6) and (3.2.7) of Theorem (3.2.1), one can conclude that

$$
\frac{\frac{\alpha}{n}\sum_{i=1}^{n} -\left(\frac{\varphi\left(\phi\left(\bar{F}_{X_i}(x)\right)\right)}{\varphi'\left(\phi\left(\bar{F}_{X_i}(x)\right)\right)}\right)\frac{1}{1-\bar{\alpha}_i\bar{F}(x)}}{\left(\alpha + \bar{\alpha}\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(\bar{F}_{X_i}(x)\right)\right)\right)\left(-\frac{\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(\bar{F}_{X_i}(x)\right)\right)}{\varphi'\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(\bar{F}_{X_i}(x)\right)\right)}\right)} \geq 1. \tag{3.2.15}
$$

Using (3.2.14) and (3.2.15), (3.2.13) gives

$$x^2 \frac{d}{dx}\left(\frac{G_1^{-1}(F_1(x))}{x}\right) \geq 0.$$

So, $\frac{G_1^{-1}(F_1(x))}{x}$ is increasing in $x \geq 0$. Hence $X_{n:n} \leq_\star Y_{n:n}$.

Remark 3.2.4. The expression $xr(x)$ is known as the proportional failure rate (also known as the generalized failure rate) Righter et al. [116]. The concerned rv is said to have the Decreasing Proportional Failure Rate (DPFR) property if $xr(x)$ is decreasing in x, and in that case, domain of the rv will be $(0, \infty)$.

It is of interest to know whether as in case of Theorem 3.2.3 it is possible to establish star ordering for $\alpha \geq 1$ when $xr(x)$ is decreasing or increasing. The following counterexample shows that with these conditions, it is not possible establish star ordering even in case of samples from independent rvs.

Counterexample 3.2.2. *Consider maximums of two samples, one having four independent and heterogeneous rvs, and another having four independent and homogeneous rvs. Consider* $\alpha_1 = 0.75$, $\alpha_2 = 0.95$, $\alpha_3 = 23$, $\alpha_4 = 43$, $\alpha = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)/4 = 16.925$, *and* $\bar{F}(x) = (1 + \frac{x}{13})^{-0.9}$, *so that* $xr(x)$ *is increasing. Therefore*

$$G_1^{-1}(F_1(x)) = \bar{F}^{-1}\left(\frac{\left(\prod_{i=1}^4 \bar{F}_{X_i}(x)\right)^{1/4}}{\alpha + \bar{\alpha}\left(\prod_{i=1}^4 \bar{F}_{X_i}(x)\right)^{1/4}}\right).$$

$G_1^{-1}(F_1(x))/x$ *plotted by substituting* $x = t/(1-t)$, *so that for* $x \in [0, \infty)$, $t \in [0, 1)$. *From the Figure 3.2(a), it is observed that* $G_1^{-1}(F_1(x))/x$ *is neither increasing nor decreasing. Next consider* $\alpha_1 = 2$, $\alpha_2 = 33$, $\alpha_3 = 63$, $\alpha_4 = 183$, $\alpha = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)/4 = 281/4$, *and* $\bar{F}(x) = \frac{1}{x^2}$, $x \in [1, \infty)$ *so that* $xr(x)$ *is decreasing.* $G_1^{-1}(F_1(x))/x$ *plotted by substituting* $x = 1/t$, *so that for* $x \in [1, \infty)$, *implies* $t \in [0, 1)$. *From the Figure 3.2(b), it is observed that* $G_1^{-1}(F_1(x))/x$ *is neither increasing nor decreasing.*

The following theorem compares the minimum of two samples, both from n dependent homogeneous rvs following the PO model and with different Archimedean copulas. The proof can be done similiar to the theorem Theorem 3.2.3, and hence omitted.

Theorem 3.2.4. *Suppose* $X \sim PO(\bar{F}, \alpha\boldsymbol{1}, \varphi_1)$ *and* $Y \sim PO(\bar{F}, \alpha\boldsymbol{1}, \varphi_2)$. *Then* $X_{1:n} \leq_* Y_{1:n}$ *if* $xr(x)$ *is decreasing,* $\varphi_2(\phi_2(w)/n)/\varphi_1(\phi_1(w)/n)$ *is increasing in* w, *and* $0 \leq \alpha \leq 1$.

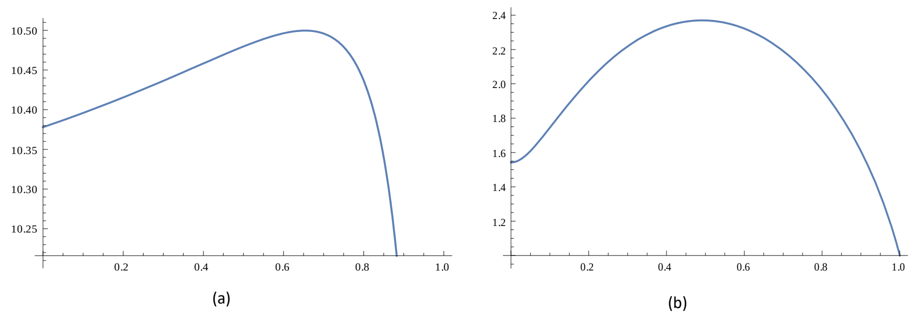

Figure 3.2: Plot of $G_1^{-1}(F_1(x))/x$ for (a) $x = t/(1-t)$ when $xr(x)$ is increasing and (b) $x = 1/t$ when $xr(x)$ is decreasing, $t \in [0, 1]$

The following corollary follows from Theorems 3.2.3 and 3.2.4.

Corollary 3.2.2. *Suppose* $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi_1)$ *and* $Y \sim PO(\bar{F}, \alpha\boldsymbol{1}, \varphi_2)$. *Then for* $\alpha \geq \frac{1}{n}\sum_{i=1}^{n}\alpha_i$, $X_{1:n} \leq_* Y_{1:n}$ *if* $xr(x)$ *is decreasing,* φ_1 *is log-convex,* $\frac{\varphi_1}{\varphi_1'}$ *is concave,* $\frac{\varphi_2(\phi_2(w)/n)}{\varphi_1(\phi_1(w)/n)}$ *is increasing in* w *and* $0 \leq \alpha \leq 1$.

3.3 Ordering for sample maximum

This section stochastically compare the maximums of two dependent samples, one formed from heterogeneous rvs and another from homogeneous rvs. The distribution function of X_i and Y_1 are $F_{X_i}(x) = \frac{F(x)}{1-\bar{\alpha}_i\bar{F}(x)}$ and $F_{Y_1}(x) = \frac{F(x)}{1-\bar{\alpha}\bar{F}(x)}$, respectively, where $\bar{\alpha}_i = 1 - \alpha_i$ for $i = 1, 2, \ldots, n$, and $\bar{\alpha} = 1 - \alpha$. The cdfs of $X_{n:n}$ and $Y_{n:n}$ are given by (3.3.1) and

$$F_{Y_{n:n}}(x) = \varphi\left(n\phi\left(F_{Y_1}(x)\right)\right), \tag{3.3.1}$$

where $\phi(u) = \varphi^{-1}(u)$, $u \in (0, 1]$.

The following theorem compare the maximums of two samples, one from n dependent heterogeneous rvs following the PO model and another from n dependent homogeneous rvs following the PO model, in terms of dispersive order when the baseline distribution function is increasing rhr (IRHR). A distribution F is said to be IHRH distribution if the rhr $\tilde{r}(\cdot)$ is increasing. If $\tilde{r}(\cdot)$ is decreasing, then F is called decreasing rhr (DRHR) distribution.

Theorem 3.3.1. *Suppose* $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi)$ *and* $Y \sim PO(\bar{F}, \alpha\boldsymbol{1}, \varphi)$. *Then for* $\alpha \geq \frac{1}{n}\sum_{i=1}^{n}\alpha_i$ *we have* $X_{n:n} \geq_{disp} Y_{n:n}$ *if the baseline distribution is IRHR,* φ *is log-concave and* $\frac{\varphi}{\varphi'}$ *is convex.*

Proof: Equations (**??**) and (3.3.1),implies the cdfs of $X_{n:n}$ and $Y_{n:n}$ are

$$F_2(x) = \varphi\left(\sum_{i=1}^{n} \phi\left(F_{X_i}(x)\right)\right)$$

and

$$G_2(x) = \varphi\left(n\phi\left(F_{Y_1}(x)\right)\right)$$

respectively, where $F_{X_i}(x) = \frac{F(x)}{\alpha_i+\bar{\alpha}_i F(x)}$ and $F_{Y_1}(x) = \frac{F(x)}{\alpha+\bar{\alpha}F(x)}$, $x \in \mathbb{R}$. The respective pdfs of $X_{n:n}$ and $Y_{n:n}$ are given by

$$f_2(x) = \varphi'\left(\sum_{i=1}^{n} \phi\left(F_{X_i}(x)\right)\right) \sum_{i=1}^{n} \frac{\varphi\left(\phi\left(F_{X_i}(x)\right)\right)}{\varphi'\left(\phi\left(F_{X_i}(x)\right)\right)} \frac{\alpha_i \tilde{r}(x)}{\alpha_i + \bar{\alpha}_i F(x)}, \tag{3.3.2}$$

$$g_2(x) = n\varphi'\left(n\phi\left(F_{Y_1}(x)\right)\right) \cdot \frac{\alpha\tilde{r}(x)}{\alpha+\bar{\alpha}F(x)} \cdot \frac{\varphi\left(\phi\left(F_{Y_1}(x)\right)\right)}{\varphi'\left(\phi\left(F_{Y_1}(x)\right)\right)},$$

Therefore,

$$G_2^{-1}(x) = F^{-1}\left(\frac{\alpha\varphi\left(\frac{1}{n}\phi(x)\right)}{1-\bar{\alpha}\varphi\left(\frac{1}{n}\phi(x)\right)}\right),$$

and hence

$$G_2^{-1}(F_2(x)) = F^{-1}\left(\frac{\alpha\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)}{1-\bar{\alpha}\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)}\right) = F^{-1}(\beta(x)), \tag{3.3.3}$$

where $\beta(x) = \frac{\alpha\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)}{1-\bar{\alpha}\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)}$.
Now

$$\begin{aligned}
g_2(G_2^{-1}(F_2(x))) &= n\varphi'\left(n\phi\left(\frac{\beta(x)}{\alpha+\bar{\alpha}\beta(x)}\right)\right) \cdot \frac{\alpha\tilde{r}\left(\bar{F}^{-1}(\beta(x))\right)}{\alpha+\bar{\alpha}\beta(x)} \cdot \frac{\varphi\left(\phi\left(\frac{\beta(x)}{\alpha+\bar{\alpha}\beta(x)}\right)\right)}{\varphi'\left(\phi\left(\frac{\beta(x)}{\alpha+\bar{\alpha}\beta(x)}\right)\right)} \\
&= n\varphi'\left(\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right) \cdot \frac{\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)}{\varphi'\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)} \\
&\quad \times \left(1-\bar{\alpha}\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)\right) \cdot \tilde{r}\left(\bar{F}^{-1}(\beta(x))\right).
\end{aligned} \tag{3.3.4}$$

Note that $\alpha_i/(\alpha_i + \bar{\alpha}_i F(x))$ is increasing and concave in α_i. It can be seen that $\phi\left(F_{X_i}(x)\right)$ is increasing and concave in α_i if φ is log-concave. First consider $\bar{\alpha} \leq 0$. Now for $\alpha \geq \frac{1}{n}\sum_{i=1}^{n}\alpha_i$, the concavity and increasing property of $\phi\left(F_{X_i}(x)\right)$ with respect to α_i, implies

$$
\begin{aligned}
\phi\left(F_{Y_1}(x)\right) &\geq \frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right) \qquad (3.3.5)\\
\Longrightarrow 1-\bar{\alpha}F_{Y_1}(x) &\leq 1-\bar{\alpha}\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right) \qquad (3.3.6)\\
\Longrightarrow 1-\frac{1}{1-\bar{\alpha}\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)} &\geq 1-\frac{1}{1-\bar{\alpha}F_{Y_1}(x)}\\
\Longrightarrow \frac{\alpha\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)}{1-\bar{\alpha}\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)} &\geq \frac{\alpha F_{Y_1}(x)}{1-\bar{\alpha}F_{Y_1}(x)}\\
\Longrightarrow \beta(x) &\geq F(x).
\end{aligned}
$$

Similarly for $\bar{\alpha} \geq 0$, (3.3.5) implies $\beta(x) \geq F(x)$. Thus $F^{-1}(\beta(x)) \geq x$. Now if $\tilde{r}(\cdot)$ is increasing then

$$
\tilde{r}(F^{-1}(\beta(x))) \geq \tilde{r}(x). \qquad (3.3.7)
$$

Next consider $\bar{\alpha} \geq 0$. As $\varphi(x)$ is decreasing and convex, therefore

$$
\begin{aligned}
\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right) &\leq \frac{1}{n}\sum_{i=1}^{n}\varphi\left(\phi\left(F_{X_i}(x)\right)\right)\\
\Longrightarrow \bar{\alpha}\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right) &\leq \bar{\alpha}\frac{1}{n}\sum_{i=1}^{n}\frac{F(x)}{\alpha_i+\bar{\alpha}_iF(x)} \qquad (3.3.8)\\
&= \frac{1}{n}\sum_{i=1}^{n}\bar{\alpha}_i\cdot\frac{1}{n}\sum_{i=1}^{n}\frac{F(x)}{\alpha_i+\bar{\alpha}_iF(x)}\\
&\leq \frac{1}{n}\sum_{i=1}^{n}\frac{\bar{\alpha}_iF(x)}{\alpha_i+\bar{\alpha}_iF(x)}\\
\Longrightarrow 1-\bar{\alpha}\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right) &\geq 1-\frac{1}{n}\sum_{i=1}^{n}\frac{\bar{\alpha}_iF(x)}{\alpha_i+\bar{\alpha}_iF(x)}\\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{\alpha_i}{\alpha_i+\bar{\alpha}_iF(x)}
\end{aligned}
$$

Now for $\bar{\alpha} \leq 0$, (3.3.6) implies

$$
1-\bar{\alpha}\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right) \geq \frac{\alpha}{\alpha+\bar{\alpha}F(x)}
$$

$$\geq \frac{1}{n}\sum_{i=1}^{n}\frac{\alpha_i}{\alpha_i+\bar{\alpha}_i F(x)}, \tag{3.3.9}$$

where the second inequality holds from the fact that $\frac{\alpha_i}{\alpha_i+\bar{\alpha}_i F(x)}$ is increasing and concave in α_i.

If $\frac{\varphi}{\varphi'}$ is convex, then

$$-\frac{\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)}{\varphi'\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)} \geq -\frac{1}{n}\sum_{i=1}^{n}\frac{\varphi\left(\phi\left(F_{X_i}(x)\right)\right)}{\varphi'\left(\phi\left(F_{X_i}(x)\right)\right)}.$$

consequently

$$\left(1-\bar{\alpha}\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)\right)\left(-\frac{\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)}{\varphi'\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)}\right)$$
$$\geq \frac{1}{n}\sum_{i=1}^{n}\left(-\frac{\varphi\left(\phi\left(F_{X_i}(x)\right)\right)}{\varphi'\left(\phi\left(F_{X_i}(x)\right)\right)}\right)\frac{1}{n}\sum_{i=1}^{n}\frac{\alpha_i}{\alpha_i+\bar{\alpha}_i F(x)} \tag{3.3.10}$$

If φ is log-concave, then $-\frac{\varphi(x)}{\varphi'(x)}$ is decreasing in x, so that $-\frac{\varphi\left(\phi\left(F_{X_i}(x)\right)\right)}{\varphi'\left(\phi\left(F_{X_i}(x)\right)\right)}$ is decreasing in α_i and $\frac{\alpha_i}{\alpha_i+\bar{\alpha}_i F(x)}$ increasing in α_i. So by Chebyshev's inequality from Lemma 1.2.9, it follows

$$\frac{1}{n}\sum_{i=1}^{n}\left(-\frac{\varphi\left(\phi\left(F_{X_i}(x)\right)\right)}{\varphi'\left(\phi\left(F_{X_i}(x)\right)\right)}\right)\cdot\frac{1}{n}\sum_{i=1}^{n}\frac{\alpha_i}{\alpha_i+\bar{\alpha}_i F(x)} \geq \frac{1}{n}\sum_{i=1}^{n}\left(-\frac{\varphi\left(\phi\left(F_{X_i}(x)\right)\right)}{\varphi'\left(\phi\left(F_{X_i}(x)\right)\right)}\right)\frac{\alpha_i}{\alpha_i+\bar{\alpha}_i F(x)} \tag{3.3.11}$$

From (3.3.7), (3.3.10), (3.3.11) and the fact that the common factor $\varphi'\left(\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)$ in (3.3.2) and (3.3.4) is negative, it implies $g_2(G_2^{-1}(F_2(x))) \geq f_2(x)$ for all $x \in \mathbb{R}$. Hence the theorem follows.

Remark 3.3.1. A rv having support $[0,\infty)$ cannot be IRHR, however, a distribution function with finite support or the support of the form $(-\infty, b]$, $0 \leq b < \infty$, can be IRHR. For example, the following distributions are IRHR:
(i) $F(x) = e^{-(-\lambda x)^{\beta}}$, $\lambda > 0$, $\beta \leq 1$ for $x \in (-\infty, 0]$.
(ii) $F(x) = \left(\frac{bq-\mu+x(1-q)}{b-\mu}\right)^{q/(1-q)}$, $q > 1$ for $x \in (-\infty, b]$.
It is to be also noted that Archimedean copula with generator $\varphi(x) = e^{-\theta x^{\gamma}}$; for $\theta > 0, \gamma \geq 1$. satisfies the condition that φ is log-concave and $\frac{\varphi}{\varphi'}$ is convex.

It is of interest to know whether in case of Theorem 3.3.1 one can establish dispersive ordering when baseline distribution is DRHR. The following counterexample shows that with these conditions, dispersive ordering cannot establish even in case of samples from independent rvs .

Counterexample 3.3.1. *Consider maximums of two samples, one having four independent and heterogeneous rvs, and another having four independent and homogeneous rvs with respective cdfs* $F_2(x) = \prod_{i=1}^{r} \left(\frac{F(x)}{1-\bar{\alpha}_i \bar{F}(x)}\right)$ *and* $G_2(x) = \left(\frac{F(x)}{1-\bar{\alpha}\bar{F}(x)}\right)^4$*, where* $\alpha_1 = 0.9$*,* $\alpha_2 = 0.95$*,* $\alpha_3 = 27$*,* $\alpha_4 = 37$*,* $\alpha = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)/4 = 16.4625$*, and* $F(x) = 1 - e^{-(5x)^{0.5}}$*, so that the baseline distribution is DRHR. Now*

$$f_2(x) = \left(\sum_{i=1}^{4} \frac{\alpha_i \tilde{r}(x)}{\alpha_i + \bar{\alpha}_i F(x)}\right) \prod_{i=1}^{4} \left(\frac{F(x)}{\alpha_i + \bar{\alpha}_i F(x)}\right)$$

and

$$g_2(G_2^{-1}(F_2(x))) = 4\left(\prod_{i=1}^{4} F_{X_i}(x)\right)\left(1 - \bar{\alpha}\prod_{i=1}^{4} (F_{X_i}(x))^{1/n}\right) \tilde{r}(F^{-1}(\beta(x))),$$

where $\beta(x) = \frac{\alpha\left(\prod_{i=1}^{4} F_{X_i}(x)\right)^{1/4}}{1-\bar{\alpha}\left(\prod_{i=1}^{4} F_{X_i}(x)\right)^{1/4}}$.
Now $g_2(G_2^{-1}(F_2(x))) - f_2(x)$ *plotted by substituting* $x = t/(1-t)$*, where* $x \in [0,\infty)$*, implies* $t \in [0,1)$*, as shown in Figure 3.3. It is observed from Figure 3.3 that* $g_2(G_2^{-1}(F_2(x))) - f_2(x) \nleq 0$ *and also* $g_2(G_2^{-1}(F_2(x))) - f_2(x) \nleq 0$.

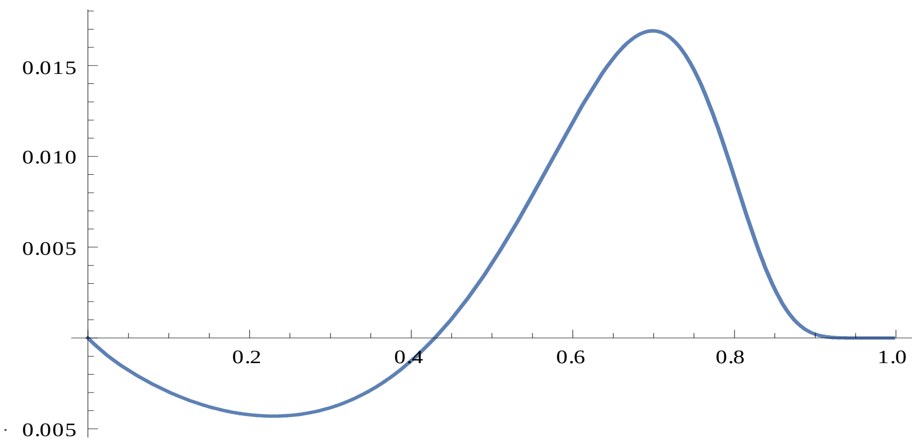

Figure 3.3: Plot of $g_2(G_2^{-1}(F_2(x))) - f_2(x)$ for $x = t/(1-t)$, $t \in [0,1]$ when baseline distribution is DRHR.

The following theorem compare the maximums of two samples, both from n dependent homogeneous rvs following the PO model and with different Archimedean copulas.

Theorem 3.3.2. *Suppose* $X \sim PO(\bar{F}, \alpha\boldsymbol{1}, \varphi_1)$ *and* $Y \sim PO(\bar{F}, \alpha\boldsymbol{1}, \varphi_2)$. *Then* $X_{n:n} \geq_{disp} Y_{n:n}$ *if baseline distribution is IRHR,* $\varphi_1(\phi_1(w)/n)/\varphi_2(\phi_2(w)/n)$ *is increasing in* w, *and* $\alpha \geq 1$.

Proof: The cdfs of $X_{n:n}$ and $Y_{n:n}$ are

$$G_1(x) = \varphi_1\left(n\phi_1\left(F_{X_1}(x)\right)\right)$$

and

$$G_2(x) = \varphi_2\left(n\phi_2\left(F_{X_1}(x)\right)\right),$$

respectively, where $F_{X_1}(x) = \frac{F(x)}{\alpha + \bar{\alpha}F(x)}$, $x \in \mathbb{R}$. Therefore the pdfs of $X_{n:n}$ and $Y_{n:n}$ are

$$g_1(x) = n\varphi_1'\left(n\phi_1\left(F_{X_1}(x)\right)\right) \cdot \frac{\alpha\tilde{r}(x)}{\alpha + \bar{\alpha}F(x)} \cdot \frac{\varphi_1\left(\phi_1\left(F_{X_1}(x)\right)\right)}{\varphi_1'\left(\phi_1\left(F_{X_1}(x)\right)\right)}, \tag{3.3.12}$$

and

$$g_2(x) = n\varphi_2'\left(n\phi_2\left(F_{X_1}(x)\right)\right) \cdot \frac{\alpha\tilde{r}(x)}{\alpha + \bar{\alpha}F(x)} \cdot \frac{\varphi_2\left(\phi_2\left(F_{X_1}(x)\right)\right)}{\varphi_2'\left(\phi_2\left(F_{X_1}(x)\right)\right)},$$

respectively. Hence

$$G_2^{-1}(G_1(x)) = F^{-1}\left(\frac{\alpha\varphi_2\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(F_{X_1}(x)\right)\right)\right)\right)}{1 - \bar{\alpha}\varphi_2\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(F_{X_1}(x)\right)\right)\right)\right)}\right) = F^{-1}(\zeta(x)) \text{ (say)},$$

$$\begin{aligned} g_2(G_2^{-1}(G_1(x))) &= n\varphi_2'\left(\phi_2\left(\varphi_1\left(n\phi_1\left(F_{X_1}(x)\right)\right)\right)\right) \cdot \frac{\varphi_2\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(F_{X_1}(x)\right)\right)\right)\right)}{\varphi_2'\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(F_{X_1}(x)\right)\right)\right)\right)} \\ &\quad \times\tilde{r}\left(F^{-1}(\zeta(x))\right)\left(1 - \bar{\alpha}\varphi_2\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(F_{X_1}(x)\right)\right)\right)\right)\right). \end{aligned} \tag{3.3.13}$$

From Since $\varphi_1(\phi_1(w)/n)/\varphi_2(\phi_2(w)/n)$ is increasing in w, using Lemma 3.9 of Fang et al. [43], it follows that

$$\begin{aligned} & \varphi_2\left(n\phi_2\left(F_{X_1}(x)\right)\right) \leq \varphi_1\left(n\phi_1\left(F_{X_1}(x)\right)\right), \\ \implies & F_{X_1}(x) \leq \varphi_2\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(F_{X_1}(x)\right)\right)\right)\right). \\ \implies & F(x) \leq \zeta(x) \\ \implies & F^{-1}(\zeta(x)) \geq x. \end{aligned}$$

Thus if $\tilde{r}(\cdot)$ is increasing, then

$$\tilde{r}(F^{-1}(\zeta(x))) \geq \tilde{r}(x). \tag{3.3.14}$$

Also for $\bar{\alpha} \leq 0$, $\varphi_2\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(F_{X_1}(x)\right)\right)\right)\right) \geq F_{X_1}(x)$ implies

$$1-\bar{\alpha}\varphi_2\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(F_{X_1}(x)\right)\right)\right)\right) \geq \frac{\alpha}{\alpha+\bar{\alpha}\bar{F}(x)}. \tag{3.3.15}$$

Again from Lemma 3.9 of Fang et al. [43] by substituting
$w = \varphi_1\left(n\phi_1\left(F_{X_1}(x)\right)\right)$ in $\varphi_1(\phi_1(w)/n)/\varphi_2(\phi_2(w)/n)$, gives

$$\begin{aligned}\frac{\varphi_2'\left(\phi_2\left(\varphi_1\left(n\phi_1\left(F_{X_1}(x)\right)\right)\right)\right)\varphi_2\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(F_{X_1}(x)\right)\right)\right)\right)}{\varphi_2'\left(\frac{1}{n}\phi_2\left(\varphi_1\left(n\phi_1\left(F_{X_1}(x)\right)\right)\right)\right)}\\ \geq \frac{\varphi_1'\left(n\phi_1\left(F_{X_1}(x)\right)\right)\varphi_1\left(\phi_1\left(F_{X_1}(x)\right)\right)}{\varphi_1'\left(\phi_1\left(F_{X_1}(x)\right)\right)}.\end{aligned} \tag{3.3.16}$$

Now using (3.3.14), (3.3.15) and (3.3.16), (3.3.13) and (3.3.12) implies $g_2(G_2^{-1}(G_1(x))) \geq g_1(x)$ for all $x \in \mathbb{R}$. This completes the proof.

Remark 3.3.2. It is to be noted that Archimedean copula with generators as given in (i) and (ii) below satisfy the condition that $\varphi_1(\phi_1(w)/n)/\varphi_2(\phi_2(w)/n)$ is increasing in w for all $n \in Z$,

(i) $\varphi_1(x) = \exp\left(\frac{(1-e^x)}{\theta_1}\right), \theta_1 \in (0,1)$ and $\varphi_2(x) = \exp\left(1-(1+x)^{1/\theta_2}\right), \theta_2 \in (0,\infty)$. Therefore, for any $n \in Z$

$$\frac{d}{dx}(\varphi_1(\phi_1(x)/n)) = \frac{\exp\left(1-(1-\theta_1 log(x))^{1/n}\right)(1-\theta_1 log(x))^{-1+1/n}}{nx}$$

is non-negative. Which implies that $\varphi_1(\phi_1(x)/n)$ is increasing in x. Again for any $n \in Z$,

$$\begin{aligned}\frac{d}{dx}(\varphi_2(\phi_2(x)/n)) &= -\exp\left(1-\left(\frac{1+n-(1-log(x))^{\theta_2}}{n}\right)^{1/\theta_2}\right)\\ &\times \frac{\left(\frac{1+n-(1-log(x))^{\theta_2}}{n}\right)^{-1+\frac{1}{\theta_2}}(1-log(x))^{-1+\theta_2}}{nx}.\end{aligned}$$

Which implies that $\varphi_2(\phi_2(x)/n)$ is decreasing in x. Now it can be easily conclude that $\varphi_1(\phi_1(w)/n)/\varphi_2(\phi_2(w)/n)$ is increasing in w for all $n \in Z$.

(ii) Suppose $\varphi_1(x) = e^{\theta_1(1-e^x)}$ and $\varphi_2(x) = e^{\theta_2(1-e^x)}$, for $0 < \theta_2 < \theta_1 < 1$. Then for any

$n \in Z$,

$$\frac{d}{dx}(\varphi_1(\phi_1(x)/n)/\varphi_2(\phi_2(x)/n)) = \left[\left(\left(1-\frac{log(x)}{\theta_1}\right)^{-1+\frac{1}{n}} - \left(1-\frac{log(x)}{\theta_2}\right)^{-1+\frac{1}{n}}\right)\right.$$
$$\left.\times \quad \exp\left(\frac{((\theta_1-\theta_2)) - \left(\left(1-\frac{log(x)}{\theta_1}\right)^{-1+\frac{1}{n}} - \left(1-\frac{log(x)}{\theta_2}\right)^{-1+\frac{1}{n}}\right)}{nx}\right)\right],$$

which is non-negative, since for $0 < \theta_2 < \theta_1 < 1$, therefore

$$\left(1-\frac{log(x)}{\theta_1}\right)^{-1+\frac{1}{n}} \geq \left(1-\frac{log(x)}{\theta_2}\right)^{-1+\frac{1}{n}}. \quad \forall x \in [0,1].$$

The following corollary follows from Theorems 3.3.1 and 3.3.2. This corollary compares the minimum of two samples, one from from n dependent heterogeneous rvs following the PO model and another from n dependent homogeneous rvs following the PO model and with different Archimedean copulas.

Corollary 3.3.1. *Suppose $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi_1)$ and $Y \sim PO(\bar{F}, \alpha\boldsymbol{1}, \varphi_2)$. Then for $\alpha \geq \frac{1}{n}\sum_{i=1}^{n}\alpha_i$, $X_{n:n} \geq_{disp} Y_{n:n}$ if the baseline distribution is IRHR, φ_1 is log-concave, $\frac{\varphi_1}{\varphi_1'}$ is convex, $\varphi_1(\phi_1(w)/n)/\varphi_2(\phi_2(w)/n)$ is increasing in w, and $\alpha \geq 1$.*

Remark 3.3.3. It is to be noted that Archimedean copula with generators $\varphi_1(x) = e^{\theta_1 x^{\gamma}}, \theta_1 \in (0,1), \gamma \in (1,\infty)$ and $\varphi_2(x) = e^{(1-(1+x)^{1/\theta_2})}, \theta_2 \in (0,\infty)$ satisfied the conditions of the corollary(3.3.1).

The following theorem compares the minimum of two samples, one from n dependent heterogeneous rvs following the PO model and another from n dependent homogeneous rvs following the PO model, in terms of star order.

Theorem 3.3.3. *Suppose $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi)$ and $Y \sim PO(\bar{F}, \alpha\boldsymbol{1}, \varphi)$. Then for $\alpha \geq \frac{1}{n}\sum_{i=1}^{n}\alpha_i$, $X_{n:n} \geq_{*} Y_{n:n}$ if $x\tilde{r}(x)$ is increasing in x, φ is log-concave, $\frac{\varphi}{\varphi'}$ is convex.*

Proof: Equations (3.3.2), (3.3.3) and (3.3.4), implies

$$x^2\frac{d}{dx}\left(\frac{G_2^{-1}(F_2(x))}{x}\right)$$
$$= \quad x\frac{d}{dx}\left(G_2^{-1}(F_2(x))\right) - G_2^{-1}(F_2(x))$$

$$= x\frac{f_2(x)}{g_2\left(G_2^{-1}(F_2(x))\right)} - G_2^{-1}(F_2(x))$$

$$= \frac{x\tilde{r}(x)\frac{1}{n}\sum_{i=1}^{n}\frac{\varphi\left(\phi\left(F_{X_i}(x)\right)\right)}{\varphi'\left(\phi\left(F_{X_i}(x)\right)\right)}\frac{\alpha_i}{\alpha_i+\bar{\alpha}_iF(x)}}{\tilde{r}\left(\bar{F}^{-1}(\beta(x))\right)\frac{\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)}{\varphi'\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)}\left(1-\bar{\alpha}\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)\right)} - F^{-1}(\beta(x)) \tag{3.3.17}$$

In Theorem 3.3.1, it's already proved that

$$F^{-1}(\beta(x)) \geq x. \tag{3.3.18}$$

Now, if $x\tilde{r}(x)$ is increasing in x, then from (3.3.18), $x\tilde{r}(x) \leq F^{-1}(\beta(x))\tilde{r}(F^{-1}(\beta(x)))$, that is

$$\frac{x\tilde{r}(x)}{\tilde{r}(F^{-1}(\beta(x)))} \leq F^{-1}(\beta(x)). \tag{3.3.19}$$

According to the equations (3.3.10) and (3.3.11) of Theorem 3.3.2, gives

$$\frac{\frac{1}{n}\sum_{i=1}^{n}\left(-\frac{\varphi\left(\phi\left(F_{X_i}(x)\right)\right)}{\varphi'\left(\phi\left(F_{X_i}(x)\right)\right)}\right)\frac{\alpha_i}{\alpha_i+\bar{\alpha}_iF(x)}}{\left(1-\bar{\alpha}\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)\right)\left(-\frac{\varphi\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)}{\varphi'\left(\frac{1}{n}\sum_{i=1}^{n}\phi\left(F_{X_i}(x)\right)\right)}\right)} \leq 1. \tag{3.3.20}$$

Using (3.3.19) and (3.3.20), (3.3.17) implies

$$x^2\frac{d}{dx}\left(\frac{G_2^{-1}(F_2(x))}{x}\right) \leq 0. \tag{3.3.21}$$

So, $\frac{G_2^{-1}(F_2(x))}{x}$ is decreasing in $x \geq 0$. Hence $X_{n:n} \geq_\star Y_{n:n}$. The following counterexample shows that one cannot establish star ordering as in case of Theorem 3.3.3 when $x\tilde{r}(x)$ is decreasing or increasing even in case of samples from independent rvs.

Counterexample 3.3.2. *Consider maximums of two samples, one having four independent and heterogeneous rvs, and another having four independent and homogeneous rvs. Consider* $\alpha_1 = 5$, $\alpha_2 = 15$, $\alpha_3 = 25$, $\alpha_4 = 45$, $\alpha = (\alpha_1+\alpha_2+\alpha_3+\alpha_4)/4 = 45/2$, *and* $F(x) = 1-(1+x)^{-0.6}$, *so that* $x\tilde{r}(x)$ *is decreasing.* $G_2^{-1}(F_2(x))/x$ *plotted by substituting* $x = t/(1-t)$, *so that for* $x \in [0,\infty)$, *implies* $t \in [0,1)$. *Therefore*

$$G_2^{-1}(F_2(x)) = F^{-1}\left(\frac{\alpha\left(\prod_{i=1}^{4} F_{X_i}(x)\right)^{1/4}}{1-\bar{\alpha}\left(\prod_{i=1}^{4} F_{X_i}(x)\right)^{1/4}}\right).$$

From the Figure 3.4, it is ovserved that $G_2^{-1}(F_2(x))/x$ is neither increasing nor decreasing.

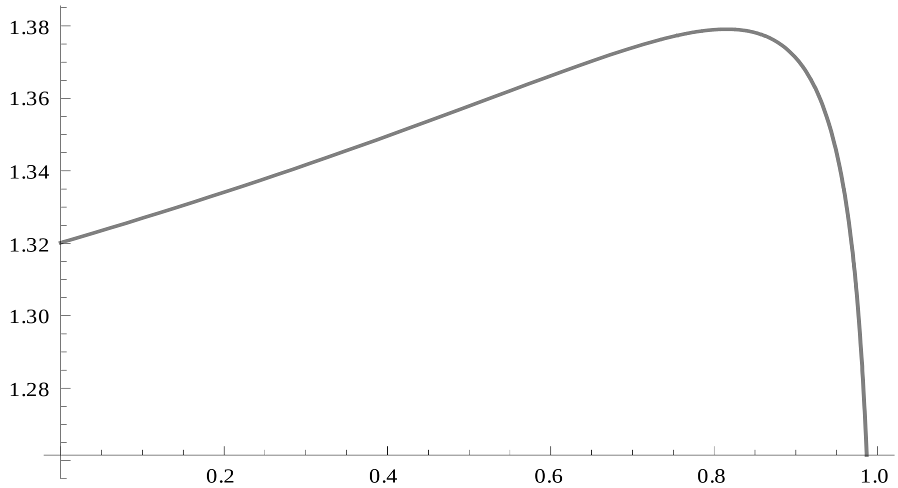

Figure 3.4: Plot of $G_2^{-1}(F_2(x))/x$ for $x = t/(1-t)$, $t \in [0,1]$

The following theorem compare the minimums of two samples, both from n dependent homogeneous rvs following the PO model and with different Archimedean copulas.

Theorem 3.3.4. *Suppose $X \sim PO(\bar{F}, \alpha\boldsymbol{1}, \varphi_1)$ and $Y \sim PO(\bar{F}, \alpha\boldsymbol{1}, \varphi_2)$. Then for $\alpha \geq \frac{1}{n}\sum_{i=1}^{n}\alpha_i$, $X_{n:n} \geq_* Y_{n:n}$ if $x\tilde{r}(x)$ is increasing in x, $\varphi_1(\phi_1(w)/n)/\varphi_2(\phi_2(w)/n)$ is increasing in w, and $\alpha \geq 1$.*

Proof: The proof can be done using the results of proof of Theorem 3.3.2 in the same line as of Theorem 3.3.3, and hence omitted.

The following corollary follows from Theorems 3.3.3 and 3.3.4.

Corollary 3.3.2. *Suppose $X \sim PO(\bar{F}, \boldsymbol{\alpha}, \varphi_1)$ and $Y \sim PO(\bar{F}, \alpha\boldsymbol{1}, \varphi_2)$. Then for $\alpha \geq \frac{1}{n}\sum_{i=1}^{n}\alpha_i$, $X_{n:n} \geq_* Y_{n:n}$ if $x\tilde{r}(x)$ is increasing in x, φ_1 is log-concave, $\frac{\varphi_1'}{\varphi'}$ is convex, $\varphi_1(\phi_1(w)/n)/\varphi_2(\phi_2(w)/n)$ is increasing in w, and $\alpha \geq 1$.*

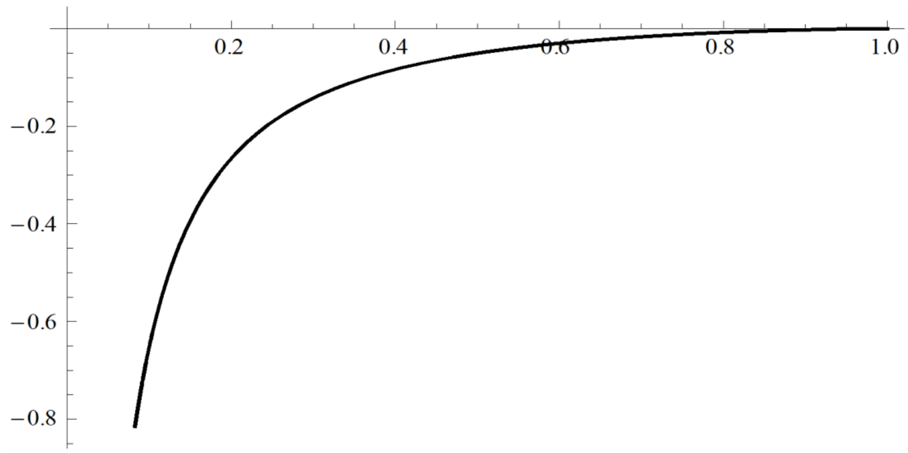

Figure 3.5: Plot of $g_1(G_1^{-1}(F_1(x))) - f_1(x)$ for $x = t/(1-t)$, $t \in [0,1]$ when baseline distribution is DFR.

3.4 Examples

This section demonstrate some of the proposed results numerically. The first example illustrates the result of Theorem 3.2.1.

Example 3.4.1. Consider the minimums of two samples, one from three dependent and heterogeneous rvs, and another from three dependent and homogeneous rvs, with respective cdfs $F_1(x) = 1 - \varphi\left(\sum_{i=1}^{3} \phi\left(\frac{\alpha_i \bar{F}(x)}{1-\bar{\alpha}_i \bar{F}(x)}\right)\right)$ and $G_1(x) = 1 - \varphi\left(3\phi\left(\frac{\alpha \bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}\right)\right)$, where $\alpha_1 = 0.34$, $\alpha_2 = 0.65$, $\alpha_3 = 1.23$, $\alpha = 0.88 > 0.74 = (\alpha_1 + \alpha_2 + \alpha_3)/3$, and $\bar{F}(x) = e^{-x^{0.3}}$, so that the baseline distribution is DFR. Let us consider $\varphi(x) = a/\log(x + e^a)$, $a \in (0, \infty)$ (4.2.19, Nelsen [112]) which satisfies all the conditions of Theorem 3.2.1. For this example let us consider $a = 5$. $g_1(G_1^{-1}(F_1(x))) - f_1(x)$ plotted by substituting $x = t/(1-t)$, so that for $x \in [0, \infty)$, implies $t \in [0, 1)$. The plot is shown in Figure 3.5 and it is observed from the plot that $g_1(G_1^{-1}(F_1(x))) \le f_1(x)$. Thus $X_{1:3} \le_{disp} Y_{1:3}$.

The following example illustrates the result of Theorem 3.2.3.

Example 3.4.2. Consider the minimums of two samples, one from four dependent and heterogeneous rvs, and another from four dependent and homogeneous rvs. Consider $\alpha_1 = 0.24$, $\alpha_2 = 0.45$, $\alpha_3 = 0.57$, $\alpha_3 = 0.57$, $\alpha_4 = 1.23$, $\alpha = 0.73 > (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)/4 = 0.6225$, and $\bar{F}(x) = 1/\sqrt{x}$, $x \in [1, \infty)$ so that $xr(x)$ is constant. Consider $\varphi(x) = a/\log(x + e^a)$, $a \in (0, \infty)$ which satisfies all the conditions of Theorem 3.2.3. For this example let us consider $a = 7$. $\left(G_1^{-1}(F_1(x))/x\right)'$ plotted by substituting $x = 1/t$, so that for $x \in [1, \infty)$, implies

$t \in (0,1]$, as shown in Figure 3.6. From the figure, it is observed that $G_1^{-1}(F_1(x))/x$ is increasing. Thus $X_{1:4} \leq_{\star} Y_{1:4}$.

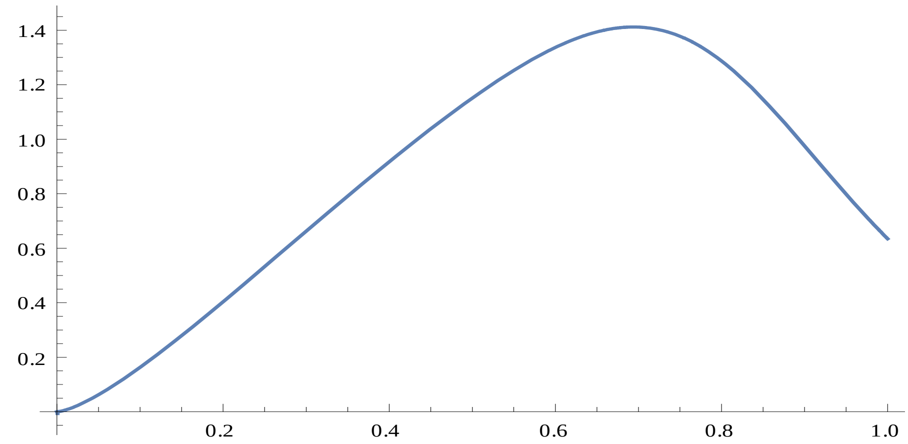

Figure 3.6: Plot of $\left(G_1^{-1}(F_1(x))/x\right)'$ for $x = 1/t$, $t \in [0,1]$when $xr(x)$ is decreasing.

The following example illustrates the result of Theorem 3.3.3.

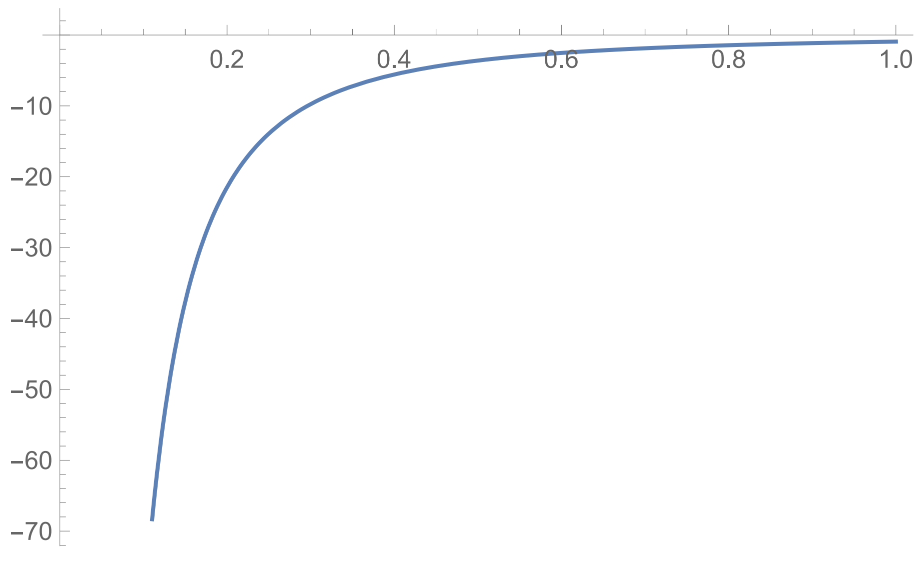

Figure 3.7: Plot of $\left(G_2^{-1}(F_2(x))/x\right)'$, $x \in [0,1]$

Example 3.4.3. Consider the maximums of two samples, one from three dependent and heterogeneous rvs, and another from three dependent and homogeneous rvs. Consider $\alpha_1 = 0.5$, $\alpha_2 = 0.8$, $\alpha_3 = 1.7$, $\alpha = 1.6 > (\alpha_1 + \alpha_2 + \alpha_3)/3 = 1$, and $F(x) = (e^x - 1)/(e - 1)$,

$x \in [0,1]$ so that $x\tilde{r}(x)$ is decreasing. Let us consider the Archimedean copula with generator $\varphi(x) = e^{\theta x^{\gamma}}, \theta \in (0,\infty), \gamma \in [1,\infty)$ which satisfies all the conditions of Theorem 3.3.3. $\left(G_2^{-1}(F_2(x))/x\right)'$ plotted in Figure 3.7 for $\theta = 2, \gamma = 2$.. It is observed from the figure that $G_2^{-1}(F_2(x))/x$ is decreasing. Thus $X_{3:3} \geq_{\star} Y_{3:3}$.

Chapter 4

Stochastic comparisons of finite mixture models [1]

4.1 Introduction

Finite mixture models appear naturally in many areas of reliability theory, survival analysis, and risk theory. In literature, some researchers have studied the properties and applications of finite mixture model for the random samples drawn from a finite number of heterogeneous populations. Consider an absolutely continuous rv X having pdf $f_X(.)$, cdf $F_X(.)$, sf $\bar{F}_X(.)$, hr function $r_X(.)$ and rhr function $\tilde{r}_X(.)$. Formally, these functions describe a homogeneous, infinite population of items meaning that if we draw an item at random from a population it's lifetime will be characterized by these functions.

However, if we have different infinite populations (to be called now subpopulations) and draw an item with certain probabilities from each subpopulation, this item will be already described by the corresponding mixed characteristics and the whole population will be heterogeneous. In practice, populations are finite and, for instance, the user draws items either from homogeneous or heterogeneous populations.

Now, we briefly discuss some practical situations where the finite mixture models play a significance role.

- Consider a component having cdf F that should operate in a specific regime or level of stress, e.g., voltage, temperature, compression and tension. However, the future regime (stress) is uncertain and its probability is given by a discrete distribution

[1] One paper based on this chapter has appeared under:

1. On stochastic comparisons of finite mixture models. *Stochastic Models*, **38(2)**, 190-213, 2022.

$p = (p_1, p_2, \ldots, p_n)$ (Hazra and Finkelstein [59]). Each regime results in a cdf F_{α_i} as function of the baseline cdf F through a parameter $\alpha_i (> 0)$ which defines relationship with the baseline distribution for regime i, $i = 1, \ldots, n$. Then mixture distribution gives a modeling of the stress influence on reliability characteristics of the component.

- There is a considerable number of research works on portfolio construction/optimization, where the stock/asset returns are modeled as a mixture of rvs. Recently, Kocuk and Cornuéjols [80] have analyzed the stocks in Standard & Poors (S&P) 500 index over a 30-year time span, and modeled the stock returns as a mixture of normal rvs to grasp the randomness of the stock returns that have typically heavier left tails which directly relates to the investment risk. In Kocuk and Cornuéjols [80] one can find many relevant references in this regard.
- For a coherent system having n i.i.d. components, the sf can be expressed as a finite mixture of sfs of k-out-of-n systems (Balakrishnan et al. [6], Samaniego [118]). If T is the lifetime of the coherent system, then $P(T > t) = \sum_{k=1}^{n} p_k P(X_{k:n} > t)$, where $X_{k:n}$ is kth order statistic from the random sample $X_1, \ldots, X_n$, and the probabilities p_k, $k = 1, \ldots, n$, are the elements of the signature vector of the system.
- Usually, there is more than one reason (Amini-Seresht and Zhang [2]) causing the failures of a component or system. Then, the overall distribution can be modelled as a finite mixture of the failure distribution of the component for each reason.

Motivated by these facts in this chapter, we consider finite mixtures of rvs are drawn from one of the very important parental family of distributions, namely, proportional odds (PO), proportional hazards (PHR) and proportional reversed hazards (PRH).

First it is considered a finite mixture of lifetimes of n different subpopulations $X_1, \ldots, X_n$ where each $X_i, i = 1, 2, .., n$ following the PO model with some baseline sf $\bar{F}$ and proportionality constant (odds ratio) α_i, denoted as $X_i \sim PO(\bar{F}, \alpha_i)$, $i = 1, \ldots, n$. Let $M_{n;\boldsymbol{\alpha},\mathbf{p}}$ be a random variable representing the lifetime of an item randomly selected from the finite mixture of $F_{\alpha_1}, F_{\alpha_2}, .., F_{\alpha_n}$ where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ is the odds ratio vector, and $\mathbf{p} = (p_1, \ldots, p_n)$, p_i (> 0) is the mixture proportion (weight) such as $\sum_{i=1}^{n} p_i = 1$. Stochastic comparisons between two such finite mixture models for the case when both, the mixing proportion vector $\mathbf{p}$ and the odds ratio vector $\boldsymbol{\alpha}$ are different for the two variants under comparison established. Next finite mixture of rvs following the PHR (PRH) model with some baseline distribution function F, in case of multiple-outlier model are considered. Then stochastic comparisons between two such finite mixture models with different mixing proportions are made.

The rest of the chapter is organized as follows. In Section 4.2, we investigate stochastic comparisons between two finite mixtures where corresponding rvs follow the PO model, PHR model or PRH model. In Section 4.3, we illustrate the theoretical results with numerical examples.

4.2 Stochastic comparisons of two finite mixtures

In this section, we compare two finite mixtures having different mixing distributions as well as with different vectors of mixture proportion in the sense of some stochastic orders. Throughout this chapter it is assumed that $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ are two sets of independent rvs.

4.2.1 Finite mixtures under the PO model

Let $X_i \sim PO(\bar{F}, \alpha_i)$ and $Y_i \sim PO(\bar{F}, \beta_i)$ with $\alpha_i > 0$ and $\beta_i > 0$, for all $i = 1, 2, \ldots, n$. Suppose $M_{n;\boldsymbol{\alpha},\mathbf{p}}$ and $M_{n;\boldsymbol{\beta},\mathbf{q}}$ are rvs representing the finite mixtures of X_i's and Y_i's respectively where $i = 1, 2, \ldots, n$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ are the odds ratio vectors, and $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_n)$ are the vectors of mixture proportion (weights) for the two variants under comparison. We establish the conditions under which one mixture dominates the other in some stochastic sense.
The following theorem provides sufficient conditions on the odds ratio vectors and the mixing proportion vectors under which the finite mixture $M_{n;\boldsymbol{\alpha},\mathbf{p}}$ is smaller than the another mixture $M_{n;\boldsymbol{\beta},\mathbf{q}}$ with respect to the usual stochastic order.

Theorem 4.2.1. *Let $M_{n;\boldsymbol{\alpha},\boldsymbol{p}}$ and $M_{n;\boldsymbol{\beta},\boldsymbol{q}}$ be two finite mixtures. Then*

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \\ p_1 & p_2 & \ldots & p_n \end{pmatrix} >^w \begin{pmatrix} \beta_1 & \beta_2 & \ldots & \beta_n \\ q_1 & q_2 & \ldots & q_n \end{pmatrix} \Rightarrow M_{n;\boldsymbol{\alpha},\boldsymbol{p}} \leq_{st} M_{n;\boldsymbol{\beta},\boldsymbol{q}},$$

provided $(\boldsymbol{\alpha}, \boldsymbol{p}) \in S_n$, $(\boldsymbol{\beta}, \boldsymbol{p}) \in S_n$ and $(\boldsymbol{\beta}, \boldsymbol{q}) \in S_n$.

Proof: Here $\bar{F}_{M_{n;\boldsymbol{\alpha},\mathbf{p}}}(t) = \sum_1^n p_i \bar{F}_{X_i}(t)$, where $\bar{F}_{X_i}(t) = \frac{\alpha_i \bar{F}(t)}{1-\bar{\alpha}_i \bar{F}(t)} = \bar{F}_{\alpha_i}(t)$ (say). Note that $\bar{F}_{\alpha_i}$ is increasing and concave in α_i. Let $\phi(\boldsymbol{\alpha}, \mathbf{p}) = -\bar{F}_{M_{n;\boldsymbol{\alpha},\mathbf{p}}}(t)$. We have

$$\frac{\partial \phi(\boldsymbol{\alpha}, \mathbf{p})}{\partial \alpha_i} = -p_i \frac{\partial \bar{F}_{\alpha_i}(t)}{\partial \alpha_i} \leq 0.$$

Now, for $1 \leq i < j \leq n$,

$$\begin{aligned} \frac{\partial \phi(\boldsymbol{\alpha}, \mathbf{p})}{\partial \alpha_i} - \frac{\partial \phi(\boldsymbol{\alpha}, \mathbf{p})}{\partial \alpha_j} &= p_j \frac{\partial \bar{F}_{\alpha_j}(t)}{\partial \alpha_j} - p_i \frac{\partial \bar{F}_{\alpha_i}(t)}{\partial \alpha_i} \\ &\leq (\geq)\ 0, \end{aligned}$$

if $\alpha_i \leq$ (resp. $\geq$) α_j and $p_i \geq$ (resp. $\leq$) p_j. Thus we have $\frac{\partial \phi(\boldsymbol{\alpha},\mathbf{p})}{\partial \alpha_k}$ is non-positive increasing (resp. decreasing) function in $\alpha_k, k = 1, 2, \ldots, n$ for $\boldsymbol{\alpha} \in \mathcal{E}_n^+$(resp. $\in \mathcal{D}_n^+$). So from Theorem 1 of Haidari et al. [57] (see also Theorem A.4 of Marshall et al. [96]), we have $\boldsymbol{\alpha} \overset{w}{\geq} \boldsymbol{\beta} \Rightarrow \phi(\boldsymbol{\alpha}, \mathbf{p}) \geq \phi(\boldsymbol{\beta}, \mathbf{p})$, whenever $(\boldsymbol{\alpha}, \mathbf{p}) \in S_n$. Thus we have if $(\boldsymbol{\alpha}, \mathbf{p}) \in S_n$, then

$$\boldsymbol{\alpha} \overset{w}{\geq} \boldsymbol{\beta} \Rightarrow \bar{F}_{M_{n;\boldsymbol{\alpha},\mathbf{p}}}(t) \leq \bar{F}_{M_{n;\boldsymbol{\beta},\mathbf{p}}}(t). \tag{4.2.1}$$

Now, let $\phi(\boldsymbol{\beta}, \mathbf{p}) = -\bar{F}_{M_{n;\boldsymbol{\beta},\mathbf{p}}}(t) = \sum_1^n p_i \bar{F}_{Y_i}(t)$, where $\bar{F}_{Y_i}(t) = \frac{\beta_i \bar{F}(t)}{1-\bar{\beta}_i \bar{F}(t)} = \bar{F}_{\beta_i}$ (say). We have $\frac{\partial \phi(\boldsymbol{\beta},\mathbf{p})}{\partial p_i} = -\bar{F}_{\beta_i} \leq 0$. Now, for $1 \leq i < j \leq n$,

$$\frac{\partial \phi(\boldsymbol{\beta}, \mathbf{p})}{\partial p_i} - \frac{\partial \phi(\boldsymbol{\beta}, \mathbf{p})}{\partial p_j} = \bar{F}_{\beta_j}(t) - \bar{F}_{\beta_i}(t) \geq (\leq)\ 0,$$

if $\beta_i \leq$ (resp. $\geq$) β_j. So from Theorem 1 of Haidari et al. [57], we have $\mathbf{p} \overset{w}{\geq} \mathbf{q} \Rightarrow \phi(\boldsymbol{\beta}, \mathbf{p}) \geq \phi(\boldsymbol{\beta}, \mathbf{q})$, whenever $(\boldsymbol{\beta}, \mathbf{p}) \in S_n$. Thus, if $(\boldsymbol{\beta}, \mathbf{p}) \in S_n$, we have

$$\mathbf{p} \overset{w}{\geq} \mathbf{q} \Rightarrow \bar{F}_{M_{n;\boldsymbol{\beta},\mathbf{p}}}(t) \leq \bar{F}_{M_{n;\boldsymbol{\beta},\mathbf{q}}}(t). \tag{4.2.2}$$

Then the theorem follows from combination of (4.2.1) and (4.2.2).

Note: The result in Theorem 4.2.1 will also hold true if the condition $(\beta, p) \in S_n$ is replaced by $(\alpha, q) \in S_n$ with all the other conditions remaining the same.

A counterexample is provided to show that the ordering result in Theorem 4.2.1 does not holds if one of the sufficient conditions is dropped.

Counterexample 4.2.1. Consider finite mixtures $M_{3;\boldsymbol{\alpha},\mathbf{p}}$ and $M_{3;\boldsymbol{\beta},\mathbf{q}}$, where $\boldsymbol{\alpha} = (0.55\ , 0.85, 2.5)$, $\boldsymbol{\beta} = (0.15, 2.5, 3.5)$, $\mathbf{p} = (0.55, 0.2, 0.25)$ and $\mathbf{q} = (0.45, 0.4, 0.15)$ so that the condition of row majorization in Theorem 4.2.1 is not satisfied. We take $\bar{F}(x) = \exp(-ax)^b$ with $a = 2$ and $b = 0.7$. We depict $\bar{F}_{M_{3;\boldsymbol{\alpha},\mathbf{p}}}(x)$ and $\bar{F}_{M_{3;\boldsymbol{\beta},\mathbf{q}}}(x)$ in Figure 4.1 by substituting $x = t/(1-t)$, so that for $x \in [0, \infty)$, we have $t \in [0, 1)$. From the figure it is observed that the stochastic ordering result in Theorem 4.2.1 is not attained.

Theorem 4.2.2. *Let $M_{2;\boldsymbol{\alpha},\boldsymbol{p}}$ and $M_{2;\boldsymbol{\beta},\boldsymbol{q}}$ be two finite mixtures. Then*

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ p_1 & p_2 \end{pmatrix} >^{row} \begin{pmatrix} \beta_1 & \beta_2 \\ q_1 & q_2 \end{pmatrix} \Rightarrow M_{2;\boldsymbol{\alpha},\boldsymbol{p}} \leq_{hr} M_{2;\boldsymbol{\beta},\boldsymbol{q}},$$

provided $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \mathcal{E}_2^+(\mathcal{D}_2^+)$, $(p_1, p_2), (q_1, q_2) \in \mathcal{D}_2^+(\mathcal{E}_2^+)$, $p_1\alpha_1^2 = p_2\alpha_2^2$ *and* $q_1\beta_1^2 =$

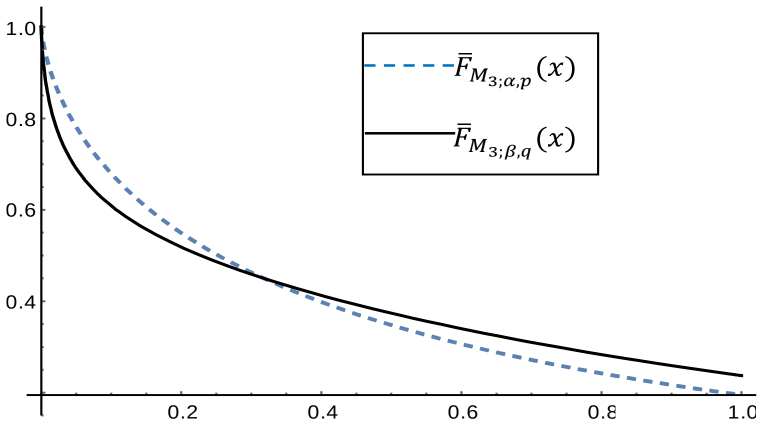

Figure 4.1: Plot of $\bar{F}_{M_{3;\boldsymbol{\alpha},\mathbf{p}}}(x)$ and $\bar{F}_{M_{3;\boldsymbol{\beta},\mathbf{q}}}(x)$ for $x = t/(1-t)$, $t \in [0,1]$.

$q_2\beta_2^2$.

Proof: We have the hr function of $M_{2;\boldsymbol{\alpha},\mathbf{p}}$ as

$$r_{M_{2;\boldsymbol{\alpha},\mathbf{p}}}(t) = r(t)\frac{\sum_{i=1}^{2} p_i \bar{F}_{\alpha_i}(t)\frac{1}{1-\bar{\alpha}_i\bar{F}(t)}}{\sum_{i=1}^{2} p_i \bar{F}_{\alpha_i}(t)}. \tag{4.2.3}$$

Now,

$$\begin{aligned}
\frac{\partial r_{M_{2;\boldsymbol{\alpha},\mathbf{p}}}}{\partial \alpha_1} &= \frac{r(t)}{\left(\sum_{i=1}^{2} p_i\bar{F}_{\alpha_i}(t)\right)^2}\left[\left(\sum_{i=1}^{2} p_i\bar{F}_{\alpha_i}(t)\right) p_1\left(\frac{F(t)\bar{F}(t)}{(1-\bar{\alpha}_1\bar{F}(t))^2}\frac{1}{1-\bar{\alpha}_1\bar{F}(t)} - \right.\right.\\
&\quad \left.\left.\bar{F}_{\alpha_1}(t)\frac{\bar{F}(t)}{(1-\bar{\alpha}_1\bar{F}(t))^2}\right) - \left(\sum_{i=1}^{2} p_i\bar{F}_{\alpha_i}(t)\frac{1}{1-\bar{\alpha}_i\bar{F}(t)}\right) p_1\frac{F(t)\bar{F}(t)}{(1-\bar{\alpha}_1\bar{F}(t))^2}\right]\\
&= A_{\alpha_1,\alpha_2}p_1\frac{\bar{F}(t)}{(1-\bar{\alpha}_1\bar{F}(t))^2}\left[\left(p_1\bar{F}_{\alpha_1}(t)+p_2\bar{F}_{\alpha_2}(t)\right)\left(F_{\alpha_1}(t)-\bar{F}_{\alpha_1}(t)\right) - \right.\\
&\quad \left.\left(p_1\bar{F}_{\alpha_1}(t)F_{\alpha_1}(t)+p_2\bar{F}_{\alpha_2}(t)F_{\alpha_2}(t)\right)\right]\\
&= A_{\alpha_1,\alpha_2}\frac{p_1\bar{F}(t)}{(1-\bar{\alpha}_1\bar{F}(t))^2}\left[-p_1\bar{F}^2_{\alpha_1}(t)+p_2\bar{F}^2_{\alpha_2}(t)-2p_2\bar{F}_{\alpha_1}(t)\bar{F}_{\alpha_2}(t)\right],
\end{aligned}$$

where $A_{\alpha_1,\alpha_2} = \frac{r(t)}{\left(\sum_{i=1}^{2} p_i\bar{F}_{\alpha_i}(t)\right)^2}$.

$$\frac{\partial r_{M_{2;\boldsymbol{\alpha},\mathbf{p}}}}{\partial \alpha_2} = A_{\alpha_1,\alpha_2}\frac{p_2\bar{F}(t)}{(1-\bar{\alpha}_2\bar{F}(t))^2}\left[-p_2\bar{F}^2_{\alpha_2}(t)+p_1\bar{F}^2_{\alpha_1}(t)-2p_1\bar{F}_{\alpha_1}(t)\bar{F}_{\alpha_2}(t)\right].$$

Then

$$\begin{aligned}\frac{\partial r_{M_{2;\boldsymbol{\alpha},\mathbf{p}}}}{\partial \alpha_1} - \frac{\partial r_{M_{2;\boldsymbol{\alpha},\mathbf{p}}}}{\partial \alpha_2} &\overset{sign}{=} \left(-p_1\bar{F}^2_{\alpha_1}(t) + p_2\bar{F}^2_{\alpha_2}(t)\right)\left(\frac{p_1}{(1-\bar{\alpha}_1\bar{F}(t))^2} + \frac{p_2}{(1-\bar{\alpha}_2\bar{F}(t))^2}\right)\\ &\quad -2p_1p_2\bar{F}_{\alpha_1}\bar{F}_{\alpha_2}\left(\frac{1}{(1-\bar{\alpha}_1\bar{F}(t))^2} - \frac{1}{(1-\bar{\alpha}_2\bar{F}(t))^2}\right)\\ &\geq (\text{resp.} \leq)\ 0,\end{aligned}$$

if $\alpha_1 \geq$ (resp. $\leq$) α_2 and $p_1\alpha_1^2 = p_2\alpha_2^2$. This inequality follows from the fact that $\frac{1}{1-\bar{\alpha}_i\bar{F}(t)}$ is decreasing in α_i. Thus from Theorem A.3 of Marshall et al. [96] and Lemma 3 of Hazra et al. [60], we have

$$(\alpha_1, \alpha_2) \overset{m}{\geq} (\beta_1, \beta_2) \Rightarrow r_{M_{2;\boldsymbol{\alpha},\mathbf{p}}} \geq r_{M_{2;\boldsymbol{\beta},\mathbf{p}}}, \tag{4.2.4}$$

if $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \mathcal{E}_2^+$ or $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \mathcal{D}_2^+$ and $p_1\alpha_1^2 = p_2\alpha_2^2$. Now

$$\begin{aligned}\frac{\partial r_{M_{2;\boldsymbol{\beta},\mathbf{p}}}}{\partial p_1} &= A_{\beta_1,\beta_2}\bar{F}_{\beta_1}(t)\Bigg[\left(p_1\bar{F}_{\beta_1}(t) + p_2\bar{F}_{\beta_2}(t)\right)\frac{1}{1-\bar{\beta}_1\bar{F}(t)} -\\ &\quad \left(p_1\bar{F}_{\beta_1}(t)\frac{1}{1-\bar{\beta}_1\bar{F}(t)} + p_2\bar{F}_{\beta_2}(t)\frac{1}{1-\bar{\beta}_2\bar{F}(t)}\right)\Bigg]\\ &= A_{\beta_1,\beta_2}p_2\bar{F}_{\beta_1}(t)\bar{F}_{\beta_2}(t)\frac{(\beta_2-\beta_1)\bar{F}(t)}{(1-\bar{\beta}_1\bar{F}(t))(1-\bar{\beta}_2\bar{F}(t))}.\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial r_{M_{2;\boldsymbol{\beta},\mathbf{p}}}}{\partial p_1} - \frac{\partial r_{M_{2;\boldsymbol{\beta},\mathbf{p}}}}{\partial p_2} &\overset{sign}{=} p_2(\beta_2-\beta_1) - p_1(\beta_1-\beta_2)\\ &= -(\beta_1-\beta_2)(p_1+p_2) \geq (\text{resp.} \leq)\ 0,\end{aligned}$$

for $\beta_1 \leq$ (resp. $\geq$) β_2. As per our assumption, $(p_1, p_2) \in \mathcal{D}_2^+$ when $(\beta_1, \beta_2) \in \mathcal{E}_2^+$, and $(p_1, p_2) \in \mathcal{E}_2^+$ when $(\beta_1, \beta_2) \in \mathcal{D}_2^+$. Thus again from Theorem A.3 of Marshall et al. [96] and Lemma 3 of Hazra et al. [60], we have

$$(p_1, p_2) \overset{m}{\geq} (q_1, q_2) \Rightarrow r_{M_{2;\boldsymbol{\beta},\mathbf{p}}} \geq r_{M_{2;\boldsymbol{\beta},\mathbf{q}}}, \tag{4.2.5}$$

if $(p_1, p_2), (q_1, q_2) \in \mathcal{D}_2^+$ and $(\beta_1, \beta_2) \in \mathcal{E}_2^+$ or $(p_1, p_2), (q_1, q_2) \in \mathcal{E}_2^+$ and $(\beta_1, \beta_2) \in \mathcal{D}_2^+$. Then the theorem follows from combination of (4.2.4) and (4.2.5).

Next, we establish star ordering result for comparing two finite mixtures in multiple-outlier model. The star order compares the skewness of probability distributions. The skewness of the distribution of finite mixture play a key role in many practical scenarios. In order to compare the skewness of the distributions of the finite mixture, it is natural to establish sufficient conditions for some transform orders between them to analyze the effects of the heterogeneity among mixture proportion and the corresponding parameters on the skewness of their distributions. With the help of star ordering results, lower bounds can be established for the coefficient of variation of the finite mixture from a set of multiple-outlier mixture proportion and corresponding parameters of interest. The star order is also known as more IFRA (increasing failure rate in average) order. If one rv is smaller than another in terms of star order, then this can be interpreted as the former rv ages faster than the later in the sense of the star ordering. The following results show how the changes among the mixture probabilities of finite mixture and corresponding parameters of interest affects the skewness of the distribution of the finite mixture model.

We consider finite mixtures of $n(=n_1+n_2)$ components where n_1 components are drawn from a particular homogeneous subpopulations and rest n_2 components are drawn from another homogeneous subpopulations. Let $X_i \sim PO(\bar{F}, \alpha_i)$ such that $\alpha_i = \alpha_1$ for $i = 1, 2, \ldots, n_1$ and $\alpha_i = \alpha_2$ for $i = n_1+1, \ldots, n$. Let $M_{n;\boldsymbol{\alpha}^{mo},\mathbf{p}^{mo}}$ be the random variable representing the finite mixtures of X_i's, and p_i are the corresponding mixture proportion such that $p_i = p_1$ for $i = 1, 2, \ldots, n_1$ and $p_i = p_2$ for $i = n_1+1, \ldots, n$, so that $n_1p_1 + n_2p_2 = 1$. Here we use the notations $\boldsymbol{\alpha}^{mo} = (\underbrace{\alpha_1, \alpha_1, ..., \alpha_1}_{n_1\ terms}, \underbrace{\alpha_2, \alpha_2, ..., \alpha_2}_{n_2\ terms})$ and $\mathbf{p}^{mo} = (\underbrace{p_1, p_1, ..., p_1}_{n_1\ terms}, \underbrace{p_2, p_2, ..., p_2}_{n_2\ terms})$. We use the following lemma which we derive from Saunders and Moran [119] to prove the Theorem 4.2.3.

Lemma 4.2.1. *[119] Let $\{F_a, a \in \mathbb{R}_+\}$ be a class of distribution functions, such that F_a is supported on some interval $(c, d) \subseteq (0, \infty)$ and has a density f_a which does not vanish on any subinterval of (c, d). Then $F_a \geq_\star F_b$ for $a \geq b$, if and only if $\frac{\partial F_a(x)/\partial a}{x f_a(x)}$ is decreasing in x.*

The following theorem provides sufficient conditions on the odds ratio vector vectors and the mixing proportion vectors under which the finite mixture $M_{n;\boldsymbol{\alpha}^{mo},\mathbf{p}^{mo}}$ is greater than the another mixture $M_{n;\boldsymbol{\beta}^{mo},\mathbf{p}^{mo}}$ with respect to the star order. In other words, this theorem can be used to compare the skewness of the distributions of two finite mixtures.

Theorem 4.2.3. *Let $M_{n;\boldsymbol{\alpha}^{mo},\boldsymbol{p}^{mo}}$ and $M_{n;\boldsymbol{\beta}^{mo},\boldsymbol{p}^{mo}}$ be two finite mixtures with same mixture proportion and $F(x)/xr(x)$ is decreasing in x. If $\alpha_1 \geq \beta_1 \geq \beta_2 \geq \alpha_2$, $p_1 \leq p_2$ and $\boldsymbol{\alpha}^{mo} \overset{w}{\preceq} \boldsymbol{\beta}^{mo}$, then $M_{n;\boldsymbol{\alpha}^{mo},\boldsymbol{p}^{mo}} \geq_\star M_{n;\boldsymbol{\beta}^{mo},\boldsymbol{p}^{mo}}$.*

Proof: We have the distribution function of $M_{n;\boldsymbol{\alpha}^{mo},\mathbf{p}^{mo}}$ as $F_{M_{n;\boldsymbol{\alpha}^{mo},\mathbf{p}^{mo}}}(x) = n_1p_1F_{\alpha_1}(x) + n_2p_2F_{\alpha_2}(x)$, where $F_{\alpha_i}(x) = F(x)/(1-\bar{\alpha}_i\bar{F}(x))$, $i = 1, 2$. We prove the theorem using the concept of Lemma 4.2.1. Under the conditions $\alpha_1 \geq \beta_1 \geq \beta_2 \geq \alpha_2$ and $\boldsymbol{\alpha}^{mo} \overset{w}{\preceq} \boldsymbol{\beta}^{mo}$, we

have $n_1\alpha_1 + n_2\alpha_2 \geq n_1\beta_1 + n_2\beta_2$.
Case I: $n_1\alpha_1 + n_2\alpha_2 = n_1\beta_1 + n_2\beta_2$. Without loss of generality we consider $n_1\alpha_1 + n_2\alpha_2 = n_1\beta_1 + n_2\beta_2 = 1$. Let $\alpha_1 = \alpha$ and $\beta_1 = \beta$, where $\alpha, \beta \in [1/(n_1+n_2), 1/n_1)$, and we write $F_{M_{n;\boldsymbol{\alpha}^{mo},\mathbf{p}^{mo}}}(x) = F_{M_\alpha}(x)$, just to indicate that it becomes an expression of α. Now,

$$\frac{\partial F_{M_\alpha}(x)}{\partial \alpha} = -F(x)\bar{F}(x)\left[\frac{n_1p_1}{(1-\bar{\alpha}\bar{F}(x))^2} - \frac{n_1p_2}{\left(1-\left(1-\frac{1-n_1\alpha}{n_2}\right)\bar{F}(x)\right)^2}\right].$$

Again

$$f_{M_\alpha}(x) = f(x)\left[\frac{n_1p_1\alpha}{(1-\bar{\alpha}\bar{F}(x))^2} + \frac{n_2p_2\left(\frac{1-n_1\alpha}{n_2}\right)}{\left(1-\left(1-\frac{1-n_1\alpha}{n_2}\right)\bar{F}(x)\right)^2}\right].$$

Now

$$\begin{aligned}
\frac{\partial F_{M_\alpha}(x)/\partial\alpha}{xf_{M_\alpha}(x)} &= -\frac{F(x)}{xr(x)}\left[\frac{n_1p_1\left(1-\left(1-\frac{1-n_1\alpha}{n_2}\right)\bar{F}(x)\right)^2 - n_1p_2(1-\bar{\alpha}\bar{F}(x))^2}{n_1p_1\alpha\left(1-\left(1-\frac{1-n_1\alpha}{n_2}\right)\bar{F}(x)\right)^2 + p_2(1-n_1\alpha)(1-\bar{\alpha}\bar{F}(x))^2}\right] \\
&= -\frac{F(x)}{xr(x)}\left[\frac{n_1p_1\alpha\left(1-\left(1-\frac{1-n_1\alpha}{n_2}\right)\bar{F}(x)\right)^2 + p_2(1-n_1\alpha)\left(1-\bar{\alpha}\bar{F}(x)\right)^2}{n_1p_1\left(1-\left(1-\frac{1-n_1\alpha}{n_2}\right)\bar{F}(x)\right)^2 - n_1p_2(1-\bar{\alpha}\bar{F}(x))^2}\right]^{-1} \\
&= -\frac{F(x)}{xr(x)}\left[\alpha + \frac{p_2(1-\bar{\alpha}\bar{F}(x))^2}{n_1p_1\left(1-\left(1-\frac{1-n_1\alpha}{n_2}\right)\bar{F}(x)\right)^2 - n_1p_2(1-\bar{\alpha}\bar{F}(x))^2}\right]^{-1} \\
&= -\frac{F(x)}{xr(x)}\left[\alpha + \left(\frac{p_2}{n_1}\right)\left(\frac{p_1\left(1-\left(1-\left(\frac{1-n_1\alpha}{n_2}\right)\right)\bar{F}(x)\right)^2}{(1-\bar{\alpha}\bar{F}(x))^2} - p_2\right)^{-1}\right]^{-1} \qquad (4.2.6) \\
&= -\frac{F(x)}{xr(x)}\Delta(x)\ (say).
\end{aligned}$$

where $\Delta(x) = \left[\alpha + \left(\frac{p_2}{n_1}\right)\left(\frac{p_1\left(1-\left(1-\left(\frac{1-n_1\alpha}{n_2}\right)\right)\bar{F}(x)\right)^2}{(1-\bar{\alpha}\bar{F}(x))^2} - p_2\right)^{-1}\right]^{-1}$. For $\alpha = \alpha_1 \geq \alpha_2 = \left(\frac{1-n_1\alpha}{n_2}\right)$, $(1-\bar{\alpha}\bar{F}(x))^2 \geq (1-\left(1-\frac{(1-n_1\alpha)}{n_2}\right)\bar{F}(x))^2$. So for $p_1 \leq p_2$, from (4.2.6) it follows that $\Delta(x) \leq 0$.
Again, $\frac{\left(1-\left(1-\frac{1-n_1\alpha}{n_2}\right)\bar{F}(x)\right)}{(1-\bar{\alpha}\bar{F}(x))}$ is increasing in x for $\alpha \geq \left(\frac{1-n_1\alpha}{n_2}\right)$, so that from (4.2.6) it follows that $\Delta(x)$ is also increasing in x. So $\frac{F(x)}{xr(x)}(-\Delta(x))$ is decreasing in x. Hence $\frac{\partial F_{M_\alpha}(x)/\partial\alpha}{f_{M_\alpha}(x)}$ is

decreasing in x.

Case II: Suppose $n_1\alpha_1 + n_2\alpha_2 \geq n_1\beta_1 + n_2\beta_2$. In this case there must exist some α_1' such that $\alpha_1 \geq \alpha_1' \geq \beta_1$ such that $n_1\alpha_1' + n_2\alpha_2 = n_1\beta_1 + n_2\beta_2$. Let us denote $\boldsymbol{\alpha}'^{mo} = (\underbrace{\alpha_1', \alpha_1', ..., \alpha_1'}_{n_1\ terms}, \underbrace{\alpha_2, \alpha_2, ..., \alpha_2}_{n_2\ terms})$. Then from previous case we have $M_{n;\boldsymbol{\alpha}'^{mo},\mathbf{p}^{mo}} \geq_\star M_{n;\boldsymbol{\beta}^{mo},\mathbf{p}^{mo}}$. Now we need to show that $M_{n;\boldsymbol{\alpha}^{mo},\mathbf{p}^{mo}} \geq_\star M_{n;\boldsymbol{\alpha}'^{mo},\mathbf{p}^{mo}}$. Let $\eta = \alpha_1 - \alpha_2, \eta' = \alpha_1' - \alpha_2$, and $\eta^\star = \alpha_1 - \eta$. then $\eta \geq \eta' \geq 0$. According to Lemma 4.2.1, it is sufficient to show that

$$\begin{aligned}\frac{\frac{\partial F_\eta}{\partial \eta}}{xf_\eta(x)} &= \frac{F(x)}{xr(x)}\left[\frac{n_2p_2(1-\bar{\alpha}_1\bar{F}(x))^2}{\alpha_1 n_1 p_1(1-\bar{\eta^*}\bar{F}(x))^2 + n_2p_2\eta^*(1-\bar{\alpha}_1\bar{F}(x))}\right]\\ &= \frac{F(x)}{xr(x)}\left[\eta^* + \frac{n_1p_1\alpha_1}{n_2p_2}\left(\frac{(1-\bar{\eta^*}\bar{F}(x))}{1-\bar{\alpha}_1\bar{F}(x)}\right)^2\right]^{-1}\\ &= \frac{F(x)}{xr(x)} \times \Omega(x)\end{aligned}$$

is decreasing in x, where $\Omega(x) = \left[\eta^* + \frac{n_1p_1\alpha_1}{n_2p_2}\left(\frac{(1-\bar{\eta^*}\bar{F}(x))}{1-\bar{\alpha}_1\bar{F}(x)}\right)^2\right]^{-1}$. Note that $\frac{d}{dx}\left(\frac{1-\bar{\eta^*}\bar{F}(x)}{1-\bar{\alpha_1}\bar{F}(x)}\right) = \frac{(\bar{\eta^*}-\bar{\alpha_1})f(x)}{(1-\bar{\alpha_1}\bar{F}(x))^2} \geq 0$ as $\alpha_1 \geq \eta^*$. So $\Omega(x)$ is non-negative and decreasing in x. Consequently $\frac{\frac{\partial F_\eta}{\partial \eta}}{xf_\eta(x)}$ is decreasing in x.

Remark 4.2.1. *It is to be noted that the condition $F(x)/xr(x)$ is decreasing in x is satisfied by many class of distributions. For example, Burr distribution with sf $\bar{F}(x) = (1+x^\alpha)^{-\beta}, \alpha > 0, \beta > 0$ satisfies the condition for $0 < \alpha < \beta < \infty$, and Gompertz distribution with sf $\bar{F}(x) = \exp([-\beta(\alpha^x - 1)]/\log(\alpha)), \alpha > 0, \beta > 0$ satisfies the condition for $1 < \alpha < \infty$ and $0 < \beta < \infty$.*

Corollary 4.2.1. *Under the same setup of Theorem 4.2.3, if $\alpha_1 \geq \beta_1 \geq \beta_2 \geq \alpha_2$ and $\boldsymbol{\alpha}^{mo} \overset{w}{\preceq} \boldsymbol{\beta}^{mo}$, we have $M_{n;\boldsymbol{\alpha}^{mo},\boldsymbol{p}^{mo}} \geq_{Lorenz} M_{n;\boldsymbol{\beta}^{mo},\boldsymbol{p}^{mo}}$.*

4.2.2 Finite mixtures under the PHR and the PRH model

Here we establish the hr order between two finite mixtures in multiple-outlier model with PHR and PRH distributed components under the majorization order between the vectors of corresponding parameters. Let us consider finite mixtures of $n(= n_1 + n_2)$ components where n_1 components are drawn from a particular homogeneous subpopulations and rest n_2 components are drawn from another homogeneous subpopulations. Let us denote $\boldsymbol{\lambda}^{mo} = (\underbrace{\lambda_1, \lambda_1, ..., \lambda_1}_{n_1\ terms}, \underbrace{\lambda_2, \lambda_2, ..., \lambda_2}_{n_2\ terms})$. First let $X_i \sim PHR(\bar{F}, \lambda_i)$ and suppose $B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}$ be the random variable representing the finite mixtures of X_i's, and p_i are the corresponding mixture proportion, where $\lambda_i = \lambda_1$, $p_i = p_1$ for $i = 1, \dots, n_1$, and $\lambda_i = \lambda_2$, $p_i = p_2$ for $i = n_1+1, \dots, n$, and $n_1p_1 + n_2p_2 = 1$. The following theorem provides sufficient conditions on the hazard ratio vectors of the components and the mixing proportion vectors under which the finite mixture $B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}$ is greater than $B_{n_1,n_2;\boldsymbol{\mu}^{mo},\mathbf{q}^{mo}}$ in terms of hr order.

Theorem 4.2.4. *Let $B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}}$ and $B_{n_1,n_2;\boldsymbol{\mu}^{mo},\boldsymbol{q}^{mo}}$ be two finite mixtures. Then*

$$\begin{pmatrix} \lambda_1 & \dots & \lambda_1 & \lambda_2 & \dots & \lambda_2 \\ p_1 & \dots & p_1 & p_2 & \dots & p_2 \end{pmatrix} >^{row} \begin{pmatrix} \mu_1 & \dots & \mu_1 & \mu_2 & \dots & \mu_2 \\ q_1 & \dots & q_1 & q_2 & \dots & q_2 \end{pmatrix}$$

$$\Rightarrow B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}} \geq_{hr} B_{n_1,n_2;\boldsymbol{\mu}^{mo},\boldsymbol{q}^{mo}}$$

provided $(\lambda_1, \lambda_2), (\mu_1, \mu_2) \in \mathcal{E}_2^+(\mathcal{D}_2^+)$, $(p_1, p_2), (q_1, q_2) \in \mathcal{D}_2^+(\mathcal{E}_2^+)$ and $n_1 \geq (\leq)\ n_2$.

Proof: The hr function of $B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}$ is given by

$$r_{B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}}(t) = r(t)\frac{\sum_{i=1}^{n} p_i\lambda_i\bar{F}^{\lambda_i}(t)}{\sum_{i=1}^{n} p_i\bar{F}^{\lambda_i}(t)} = r(t)\frac{\sum_{i=1}^{n_1} p_i\lambda_i\bar{F}^{\lambda_i}(t) + \sum_{i=n_1+1}^{n} p_i\lambda_i\bar{F}^{\lambda_i}(t)}{\sum_{i=1}^{n_1} p_i\bar{F}^{\lambda_i}(t) + \sum_{i=n_1+1}^{n} p_i\bar{F}^{\lambda_i}(t)}. \tag{4.2.7}$$

For $1 \leq i \leq n_1$,

$$\frac{\partial r_{B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}}}{\partial \lambda_i} = \frac{r(t)n_1p_1\bar{F}^{\lambda_1}(t)}{\left(n_1p_1\bar{F}^{\lambda_1}(t) + n_2p_2\bar{F}^{\lambda_2}(t)\right)^2}\Big[\Big(n_1p_1\bar{F}^{\lambda_1}(t) + n_2p_2\bar{F}^{\lambda_2}(t)\Big) + n_2p_2(\lambda_1 - \lambda_2)\bar{F}^{\lambda_2}(t)\log\bar{F}(t)\Big],$$

and for $n_1 + 1 \leq j \leq n$,

$$\frac{\partial r_{B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}}}{\partial \lambda_j} = \frac{r(t)n_2p_2\bar{F}^{\lambda_2}(t)}{\left(n_1p_1\bar{F}^{\lambda_1}(t) + n_2p_2\bar{F}^{\lambda_2}(t)\right)^2}\Big[\Big(n_1p_1\bar{F}^{\lambda_1}(t) + n_2p_2\bar{F}^{\lambda_2}(t)\Big) + n_1p_1(\lambda_2 - \lambda_1)\bar{F}^{\lambda_1}(t)\log\bar{F}(t)\Big].$$

For $1 \leq i, j \leq n_1$ or $n_1 + 1 \leq i, j \leq n$, $\frac{\partial r_{B_{n;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}}}{\partial \lambda_i} - \frac{\partial r_{B_{n;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}}}{\partial \lambda_j} = 0$. Again for $1 \leq i \leq n_1$ and $n_1 + 1 \leq j \leq n$,

$$\frac{\partial r_{B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}}}{\partial \lambda_i} - \frac{\partial r_{B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}}}{\partial \lambda_j} \overset{sign}{=} \Big(n_1p_1\bar{F}^{\lambda_1}(t) + n_2p_2\bar{F}^{\lambda_2}(t)\Big) \times \Big(n_1p_1\bar{F}^{\lambda_1}(t) - n_2p_2\bar{F}^{\lambda_2}(t)\Big)$$

$$\begin{aligned} &+ \quad 2n_1n_2p_1p_2\bar{F}^{\lambda_1}(t)\bar{F}^{\lambda_2}(t)(\lambda_1-\lambda_2)\log\bar{F}(t) \\ &\leq \quad (\text{resp. } \geq)\ 0, \end{aligned}$$

if $\lambda_1 \geq$ (resp. $\leq$) λ_2, $p_1 \leq$ (resp. $\geq$) p_2 and $n_1 \leq$ (resp. $\geq$) n_2. Thus we have from Theorem A.3 of Marshall et al. [96] and Lemma 3 of Hazra et al. [60],

$$\boldsymbol{\lambda}^{mo} \overset{m}{\geq} \boldsymbol{\mu}^{mo} \Rightarrow r_{B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}} \leq r_{B_{n_1,n_2;\boldsymbol{\mu}^{mo},\mathbf{p}^{mo}}}, \tag{4.2.8}$$

if $(\lambda_1,\lambda_2),(\mu_1,\mu_2) \in \mathcal{E}_2^+(\mathcal{D}_2^+)$, $(p_1,p_2) \in \mathcal{D}_2^+(\mathcal{E}_2^+)$ and $n_1 \geq (\leq)\ n_2$. For $1 \leq i \leq n_1$,

$$\frac{\partial r_{B_{n_1,n_2;\boldsymbol{\mu}^{mo},\mathbf{p}^{mo}}}}{\partial p_i} = \frac{r(t)}{\left(n_1p_1\bar{F}^{\mu_1}(t)+n_2p_2\bar{F}^{\mu_2}(t)\right)^2}n_1n_2p_2\bar{F}^{\mu_1}(t)\bar{F}^{\mu_2}(t)(\mu_1-\mu_2),$$

and for $n_1+1 \leq j \leq n$,

$$\frac{\partial r_{B_{n_1,n_2;\boldsymbol{\mu}^{mo},\mathbf{p}^{mo}}}}{\partial p_j} = \frac{r(t)}{\left(n_1p_1\bar{F}^{\mu_1}(t)+n_2p_2\bar{F}^{\mu_2}(t)\right)^2}n_1n_2p_1\bar{F}^{\mu_1}(t)\bar{F}^{\mu_2}(t)(\mu_2-\mu_1).$$

For $1 \leq i,j \leq n_1$ or $n_1+1 \leq i,j \leq n$, $\frac{\partial r_{B_{n_1,n_2;\boldsymbol{\mu}^{mo},\mathbf{p}^{mo}}}}{\partial p_i} - \frac{\partial r_{B_{n_1,n_2;\boldsymbol{\mu}^{mo},\mathbf{p}^{mo}}}}{\partial p_j} = 0$. Again for $1 \leq i \leq n_1$ and $n_1+1 \leq j \leq n$,

$$\begin{aligned} \frac{\partial r_{B_{n_1,n_2;\boldsymbol{\mu}^{mo},\mathbf{p}^{mo}}}}{\partial p_i} - \frac{\partial r_{B_{n_1,n_2;\boldsymbol{\mu}^{mo},\mathbf{p}^{mo}}}}{\partial p_j} &\overset{sign}{=} n_1n_2\bar{F}^{\mu_1+\mu_2}(t)(\mu_1-\mu_2)(p_1+p_2) \\ &\leq (\text{resp. } \geq)\ 0, \end{aligned}$$

if $\mu_1 \leq$ (resp. $\geq$) μ_2. As per our assumption, $p_1 \geq$ (resp. $\leq$) p_2 when $\mu_1 \leq$ (resp. $\geq$) μ_2. Thus we have from Theorem A.3 of Marshal et al. Marshall et al. [96] and Lemma 3 of Hazra et al. Hazra et al. [60],

$$\mathbf{p}^{mo} \overset{m}{\geq} \mathbf{q}^{mo} \Rightarrow r_{B_{n_1,n_2;\boldsymbol{\mu}^{mo},\mathbf{p}^{mo}}} \leq r_{B_{n_1,n_2;\boldsymbol{\mu}^{mo},\mathbf{q}^{mo}}}, \tag{4.2.9}$$

if $(p_1,p_2),(q_1,q_2) \in \mathcal{D}_2^+(\mathcal{E}_2^+)$, and $(\mu_1,\mu_2) \in \mathcal{E}_2^+(\mathcal{D}_2^+)$. Then we have the desired result by combining (4.2.8) and (4.2.9).

By taking $n_1 = n_2 = 1$ in the above theorem, we get the following corollary.

Corollary 4.2.2. *Let $B_{2;\boldsymbol{\lambda},\boldsymbol{p}}$ and $B_{2;\boldsymbol{\mu},\boldsymbol{q}}$ be two finite mixtures. Then*

$$\begin{pmatrix} \lambda_1 & \lambda_2 \\ p_1 & p_2 \end{pmatrix} >^{row} \begin{pmatrix} \mu_1 & \mu_2 \\ q_1 & q_2 \end{pmatrix} \Rightarrow B_{2;\boldsymbol{\lambda},\boldsymbol{p}} \geq_{hr} B_{2;\boldsymbol{\mu},\boldsymbol{q}},$$

provided $(\lambda_1, \lambda_2), (\mu_1, \mu_2) \in \mathcal{E}_2^+(\mathcal{D}_2^+)$ *and* $(p_1, p_2), (q_1, q_2) \in \mathcal{D}_2^+(\mathcal{E}_2^+)$.

The corollary follows by taking $n_1 = n_2 = 1$ in Theorem 4.2.4. Next we extend the Theorem 4.2.4 to compare $B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}$ and $B_{n_1^*,n_2^*;\boldsymbol{\mu}^{mo},\mathbf{q}^{mo}}$, where n_1, n_1^* and n_2, n_2^* may be different but $n_1 + n_2 = n_1^* + n_2^* = n$.

Corollary 4.2.3. *Let* $B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}}$ *and* $B_{n_1^*,n_2^*;\boldsymbol{\mu}^{mo},\boldsymbol{q}^{mo}}$ *be two finite mixtures. Then*

$$B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}} \geq_{hr} B_{n_1^*,n_2^*;\boldsymbol{\mu}^{mo},\boldsymbol{q}^{mo}},$$

provided all the conditions of Theorem 4.2.4 hold along with the condition $(n_1, n_2) \overset{m}{\succeq} (n_1^*, n_2^*)$.

Proof: From equation (4.2.7), we have

$$r_{B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}}(t) = r(t)\frac{n_1p_1\lambda_1\bar{F}^{\lambda_1}(t) + n_2p_2\lambda_2\bar{F}^{\lambda_2}(t)}{n_1p_1\bar{F}^{\lambda_1}(t) + n_2p_2\bar{F}^{\lambda_2}(t)}. \tag{4.2.10}$$

Then

$$\begin{aligned}\frac{\partial r_{B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}}}{\partial n_1} - \frac{\partial r_{B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}}}{\partial n_2} &= r(t)\frac{p_1p_2\bar{F}^{\lambda_1+\lambda_2}(t)(\lambda_1 - \lambda_2)(n_1 + n_2)}{\left(n_1p_1\bar{F}^{\lambda_1}(t) + n_2p_2\bar{F}^{\lambda_2}(t)\right)^2} \\ &\leq \text{ (resp. } \geq \text{) } 0,\end{aligned}$$

if $\lambda_1 \leq$ (resp. $\geq$) λ_2. As per our assumption in Theorem 4.2.4, $n_1 \geq$ (resp. $\leq$) n_2 when $\lambda_1 \leq$ (resp. $\geq$) λ_2. Thus we have from Lemma 3 of Hazra et al. Hazra et al. [60] (see also Marshal et al. Marshall et al. [96], pp. 83-84),

$$(n_1, n_2) \overset{m}{\geq} (n_1^*, n_2^*) \Rightarrow r_{B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}} \leq r_{B_{n_1^*,n_2^*;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}}. \tag{4.2.11}$$

So

$$\begin{aligned}B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}} &\geq_{hr} B_{n_1^*,n_2^*;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}} \\ &\geq_{hr} B_{n_1^*,n_2^*;\boldsymbol{\mu}^{mo},\mathbf{q}^{mo}},\end{aligned}$$

where the first inequality follows from the equation (4.2.11) and the second inequality follows from Theorem 4.2.4. This completes the proof.

The following theorem provides sufficient conditions on the baseline distribution, hazard ratio vectors and the mixing proportion vectors under which the finite mixture $B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}$ is greater than the another mixture $B_{n_1,n_2;\boldsymbol{\mu}^{mo},\mathbf{p}^{mo}}$ with respect to the star order. In other words, this theorem can be used to compare the skewness of the distributions of two finite mixtures under study.

Theorem 4.2.5. *Let* $B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}}$ *and* $B_{n_1,n_2;\boldsymbol{\mu}^{mo},\boldsymbol{p}^{mo}}$ *be two finite mixtures with same mixture proportion and* $\log(\bar{F}(x))/xr(x)$ *is decreasing in* x*. If* $\lambda_1 \geq \mu_1 \geq \mu_2 \geq \lambda_2$*,* $p_1 \leq p_2$ *and* $\boldsymbol{\lambda}^{mo} \overset{w}{\preceq} \boldsymbol{\mu}^{mo}$*, then* $B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}} \geq_{\star} B_{n_1,n_2;\boldsymbol{\mu}^{mo},\boldsymbol{p}^{mo}}$*.*

Proof: We have the cdf of $B_{n_1,n_2;\boldsymbol{\alpha}^{mo},\mathbf{p}^{mo}}$ as

$$F_{B_{n_1,n_2;\boldsymbol{\alpha}^{mo},\mathbf{p}^{mo}}}(x) = 1 - \left[n_1 p_1 \bar{F}^{\lambda_1}(x) + n_2 p_2 \bar{F}^{\lambda_2}(x)\right]$$

The theorem is proved using the concept of Lemma 4.2.1. Under the conditions $\lambda_1 \geq \mu_1 \geq \mu_2 \geq \lambda_2$ and $\boldsymbol{\lambda}^{mo} \overset{w}{\preceq} \boldsymbol{\mu}^{mo}$, we have $n_1\lambda_1 + n_2\lambda_2 \geq n_1\mu_1 + n_2\mu_2$.
Case I: $n_1\lambda_1 + n_2\lambda = n_1\mu_1 + n_2\mu_2$. Without loss of generality we consider $n_1\lambda_1 + n_2\lambda_2 = n_1\mu_1 + n_2\mu_2 = 1$. Let $\lambda_1 = \lambda$ and $\mu_1 = \mu$, where $\lambda, \mu \in (1/(n_1+n_2), 1/n_1]$, and we write $F_{B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}}(x) = F_{B_\lambda}(x)$, just to indicate that it becomes an expression of λ. Now,

$$\frac{\partial F_{B_\lambda}(x)}{\partial \lambda} = -\log(\bar{F}(x))\left[n_1 p_1 \bar{F}^{\lambda}(x) - n_1 p_2 \bar{F}^{\left(\frac{1-n_1\lambda}{n_2}\right)}(x)\right].$$

Again

$$f_{B_\lambda}(x) = r(x)\left[n_1 p_1 \lambda \bar{F}^{\lambda}(x) + p_2(1-n_1\lambda)\bar{F}^{\left(\frac{1-n_1\lambda}{n_2}\right)}(x)\right].$$

Now

$$\begin{aligned}
\frac{\partial F_{B_\lambda}(x)/\partial\lambda}{x f_{B_\lambda}(x)} &= -\frac{\log(\bar{F}(x))}{xr(x)}\left[\frac{n_1 p_1 \lambda \bar{F}^{\lambda}(x) - n_1 p_2 \bar{F}^{\left(\frac{1-n_1\lambda}{n_2}\right)}(x)}{n_1 p_1 \lambda \bar{F}^{\lambda}(x) + p_2(1-n_1\lambda)\bar{F}^{\left(\frac{1-n_1\lambda}{n_2}\right)}(x)}\right] \\
&= -\frac{\log(\bar{F}(x))}{xr(x)}\left[\lambda + \left(\frac{p_2}{n_1}\right)\left(\frac{p_1}{p_2}\cdot \bar{F}^{\lambda-\frac{1-n_1\lambda}{n_2}}(x) - 1\right)^{-1}\right]^{-1} \\
&= -\frac{\log(\bar{F}(x))}{xr(x)} \times \Theta(x)\ (say).
\end{aligned}$$

where $\Theta(x) = \left[\lambda + \left(\frac{p_2}{n_1}\right)\left(\frac{p_1}{p_2}\cdot \bar{F}^{\lambda-\frac{1-n_1\lambda}{n_2}}(x) - 1\right)^{-1}\right]^{-1}$. Since $\lambda \geq \frac{1-n_1\lambda}{n_2}$, so for $p_1 \leq p_2$ we have $\Theta(x)$ is non-positive (follows from the first equation) and also increasing in x (follows from the second equation). Consequently we have $\frac{\partial F_{B_\lambda}(x)/\partial\lambda}{f_{B_\lambda}(x)}$ is decreasing in x.

Case II: Suppose $n_1\lambda_1 + n_2\lambda_2 \geq n_1\mu_1 + n_2\mu_2$. In the same line as of Case II in Theorem 4.2.3, it can be shows that the theorem holds for this case also.

Remark 4.2.2. *It is to be noted that the condition* $\frac{\log(\bar{F}(x))}{xr(x)}$ *is decreasing in* x *is satisfied by many class of distributions. For example, Burr distribution with sf* $\bar{F}(x) = (1+x^{\alpha})^{-\beta}, \alpha > 0, \beta > 0$ *satisfies the conditions for* $0 < \alpha < \beta < \infty$*, and log-logistic distributions with sf* $\bar{F}(x) = 1/(1+(\alpha x)^{\beta}), \alpha > 0, \beta > 0$ *satisfies the condition for* $0 < \alpha, \beta < \infty$*. The term* $\frac{\log(\bar{F}(x))}{xr(x)}$ *can be expanded as*

$$\frac{\log(\bar{F}(x))}{xr(x)} = \frac{-\int_0^x r(u)du}{xr(x)} = -\left[x\frac{\Lambda'(x)}{\Lambda(x)}\right]^{-1},$$

Hence the condition $\frac{\log(\bar{F}(x))}{xr(x)}$ *is decreasing in* x *equivalent to* $x\frac{\Lambda'(x)}{\Lambda(x)}$ *is decreasing in* x*, where* $\Lambda(x)$ *is the cumulative hazard function. It is also to be noted that if* F *is DFR (decreasing failure rate) distribution, then* $\frac{\log(\bar{F}(x))}{xr(x)}$ *is decreasing in* x*. It follows from the fact that, DFR* $\Longrightarrow$ *DFRA (decreasing failure rate average), which implies* $\frac{log(\bar{F}(x))}{x}$ *is decreasing in* x*. Hence as an immediate consequence we can say that the above theorem holds for all DFR baseline distribution.*

Corollary 4.2.4. *Under the same setup of Theorem 4.2.5 if* $\lambda_1 \geq \mu_1 \geq \mu_2 \geq \lambda_2$*,* $p_1 \leq p_2$ *and* $\boldsymbol{\lambda}^{mo} \overset{w}{\preceq} \boldsymbol{\mu}^{mo}$*, then* $B_{n_1,n_2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}} \geq_{Lorenz} B_{n_1,n_2;\boldsymbol{\mu}^{mo},\boldsymbol{p}^{mo}}$*.*

Next let $X_i \sim PRH(F, \lambda_i)$ and suppose $J_{n_1,n_2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}$ be the random variable representing the finite mixtures of X_i's. The following theorem compares $J_{n_1,n_2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}$ and $J_{n_1,n_2;\boldsymbol{\mu}^{mo},\mathbf{q}^{mo}}$ with respect to rhr order. The proof is similar to that of Theorem 4.2.4 and thus omitted.

Theorem 4.2.6. *Let* $J_{n_1,n_2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}}$ *and* $J_{n_1,n_2;\boldsymbol{\mu}^{mo},\boldsymbol{q}^{mo}}$ *be two finite mixtures. Then*

$$\begin{pmatrix} \lambda_1 & \dots & \lambda_1 & \lambda_2 & \dots & \lambda_2 \\ p_1 & \dots & p_1 & p_2 & \dots & p_2 \end{pmatrix} >^{row} \begin{pmatrix} \mu_1 & \dots & \mu_1 & \mu_2 & \dots & \mu_2 \\ q_1 & \dots & q_1 & q_2 & \dots & q_2 \end{pmatrix}$$
$$\Longrightarrow J_{n_1,n_2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}} \leq_{rhr} J_{n_1,n_2;\boldsymbol{\mu}^{mo},\boldsymbol{q}^{mo}},$$

provided $(\lambda_1, \lambda_2), (\mu_1, \mu_2) \in \mathcal{E}_2^+(\mathcal{D}_2^+)$*,* $(p_1, p_2), (q_1, q_2) \in \mathcal{D}_2^+(\mathcal{E}_2^+)$ *and* $n_1 \geq (\leq)\ n_2$*.*

The following corollary follows from the Theorem 4.2.6 by taking $n_1 = n_2 = 1$.

Corollary 4.2.5. *Let* $J_{2;\boldsymbol{\lambda},\boldsymbol{p}}$ *and* $J_{2;\boldsymbol{\mu},\boldsymbol{q}}$ *be two finite mixtures. Then*

$$\begin{pmatrix} \lambda_1 & \lambda_2 \\ p_1 & p_2 \end{pmatrix} >^{row} \begin{pmatrix} \mu_1 & \mu_2 \\ q_1 & q_2 \end{pmatrix} \Rightarrow J_{2;\boldsymbol{\lambda},\boldsymbol{p}} \leq_{rhr} J_{2;\boldsymbol{\mu},\boldsymbol{q}},$$

provided $(\lambda_1, \lambda_2), (\mu_1, \mu_2) \in \mathcal{E}_2^+(\mathcal{D}_2^+)$ *and* $(p_1, p_2), (q_1, q_2) \in \mathcal{D}_2^+(\mathcal{E}_2^+)$.

Remark 4.2.3. Hazra and Finkelstein [59] have obtained the hr ordering and reversed hr ordering results for two-component mixture as in corollaries 4.2.2 and 4.2.5 respectively under the chain majorization. As chain majorization is stronger condition than row majorization, so Theorems 4.2.4 and 4.2.6 serve as improvement of the previous results in terms of multiple-outlier model and row majorization.

Next we extend the Theorem 4.2.6 to compare $J_{n_1,n_2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}$ and $J_{n_1^*,n_2^*;\boldsymbol{\mu}^{mo},\mathbf{q}^{mo}}$, where n_1, n_1^* and n_2, n_2^* may be different but $n_1 + n_2 = n_1^* + n_2^* = n$. The proof is similar to that of Corollary 4.2.3 and thus omitted.

Corollary 4.2.6. *Let* $J_{n_1,n_2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}}$ *and* $J_{n_1^*,n_2^*;\boldsymbol{\mu}^{mo},\boldsymbol{q}^{mo}}$ *be two finite mixtures. Then*

$$J_{n_1.n_2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}} \leq_{rhr} J_{n_1^*,n_2^*;\boldsymbol{\mu}^{mo},\boldsymbol{q}^{mo}},$$

provided all the conditions of Theorem 4.2.6 hold along with the condition $(n_1, n_2) \overset{m}{\succeq} (n_1^*, n_2^*)$.

Theorem 4.2.7. *Let* $J_{n_1,n_2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}}$ *and* $J_{n_1,n_2;\boldsymbol{\mu}^{mo},\boldsymbol{p}^{mo}}$ *be two finite mixtures with same mixture proportion and* $\frac{log(F(x))}{x\tilde{r}(x)}$ *is increasing in* x. *If* $\lambda_1 \geq \mu_1 \geq \mu_2 \geq \mu_2$, $p_1 \leq p_2$ *and* $\boldsymbol{\lambda}^{mo} \overset{w}{\preceq} \boldsymbol{\mu}^{mo}$, *then* $J_{n_1,n_2;\boldsymbol{\alpha}^{mo},\boldsymbol{p}^{mo}} \geq_{\star} J_{n_1,n_2;\boldsymbol{\beta}^{mo},\boldsymbol{p}^{mo}}$.

Proof: The proof can be done in the same line as of Theorem 4.2.5 using the Lemma 4.2.1.

Remark 4.2.4. *The condition* $\frac{log(F(x))}{x\tilde{r}(x)}$ *is increasing in* x *is satisfied by many class of distributions. For example Weibull distribution with sf* $\bar{F}(x) = \exp(-(\alpha x)^{\beta}), \alpha > 0, \beta > 0$ *satisfies the condition for* $0 < \alpha < \beta < 1$, *and Gompertz distribution with sf* $\bar{F}(x) = \exp([-\beta(\alpha^x - 1)]/[\log(\alpha)]), \alpha > 0, \beta > 0$ *satisfies the condition for* $1 < \alpha < \infty$ *and* $0 < \beta < \infty$. *The term* $\frac{log(F(x))}{x\tilde{r}(x)}$ *can be expanded as*

$$\frac{log(F(x))}{x\tilde{r}(x)} = \frac{\int_0^x \tilde{r}(u)du}{x\tilde{r}(x)} = \left[x\frac{\Delta'(x)}{\Delta(x)}\right]^{-1},$$

so that the condition becomes $x\frac{\Delta'(x)}{\Delta(x)}$ *is decreasing in* x, *where* $\Delta(x)$ *is the cumulative rhr function.*

Corollary 4.2.7. *Under the same setup of Theorem 4.2.7 if* $\lambda_1 \geq \mu_1 \geq \mu_2 \geq \mu_2$, $p_1 \leq p_2$ *and* $\boldsymbol{\lambda}^{mo} \overset{w}{\preceq} \boldsymbol{\mu}^{mo}$, *then* $J_{n_1,n_2;\boldsymbol{\alpha}^{mo},\boldsymbol{p}^{mo}} \geq_{Lorenz} J_{n_1,n_2;\boldsymbol{\beta}^{mo},\boldsymbol{p}^{mo}}$.

4.3 Examples

Here we demonstrate some of the proposed results numerically. The first example illustrates the result of Theorem 4.2.1.

Example 4.3.1. *Consider two finite mixtures $M_{3;\boldsymbol{\alpha},\boldsymbol{p}}$ and $M_{3;\boldsymbol{\beta},\boldsymbol{q}}$, where $\boldsymbol{\alpha} = (0.2, 0.6, 1.7)$, $\boldsymbol{\beta} = (0.5, 0.95, 2.5)$, $\boldsymbol{p} = (0.5, 0.3, 0.2)$ and $\boldsymbol{q} = (0.6, 0.3, 0.1)$ so that all the conditions of Theorem 4.2.1 are satisfied. We take $\bar{F}(x) = \exp(-ax)^b$ with $a = 2$ and $b = 0.7$. We plot $\bar{F}_{M_{3;\boldsymbol{\alpha},\boldsymbol{p}}}(x)$ and $\bar{F}_{M_{3;\boldsymbol{\beta},\boldsymbol{q}}}(x)$ by substituting $x = t/(1-t)$, so that for $x \in [0,\infty)$, $t \in [0,1)$. From Figure 4.2 it is observed that $\bar{F}_{M_{3;\boldsymbol{\alpha},\boldsymbol{p}}}(x) \leq \bar{F}_{M_{3;\boldsymbol{\beta},\boldsymbol{q}}}(x)$.*

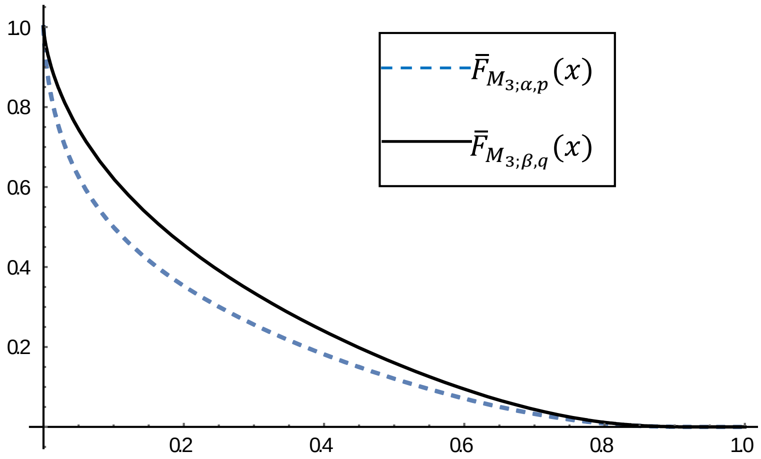

Figure 4.2: Plot of $\bar{F}_{M_{3;\boldsymbol{\alpha},\mathbf{p}}}(x)$ and $\bar{F}_{M_{3;\boldsymbol{\beta},\mathbf{q}}}(x)$ for $x = t/(1-t)$, $t \in [0,1]$.

The following example illustrates the result of Theorem 4.2.2.

Example 4.3.2. *Consider two finite mixtures $M_{2;\boldsymbol{\alpha},\boldsymbol{p}}$ and $M_{2;\boldsymbol{\beta},\boldsymbol{q}}$, where $\boldsymbol{\alpha} = (4, 2)$, $\boldsymbol{\beta} = (3.5, 2.5)$, $\boldsymbol{p} = (0.2, 0.8)$ and $\boldsymbol{q} = (25/74, 49/74)$ so that all the conditions of Theorem 4.2.2 are satisfied. We take $\bar{F}(x) = \exp(-ax)^b$ with $a = 5$ and $b = 0.5$. We plot $r_{M_{2;\boldsymbol{\alpha},\boldsymbol{p}}}(x)$ and $r_{M_{2;\boldsymbol{\beta},\boldsymbol{q}}}(x)$ by substituting $x = t/(1-t)$, so that for $x \in [0,\infty)$, $t \in [0,1)$. From Figure 4.3 it is observed that $r_{M_{2;\boldsymbol{\alpha},\boldsymbol{p}}}(x) \geq r_{M_{2;\boldsymbol{\beta},\boldsymbol{q}}}(x)$.*

Next example illustrates Theorem 4.2.3.

Example 4.3.3. *Consider two finite mixtures $M_{3,2;\boldsymbol{\alpha}^{mo},\boldsymbol{p}^{mo}}$ and $M_{3,2;\boldsymbol{\beta}^{mo},\boldsymbol{p}^{mo}}$ with $p_1 = 1/6$, $p_2 = 1/4$, $\alpha_1 = 8$, $\alpha_2 = 2$, $\beta_1 = 5$, $\beta_2 = 3$, $n_1 = 3$, $n_2 = 2$ and $\bar{F}(x) = 1/(1 + x^a)^b$ with $a = 1$, $b = 4$, so that all the conditions of Theorem 4.2.3 are satisfied. Let us*

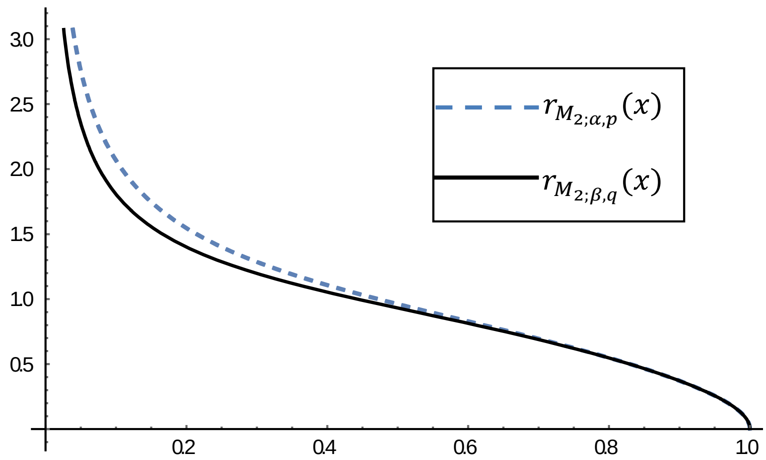

Figure 4.3: Plot of $r_{M_{2;\boldsymbol{\alpha},\mathbf{p}}}(x)$ and $r_{M_{2;\boldsymbol{\beta},\mathbf{q}}}(x)$ for $x = t/(1-t)$, $t \in [0,1]$.

denote the respective cdfs of the two mixtures as $F_1(x)$ *and* $F_2(x)$ *respectively. We plot the derivative of the function* $F_1^{-1}(F_2(x))$ *in Figure 4.4 by applying the transformation* $x = t/(1-t),\ t \in [0,1]$. *Figure 4.4 indicates that* $F_1^{-1}(F_2(x))$ *is increasing in* x *which implies that* $M_{3,2;\boldsymbol{\alpha}^{mo},\boldsymbol{p}^{mo}}(x) \geq_{\star} M_{3,2;\boldsymbol{\beta}^{mo},\boldsymbol{p}^{mo}}(x)$. *We also calculate coefficients of variations of the two mixtures as 1.12515 and 1.09363 respectively. So,* $cv\left(M_{3,2;\boldsymbol{\alpha}^{mo},\boldsymbol{p}^{mo}}\right) > cv\left(M_{3,2;\boldsymbol{\beta}^{mo},\boldsymbol{p}^{mo}}\right)$.

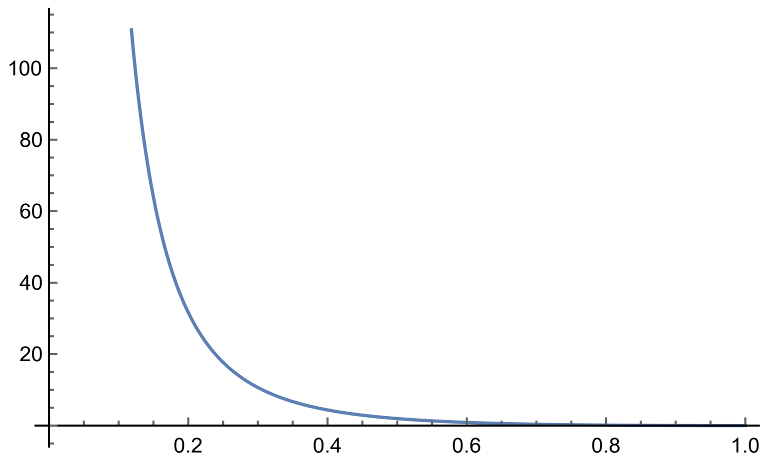

Figure 4.4: Plot of derivative of $F_1^{-1}(F_2(x))$ for $x = t/(1-t),\ t \in [0,1]$.

Next example illustrates Theorem 4.2.4.

Example 4.3.4. *Consider two finite mixtures* $B_{4,2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}}$ *and* $B_{4,2;\boldsymbol{\mu}^{mo},\boldsymbol{q}^{mo}}$, *where* $\lambda_1 = 0.38,\ \lambda_2 = 1.74,\ \mu_1 = 0.5,\ \mu_2 = 1.5\ p_1 = 0.175,\ p_2 = 0.15,\ q_1 = 0.18,\ q_2 = 0.14,$

$n_1 = 4$, and $n_2 = 2$ so that all the conditions of Theorem 4.2.4 are satisfied. We take $\bar{F}(x) = \exp(-(ax)^b)$ with $a = 5$, $b = 0.8$. We plot $r_{B_{4,2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}}}(x)$ and $r_{B_{4,2;\boldsymbol{\mu}^{mo},\boldsymbol{q}^{mo}}}(x)$ by substituting $x = t/(1-t)$, so that for $x \in [0, \infty)$, we have $t \in [0, 1)$. From Figure 4.5 it is observed that $r_{B_{4,2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}}}(x) \leq r_{B_{4,2;\boldsymbol{\mu}^{mo},\boldsymbol{p}^{mo}}}(x)$.

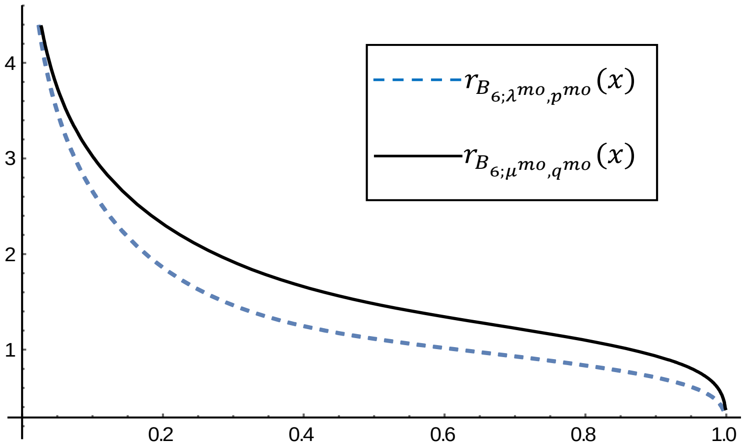

Figure 4.5: Plot of $r_{B_{4,2;\boldsymbol{\lambda}^{mo},\mathbf{p}^{mo}}}(x)$ and $r_{B_{4,2;\boldsymbol{\mu}^{mo},\mathbf{q}^{mo}}}(x)$ for $x = t/(1-t)$, $t \in [0, 1]$.

Next example illustrates Theorem 4.2.5.

Example 4.3.5. *Consider two finite mixtures $B_{3,2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}}$ and $B_{3,2;\boldsymbol{\mu}^{mo},\boldsymbol{q}^{mo}}$, where $p_1 = 1/6$, $p_2 = 1/4$, $\lambda_1 = 9$, $\lambda_2 = 2$, $\mu_1 = 5$, $\mu_2 = 3$, $n_1 = 3$, $n_2 = 2$ and $\bar{F}(x) = 1/(1 + (ax)^b)$,with $a = 3; b = 5$ so that all the conditions of Theorem 4.2.5 are satisfied. Let us denote the respective cdfs of the two mixtures as $G_1(x)$ and $G_2(x)$ respectively. We plot the derivative of the function $G_1^{-1}(G_2(x))$ in Figure 4.6 by applying the transformation $x = t/(1-t)$, $t \in [0, 1]$. Figure 4.6 indicates that $G_1^{-1}(G_2(x))$ is increasing in x which implies that $B_{3,2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}}(x) \geq_\star B_{3,2;\boldsymbol{\mu}^{mo},\boldsymbol{q}^{mo}}(x)$. We also calculate the coefficients of variations of the two mixtures as 0.329296 and 0.266123 respectively. So $cv(B_{3,2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}}) > cv(B_{3,2;\boldsymbol{\mu}^{mo},\boldsymbol{q}^{mo}})$.*

Example 4.3.6. *This example for Theorem 4.2.7.Consider two finite mixtures $J_{3,2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}}$ and $J_{3,2;\boldsymbol{\mu}^{mo},\boldsymbol{q}^{mo}}$, where $p_1 = 1/6$, $p_2 = 1/4$, $\lambda_1 = 12$, $\lambda_2 = 3$, $\mu_1 = 9$, $\mu_2 = 6$, $n_1 = 3$, $n_2 = 2$ and $\bar{F}(x) = \exp(-(ax)^b)$ with $a = 0.5, b = 0.9$ so that all the conditions of Theorem 4.2.7 are satisfied. Let us denote the respective cdfs of the two mixtures as $H_1(x)$ and $H_2(x)$ respectively. We plot the derivative of the function $H_1^{-1}(H_2(x))$ in Figure 4.7 by*

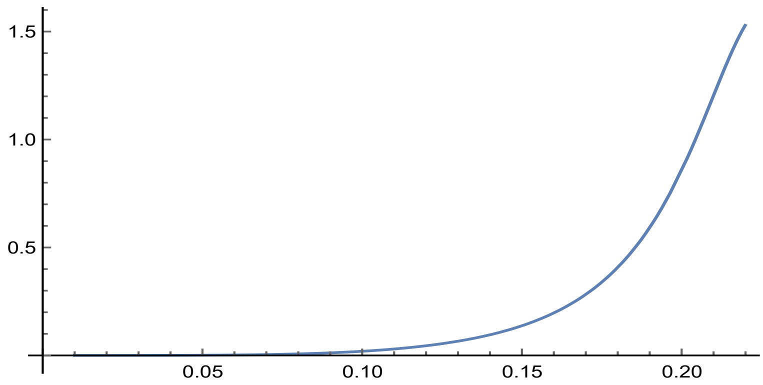

Figure 4.6: Plot of derivative of $G_1^{-1}(G_2(x))$ for $x = t/(1-t),\ t \in [0,1]$.

applying the transformation $x = t/(1-t),\ t \in [0,1]$. *Figure 4.7 indicates that* $H_1^{-1}(H_2(x))$ *is increasing in* x *which implies that* $J_{3,2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}}(x) \geq_{\star} J_{3,2;\boldsymbol{\mu}^{mo},\boldsymbol{q}^{mo}}(x)$. *We also calculate the coefficients of variations of the two mixtures as 0.688144 and 0.583667 respectively. So,* $cv\left(J_{3,2;\boldsymbol{\lambda}^{mo},\boldsymbol{p}^{mo}}\right) > cv\left(J_{3,2;\boldsymbol{\mu}^{mo},\boldsymbol{q}^{mo}}\right)$.

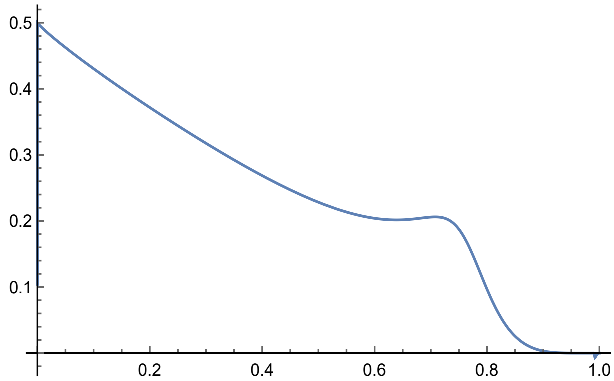

Figure 4.7: Plot of derivative of $H_1^{-1}(H_2(x))$ for $x = t/(1-t),\ t \in [0,1]$.

Chapter 5

Stochastic comparisons of continuous mixture models

5.1 Introduction:

Heterogeneity is a very common issue in many areas including reliability, survival analysis, demography and epidemiology. For instance, in mechanical systems, heterogeneity occurs due to unit-to-unit variability, changes in operating environments, the diversity of tasks and workloads during its lifetime. Özekici and Soyer [113] mentioned that a complex device like an airplane has large number of components where the failure structure of each component depends on a set of environmental conditions (e.g. the levels of vibration, atmospheric pressure, temperature, etc.) that vary during take-off, cruising and landing. So incorporating heterogeneity into hazard (failure) rate modeling is a common practice to achieve accuracy in the estimation. The proportional hazard rate (PHR) model is the most applied model in the case where factors (covariates) influencing the environment/operating condition are known and can be quantified. In such case, hazard rate of an individual is considered to be constant multiplicative to the baseline hazard. However, in many practical situation it may happen that some factors influencing the operating condition are unknown, and heterogeneity occurs in an unpredicted and unexplained manner. Proschan [115] pointed out that observed decreasing failure rates could be caused by unobserved heterogeneity. A component may be subject to different levels of operating environment (e.g. voltage, temperature) which is not fixed but changes over time. Component lifetimes and reliability depend on these random environmental variations. Frailty models (Cha and Finkelstein [30], Da and

Ding [36], Hougaard and Hougaard [65], Gupta et al. [52], Gupta and Peng [56], Li and Li [89], Vaupel et al. [129], Zaki et al. [134]) provide a way to introduce random effects in the model by a rv, called frailty rv, to account for unobserved (unexplained) heterogeneity among experimental units in their hazard (failure) rates. For instance,Vaupel et al. [129] discussed that in survival analysis, mortality of individuals differ due to large number of factors beyond age, e.g. the individual's susceptibility to causes of death, response to treatment and various risk factors. They considered a frailty rv to cope with the unobserved individual differences in mortality rates while defining the force of mortality of individuals. Cha et al. [31] considered a frailty rv in the model for mission abort/continuation policy for heterogeneous systems, to justify heterogeneity which may occur due to various reasons such as quality of resources used in the production process, operation and maintenance history, and human errors.

Ageing properties and stochastic comparisons of frailty models, arising from different choices of frailty/baseline distributions, have been studied in (Gupta and Kirmani [55], Kayid et al. [69], Misra et al. [99], Xie et al. [131]). On the other hand ageing properties and stochastic comparison of resilience models have been studied in Gupta and Kirmani [55], Li and Li [89] considering different baseline distributions and/or resilience distributions. He and Xie [64] derived comparison results for general weighted frailty models with respect to some relative stochastic orders. The frailty model is also regarded as mixture (continuous) distribution of the PHR model with baseline cdf F, and mixing rv Λ (Da and Ding [36]). Similarly resilience model is regarded as the mixture distribution of the PRH model (Li and Li [89]).

In this chapter we study the effect of frailty and resilience rvs on the baseline rv (X) using some shifted stochastic orders based on some ageing properties of X. In Section 5.2, we study the effect of frailty rv on the baseline rv, with respect to some shifted stochastic orders, where in Section 5.3, a similar study is carried out in case of resilience model. In Section 5.4, we illustrate some of our derived results with real-world data.

5.2 Results for frailty model

Here we study the effect of frailty rv on the baseline rv with respect to some shifted stochastic ordering based on some ageing properties of concerned baseline rvs. Throughout this section, we consider X and X^* be two rvs having sfs $\bar{F}, \bar{F}^*$ respectively, with corresponding pdfs f and f^*. The sf of X^* is given by the equation (1.2.15). Also consider that X an absolutely continuous non-negative rv.

In the following theorem we derived that, for a baseline rv X with ILR (resp. DLR) property, effect of a frailty rv Λ with $P(0 < \Lambda \leq 1) = 1$ (resp. $P(\Lambda \geq 1) = 1$) on X is that,

X^* will be greater than (resp. less than) X in the sense of the up shifted likelihood ratio order.

Theorem 5.2.1.

(i) $X^* \geq_{lr\uparrow} X$ *if* X *is ILR, provided* $0 < \Lambda \leq 1$ *with probability 1;*

(ii) $X^* \leq_{lr\uparrow} X$ *if* X *is DLR, provided* $\Lambda \geq 1$ *with probability 1.*

Proof:

(i) We have

$$\begin{aligned} \frac{f^*(x)}{f(x+t)} &= \frac{f(x)}{f(x+t)} \times \int_0^\infty \lambda \bar{F}^{\lambda-1}(x) dH(\lambda) \\ &= E\left[\frac{f(x)\Lambda \bar{F}^{\Lambda-1}(x)}{f(x+t)}\right]. \end{aligned} \tag{5.2.1}$$

Now X is ILR implies $\frac{f(x)}{f(x+t)}$ is increasing in x for any $t > 0$. Again $\lambda \bar{F}^{\lambda-1}(x)$ will be increasing in x for any $0 < \lambda \leq 1$. Now if we consider Λ such that $P(0 < \Lambda \leq 1) = 1$ the result follows immediately.

(ii) Similarly X is DLR implies $\frac{f(x)}{f(x+t)}$ is decreasing in x. Again $\lambda \bar{F}^{\lambda-1}(x)$ will be decreasing in x for any $\lambda \geq 1$. Now if we consider Λ such that $P(\Lambda \geq 1) = 1$, the result follows immediately.

Examples 5.2.1 and 5.2.2 illustrate (i) and (ii) of the Theorem 5.2.1 respectively.

Example 5.2.1. *Suppose* X *follows gamma distribution with pdf* $f(x) = xe^{-x},\ x \geq 0$. *Then clearly* X *is ILR. Consider the frailty rv* Λ *to be uniformly distributed on* $[0, 1]$. *Then it is easy to check that* $f^*(x)/f(x+t)$ *is increasing in* x *for all* $t > 0$, *giving* $X \leq_{lr\uparrow} X^*$.

Example 5.2.2. *Suppose* X *follows Weibull distribution with pdf* $f(x) = 3x^2 e^{-x^3},\ x \geq 0$. *Then clearly* X *is ILR. Consider the frailty rv* Λ *to be uniformly distributed on* $[1, 3]$. *Then it is easy to check that* $f^*(x)/f(x+t)$ *is decreasing in* x *for all* $t > 0$, *giving* $X \geq_{lr\uparrow} X^*$.

The following corollary follows immediately in case Λ is a degenerate rv

Corollary 5.2.1.

(i) $X^* \geq_{lr\uparrow} X$ *if* X *is ILR, provided* $0 < \lambda \leq 1$ *;*

(ii) $X^* \leq_{lr\uparrow} X$ *if* X *is DLR, provided* $\lambda \geq 1$ *.*

Theorem 5.2.2.

(i) $X^* \geq_{lr\downarrow} X$ *if* X *is DLR, provided* $0 < \Lambda \leq 1$ *with probability 1;*

(ii) $X^* \leq_{lr\downarrow} X$ *if* X *is ILR, provided* $\Lambda \geq 1$ *with probability 1.*

Proof:

(i) We have

$$\begin{aligned} \frac{f^*(x+t)}{f(x)} &= \frac{f(x+t)}{f(x)} \times \int_0^\infty \lambda \bar{F}^{\lambda-1}(x+t) dH(\lambda) \\ &= E\left[\frac{f(x+t)\Lambda \bar{F}^{\Lambda-1}(x+t)}{f(x)}\right]. \end{aligned} \quad (5.2.2)$$

Now X is DLR implies $\frac{f(x+t)}{f(x)}$ is increasing in x for any $t > 0$. Again $\lambda \bar{F}^{\lambda-1}(x+t)$ will be increasing in x for any $0 < \lambda \leq 1$. Now if we consider Λ such that $P(0 < \Lambda \leq 1) = 1$ the result follows immediately.

(ii) Similarly X is ILR implies $\frac{f(x+t)}{f(x)}$ is decreasing in x. Again $\lambda \bar{F}^{\lambda-1}(x+t)$ will be decreasing in x for any $\lambda \geq 1$. Now if we consider Λ such that $P(\Lambda \geq 1) = 1$ the result follows immediately.

Remark 5.2.1. *Theorem 5.2.2(i) implies that under the stated assumptions on* X *and* Λ, $\kappa_{X^*}(t) \geq \kappa_X(t')$ *for* $t \geq t' \geq 0$*. Similarly, Theorem 5.2.2(ii) implies that* $\kappa_{X^*}(t) \leq \kappa_X(t')$ *for* $t' \geq t \geq 0$.

The following theorem shows that, for a baseline rv X with IFR (resp. DFR) property, effect of a frailty rv Λ with $P(0 < \Lambda \leq 1) = 1$ (resp. $P(\Lambda \geq 1) = 1$) on X is that, X^* will be greater than (resp. less than) X under up shifted hr order.

Theorem 5.2.3.

(i) $X^* \geq_{hr\uparrow} X$ *if* X *is IFR, provided* $0 < \Lambda \leq 1$ *with probability 1;*

(ii) $X^* \leq_{hr\uparrow} X$ *if* X *is DFR, provided* $\Lambda \geq 1$ *with probability 1.*

Proof:

(i) We have

$$\begin{aligned} \frac{\bar{F}^*(x)}{\bar{F}(x+t)} &= \frac{\bar{F}(x)}{\bar{F}(x+t)} \times \int_0^\infty \bar{F}^{\lambda-1}(x) dH(\lambda) \\ &= E\left[\frac{\bar{F}(x)\bar{F}^{\Lambda-1}(x)}{\bar{F}(x+t)}\right]. \end{aligned} \quad (5.2.3)$$

Now X is IFR implies $\frac{\bar{F}(x)}{\bar{F}(x+t)}$ is increasing in x for any $t > 0$. Again $\bar{F}^{\lambda-1}(x)$ will be increasing in x for any $0 < \lambda \leq 1$. Now if we consider Λ such that $P(0 < \Lambda \leq 1) = 1$ the result follows immediately.

(ii) Similarly X is DFR implies $\frac{\bar{F}(x)}{\bar{F}(x+t)}$ is decreasing in x. Again $\bar{F}^{\lambda-1}(x)$ will be decreasing in x for any $\lambda \geq 1$. Now if we consider Λ such that $P(\Lambda \geq 1) = 1$ the result follows immediately.

Remark 5.2.2. *Theorem 5.2.3(i) implies that under the stated assumptions on X and Λ, $r_{X^*}(t) \leq r_X(t')$ for $t' \geq t \geq 0$. Similarly, Theorem 5.2.3(ii) implies that $r_{X^*}(t) \geq r_X(t')$ for $t \geq t' \geq 0$.*

Examples 5.2.3 and 5.2.4 (i) and (ii) of the Theorem 5.2.3 respectively.

Example 5.2.3. *Let X follows Weibull distribution with sf $\bar{F}(x) = e^{-x^2}$, $x \geq 0$. Clearly, X is IFR. Let the frailty rv Λ be uniformly distributed on $[0, 1]$. Then it follows that $\bar{F}^*(x)/\bar{F}(x+t)$ is increasing in x for all $t > 0$.*

Example 5.2.4. *Suppose X follows Weibull distribution with sf $\bar{F}(x) = e^{-x^{0.5}}$, $x \geq 0$. Clearly, X is DFR. Let the frailty rv Λ be uniformly distributed on $[2, 5]$. Then it is easy to check that $\bar{F}^*(x)/\bar{F}(x+t)$ is decreasing in x for all $t > 0$.*

Theorem 5.2.4.

(i) $X^* \geq_{hr\downarrow} X$ if X is DFR, provided $0 < \Lambda \leq 1$ with probability 1;

(ii) $X^* \leq_{hr\downarrow} X$ if X is IFR, provided $\Lambda \geq 1$ with probability 1.

Proof:

(i) We have

$$\begin{aligned}
\frac{\bar{F}^*(x+t)}{\bar{F}(x)} &= \frac{\bar{F}(x+t)}{\bar{F}(x)} \times \int_0^\infty \bar{F}^{\lambda-1}(x+t)dH(\lambda) \\
&= E\left[\frac{\bar{F}(x+t)\bar{F}^{\Lambda-1}(x+t)}{\bar{F}(x)}\right]. \qquad (5.2.4)
\end{aligned}$$

Now X is DFR implies $\frac{\bar{F}(x+t)}{\bar{F}(x)}$ is increasing in x for any $t > 0$. Again $\bar{F}^{\lambda-1}(x+t)$ will be increasing in x for any $0 < \lambda \leq 1$. Now if we consider Λ such that $P(0 < \Lambda \leq 1) = 1$ the result follows immediately.

(ii) Similarly X is IFR implies $\frac{\bar{F}(x+t)}{\bar{F}(x)}$ is decreasing in x. Again $\bar{F}^{\lambda-1}(x+t)$ will be decreasing in x for any $\lambda \geq 1$. Now if we consider Λ such that $P(\Lambda \geq 1) = 1$ the result follows immediately.

Remark 5.2.3. *Theorem 5.2.4(i) implies that under the stated assumptions on X and Λ, $r_{X^*}(t) \leq r_X(t')$ for $t \geq t' \geq 0$. Similarly, Theorem 5.2.4(ii) implies that $r_{X^*}(t) \geq r_X(t')$ for $t' \geq t \geq 0$.*

Theorem 5.2.5.

(i) $X^* \geq_{mrl\uparrow} X$ if X is IMRL, provided $0 < \Lambda \leq 1$ with probability 1;

(ii) $X^* \leq_{mrl\uparrow} X$ if X is DMRL, provided $\Lambda \geq 1$ with probability 1.

Proof:

(i) We have

$$\begin{aligned}\int_{x+t}^{\infty} \bar{F}^*(u)du \Big/ \int_{x}^{\infty} \bar{F}(u)du &= \frac{\int_0^\infty \int_{x+t}^\infty \bar{F}^\lambda(u)dH(\lambda)}{\int_x^\infty \bar{F}(u)du} \\ &= E\left[\int_{x+t}^{\infty} \bar{F}^\Lambda(u)du \Big/ \int_{x}^{\infty} \bar{F}(u)du\right]. \end{aligned} \tag{5.2.5}$$

Now if X is IMRL then $\int_{x+t}^\infty \bar{F}(u)du/\int_x^\infty \bar{F}(u)du$ increasing in x for any $t > 0$. That is we have

$$\frac{\bar{F}(x+t)}{\int_{x+t}^\infty \bar{F}(u)du} \leq \frac{\bar{F}(x)}{\int_x^\infty \bar{F}(u)du}. \tag{5.2.6}$$

Let us define a function $\alpha(\lambda) = \frac{\bar{F}^\lambda(x+t)}{\int_{x+t}^\infty \bar{F}^\lambda(u)du}$. $\lambda > 0$.

$$\begin{aligned}\alpha'(\lambda) &\overset{sgn}{=} \int_{x+t}^{\infty} \bar{F}^\lambda(u)[\log(\bar{F}(x+t)) - \log(\bar{F}(u))]du \\ &\overset{sgn}{=} \geq 0. \end{aligned} \tag{5.2.7}$$

Therefore from (5.2.6) and (5.2.7) we have for any $0 < \lambda \leq 1$ we have

$$\frac{\bar{F}^\lambda(x+t)}{\int_{x+t}^\infty \bar{F}^\lambda(u)du} \leq \frac{\bar{F}(x+t)}{\int_{x+t}^\infty \bar{F}(u)du} \leq \frac{\bar{F}(x)}{\int_x^\infty \bar{F}(u)du}. \tag{5.2.8}$$

Hence from (5.2.8) we can easily conclude that (5.2.5) is increasing in x if $P(0 < \Lambda \leq 1) = 1$.

(ii) Since X is DMRL, hence $\int_{x+t}^{\infty} \bar{F}(u)du / \int_{x}^{\infty} \bar{F}(u)du$ decreasing in x. This implies

$$\frac{\bar{F}(x+t)}{\int_{x+t}^{\infty} \bar{F}(u)du} \geq \frac{\bar{F}(x)}{\int_{x}^{\infty} \bar{F}(u)du}. \tag{5.2.9}$$

Therefore from (5.2.9) and (5.2.7) we have for any $\lambda \geq 1$,

$$\frac{\bar{F}^{\lambda}(x+t)}{\int_{x+t}^{\infty} \bar{F}^{\lambda}(u)du} \geq \frac{\bar{F}(x+t)}{\int_{x+t}^{\infty} \bar{F}(u)du} \geq \frac{\bar{F}(x)}{\int_{x}^{\infty} \bar{F}(u)du}. \tag{5.2.10}$$

Hence from (5.2.9) we can easily conclude that (5.2.5) is decreasing in x if $P(\Lambda \geq 1) = 1$.

Remark 5.2.4. *Theorem 5.2.5(i) implies that under the stated assumptions on X and Λ, $m_{X^*}(t) \geq m_X(t')$ for $t' \geq t \geq 0$. Similarly, Theorem 5.2.5(ii) implies that $m_{X^*}(t) \leq m_X(t')$ for $t \geq t' \geq 0$.*

Theorem 5.2.6.

(i) $X^* \geq_{mrl\downarrow} X$ if X is DMRL, provided $0 < \Lambda \leq 1$ with probability 1.

(ii) $X^* \leq_{mrl\downarrow} X$ if X is IMRL, provided $\Lambda \geq 1$ with probability 1;

Proof: We have

(i)

$$\begin{aligned} \int_{x}^{\infty} \bar{F}^*(u)du \Big/ \int_{x+t}^{\infty} \bar{F}(u)du &= \frac{\int_0^{\infty} \int_x^{\infty} \bar{F}^{\lambda}(u)dH(\lambda)}{\int_{x+t}^{\infty} \bar{F}(u)du} \\ &= E\left[\int_{x}^{\infty} \bar{F}^{\Lambda}(u)du \Big/ \int_{x+t}^{\infty} \bar{F}(u)du\right]. \end{aligned} \tag{5.2.11}$$

Now, if X is DMRL then $\int_{x+t}^{\infty} \bar{F}(u)du / \int_{x}^{\infty} \bar{F}(u)du$ is decreasing in x. This implies

$$\frac{\bar{F}(x+t)}{\int_{x+t}^{\infty} \bar{F}(u)du} \geq \frac{\bar{F}(x)}{\int_{x}^{\infty} \bar{F}(u)du}. \tag{5.2.12}$$

Therefore from (5.2.12) and (5.2.15) we have, for any $0 < \lambda \leq 1$,

$$\frac{\bar{F}(x+t)}{\int_{x+t}^{\infty} \bar{F}(u)du} \geq \frac{\bar{F}(x)}{\int_{x}^{\infty} \bar{F}(u)du} \geq \frac{\bar{F}^{\lambda}(x)}{\int_{x}^{\infty} \bar{F}^{\lambda}(u)du}. \tag{5.2.13}$$

Hence from (5.2.12) we can easily conclude that (5.2.11) is increasing in x if $P(0 < \Lambda \leq 1)$.

(ii) If X is IMRL then $\int_{x+t}^{\infty}\bar{F}(u)du/\int_{x}^{\infty}\bar{F}(u)du$ increasing in x. This implies

$$\frac{\bar{F}(x+t)}{\int_{x+t}^{\infty}\bar{F}(u)du} \leq \frac{\bar{F}(x)}{\int_{x}^{\infty}\bar{F}(u)du}. \tag{5.2.14}$$

Let us define a function $\beta(\lambda) = \frac{\bar{F}^{\lambda}(x)}{\int_{x}^{\infty}\bar{F}^{\lambda}(u)du} \quad \lambda > 0.$

$$\begin{aligned}\beta'(\lambda) &\overset{sgn}{=} \int_{x}^{\infty}\bar{F}^{\lambda}(u)[\log(\bar{F}(x)) - \log(\bar{F}(u))]du \\ &\overset{sgn}{=} \geq 0.\end{aligned} \tag{5.2.15}$$

Therefore from (5.2.14) and (5.2.15) we have, for any $\lambda \geq 1$,

$$\frac{\bar{F}(x+t)}{\int_{x+t}^{\infty}\bar{F}(u)du} \leq \frac{\bar{F}(x)}{\int_{x}^{\infty}\bar{F}(u)du} \leq \frac{\bar{F}^{\lambda}(x)}{\int_{x}^{\infty}\bar{F}^{\lambda}(u)du}. \tag{5.2.16}$$

Hence from (5.2.16) we can easily conclude that (5.2.11) is decreasing in x if $P(\Lambda \geq 1) = 1$.

Remark 5.2.5. *Theorem 5.2.6(i) implies that under the stated assumptions on X and Λ, $m_{X^*}(t) \geq m_X(t')$ for $t \geq t' \geq 0$. Similarly, Theorem 5.2.6(ii) implies that $m_{X^*}(t) \leq m_X(t')$ for $t' \geq t \geq 0$.*

5.3 Results for resilience model:

Here we study some shifted stochastic ordering of resilience models based on some ageing properties of concerned baseline rvs. Let X^* follow resilience model with baseline distribution G, and resilience rv Ω having cdf K so that the cdf of X^* is given by

$$G^*(x) = \int_0^{\infty} G^{\omega}(x)dK(\omega). \tag{5.3.1}$$

Throughout this section, we consider X be a rv with cdf G and X^* be the rv as defined above for which the cdf is given by equation (5.3.1). Also consider that X be an absolutely continuous non-negative rv

Following theorem shows that for a baseline rv X with ILR (resp. DLR) property, effect of a resilience rv Ω with $P(\Omega \geq 1) = 1$ (resp. $P(0 < \Omega \leq 1) = 1$) on X is that, X^* is greater than (resp. less than) X in the sense of the up shifted likelihood ratio order.

Theorem 5.3.1.

(i) $X^* \geq_{lr\uparrow} X$ *if* X *is ILR, provided* $\Omega \geq 1$ *with probability 1;*

(ii) $X^* \leq_{lr\uparrow} X$ *if* X *is DLR, provided* $0 < \Omega \leq 1$ *with probability 1.*

Proof:

(i) We have

$$\begin{aligned} \frac{g^*(x)}{g(x+t)} &= \frac{g(x)}{g(x+t)} \times \int_0^\infty \omega G^{\omega-1}(x) dK(\omega) \\ &= E\left[\frac{g(x)\Omega G^{\Omega-1}(x)}{g(x+t)}\right]. \end{aligned} \tag{5.3.2}$$

Now X is ILR implies $\frac{g(x)}{g(x+t)}$ is increasing in x for any $t > 0$. Again $\omega G^{\omega-1}(x)$ is increasing in x for any $\omega \geq 1$. Now if we consider Ω such that $P(\Omega \geq 1) = 1$ the result follows immediately.

(ii) Similarly X is DLR implies $\frac{g(x)}{g(x+t)}$ is decreasing in x for any $t > 0$. Again $\omega G^{\omega-1}(x)$ is decreasing in x for any $0 < \omega \leq 1$. Now if we consider Ω such that $P(0 < \Omega \leq 1) = 1$ the result follows immediately.

The following corollary follows immediately in case Ω is a degenerate rv.

Corollary 5.3.1.

(i) $X^* \geq_{lr\uparrow} X$ *if* X *is ILR, provided* $\omega \geq 1$ *;*

(ii) $X^* \leq_{lr\uparrow} X$ *if* X *is DLR, provided* $0 < \omega \leq 1$ *.*

Theorem 5.3.2.

(i) $X^* \geq_{lr\downarrow} X$ *if* X *is DLR, provided* $\Omega \geq 1$ *with probability 1;*

(ii) $X^* \leq_{lr\downarrow} X$ *if* X *is ILR, provided* $0 < \Omega \leq 1$ *with probability 1.*

Proof:

(i) We have

$$\begin{aligned} \frac{g^*(x+t)}{g(x)} &= \frac{g(x+t)}{g(x)} \times \int_0^\infty \omega G^{\omega-1}(x+t) dK(\omega) \\ &= E\left[\frac{g(x+t)\Omega G^{\Omega-1}(x+t)}{g(x)}\right]. \end{aligned} \tag{5.3.3}$$

Now X is DLR implies $\frac{g(x+t)}{g(x)}$ is increasing in x for any $t > 0$. Again $\omega G^{\omega-1}(x)$ is increasing in x for any $\omega \geq 1$. Now if we consider Ω such that $P(\Omega \geq 1) = 1$, the result follows immediately.

(ii) Similarly X is ILR implies $\frac{g(x+t)}{g(x)}$ is decreasing in x. Again $\omega G^{\omega-1}(x)$ is decreasing in x for any $0 < \omega \leq 1$. Now if we consider Ω such that $P(0 < \Omega \leq 1) = 1$, the result follows immediately.

Following theorem shows that, for a baseline rv X with DRFR (resp. IRFR) property, effect of a resilience rv Ω with $P(\Omega \geq 1) = 1$ (resp. $P(0 < \Omega \leq 1) = 1$) on X is that, X^* is greater than (resp. less than) X in the sense of the up shifted rhr order.

Theorem 5.3.3.

(i) $X^* \geq_{rh\uparrow} X$ *if* X *is DRFR, provided* $\Omega \geq 1$ *with probability 1;*

(ii) $X^* \leq_{rh\uparrow} X$ *if* X *is IRFR, provided* $0 < \Omega \leq 1$ *with probability 1.*

Proof:

(i) We have

$$\begin{aligned}\frac{G^*(x)}{G(x+t)} &= \frac{G(x)}{G(x+t)} \times \int_0^\infty G^{\omega-1}(x)dK(\omega) \\ &= E\left[\frac{G(x)G^{\Omega-1}(x)}{G(x+t)}\right]. \end{aligned} \tag{5.3.4}$$

Now X is DRFR implies $\frac{G(x)}{G(x+t)}$ is increasing in x for any $t > 0$. Again $G^{\omega-1}(x)$ is increasing in x for any $\omega \geq 1$. Now if we consider Ω such that $P(\Omega \geq 1) = 1$ the result follows immediately.

(ii) Similarly X is IRFR implies $\frac{G(x)}{G(x+t)}$ is decreasing in x. Again $G^{\omega-1}(x)$ is decreasing in x for any $0 < \omega \leq 1$. Now if we consider Ω such that $P(0 < \Omega \leq 1) = 1$ the result follows immediately.

Remark 5.3.1. *Theorem 5.3.3(i) implies that under the stated assumptions on* X *and* Ω*,* $\tilde{r}_{X^*}(t) \geq \tilde{r}_X(t')$ *for* $t' \geq t \geq 0$*. Similarly, Theorem 5.3.3(ii) implies that* $\tilde{r}_{X^*}(t) \leq \tilde{r}_X(t')$ *for* $t \geq t' \geq 0$.

Example 5.3.1. *Let* X *follows Weibull distribution with cdf* $G(x) = 1 - e^{-2x^2}$*,* $x \geq 0$*. Clearly,* X *is DRFR. Let* Ω *to be uniformly distributed on* $[2, 5]$*. Then it is easy to check that* $G^*(x)/G(x+t)$ *is increasing in* x *for all* $t > 0$.

Theorem 5.3.4.

(i) $X^* \geq_{rh\downarrow} X$ *if* X *is IRFR, provided* $\Omega \geq 1$ *with probability 1;*

(ii) $X^* \leq_{rh\downarrow} X$ *if* X *is DRFR, provided* $0 < \Omega \leq 1$ *with probability 1.*

Proof:

(i) We have

$$\begin{aligned} \frac{G^*(x+t)}{G(x)} &= \frac{G(x+t)}{G(x)} \times \int_0^\infty G^{\omega-1}(x+t)dK(\omega) \\ &= E\left[\frac{G(x+t)G^{\Omega-1}(x+t)}{G(x)}\right]. \end{aligned} \tag{5.3.5}$$

Now X is IRFR implies $\frac{G(x+t)}{G(x)}$ is increasing in x for any $t > 0$. Again $G^{\omega-1}(x)$ is increasing in x for any $\omega \geq 1$. Now if we consider Ω such that $P(\Omega \geq 1) = 1$, the result follows immediately.

(ii) Similarly X is DRFR implies $\frac{G(x+t)}{G(x)}$ is decreasing in x. Again $\omega G^{\omega-1}(x)$ is decreasing in x for any $0 < \omega \leq 1$. Now if we consider Ω such that $P(0 < \Omega \leq 1) = 1$, the result follows immediately.

Remark 5.3.2. *Theorem 5.3.4(i) implies that under the stated assumptions on* X *and* Ω, $\tilde{r}_{X^*}(t) \geq \tilde{r}_X(t')$ *for* $t \geq t' \geq 0$. *Similarly, Theorem 5.3.4(ii) implies that* $\tilde{r}_{X^*}(t) \leq \tilde{r}_X(t')$ *for* $t' \geq t \geq 0$.

Example 5.3.2. *Let* X *follows Weibull distribution with cdf* $G(x) = 1 - e^{-x^3}$, $x \geq 0$ *so that* X *is DRFR. Let* Ω *be uniformly distributed on* $[0,1]$. *Then it is easy to check that* $G^*(x+t)/G(x)$ *is decreasing in* x *for all* $t > 0$.

Theorem 5.3.5.

(i) $X^* \leq_{mit\uparrow} X$ *if* X *is IMIT, provided* $\Omega \geq 1$ *with probability 1;*

(ii) $X^* \geq_{mit\downarrow} X$ *if* X *is IMIT, provided* $0 < \Omega \geq 1$ *with probability 1.*

Proof:

(i) We have

$$\int_0^{x+t} G^*(u)du \Bigg/ \int_0^x G(u)du = \frac{\int_0^{x+t}\int_0^\infty G^\omega(u)dK(\omega)du}{\int_0^x G(u)du}$$

$$= E\left[\frac{\int_0^{x+t} G^{\Omega}(u)du}{\int_0^x G(u)du}\right]. \tag{5.3.6}$$

Now X is IMIT implies $\frac{\int_0^{x+t} G(u)du}{\int_0^x G(u)du}$ is decreasing in x for any $t > 0$. That is, for any $t > 0$

$$\frac{G(x+t)}{\int_0^{x+t} G(u)du} \leq \frac{G(x)}{\int_0^x G(u)du}. \tag{5.3.7}$$

Also it is easy to verify that for any $\omega > 0$, $\frac{G^{\omega}(x)}{\int_0^x G^{\omega}(u)du}$ is increasing function of ω. So from (5.3.7) for any $0 < \omega \leq 1$

$$\frac{G^{\omega}(x+t)}{\int_0^{x+t} G^{\omega}(u)du} \leq \frac{G(x+t)}{\int_0^{x+t} G(u)du} \leq \frac{G(x)}{\int_0^x G(u)du}. \tag{5.3.8}$$

Hence from (5.3.8) we can conclude that (5.3.6) is decreasing in x.

(ii). Again we have

$$\begin{aligned}\int_0^x G(u)du \Big/ \int_0^{x+t} G^*(u)du &= \frac{\int_0^x G(u)du}{\int_0^{x+t}\int_0^{\infty} G^{\omega}(u)dH(\omega)du} \\ &= E\left[\frac{\int_0^x G(u)du}{\int_0^{x+t} G^{\Omega}(u)du}\right]. \end{aligned}\tag{5.3.9}$$

As $\frac{G^{\omega}(x)}{\int_0^x G^{\omega}(u)du}$ is increasing function of ω for any $\omega \geq 1$ we have

$$\frac{G(x)}{\int_0^x G(u)du} \leq \frac{G(x+t)}{\int_0^{x+t} G(u)du} \leq \frac{G^{\omega}(x+t)}{\int_0^{x+t} G^{\omega}(u)du}. \tag{5.3.10}$$

Consequently from (5.3.10) we can conclude that (5.3.9) is decreasing in x.

Remark 5.3.3. *Theorem 5.3.5(i) implies that under the stated assumptions on X and Ω, $mit_{X^*}(t) \geq mit_X(t')$ for $t \geq t' \geq 0$. Similarly, Theorem 5.3.5(ii) implies that $mit_{X^*}(t) \leq mit_X(t')$ for $t \geq t' \geq 0$.*

5.4 Data analysis

Here we illustrate some of our results in two real scenarios considering two data sets, namely Survival times in leukaemia and Fatigue-life failures (Hand et al. [58]) data. In scenario I, we illustrate results (Theorems 5.2.2 and 5.2.4) for frailty model. In scenario II, we illustrate results for resilience model (Theorems 5.3.1 and 5.3.3).

Scenario I: We consider the data set Survival times in leukaemia (Hand et al. [58]) which contains the survival times of 43 patients suffering from chronic granulocytic leukaemia, measured in days from the time of diagnosis. From the quantile-quantile (Q-Q) plot (Figure 5.1) and results of Anderson-Darling test (Table 5.1) for the observed samples, it is observed that Weibull distribution fits well. Estimated values of parameters of the fitted baseline Weibull (X) with the cdf $F(x) = 1 - e^{(-x/\beta)^k}$, $x \geq 0, \beta > 0, k > 0$ are presented in Table 5.2.

Table 5.1: **Results of Anderson-Darling test**

AD-value	p-value	Critical value(cv)
0.3616	0.8852	2.4978

Table 5.2: **Estimated parameters of Weibull distribution**

Parameters	Estimated value	95% confidence interval
Scale (β)	986.672	$[766.52, 1270.06]$
Shape (k)	1.24044	$[0.973535, 1.58052]$

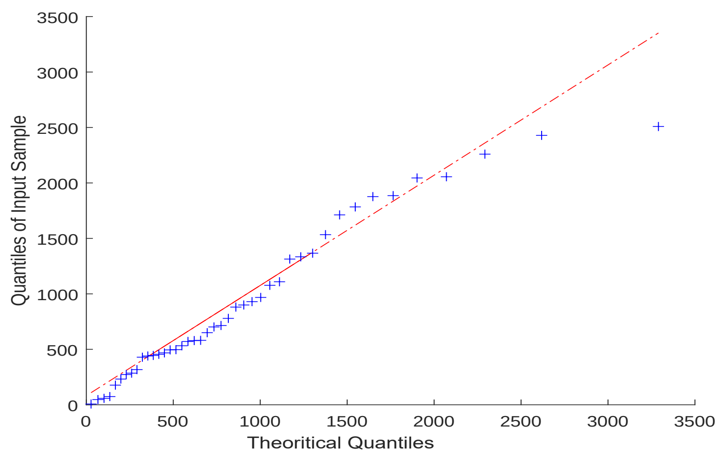

Figure 5.1: QQ plot of sample data vs Weibull distribution

With shape parameter $k > 1$, this baseline Weibull distribution is ILR and and so it is IFR. Next we consider well known Gamma-frailty i.e. $\Lambda \sim \Gamma(1/a^2, 1/a^2)$ where $\Lambda \geq 1$ with probability 1. According to Theorem 5.2.2(ii), the effect of considered gamma frailty on X is that, $X^* \leq_{lr\downarrow} X$, which implies that $\kappa_{X^*}(t) \leq \kappa_X(t')$ for $t' \geq t \geq 0$. Similarly, according to Theorem 5.2.4(ii), $X^* \leq_{hr\downarrow} X$, which implies that $r_X^*(t) \geq r_X(t')$ for $t' \geq t \geq 0$. Also, we have $X^* \geq_{disp} X$, where 'disp' stands for dispersive order (Shaked and Shanthikumar [122]).

It follows from the fact that for two non-negative rvs X and Y, $X \leq_{hr\downarrow} Y \Rightarrow X \leq_{disp} Y$ (Lillo et al. [90]).

To demonstrate the above mentioned stochastic orders, we proceed as follows. The sf and pdf of the above frailty model (Gamma-frailty Weibull-baseline) are

$$\begin{aligned}
\bar{F}^*(t) &= \frac{(1/a^2)^{-1/a^2}(a^2+\frac{t^k}{\beta^k})^{-1/a^2}\zeta_1\left(\frac{1}{a^2},a^2+\frac{t^k}{\beta^k}\right)}{\Gamma(\frac{1}{a^2})\left(1-\zeta_2(\frac{1}{a^2},0,a^2)\right)} \qquad (5.4.1)\\
\text{and } f^*(t) &= \frac{\frac{kt^{k-1}}{\beta^k}(1/a^2)^{-1/a^2}(a^2+\frac{t^k}{\beta^k})^{-1-1/a^2}\zeta_1\left(\frac{1}{a^2},a^2+\frac{t^k}{\beta^k}\right)}{\Gamma(\frac{1}{a^2})\left(1-\zeta_2(\frac{1}{a^2},0,a^2)\right)}, \qquad (5.4.2)
\end{aligned}$$

respectively. Let $t_1, t_2, ..., t_n$ be the observations under consideration. We now obtain maximum likelihood estimation of the parameter a under the Gamma-frailty Weibull-baseline. The likelihood function is given by

$$\begin{aligned}
\mathcal{L}(a|t_1,t_2,..,t_n) &= \left(\frac{(\frac{1}{a^2})^{-1/a^2}}{\Gamma(\frac{1}{a^2})\left(1-\zeta_2(\frac{1}{a^2},0,a^2)\right)}\right)^n k^n\prod_{i=1}^n (t_i)^{k-1}\prod_{i=1}^n(a^2+\frac{t_i^k}{\beta^k})^{-1-1/a^2}\\
&\quad\times\prod_{i=1}^n\zeta_1\left(\frac{1}{a^2},a^2+\frac{t_i^k}{\beta^k}\right), \qquad (5.4.3)
\end{aligned}$$

where $\zeta_1(a,x) = \int_x^\infty t^{a-1}e^{-t}dt$ and $\zeta_2(a,x) = \frac{\int_0^x t^{a-1}e^{-t}dt}{\Gamma(a)}$ are upper incomplete gamma functions and regularized lower incomplete gamma functions respectively. Estimated value of a is obtained as 0.784 with $\mathbb{P}(\Gamma \geq 1) = 1$.
We then plotted $f^*(x+t)\Big/f(x)$ taking some finite range of x and t as shown in Figure 5.2, which is clearly showing that the ratio is decreasing in x, giving $X^* \leq_{lr\downarrow} X$. To demonstrate that $X^* \leq_{hr\downarrow} X$, we plotted $\bar{F}^*(x+t)\Big/\bar{F}(x)$ in Figure 5.3 showing that it is decreasing in x.

Scenario II: Here we consider the data set Fatigue-life failures (Hand et al. [58]) on the fatigue-life failures of ball-bearings. The data give the number of cycles to failure. From the quantile-quantile (Q-Q) plot (Figure 5.4) and the results of Anderson-Darling test (Table 5.3) for the observed samples, it is observed that the samples can taken to be from Weibull distribution. Estimated values of parameters of baseline Weibull with the cdf $G(x) = 1 - e^{(-x/\beta)^k}$, $x \geq 0, \beta > 0, k > 0$ are given in Table 5.4.

Table 5.3: **Results of Anderson-Darling test**

AD-value	p-value	Critical value(cv)
0.1496	0.99	2.503

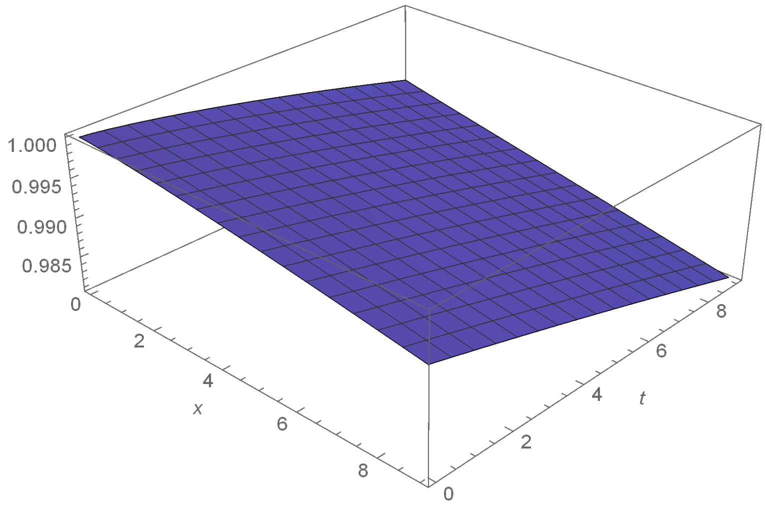

Figure 5.2: Plot of $f^*(x+t)\Big/f(x)$

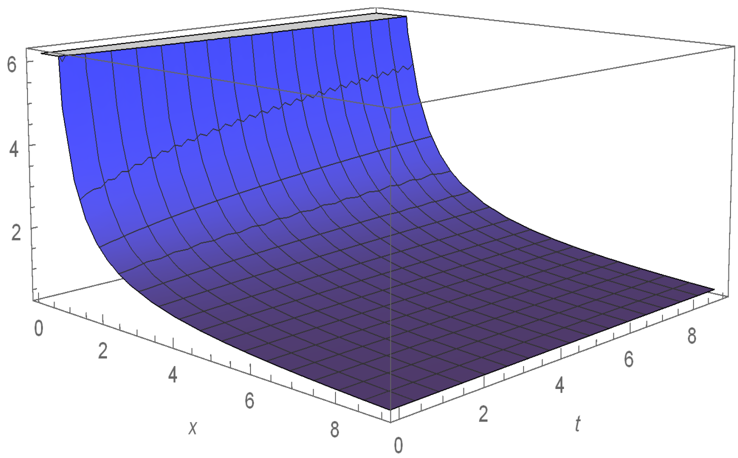

Figure 5.3: Plot of $\bar{F}^*(x+t)\Big/\bar{F}(x)$

Table 5.4: **Estimated parameters of Weibull distribution**

Parameters	Estimated value	95% confidence interval
Scale	232.9	$[198.758, 272.906]$
Shape	3.0721	$[2.13732, 4.41572]$

With shape parameter $k > 1$, this baseline Weibull distribution is ILR and also is DRFR. Next we consider Gamma resilience i.e. $\Omega \sim \Gamma(1/a^2, 1/a^2)$ where $\Omega \geq 1$ with probability 1. According to Theorem 5.3.1(i), the effect of considered gamma resilience on X is that,

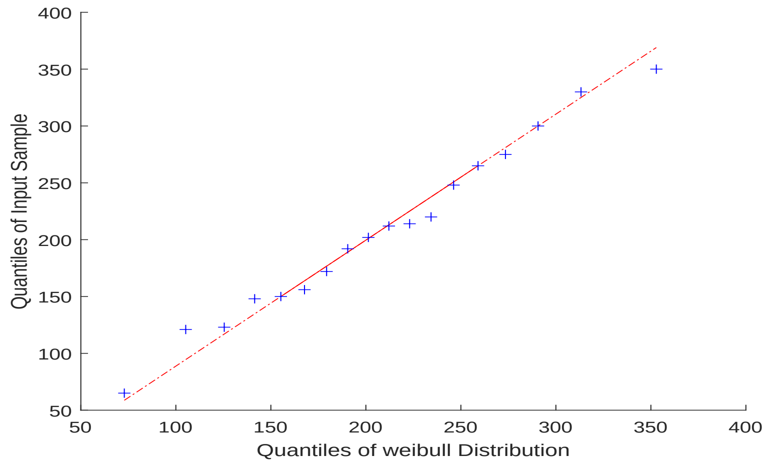

Figure 5.4: QQ plot of sample data vs Weibull distribution

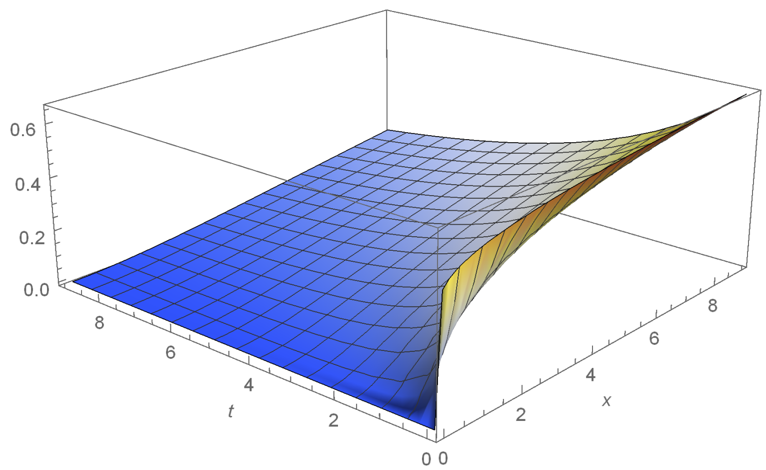

Figure 5.5: Plot of $g^*(x)\big/g(x+t)$

$X^* \leq_{lr\downarrow} X$. Similarly, according to Theorem 5.3.3(i), $X^* \leq_{rh\uparrow} X$, which indicates that $\tilde{r}^*_X(t) \leq \tilde{r}_X(t')$ for $t \geq t' \geq 0$.

To demonstrate the above mentioned stochastic orders, we proceed as follows. The cdf and pdf of the above resilience model (Gamma-resilience Weibull-baseline) are

$$G^*(t) = \frac{(1/a^2)^{-1/a^2}(a^2 - \ln(1-e^{(\frac{t}{\beta})^k}))^{-1/a^2}\zeta_1\left(\frac{1}{a^2}, a^2 - \ln(1-e^{(\frac{t}{\beta})^k})\right)}{\Gamma(\frac{1}{a^2})\left(1-\zeta_2(\frac{1}{a^2},0,a^2)\right)}$$

$$\text{and } g^*(t) = \frac{\frac{kt^{k-1}}{\beta^k}(1/a^2)^{-1/a^2}(a^2 - \ln(1-e^{(\frac{t}{\beta})^k}))^{-1-1/a^2}\zeta_1\left(\frac{1}{a^2}, a^2 - \ln(1-e^{(\frac{t}{\beta})^k})\right)}{\Gamma(\frac{1}{a^2})\left(1-\zeta_2(\frac{1}{a^2}, 0, a^2)\right)},$$

respectively. Let $t_1, t_2, ..., t_n$ be the observations under consideration. We now obtain maximum likelihood estimate of the parameter a under the Gamma-resilience Weibull-baseline. The likelihood function is given by

$$\mathcal{L}(a|t_1, t_2, .., t_n) = \left(\frac{(\frac{1}{a^2})^{-1/a^2}}{\Gamma(\frac{1}{a^2})\left(1-\zeta_2(\frac{1}{a^2}, 0, a^2)\right)}\right)^n \prod_{i=1}^{n}(t_i)^{k-1}\prod_{i=1}^{n}(a^2 - \ln(1-e^{(\frac{t_i}{\beta})^k}))^{-1-1/a^2}$$
$$\times \prod_{i=1}^{n}\zeta_1\left(\frac{1}{a^2}, a^2 - \ln(1-e^{(\frac{t_i}{\beta})^k})\right),$$

where $\zeta_1(a, x)$ and $\zeta_2(a, x)$ are defined in previous case. Estimated value of the parameter a is obtained as 4.0558 with $\mathbb{P}(\Omega \geq 1) = 1$.

Then we plotted $g^*(x)\Big/g(x+t)$ taking some finite range of x and t as shown in Figure 5.5, which is clearly showing that the ratio is increasing in x, giving $X^* \leq_{lr\uparrow} X$. To demonstrate that $X^* \leq_{rh\uparrow} X$, we plotted $G^*(x)\Big/G(x+t)$ in Figure 5.6 showing that it is increasing in x.

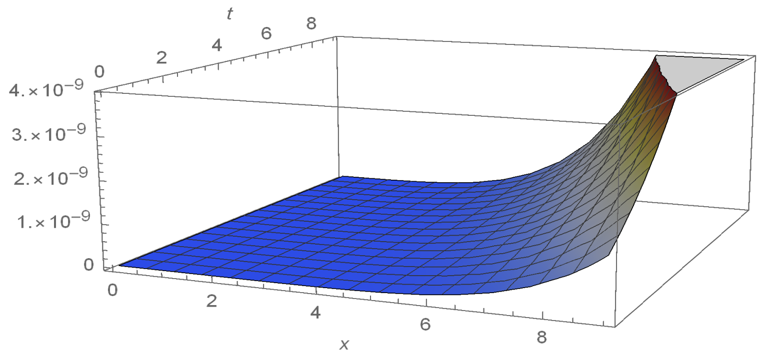

Figure 5.6: Plot of $G^*(x)\Big/G(x+t)$

Chapter 6

Stochastic comparisons with active redundancy allocation

Incorporating redundancies (standby components/spares) into a system is an effective way to enhance system reliability. Among the various types of redundancies, e.g. hot, cold and warm standby, the commonly used redundancy is active redundancy. In active redundancy, standby components are allocated with the original components in parallel, and start functioning along with the original components of the system. In practice, while a system is running, when the replacement of a failed component is not possible or replacement is time-consuming and will result in a huge cost, in such cases active redundancy allocation is economical. In general, the matching and non-matching aspects of active redundancy allocations are considered. In matching allocation standby components are identically distributed as of the original components, whereas in non-matching, standby and original components are non-identically distributed. Several researchers studied active redundancy allocation focusing on how to allocate the standby components into the system at component or system levels so that the reliability of the system is improved in some stochastic sense (see Brito et al. [26], Da and Ding [36], Hazra and Nanda [63], Misra et al. [99], Zhao et al. [141] and references therein).

It is also to be noted that in all of the aforementioned works, components are assumed to be statistically independent. However, in many practical scenarios, components of a system may not be independent but are dependent because of various factors like different environmental factors (stress, load, voltage, etc.) and system design (Ghoraf [50], Gupta and Gupta [54], Yang et al. [133]). Hence it is of natural interest to study active redundancy allocations policy when system components are statistically dependent. Gupta and Gupta [54] first investigated component and system level active redundancy allocations policy with respect to some stochastic orders for matching spares when system components are statistically d.i.d. Later on, [135] further investigated in that direction and provided more conditions for comparing component and system redundancies (considering non-matching spares). Hazra and Misra [62] considered the comparison of coherent systems with active redundancy at the component level versus at the system level for d.i.d. components and matching spares. However, for statistically dependent components and non-matching spares, not many works are available may be due to the complexity of structure function.

On allocating active redundancies, it is of interest to know which sets of redundant components will provide more improved reliability of the system. Here we consider the case when the original components of a coherent system are d.i.d., and redundant components are non-identical (non-matching spares). Also, we discuss active redundancy at the component level as well as redundancy at the system level.

Our aim to investigate the optimal selection of redundant components in coherent systems based on the underlying distribution of component lifetime. To the best of our knowledge except Kelkinnama [71] there is no considerable work done in this direction. Kelkinnama [71] considered that the lifetime distributions of the original and redundant components follow the PHR or PRH model. In this chapter we provide sufficient conditions to optimal selection of redundant components in a coherent system based on the underlying distribution of the components lifetime. We consider that component lifetime follow two important semi-parametric models namely AL and PO models.

In Section 6.1, we derive stochastic comparison results for coherent systems with redundancy at the component level when components lifetimes follows AL model. In Section **??**, we derive the same for redundancy at the system level when components lifetimes follows AL model. In Section 6.3, we derive stochastic comparison results for coherent systems with redundancy at the component level when components lifetimes follows PO model. In Section 6.4, we derive the same for redundancy at the system level when components lifetimes follows PO model. In section, 6.5 we demonstrate some of the derived results with real data.

6.1 Component redundancy: AL model:

Here we present stochastic comparison results for coherent systems of dependent and identically distributed components with redundancy at the component level, where original and redundant components follow AL model. Suppose that each of the n original identically distributed components of a coherent system is connected with m redundant components in parallel, where the original and m redundant components follow AL model with a baseline distribution function F, and corresponding scale parameters $a_0, a_1, \cdots, a_m$ respectively. Let

S_c denotes the lifetime for this coherent system with active redundancy at the component level. The reliability function can be written as

$$\bar{F}_{S_c}(t) = h_\theta \left(1 - \prod_{j=0}^{m} (F(a_j t))\right), \tag{6.1.1}$$

where, $h_\theta : [0,1] \to [0,1]$ is the distorted function. It may be noted that as along with the structure function of the system, h also depends on the dependence structure (associated copula) of the components, here θ represents the dependence parameter of the associated copula. For convenient we denote this system as $(F, \boldsymbol{a}, h_\theta)$, where $\boldsymbol{a} = \{a_0, a_1, \cdots, a_m\}$. We also denote the survival, hazard rate and reversed hazard rate function of the baseline distribution as $\bar{F}(\cdot)$, $r(\cdot)$ and $\tilde{r}(\cdot)$, respectively.

Lemma 6.1.1. *The function* $\log(F(at))$ *is increasing and concave in* a *if* $\tilde{r}(x)$ *is decreasing in* x.

Proof. Let $\mathcal{D}(a) = \log(F(at))$. Then we have

$$\mathcal{D}'(a) = t\tilde{r}(at) \tag{6.1.2}$$

$$\mathcal{D}''(a) = t^2\tilde{r}'(at) \tag{6.1.3}$$

From (6.1.2) and (6.1.3) we can easily conclude that $\log(F(at))$ is increasing and concave in a if $\tilde{r}(x)$ is decreasing in x. □

We now derive some results that could help us design more reliable systems by allocating appropriate redundant components from the available spare components. Theorem 6.1.1 provides us the sufficient conditions under which survival (reliability) function of a coherent system of dependent and identically distributed components with a set of redundant components is larger than that of the same system with another set of redundant components.

Theorem 6.1.1. *Let the coherent system with component level redundancy following the AL model* $(F, \boldsymbol{a}, h_{\theta_1})$*, has the lifetime* $\mathcal{S}_c$ *and following the AL model* $(F, \boldsymbol{a}^*, h_{\theta_2})$ *has the lifetime* $\mathcal{S}_c^*$*. If* $h_\theta(u)$ *is increasing (decreasing) in* θ *and* $\tilde{r}(x)$ *is decreasing in* x*, then for* $\theta_1 \leq (\geq)\ \theta_2$*,*

$$\boldsymbol{a} \overset{w}{\preceq} \boldsymbol{a}^* \implies \mathcal{S}_c \leq_{st} (\geq_{st})\ \mathcal{S}_c^*.$$

Proof. Since $\tilde{r}(x)$ is decreasing, using Lemma 1.2.7(ii) and Lemma 6.1.1 we can conclude that if $\boldsymbol{a} \overset{w}{\preceq} \boldsymbol{a^*}$ then

$$(\ln(F(a_0 t)), \ln(F(a_1 t)), ..., \ln(F(a_m t))) \preceq^w (\ln(F(a_0^* t)), \ln(F(a_1^* t)), ..., \ln(F(a_m^* t)))$$

which implies

$$((F(a_0 t)), (F(a_1 t)), ..., (F(a_m t))) \preceq^p ((F(a_0^* t)), (F(a_1^* t)), ..., (F(a_m^* t)))$$

which implies

$$1 - \prod_{j=0}^{m} F(a_j t) \leq 1 - \prod_{j=0}^{m} F(a_j^* t) \tag{6.1.4}$$

Since $h_\theta(x)$ is an increasing in x, we have for any $\theta > 0$

$$h_\theta \left(1 - \prod_{j=0}^{m} F(a_j t)\right) \leq h_\theta \left(1 - \prod_{j=0}^{m} F(a_j^* t)\right). \tag{6.1.5}$$

Now for $\theta_1 \leq \theta_2$, as $h_\theta(x)$ is an increasing in θ, we have from (6.1.5)

$$h_{\theta_1} \left(1 - \prod_{j=0}^{m} F(a_j t)\right) \leq h_{\theta_2} \left(1 - \prod_{j=0}^{m} F(a_j^* t)\right). \tag{6.1.6}$$

Reverse equality follows in similar way when $h_\theta(x)$ is decreasing in θ and $\theta_1 \geq \theta_2$. Hence the desired result holds. □

The following example provides some copulas and coherent structures for which the distortion functions satisfy the conditions of Theorem 6.1.1.

Example 6.1.1. *It is to be noted that the distributions having $\tilde{r}(x)$ decreasing in x belong to the decreasing reversed hazard rate (DRHR) class. Some well-known DRHR distributions under suitable parameter restrictions are given in Table 6.1. In Table 6.2, we present some well-known copulas and coherent structures for which the condition (ii) of Theorem 6.1.2 is satisfied.*

Theorem 6.1.2 provides us the sufficient conditions under which hazard rate (failure rate) function of a coherent system with redundancy at the component level is smaller than that of another such system.

Table 6.1: Some distributions having the property that $\tilde{r}(x)$ is decreasing in x

Distribution	CDF	Parameter restriction
Weibull	$1-\exp(-(\alpha x)^{\beta}), x>0, \alpha>0, \beta>0$	$\alpha>0, 0<\beta<1.$
Generalised Gamma	$\frac{\gamma(d/p,(x/\alpha)^p)}{\Gamma(d/p)},\ x>0, \alpha, d, p>0$	$\alpha, d, p>0, d\neq p.$
Generalised Pareto	$1-\left(1-\frac{kx}{\sigma}\right)^{1/k},\ 0<x\leq \sigma/k, \sigma, k>0,$	$0<k<1, \sigma>0.$

Table 6.2: Some copulas and coherent structures for which $h_\theta(u)$ is increasing in θ

Copula	Coherent system	$h_\theta(u)$
(i), (ii)	$X_{2:3}$ (2-out-of-3)	Increasing in $\theta\geq 1$
(iv)	$X_{2:4}$(3-out-of-4)	Increasing in $0<\theta\leq 1$

Theorem 6.1.2. *Let the coherent system with component level redundancy following the AL model $(F, \boldsymbol{a}, h_{\theta_1})$, has the lifetime $\mathcal{S}_c$ and following AL model $(F, \boldsymbol{a}^{*}, h_{\theta_2})$ has the lifetime $\mathcal{S}_c^*$. If*

(i) $x\tilde{r}(x)$ is decreasing and concave in x.
(ii) $\frac{(1-u)h'_\theta(u)}{h_\theta(u)}$ is decreasing in u and is increasing (decreasing) in θ,

then for $\theta_1 \leq (\geq)\ \theta_2$,

$$\boldsymbol{a} \overset{m}{\preceq} \boldsymbol{a}^{*} \implies \mathcal{S}_c \geq_{hr} \mathcal{S}_c^*.$$

Proof. Let us define

$$\xi(t) = \frac{h_{\theta_2}\left(1-\prod_{j=0}^{m}(F(a_j^*t)\right)}{h_{\theta_1}\left(1-\prod_{j=0}^{m}(F(a_jt)\right)}$$

Now differentiating $\xi(t)$ w.r.t. t we have

$$\begin{aligned}\xi'(t) \overset{sgn}{=} & \frac{h'_{\theta_1}\left(1-\prod_{j=0}^{m}F(a_jt)\right)}{h_{\theta_1}\left(1-\prod_{j=0}^{m}F(a_jt)\right)}\prod_{j=0}^{m}(F(a_jt))\sum_{j=0}^{m}a_j\tilde{r}(a_jt) \\ & - \frac{h'_{\theta_2}\left(1-\prod_{j=0}^{m}F(a_j^*t)\right)}{h_{\theta_2}\left(1-\prod_{j=0}^{m}F(a_j^*t)\right)}\prod_{j=0}^{m}(F(a_j^*t))\sum_{j=0}^{m}a_j^*\tilde{r}(a_j^*t)\end{aligned} \tag{6.1.7}$$

Since $x\tilde{r}(x)$ is decreasing and concave, we have by using Lemma 1.2.7(iv), if $a \preceq_w a^*$ then

$$\sum_{j=0}^{m}a_j\tilde{r}(a_jt) \geq \sum_{j=0}^{m}a_j^*\tilde{r}(a_j^*t) \tag{6.1.8}$$

Again, since $\frac{(1-u)h'_\theta(u)}{h_\theta(u)}$ is decreasing in u and $\tilde{r}(x)$ is decreasing in x (as $x\tilde{r}(x)$ is decreasing in x), applying (7.2.2) we have if $a \overset{w}{\preceq} a^*$ then

$$\frac{h'_{\theta_1}\left(1-\prod_{j=0}^m F(a_j t)\right)}{h_{\theta_1}\left(1-\prod_{j=0}^m F(a_j t)\right)}\prod_{j=0}^m (F(a_j t)) \geq \frac{h'_{\theta_1}\left(1-\prod_{j=0}^m F(a_j^* t)\right)}{h_{\theta_1}\left(1-\prod_{j=0}^m F(a_j^* t)\right)}\prod_{j=0}^m (F(a_j^* t)), \tag{6.1.9}$$

for all $t > 0$. Now if $\frac{(1-u)h'_\theta(u)}{h_\theta(u)}$ is increasing (decreasing) in θ, then for $\theta_1 \leq \theta_2$ ($\theta_1 \geq \theta_2$), using (6.1.9) and (6.1.8), from (6.1.7) it follows that $\boldsymbol{a} \preceq^m \boldsymbol{a}^*$ implies $\xi(t)$ is decreasing in t, so that $\mathcal{S}_c \geq_{hr} \mathcal{S}_c^*$. □

Example 6.1.2. *It is to be noted that $x\tilde{r}(x)$ belongs to the proportional reversed hazard rate class. The condition $x\tilde{r}(x)$ is decreasing and concave, satisfied by some well-known distribution under suitable parameter restrictions as given in Table 6.3. In Table 6.4, we present some well-known copulas and coherent structures for which the condition (ii) of Theorem 6.1.2 is satisfied.*

Table 6.3: Some distributions having the property that $x\tilde{r}(x)$ is decreasing and concave

Distribution	CDF	Parameter restriction
Generalised Gamma	$\frac{\gamma(d/p,(x/\alpha)^p)}{\Gamma(d/p)}$, $x > 0, \alpha, d, p > 0$	$1 < p < d < \alpha$
Generalised Pareto	$1-\left(1-\frac{kx}{\sigma}\right)^{1/k}$, $0 < x \leq \sigma/k, \sigma, k > 0,$	$0 < k < 1, \sigma > 0.$

Table 6.4: Some copulas and coherent structures for which $(1-u)h'_\theta(u)/h_\theta(u)$ is decreasing in u and θ

Copula	Coherent system	$\frac{(1-u)h'_\theta(u)}{h_\theta(u)}$
(ii), (iii)	$X_{2:4}$(3-out-of-4)	Decreasing in u and θ
(ii), (iii)	$\min(X_{2:4}, X_4)$	Decreasing in u and θ

Theorem 6.1.3 provides us the sufficient conditions under which a coherent system with redundancy at the component level is better than that of another such system with respect to reversed hazard rate function.

Theorem 6.1.3. *Let the coherent system with component level redundancy following the AL model $(F, \boldsymbol{a}, h_{\theta_1})$, has the lifetime $\mathcal{S}_c$ and following AL model $(F, \boldsymbol{a}^*, h_{\theta_2})$ has lifetime $\mathcal{S}_c^*$. If* *(i)* $x\tilde{r}(x)$ *is decreasing and convex in* x.

(ii) $\frac{(1-u)h'_\theta(u)}{1-h_\theta(u)}$ *is increasing in* u *and increasing (decreasing) in* θ,

then for $\theta_1 \leq (\geq)\ \theta_2$,

$$\boldsymbol{a} \overset{w}{\preceq} \boldsymbol{a^*} \implies \mathcal{S}_c \leq_{rhr} \mathcal{S}_c^*.$$

Proof. Let us define

$$\chi(t) = \frac{1 - h_{\theta_2}\left(1 - \prod_{j=0}^m (F(a_j^* t)\right)}{1 - h_{\theta_1}\left(1 - \prod_{j=0}^m (F(a_j t)\right)}.$$

Now differentiating $\chi(t)$ w.r.t. t we have

$$\begin{aligned} \chi^{'}(t) &= \frac{h'_{\theta_2}\left(1 - \prod_{j=0}^m (F(a_j^* t)\right)}{1 - h_{\theta_2}\left(1 - \prod_{j=0}^m (F(a_j^* t)\right)} \prod_{j=0}^m (F(a_j^* t) \sum_{j=0}^m a_j^* \tilde{r}(a_j^* t) \\ &- \frac{h'_{\theta_1}\left(1 - \prod_{j=0}^m (F(a_j t)\right)}{1 - h_{\theta_1}\left(1 - \prod_{j=0}^m (F(a_j t)\right)} \prod_{j=0}^m (F(a_j t) \sum_{j=0}^m a_j \tilde{r}(a_j t) \end{aligned} \tag{6.1.10}$$

Since $x\tilde{r}(x)$ is decreasing & convex we have by using Lemma 1.2.7(iii) if $\boldsymbol{a} \preceq^w \boldsymbol{a^*}$ then

$$\sum_{j=0}^m a_j^* \tilde{r}(a_j^* t) \geq \sum_{j=0}^m a_j \tilde{r}(a_j t) \tag{6.1.11}$$

Again, since $\frac{(1-u)h'_\theta(u)}{1-h_\theta(u)}$ is increasing in u and $\tilde{r}(x)$ is decreasing in x, applying (7.2.2) we have if $\boldsymbol{a} \overset{w}{\preceq} \boldsymbol{a^*}$

$$\frac{h'_{\theta_2}\left(1 - \prod_{j=0}^m (F(a_j^* t)\right)}{1 - h_{\theta_2}\left(1 - \prod_{j=0}^m (F(a_j^* t)\right)} \prod_{j=0}^m (F(a_j^* t) \geq \frac{h'_{\theta_2}\left(1 - \prod_{j=0}^m (F(a_j t)\right)}{1 - h_{\theta_2}\left(1 - \prod_{j=0}^m (F(a_j t)\right)} \prod_{j=0}^m (F(a_j t), \tag{6.1.12}$$

for all $t > 0$. Now if $\frac{(1-u)h'_\theta(u)}{1-h_\theta(u)}$ is increasing (decreasing) in θ, then for $\theta_1 \leq \theta_2$ ($\theta_1 \geq \theta_2$), using (6.4.18) and (6.4.19), from (6.4.17) we have $\boldsymbol{a} \preceq^w \boldsymbol{a^*}$ implies $\chi(t)$ is increasing in t, so that $\mathcal{S}_c \leq_{rhr} \mathcal{S}_c^*$. □

Example 6.1.3. *The condition* $x\tilde{r}(x)$ *is decreasing and convex, satisfied by some well-known distribution under suitable parameter restrictions as given in Table 6.5. In Table 6.6, we present some well-known copulas and coherent structures for which the condition* *(ii) of Theorem 6.1.3 is satisfied.*

Table 6.5: Some distributions having the property that $x\tilde{r}(x)$ is decreasing and convex

Distribution	CDF	Parameter restriction
Weibull	$1-\exp\left(-(\alpha x)^{\beta}\right),\ x>0,\alpha>0,\beta>0.$	$\alpha>0, 0<\beta<1.$
Generalised Gamma	$\frac{\gamma(d/p,(x/\alpha)^p)}{\Gamma(d/p)},\ x>0,\alpha,d,p>0$	$0<p<1; d,\alpha>0;$
Generalised Pareto	$1-\left(1-\frac{kx}{\sigma}\right)^{1/k},\ 0<x\leq\sigma/k,\sigma,k>0,$	$k>1,\sigma>0.$

Table 6.6: Some copulas and coherent structures for which $(1-u)h'_{\theta}(u)/(1-h_{\theta}(u))$ is increasing in u and increasing/decreasing in θ

Copula	Coherent system	$\frac{(1-u)h'_{\theta}(u)}{1-h_{\theta}(u)}$
(ii)	$X_{2:3}$(3-out-of-4)	Increasing in u and increasing in $\theta\geq 1$
(i)	$X_{3:4}$(3-out-of-4)	Increasing in u and decreasing in $\theta>0$
(i)	$\min(X_1,\max(X_2,X_4))$	Increasing in u and decreasing in $\theta>0$

6.2 Systems redundancy:AL model

Here we present the stochastic comparison results for coherent systems of dependent and identically distributed components with redundancy at the system level, where original and redundant components follow AL model. Suppose that the original system is connected with m same structured coherent systems (redundant systems) in parallel, where all the components of the original and m redundant systems follow AL model with a baseline distribution function F, and corresponding scale parameters $a_0, a_1, \cdots, a_m$ respectively. Let S_s denotes the lifetime for this coherent system with active redundancy at the system level. The reliability function can be written as

$$\bar{F}_{S_s}(t)=1-\prod_{j=0}^{m}\left(1-h_{\theta_j}(1-F(a_j t))\right), \tag{6.2.1}$$

where h_{θ_0} and h_{θ_j}, $j=1,\cdots,m$ are the distorted functions of the original and jth redundant systems, respectively. Here we consider different parameters θ_j for the original and each redundant system, as, the distorted function also depends on the dependence structure (associated copula) of the components, so for different distributions, the dependence parameter of the copulas may be different. For convenient we denote this system as $(F,\boldsymbol{a},\boldsymbol{h_\theta})$, where $\boldsymbol{a}=\{a_0,a_1,\cdots,a_m\}$ and $\boldsymbol{h_\theta}=\{h_{\theta_0},h_{\theta_1},\cdots,h_{\theta_m}\}$.

The derived results in this section could help us to design more reliable systems by assigning appropriate system level redundancy from the available options.

Theorem 6.2.1 provides sufficient conditions under which survival function of a coherent system of dependent and identically distributed components with a set of redundant systems is larger than that of the same system with another set of redundant systems. This theorem also applies if scale parameter of the original component of a system is different from that of the other system.

Theorem 6.2.1. *Let the coherent system with system level redundancy following the AL model* $(F, \boldsymbol{a}, \boldsymbol{h_\theta})$*, has the lifetime* $\mathcal{S}_s$ *and following AL model* $(F, \boldsymbol{a^*}, \boldsymbol{h_{\theta^*}})$ *has the lifetime* $\mathcal{S}_s^*$ *with*

(i) $r(u)$ *is decreasing in* u*,*

(ii) $h'_\theta(u)$ *is increasing in* u *and decreasing in* θ*;* $h_\theta(u)$ *is decreasing in* θ*,*

(iii) $\frac{\partial h_\theta(u)}{\partial \theta}$ *is decreasing in* u *and increasing in* θ*.*

Then for $\boldsymbol{a}, \boldsymbol{a^*}, \boldsymbol{\theta}, \boldsymbol{\theta^*} \in \mathcal{D}_n^+$*,*

$$\boldsymbol{a} \overset{m}{\preceq} \boldsymbol{a^*} \text{ and } \boldsymbol{\theta} \overset{m}{\preceq} \boldsymbol{\theta^*} \implies \mathcal{S}_s \leq_{st} \mathcal{S}_s^*.$$

Proof. Let us write

$$F_{\boldsymbol{a},\boldsymbol{\theta}}(t) = \log(F_{\mathcal{S}_s}(t)) = \sum_{i=1}^{m} \log(1 - h_{\theta_i}(1 - F(a_i t))) \tag{6.2.2}$$

Then we have

$$\frac{\partial F_{\boldsymbol{a},\boldsymbol{\theta}}(t)}{\partial a_j} = r(a_j t)\frac{((1 - F(a_j t))h'_{\theta_j}(1 - F(a_j t))}{1 - h_{\theta_j}(1 - F(a_j t))} \tag{6.2.3}$$

Now

$$\begin{aligned} & \frac{\partial F_{\boldsymbol{a},\boldsymbol{\theta}}(t)}{\partial a_i} - \frac{\partial F_{\boldsymbol{a},\boldsymbol{\theta}}(t)}{\partial a_j} \\ = \quad & r(a_i t)\frac{((1 - F(a_i t))h'_{\theta_i}(1 - F(a_i t))}{1 - h_{\theta_i}(1 - F(a_i t))} - r(a_j t)\frac{((1 - F(a_j t))h'_{\theta_j}(1 - F(a_j t))}{1 - h_{\theta_j}(1 - F(a_j t))} \end{aligned}$$

From condition (i), we have

$$a_i \geq a_j \implies r(a_i t) \leq r(a_j t) \tag{6.2.4}$$

Applying (ii), we have

$$a_i \geq a_j \implies \frac{((1-F(a_it))h'_{\theta_i}(1-F(a_it))}{1-h_{\theta_i}(1-F(a_it))} \leq \frac{((1-F(a_jt))h'_{\theta_i}(1-F(a_jt))}{1-h_{\theta_i}(1-F(a_jt))} \tag{6.2.5}$$

and

$$\theta_i \geq \theta_j \implies \frac{((1-F(a_jt))h'_{\theta_i}(1-F(a_jt))}{1-h_{\theta_i}(1-F(a_jt))} \leq \frac{((1-F(a_jt))h'_{\theta_j}(1-F(a_jt))}{1-h_{\theta_j}(1-F(a_jt))} \tag{6.2.6}$$

Now for $1 \leq i < j \leq m$, using (6.2.4)-(6.2.6) we have if $a_i \geq a_j$ and $\theta_i \geq \theta_j$,

$$\frac{\partial F_{\boldsymbol{a},\boldsymbol{\theta}}(t)}{\partial a_i} - \frac{\partial F_{\boldsymbol{a},\boldsymbol{\theta}}(t)}{\partial a_j} \leq 0.$$

So we can conclude that $\frac{\partial F_{\boldsymbol{a},\boldsymbol{\theta}}(t)}{\partial a_k}$ is increasing in k, $k = 1, 2.., m$ for $\boldsymbol{a}, \boldsymbol{\theta} \in \mathcal{D}_n^+$. Hence from Lemma 2 of [?] we have for $\boldsymbol{a}, \boldsymbol{a^*}, \boldsymbol{\theta}, \boldsymbol{\theta^*} \in \mathcal{D}_n^+$,

$$\boldsymbol{a} \overset{m}{\preceq} \boldsymbol{a^*} \implies F_{\boldsymbol{a},\boldsymbol{\theta}}(t) \geq F_{\boldsymbol{a^*},\boldsymbol{\theta}}(t) \tag{6.2.7}$$

Again,

$$\frac{\partial F_{\boldsymbol{a},\boldsymbol{\theta}}(t)}{\partial \theta_j} = \frac{-\frac{\partial h_{\theta_j}(1-F(a_jt))}{\partial \theta_j}}{1-h_{\theta_j}(1-F(a_jt))} \tag{6.2.8}$$

$$\frac{\partial F_{\boldsymbol{a},\boldsymbol{\theta}}(t)}{\partial \theta_i} - \frac{\partial F_{\boldsymbol{a},\boldsymbol{\theta}}(t)}{\partial \theta_j} = \frac{-\frac{\partial h_{\theta_i}(1-F(a_it))}{\partial \theta_i}}{1-h_{\theta_i}(1-F(a_it))} - \frac{-\frac{\partial h_{\theta_j}(1-F(a_jt))}{\partial \theta_j}}{1-h_{\theta_j}(1-F(a_jt))}$$

From condition (iii), along with the condition $h_\theta(u)$ is decreasing in θ we have

$$\theta_i \geq \theta_j \implies \frac{-\frac{\partial h_{\theta_i}(1-F(a_it))}{\partial \theta_i}}{1-h_{\theta_i}(1-F(a_it))} \leq \frac{-\frac{\partial h_{\theta_j}(1-F(a_it))}{\partial \theta_j}}{1-h_{\theta_j}(1-F(a_it))} \tag{6.2.9}$$

and

$$a_i \geq a_j \implies \frac{-\frac{\partial h_{\theta_i}(1-F(a_it))}{\partial \theta_j}}{1-h_{\theta_j}(1-F(a_it))} \leq \frac{-\frac{\partial h_{\theta_j}(1-F(a_jt))}{\partial \theta_j}}{1-h_{\theta_j}(1-F(a_jt))} \tag{6.2.10}$$

Now for $1 \leq i < j \leq m$, we have if $a_i \geq a_j$ and $\theta_i \geq \theta_j$,

$$\frac{\partial F_{\boldsymbol{a},\boldsymbol{\theta}}(t)}{\partial \theta_i} - \frac{\partial F_{\boldsymbol{a},\boldsymbol{\theta}}(t)}{\partial \theta_j} \leq 0.$$

Hence from Lemma 2 of [**?**] we have for $\boldsymbol{a}, \boldsymbol{a}^*, \boldsymbol{\theta}, \boldsymbol{\theta}^* \in \mathcal{D}_n^+$,

$$\boldsymbol{\theta} \overset{m}{\preceq} \boldsymbol{\theta}^* \implies F_{\boldsymbol{a},\boldsymbol{\theta}}(t) \geq F_{\boldsymbol{a},\boldsymbol{\theta}^*}(t) \tag{6.2.11}$$

From (6.2.7) and (6.2.11), we have the desired result. □

Example 6.2.1. *It is to be noted that distributions having $r(x)$ decreasing in x belong to the decreasing failure (hazard) rate (DFR) class. Some well-known DFR distributions under suitable parameter restrictions are given in Table 6.7. In Table 6.8, we present some well-known copulas and coherent structures for which the condition (i)-(iv) of Theorem 6.2.1 is satisfied.*

Table 6.7: Some distributions having the property that $r(x)$ is decreasing in x

Distribution	CDF	Parameter restriction
Weibull	$1 - \exp(-(\alpha x)^{\beta}), x > 0, \alpha > 0, \beta > 0$	$\alpha > 0, 0 < \beta < 1.$
Burr	$1 - (1 + x^c)^{-\alpha},\ x \geq 0, \alpha, c > 0,$	$\alpha > 0, 0 < c < 1.$
Generalised Gamma	$\frac{\gamma(d/p,(x/\alpha)^p)}{\Gamma(d/p)},\ x > 0, \alpha, d, p > 0$	$\alpha > 0, 0 < d, p < 1.$

Table 6.8: A copula and a coherent structure for which $h_\theta(u)$ is decreasing in θ, and $h'_\theta(u)$ is increasing in u and decreasing in θ

Copula	Coherent system	$h_\theta(u)$	$h'_\theta(u)$
(i)	$\max(X_1, \min(X_2, X_3, X_4))$	Decreasing in θ	Increasing in u and decreasing in θ

Next, we provide an example to show that under sufficient conditions in Theorem 6.2.1, the hazard rate ordering would not hold in general.

Example 6.2.2. *Consider $\bar{F}(x) = \exp(-(\alpha x)^k)$ where $\alpha = 3, k = 0.8$. It is easy to check that $r(x)$ is decreasing in x. Next, we consider the following one-parameter copulas*

$$C_3(u_1, u_2, u_3, u_4, \theta) = (u_1^{-\theta} + u_2^{-\theta} + u_3^{-\theta} + u_4^{-\theta} - 3)^{-1/\theta}, \quad \theta \in [-1, \infty) \setminus \{0\}.$$

Now consider the following coherent structure with distortion functions underlying the above *one parameter copula* $C_3(u_1, u_2, u_3, u_4, \theta)$

$$\phi_1(X) = \max(X_1, \min(X_2, X_3, X_4))$$

and

$$h_\theta(u) = C_3[u, 1, 1, 1, \theta] + C_3[u, u, u, 1, \theta] - C_3[u, u, u, u, \theta]$$

Here $h_\theta(u)$ *satisfies the condition (ii) and (iii) of Theorem 6.2.1.*
Now consider $\boldsymbol{a} = \{a_0, a_1, a_2, a_3\} = \{1.8, 1.2, 0.8, 0.2\}$, $\boldsymbol{\theta} = \{\theta_0, \theta_1, \theta_2, \theta_3\} = \{10, 7, 6, 4\}$, $\boldsymbol{a}^* = \{1.8, 1.5, 0.5, 0.2\}$, $\boldsymbol{\theta}^* = \{10, 9, 7, 1\}$. *Here* $\boldsymbol{a} \overset{m}{\preceq} \boldsymbol{a}^*$ *and* $\boldsymbol{\theta} \overset{m}{\preceq} \boldsymbol{\theta}^*$. *In Figure 6.1(a), it is shown that* $\mathcal{S}_s \leq_{st} \mathcal{S}_s^*$; *however, from Figure 6.1(b) it is evident that hazard rate order is not satisfied.*

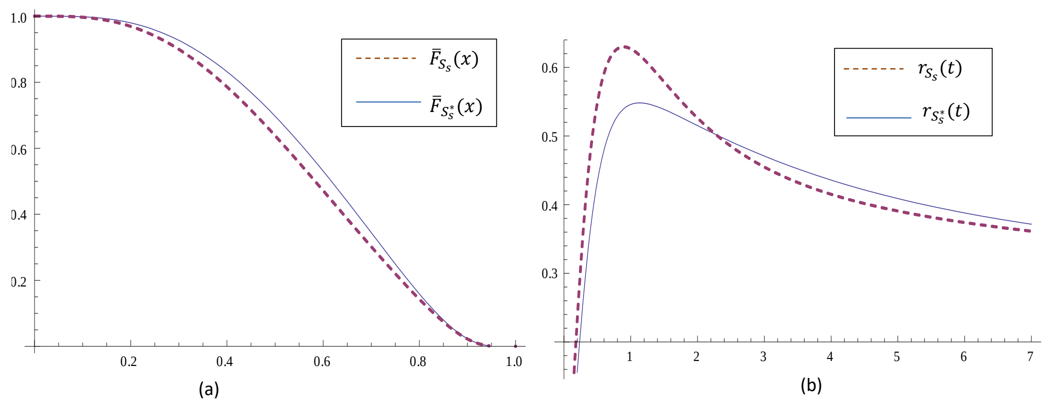

Figure 6.1: Plots of (a) $\bar{F}_{\mathcal{S}_s}(x)$, $\bar{F}_{\mathcal{S}_s^*}(x)$, $t = x/(1-x)$ and (b) $r_{\mathcal{S}_s}(t)$, $r_{\mathcal{S}_s^*}(t)$

Theorem 6.2.2 compares two coherent systems of dependent and identically distributed components with different sets of redundant systems with respect to the reversed hazard rate order. Here we consider that the systems have the same dependent structures (i.e. $\theta_j = \theta, \forall j.$).

Theorem 6.2.2. *Let the coherent system with system level redundancy following the AL model* $(F, \boldsymbol{a}, h_\theta)$, *has the lifetime* $\mathcal{S}_{s,\boldsymbol{a}}$ *and following AL model* $(F, \boldsymbol{a}^*, h_\theta)$ *has the lifetime* $\mathcal{S}^*_{s,\boldsymbol{a}^*}$. *Suppose the following conditions hold.*

(i) $ur(u)$ *is decreasing and convex in* u.

(ii) $\frac{uh_\theta'(u)}{1-h_\theta(u)}$ *is increasing and convex in* u.

Then

$$\boldsymbol{a} \preceq^{w} \boldsymbol{a^*} \implies \mathcal{S}_{s,\boldsymbol{a}} \leq_{rhr} S^*_{s,\boldsymbol{a^*}}.$$

Proof. We have

$$\tilde{r}_{\boldsymbol{S},\boldsymbol{a}}(t) = \sum_{j=0}^{m} a_j r(a_j t) \frac{(1-F(a_j t))h'_\theta(1-F(a_j t))}{1-h_\theta(1-F(a_j t))} \tag{6.2.12}$$

From the conditions (i) and (ii), we can conclude that $\frac{\partial r_{\boldsymbol{S},\boldsymbol{a}}(t)}{\partial a_j}$ is non-positive and increasing in a_j. Now for for $1 \leq i < j \leq n$, we have

$$(a_i - a_j)\left(\frac{\partial \tilde{r}_{\boldsymbol{S},\boldsymbol{a}}(t)}{\partial a_i} - \frac{\partial \tilde{r}_{\boldsymbol{S},\boldsymbol{a}}(t)}{\partial a_j}\right) \geq 0. \tag{6.2.13}$$

So from Theorem A.4 of [95], $\tilde{r}_{\boldsymbol{S},\boldsymbol{a}}(t)$ is Schur-convex in $\boldsymbol{a}$. Hence $\tilde{r}_{\boldsymbol{S},\boldsymbol{a}}(t)$ is decreasing in a_i, $i = 1, 2, ..., n$ and Schur-convex in $\boldsymbol{a}$. Thus from Theorem A.8 of [95], we have

$$\boldsymbol{a} \overset{w}{\preceq} \boldsymbol{a^*} \implies \tilde{r}_{\boldsymbol{S},\boldsymbol{a}}(t) \leq \tilde{r}_{\boldsymbol{S},\boldsymbol{a^*}}(t) \tag{6.2.14}$$

□

Example 6.2.3. *The condition $u\tilde{r}(u)$ is decreasing and convex, satisfied by some well-known distribution under suitable parameter restrictions as given in Table. In Table 6.9, we present some well-known copulas and coherent structures for which the condition (ii) of Theorem 6.2.2 is satisfied. Consider $F(x) = 1 - x^{-k},\ x \geq 1, k > 0$. Then it is easy to check that $xr(x)$ is decreasing and convex in $x \geq 1$ for all $k > 0$.*

Table 6.9: Some well-known copulas and coherent structures for which $uh'_\theta(u)/(1-h_\theta(u))$ is increasing and convex in u

Copula	Coherent system	$\frac{uh'_\theta(u)}{1-h_\theta(u)}$
1	$X_{2:3}$(2-out-of-3)	Increasing and convex in u, $\forall\theta \geq 0$.
1	$\min(X_1, \max(X_2, X_4))$	Increasing and convex in u, $\forall\theta \geq 0$.
4	$X_{3:4}$(3-out-of-4)	Increasing and convex in u, $\forall\theta \geq 1$.
4	$X_{2:4}$(2-out-of-4)	Increasing and convex in u, $\forall\theta \geq 1$.

6.3 Component redundancy under PO model:

Here we present the stochastic comparison results for coherent systems of d.i.d. components with redundancy (non-matching) at the component level. Suppose that each of the n original identically distributed components of a coherent system is connected with m redundant components in parallel, where the original and redundant components follow the PO model with a baseline survival function $\bar{F}$ with corresponding odds ratio parameters as ρ_0 and $\rho_1, \cdots, \rho_m$, respectively. So the survival functions of the original and m redundant components are given by $\bar{F}_{\rho_j}(t) = \frac{\rho_j \bar{F}(t)}{1-\bar{\rho}_j \bar{F}(t)}$, $j = 0, 1, \cdots, m$. For this coherent system with active redundancy at the component level (let us denote the lifetime by $\mathcal{T}_c$), the reliability function can be written as

$$\bar{F}_{\mathcal{T}_c}(t) = \ell_\theta \left(1 - \prod_{j=0}^{m} (1 - \bar{F}_{\rho_j}(t)) \right) \tag{6.3.1}$$

where, $\ell_\theta : [0, 1] \to [0, 1]$ is the distorted function. Here θ indicates the parameter of the dependence structure (associated copula) of the dependent components. For convenience, we say that this system is following the PO model $(\bar{F}, \boldsymbol{\rho}, \ell_\theta)$, where $\boldsymbol{\rho} = \{\rho_0, \rho_1, \cdots, \rho_m\}$.

In Theorem 6.3.1, we derive sufficient conditions under which the survival function of a coherent system of d.i.d. components following the PO model with a set of redundant components at the component level is larger than the same system with another set of redundant components. In practice, the quality of manufactured components varies due to various factors (e.g., human errors, defective resources, instability of production processes, etc.). Hence the optimal choice of spares will provide improved reliability. Theorem 6.3.1 will be useful in determining the optimal set of redundant components from some available options concerning the reliability function of the system lifetime. Based on the characteristics of component lifetimes (odds ratios/tilt parameters) and system design (distortion), a design engineer can choose a set of redundant components that optimize system reliability.

Theorem 6.3.1. *Let the system with redundancy at the component level following the PO model $(\bar{F}, \boldsymbol{\rho}, \ell_{\theta_1})$, has the lifetime $\mathcal{T}_c$ and under PO model $PO(\bar{F}, \boldsymbol{\rho}^*, \ell_{\theta_2})$ has lifetime $\mathcal{T}_c^*$. If $\ell_\theta(u)$ is increasing (decreasing) in θ, then for $\theta_1 \geq (\leq)\ \theta_2$,*

$$\boldsymbol{\rho} \overset{w}{\preceq} \boldsymbol{\rho}^* \implies \mathcal{T}_c \geq_{st} \mathcal{T}_c^*.$$

Proof: Let us consider the function $g(\rho) = \ln(F_\rho(t)) = \ln\left(\frac{F(t)}{1-\bar{\rho}\bar{F}(t)}\right)$. Then it is easy to check that $g(\rho)$ is decreasing and convex in ρ. Therefore from Lemma 1.2.7(iii) we have

$\boldsymbol{\rho} \overset{w}{\preceq} \boldsymbol{\rho}^*$ implies

$$(g(\rho_0), g(\rho_1), g(\rho_2), ..., g(\rho_m)) \preceq_w (g(\rho_0^*), g(\rho_1^*), g(\rho_2^*), ..., g(\rho_m^*))$$

which implies

$$((F(\rho_0 t)), (F(\rho_1 t)), (F(\rho_2 t)), ..., (F(\rho_n t))) \overset{p}{\preceq} ((F(\rho_0^* t)), (F(\rho_1^* t)), (F(\rho_2^* t)), ..., (F(\rho_n^* t)))$$

which gives

$$\left(1 - \prod_{j=0}^{m}(1 - \bar{F}_{\rho_j}(t))\right) \geq \left(1 - \prod_{j=0}^{m}(1 - \bar{F}_{\rho_j^*}(t))\right) \tag{6.3.2}$$

Since $\ell_\theta(u)$ is an increasing function in u for all $\theta > 0$, using equation (6.3.2) we immediately have

$$\ell_\theta\left(1 - \prod_{j=0}^{m}(1 - \bar{F}_{\rho_j}(t))\right) \geq \ell_\theta\left(1 - \prod_{j=0}^{m}(1 - \bar{F}_{\rho_j^*}(t))\right) \tag{6.3.3}$$

Now if $\ell_\theta(u)$ is increasing (decreasing) θ, then for $\theta_1 \geq (\leq)\theta_2$, from (6.3.3) we can easily conclude that $\mathcal{T}_c \geq_{st} \mathcal{T}_c^*$.

Theorem 6.3.2 compares the hazard (failure) rates of two coherent systems with different sets of active redundancy at the component level. According to the definition of hazard rate, this theorem enables us to compare the failure or hazard rate of a system with two different sets of active redundant components, whether at the start or after a certain time of successful system operation. Another interpretation is that it allows us to compare the failure rate of two used systems. Based on the characteristics of component lifetimes (odds ratios/tilt parameters) and system design, this theorem will be helpful in determining the optimal set of redundant components from some available options concerning the failure rate of the system.

Theorem 6.3.2. *Let the system with redundancy at the component level following the PO model $(\bar{F}, \boldsymbol{\rho}, \ell_{\theta_1})$, has the lifetime $\mathcal{T}_c$ and under PO model $(\bar{F}, \boldsymbol{\rho}^*, \ell_{\theta_2})$ has lifetime $\mathcal{T}_c^*$. If*

(i) $\frac{(1-u)\ell_\theta'(u)}{\ell_\theta(u)}$ *is increasing in u,*

(ii) $\frac{\ell_\theta'(u)}{\ell_\theta(u)}$ *increasing (decreasing) in θ,*

then for $\theta_1 \geq (\leq)\ \theta_2$,

$$\boldsymbol{\rho} \overset{w}{\preceq} \boldsymbol{\rho}^* \implies \mathcal{T}_c \leq_{hr} \mathcal{T}_c^*.$$

Proof: Let $\phi(t) = \frac{\ell_{\theta_2}\left(1-\prod_{j=0}^{m}(1-\bar{F}_{\rho_j^*}(t))\right)}{\ell_{\theta_1}\left(1-\prod_{j=0}^{m}(1-\bar{F}_{\rho_j}(t))\right)}$. Now differentiating w.r.t. t we have

$$\begin{aligned}\phi'(t) \overset{sgn}{=} & -\prod_{j=0}^{m}(1-\bar{F}_{\rho_j^*}(t))\frac{\ell'_{\theta_2}\left(1-\prod_{j=0}^{m}(1-\bar{F}_{\rho_j^*}(t))\right)}{\ell_{\theta_2}\left(1-\prod_{j=0}^{m}(1-\bar{F}_{\rho_j^*}(t))\right)}\sum_{j=0}^{m}\frac{\rho_j^* r(t)}{1-\bar{\rho}_j^*\bar{F}(t)} \\ & +\prod_{j=0}^{m}(1-\bar{F}_{\rho_j}(t))\frac{\ell'_{\theta_1}\left(1-\prod_{j=0}^{m}(1-\bar{F}_{\rho_j}(t))\right)}{\ell_{\theta_1}\left(1-\prod_{j=0}^{m}(1-\bar{F}_{\rho_j}(t))\right)}\sum_{j=0}^{m}\frac{\rho_j r(t)}{1-\bar{\rho}_j\bar{F}(t)} \end{aligned} \quad (6.3.4)$$

Let us now consider the function $\xi(\rho) = \frac{\rho}{1-\bar{\rho}\bar{F}(t)}$. It is easy to check that $\xi(\rho)$ is increasing and concave in u. Hence we have from Lemma 1.2.7(ii)

$$\begin{aligned}\boldsymbol{\rho} \preceq^{w} \boldsymbol{\rho^*} & \implies (\xi(\rho_0),\xi(\rho_1),\xi(\rho_2),..,\xi(\rho_m)) \preceq^{w} (\xi(\rho_0^*),\xi(\rho_1^*),\xi(\rho_2^*),..,\xi(\rho_m^*)) \\ & \implies \sum_{j=0}^{m}\frac{\rho_j^* r(t)}{1-\bar{\rho}_j^*\bar{F}(t)} \le \sum_{j=0}^{m}\frac{\rho_j r(t)}{1-\bar{\rho}_j\bar{F}(t)} \end{aligned} \quad (6.3.5)$$

Since $\frac{(1-u)\ell'_{\theta}(u)}{\ell_{\theta}(u)}$ is increasing in u we have from (6.3.2)

$$\left(\prod_{j=0}^{m}(1-\bar{F}_{\rho_j^*}(t))\right)\frac{\ell'_{\theta_2}\left(1-\prod_{j=0}^{m}(1-\bar{F}_{\rho_j^*}(t))\right)}{\ell_{\theta_2}\left(1-\prod_{j=0}^{m}(1-\bar{F}_{\rho_j^*}(t))\right)} \le \left(\prod_{j=0}^{m}(1-\bar{F}_{\rho_j}(t))\right)\frac{\ell'_{\theta_2}\left(1-\prod_{j=0}^{m}(1-\bar{F}_{\rho_j}(t))\right)}{\ell_{\theta_2}\left(1-\prod_{j=0}^{m}(1-\bar{F}_{\rho_j}(t))\right)} \quad (6.3.6)$$

If $\frac{\ell'_{\theta}(u)}{\ell_{\theta}(u)}$ increasing (decreasing) in θ, then for $\theta_1 \ge (\le)\theta_2$ from (6.3.6) we can easily conclude that $\mathcal{T}_c \le_{hr} \mathcal{T}_c^*$.

Example 6.3.1. *Consider any survival function $\bar{F}(x)$. Next, we consider the following four-dimensional one-parameter copulas*

$$C_1(u_1,u_2,u_3,u_4,\theta) = \frac{\theta}{\ln(e^{\theta/u_1}+e^{\theta/u_2}+e^{\theta/u_3}-2e^{\theta})},\ \theta > 0$$

Now consider the coherent structure $\max(X_{2:3},X_4)$, *where* $X_{2:3}$ *denotes a* 2*-out-of-*3 *system. Then the distortion function of this coherent structure underlying four-dimensional one-parameter copula* $C_1(u_1,u_2,u_3,u_4,\theta)$ *can be written as*

$$\ell_{\theta}(u) = C_1(u,1,1,1,\theta) + 3C_1(u,u,1,1,\theta) - 5C_1(u,u,u,1,\theta) + 2C_1(u,u,u,u,\theta).$$

Here $\ell_\theta(u)$ satisfied the conditions $(i)-(ii)$ of Theorem 6.3.2.

Theorem 6.3.3 compares the reversed hazard (failure) rates of two coherent systems with different sets of active redundancy at the component level. In many situations, it happens that when a system is running, it is not monitored continuously due to the complexity of the system, cost of monitoring, etc. Suppose the systems failed at time t or sometimes before time t. For such cases, the exact failure time of individual components can not be observed, which sometimes refer as Black Box. Engineers and reliability analysts often need to make inferences on the inactivity time $t-X|(X \le t)$, the time elapsed since the system's failure. Now if a system fails at time t or sometimes before time t, the concept of reversed hazard rate provides us an estimate of the system's failure rate just before the time t. Based on the characteristics of component lifetimes and system design, Theorem 6.3.3 will be useful to compare the reversed hazard rate of a system with different sets of active redundancies.

Theorem 6.3.3. *Let the system with redundancy at the component level following the PO model $(\bar{F}, \boldsymbol{\rho}, \ell_{\theta_1})$, has the lifetime $\mathcal{T}_c$ and under PO model $(\bar{F}, \boldsymbol{\rho}^*, \ell_{\theta_2})$ has lifetime $\mathcal{T}_c^*$. If*

(i) $\frac{(1-u)\ell_\theta'(u)}{1-\ell_\theta(u)}$ *is increasing in u,*

(ii) $\frac{\ell_\theta'(u)}{1-\ell_\theta(u)}$ *increasing (decreasing) in θ,*

then for $\theta_1 \ge (\le)\theta_2$

$$\boldsymbol{\rho} \overset{w}{\preceq} \boldsymbol{\rho}^* \implies \mathcal{T}_c \ge_{rhr} \mathcal{T}_c^*.$$

Proof: Let $\phi(t) = \frac{1-\ell_{\theta_2}\left(1-\prod_{j=0}^m(1-\bar{F}_{\rho_j^*}(t))\right)}{1-\ell_{\theta_1}\left(1-\prod_{j=0}^m(1-\bar{F}_{\rho_j}(t))\right)}$. To prove our theorem, it is sufficient to prove that $\phi(t)$ is decreasing. Now differentiating w.r.t. t we have

$$\begin{aligned}\phi'(t) \overset{sgn}{=} \quad & \prod_{j=0}^m(1-\bar{F}_{\rho_j^*}(t))\frac{\ell_{\theta_2}'\left(1-\prod_{j=0}^m(1-\bar{F}_{\rho_j^*}(t))\right)}{1-\ell_{\theta_2}\left(1-\prod_{j=0}^m(1-\bar{F}_{\rho_j^*}(t))\right)}\sum_{j=0}^m\frac{\rho_j^* r(t)}{1-\bar{\rho}_j^*\bar{F}(t)} \\ & -\prod_{j=0}^m(1-\bar{F}_{\rho_j}(t))\frac{\ell_{\theta_1}'\left(1-\prod_{j=0}^m(1-\bar{F}_{\rho_j}(t))\right)}{1-\ell_{\theta_1}\left(1-\prod_{j=0}^m(1-\bar{F}_{\rho_j}(t))\right)}\sum_{j=0}^m\frac{\rho_j r(t)}{1-\bar{\rho}_j\bar{F}(t)} \quad (6.3.7)\end{aligned}$$

Since $\frac{(1-u)\ell_\theta'(u)}{1-\ell_\theta(u)}$ is increasing in u we have from (6.3.2)

$$\prod_{j=0}^m(1-\bar{F}_{\rho_j^*}(t))\frac{\ell_{\theta_2}'\left(1-\prod_{j=0}^m(1-\bar{F}_{\rho_j^*}(t))\right)}{1-\ell_{\theta_2}\left(1-\prod_{j=0}^m(1-\bar{F}_{\rho_j^*}(t))\right)} \le \prod_{j=0}^m(1-\bar{F}_{\rho_j}(t))\frac{\ell_{\theta_2}'\left(1-\prod_{j=0}^m(1-\bar{F}_{\rho_j}(t))\right)}{1-\ell_{\theta_2}\left(1-\prod_{j=0}^m(1-\bar{F}_{\rho_j}(t))\right)} \quad (6.3.8)$$

If $\frac{\ell_\theta'(u)}{1-\ell_\theta(u)}$ increasing (decreasing) in θ then for $\theta_1 \geq (\leq)\theta_2$ from (6.3.8) we can easily conclude that $\mathcal{T}_s \geq_{rhr} \mathcal{T}_s^*$.

Example 6.3.2. *Consider any survival function $\bar{F}(x)$. Next, we consider the following three-dimensional one-parameter copulas (Gumbel family)*

$$C_2(u_1, u_2, u_3, \theta) = \exp(-[(-\ln u_1)^\theta + (-\ln u_2)^\theta + (-\ln u_3)^\theta]^{1/\theta}),\ \theta > 0.$$

Now consider a $X_{2:3}$ (2-out-of-3) coherent structure. Then the distortion function of this coherent structure underlying three-dimensional one parameter copula $C_2(u_1, u_2, u_3, \theta)$ can be written as $\ell_\theta(u) = 3C_2(u, u, 1, \theta) - 2C_2(u, u, u, \theta)$. Here $\ell_\theta(u)$ satisfied the conditions $(i) - (ii)$ of Theorem 6.3.3.

6.4 System redundancy under PO model:

Here we present the stochastic comparison results for coherent systems of d.i.d. components with redundancy at the system level. Suppose the original system is connected with m same structured coherent systems (redundant systems) of d.i.d. components in parallel. Let the identical components of the original and each m redundant systems follow the PO model with a baseline survival function $\bar{F}$, and corresponding odds ratios ρ_0 and $\rho_1, \cdots, \rho_m$, respectively. So for this coherent system with active redundancy at the system level (let us denote the lifetime by $\mathcal{T}_s$), the reliability function can be written as

$$\bar{F}_{\mathcal{T}_s}(t) = 1 - \prod_{j=0}^{m} \left(1 - \ell_{\theta_j}\left(\bar{F}_{\rho_j}(t)\right)\right) \tag{6.4.1}$$

where, ℓ_{θ_0} and ℓ_{θ_j}, $j = 1, \cdots, m$ are the distorted functions of the original and jth redundant systems, respectively. Here θ_0 and θ_j indicate the parameter of the dependence structure (associated copula) of the dependent components of the original system, and m redundant systems, respectively. For convenient we denote this system as $(\bar{F}, \boldsymbol{\rho}, \boldsymbol{\ell_\theta})$, where $\boldsymbol{\rho} = \{\rho_0, \rho_1, \cdots, \rho_m\}$, $\boldsymbol{\theta} = \{\theta_0, \theta_1, \cdots, \theta_m\}$ and $\boldsymbol{\ell_\theta} = \{\ell_{\theta_0}, \ell_{\theta_1}, \cdots, \ell_{\theta_m}\}$.

In Theorems 6.4.1, we derive two sets of sufficient conditions under which the survival or reliability function of a coherent system of d.i.d. components following the PO model with a set of system-level redundancies is larger or smaller than the system with another set of system-level redundancies. These theorems enable us to select the most favourable system-level redundancies among available options to enhance system reliability.

Theorem 6.4.1. *Let the system with redundancy at the system level following the PO model $(\bar{F}, \boldsymbol{\rho}, \boldsymbol{\ell_\theta})$, has the lifetime $\mathcal{T}_s$ and under PO model $(\bar{F}, \boldsymbol{\rho^*}, \boldsymbol{\ell_{\theta^*}})$ has lifetime $\mathcal{T}_s^*$. Then for all reliability functions $\bar{F}$ with*

(i) $\frac{(1-u)^2\ell_\theta^{'}(u)}{1-\ell_\theta(u)}$ *is decreasing in* u,
(ii) $\frac{\ell_\theta^{'}(u)}{1-\ell_\theta(u)}$ *decreasing in* θ,
(iii) $\ell_\theta(u)$ *is increasing* θ,
(iv) $\frac{\frac{\partial \ell_\theta(\bar{F}_\rho(t))}{\partial \theta}}{1-\ell_\theta(\bar{F}_\rho(t))}$ *decreasing in* u *and* θ,

and $\boldsymbol{\rho}, \boldsymbol{\rho^*}, \boldsymbol{\theta}, \boldsymbol{\theta^*} \in \mathcal{E}_n^+$ *(or* $\mathcal{D}_n^+$*),*

$$\boldsymbol{\rho} \preceq^w \boldsymbol{\rho^*} \text{ and } \boldsymbol{\theta} \preceq^w \boldsymbol{\theta^*} \implies \mathcal{T}_s \geq_{st} \mathcal{T}_s^*.$$

Proof: Let us define

$$F_{\boldsymbol{S},\boldsymbol{\rho},\boldsymbol{\theta}}(t) = \sum_{j-1}^{n} \log(1 - \ell_{\theta_j}(\bar{F}_{\rho_j}(t)), \tag{6.4.2}$$

so that $F_{\boldsymbol{S},\boldsymbol{\rho},\boldsymbol{\theta}}(t) = \log(F_{\mathcal{T}_s}(t))$. Now,

$$\frac{\partial F_{\boldsymbol{S},\boldsymbol{\rho},\boldsymbol{\theta}}(t)}{\partial \rho_j} \overset{sgn}{=} \left(1 - \frac{\rho_j \bar{F}(t)}{1-\bar{\rho}_j \bar{F}(t)}\right)^2 \times \frac{-\ell'_{\theta_j}\left(\frac{\rho_j \bar{F}(t)}{1-\bar{\rho}_j \bar{F}(t)}\right)}{1-\ell_{\theta_j}\left(\frac{\rho_j \bar{F}(t)}{1-\bar{\rho}_j \bar{F}(t)}\right)} \tag{6.4.3}$$

$$\begin{aligned} &\frac{\partial F_{\boldsymbol{S},\boldsymbol{\rho},\boldsymbol{\theta}}(t)}{\partial \rho_i} - \frac{\partial F_{\boldsymbol{S},\boldsymbol{\rho},\boldsymbol{\theta}}(t)}{\partial \rho_j} \\ \overset{sgn}{=}\ & -\left(1 - \bar{F}_{\rho_i}(t)\right)^2 \times \frac{\ell'_{\theta_i}\left(\bar{F}_{\rho_i}(t)\right)}{1-\ell_{\theta_i}\left(\bar{F}_{\rho_i}(t)\right)} + \left(1 - \bar{F}_{\rho_j}(t)\right)^2 \times \frac{\ell'_{\theta_j}\left(\bar{F}_{\rho_j}(t)\right)}{1-\ell_{\theta_j}\left(\bar{F}_{\rho_j}(t)\right)} \end{aligned} \tag{6.4.4}$$

Now let us define $\mathcal{L}(\rho, \theta) = \left(1 - \bar{F}_\rho(t)\right)^2 \times \frac{\ell'_\theta\left(\bar{F}_\rho(t)\right)}{1-\ell_\theta\left(\bar{F}_\rho(t)\right)}$

Now let $\boldsymbol{\rho} \in \mathcal{E}_n^+$. Then for $\leq i < j \leq n$, we have from the fact that $\bar{F}_\rho(t)$ is increasing in ρ and from (i),

$$\rho_i \leq \rho_j \implies \bar{F}_{\rho_i}(t) \leq \bar{F}_{\rho_j}(t) \implies \mathcal{L}(\rho_i, \theta) \geq \mathcal{L}(\rho_j, \theta). \tag{6.4.5}$$

From (ii) for $\frac{\ell'_\theta(u)}{1-\ell_\theta(u)}$ is decreasing in θ and $\boldsymbol{\theta} \in \mathcal{E}_n^+$ we have

$$\theta_i \leq \theta_j \implies \mathcal{L}(\rho, \theta_i) \geq \mathcal{L}(\rho, \theta_j). \tag{6.4.6}$$

Combining (6.4.5) and (6.4.6) we have for $\boldsymbol{\rho}, \boldsymbol{\theta} \in \mathcal{E}_n^+$, $\frac{\partial F_S(t)}{\partial \rho_j}$ is non-positive and increasing in $\rho_k, k = 1, \cdots, m$. In a similar way, we can show that for $\boldsymbol{\rho}, \boldsymbol{\theta} \in \mathcal{D}_n^+$, $\frac{\partial F_S(t)}{\partial \rho_j}$ is non-positive and decreasing in $\rho_k, k = 1, \cdots, m$. Hence by Theorem 1 of Haidari et al. [57] for $\boldsymbol{\rho}, \boldsymbol{\rho^*} \in \mathcal{E}_n^+$ and $\boldsymbol{\theta} \in \mathcal{E}_n^+$ (or $\boldsymbol{\rho}, \boldsymbol{\rho^*} \in \mathcal{D}_n^+$ and $\boldsymbol{\theta} \in \mathcal{D}_n^+$) we have

$$\boldsymbol{\rho} \preceq^w \boldsymbol{\rho^*} \implies \bar{F}_{\boldsymbol{S},\boldsymbol{\rho},\boldsymbol{\theta}}(t) \geq \bar{F}_{\boldsymbol{S},\boldsymbol{\rho^*},\boldsymbol{\theta}}(t) \tag{6.4.7}$$

Now we have

$$\frac{\partial F_S(t)}{\partial \theta_j} \overset{sgn}{=} -\frac{\frac{\partial \ell_{\theta_j}(\bar{F}_{\rho_j})}{\partial \theta_j}}{1 - \ell_{\theta_j}(\bar{F}_{\rho_j}(t))} \tag{6.4.8}$$

consequently

$$\frac{\partial F_S(t)}{\partial \theta_i} - \frac{\partial F_S(t)}{\partial \theta_j} \overset{sgn}{=} -\frac{\frac{\partial \ell_{\theta_i}(\bar{F}_{\rho_i})}{\partial \theta_i}}{1 - \ell_{\theta_i}(\bar{F}_{\rho_i}(t))} + \frac{\frac{\partial \ell_{\theta_j}(\bar{F}_{\rho_j})}{\partial \theta_j}}{1 - \ell_{\theta_j}(\bar{F}_{\rho_j}(t))} \tag{6.4.9}$$

Let us define $\mathcal{H}(\rho, \theta) = \frac{\frac{\partial \ell_\theta(\bar{F}_\rho(t))}{\partial \theta}}{1-\ell_\theta(\bar{F}_\rho(t))}$. By (iv) we have

$$\theta_i \leq \theta_j \implies \mathcal{H}(\rho, \theta_i) \geq \mathcal{H}(\rho, \theta_j), \tag{6.4.10}$$

and

$$\rho_i \leq \rho_j \implies \mathcal{H}(\rho_i, \theta) \geq \mathcal{H}(\rho_j, \theta). \tag{6.4.11}$$

From (6.4.10) and (6.4.11) we have for $\boldsymbol{\theta} \in \mathcal{E}_n^+$ and $\boldsymbol{\rho} \in \mathcal{E}_n^+$, $\frac{\partial F_S(t)}{\partial \theta_k}$ is non-positive increasing in k. In a similar way, we can show that for $\boldsymbol{\theta} \in \mathcal{D}_n^+$ and $\boldsymbol{\rho} \in \mathcal{D}_n^+$, $\frac{\partial F_S(t)}{\partial \theta_k}$ is non-positive decreasing in k. Hence by Theorem 1 of Haidari et al. [57] for $\boldsymbol{\theta}, \boldsymbol{\theta^*} \in \mathcal{E}_n^+$ and $\boldsymbol{\rho} \in \mathcal{E}_n^+$ (or $\boldsymbol{\theta}, \boldsymbol{\theta^*} \in \mathcal{D}_n^+$ and $\boldsymbol{\rho} \in \mathcal{D}_n^+$), we have

$$\boldsymbol{\theta} \preceq^w \boldsymbol{\theta^*} \implies \bar{F}_{\boldsymbol{S},\boldsymbol{\rho},\boldsymbol{\theta}}(t) \geq \bar{F}_{\boldsymbol{S},\boldsymbol{\rho},\boldsymbol{\theta^*}}(t) \tag{6.4.12}$$

Combining (6.4.7) and (6.4.12) we have for $\boldsymbol{\rho}, \boldsymbol{\rho}^*, \boldsymbol{\theta}, \boldsymbol{\theta}^* \in \mathcal{E}_n^+$ (or $\mathcal{D}_n^+$) we have

$$\boldsymbol{\rho} \preceq^w \boldsymbol{\rho}^*, \boldsymbol{\theta} \preceq^w \boldsymbol{\theta}^* \implies \bar{F}_{\boldsymbol{S},\boldsymbol{\rho},\boldsymbol{\theta}}(t) \geq \bar{F}_{\boldsymbol{S},\boldsymbol{\rho}^*,\boldsymbol{\theta}^*}(t), \tag{6.4.13}$$

which proves the theorem.

Theorem 6.4.2 provides sufficient conditions on the characteristics of component lifetimes (odds ratios) and system design to compare the reversed hazard (failure) rates of two coherent systems with different sets of system-level active redundancies, when the two systems have the same dependent structure.

Theorem 6.4.2. *Let the system with redundancy at the system level following the PO model* $(\bar{F}, \boldsymbol{\rho}, \boldsymbol{\ell_\theta})$, *has the lifetime* $\mathcal{T}_{s,\rho}$ *and under PO model* $(\bar{F}, \boldsymbol{\rho}^*, \boldsymbol{\ell_\theta})$ *has lifetime* $\mathcal{T}_{s,\rho^*}$. *Then for all reliability functions* $\bar{F}$ *with*

(i) $u(1-u)\frac{\ell'(u)}{1-\ell(u)}$ *is decreasing and convex in* u,
(ii) $\frac{\ell'_\theta(u)}{1-\ell_\theta(u)}$ *is increasing in* θ,

and $\boldsymbol{\rho}, \boldsymbol{\rho}^*, \boldsymbol{\theta} \in \mathcal{E}_n^+$ *(or* $\mathcal{D}_n^+$*),*

$$\boldsymbol{\rho} \preceq_w \boldsymbol{\rho}^* \implies \mathcal{T}_{s,\rho} \leq_{rhr} \mathcal{T}_{s,\rho^*}.$$

Proof: Consider the function $\phi_\theta(t) = \frac{\prod_{j=0}^m\left(1-\ell_{\theta_j}\left(\frac{\rho_j^*\bar{F}(t)}{1-\bar{\rho}_j^*\bar{F}(t)}\right)\right)}{\prod_{j=0}^m\left(1-\ell_{\theta_j}\left(\frac{\rho_j\bar{F}(t)}{1-\bar{\rho}_j\bar{F}(t)}\right)\right)}$. We prove that $\phi_\theta(t)$ is increasing in t. Differentiating $\phi_\theta(t)$ w.r.t. t we have

$$\begin{aligned}\phi'_\theta(t) \quad &\stackrel{sgn}{=} \quad \sum_{j=0}^m \left(\frac{\rho_j^*\bar{F}(t)}{1-\bar{\rho}_j^*\bar{F}(t)}\right)\left(1-\frac{\rho_j^*\bar{F}(t)}{1-\bar{\rho}_j^*\bar{F}(t)}\right)\frac{\ell'_{\theta_j}\left(\frac{\rho_j^*\bar{F}(t)}{1-\bar{\rho}_j^*\bar{F}(t)}\right)}{1-\ell_{\theta_j}\left(\frac{\bar{F}}{1-\bar{\rho}_j^*\bar{F}(t)}\right)} \\ &- \quad \sum_{j=0}^m \left(\frac{\rho_j\bar{F}(t)}{1-\bar{\rho}_j\bar{F}(t)}\right)\left(1-\frac{\rho_j\bar{F}(t)}{1-\bar{\rho}_j\bar{F}(t)}\right)\frac{\ell'_{\theta_j}\left(\frac{\rho_j\bar{F}(t)}{1-\bar{\rho}_j\bar{F}(t)}\right)}{1-\ell_{\theta_j}\left(\frac{\bar{F}}{1-\bar{\rho}_j\bar{F}(t)}\right)}\end{aligned}$$

Let us define

$$\mathcal{A}(\boldsymbol{\rho}, \boldsymbol{\theta}) = \sum_{j=1}^m \frac{\rho_j\bar{F}(t)}{1-\bar{\rho}_j\bar{F}(t)}\left(1-\frac{\rho_j\bar{F}(t)}{1-\bar{\rho}_j\bar{F}(t)}\right)\frac{\ell'_{\theta_j}\left(\frac{\rho_j\bar{F}(t)}{1-\bar{\rho}_j\bar{F}(t)}\right)}{1-\ell_{\theta_j}\left(\frac{\rho_j\bar{F}(t)}{1-\bar{\rho}_j\bar{F}(t)}\right)} \tag{6.4.14}$$

Now for $k = 1, \cdots, m$

$$\mathcal{B}(\rho_k, \theta_k) = \frac{\partial \mathcal{A}}{\partial \rho_k} = \frac{\partial}{\partial \rho_k} \left(\frac{\rho_k \bar{F}(t)}{1 - \bar{\rho}_k \bar{F}(t)} \left(1 - \frac{\rho_k \bar{F}(t)}{1 - \bar{\rho}_k \bar{F}(t)} \right) \frac{\ell'_{\theta_k} \left(\frac{\rho_k \bar{F}(t)}{1 - \bar{\rho}_k \bar{F}(t)} \right)}{1 - \ell_{\theta_k} \left(\frac{\rho_k \bar{F}(t)}{1 - \bar{\rho}_k \bar{F}(t)} \right)} \right) \quad (6.4.15)$$

Hence

$$\frac{\partial \mathcal{A}}{\partial \rho_i} - \frac{\partial \mathcal{A}}{\partial \rho_j} = \mathcal{B}(\rho_i, \theta_i) - \mathcal{B}(\rho_j, \theta_j) \quad (6.4.16)$$

If $u(1-u)\frac{\ell'_\theta(u)}{1-\ell_\theta(u)}$ is decreasing and convex in u, then from the fact that $\frac{\rho \bar{F}(t)}{1-\bar{\rho}\bar{F}(t)}$ is increasing and concave in ρ, we have for $\boldsymbol{\rho} \in \mathcal{E}_n^+$,

$$i < j \implies \rho_i \leq \rho_j \implies \mathcal{B}(\rho_i, \theta) \leq \mathcal{B}(\rho_j, \theta) \quad (6.4.17)$$

Again if $\frac{\ell'_\theta(u)}{1-\ell_\theta(u)}$ increasing in θ, then for $\boldsymbol{\theta} \in \mathcal{E}_n^+$ we have

$$i < j \implies \theta_i \leq \theta_j \implies \mathcal{B}(\rho, \theta_i) \leq \mathcal{B}(\rho, \theta_j) \quad (6.4.18)$$

From (6.4.17) and (6.4.18) we have for $\boldsymbol{\rho}, \boldsymbol{\theta} \in \mathcal{E}_n^+$,

$$i < j \implies \mathcal{B}(\rho_i, \theta_i) \leq \mathcal{B}(\rho_j, \theta_j) \quad (6.4.19)$$

i.e. $\frac{\partial \mathcal{A}}{\partial \rho_k}$ is non-negative and increasing in k. In a similar way, we can show that for $\boldsymbol{\rho}, \boldsymbol{\theta} \in \mathcal{D}_n^+$, $\frac{\partial \mathcal{A}}{\partial \rho_k}$ is non-negative decreasing in k. Hence by Theorem 1 of Haidari et al. [57] for $\boldsymbol{\rho}, \boldsymbol{\rho}^*, \boldsymbol{\theta} \in \mathcal{E}_n^+$ (or $\mathcal{D}_n^+$) we have

$$\boldsymbol{\rho} \preceq_w \boldsymbol{\rho}^* \implies \mathcal{A}(\boldsymbol{\rho}, \boldsymbol{\theta}) \leq \mathcal{A}(\boldsymbol{\rho}^*, \boldsymbol{\theta}), \quad (6.4.20)$$

showing $\phi_\theta(t)$ is increasing in t. Thus $\boldsymbol{\rho} \preceq_w \boldsymbol{\rho}^* \implies \mathcal{T}_{s,\rho} \leq_{rhr} \mathcal{T}_{s,\rho^*}$.

6.5 Data Analysis:

We consider the data set "Strength of cables" [58], which gives the tensile strengths (in kg) of 12 types of wires (Wire #1-Wire #12) that are used to make cables. The data contains the tensile strengths of a sample of 9 wires for each type of wire. Uniform cables of high tensile strength are essential for a high-voltage electricity transmission network. Also, a reliable, real-time signal transmission network is crucial for an efficient power network. Failure of critical signal transmission network components, e.g., cables, can result in signal disruptions that can lead to power outages. Signal transmission redundancy (Miclot, 2011), redundancy in power circuits (Csanyi, 2019) are common practices to ensure continuous operation and high availability. Here we have demonstrated how our derived results can be applied to compare two cable networks with redundancies formed by different wires, taking into account various reliability concepts such as the reliability function and reversed hazard rate function.

To choose baseline distribution for strengths of cables, we first consider some well known distributions:

(i) Exponential with rate parameter μ. The pdf is given by

$$f(x;\mu) = \frac{1}{\mu}e^{-x/\mu} \qquad x \geq 0, \mu > 0.$$

(ii) Weibull with scale parameter σ and shape parameter a. The pdf is given by

$$f(x;\sigma,a) = (a/\sigma)(x/\sigma)^{a-1}\exp\left(-(x/\sigma)^{a}\right) \qquad a > 0, \sigma > 0, x > 0.$$

(iii) Pareto with location parameter η and shape parameter θ. The pdf is given by

$$f(x;\eta,\theta) = \frac{\theta\eta^{\theta}}{x^{\theta+1}}, \quad \eta > 0, \theta > 0, x \geq \eta.$$

(iv) Frechet with location parameter a, scale parameter b and shape parameter s. The pdf is given by

$$f(x;,b,s) = \frac{s}{b}\left(\frac{x}{b}\right)^{-1-s}\exp\left\{-\left(\frac{x}{b}\right)^{-s}\right\}, \quad ,b > 0, s > 0, x > 0.$$

(v) Burr with shape parameters α and γ and scale parameter θ. The pdf is given by

$$f(x;\alpha,\gamma,\theta) = \frac{\alpha\gamma(x/\theta)^{\gamma}}{x[1+(x/\theta)^{\gamma}]^{\alpha+1}}, \quad \alpha > 0, \gamma > 0, \theta > 0, x > 0.$$

Goodness-of-fit test are performed and results are tabulated in Tables 6.10. Among them Weibull is the best fitted according to AIC and BIC values. Estimated scale and shape parameters are obtained as $\hat{\sigma} = 342.3578, \hat{a} = 57.0885$.

Scenario I: In this scenario, we illustrate some theoretical results for the system with redundancy at the component level following the AL model.Let us consider a three-component cable network system made by Wire #3, which follows a $X_{2:3}$ (2-out-of-3) coherent structure.

Table 6.10: Goodness-of-fit criteria

Distribution	Parameter estimates	AIC	BIC
Exponential	$\hat{\mu} = 0.00294784$	1476.563	1479.246
Weibull	$\hat{\sigma} = 342.3578, \hat{a} = 57.0885$	720.2729	725.6371
Pareto	$\hat{\eta} = 1047978942, \hat{\theta} = 3090327$	1478.564	1483.928
Frechet	$\hat{b} = 335.8897, \hat{s} = 53.2325$	731.7465	737.1107
Burr	$\hat{\alpha} = 128.6677, \hat{\gamma} = 57.43633, \hat{\theta} = 2684.366$	722.0600	730.1064

Where components lifetime follows random identical but dependent variables X_1, X_2, X_3 respectively. The distortion of this coherent structure $\phi(X)$ under any three-dimensional Archimedean copula $C(u_1, u_2, u_3)$ is given by $h_\theta(u) = 3C[u, u, 1; \theta] - 2C[u, u, u; \theta]$. Next we consider two different sets of active redundancy (cables), where the first set of two cables made by Wire #1 and Wire #2, respectively, and the second set of two cables made by Wire #10 and Wire #12, respectively. Here we reasonably assume that all components follow the AL model for which the corresponding Al-constants (scale parameters) corresponding to the baseline distribution function $F(x) = 1 - e^{-(\frac{x}{342.3578})^{57.0885}}$ are given in Table 6.11.

Table 6.11: Estimated parameters under $\bar{F}(x) = e^{-(\frac{x}{342.3578})^{57.0885}}$

wrie	Al- constant	odds ratio
wrie 1	1.005192	0.8098226
wrie 2	1.006458	0.6562225
wrie 3	1.000894	0.9917941
wrie 4	0.9942	1.932896
wrie 5	0.9887188	1.225557
wrie 6	0.9967861	0.9702483
wrie 7	1.005974	0.5752256
wrie 8	0.9994444	1.528433
wrie 9	1.000404	1.032759
wrie 10	1.002343	1.070463
wrie 11	1.005716	0.51783
wrie 12	1.004117	0.8152945

Our aim is to compare the reliability of the considered system having the first set of standby components at the component level with the same system having the second set of standby components at the component level. Let us denote their lifetimes as $\mathcal{S}_c$ and $\mathcal{S}_c^*$, respectively. To capture the dependency among the original and standby components for component label allocations, we consider three commonly used Archimedean copulas (i)-(iii).

The estimated parameters, test statistics, and p-values of these three copulas for the original and two different sets of allocated standby components are reported in Table 6.12. We adopt a goodness-of-fit test that uses the rank-based analogue of the Cramér-von Mises statistic to select the best copula. The technical details for Cramér-von Mises test are taken from Fermanian [45] and Genest et al. [48] . From both the tables it can be seen that the Gumbel copula is the best candidate.

Table 6.12: Goodness of fit

Set of Wires	Family	Parameters estimate ($\hat{\theta}$)	Test statistics	p-value
(Wire : 3,1,2)	Clayton	0.8405	0.1194	0.68
	Gumbel	1.6074	0.0761	0.78
	Frank	3.1241	0.1024	0.65
(Wire : 3,10,12)	Clayton	1.5770	0.0558	0.85
	Gumbel	1.8052	0.0524	0.89
	Frank	4.5964	0.0566	0.89

It is easy to check that $h_\theta(u)$ is increasing in θ, and also $(1.000894, 1.006458, 1.005192) \overset{w}{\preceq} (1.000894, 1.004117, 1.002343)$. Therefore conditions of the Theorem 3.1 are well satisfied, which gives us $\mathcal{S}_c \leq_{st} \mathcal{S}_c^*$. In Figure 6.2, we have plotted the survival functions of the two systems, denoted as $\bar{S}_1$ and $\bar{S}_2$, respectively, which clearly shows this fact.

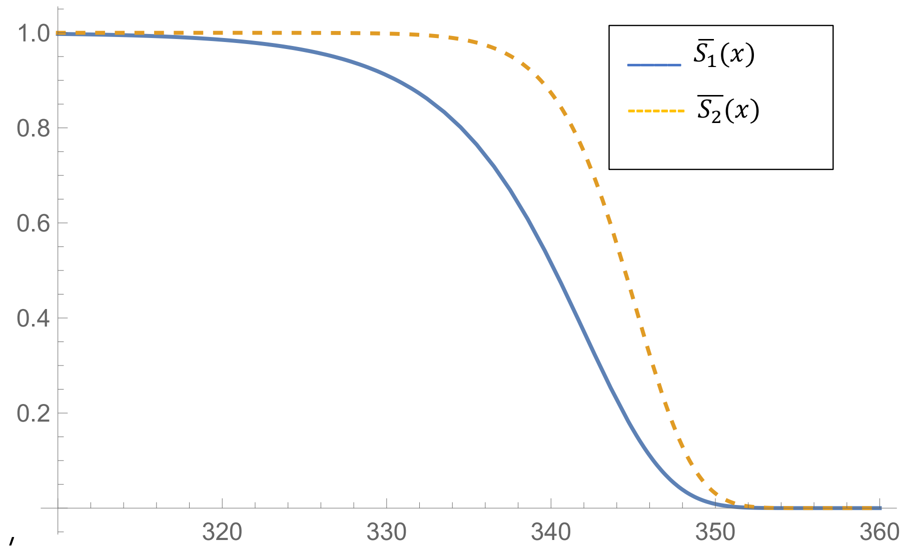

Figure 6.2: Plots of $\bar{S}_1(x)$ and $\bar{S}_2(x)$

Also, $\frac{(1-u)h'_\theta(u)}{1-h_\theta(u)}$ is increasing in both u and θ. Therefore conditions of the Theorem 3.3 are also satisfied. Therefore we have $\mathcal{S}_c \leq_{rhr} \mathcal{S}_c^*$. In Figure 6.3, we have plotted the reversed hazard rate functions of the two systems, denoted as $\tilde{r}_{S_1}$ and $\tilde{r}_{S_2}$, respectively, which validate this result.

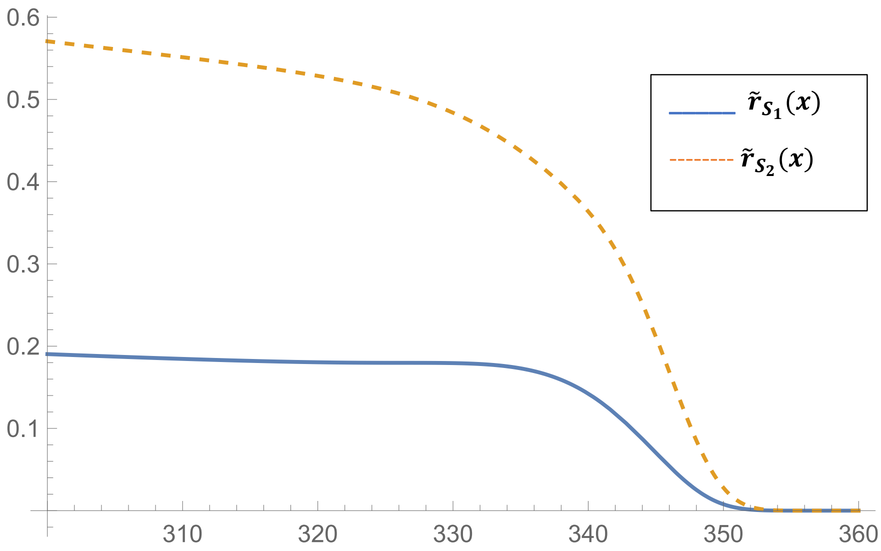

Figure 6.3: Plot of $(1 - S_2(x))/(1 - S_1(x))$

Scenario II: In this scenario, we illustrate some theoretical results for the system with redundancy at the component level following the PO model with real data set. Let us now consider four-component cable network made by wire #7 having the structure function $\phi(X) = \max(X_1, \min(X_2, X_3, X_4))$ where X_1, X_2, X_3, X_4 are d.i.d. The distorted function of this coherent structure $\phi(X)$ under any four-dimensional Archimedean copula $C(u_1, u_2, u_3, u_4, \theta)$ is given by $\ell_\theta(u) = C(u, 1, 1, 1, \theta) + C(u, u, u, 1, \theta) - C(u, u, u, u, \theta)$. Next, we consider two different sets of standby components (cables), the first set of three cables made by wire #8, wire #9 and wire #10, respectively, and the second set of three cables made by wire #1, wire #2 and wire #3, respectively. With the fitted baseline Weibull distribution, we then model the distribution of individual wires as extended Weibull distributions, incorporating estimated tilt parameters to leverage the flexibility and improve the fit to the data. This also gives us another realization that the odds function of the distribution of an individual wire is related to the baseline distribution with a proportionality constant (ρ). Under this consideration of the PO model for the distribution of the considered wires, the odds ratios (tilt parameters) corresponding to the baseline distribution function $F(x) = 1 - e^{-(\frac{x}{342.3578})^{57.0885}}$ are given in Table 6.11. In Table 6.13, we present the Cramer-von Mises (CVM) test statistic of goodness-of-fit and corresponding p-values while fitting the distribution of the individual weirs as Marshall-Olkin extended Weibull (MOEW) distributions and Weibull distributions, respectively. This table clearly shows the superiority of MOEW over Weibull in fitting the data.

Table 6.13: Cramér–von Mises test

Wire #	Test statistic (MOEW)	p-value (MOEW)	Test statistic (Weibull)	p-value (Weibull)
1	0.1154957	0.52	0.1285011	0.47
2	0.08867033	0.65	0.1214296	0.50
3	0.05176394	0.88	0.05168517	0.88
4	0.04806504	0.90	0.1282451	0.47
5	0.180623	0.31	0.1519284	0.38
6	0.07582544	0.73	0.07711442	0.72
7	0.04046622	0.94	0.1283638	0.47
8	0.1089786	0.55	0.195199	0.28
9	0.1063032	0.56	0.1055502	0.57
10	0.1102138	0.55	0.1111536	0.54
11	0.04884404	0.89	0.2151886	0.24
12	0.02378149	0.99	0.036573	0.96

Our intensity is to compare the reliability of the considered system having the first set of active standby components at the component level with the same system having the second set of standby components at the component level. Let us denote their lifetimes as $\mathcal{T}_c$ and $\mathcal{T}_c^*$, respectively.

Active standby components started working immediately after allocating them in parallel to the original components. Lifetimes of spares are independent of original components while allocated parallel to the original components. After this allocation, our newly allocated system's lifetime will be governed by both original and standby components. As the original components of the considered system are dependent, after active standby allocation in parallel to the original components, the original components coupled with their allocated spares will be dependent for the system lifetime. To capture the dependency structure for allocated system lifetime based on the original and standby components, here we consider three commonly used Archimedean copulas: Clayton, Gumbel and Frank. We adopt a goodness-of-fit test that uses the rank-based analogue of the Cramér-von Mises statistic to select the best-fitted copula. We refer interested readers to Genest et al. [49] for more technical details. Table 6.14 and Table **??** collect the estimated parameters, test statistics, and p-values of these three copulas for the original and two different sets of three allocated components. From both the tables, it can be seen that the Clayton copula is the best candidate.

It is easy to check that $\ell_\theta(u)$ as given above under the four-dimensional Clayton copula $C(u_1, u_2, u_3, u_4, \theta) = \left(u_1^{-\theta} + u_2^{-\theta} + u_3^{-\theta} + u_4^{-\theta} - 3\right)^{-1/\theta}, \theta > 0$ is increasing in θ. Also we have $\theta_1 = 1.1436 > 0.8651 = \theta_2$ and

$$(0.5752256, 1.528433, 1.032759, 1.070463) \overset{w}{\preceq} (0.5752256, 0.8098226, 0.6562225, 0.9917941)$$

. Therefore conditions of Theorem 6.3.1 are well satisfied, which gives us $\mathcal{T}_c \geq_{st} \mathcal{T}_c^*$. In Figure 6.4, we have plotted the survival functions of the two systems, which clearly shows this fact.

Also, $\frac{(1-u)\ell'_\theta(u)}{1-\ell_\theta(u)}$ is increasing in both u and θ. Therefore the conditions of Theorem 6.3.3 are also satisfied. Therefore we have $\mathcal{T}_c \geq_{rhr} \mathcal{T}_c^*$. In Figure 6.5, we have plotted $(1-\bar{F}_{\mathcal{T}_c^*}(x))/(1-\bar{F}_{\mathcal{T}_c}(x))$ showing that this ratio is decreasing, and thus validating this result.

Table 6.14: Goodness of fit

Set of Wires	Family	Parameters estimate $(\hat{\theta})$	Test statistics	p-value
(Wire : 7,8,9,10)	Clayton	1.1436	0.09638418	0.68
	Gumbel	1.5056	0.133003	0.58
	Frank	2.9624	0.1359992	0.52
(Wire : 3,10,12)	Clayton	0.8651	0.08261695	0.79
	Gumbel 1.4382	0.09410233	0.63	
	Frank	2.4498	0.1060551	0.62

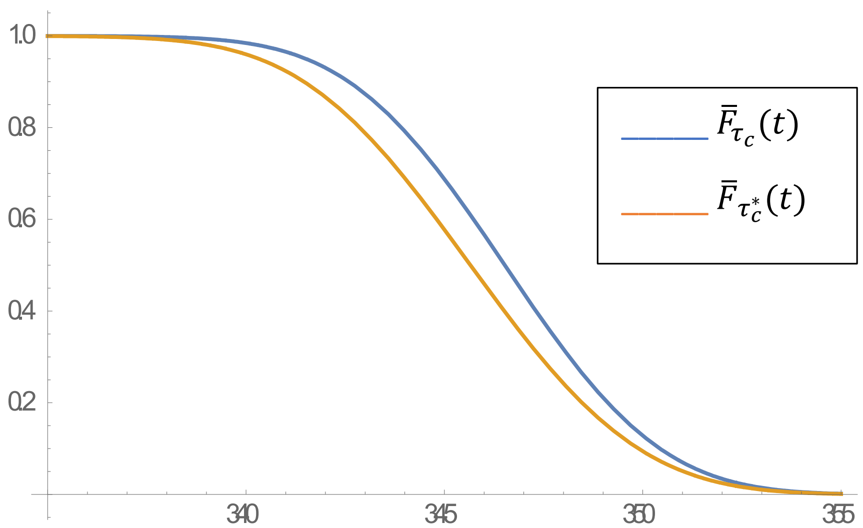

Figure 6.4: Plots of $\bar{F}_{\mathcal{T}_c}(x)$ and $\bar{F}_{\mathcal{T}_c^*}(x)$

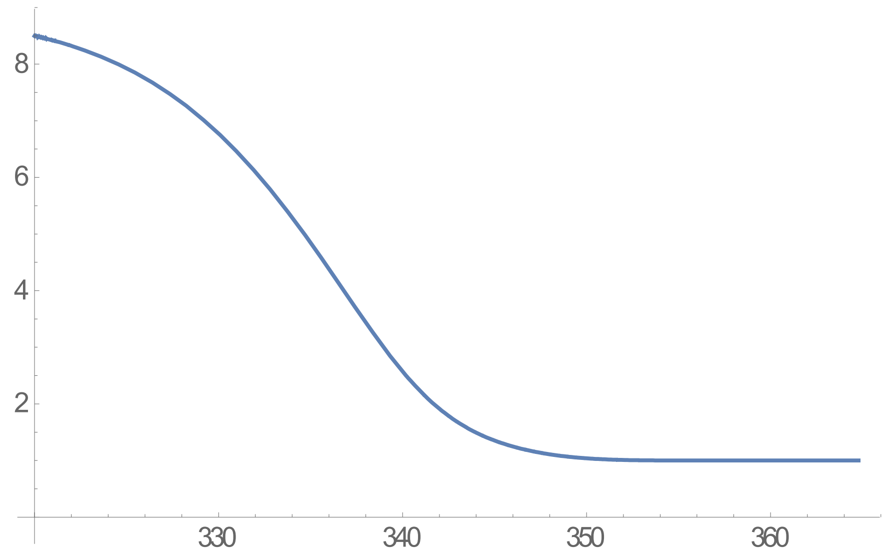

Figure 6.5: Plot of $(1 - \bar{F}_{\mathcal{T}_c^*}(x))/(1 - \bar{F}_{\mathcal{T}_c}(x))$

Chapter 7

Stochastic comparisons of claim amounts[1]

7.1 Introduction

An insurance policy is an agreement between the insurer and the insured. Consequently, there are two thought processes in any insurance policy, namely, one from insurer side and the other one from insured side. An insured always looks into the plan (that contains the annual premium amount, total time period, whether it is the individual or the group insurance policy, etc.) and the key benefits (namely, sum insured amount, withdrawal facility, tax saving facility, etc.) of a policy before having it. On the other hand, the insurer comes up with a policy whose existence and upgradation (as and when necessary) depend on different key factors, namely, number of claims in a given time frame, size of the portfolio, aggregate claim amount, largest claim amount, smallest claim amount, etc. Thus, numerous researchers have shown their keen interest in studying useful characteristics of these key factors.

It is also important for an actuary to be able to compare different portfolios of risks according to these important information. In this prospect, stochastic comparisons of maximum, minimum and aggregate claim amounts arising from two sets of portfolios have great importance in actuarial science on both theoretical and practical grounds (see,Barmalzan

[1] One paper based on this chapter has appeared under:

1. Stochastic comparisons of largest claim and aggregate claim amounts. *Probability in the Engineering and Informational Sciences,* 2024, DOI: 10.1017/S0269964823000104.

and Najafabadi [14], Barmalzan et al. [15], Balakrishnan et al. [10], Torrado and Navarro [127], Zhang et al. [140, 136]).

In this chapter, we investigate stochastic comparisons of the largest claim amounts from two sets of heterogeneous portfolios in the sense of some stochastic orderings under the setup of the PO model. We also investigate stochastic comparisons of aggregate claim amounts. It's worth noting that our results are not limited to be applied in actuarial science. For instance, our proposed results can be used to compare the lifetimes of two parallel systems whose components are subject to random shocks instantaneously. Suppose that the r.v. X_i denotes the lifetime of the i-th component of a parallel system which may receive a random shock defined by the Bernoulli r.v. I_{p_i} where $I_{p_i} = 1$ if the shock does not occur with $p_i = P(I_{p_i} = 1)$ and 0 if the shock occurs. Then $X^*_{n:n}$ represents the lifetime of a parallel system whose components are subject to random shocks instantaneously.

The rest of the chapter is organized as follows. Section 7.2 presents some stochastic comparisons results for largest claim amounts of two sets of independent and also for inter-dependent portfolios under the setup of the PO model. Section 7.3 presents star ordering result for two sets of independent multiple-outlier claims. Section 7.4 presents comparisons results on aggregate claim amounts under two sets of independent portfolios.

7.2 Comparisons of largest claim amounts

In this section, we derive some stochastic comparisons results for largest claim amounts of two different portfolios of risks. Unless otherwise specified, we assume that $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ are two sets of independent r.v.'s.

Assume that I_{p_i}, $i = 1, \ldots, n$, are independent Bernoulli r.v.'s, independent of X_i's, with $E(I_{p_i}) = p_i$. Denote multivariate Bernoulli random vector $\boldsymbol{I} = (I_{p_1}, \ldots, I_{p_n})$. Let $X^*_i = X_i I_{p_i}$, $i = 1, \ldots, n$, and denote $X^*_{n:n} = \max(X^*_1, \ldots, X^*_n)$. Then $X^*_{n:n}$ corresponds to the largest claim amount in a portfolio of risks, where X_i's represent random claim amount that can be made by a policy in an insurance period, and I_{p_i}'s indicate the occurrence of these claims. Further suppose odds function of each X_i in X is proportional to that of a baseline r.v. with proportionality constant (odds ratio) $\alpha_i > 0$, i.e. $X_i \sim PO(\bar{F}, \alpha_i)$, $i = 1, \ldots, n$ where $\bar{F}$ denotes the sf of the baseline r.v. Let us denote $X^\circ_{n:n} = \max(X^\circ_1, \ldots, X^\circ_n)$, where $X^\circ_i = X_i I_{q_i}$ and I_{q_i}, $i = 1, \ldots, n$, are independent Bernoulli r.v.'s, independent of X_i's, with $E(I_{q_i}) = q_i$.

Similarly suppose $Y_i \sim PO(\bar{F}, \beta_i)$, $\beta_i > 0$, $i = 1, \ldots, n$. Denote $Y^*_{n:n} = \max(Y^*_1, \ldots, Y^*_n)$, where $Y^*_i = Y_i I_{p_i}$, and $Y^\circ_{n:n} = \max(Y^\circ_1, \ldots, Y^\circ_n)$, where $Y^\circ_i = Y_i I_{q_i}$, $i = 1, \ldots, n$.

Theorem 7.2.1 established that more heterogeneity among the odds of claim in terms of the weakly supermajorization order results in less largest claim amount in the sense of the usual stochastic orders when both portfolios having common occurrence of claim **p**. Theorem 7.2.1 established that more heterogeneity among the odds of claim in terms of the weakly supermajorization order results in less largest claim amount in the sense of the usual stochastic orders when both portfolios having common occurrence of claim **p**. By the symbol $a \overset{sign}{=} b$ we mean that a and b have the same sign.

Theorem 7.2.1. *Suppose $\kappa : [0,1] \to R_+$ be a differentiable and strictly decreasing function. Let I_{p_i}, $i = 1, \ldots, n$, be independent Bernoulli r.v.'s, independent of X_i's, with $E(I_{p_i}) = p_i$. Then, for $(\kappa(\boldsymbol{p}), \boldsymbol{\alpha}) \in \mathcal{U}_n$ and $(\kappa(\boldsymbol{p}), \boldsymbol{\beta}) \in \mathcal{U}_n$,*

$$\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta} \implies X^*_{n:n} \leq_{st} Y^*_{n:n}.$$

Proof: We have $F_{X^*_{n:n}}(x) = \prod_{i=1}^{n}(1 - \kappa^{-1}(u_i)\bar{F}_{X_i}(x))$, where $\bar{F}_{X_i}(x) = \frac{\alpha_i \bar{F}(x)}{1-\bar{\alpha}_i \bar{F}(x)}$, and $u_i = \kappa(p_i)$, $i = 1, 2, \ldots, n$. Note that $\bar{F}_{X_i}$ is increasing and concave in α_i. Now,

$$\begin{aligned}
\frac{\partial F_{X^*_{n:n}}(x)}{\partial \alpha_i} &= \frac{-\kappa^{-1}(u_i)\frac{\partial \bar{F}_{X_i}}{\partial \alpha_i}}{1 - \kappa^{-1}(u_i)\bar{F}_{X_i}(x)} F_{X^*_{n:n}}(x) \\
&= -\frac{\kappa^{-1}(u_i)\frac{\bar{F}(x)F(x)}{(1-\bar{\alpha}_i\bar{F}(x))^2}}{1 - \kappa^{-1}(u_i)\frac{\alpha_i\bar{F}(x)}{1-\bar{\alpha}_i\bar{F}(x)}} F_{X^*_{n:n}}(x) \\
&= -\frac{\kappa^{-1}(u_i)\bar{F}(x)F(x)}{(1-\bar{\alpha}_i\bar{F}(x))^2 - \kappa^{-1}(u_i)\alpha_i(1-\bar{\alpha}_i\bar{F}(x))\bar{F}(x)} F_{X^*_{n:n}}(x) \\
&= -g(z_i, \alpha_i)\bar{F}(x)F(x)F_{X^*_{n:n}}(x) \ (say),
\end{aligned}$$

where $z_i = \kappa^{-1}(u_i)$. Again,

$$\frac{\partial g}{\partial \alpha_i} \overset{sign}{=} -\kappa^{-1}(u_i)\bar{F}(x)\left[(2-\kappa^{-1}(u_i))(1-\bar{F}(x)) + 2(1-\kappa^{-1}(u_i))\alpha_i\bar{F}(x)\right] \leq 0.$$

So $g(z_i, \alpha_i)$ is decreasing in α_i. Further,

$$\frac{\partial g}{\partial z_i} \overset{sign}{=} (1-\bar{\alpha}_i\bar{F}(x))^2 \geq 0,$$

so $g(z_i, \alpha_i)$ is increasing in $z_i = \kappa^{-1}(u_i)$, and so it is decreasing in u_i as $z_i = \kappa^{-1}(u_i)$ is decreasing in u_i. Without loss of generality we assume that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ so that $(\boldsymbol{\kappa}(\mathbf{p}), \boldsymbol{\alpha}) \in \mathcal{U}_n$ implies $\kappa(p_1) \geq \kappa(p_2) \geq \cdots \geq \kappa(p_n)$. Now for any pair $((i,j))$ such that

$1 \le i < j \le n$, we have $\alpha_i \ge \alpha_j$ and $u_i \ge u_j$. Thus we have

$$g(z_i, \alpha_i) \le g(z_j, \alpha_i) \le g(z_j, \alpha_j) \implies \frac{\partial F_{X^*_{n:n}}(x)}{\partial \alpha_j} \le \frac{\partial F_{X^*_{n:n}}(x)}{\partial \alpha_i} \le 0. \tag{7.2.1}$$

So from Lemma 1.2.6, we get

$$\boldsymbol{\alpha} \overset{w}{\succeq} \boldsymbol{\beta} \implies X^*_{n:n} \le_{st} Y^*_{n:n}.$$

The following example demonstrates the Theorem 7.2.1.

Example 7.2.1. Suppose that $\{X_1, X_2, X_3, X_4\}$ and $\{Y_1, Y_2, Y_3, Y_4\}$ are two sets of independent non-negative r.v.'s with $X_i \sim PO(\bar{F}(x), \alpha_i)$ and $Y_i \sim PO(\bar{F}(x), \beta_i),\ i = 1, 2, 3, 4$, where $\bar{F}(x) = e^{-(0.5x)^2},\ x > 0$. Further, suppose that $\{I_{p_1}, I_{p_2}, I_{p_3}, I_{p_4}\}$ is a set of Bernoulli r.v.'s, independent of X_i's and Y_i's. Set $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.2, 0.6, 1.5, 2.8)$, $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.5, 0.8, 2.5, 4.8)$, $(p_1, p_2, p_3, p_4) = (0.95, 0.65, 0.5, 0.35)$ and $\kappa(p) = 1/p^2$, satisfying all the conditions of Theorem 7.2.1. We consider the transformation $x = t/(1-t)$ so that, for $t \in [0, 1)$, we have $x \in [0, \infty)$. Then denote the cdfs of $X^*_{n:n}$ and $Y^*_{n:n}$ by $F_{X^*_{n:n}}(t/(1-t)) = \zeta_1(t)$ and $F_{Y^*_{n:n}}(t/(1-t)) = \zeta_2(t)$, respectively. Figure 7.1 shows that $\zeta_1(t) \ge \zeta_2(t)$, for all $t \in [0, 1)$, and hence, $X^*_{n:n} \le_{st} Y^*_{n:n}$.

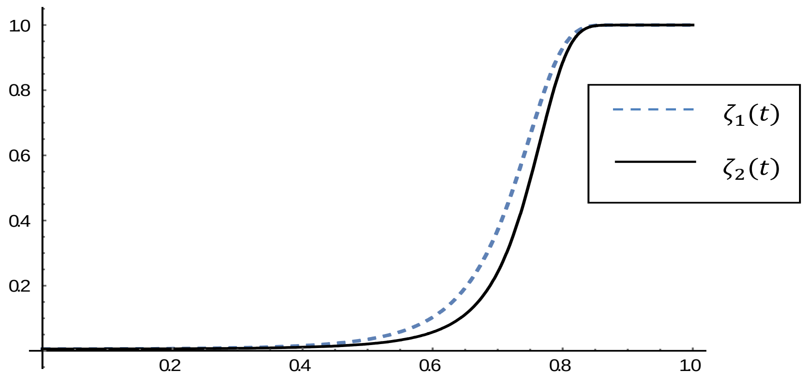

Figure 7.1: Plots of $\zeta_1(t)$ and $\zeta_2(t)$, $t \in [0, 1]$

Next we provide a counterexample to show that the stochastic ordering result in Theorem 7.2.1 may not hold if we relax the weakly supermajorize condition under a strictly decreasing function.

Counterexample 7.2.1. *In Example 7.2.1, let us take* $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.2, 0.9, 1.5, 4.5)$, $(\beta_1, \beta_2, \beta_3, \beta_4) = (0.35, 0.4, 2.9, 3.8)$ *so that* $\boldsymbol{\alpha} \overset{w}{\not\succeq} \boldsymbol{\beta}$. *In Figure 7.2 we have plotted* $\zeta_1(t) - \zeta_2(t)$ *for all* $t \in [0, 1)$, *from which it is clear that stochastic ordering result of Theorem 7.2.1 does not hold.*

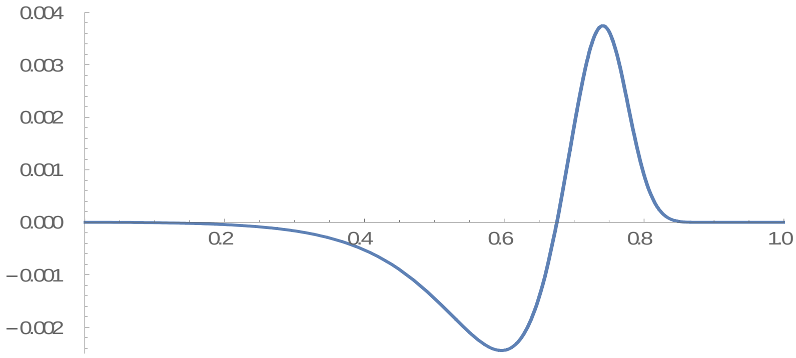

Figure 7.2: Plot of $\zeta_1(t) - \zeta_2(t)$, $t \in [0, 1]$

Theorem 7.2.2 establishes that largest claim amounts of two portfolios of risks might be increased in terms of the usual stochastic order with the increased heterogeneity among the probabilities of occurrence of claims, when both the portfolio of risks having common odds of claim vector $\boldsymbol{\alpha}$.

Theorem 7.2.2. *Suppose* $\kappa : [0, 1] \to R_+$ *be a differentiable and strictly increasing concave function. Let* $I_{p_i}(I_{q_i})$, $i = 1, \ldots, n$, *be independent Bernoulli r.v.'s, independent of* $X_i(Y_i)$*'s, with* $E(I_{p_i}) = p_i(E(I_{q_i}) = q_i)$. *Then, for* $(\boldsymbol{\kappa}(\boldsymbol{p}), \boldsymbol{\alpha}) \in \mathcal{U}_n$ *and* $(\boldsymbol{\kappa}(\boldsymbol{q}), \boldsymbol{\alpha}) \in \mathcal{U}_n$,

$$(\kappa(p_1), \kappa(p_2), ..., \kappa(p_n)) \succeq_w (\kappa(q_1), \kappa(q_2), ..., \kappa(q_n)) \implies X^*_{n:n} \geq_{st} X^\circ_{n:n}.$$

Proof: Here $F_{X^*_{n:n}}(x) = \prod_{i=1}^n (1 - \kappa^{-1}(u_i)\bar{F}_{X_i}(x))$, where $\bar{F}_{X_i}(x) = \frac{\alpha_i \bar{F}(x)}{1 - \bar{\alpha}_i \bar{F}(x)}$. It is to be noted that $\bar{F}_{X_i}(x)$ is increasing in α_i. Also κ^{-1} is strictly increasing and convex. Let $\phi(\mathbf{u}) = -F_{X^*_{n:n}}(x)$. We have

$$\frac{\partial \phi(\mathbf{u})}{\partial u_i} = \frac{\bar{F}_{X_i}(x) \frac{d\kappa^{-1}(u_i)}{du_i}}{1 - \kappa^{-1}(u_i)\bar{F}_{X_i}(x)} F_{X^*_{n:n}}(x) \geq 0$$

as $\kappa^{-1}(.)$ is increasing.

Let $\ell(\tau_i, u_i) = \frac{\tau_i \frac{d\kappa^{-1}(u_i)}{du_i}}{1-\kappa^{-1}(u_i)\tau_i}$, where $\tau_i = \bar{F}_{X_i}(x)$. Then

$$\frac{\partial \ell}{\partial u_i} \overset{sign}{=} \tau_i(1-\tau_i\kappa^{-1}(u_i))\frac{d^2\kappa^{-1}(u_i)}{du_i^2} + \tau_i^2\left(\frac{d\kappa^{-1}(u_i)}{du_i}\right)^2 \geq 0,$$

which holds as $\kappa^{-1}(u_i)$ is convex. So, $\ell(\tau_i, u_i)$ is increasing in u_i. Further,

$$\frac{\partial \ell}{\partial \tau_i} \overset{sign}{=} (1-\tau_i\kappa^{-1}(u_i))\frac{d\kappa^{-1}(u_i)}{du_i} - \tau_i\frac{d\kappa^{-1}(u_i)}{du_i}(-\kappa^{-1}(u_i)) = \frac{d\kappa^{-1}(u_i)}{du_i} \geq 0,$$

since $\kappa^{-1}(u_i)$ is increasing in u_i. Then $\ell(\tau_i, u_i)$ is increasing in τ_i (i.e., in $\bar{F}_{X_i}(x)$), so that it is increasing in α_i as $\bar{F}_{X_i}(x)$ is increasing in α_i.
Without loss of generality we assume that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ so that $(\boldsymbol{\kappa}(\mathbf{p}), \boldsymbol{\alpha}) \in \mathcal{U}_n$ implies $\kappa(p_1) \geq \kappa(p_2) \geq \cdots \geq \kappa(p_n)$. Now for any pair (i,j) with $1 \leq i < j \leq n$, we have $\alpha_i \geq \alpha_j$ & $u_i \geq u_j$. Then if $\kappa^{-1}(u_i)$ is increasing and convex in u_i we have

$$\begin{aligned} &\ell(\tau_i, u_i) \geq \ell(\tau_j, u_i) \geq \ell(\tau_j, u_j) \\ &\text{i.e. } \frac{\partial \phi(\mathbf{u})}{\partial u_i} \geq \frac{\partial \phi(\mathbf{u})}{\partial u_j} \geq 0. \end{aligned} \tag{7.2.2}$$

So from Lemma 1.2.6, we have

$$(\kappa(p_1), \kappa(p_2), ..., \kappa(p_n)) \succeq_w (\kappa(q_1), \kappa(q_2), ..., \kappa(q_n)) \implies X^*_{n:n} \geq_{st} X^\circ_{n:n}.$$

We illustrate Theorem 7.2.2 with the following example.

Example 7.2.2. Suppose that $\{X_1, X_2, X_3, X_4\}$ is a set of independent non-negative r.v.'s with $X_i \sim PO(\bar{F}(x), \alpha_i), i = 1,2,3,4$, where $\bar{F}(x) = e^{-(x/2)^{1.5}},\ x > 0$. Set $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ $= (0.9, 1.36, 2.55, 3.5)$,$(p_1, p_2, p_3, p_4) = (0.35, 0.5, 0.8, 0.9)$, $(q_1, q_2, q_3, q_4) = (0.2, 0.4, 0.65, 0.8)$ and $\kappa(p) = p/(1+p)$, satisfying all the conditions of Theorem 7.2.2. We consider the transformation $x = t/(1-t)$ so that, for $t \in [0,1)$, we have $x \in [0,\infty)$. After this substitution, let us denote the respective distribution functions of $X^*_{n:n}$ and $X^\circ_{n:n}$ by $F_{X^*_{n:n}}(t/(1-t)) = \xi_1(t)$ and $F_{X^\circ_{n:n}}(t/(1-t)) = \xi_2(t)$. From Figure 7.3, it is clear that $\xi_1(t) \leq \xi_2(t),\ \forall t \in [0,1)$, and hence, $X^*_{n:n} \geq_{st} X^\circ_{n:n}$.

Next we provide a counterexample to show that the stochastic ordering result in Theorem 7.2.2 may not hold if we relax the weakly submajorize condition under an increasing

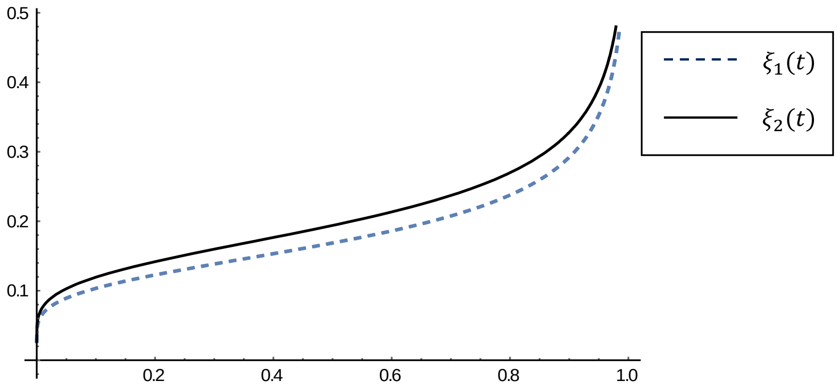

Figure 7.3: Plots of $\xi_1(t)$ and $\xi_2(t)$, $t \in [0, 1]$

concave function.

Counterexample 7.2.2. *In Example 7.2.2, let us take* $(p_1, p_2, p_3, p_4) = (0.1, 0.2, 0.85, 0.95)$ *and* $(q_1, q_2, q_3, q_4) = (0.5, 0.55, 0.75, 0.8)$ *so that*

$$(\kappa(p_1), \kappa(p_2), \kappa(p_3), \kappa(p_4)) \not\succeq_w (\kappa(q_1), \kappa(q_2), \kappa(q_3), \kappa(q_4))$$

. In Figure 7.4 we have plotted $\xi_1(t) - \xi_2(t)$ $\forall t \in [0, 1)$, *from which it is clear that none of these distribution functions dominating each other.*

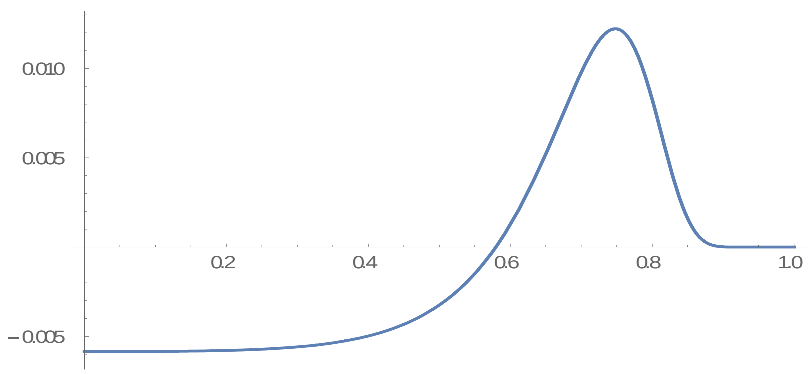

Figure 7.4: Plot of $\xi_1(t) - \xi_2(t)$, $t \in [0, 1]$

Cai and Wei [29] proposed some multivariate dependence notions based on SAI (stochastic arrangement increasing) notion, including LWSAI (weakly SAI through left tail probability), to model multivariate dependent risks. Since then it has been applied in finance and actuarial science to model dependent stochastic returns and risks (Cai and Wei [29], Zhang et al. [139, 140]). For a random vector $\boldsymbol{X} = (X_1, X_2, ..., X_n)$, one of the ways to define or describe its dependence notion is to characterize the expectations of the transformations of the random vector (Cai and Wei [28]). For any (i, j) with $1 \leq i < j \leq n$, denote

$$\mathcal{G}^{i,j}_{\text{lwsai}}(n) = \{g(\boldsymbol{x}) : g(\boldsymbol{x}) - g(\pi_{ij}(\boldsymbol{x})) \text{ is decreasing in } x_i \leq x_j\},$$

where $\pi_{i,j}$ is the special permutation of transposition defined as $\pi_{i,j}(\boldsymbol{x}) = (x_1, ., x_j, ., x_i, ., x_n)$. A random vector $\boldsymbol{X} = (X_1, X_2, ..., X_n)$ or its distribution is said to be the LWSAI (Cai and Wei [29]), if $\mathbb{E}[g(\boldsymbol{X})] \geq \mathbb{E}[g(\tau_{i,j}(\boldsymbol{X}))]$ for any $g(\boldsymbol{x}) \in \mathcal{G}^{i,j}_{lwsai}(n)$ and any $1 \leq i < j \leq n$.

Next, we present a stochastic ordering result when the occurrence probabilities are interdependent in terms of LWSAI. Let us denote $S_k = \{\boldsymbol{\chi} | \chi_i = 0 \text{ or } 1,\ i = 1, 2, \ldots, n,\ \chi_1 + \ldots + \chi_n = k\}$, $k = 0, \ldots, n$, and $S^{i,j}_k(\eta_i, \eta_j) = \{\boldsymbol{\chi} \in S_k | \chi_i = \eta_i, \chi_j = \eta_j,\ \eta_i, \eta_j \in \{0, 1\}\}$, for any $1 \leq i \neq j \leq n$, $k = 1, \ldots, n-1$. Then $S_k = \bigcup_{\eta_i, \eta_j \in \{0,1\}} S^{i,j}_k(\eta_i, \eta_j)$. Also denote $p(\boldsymbol{\chi}) = \mathbb{P}(\boldsymbol{I} = \boldsymbol{\chi}) = \mathbb{P}(I_{p_1} = \chi_1, \ldots, I_{p_n} = \chi_n)$.

Lemma 7.2.1. *([29];[10]) A multivariate Bernoulli random vector $\boldsymbol{I}$ is LWSAI iff $p(\boldsymbol{\chi}) \geq p(\pi_{ij}(\boldsymbol{\chi}))$ for all $\boldsymbol{\chi} \in S^{i,j}_k(0, 1)$, $1 \leq i < j \leq n$, and $k = 1, \ldots, n-1$.*

Theorem 7.2.3. *Suppose that $\boldsymbol{I} = (I_{p_1}, \ldots, I_{p_n})$ is LWSAI. If*

$$\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\beta} \text{ such that } \alpha_1 \leq \alpha_2 \leq ... \leq \alpha_n \text{ and } \beta_1 \leq \beta_2 \leq ... \leq \beta_n,$$

then $X^*_{n:n} \leq_{st} Y^*_{n:n}$.

Proof: From Theorem 4.1 of Kundu et al. Kundu et al. [81], it follows that $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\beta} \implies X_{n:n} \leq_{st} Y_{n:n}$, i.e., $F_{X_{n:n}}(t) \geq F_{Y_{n:n}}(t)$ for all t, where $X_{n:n} = \max(X_1, X_2, ..., X_n)$. We desire to show that $F_{X^*_{n:n}}(t) \geq F_{Y^*_{n:n}}(t)$, for all $t \in \Re_+$. By the nature of majorization order, it suffices to prove the result when $(\alpha_i, \alpha_j) \overset{m}{\succeq} (\beta_i, \beta_j)$ for some pair $1 \leq i < j \leq n$, and $\alpha_r = \beta_r$ for all $r \neq i, j$. Now, we have

$$\begin{aligned}
F_{X^*_{n:n}}(t) &= \mathbb{P}(\max\{I_{p_1}X_1, ..., I_{p_n}X_n\} \leq t) \\
&= \sum_{k=0}^{n} \sum_{\chi \in S_k} \mathbb{P}(\max\{I_{p_1}X_1, ..., I_{p_n}X_n\} \leq t | \boldsymbol{I} = \boldsymbol{\chi}) p(\boldsymbol{\chi}) \\
&= p(\mathbf{0}) + p(\mathbf{1})\mathbb{P}(\max\{X_1, ..., X_n\} \leq t) + \sum_{k=1}^{n-1} \sum_{\boldsymbol{\chi} \in S_k} p(\boldsymbol{\chi}) \mathbb{P}(\max\{\chi_1 X_1, ..., \chi_n X_n\} \leq t)
\end{aligned}$$

$$
\begin{aligned}
&= p(\mathbf{0}) + p(\mathbf{1})F_{X_{n:n}}(t) + \sum_{k=1}^{n-1}\left\{ \sum_{\boldsymbol{\chi}\in S_k^{i,j}(0,0)} p(\boldsymbol{\chi}) \prod_{r\neq i,j}^{n} \mathbb{P}(\chi_r X_r \le t) \right.\\
&+ \sum_{\boldsymbol{\chi}\in S_k^{i,j}(0,1)} p(\boldsymbol{\chi})F_{X_j}(t) \prod_{r\neq i,j}^{n} \mathbb{P}(\chi_r X_r \le t) + \sum_{\boldsymbol{\chi}\in S_k^{i,j}(0,1)} p(\tau_{i,j}(\boldsymbol{\chi}))F_{X_i}(t) \prod_{r\neq i,j}^{n} \mathbb{P}(\chi_r X_r \le t)\\
&\left. + \sum_{\boldsymbol{\chi}\in S_k^{i,j}(1,1)} p(\boldsymbol{\chi})F_{X_i}(t)F_{X_j}(t) \prod_{r\neq i,j}^{n} \mathbb{P}(\chi_r X_r \le t) \right\}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
F_{Y^*_{n:n}}(t) &= p(\mathbf{0}) + p(\mathbf{1})F_{Y_{n:n}}(t) + \sum_{k=1}^{n-1}\left\{ \sum_{\boldsymbol{\chi}\in S_k^{i,j}(0,0)} p(\boldsymbol{\chi}) \prod_{r\neq i,j}^{n} \mathbb{P}(\chi_r Y_r \le t) \right.\\
&+ \sum_{\boldsymbol{\chi}\in S_k^{i,j}(0,1)} p(\boldsymbol{\chi})F_{Y_j}(t) \prod_{r\neq i,j}^{n} \mathbb{P}(\chi_r Y_r \le t) + \sum_{\boldsymbol{\chi}\in S_k^{i,j}(0,1)} p(\tau_{i,j}(\boldsymbol{\chi}))F_{Y_i}(t) \prod_{r\neq i,j}^{n} \mathbb{P}(\chi_r Y_r \le t)\\
&\left. + \sum_{\boldsymbol{\chi}\in S_k^{i,j}(1,1)} p(\boldsymbol{\chi})F_{Y_i}(t)F_{Y_j}(t) \prod_{r\neq i,j}^{n} \mathbb{P}(\chi_r Y_r \le t) \right\}.
\end{aligned}
$$

Upon using the condition that $\mathbb{P}(\chi_r X_r \le t) = \mathbb{P}(\chi_r Y_r \le t)$ for all $r \neq i, j$, we have

$$
\begin{aligned}
F_{X^*_{n:n}}(t) - F_{Y^*_{n:n}}(t) &= p(\mathbf{1})[F_{X_{n:n}}(t) - F_{Y_{n:n}}(t)]\\
&\sum_{k=1}^{n-1}\left\{ \sum_{\boldsymbol{\chi}\in S_k^{i,j}(0,1)} p(\boldsymbol{\chi})[F_{X_j}(t) - F_{Y_j}(t)] \prod_{r\neq i,j}^{n} \mathbb{P}(\chi_r X_r \le t) \right.\\
&+ \sum_{\boldsymbol{\chi}\in S_k^{i,j}(0,1)} p(\tau_{i,j}(\boldsymbol{\chi}))[F_{X_i}(t) - F_{Y_i}(t)] \prod_{r\neq i,j}^{n} \mathbb{P}(\chi_r X_r \le t)\\
&\left. + \sum_{\boldsymbol{\chi}\in S_k^{i,j}(1,1)} p(\boldsymbol{\chi})[F_{X_i}(t)F_{X_j}(t) - F_{Y_i}(t)F_{Y_j}(t)] \prod_{r\neq i,j}^{n} \mathbb{P}(\chi_r X_r \le t) \right\}\\
&\ge \sum_{k=1}^{n-1}\left\{ \sum_{\boldsymbol{\chi}\in S_k^{i,j}(0,1)} p(\tau_{i,j}(\boldsymbol{\chi}))[F_{X_j}(t) - F_{Y_j}(t)] \prod_{r\neq i,j}^{n} \mathbb{P}(\chi_r X_r \le t) \right.\\
&+ \sum_{\boldsymbol{\chi}\in S_k^{i,j}(0,1)} p(\tau_{i,j}(\boldsymbol{\chi}))[F_{X_i}(t) - F_{Y_i}(t)] \prod_{r\neq i,j}^{n} \mathbb{P}(\chi_r X_r \le t)
\end{aligned}
$$

$$+ \sum_{\boldsymbol{\chi} \in S_k^{i,j}(1,1)} p(\boldsymbol{\chi})[F_{X_i}(t)F_{X_j}(t) - F_{Y_i}(t)F_{Y_j}(t)] \prod_{r \neq i,j}^{n} \mathbb{P}(\chi_r X_r \leq t) \Bigg\}$$

$$= \sum_{k=1}^{n-1} \Bigg\{ \sum_{\boldsymbol{\chi} \in S_k^{i,j}(0,1)} \Bigg(p(\tau_{i,j}(\boldsymbol{\chi}))[F_{X_i}(t) + F_{X_j}(t) - F_{Y_i}(t) - F_{Y_j}(t)]$$

$$\prod_{r \neq i,j}^{n} \mathbb{P}(\chi_r X_r \leq t) \Bigg) + \sum_{\boldsymbol{\chi} \in S_k^{i,j}(1,1)} \Bigg(p(\boldsymbol{\chi})[F_{X_i}(t)F_{X_j}(t) - F_{Y_i}(t)F_{Y_j}(t)]$$

$$\prod_{r \neq i,j}^{n} \mathbb{P}(\chi_r X_r \leq t) \Bigg) \Bigg\} \geq 0,$$

where the first inequality follows from the fact $F_{X_{n:n}}(t) \geq F_{Y_{n:n}}(t)$, Lemma 7.2.1 and the fact that $F_{X_i}(t) = \frac{F(t)}{1-\bar{\alpha}_i \bar{F}(t)}$ is decreasing in α_i. For the last inequality we have the following explanation. Since $F_{X_i}(t)$ is convex in α_i, it follows that $F_{X_i}(t) + F_{X_j}(t) \geq F_{Y_i}(t) + F_{Y_j}(t)$. Let

$$\phi(\alpha_i, \alpha_j) = F_{X_i}(t)F_{X_j}(t) = \frac{F^2(t)}{(1-\bar{\alpha}_i \bar{F}(t))(1-\bar{\alpha}_j \bar{F}(t))}.$$

Then, for $1 \leq i < j \leq n$,

$$\frac{\partial \phi}{\partial \alpha_i} - \frac{\partial \phi}{\partial \alpha_j} = \frac{(\alpha_i - \alpha_j)\bar{F}(x)}{(1-\bar{\alpha}_i \bar{F}(t))^2(1-\bar{\alpha}_j \bar{F}(t))^2} \leq 0.$$

So, from Lemma 1.2.4, we get that $(\alpha_i, \alpha_j) \overset{m}{\succeq} (\beta_i, \beta_j) \Rightarrow \phi(\alpha_i, \alpha_j) \geq \phi(\beta_i, \beta_j)$, and thus the proof is completed.

Remark 7.2.1. *Here it is to be noted that in Theorem 3.11 of Balakrishnan et al. Barmalzan et al. [15] they derived the similar results under the assumption that the sf $\bar{F}(x;\alpha)$ is decreasing and convex in $\alpha > 0$ which does not satisfy by the PO model. Here our established results in Theorem 7.2.3 can be generalised for any semi-parametric model for which $\bar{F}(x;\alpha)$ is increasing and concave in $\alpha > 0$.*

We illustrate Theorem 7.2.3 with the following example.

Example 7.2.3. Suppose that $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ are two sets of independent non-negative r.v.'s with $X_i \sim PO(\bar{F}(x), \alpha_i), i = 1, 2,$ and $Y_i \sim PO(\bar{F}(x), \beta_i), i = 1, 2,$ where $\bar{F}(x) = e^{-(0.08x)^{0.08}}$, $x > 0$. Set $(\alpha_1, \alpha_2) = (0.55, 1.45), (\beta_1, \beta_2) = (0.65, 1.35), p(0,0) = P(I_{p_1} = 0, I_{p_2} = 0) = 0.14, p(0,1) = 0.47, p(1,0) = 0.25, p(1,1) = 0.14$. Then $I = \{I_{p_1}, I_{p_2}\}$

is LWSAI. We consider the transformation $x = t/(1-t)$. After this substitution, let us denote the respective cdfs of $X^*_{n:n}$ and $Y^*_{n:n}$ by $F_{X^*_{n:n}}(t/(1-t)) = \varphi_1(t)$ and $F_{Y^*_{n:n}}(t/(1-t)) = \varphi_2(t)$. Figure 7.5 shows that $\varphi_1(t) \geq \varphi_2(t)$ for all $t \in [0,1)$. Hence $X^*_{n:n} \leq_{st} Y^*_{n:n}$.

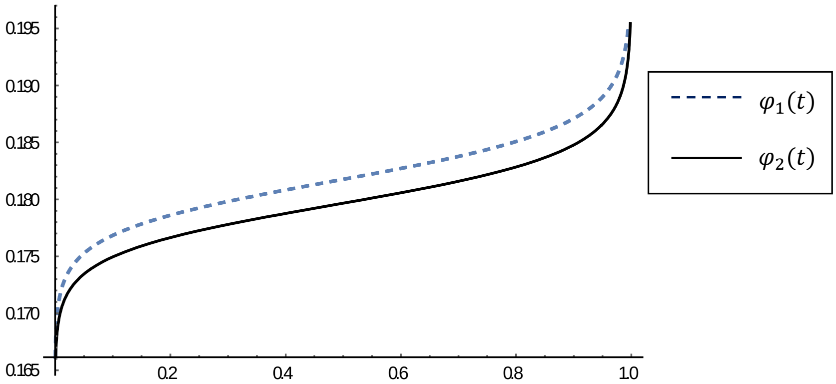

Figure 7.5: Plots of $\varphi_1(t)$ and $\varphi_2(t)$, $t \in [0,1]$

Theorems 7.2.4-7.2.6 compare the largest claim amounts of two interdependent heterogeneous portfolios of risks where the joint cdfs are modeled using copulas.

Theorem 7.2.4. *Let* $X_1, X_2, ..., X_n$ $(Y_1, Y_2, ..., Y_n)$ *be non-negative r.v.'s with* $X_i \sim PO(\bar{F}, \alpha_i)$ $(Y_i \sim PO(\bar{F}, \beta_i))$, $i = 1, 2, ..., n$, *and let the associated copula be* C. *Further, suppose that* $\{I_{p_1}, I_{p_2}, ..., I_{p_n}\}$ *is a set of independent Bernoulli r.v.'s, independent of* X_i*'s* (Y_i*'s*). *Then*

$$\alpha_i \leq \beta_i, \forall i = 1, 2, ..., n \implies X^*_{n:n} \leq_{st} Y^*_{n:n}.$$

Proof: The cdf of $X^*_{n:n}$ can be obtain as

$$\begin{aligned}
G_{X^*_{n:n}}(t) &= \mathbb{P}(X^*_1 \leq t, X^*_2 \leq t, ..., X^*_n \leq t) \\
&= \mathbb{P}(I_{p_1}X_1 \leq t, ..., I_{p_n}X_n \leq t) \\
&= \sum_{\boldsymbol{\chi} \in \{0,1\}^n} \mathbb{P}\left(I_{p_1}X_1 \leq t, ..., I_{p_n}X_n \leq t | \boldsymbol{I} = \boldsymbol{\chi}\right) p(\boldsymbol{\chi}) \\
&= \sum_{\boldsymbol{\chi} \in \{0,1\}^n} p(\boldsymbol{\chi}) \mathbb{P}\left(\chi_1 X_1 \leq t, ..., \chi_n X_n \leq t\right) \\
&= \sum_{\boldsymbol{\chi} \in \{0,1\}^n} p(\boldsymbol{\chi}) C\left([F_{X_1})]^{\chi_1}, ..., [F_{X_n}]^{\chi_n}\right).
\end{aligned}$$

Since $F_{X_i}(x) = \frac{F(x)}{1-\bar{\alpha}_i \bar{F}(x)}$ is decreasing in α_i and the copula is component-wise increasing, we have that $G_{X^*_{n:n}}(x)$ is decreasing in α_i, for $i = 1, 2, \ldots, n$. Hence, the desired result follows.

Theorem 7.2.5. *Let $X_1, X_2, ..., X_n$ be non-negative r.v.'s with $X_i \sim PO(\bar{F}, \alpha_i)$, $i = 1, \dots, n$, and let the associated copula be C (C'). Further, suppose that $\{I_{p_1}, I_{p_2}, ..., I_{p_n}\}$ is a set of independent Bernoulli r.v.'s, independent of X_i's. Then*

$$C' \prec C \implies X^*_{n:n} \leq_{st} X^{*'}_{n:n}$$

where the rvs $X^*_{n:n}$ and $X^{*'}_{n:n}$ represent the largest claim amount with the associated copula C (C').

Proof: The proof follows from Theorem 7.2.4 and the fact C' is less positively lower orthant dependent (PLOD) than C".

For the next theorem, proof follows from Theorems 7.2.4 and 7.2.5 and hence, omitted.

Theorem 7.2.6. *Let $X_1, X_2, ..., X_n$ $(Y_1, Y_2, ..., Y_n)$ be non-negative r.v.'s with $X_i \sim PO(\bar{F}, \alpha_i)$ $(Y_i \sim PO(\bar{F}, \beta_i))$, $i = 1, 2, ..., n$, and let the associated copula be C (C'). Further, suppose that $\{I_{p_1}, I_{p_2}, ..., I_{p_n}\}$ is a set of independent Bernoulli r.v.'s, independent of X_i's (Y_i's). Then*

$$\alpha_i \leq \beta_i, \forall i = 1, 2, ..., n, \text{ and } C' \prec C \implies X^*_{n:n} \leq_{st} Y^*_{n:n}.$$

The following example demonstrates the result given in the above theorem.

Example 7.2.4. Suppose $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ are two sets of independent non-negative r.v.'s with $X_i \sim PO(\bar{F}(x), \alpha_i)$, $i = 1, 2$, and $Y_i \sim PO(\bar{F}(x), \beta_i)$, $i = 1, 2$, where $\bar{F}(x) = e^{-(0.05x)^{0.5}}$, $x > 0$. Set $(\alpha_1, \alpha_2) = (0.5, 1.25)$, $(\beta_1, \beta_2) = (0.75, 1.55)$, $p(0,0) = 0.89$, $p(0,1) = 0.06$, $p(1,0) = 0.04$, $p(1,1) = 0.01$. Here we take $C(x_1, x_2) = e^{-\left\{(\log(x_1))^{\theta_1} + (\log(x_2))^{\theta_1}\right\}^{1/\theta_1}}$ and $C'(x_1, x_2) = e^{-\left\{(\log(x_1))^{\theta_2} + (\log(x_2))^{\theta_2}\right\}^{1/\theta_2}}$, where $\theta_1 = 2$ and $\theta_2 = 5$. We consider the transformation $x = t/(1-t)$. After this substitution, we denote the cdfs of $X^*_{n:n}$ and $Y^*_{n:n}$ by $F_{X^*_{n:n}}(t/(1-t)) = \nu_1(t)$ and $F_{Y^*_{n:n}}(t/(1-t)) = \nu_2(t)$, respectively. Figure 7.6 shows that $\nu_1(t) \geq \nu_2(t)$ for all $t \in [0, 1)$. Hence $X^*_{n:n} \leq_{st} Y^*_{n:n}$.

The following theorem compares the largest claim amounts of two sets of heterogeneous portfolios of risks in terms of the rhr order. It is assumed that the odds ratios are the same but the probabilities of occurrences of claims are different.

Theorem 7.2.7. *Let $X_1, \dots, X_n$ be independent r.v.'s with $X_i \sim PO(\bar{F}, \alpha)$, $i = 1, \dots, n$, and let I_{p_i} (I_{q_i}), $i = 1, \dots, n$, be independent Bernoulli r.v.'s, independent of X_i's. Further,*

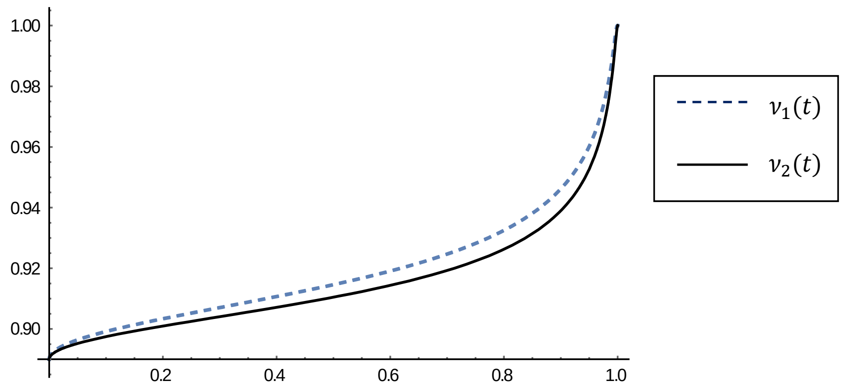

Figure 7.6: Plots of $\nu_1(t)$ and $\nu_2(t)$, $t \in [0,1]$

let $X_i^* = X_i I_{p_i}$ *and* $X_i^\circ = X_i I_{q_i}$, $i = 1, \ldots, n$. *Let* $\kappa : [0,1] \to R_+$ *be a differentiable function. Then*

(i) $(\kappa(p_1), \kappa(p_2), ..., \kappa(p_n)) \overset{w}{\succeq} (\kappa(q_1), \kappa(q_2), ..., \kappa(q_n)) \implies X_{n:n}^* \geq_{rh} X_{n:n}^\circ$, *if* $\kappa(x)$ *is strictly decreasing and convex in* x*;*

(ii) $(\kappa(p_1), \kappa(p_2), ..., \kappa(p_n)) \succeq_w (\kappa(q_1), \kappa(q_2), ..., \kappa(q_n)) \implies X_{n:n}^* \geq_{rh} X_{n:n}^\circ$, *if* $\kappa(x)$ *is strictly increasing and concave in* x.

Proof: We have $F_{X_{n:n}^*}(t) = \prod_{i=1}^n (1 - \kappa^{-1}(u_i)\bar{F}_\alpha(t))$, where $\kappa(p_i) = u_i$, $i = 1, \ldots, n$. Since $X_i \sim PO(\bar{F}, \alpha)$ for $i = 1, .., n$. We have $\bar{F}_\alpha(x) = \frac{\alpha \bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}$.

Now $f_{X_{n:n}^*}(t) = \sum_{i=1}^n \left(\frac{\kappa^{-1}(u_i) f_\alpha(t)}{1-\kappa^{-1}(u_i)\bar{F}_\alpha(t)} \right) F_{X_{n:n}^*}(t)$ and therefore

$$\tilde{r}_{X_{n:n}^*}(t) = \sum_{i=1}^n \frac{\kappa^{-1}(u_i) f_\alpha(t)}{1 - \kappa^{-1}(u_i)\bar{F}_\alpha(t)}.$$

So, we have

$$\frac{\partial \tilde{r}_{X_{n:n}^*}(t)}{\partial u_i} = \frac{\frac{d\kappa^{-1}(u_i)}{du_i} f_\alpha(t)}{(1 - \kappa^{-1}(u_i)\bar{F}_\alpha(t))} + \frac{\frac{d\kappa^{-1}(u_i)}{du_i} \kappa^{-1}(u_i) f_\alpha(t) \bar{F}_\alpha(t)}{(1 - \kappa^{-1}(u_i)\bar{F}_\alpha(t))^2}$$

and

$$\frac{\partial \tilde{r}_{X_{n:n}^*}(t)}{\partial u_j} = \frac{\frac{d\kappa^{-1}(u_j)}{du_j} f_\alpha(t)}{(1 - \kappa^{-1}(u_j)\bar{F}_\alpha(t))} + \frac{\frac{d\kappa^{-1}(u_j)}{du_j} \kappa^{-1}(u_j) f_\alpha(t) \bar{F}_\alpha(t)}{(1 - \kappa^{-1}(u_j)\bar{F}_\alpha(t))^2}.$$

Now, consider the following two cases.

Case-I: Let κ be strictly decreasing and convex. Then κ^{-1} is strictly decreasing and convex.

Consequently, $\tilde{r}_{X^*_{n:n}}(t)$ is decreasing in u_i. Further,

$$(u_i - u_j)\left(\frac{\partial \tilde{r}_{X^*_{n:n}}(t)}{\partial u_i} - \frac{\partial \tilde{r}_{X^*_{n:n}}(t)}{\partial u_j}\right) \stackrel{sign}{=} (u_i - u_j)\left[\left(\frac{\frac{d\kappa^{-1}(u_i)}{du_i}}{1-\kappa^{-1}(u_i)\bar{F}_\alpha(t)} - \frac{\frac{d\kappa^{-1}(u_j)}{du_j}}{1-\kappa^{-1}(u_j)\bar{F}_\alpha(t)}\right) + \left(\frac{\kappa^{-1}(u_i)\frac{d\kappa^{-1}(u_i)}{du_i}}{1-\kappa^{-1}(u_i)\bar{F}_\alpha(t)} - \frac{\kappa^{-1}(u_j)\frac{d\kappa^{-1}(u_j)}{du_j}}{1-\kappa^{-1}(u_j)\bar{F}_\alpha(t)}\right)\right] \geq 0,$$

which follows from the fact that κ^{-1} is decreasing and convex. Consequently, we have $\tilde{r}_{X^*_{n:n}}(t)$ is decreasing and Schur-convex in u_i. So, from Lemma 1.2.5, the first part of the theorem follows.

Case II: Let κ be strictly increasing and concave. Then κ^{-1} is strictly increasing and convex. Consequently, $\tilde{r}_{X^*_{n:n}}(t)$ is increasing in u_i. Further,

$$(u_i - u_j)\left(\frac{\partial \tilde{r}_{X^*_{n:n}}(t)}{\partial u_i} - \frac{\partial \tilde{r}_{X^*_{n:n}}(t)}{\partial u_j}\right) \stackrel{sign}{=} (u_i - u_j)\left[\left(\frac{\frac{d\kappa^{-1}(u_i)}{du_i}}{1-\kappa^{-1}(u_i)\bar{F}_\alpha(t)} - \frac{\frac{d\kappa^{-1}(u_j)}{du_j}}{1-\kappa^{-1}(u_j)\bar{F}_\alpha(t)}\right) + \left(\frac{\kappa^{-1}(u_i)\frac{d\kappa^{-1}(u_i)}{du_i}}{1-\kappa^{-1}(u_i)\bar{F}_\alpha(t)} - \frac{\kappa^{-1}(u_j)\frac{d\kappa^{-1}(u_j)}{du_j}}{1-\kappa^{-1}(u_j)\bar{F}_\alpha(t)}\right)\right] \geq 0,$$

which follows from the fact that κ^{-1} is increasing and convex. Consequently, we have $\tilde{r}_{X^*_{n:n}}(t)$ is increasing and Schur-convex in u_i. So, from Lemma 1.2.5, the second part of the theorem follows. We illustrate Theorem 7.2.7 with the following example.

Example 7.2.5. *Suppose that $\{X_1, X_2, X_3, X_4\}$ is a set of independent non-negative r.v.'s with $X_i \sim PO(\bar{F}(x), \alpha), i = 1,2,3,4,$ where $\bar{F}(x) = e^{-(0.5x)^{1.5}},\ x > 0,$ and $\alpha = 0.75$. Further, suppose that $\{I_{p_1}, I_{p_2}, I_{p_3}, I_{p_4}\}$ and $\{I_{q_1}, I_{q_2}, I_{q_3}, I_{q_4}\}$ are two sets of Bernoulli r.v.'s, independent of X_i's, $i = 1,2,3,4$. Set $(p_1,p_2,p_3,p_4) = (0.35, 0.65, 0.85, 0.96), (q_1,q_2,q_3,q_4) = (0.15, 0.35, 0.55, 0.82)$. Let $\kappa(x) = \log(1+x)$ which is strictly increasing and concave. We consider the transformation $x = t/(1-t)$. After this substitution, let us denote the respective reverse hazard functions by $\tilde{r}_{X^*_{n:n}}(t/(1-t) = \tilde{r}_1(t)$ and $\tilde{r}_{X^\circ_{n:n}}(t/(1-t)) = \tilde{r}_2(t)$. From Figure 7.7, it is seen that that $\tilde{r}_1(t) \geq \tilde{r}_2(t)$ for all $t \in [0,1)$. Hence $X^*_{n:n} \geq_{rh} X^\circ_{n:n}$.*

Next, we provide a counterexample to show that the ordering result in Theorem 7.2.7 may not hold if we relax the stated majorization conditions.

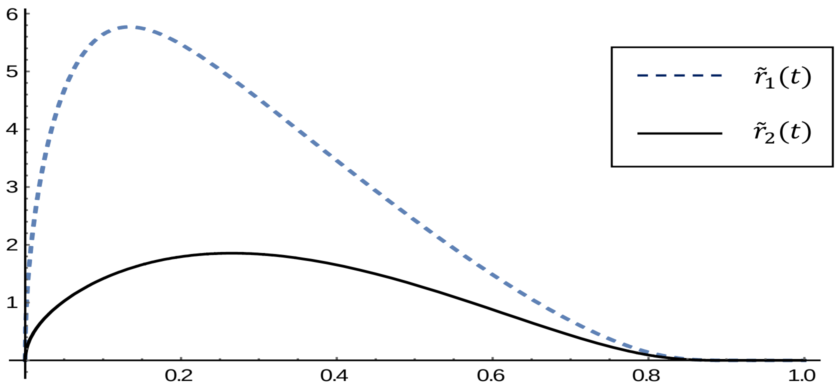

Figure 7.7: Plots of $\tilde{r}_1(t)$ and $\tilde{r}_2(t)$, $t \in [0, 1]$

Counterexample 7.2.3. *In Example 7.2.5, let us take* $(p_1, p_2, p_3, p_4) = (0.1, 0.2, 0.85, 0.95)$ *and* $(q_1, q_2, q_3, q_4) = (0.5, 0.65, 0.8, 0.85)$ *so that* $(\kappa(p_1), \kappa(p_2), \kappa(p_3), \kappa(p_4)) \not\succeq_w (\kappa(q_1), \kappa(q_2), \kappa(q_3), \kappa(q_4))$. *In Figure 7.8 we have plotted* $\tilde{r}_1(t) - \tilde{r}_2(t)$ *for all* $t \in [0, 1)$, *which shows that the hr ordering result of Theorem 7.2.7(ii) does not hold in this case.*

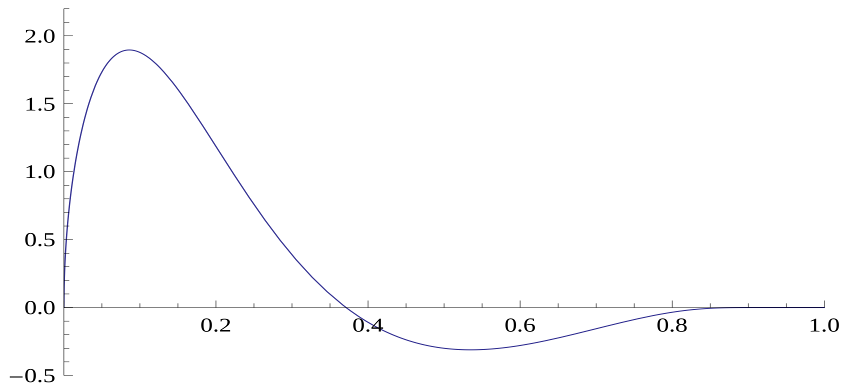

Figure 7.8: Plot of $\tilde{r}_1(t) - \tilde{r}_2(t)$, $t \in [0, 1]$

7.3 Star order for multiple-outlier claim

Let X_i, $i = 1, 2, \ldots, r$ have a common distribution F, and X_j, $j = r+1, r+2, \ldots, n$ have a common distribution G, where $r = 1, 2, \ldots, n-1$. This type of model is known as outlier model, where F is called the original distribution whereas the G is called the outlier distribution. For $r = 1, 2, \ldots, n-2$, it is called multiple-outlier model. In actuarial practice, even for a portfolio of risks consisting of similar kind of insureds, it may happen that some insureds have different (higher/lower) probabilities of occurrence of claims, claim sizes or odds of claims than the rest. Then this phenomena falls in the multiple-outlier claims model.

Star order is one of the most important transform order used to compare the skewness of probability distributions. Since in general the insurance claims follow positively skewed and heavy-tailed distributions, it is therefore of interest to establish sufficient conditions for star order between them to analyze the effects of the heterogeneity among occurrence probabilities and claim severity parameters (e.g. the odds ratio in our considered model) on the skewness of their distributions.

In Theorem 7.3.1, we derive stochastic comparisons on the largest claim amounts in case of multiple-outlier claims model with respect to star order.

The following lemma is derived from Saunders and Moran [119] which will be used in proving Theorem 7.3.1.

Lemma 7.3.1. *Let $\{G_\lambda | \lambda \in \mathbb{R}_+\}$ be a class of cdfs such that G_λ is supported on some interval $\mathbf{I} \subseteq \mathbb{R}_+$. Then, $G_\lambda \geq_\star G_{\lambda^\star}$ for $\lambda \leq \lambda^\star$ iff $\frac{\frac{\partial G_\lambda(x)}{\partial \lambda}}{x g_\lambda(x)}$ is increasing in x, where the density g_λ of G_λ does not vanishes on any subinterval of $\mathbf{I}$.*

Theorem 7.3.1. *Let $X_i \sim PO(\bar{F}, \alpha_1)$ ($Y_i \sim PO(\bar{F}, \beta_1)$), for $i = 1, 2, \ldots, n_1$, and let $X_j \sim PO(\bar{F}, \alpha_2)$ ($Y_j \sim PO(\bar{F}, \beta_2)$), for $j = n_1 + 1, n_1 + 2, \ldots, n_1 + n_2$ $(= n)$. Assume that X_i's are independent and that the Y_j's are independent. Further, let I_{p_i}, $i = 1, 2, \ldots, n_1$, be independent Bernoulli r.v.'s such that $\mathbb{E}[I_{p_i}] = p_1$, and let I_{p_j}, $j = n_1 + 1, \ldots, n$, be another set of independent Bernoulli r.v.'s such that $\mathbb{E}[I_{p_j}] = p_2$. Then, for $n_1 p_1 \geq n_2 p_2$, $p_1 \geq p_2$, $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$,*

$$\frac{\alpha_1}{\alpha_2} \leq \frac{\beta_1}{\beta_2} \implies X^*_{n:n} \geq_\star X^*_{n:n}.$$

Proof: Consider the following two cases.

Case I: Let $\alpha_1 + \alpha_2 = \beta_1 + \beta_2 = c$ (say). Further, let $\alpha_1 = \alpha \leq \alpha_2$ and $\beta_1 = \beta \leq \beta_2$ so that $\alpha \in [0, c/2]$. Then the cdf of $X^*_{n:n}$ is given by

$$F_{n,\alpha}(x) = \left[1 - p_1 \bar{F}_\alpha(x)\right]^{n_1} \left[1 - p_2 \bar{F}_{c-\alpha}(x)\right]^{n_2}. \tag{7.3.1}$$

Here $\bar{F}_\alpha(x) = \frac{\alpha \bar{F}(x)}{1 - \bar{\alpha}\bar{F}(x)}$ and $\bar{F}_{c-\alpha}(x) = \frac{(c-\alpha)\bar{F}(x)}{1 - \overline{(c-\alpha)}\bar{F}(x)}$

The pdf corresponding to (7.3.1) is

$$\begin{aligned} f_{n,\alpha}(x) &= \left[1 - p_1 \bar{F}_\alpha(x)\right]^{n_1-1} \left[1 - p_2 \bar{F}_{c-\alpha}(x)\right]^{n_2-1} f(x) \\ &\quad \times \left[\frac{n_1 p_1 \alpha (1 - p_2 \bar{F}_{c-\alpha}(x))}{(1 - \bar{\alpha}\bar{F}(x))^2} + \frac{n_2 p_2 (c-\alpha)(1 - p_1 \bar{F}_\alpha(x))}{(1 - \overline{(c-\alpha)}\bar{F}(x))^2} \right]. \end{aligned}$$

Now,

$$\begin{aligned} \frac{\partial F_{n,\alpha}(x)}{\partial \alpha} &= F(x)\bar{F}(x) \left[1 - p_1 \bar{F}_\alpha(x)\right]^{n_1-1} \left[1 - p_2 \bar{F}_{c-\alpha}(x)\right]^{n_2-1} \\ &\quad \times \left[\frac{-n_1 p_1 (1 - p_2 \bar{F}_{c-\alpha}(x))}{(1 - \bar{\alpha}\bar{F}(x))^2} + \frac{n_2 p_2 (1 - p_1 \bar{F}_\alpha(x))}{(1 - \overline{(c-\alpha)}\bar{F}(x))^2} \right] \end{aligned}$$

Let $\Lambda_1(x) = \frac{1-p_2\bar{F}_{c-\alpha}(x)}{(1-\bar{\alpha}\bar{F}(x))^2}$ and $\Lambda_2(x) = \frac{1-p_1\bar{F}_\alpha(x)}{(1-\overline{(c-\alpha)}\bar{F}(x))^2}$. Then, by using Lemma 7.3.1, it suffices to show that

$$\begin{aligned} \frac{\frac{\partial F_{n,\alpha}(x)}{\partial \alpha}}{x f_{n,\alpha}(x)} &= \frac{F(x)\bar{F}(x)}{x f(x)} \left[\frac{-n_1 p_1 \Lambda_1(x) + n_2 p_2 \Lambda_2(x)}{n_1 p_1 \alpha \Lambda_1(x) + n_2 p_2 (c-\alpha) \Lambda_2(x)} \right] \\ &= \frac{F(x)\bar{F}(x)}{x f(x)} \times \Lambda(x) \end{aligned}$$

is increasing in $x \in \mathbb{R}_+$, for $\alpha \in [0, c/2]$, where

$$\begin{aligned} \Lambda(x) &= \left[\frac{n_1 p_1 \alpha \Lambda_1(x) + n_2 p_2 (c-\alpha) \Lambda_2(x)}{-n_1 p_1 \Lambda_1(x) + n_2 p_2 \Lambda_2(x)} \right]^{-1} \\ &= \left[\frac{c n_2 p_2 \Lambda_2(x)}{n_2 p_2 \Lambda_2(x) - n_1 p_1 \Lambda_1(x)} - \alpha \right]^{-1} \\ &= \left[c \left(1 - \frac{n_1 p_1}{n_2 p_2} \frac{\Lambda_1(x)}{\Lambda_2(x)} \right)^{-1} - \alpha \right]^{-1}. \end{aligned}$$

Further, let

$$\begin{aligned} \Lambda_3(x) &= \frac{\Lambda_1(x)}{\Lambda_2(x)} \\ &= \left(\frac{1 - p_2 \bar{F}_{c-\alpha}(x)}{(1 - \bar{\alpha}\bar{F}(x))^2} \right) \Big/ \left(\frac{1 - p_1 \bar{F}_\lambda(x)}{(1 - \overline{(c-\alpha)}\bar{F}(x))^2} \right) \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{[1-\overline{(c-\alpha)}\bar{F}(x)-p_2(c-\alpha)\bar{F}(x)]}{(1-\bar{\alpha}\bar{F}(x))}\right)\left(\frac{(1-\bar{\alpha}\bar{F}(x))(1-\overline{(c-\alpha)}\bar{F}(x))^2}{[1-\bar{\alpha}\bar{F}(x)-p_1\alpha\bar{F}(x)]}\right)\\
&= \left(\frac{[F(x)+(1-p_2)(c-\alpha)\bar{F}(x)]}{[F(x)+(1-p_1)\alpha\bar{F}(x)}\right)\left(\frac{1-\overline{(c-\alpha)}\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}\right)\\
&= \Delta_1(x)\times\Delta_2(x).
\end{aligned}
$$

where $\Delta_1(x) = \left(\frac{[F(x)+(1-p_2)(c-\alpha)\bar{F}(x)]}{[F(x)+(1-p_1)\alpha\bar{F}(x)}\right)$ and $\Delta_2(x) = \left(\frac{1-\overline{(c-\alpha)}\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}\right)$. It is clear that $\Delta_1(x) \geq 0$ and $\Delta_2(x) \geq 0\ \forall x \in \mathbb{R}_+$. Now we have

$$
\begin{aligned}
\Delta_1'(x) &\overset{sign}{=} [1-(1-p_2)(c-\alpha)][F(x)+\alpha(1-p_1)\bar{F}(x)]\\
&- [1-\alpha(1-p_1)][F(x)+(1-p_2)(c-\alpha)\bar{F}(x)]\\
&\overset{sign}{=} [\alpha(1-p_1)-(1-p_2)(c-\alpha)](\bar{F}(x)+F(x))\\
&\overset{sign}{=} [\alpha(1-p_1)-(1-p_2)(c-\alpha)]\\
&\leq 0,
\end{aligned}
$$

which holds as $\alpha \leq c-\alpha$ and $p_1 \geq p_2$. Further,

$$
\begin{aligned}
\Delta_2'(x) &\overset{sign}{=} (\overline{c-\alpha})(1-\bar{\alpha}\bar{F}(x))(1-c\bar{-}\alpha\bar{F}(x))\\
&\overset{sign}{=} (c\bar{-}\alpha)-\bar{\alpha} \leq 0.
\end{aligned}
$$

Hence ultimately we have $\Lambda_3'(x) \leq 0$. Consequently $\Lambda_3(x)$ is non-negative and decreasing in x. Now

$$
\begin{aligned}
&\alpha \leq (c-\alpha)\\
&\implies \frac{1-\overline{(c-\alpha)}\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)} \geq 1,
\end{aligned}
$$

and

$$
\begin{aligned}
&(c-\alpha) \geq \alpha, p_1 \geq p_2\\
&\implies F(x)+(1-p_2)(c-\alpha)\bar{F}(x) \geq F(x)+(1-p_1)\alpha\bar{F}(x)
\end{aligned}
$$

and hence, $\Lambda_3(x) \geq 1$.

Now if $n_1 \geq n_2$ then $n_1p_1 \geq n_2p_2$. On combining all of these results, we have

$$\frac{n_1p_1}{n_2p_2}\Lambda_3(x) \geq 1$$
$$\implies \left(1 - \frac{n_1p_1}{n_2p_2}\Lambda_3(x)\right) \leq 0.$$

Hence $\left(1 - \frac{n_1p_1}{n_2p_2}\Lambda_3(x)\right)$ is increasing in x, which implies $\left[\left(1 - \frac{n_1p_1}{n_2p_2}\Lambda_3(x)\right)^{-1} - \alpha\right]^{-1}$ is increasing in x. So ultimately we have $\Lambda(x)$ is increasing in x. This completes the proof.
Case II: Let $\alpha_1+\alpha_2 \neq \beta_1+\beta_2$. In this case there exists some $\kappa > 0$ such that $\alpha_1+\alpha_2 = \kappa(\beta_1+\beta_2)$. Now, let $Z_{n:n}$ be the largest claim amount from $I_1Z_1, ..., I_{n_1}Z_{n_1}, I_{n_1+1}Z_{n_1+1}, ..., I_nZ_n$, where $Z_1, ..., Z_{n_1}$ have the distribution $F_{\kappa\mu_1}$ and $Z_{n_1+1}, ..., Z_n$ have the distribution $F_{\kappa\mu_2}$. Finally, on using the result of Case I and the scale invariant property of the star order, the desire result follows.

Example 7.3.1. Suppose that $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ are two sets of independent non-negative r.v.'s with $X_i \sim PO(\bar{F}(x), \alpha_i), i = 1, 2,$ and $Y_i \sim PO(\bar{F}(x), \beta_i), i = 1, 2,$ where $\bar{F}(x) = e^{-x},\ x > 0$. Set $(\alpha_1, \alpha_2) = (0.5, 1.9), (\beta_1, \beta_2) = (0.8, 1.2), (p_1, p_2) = (1/4, 1/8)$, $(n_1, n_2) = (3, 2)$. Then all the conditions of Theorem 7.3.1 are satisfied. In Figure 7.9, we have plotted $\frac{d}{dt}\left(\frac{F^{-1}_{X^*_{n:n}}(t)}{F^{-1}_{Y^*_{n:n}}(t)}\right)$ with respect to t from which it is clear that $\frac{F^{-1}_{X^*_{n:n}}(t)}{F^{-1}_{Y^*_{n:n}}(t)}$ is increasing for $t \in (0, 1)$. Hence $X^*_{n:n} \geq_\star Y^*_{n:n}$.

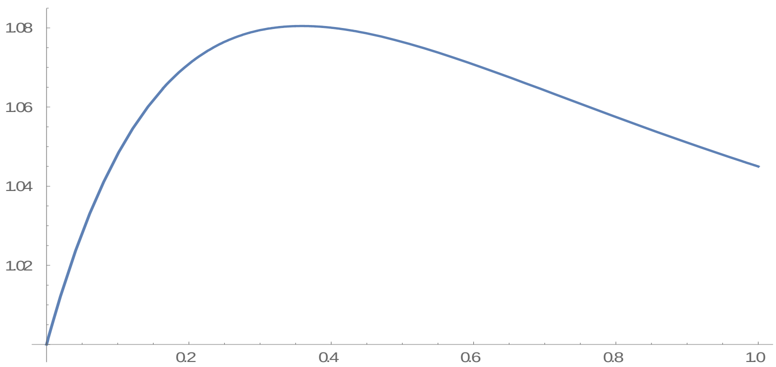

Figure 7.9: Plots of derivative of $\frac{F^{-1}_{X^*_{n:n}}(t)}{F^{-1}_{Y^*_{n:n}}(t)}$ with respect to t for $t \in (0, 1)$

7.4 Comparison of aggregate claim amount

The aggregate claim of a portfolio is the sum of all amounts payable during the reference period. Our next theorem derives sufficient conditions that the aggregate claim amount increases on reducing the heterogeneity in the sense of majorization among the concerned parameters of a considered semiparametric family of distributions when they are in ascending order. Here we assume that occurrence probabilities are arranged according to LWSAI.

Theorem 7.4.1. *Suppose that* $\boldsymbol{I} = (I_{p_1}, \ldots, I_{p_n})$ *is LWSAI. Let* $X_{\alpha_i} \sim \bar{F}(x;\alpha_i)$ $(X_{\beta_i} \sim \bar{F}(x;\beta_i)), i = 1, ..., n$, *be independent r.v.'s. Suppose that the following conditions hold:*

(i) $\bar{F}(x;\alpha)$ *is increasing and concave in* $\alpha > 0$*; and*

(ii) the sf of $X_{\mu_1} + X_{\mu_2}$ *is Schur-concave in* (μ_1, μ_2), $\mu_1, \mu_2 > 0$.

If $\boldsymbol{\alpha} \overset{m}{\succeq} \boldsymbol{\beta}$ *such that* $\alpha_1 \le \alpha_2 \le ... \le \alpha_n$ *and* $\beta_1 \le \beta_2 \le ... \le \beta_n$*; then* $\sum_{i=1}^n I_{p_i} X_{\alpha_i} \le_{st} \sum_{i=1}^n I_{p_i} X_{\beta_i}$.

Proof: Let $A(\boldsymbol{I}, \alpha) = \sum_{i=1}^n I_i X_{\alpha_i}$ and $A(\boldsymbol{I}, \boldsymbol{\beta}) = \sum_{i=1}^n I_i X_{\beta_i}$. We have to prove that $F_{A(\boldsymbol{I},\boldsymbol{\alpha})}(t) \ge F_{A(\boldsymbol{I},\boldsymbol{\beta})}(t)$ $\forall t \in \Re_+$. By the nature of majorization order, it suffices to prove it when $(\alpha_i, \alpha_j) \overset{m}{\succeq} (\beta_i, \beta_j)$ for some pair $1 \le i < j \le n$, and $\alpha_r = \beta_r$ for all $r \ne i, j$. The cdf of $A(\boldsymbol{I}, \boldsymbol{\alpha})$ is

$$
\begin{aligned}
F_{A(\boldsymbol{I},\boldsymbol{\alpha})}(t) &= \mathbb{P}\left(\sum_{i=1}^n I_{p_i} X_{\alpha_i} \le t\right) \\
&= \sum_{k=0}^n \sum_{\boldsymbol{\chi} \in S_k} \mathbb{P}\left(\sum_{i=1}^n I_{p_i} X_{\alpha_i} \le t \mid \boldsymbol{I} = \boldsymbol{\chi}\right) p(\boldsymbol{\chi}) \\
&= p(\mathbf{0}) + p(\mathbf{1})\mathbb{P}\left(\sum_{i=1}^n X_{\alpha_i} \le t\right) + \sum_{k=1}^n \sum_{\boldsymbol{\chi} \in S_k} p(\boldsymbol{\chi})\mathbb{P}\left(\sum_{i=1}^n \chi_i X_{\alpha_i} \le t\right) \\
&= p(\mathbf{0}) + p(\mathbf{1})\mathbb{P}\left(\sum_{i=1}^n X_{\alpha_i} \le t\right) + \sum_{k=1}^{n-1} \left\{ \sum_{\chi \in S_k^{i,j}(0,0)} p(\boldsymbol{\chi})\mathbb{P}\left(\sum_{r \ne i,j}^n \chi_r X_{\alpha_r} \le t\right) \right. \\
&+ \sum_{\chi \in S_k^{i,j}(0,1)} p(\boldsymbol{\chi})\mathbb{P}\left(X_{\alpha_j} + \sum_{r \ne i,j}^n \chi_r X_{\alpha_r} \le t\right) \\
&+ \sum_{\chi \in S_k^{i,j}(0,1)} p(\tau_{i,j}(\boldsymbol{\chi}))\mathbb{P}\left(X_{\alpha_i} + \sum_{r \ne i,j}^n \chi_r X_{\alpha_r} \le t\right)
\end{aligned}
$$

$$+ \sum_{\chi \in S_k^{i,j}(1,1)} p(\boldsymbol{\chi}) \mathbb{P}\left(X_{\alpha_i} + X_{\alpha_j} + \sum_{r \neq i,j}^{n} \chi_r X_{\alpha_r} \leq t \right) \Bigg\}.$$

Similarly,

$$\begin{aligned} F_{A(\boldsymbol{I},\boldsymbol{\beta})}(t) &= \mathbb{P}\left(\sum_{i=1}^{n} I_i X_{\beta_i} \leq t \right) \\ &= p(\mathbf{0}) + p(\mathbf{1}) \mathbb{P}\left(\sum_{i=1}^{n} X_{\beta_i} \leq t \right) + \sum_{k=1}^{n-1} \Bigg\{ \sum_{\chi \in S_k^{i,j}(0,0)} p(\boldsymbol{\chi}) \mathbb{P}\left(\sum_{r \neq i,j}^{n} \chi_r X_{\beta_r} \leq t \right) \\ &+ \sum_{\chi \in S_k^{i,j}(0,1)} p(\boldsymbol{\chi}) \mathbb{P}\left(X_{\beta_j} + \sum_{r \neq i,j}^{n} \chi_r X_{\beta_r} \leq t \right) \\ &+ \sum_{\chi \in S_k^{i,j}(0,1)} p(\tau_{i,j}(\boldsymbol{\chi})) \mathbb{P}\left(X_{\beta_i} + \sum_{r \neq i,j}^{n} \chi_r X_{\beta_r} \leq t \right) \\ &+ \sum_{\chi \in S_k^{i,j}(1,1)} p(\boldsymbol{\chi}) \mathbb{P}\left(X_{\beta_i} + X_{\beta_j} + \sum_{r \neq i,j}^{n} \chi_r X_{\beta_r} \leq t \right) \Bigg\}. \end{aligned}$$

Under assumption (ii), it holds that

$$\mathbb{P}\left(\sum_{i=1}^{n} X_{\alpha_i} \leq t \right) \geq \mathbb{P}\left(\sum_{i=1}^{n} X_{\beta_i} \leq t \right) \tag{7.4.1}$$

and, for any $\boldsymbol{\chi} \in S_k^{i,j}(1,1), k = 1, ..., n-1$,

$$\mathbb{P}\left(X_{\alpha_i} + X_{\alpha_j} + \sum_{r \neq i,j}^{n} \chi_r X_{\alpha_i} \leq t \right) \geq \mathbb{P}\left(X_{\beta_i} + X_{\beta_j} + \sum_{r \neq i,j}^{n} \chi_r X_{\beta_i} \leq t \right). \tag{7.4.2}$$

Then combining above two we have

$$\begin{aligned} &F_{A(\boldsymbol{I},\boldsymbol{\alpha})}(t) - F_{A(\boldsymbol{I},\boldsymbol{\beta})}(t) \\ &= p(\mathbf{1}) \left[\mathbb{P}\left(\sum_{i=1}^{n} X_{\alpha_i} \leq t \right) - \mathbb{P}\left(\sum_{i=1}^{n} X_{\beta_i} \leq t \right) \right] \\ &+ \sum_{k=1}^{n-1} \Bigg\{ \sum_{\boldsymbol{\chi} \in S_k^{i,j}(0,1)} p(\boldsymbol{\chi}) \left[\mathbb{P}\left(X_{\alpha_j} + \sum_{r \neq i,j}^{n} \chi_r X_{\alpha_r} \leq t \right) - \mathbb{P}\left(X_{\beta_j} + \sum_{r \neq i,j}^{n} \chi_r X_{\alpha_r} \leq t \right) \right] \end{aligned}$$

$$
\begin{aligned}
&+\sum_{\boldsymbol{\chi}\in S_k^{i,j}(0,1)} p(\tau_{i,j}\boldsymbol{\chi})\left[\mathbb{P}\left(X_{\alpha_i}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\le t\right)-\mathbb{P}\left(X_{\beta_i}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\le t\right)\right]\\
&+\sum_{\chi\in S_k^{i,j}(1,1)} p(\boldsymbol{\chi})\left[\mathbb{P}\left(X_{\alpha_i}+X_{\alpha_j}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_i}\le t\right)\ge\mathbb{P}\left(X_{\beta_i}+X_{\beta_j}+\sum_{r\neq i,j}^{n}\chi_r X_{\beta_i}\le t\right)\right]\Bigg\}\\
&\ge\sum_{k=1}^{n-1}\Bigg\{\sum_{\boldsymbol{\chi}\in S_k^{i,j}(0,1)} p(\boldsymbol{\chi})\left[\mathbb{P}\left(X_{\alpha_j}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\le t\right)-\mathbb{P}\left(X_{\beta_j}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\le t\right)\right]\\
&+\sum_{\boldsymbol{\chi}\in S_k^{i,j}(0,1)} p(\tau_{i,j}\boldsymbol{\chi})\left[\mathbb{P}\left(X_{\alpha_i}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\le t\right)-\mathbb{P}\left(X_{\beta_i}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\le t\right)\right]\Bigg\}\\
&\ge\sum_{k=1}^{n-1}\Bigg\{\sum_{\boldsymbol{\chi}\in S_k^{i,j}(0,1)} p(\tau_{i,j}\boldsymbol{\chi})\left[\mathbb{P}\left(X_{\alpha_j}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\le t\right)-\mathbb{P}\left(X_{\beta_j}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\le t\right)\right]\\
&+\sum_{\boldsymbol{\chi}\in S_k^{i,j}(0,1)} p(\tau_{i,j}\boldsymbol{\chi})\left[\mathbb{P}\left(X_{\alpha_i}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\le t\right)-\mathbb{P}\left(X_{\beta_i}+\sum_{r\neq i,j}^{n}\chi_r X_{\alpha_r}\le t\right)\right]\Bigg\}\\
=&\sum_{k=1}^{n-1}\Bigg\{\sum_{\boldsymbol{\chi}\in S_k^{i,j}(0,1)} p(\tau_{i,j}\boldsymbol{\chi})\int\cdots\int_{\mathbb{R}_+^{n-2}}\left[\mathbb{P}\left(X_{\alpha_i}\le t-\sum_{r\neq i,j}^{n}\chi_r x_{\alpha_r}\right)+\mathbb{P}\left(X_{\alpha_j}\le t-\sum_{r\neq i,j}^{n}\chi_r x_{\alpha_r}\right)\right.\\
&\left.-\mathbb{P}\left(X_{\beta_i}\le t-\sum_{r\neq i,j}^{n}\chi_r x_{\alpha_r}\right)-\mathbb{P}\left(X_{\beta_j}\le t-\sum_{r\neq i,j}^{n}\chi_r x_{\alpha_r}\right)\right]\prod_{r\neq i,j}^{n} g_{X_{\alpha_r}}(x_{\alpha_r})dx_{\alpha_r}\Bigg\}\\
=&\sum_{k=1}^{n-1}\Bigg\{\sum_{\boldsymbol{\chi}\in S_k^{i,j}(0,1)} p(\tau_{i,j}\boldsymbol{\chi})\int\cdots\int_{\mathbb{R}_+^{n-2}}\left[\bar{F}_{X_{\beta_i}}\left(t-\sum_{r\neq i,j}^{n}\chi_r x_{\alpha_r}\right)+\bar{F}_{X_{\beta_j}}\left(t-\sum_{r\neq i,j}^{n}\chi_r x_{\alpha_r}\right)\right.\\
&\left.-\bar{F}_{X_{\alpha_i}}\left(t-\sum_{r\neq i,j}^{n}\chi_r x_{\alpha_r}\right)-\bar{F}_{X_{\alpha_j}}\left(t-\sum_{r\neq i,j}^{n}\chi_r x_{\alpha_r}\right)\right]\prod_{r\neq i,j}^{n} g_{X_{\alpha_r}}(x_{\alpha_r})dx_{\alpha_r}\Bigg\}\\
&\ge 0,
\end{aligned}
$$

where $g_{X_{\alpha_r}}(x)$ is the pdf of X_{α_r}. The first inequality follows from (7.4.1) and (7.4.2), the second inequality from Lemma 7.2.1, and finally the last inequality is due to the fact that $\bar{F}_{X_\alpha}$ is concave in $\alpha\in\mathbb{R}_+$ as per the assumption (i).

Remark 7.4.1. *Theorem 7.4.1 hold true for the PO model, i.e. for* $X_{\alpha_i}\sim PO(\bar{F},\alpha_i)$ $(X_{\beta_i}\sim PO(\bar{F},\beta_i)), i=1,...,n,$ *with* $\bar{F}(t)=e^{-\lambda t}$, $\lambda>0$. *This family of distribution is known as Marshall–Olkin extended exponential (MOEE) distribution. Note that* $\bar{F}_{X_\alpha}(t)=\frac{\alpha\bar{F}(t)}{1-\bar{\alpha}\bar{F}(t)}$ *is increasing and concave in* α. *The condition (ii), i.e. the sf of* $X_{\mu_1}+X_{\mu_2}$ *is Schur-concave in* (μ_1,μ_2) *follows from the Corrollary F.12.a. (p. 235) of Marshall et al. [96] with the fact that both the sf* $\bar{F}_{X_\mu}(t)=\frac{\mu e^{-\lambda t}}{1-\bar{\mu}e^{-\lambda t}}$ *and pdf* $f_{X_\mu}(t)=\frac{\mu\cdot\lambda e^{-\lambda t}}{(1-\bar{\mu}e^{-\lambda t})^2}$ *are concave in* μ.

It is to be noted that exponentiated Weibull distribution having the cdf $F(t;\alpha,\beta)) = \left(1-e^{-t^{\beta}}\right)^{\alpha}$*,* $\alpha,\beta > 0$ *also satisfy both the conditions (i) and (ii) of Theorem 7.4.1 with respect to the parameter* α*.*

Remark 7.4.2. *It is worth to be mention that in Theorem 4.7 of Zhang et al. [136], they compared aggregate claim amounts of two sets of heterogeneous portfolios under the assumption that the survival function* $\bar{F}(x;\alpha)$ *is decreasing and convex in* $\alpha > 0$*.*

The following example illustrates the result given in Theorem 7.4.1.

Example 7.4.1. Suppose that $\{X_1, X_2\}$ and $\{Y_1, Y_2\}$ are two sets of independent non-negative r.v.'s with $X_i \sim PO(\bar{F}(x), \alpha_i), i = 1,2,$ and $Y_i \sim PO(\bar{F}(x), \beta_i), i = 1,2,$ where $\bar{F}(x) = e^{-0.3x},\ x > 0$. Set $(\alpha_1,\alpha_2) = (0.4, 2.6), (\beta_1,\beta_2) = (0.8, 2.2), p(0,0) = P(I_{p_1} = 0, I_{p_2} = 0) = 0.15, p(0,1) = 0.46, p(1,0) = 0.34, p(1,1) = 0.05$. Then $I = \{I_{p_1}, I_{p_2}\}$ is LWSAI. We consider the transformation $x = t/(1-t)$ so that, for $t \in [0,1)$, we have $x \in [0,\infty)$. After this substitution, we denote the cdfs of $\sum_{i=1}^{2} I_{p_i} X_{\alpha_i}$ and $\sum_{i=1}^{2} I_{p_i} X_{\beta_i}$ by $F_{A(I,\alpha)}$ and $F_{A(I,\beta)}$ respectively. $F_{A(I,\alpha)}(t/(1-t)) = \psi_1(t)$ and $F_{A(I,\beta)}(t/(1-t)) = \psi_2(t)$, respectively. From Figure 7.10, it is clear that $\psi_1(t) \geq \psi_2(t)$ for all $t \in [0,1)$. Hence $\sum_{i=1}^{2} I_{p_i} X_{\alpha_i} \leq_{st} \sum_{i=1}^{2} I_{p_i} X_{\beta_i}$.

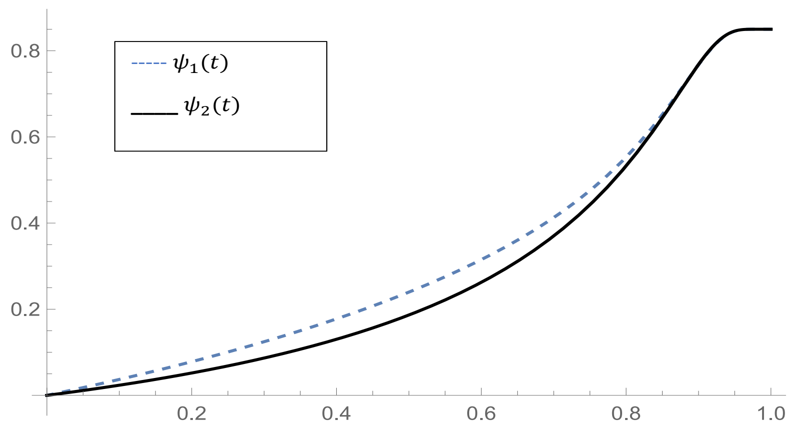

Figure 7.10: Plots of $\psi_1(t)$ and $\psi_2(t)$, $t \in [0,1]$

Chapter 8

Future Research Direction

In this chapter we discuss some specific research problems, which can be taken up as future research work.

(i) Stochastic orders are very useful tool to compare the lifetimes of two systems. Many different kinds of stochastic orders have been developed in the literature, for example, likelihood ratio order, hazard rate order, usual stochastic order, etc. (cf. Shaked and Shanthikumar [122]). Blyth [20] proposed a new stochastic order called stochastic precedence order (or probabilistic order). Many applications of this order are found in reliability theory (cf. Singh and Misra [125], Boland *et al.* [24]). But the detailed properties of this order are not studied yet. Thus, our aim is to study this order in detail.

(ii) Order statistics play an important role in reliability theory. Some comparison results of smallest and largest order statistics from exponential distribution, gamma distribution and Weibull distribution are studied in the literature (cf. Balakrishnan and Balakrishnan and Zhao [11], and the references there in). We want to study the properties of k-th order statistic so that results related to the smallest and the largest order statistics will come as particular cases. In addition to this, our goal is to study the order statistics from some other distributions, namely, generalized gamma distribution, power generalized Weibull distribution, etc. Order statistics from scale model, proportional hazards model, proportional reversed hazards model are also another prime interest of our research.

(iii) The concept of sequential order statistics is a generalized concept of order statistics (cf. Kamps [68]).Bairamov and Arnold [5] studied various stochastic orderings and ageing properties of residual life lengths of live components in a k-out-of-n system. We want to study the same problem under the sequential order statistics framework.

(iv) Design engineers always try to find out different strategies to allocate the standby component(s) into the system so that the system reliability becomes optimum. Cha et al. [32] proposed a new technique to handle the general standby (or warm standby) based on the concept of accelerated life model and virtual age model. We want to study different strategies to allocate the redundant component(s) into the system (for example, series system, parallel system and k-out-of-n system) under the different environmental set up, for example, perfect switching, imperfect switching, random warm-up period case.

(v) Multivariate hazard rate function is well studied in the literature (cf. Marshall [93]). A similar kind of study for multivariate reversed hazard rate function may be taken up as a future research project.

(vi) In the field of reliability and survival analysis, mean residual life (MRL) is a very well known and central concept. It plays an important role in reliability theory and survival analysis. To model parametric and/or nonparametric lifetime data, the lifetime distributions having decreasing, increasing or bathtub-shaped MRL are used. We want to model different lifetime distributions through MRL function and study their properties.

(vii) It must be mentioned here that the dependent structure of systems are more practical in nature, and where ever, in the above problems, dependency makes sense, we also shall study the same for the dependent components.

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List of Publications from the Thesis

- **Published:**

  1. Panja, A., Kundu, P., & Pradhan, B. (2021). Stochastic comparisons of lifetimes of series and parallel systems with dependent and heterogeneous components. Operations Research Letters, 49(2), 176-183.

  2. Panja, A., Kundu, P., & Pradhan, B. (2022). On stochastic comparisons of finite mixture models. Stochastic Models, 38(2), 190-213.

  3. Panja, A., Kundu, P., & Pradhan, B. (2022). Dispersive and star ordering of sample extremes from dependent random variables following the proportional odds model. Communications in Statistics-Theory and Methods, DOI: 10.1080/03610926.2022.2037643.

  4. Panja, A., Kundu, P., Hazra, N. K., & Pradhan, B. (2023). Stochastic comparisons of largest claim and aggregate claim amounts. Probability in the Engineering and Informational Sciences, DOI: 10.1017/S0269964823000104.

  5. Panja, A., Kundu, P., & Pradhan, B. (2023).Stochastic comparisons of coherent systems with active redundancy at the component or system levels and component lifetimes following the accelerated life model. Applied Stochastic Models in Business and Industry, DOI: 10.1002/asmb.2822.

- **Submitted:**

  1. Panja, A., Kundu, P., & Pradhan, B. (2022). Some stochastic comparison results for frailty and resilience models.

  2. Panja, A., Kundu, P., & Pradhan, B. (2022).Comparisons of coherent systems with active redundancy and component lifetimes following the proportional odds model.