Quantum symmetry in multigraphs and its applications in physical models

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Dedicated to the memory of Prof. V.R. Padmawar

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Chapter 1

Introduction

The idea of quantum groups was introduced by Drinfeld and Jimbo ([Dri87], [Dri89],[Jim85]). It was done on an algebraic level where quantum groups were viewed as Hopf algebras typically arising as deformations of semisimple Lie algebras. The analytic version of quantum groups was first described by Woronowicz ([Wor87], [Wor98]) who formulated the notion of **compact quantum group** as a generalization of a compact topological group in the non-commutative realm. Let us start by a brief description of the notion of a compact quantum group (CQG in short).

Let G be a compact topological group and let us consider the unital algebra $C(G)$ consisting of continuous complex valued functions on G. Instead of looking at the multiplication in G, we look at the co-multiplication map $\Delta : C(G) \to C(G) \otimes C(G) (\cong C(G \times G))$ which is a C* algebra homomorphism obtained by dualizing the multiplication map in G. The associativity of the multiplication map induces a co-associativity condition on the C* algebra homomorphism Δ and the inversion map in G can be characterised by two density conditions imposed on Δ. In the spirit of noncommutative geometry, a compact quantum group is pair $(\mathcal{A}, \Delta)$ where $\mathcal{A}$ is a unital C* algebra (possibly noncommutative) along with a co-product $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ satisfying co-associativity and the density conditions we have talked about before. If $\mathcal{A}$ is a commutative C* algebra, then from Gelfand Neimark construction it follows that, there exists a compact topological group G such that $\mathcal{A} \cong C(G)$ and the co-product Δ on $\mathcal{A}$ arises precisely by dualizing the multiplication in G.

Groups are often viewed as "symmetry objects", in a similar way, quantum groups correspond to some kind of "generalized symmetry" of physical systems and mathematical structures. Indeed, the idea of a group acting on a space can be extended to the idea of a quantum group co-acting on a noncommutative space (that is, possibly a noncommutative C*algebra). The question of defining and finding "all quantum symmetries" arises naturally in this context.

Such an approach was first taken by Manin though purely in an algebraic framework ([Man88], [Man87]). Study of quantum symmetry in analytic setting, that is, in the framework of compact quantum groups was first started by Shouzhou Wang. In 1997, Wang introduced the notion of quantum symmetry in finite spaces ([Wan98]), that is, finite dimensional C* algebras. In particular, he described the notion of **quantum permutations** (in the category of compact quantum groups) of n objects and defined **quantum permutation group** S_n^+ as the universal object in the category of all such **quantum permutations**. S_n^+ is indeed the compact quantum analogue of the standard permutation group S_n on n elements.

After describing **quantum automorphisms** on a finite space consisting of n points, it was natural to look into the notions of **quantum automorphisms** in finite graphs and small metric spaces. In 2003, the notion of **quantum automorphism** in a finite directed simple graph (V, E) was first introduced by Bichon ([Bic03]). It was formulated in terms of simultaneous **quantum permutations** of both edge set E and vertex set V which were compatible through source and target maps of a directed graph. Here by **"simple"** we mean only the absence of multiple edges between a fixed pair of vertices (not necessarily a distinct pair), not the absence of loops (a loop is an edge with a single endpoint vertex). We will maintain this convention about **simple graphs** throughout this thesis. Two years later, in [Ban05a] Banica gave a more general notion of quantum symmetry in a simple graph in terms of its adjacency matrix. Any **quantum permutation** of vertex set which commutes with adjacency matrix is a **quantum automorphism** of the simple graph in Banica's sense. As there was absolutely no restriction on the entries of the adjacency matrix, this construction was generalised easily to produce **quantum automorphisms** in the context of weighted simple graphs and small metric spaces ([Ban05b]). For a simple graph (V, E), the categories of **quantum automorphisms** described by Bichon and Banica will be denoted as $\mathcal{D}^{Ban}_{(V,E)}$ and $\mathcal{D}^{Bic}_{(V,E)}$ respectively. It turned out that $\mathcal{D}^{Bic}_{(V,E)}$ is always a full subcategory of $\mathcal{D}^{Ban}_{(V,E)}$.

Starting from the existence of quantum symmetry in simple graphs to explicit computations of quantum automorphism groups, there has been quite a lot of work done in this direction in recent years. We review some results here. In [BB07b] Banica computed the **quantum automorphism groups** of **vertex-transitive** graphs with less than or equal to eleven vertices except for **Petersen graph**. Later in [Sch18], Schmidt showed that Petersen graph does not have any quantum symmetry. In [BBC07b] a new universal quantum group, **hyperoctahedral quantum group** H_n^+ were added to the Wang's series of **universal quantum groups** ([Wan98],[Wan95]), which arose as **quantum automorphism groups** of **hypercubes** (n-dimensional analogue of a square or a cube). In [BBC07a] Banica and collaborators showed that there is no quantum symmetry in **circulant graphs** where number of vertices is prime and satisfies a certain condition. Quantum symmetries in **circulant** and **vertex-transitive** graphs were also studied by Chassaniol in [Cha]. In [Cha16], he described the **qauntum automorphism groups** of **lexicographic product** of two finite regular graphs. In [Ful06], Fulton computed **quantum automorphism groups** of undirected trees with some specific classical automorphism groups. Later various computations for interesting classes of graphs were done by Schmidt in [Sch20c], [Sch20a]. In [LMR20], Lupini, Mancinska and Roberson showed that almost all graphs have trivial quantum automorphism group which is quantum analogue of a well-known result in [ER63] proved by Erdos and Renyi. There was also a contemporary result in [ER63] saying that "almost all trees have non-trivial symmetry" which was generalised in quantum sense by Junk, Schmidt and Weber in [JSW20]. There has also been some recent developments in finding perfect **quantum strategies** for the **graph isomorphism game** using quantum symmetry in graphs leading to a deep connection between **quantum information theory** and the theory of **quantum groups**. For an interested reader, we list some references here, [MR14],[MRV19],[BCE+20][AMR+19], [LMR20]. Recently, Vaes and Rollier in [RV22] introduced the notion of quantum symmetry and described **quantum automorphism groups** of **connected locally finite infinite graphs**.

It is natural to ask whether Banica and Bichon's notions of **quantum automorphisms** can be generalised in the context of multigraphs. A multigraph or a finite quiver (V,E) consists of a finite vertex set V and a finte edge set E with source and target maps $s : E \to V$ and $t : E \to V$. Classically an automorphism of a multigraph is pair (f_V, f_E) where f_V and f_E are permutations of vertex set and edge set respectively which are compatible via source and target maps s and t. In case of simple graphs, the formulations of quantum symmetry were done in terms of permutation of vertices and adjacency relations between two vertices. Same technique fails to work for multigraphs as there can be multiple edges between any two vertices.

We have reformulated the notions of quantum symmetry in a simple graph in terms of "permutations" of edges instead of permutations of vertices which can be easily generalised in the context of multigraphs.

For a multigraph (V,E), we have constructed three different categories $\mathcal{C}^{Ban}_{(V,E)}$, $\mathcal{C}^{sym}_{(V,E)}$ and $\mathcal{C}^{Bic}_{(V,E)}$ consisting of compact quantum groups co-acting by preserving different levels of quantum symmetry in (V,E). If (V,E) is simple, then it turns out that $\mathcal{C}^{Ban}_{(V,E)} = \mathcal{C}^{sym}_{(V,E)} = \mathcal{D}^{Ban}_{(V,E)}$ and $\mathcal{C}^{Bic}_{(V,E)} = \mathcal{D}^{Bic}_{(V,E)}$. For a multigraph (V,E), we have the following :

$$\mathcal{C}^{Bic}_{(V,E)} \subseteq \mathcal{C}^{sym}_{(V,E)} \subseteq \mathcal{C}^{Ban}_{(V,E)}.$$

We have shown that the categories $\mathcal{C}^{Bic}_{(V,E)}$ and $\mathcal{C}^{Ban}_{(V,E)}$ admit universal objects namely $Q^{Bic}_{(V,E)}$ and $Q^{Ban}_{(V,E)}$. However that is not the case for $\mathcal{C}^{sym}_{(V,E)}$. It is still unclear whether for an arbitrary multigraph (V,E), the category $\mathcal{C}^{sym}_{(V,E)}$ admits a universal object or not. The compact quantum group $Q^{Bic}_{(V,E)}$ is the **quantum automorphism group** of (V,E) which is a quantum analogue of the classical automorphism group of (V,E). On the other hand, $Q^{Ban}_{(V,E)}$ is too large to be called an **"automorphism group of (V,E)"** and therefore will be referred to as **universal quantum group associated with (V,E)**.

We have also described these **quantum automorphisms** in the context of undirected multigraphs. An undirected multigraph (V,E) consists of an edge set E and vertex set V and a map $r : E \to \{\{x,y\}|x,y \in V\}$ which assigns each edge to an unordered pair of endpoint vertices.

It is natural to ask for which class of multigraphs, the two categories $\mathcal{C}^{sym}_{(V,E)}$ and $\mathcal{C}^{Bic}_{(V,E)}$ coincide. We have provided a necessary and sufficient condition in terms of weighted symmetry of the **underlying simple graph**. We have shown that for a uniform multigraph (V,E)(i.e. either 0 or a fixed number of edges between any two vertices) the categories $\mathcal{C}^{sym}_{(V,E)}$ and $\mathcal{C}^{Bic}_{(V,E)}$ coincide if and only if $\mathcal{D}^{Ban}_{(V,\overline{E})}$ and $\mathcal{D}^{Bic}_{(V,\overline{E})}$ coincide where $(V,\overline{E})$ is the **underlying simple graph** of (V,E). For this class of multigraphs, the compact quantum group $Q^{Bic}_{(V,E)}$ does act as a universal object in $\mathcal{C}^{sym}_{(V,E)}$. We have expressed $Q^{Bic}_{(V,E)}$ as **free wreath product by quantum permutation groups** ([Bic04], [BB07a]) where the co-action corresponding to the wreath product comes from a permutation of pairs of vertices induced by the weighted symmetry of the **underlying simple graph**. This wreath product formula for $Q^{Bic}_{(V,E)}$ also emphasizes that any multigraph which have at least two pairs of vertices with multiple edges among them possesses genuine quantum symmetry.

In search of members of $\mathcal{C}^{sym}_{(V,E)}$ which are essentially **non-Bichon** (that is, not a member of $\mathcal{C}^{Bic}_{(V,E)}$), we have stumbled upon a particular class of automorphisms of a multigraph (V,E), namely **source and target dependent automorphisms of (V,E)**. Vaguely speaking, an automorphism of a multigraph (V,E) is source dependent if permutation of an edge depends on the source of that edge and similar description goes for the target dependent case. We have constructed a series of compact quantum groups $Q^{s}_{(V,E)}$, $Q^{t}_{(V,E)}$ and $Q^{s,t}_{(V,E)}$ consisting of **source and target dependent quantum automorphisms of (V,E)** which are all **non-Bichon** members of $\mathcal{C}^{Ban}_{(V,E)}$. Among all these quantum groups, $Q^{s,t}_{(V,E)}$ turns out to be a member of $\mathcal{C}^{sym}_{(V,E)}$.

There has been an extensive study of quantum symmetry in **graph C* algebras** in recent times (see [PR06], [BS13],[JM18], [SW18],[BEVW22]). Following the line of [SW18] we have shown that our notions of quantum symmetry in multigraphs in fact lift to the level of graph

C* algebras. We have also shown that our ideas of quantum symmetry fit the framework of **co-actions on C* correspondences** (see [Kat04], [BJ],[KQR15]) coming naturally from multigraphs.

Apart from mathematical structures, multigraphs are also important in many physical models such as lattices of atoms with double or triple bonds. We have extended our work of **Quantum symmetry on q-state Potts model** ([GAH22]) in the context of multigraphs.

Potts model, or more general vertex and spin models are some of the most popular and useful models arising primarily in statistical (including quantum statistical) mechanics, but they have found wide applications in many other areas of physics and even other scientific (including social sciences) disciplines. Typically, Potts model is considered on infinite lattices or infinite simple graphs where thermodynamic properties are analysed. However, we have considered the notion of **Potts model on a finite multigraph**. As we have mentioned before, physically a finite multigraph can correspond to molecules consisting of double or triple bonds.

Now we look into the theory of **phase transitions**. Instead of looking at phase transition as breaking of continuity or smoothness of suitable thermodynamic properties, we will be considering the **symmetry viewpoint** of Landau in [LL69] which says that a change of the group of symmetry of the underlying physical system signifies a change of phase. For example, gaseous phase has lot more symmetry than liquid which is constrained to have a fixed volume. Similarly, liquid has more symmetry than solid. There are also several models in condensed matter physics where there are theoretical and experimental explanations of effect of doping-induced phase transition in terms of change of the point symmetry group of the underlying crystal structures (see for example, [PMAO12]). We are proposing an extension of such ideas in the context of quantum group symmetry seen as generalised group symmetry.

It should be noted that the interplay of quantum groups and operator algebras with the models (including Potts model) of statistical mechanics goes back to the seminal work of Jones, [Jon89], (see also [Jon94a]) leading to a theory which connects physical models with subfactor theory, quantum field theory and so on. Later in [Ban98], Banica showed that spin and vertex models of Jones do come from quantum groups. Our approach to quantum symmetry in Potts model is different in a way that we are considering a compact quantum group co-acting on the set of vertices of the graph commuting with the Hamiltonian of the model instead of an on-site symmetry coming from the permutation of the set of states or the commutator with the site-to-site transfer matrices as in [Jon94b].

Now we give a brief overview of the chapters in this thesis.

In chapter 2, we recollect all the basic concepts we will need in the later chapters to make this thesis as self-content as possible. In section 2.1 we describe the basic notions related to simple graphs, multigraphs, adjacency relations and automorphism of graphs. As we have mentioned earlier, our simple graphs may have loops (an edge with same endpoints) but no multiple edges between a pair of vertices. We also describe "undirectedness" in graphs in such a way that will be helpful to us in later chapters. In section 2.2 we provide a very brief discussion about **q-state Potts model**. Next two sections are about basic concepts and examples related to compact quantum groups. In section 2.5, we discuss two different formulations of quantum symmetry in a simple graph given by Bichon and Banica ([Bic03], [Ban05a]) and prove some useful results related to them. Along with all the notations introduced in previous sections, in section 2.7, we introduce some additional notations related to multigraphs which we will be using extensively throughout this thesis.

In chapter 3 we reformulate previously described notions of quantum symmetry in a simple graph in terms of "permutations" of edges (instead of vertices) which can be easily generalised in the context of a multigraph. In section 3.1 we formulate quantum automorphisms in terms of **left and right equivariant bi-unitary co-representations**. The next two sections provide another equivalent description of **left and right equivariant co-representations** in terms of **implemented co-actions of a bi-unitary** (see definition 2.3.6) which helps us see how a "permutation" of edges preserving the symmetry of the multigraph induces permutations on the sets of initial and final vertices. With the theorems discussed in previous sections, in section 3.5, we find another additional constraint to capture the complete picture of already known notions of quantum symmetry in a simple graph. Section 3.6 connects bi-unitarity with our "inversion" map in an undirected multigraph (see definition 2.1.7). Finally, in section 3.7 we give our equivalent definitions of quantum symmetry in a simple graph in terms of a bi-unitary map on a suitable Hilbert space and its induced co-actions on the algebra of operators on that Hilbert space.

In chapter 4 we define the notions of quantum symmetry in a multigraph. In section 4.1, we see that most of the results derived in chapter 3 hold in case of a multigraph with verbatim proofs. Using these, in the section 4.2, we define three levels of quantum symmetry in a finite multigraph. In subsection 4.2.3, the propositions 4.2.7 and 4.2.9 tell us that our description of quantum symmetry is a correct generalization of already existing notions of quantum symmetry in simple graphs. In subsection 4.2.4 we look into the condition (5) of definition 4.2.3 and investigate whether it can be replaced with a nicer one. The next two sections 4.4 and 4.5 introduce different **quantum automorphism groups** of a multigraph corresponding to different levels of quantum symmetry. In section 4.6 we see that "bi-unitarity" is closely related to inversion in an undirected multigraph which has been already observed for simple graphs in chapter 3. From theorem 4.6.11, we see that quantum automorphism of any directed multigraph can be realised as a quantum automorphism of the "underlying undirected multigraph" preserving the set of directed edges.

In chapter 5, we dive deep into the structures of objects in $\mathcal{C}^{sym}_{(V,E)}$ and $\mathcal{C}^{Bic}_{(V,E)}$. From propositions 5.2.1 and 5.2.2, it follows that co-actions in both categories $\mathcal{C}^{sym}_{(V,E)}$ and $\mathcal{C}^{Bic}_{(V,E)}$ preserve **uniform components** of (V,E) (see notation 5.1.2). Hence it suffices to look into the **co-actions on uniform multigraphs**. In subsection 5.3.2, proposition 5.3.7 gives us more insight into the modified notion of "permutation" of edges we have been considering till now. The rest of the chapter concerns with some interesting structural results about $Q^{Bic}_{(V,E)}$. In theorem 5.3.13, we see that for a directed uniform multigraph (V,E), $Q^{Bic}_{(V,E)}$ is the **free wreath product by** $Q^{Bic}_{(V,\overline{E})}$ (see subsection 5.3.4) where the underlying co-action of the wreath product comes from quantum symmetry of the **underlying simple graph** $(V,\overline{E})$. Theorem 5.3.9 in subsection 5.3.3 provides us with a large class of multigraphs where the categories $\mathcal{C}^{sym}_{(V,E)}$ and $\mathcal{C}^{Bic}_{(V,E)}$ coincide making computations possible. However, outside this class, existence of a universal object in $\mathcal{C}^{sym}_{(V,E)}$ remains unknown.

In chapter 6, we focus on constructing **quantum automorphisms** of an arbitrary multigraph (V,E) which are essentially of **non-Bichon** type, that is, not necessarily a member of $\mathcal{C}^{Bic}_{(V,E)}$. In section 6.1 we consider three special classes of automorphisms namely, **source dependent**, **target dependent** and **both source and target dependent** automorphisms of a directed multigraph (V,E). By replacing condition (5) in definition 4.2.3 with the more suitable ones we construct quantum analogues of the above mentioned automorphisms in section 6.2. In subsection 6.2.4 we rewrite the above mentioned quantum automorphisms in forms resembling to their classical counterparts. In the process of that, we see that **source dependent** quantum automorphisms can be formulated as a **free wreath product** where the underlying co-action is on the set of initial vertices. Similar phenomena happens for the case of **target dependent** quantum automorphisms where the underlying co-action of the **free wreath product** is on the set of final vertices. In section 6.4 we define notions of **source and target dependent quantum automorphism groups** using the results of subsection 6.2.4. They exist as universal objects in their respective categories of **source dependent, target dependent** and **both source and target dependent co-actions** as we have shown in section 6.3. Section 6.5 is about extension of these ideas for undirected multigraphs. As there are no inherent differences between source and target maps for an undirected multigraph, we see that all the quantum groups $Q^{s}_{(V,E,j)}$,

$Q^t_{(V,E,j)}$ and $Q^{s,t}_{(V,E,j)}$ coincide for an undirected multigraph (V, E, j) .

In chapter 7 we compute quantum automorphism groups for a few multigraphs. We have considered only undirected multigraphs as quantum automorphism of any directed multigraph do originate from its underlying undirected part (see theorem 4.6.11). However it is worth mentioning that similar phenomena is not true for **source and target dependent** quantum symmetries. In section 7.3, we show that our notions of quantum automorphisms also fit in the pictures of existing notions of quantum symmetry in various mathematical structures related to multigraphs.

In chapter 8, we describe the notion of quantum symmetry in a q-state Potts model on a finite undirected multigraph with a finite state space. The original work done by us in [GAH22] was in the context of simple graphs and we will do a similar treatment here. We start by extending the definition of Hamiltonian in the context of a multigraph. We identify the elements of finite state space with a suitable cyclic group and formulate a notion of co-action of compact quantum groups on the space of functions on the vertex set of the graph such that the co-action commutes with the Hamiltonian (that is, it gives a symmetry of the underlying model). Following the line of [Ban05b] we prove the existence of a universal compact quantum group which gives symmetry of the Potts model in the above sense, calling it the quantum group of symmetry of the model. Then we compute this quantum group in a few examples and show how a slight change of the parameters of the model can result in a rather remarkable change of quantum symmetry. More interestingly, we give an example where classical symmetry group remains the same even after the change of the model parameters but quantum symmetry group changes. This makes a strong case for studying quantum group symmetry in physical models.

Except chapter 2 which contains necessary preliminaries, from chapter 3 to chapter 7 we refer to [GH23] and for chapter 8 we refer to [GAH22] (see also [GH21] which is a corrected version).

Chapter 2

Preliminaries

2.1 Graphs

2.1.1 Simple graphs and weighted simple graphs

Definition 2.1.1. *A* **simple graph** (V, E) *consists of a finite set of vertices* V *and a finite set of edges* $E \subseteq V \times V$. *We define source and target maps* $s : E \to V$ *and* $t : E \to V$ *by*

$$s(i, j) = i \quad \text{and} \quad t(i, j) = j \quad \text{where} \quad (i, j) \in E.$$

The adjacency matrix $W = (W_{ij})_{i,j \in V}$ *is given by* $W_{ij} = 1$ *if* $(i, j) \in E$ *and* $W_{ij} = 0$ *otherwise.*

Notation 2.1.2. *In later sections, we might also write the adjacency matrix as* $(W^i_j)_{i,j \in V}$ *instead of* $(W_{ij})_{i,j \in V}$ *for notational ease and consistency.*

Definition 2.1.3. *A* **weighted simple graph** (V, E, w) *is a simple graph* (V, E) *with a weight function* $w : E \to \mathbb{C}$. *In this case, the adjacency matrix* $W = (W_{ij})_{i,j \in V}$ *is given by* $W_{ij} = w((i, j))$ *if* $(i, j) \in E$ *and* $W_{ij} = 0$ *otherwise.*

Definition 2.1.4. *A simple graph* (V, E) *or a weighted simple graph* (V, E, w) *is said to be* **undirected** *if its adjacency matrix* W *is a symmetric matrix, that is,* $W_{ij} = W_{ji}$ *for all* $i, j \in V$.

2.1.2 Finite quivers or multigraphs

We recall the notions of finite quivers and morphisms among them. For more details on quivers and path algebras see [GMVY18].

Definition 2.1.5. *A* **finite quiver** *or a* **multigraph** (V, E) *consists of a finite set of vertices* V *and a finite set of edges* E *with source and target maps* $s : E \to V$ *and* $t : E \to V$.

An edge $\tau \in E$ is called a **loop** *if $s(\tau) = t(\tau)$. We will denote $L \subseteq E$ to be the set of all "loops" in (V, E).*

The adjacency matrix $W = (W^i_j)_{i,j \in V}$ is given by $W^i_j = |\{\tau \in E | s(\tau) = i, t(\tau) = j\}|$. Here $|.|$ denotes cardinality of a set.

Definition 2.1.6. *A* **doubly directed** *multigraph (V, E) is a multigraph (V, E) such that its adjacency matrix W is symmetric, that is, $W^i_j = W^j_i$ for all $i, j \in V$.*

Definition 2.1.7. *An* **undirected** *multigraph (V, E, j) is a "doubly directed" multigraph (V, E) with an* **inversion map** *$j : E \to E$ such that the following hold:*

1. *$j^2 = id_E$.*
2. *$j(\tau) = \tau$ for all $\tau \in L$, where L is the set of "loops" in (V, E).*
3. *For all $\tau \in E$,*
$$s(j(\tau)) = t(\tau) \quad \text{and} \quad t(j(\tau)) = s(\tau).$$

Remark 2.1.8. *By fixing an inversion map in a doubly directed graph, we are clubbing two oppositely directed edges (***loops*** are excluded as they are inherently oppositely directed) between two points to produce an* **undirected** *edge. The main advantage of our* **undirectedness** *is not losing the sense of source and target of an edge even in an undirected setting which will be crucial to our construction later.*

It should be noted that in case of a simple graph, the notion of doubly directedness and undirectedness coincide as there can exist at most one inversion map in a "doubly directed" simple graph, hence making sense of definition 2.1.4.

Definition 2.1.9. *For a multigraph (V, E), the* **underlying simple graph** *$(V, \overline{E})$ is the simple graph with the same vertex set V and the set of edges $\overline{E}$ given by,*

$$\overline{E} := \{(i, j) \in V \times V | W^i_j \neq 0\}.$$

Definition 2.1.10. *For a multigraph (V, E), the* **underlying weighted graph** *is the underlying simple graph $(V, \overline{E})$ with the weight function $w : \overline{E} \to \mathbb{C}$ given by $w((i, j)) = W^i_j$ for $(i, j) \in \overline{E}$ where W is the adjacency matrix of (V, E).*

2.1.3 Morphisms of finite quivers or multigraphs

We recall definition 2.3 from [GMVY18]. For more detailed discussion on different automorphisms of a multigraph, see also [Gel82].

Definition 2.1.11. *Let (V, E) and (V', E') be two finite quivers with pairs of source and target maps given by (s, t) and (s', t') respectively. A* **morphism of quivers** *$f : (V, E) \to (V', E')$ is a pair of maps (f_V, f_E) where $f_V : V \to V'$ is a map of vertices and $f_E : E \to E'$ is a map of edges satisfying,*

$$f_V(s(\tau)) = s'(f_E(\tau)) \quad \text{and} \quad f_V(t(\tau)) = t'(f_E(\tau)) \quad \text{for all} \quad \tau \in E.$$

Furthermore if (V, E) and (V', E') are undirected with inversion maps j and j' then we also assume that,

$$f_E(j(\tau)) = j'(f_E(\tau)) \quad \text{for all} \quad \tau \in E.$$

An **automorphism of a finite quiver or a multigraph** *(V, E) is an invertible morphism from (V, E) to (V, E). The collection of all such automorphisms is said to be the* **classical automorphism group** *of (V, E) and will be denoted as $G^{aut}_{(V,E)}$.*

2.2 Review of Potts model in statistical mechanics

The fundamental idea of statistical mechanics is to consider an "ensemble" or totality of all possible states of a physical system in equilibrium and assign a probability distribution, usually the so-called Boltzmann (or Gibbs) distribution. The probability distribution (or the probability density) of a state depends on the corresponding energy level (typically given by the Hamiltonian, say H) and the absolute temperature of the system (T).

Potts model or q-state Potts model was first introduced by Renfrey Potts in 1952 ([Pot52]) as a generalization of already existing Ising model where the state space has q points instead of only "spin up" and "spin down" symmetry. Though these structures are described on infinite graphs or infinite lattices (see for instance, [Wu82]), we will look into the "finite" case as that will be enough for our purpose.

Let us consider (V, E) to be an **undirected finite simple graph** without loops and X_q be a finite set $\{1, 2, .., q\}$. A q-state Potts model on (V, E) consists of a configuration space Ω_P and a Hamiltonian H_P.

Definition 2.2.1. *A* **configuration** *ω is a function from the vertex set V to the finite set X_q. The set of all configuration is denoted by Ω_P. The* **Hamiltonian** *H_P is a function from Ω_P to $\mathbb{C}$ given by,*

$$H_P(\omega) = -\sum_{(i,j)\in E} J_{ij}\delta_{\omega(i),\omega(j)} - \sum_{i\in V} \xi_{\omega(i)}(i)$$

where $\delta_{x,y}$ is the Kronecker delta symbol (that is, $\delta_{x,y} = 1$ if $x = y$ and is 0 otherwise) and $\xi(i) := (\xi_1(i), ..., \xi_q(i)) \in \mathbb{R}^q$ is an external (possibly random) field.

Now we briefly recall the notion of **Gibbs measure** in our context which gives us the probability of occurence of a configuration ω at an absolute temperature T.

Definition 2.2.2. *The* **partition function** *$Z(\beta)$ at an inverse temperature β (that is, $\beta = 1/T$) for a q-state Potts model is given by,*

$$Z(\beta) = \sum_{\omega \in \Omega_P} e^{-\beta H_P(\omega)}.$$

The probability of occurence of a configuration ω at an inverse temperature β is given by

$$P(\omega) = \frac{1}{Z(\beta)} e^{-\beta H_P(\omega)}.$$

The measure described by $P(\omega)$ is the **Gibbs measure** *on the configuration space Ω_P at inverse temperature β.*

2.3 Compact quantum group

2.3.1 C* algebras and Hilbert C* modules

Before going to the compact quantum groups, we will briefly recall the notions of C* algebras and Hilbert C* modules. All algebras and tensor product of algebras are considered to be over the field of complex numbers unless explicitly mentioned otherwise.

Definition 2.3.1. *Let $\mathcal{A}$ be an algebra. A norm $\|.\|$ on $\mathcal{A}$ is called sub-multiplicative if the following identity holds:*

$$\|ab\| \leq \|a\|\|b\| \quad \text{for all} \quad a, b \in \mathcal{A}.$$

An algebra $\mathcal{A}$ with a sub-multiplicative norm $\|.\|$ is a normed algebra and denoted by $(\mathcal{A}, \|.\|)$. Moreover, if $\mathcal{A}$ has a multiplicative unit 1 such that $\|1\| = 1$, then $(\mathcal{A}, \|.\|)$ is called a unital normed algebra. A **Banach algebra** *is a normed algebra which is complete with respect to the metric induced by the norm.*

Definition 2.3.2. *A* **Banach * algebra** *is a triplet $(\mathcal{A}, \|.\|, *)$ where $* : \mathcal{A} \to \mathcal{A}$ is an antihomomorphism satisfying the following conditions:*

1. *$\|a^*\| = \|a\|$ for all $a \in \mathcal{A}$.*

2. $(\lambda a + b)^* = \overline{\lambda} a^* + b^*$ *for all* $a, b \in \mathcal{A}$ *and* $\lambda \in \mathbb{C}$.

A **C* algebra** *is a Banach * algebra satisfying the C* identity:*

$$\|a^* a\| = \|a\|^2.$$

Definition 2.3.3. *Let* $\mathcal{A}$ *be a C* algebra. An element* $x \in \mathcal{A}$ *is said to be* **positive** *if there exists* $y \in \mathcal{A}$ *such that* $x = y^* y$. *The set of all positive elements in* $\mathcal{A}$ *is denoted by* $\mathcal{A}_+$.

A **linear functional** $\phi : \mathcal{A} \to \mathbb{C}$ *is a continuous* $\mathbb{C}$*-linear map from* $\mathcal{A}$ *to* $\mathbb{C}$*.The map* ϕ *is said to be* **positive** *if* $\phi(x) \geq 0$ *for all* $x \in \mathcal{A}_+$.

A positive linear functional ϕ *on a unital C* algebra* $\mathcal{A}$ *is called a* **state** *if* $\phi(1) = 1$.

The Gelfand-Naimark theorem states that any unital commutative C*-algebra is isometrically isomorphic to $C(X)$ for some compact Hausdorff space X. On the other hand, any C*-algebra is isometrically isomorphic to a norm closed * subalgebra of $B(H)$ for some Hilbert space H.

Now we recall the concept of Hilbert C* modules. For detailed discussion see [Lan95].

Definition 2.3.4. *Let* $\mathcal{A}$ *be a C* algebra. A* **pre-Hilbert** $\mathcal{A}$**-module** E *is a right* $\mathcal{A}$*-module with an* $\mathcal{A}$*-valued inner product* $< .,. >_{\mathcal{A}}: E \times E \to \mathcal{A}$ *which is* $\mathbb{C}$ *linear in second variable and satisfy the following conditions:*

1. $< \xi, \eta x >_{\mathcal{A}} = < \xi, \eta >_{\mathcal{A}} x$ *for all* $\xi, \eta \in E$ *and* $x \in \mathcal{A}$.

2. $< \xi, \eta >_{\mathcal{A}}^* = < \eta, \xi >_{\mathcal{A}}$ *for all* $\xi, \eta \in E$.

3. $< \xi, \xi >_{\mathcal{A}} \geq 0$ *for all* $\xi \in E$.

4. *For any* $\xi \in E$*, if* $< \xi, \xi >_{\mathcal{A}} = 0$*, then* $\xi = 0$.

A pre-Hilbert module E *is called a* **Hilbert C* module** *if it is complete under the norm* $\|.\|$ *where the* $\|.\|$ *is given by* $\|\xi\| = \| < \xi, \xi >_{\mathcal{A}} \|^{1/2}$.

Let E be a Hilbert $\mathcal{A}$-module where $\mathcal{A}$ is a C*algebra. We say that a $\mathbb{C}$-linear map $L : E \to E$ is adjointable if $< \xi, L^*(\eta) >_{\mathcal{A}} = < L(\xi), \eta >_{\mathcal{A}}$ for all $\xi, \eta \in E$. The set of all adjointable operators is denoted by $\mathcal{L}_{\mathcal{A}}(E)$. An adjointable operator L is automatically $\mathcal{A}$-linear and a bounded map between Banach spaces. With respect to the operator norm, $\mathcal{L}_{\mathcal{A}}(E)$ is a C* algebra. Moreover, any $L \in \mathcal{L}_{\mathcal{A}}(E)$ is said to be "unitary" if L is an isometry, that is, $< L(\xi), L(\eta) >_{\mathcal{A}} = < \xi, \eta >_{\mathcal{A}}$ for all $\xi, \eta \in E$, and its range is whole of E.

For $\xi, \eta \in E$, let us define $\theta_{\xi,\eta} \in \mathcal{L}_{\mathcal{A}}(E)$ by $\theta_{\xi,\eta}(\gamma) = \eta < \xi, \gamma >_{\mathcal{A}}$ for all $\gamma \in E$. The norm closure of linear span of the set of operators $\theta_{\xi,\eta}$ in $\mathcal{L}_{\mathcal{A}}(E)$ is the set of compact operators on E and is denoted by $\mathcal{K}_{\mathcal{A}}(E)$. We recall the following result from the theory of Hilbert C* modules.

Lemma 2.3.5. *The multiplier algebra of $\mathcal{K}_{\mathcal{A}}(E)$, $M(\mathcal{K}_{\mathcal{A}}(E))$, is isomorphic to $\mathcal{L}_{\mathcal{A}}(E)$ for any Hilbert C* module E.*

Given a Hilbert space H and a C* algebra $\mathcal{A}$, $H \otimes \mathcal{A}$ is a Hilbert C* module with right $\mathcal{A}$-module map and $\mathcal{A}$-valued inner product $<,>_{\mathcal{A}}$ given by

$$(\xi \otimes x).y = \xi \otimes (xy) \quad \text{and} \quad <\xi \otimes x, \eta \otimes y>_{\mathcal{A}} = <\xi, \eta> x^* y$$

for all $\xi, \eta \in E$ and $x, y \in \mathcal{A}$. $\mathcal{K}_{\mathcal{A}}(H \otimes \mathcal{A})$ is canonically isomorphic to $K(H) \otimes \mathcal{A}$ where $K(H)$ is the set of compact operators on H. From lemma 2.3.5 it follows that $\mathcal{L}_{\mathcal{A}}(H \otimes \mathcal{A}) \cong M(K(H) \otimes \mathcal{A})$. we will often identify an element L of $\mathcal{L}_{\mathcal{A}}(H \otimes \mathcal{A})$ with a map from H to $H \otimes \mathcal{A}$ which sends a vector ξ to $L(\xi \otimes 1)$ in $H \otimes \mathcal{A}$. We might also use the same notation in both cases.

Definition 2.3.6. *Let H be a Hilbert space and $\mathcal{A}$ be a C* algebra. By $B(H)$, we will denote the set of all bounded operators on H. For a unitary $U \in \mathcal{L}_{\mathcal{A}}(H \otimes \mathcal{A})$, there is a * homomorphism $Ad_U : B(H) \to \mathcal{L}_{\mathcal{A}}(H \otimes \mathcal{A})$ which is given by,*

$$Ad_U(T) = U(T \otimes 1)U^* \quad \textit{where} \quad T \in B(H).$$

Let us assume that H is a finite dimensional Hilbert space of dimension n. Again from lemma 2.3.5 it follows that $\mathcal{L}_{\mathcal{A}}(H \otimes \mathcal{A}) \cong M(K(H) \otimes \mathcal{A}) \cong B(H) \otimes \mathcal{A}$ where $B(H)$ is the set of all bounded operators on H. As H is an n dimensional Hilbert space, $B(H)$ can be identified with $M_n(\mathbb{C})$, the algebra of $n \times n$ matrices, by choosing a suitable orthonormal basis of H.

2.3.2 Basic definitions

We give a brief description of compact quantum groups and related concepts. For detailed discussion on quantum groups, see [CP95], [MVD98], [Tim08], [NT13] and [GB16]. All C* algebras here will be assumed to be unital and all tensor products will be minimal tensor product of C* algebras unless explicitly mentioned otherwise.

Definition 2.3.7. *A compact quantum group or a CQG (in short) is a pair $(\mathcal{A}, \Delta)$ where $\mathcal{A}$ is a unital C* algebra and $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is a homomorphism of C* algebras satisfying the following conditions:*

1. $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ *(coassociativity).*

2. *Each of the linear spans* $\Delta(\mathcal{A})(1 \otimes \mathcal{A})$ *and* $\Delta(\mathcal{A})(\mathcal{A} \otimes 1)$ *is norm-dense in* $\mathcal{A} \otimes \mathcal{A}$.

It is known that there exists a unique Haar state on a compact quantum group which is the non-commutative analogue of Haar measure on a classical compact group.

Definition 2.3.8. *The Haar state* h *on a compact quantum group* $(\mathcal{A}, \Delta)$ *is the unique state on* $\mathcal{A}$ *which satisfies the following conditions:*

$$(h \otimes id)\Delta(a) = h(a)1_{\mathcal{A}} \quad \textit{and} \quad (id \otimes h)\Delta(a) = h(a)1_{\mathcal{A}}$$

for all $a \in \mathcal{A}$.

Definition 2.3.9. *A quantum group homomorphism* Φ *among two compact quantum groups* $(\mathcal{A}_1, \Delta_1)$ *and* $(\mathcal{A}_2, \Delta_2)$ *is a C* algebra homomorphism* $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ *satisfying the following condition:*

$$(\Phi \otimes \Phi) \circ \Delta_1 = \Delta_2 \circ \Phi.$$

Definition 2.3.10. *A Woronowicz C* subalgebra of a compact quantum group* $(\mathcal{A}, \Delta)$ *is a C* subalgebra* $\mathcal{A}'$ *such that* $(\mathcal{A}'\Delta|_{\mathcal{A}'})$ *is a compact quantum group and the inclusion map* $i : \mathcal{A}' \to \mathcal{A}$ *is a homomorphism of compact quantum groups.*

Definition 2.3.11. *A Woronowicz C* ideal of a compact quantum group* $(\mathcal{A}, \Delta)$ *is a two sided C*ideal* $\mathcal{I}$ *such that* $\Delta(\mathcal{I}) \subseteq ker(\pi \otimes \pi)$ *where* π *is the natural quotient map* $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$.

Proposition 2.3.12. *The quotient of a compact quantum group* $(\mathcal{A}, \Delta)$ *by a Woronowicz C* ideal* $\mathcal{I}$ *has a unique compact quantum group structure such that the quotient map* π *is a homomorphism of compact quantum groups. More precisely, the co-product* $\tilde{\Delta}$ *on* $\mathcal{A}/\mathcal{I}$ *is given by,*

$$\tilde{\Delta}(a + \mathcal{I}) = (\pi \otimes \pi)\Delta(a)$$

where $a \in \mathcal{A}$.

Definition 2.3.13. *A compact quantum group* $(\mathcal{A}', \Delta')$ *is said to be quantum subgroup of another compact quantum group* $(\mathcal{A}, \Delta)$ *if there exists a Woronowicz C* ideal* $\mathcal{I}$ *such that* $(\mathcal{A}', \Delta') \cong (\mathcal{A}, \Delta)/\mathcal{I}$.

2.3.3 Co-actions and co-representations

Definition 2.3.14. *Let* H *be a finite dimensional Hilbert space and* $(\mathcal{A}, \Delta)$ *be a compact quantum group. We consider the Hilbert* $\mathcal{A}$*-module* $H \otimes \mathcal{A}$ *with induced* $\mathcal{A}$*-valued inner product* *from H. A finite dimensional co-representation of $(\mathcal{A},\Delta)$ on H is a $\mathbb{C}$-linear map $\delta : H \to H\otimes\mathcal{A}$ such that $\tilde{\delta} \in B(H)\otimes\mathcal{A}$ given by $\tilde{\delta}(\xi\otimes a) = \delta(\xi)a$ ($\xi \in H$,$a \in \mathcal{A}$) satisfies the following condition:*

$$(id\otimes\Delta)\tilde{\delta} = \tilde{\delta}_{(12)}\tilde{\delta}_{(13)}$$

where $\tilde{\delta}_{(12)}$ and $\tilde{\delta}_{(13)}$ are common leg notations defined in section 5 of [MVD98].

Remark 2.3.15. *By choosing an orthonormal basis $\{e_1,..,e_n\}$ of H we can identify H with $\mathbb{C}^n$ and $B(H)$ with $M_n(\mathbb{C})$. For a $\mathbb{C}$-linear map $\delta : H \to H\otimes\mathcal{A}$, we define $U^\delta \in M_n(\mathcal{A})$ by $(U^\delta)_{ij} =< e_i\otimes 1_{\mathcal{A}}, \delta(e_j) >_{\mathcal{A}}$. It is clear that δ is uniquely determined by the matrix U^δ and is a co-representation if and only if*

$$\Delta(U^\delta_{ij}) = \sum_{k=1}^{n} U^\delta_{ik}\otimes U^\delta_{kj}.$$

U^δ is said to be the **co-representation matrix** *of δ. Later in this thesis we might also write the coefficients of a co-representation matrix as $(U^\delta)^i_j$ instead of $(U^\delta)_{ij}$ for notational ease and convenience.*

A co-representation δ is said to be **non-degenerate** *if U^δ is invertible in $M_n(\mathcal{A})$ and* **unitary** *if the matrix U^δ is unitary in $M_n(\mathcal{A})$, that is, $U^\delta U^{\delta*} = U^{\delta*}U^\delta = Id_{M_n(\mathcal{A})}$.*

Definition 2.3.16. *For a finite dimensional co-representation δ of a compact quantum group $(\mathcal{A},\Delta)$ the adjoint co-representation $\overline{\delta}$ is defined by the corepresentation matrix $\overline{U^\delta}$, where $\overline{U^\delta_{ij}} = {U^\delta_{ij}}^*$.*

It is known from representation theory of compact quantum groups that for a compact quantum group $(\mathcal{A},\Delta)$, there is a dense subalgebra $\mathcal{A}_0$ generated by the matrix elements of its finite dimensional co-representations. This subalgebra $\mathcal{A}_0$ with the co-product $\Delta|_{\mathcal{A}_0}$ is a Hopf * algebra in its own right and referred to as **underlying Hopf* algebra of matrix elements** of $(\mathcal{A},\Delta)$. It is also worth mentioning that the Haar state h is faithful on $\mathcal{A}_0$ and is tracial if $(\mathcal{A},\Delta)$ is a compact quantum group of Kac type (see proposition 1.7.9 in [NT13]).

Now we describe the notion of a co-action of a compact quantum group on a unital C* algebra.

Definition 2.3.17. *Let $\mathcal{B}$ be a unital C* algebra. A co-action of a compact quantum group $(\mathcal{A},\Delta)$ on $\mathcal{B}$ is a C* homomorphism $\alpha : \mathcal{B}\to\mathcal{B}\otimes\mathcal{A}$ satisfying the following conditions:*

1. *$(\alpha\otimes id)\alpha = (id\otimes\Delta)\alpha$.*

2. *Linear span of $\alpha(\mathcal{B})(1_{\mathcal{B}}\otimes\mathcal{A})$ is norm-dense in $\mathcal{B}\otimes\mathcal{A}$.*

A co-action α is said to be **faithful** *if there does not exist a proper Woronowicz C* algebra $\mathcal{A}'$ of $(\mathcal{A}, \Delta)$ such that α is also a co-action of $(\mathcal{A}', \Delta|_{\mathcal{A}'})$ on $\mathcal{B}$.*

For a unital C* algebra $\mathcal{B}$, we consider the category of **quantum transformation groups** whose objects are compact quantum groups co-acting on $\mathcal{B}$ and morphisms are quantum group homomorphisms intertwining such co-actions. The universal object in this category, if it exists (it might not, see [Wan98] for example), is said to be **quantum automorphism group** of $\mathcal{B}$. The next proposition is easy to see and will be crucial to our constructions later on.

Proposition 2.3.18. *For a finite dimensional unitary co-representation δ of a compact quantum group $(\mathcal{A}, \Delta)$, the * homomorphism defined in definition 2.3.6, $Ad_\delta : B(H) \to B(H) \otimes \mathcal{A}$ is a co-action on the algebra $B(H)$.*

2.3.4 Examples of compact quantum groups

Now we will be looking into some examples of compact quantum groups which are used in this thesis.

Compact matrix quantum groups:

The theory of compact matrix quantum groups was first defined and developed by Woronowicz in [Wor87] and it precceded the formalism of compact quantum groups. All compact quantum groups constructed in this thesis are in fact compact matrix quantum groups. For detailed discussion on compact matrix quantum groups, see also [Web17], [Tim08].

Definition 2.3.19. *A compact matrix quantum group (A, Δ) consists of a C* algebra $\mathcal{A}$ equipped with a C* algebra homomorphism $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that there is a unitary matrix $U = (u_{ij})_{i,j=1,..,n} \in M_n(\mathcal{A})$ for some $n \in \mathbb{N}$ satisfying the following conditions:*

1. *$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$ for all $i, j = 1, .., n$.*
2. *The matrix $\overline{U} = (u^*_{ij})_{i,j=1,..,n}$ is invertible.*
3. *The elements $\{u_{ij} | i, j = 1, .., n\}$ generate $\mathcal{A}$ as a C* algebra.*

The matrix of generators $U = (u_{ij})_{i,j=1,..,n}$ is referred to as **fundamental co-representation matrix** *of $(\mathcal{A}, \Delta)$.*

Free unitary and free orthogonal quantum group

We recall the definition of free unitary and free orthogonal quantum groups from [Wan95] and [DW96] (see also [Wan02]) .

Definition 2.3.20. *Let $n \in \mathbb{N}$. The* **free unitary quantum group** $A_u(n)$ *is the universal C* algebra generated by the elements of the matrix $U = (u_{ij})_{i,j=1,..,n}$ satisfying the following relations:*

$$UU^* = U^*U = I_n \quad \text{and} \quad \overline{U}U^t = U^t\overline{U} = I_n.$$

where $\overline{U} = ({u_{ij}}^*)_{i,j=1,..,n}$ and $U^t = (u_{ji})_{i,j=1,..,n}$. There exists a co-product Δ_U on $A_u(n)$ satisfying

$$\Delta_U(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}.$$

The **free orthogonal quantum group** $A_o(n)$ *is the quotient of $A_u(n)$ by a C* ideal generated by the set of relations $\{{u_{ij}}^* = u_{ij} | i,j = 1,..,n\}$.*

Remark 2.3.21. *$A_u(n)$ and $A_o(n)$ are both compact quantum groups of Kac type.*

Let us consider a Hilbert space H with a chosen orthonormal basis $\{e_1,..,e_n\}$. The compact quantum groups $A_u(n)$ and $A_o(n)$ both have canonical co-representations on H given by the same description:

$$\alpha(e_i) = \sum_{j=1}^{n} e_j \otimes u_{ji} \quad \text{where} \quad i = 1,..,n.$$

Abelianization of $A_u(n)$ and $A_o(n)$ gives function algebras of $U(n)$ and $O(n)$, where $U(n)$ and $O(n)$ are groups of unitary and orthogonal $n \times n$ matrices respectively.

We introduce a notation which we will be using extensively for the rest of this thesis.

Notation 2.3.22. *Let X be a finite set. For $i \in X$, let us denote the characteristic function on i as χ_i, that is, $\chi_i(j) = \delta_{i,j}$ for all $j \in X$. The function algebra on X, that is, set of all functions from X to $\mathbb{C}$, is the $\mathbb{C}$-linear span of the elements $\{\chi_i | i \in X\}$. This function algebra will be treated as both an algebra (with multiplication given by, $\chi_i.\chi_j = \delta_{i,j}$) and a Hilbert space (with inner product given by, $< \chi_i, \chi_j >= \delta_{i,j}$). We will denote the function algebra by $C(X)$ when we will treat it as an algebra and $L^2(X)$ when we will treat it as a Hilbert space.*

Quantum permutation groups

We describe the quantum analogue of permutation group on n elements. The following definition is due to Wang in [Wan98].

Definition 2.3.23. *Let $X_n = \{1, 2, .., n\}$ be a finite set. The* **quantum permutation group** *on n elements, S_n^+ is the universal C* algebra generated by the elements of the matrix $(x_{ij})_{i,j=1,..,n}$ satisfying the following relations:*

1. $x_{ij}^2 = x_{ij} = {x_{ij}}^*$ *for all* $i, j = 1, .., n$.

2. $\sum_{i=1}^{n} x_{ij} = 1 = \sum_{i=1}^{n} x_{ji}$ *for all* $j = 1, .., n$.

The co-product Δ_n *on* S_n^+ *is given by* $\Delta_n(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}$.

Remark 2.3.24. S_n^+ *is a quantum subgroup of* $A_u(n)$ *and hence of Kac type.*

The compact quantum group S_n^+ *has a canonical co-action* α_n *on* $C(X_n)$ *given by,*

$$\alpha_n(\chi_i) = \sum_{j=1}^{n} \chi_j \otimes x_{ji} \quad \text{where} \quad i = 1, .., n.$$

It also follows that S_n^+ *is the universal object in this category of compact quantum groups co-acting on the algebra* $C(X_n)$. *Moreover, abelianization of* S_n^+ *gives us* $C(S_n)$, *where* S_n *is the standard permutation group on* n *elements.*

Notation 2.3.25. *For any co-action* α *of a compact quantum group* $(\mathcal{A}, \Delta)$ *on* $C(X_n)$, *we can write*

$$\alpha(\chi_i) = \sum_{j=1}^{n} \chi_j \otimes q_{ji}, \quad \text{where} \;\; i = 1, .., n \;\; \text{and} \;\; (q_{ij})_{i,j=1,..,n} \in M_n(\mathcal{A}).$$

As α *can also be treated as a unitary co-representation on* $L^2(X)$ *we will refer to the matrix* $(q_{ij})_{i,j=1,..,n}$ *as the* **co-representation matrix of** α.

The relations listed in definition 2.3.23 will be referred to as **quantum permutation relations**.

2.4 Free products of compact quantum groups

Free products of compact quantum groups were described in [Wan95]. Let $(\mathcal{A}_1, \Delta_1)$ and $(\mathcal{A}_2, \Delta_2)$ be two compact quantum groups. Let us cosider the free product of two C* algebras $\mathcal{A}_1, \mathcal{A}_2$ with canonical inclusion maps $\nu_1 : \mathcal{A}_1 \to \mathcal{A}_1 * \mathcal{A}_2$ and $\nu_2 : \mathcal{A}_2 \to \mathcal{A}_1 * \mathcal{A}_2$. We have the following results from [Wan95].

Proposition 2.4.1. *The following statements are true.*

1. *There exists a co-product* Δ *on* $\mathcal{A}_1 * \mathcal{A}_2$ *satisfying*

$$\Delta \circ \nu_1 = (\nu_1 \otimes \nu_1) \circ \Delta_1 \quad \text{and} \quad \Delta \circ \nu_2 = (\nu_2 \otimes \nu_2) \circ \Delta_2$$

and making $(\mathcal{A}_1 * \mathcal{A}_2, \Delta)$ *a compact quantum group.*

2. *If* $(\mathcal{A}_1, \Delta_1)$ *and* $(\mathcal{A}_2, \Delta_2)$ *are* **compact matrix quantum groups** *with fundamental co-representation matrices* $U = (u_{ij})_{i,j=1,..,n}$ *and* $V = (v_{kl})_{k,l=1,..,m}$ *then* $(\mathcal{A}_1 * \mathcal{A}_2, \Delta)$ *is*

also a compact matrix quantum group with the fundamental co-representation matrix $U \oplus V$ where

$$U \oplus V = \begin{bmatrix} \nu_1(U) & 0 \\ 0 & \nu_2(V) \end{bmatrix} \tag{2.4.1}$$

Here $\nu_1(U) = (\nu_1(u_{ij}))_{i,j=1,..,n}$ and $\nu_2(V) = (\nu_2(v_{kl}))_{k,l=1,.,m}$.

3. *For any two finite dimensional co-representations $\alpha_1 : H_1 \to H_1 \otimes \mathcal{A}_1$ and $\alpha_2 : H_2 \to H_2 \otimes \mathcal{A}_2$ with co-representation matrices U_1 and U_2, there exists a co-representation $\alpha : H_1 \oplus H_2 \to (H_1 \oplus H_2) \otimes (\mathcal{A}_1 * \mathcal{A}_2)$ with the co-representation matrix $U_1 \oplus U_2$ described in the same way as in equation 2.4.1.*

2.5 Quantum automorphisms of simple and weighted graphs

There are two different existing notions of quantum symmetry in a simple graph, one was introduced by Bichon (see [Bic03]) and the other was introduced by Banica ([Ban05a],[BBC07a]). We will recall both of them in this section and discuss relations between them. We start with introducing a notation which is standard in this context:

Notation 2.5.1. *Let V be a finite set and $\alpha : C(V) \to C(V) \otimes \mathcal{A}$ be a co-action of a compact quantum group $(\mathcal{A}, \Delta)$ with co-representation matrix $Q = (q_{ij})_{i,j \in V}$. Then we define $\alpha^{(2)} = (id \otimes id \otimes m)(id \otimes \Sigma_{23} \otimes id)(\alpha \otimes \alpha)$ where m is the multiplication map in $\mathcal{A}$ and Σ_{23} is the standard flip map on 2nd and 3rd coordinates of the tensor product. For $i, j, k, l \in V$, we observe that,*

$$\alpha^{(2)}(\chi_k \otimes \chi_l) = \sum_{i,j \in V} \chi_i \otimes \chi_j \otimes q_{ik} q_{jl}.$$

It is easy to check using **quantum permutation relations** *that $\alpha^{(2)}$ is actually a unitary co-representation of $(\mathcal{A}, \Delta)$ on the Hilbert space $L^2(V) \otimes L^2(V)$. It follows that the adjoint co-representation $\overline{\alpha^{(2)}}$ is also unitary.*

2.5.1 Bichon's notion of quantum symmetry

We recall definition 3.1 given by Bichon in [Bic03].

Definition 2.5.2. *A co-action of compact quantum group $(\mathcal{A}, \Delta)$ on a simple graph (V, E) preserving its quantum symmetry in Bichon's sense consists of a co-action $\alpha : C(V) \to C(V) \otimes \mathcal{A}$ on the algebra $C(V)$ and a co-action $\beta : C(E) \to C(E) \otimes \mathcal{A}$ on the algebra $C(E)$ such that the following diagram commutes:*

$$\begin{array}{ccc} C(V)\otimes C(V) & \xrightarrow{\alpha^{(2)}} & C(V)\otimes C(V)\otimes \mathcal{A} \\ \Big\downarrow{\scriptstyle m(s^*\otimes t^*)} & & \Big\downarrow{\scriptstyle m(s^*\otimes t^*)\otimes id_{\mathcal{A}}} \\ C(E) & \xrightarrow{\quad\beta\quad} & C(E)\otimes \mathcal{A} \end{array}$$

where m is the multiplication map on the algebra $C(E)$, $s^* : C(V) \to C(E)$ and $t^* : C(V) \to C(E)$ are pullbacks of source and target maps s and t on (V, E).

It follows from theorem 3.2 in [Bic03] that the quantum permutation of edges β is completely determined by the quantum permutation of the vertices α. In fact, $\beta = \alpha^{(2)}|_{C(E)}$ where $C(E)$ is identified as a subalgebra of $C(V)\otimes C(V)$ via the identification $\chi_{(k,l)} \to \chi_k \otimes \chi_l$. In light of this we give the following proposition:

Proposition 2.5.3. *Let $\alpha : C(V) \to C(V) \otimes \mathcal{A}$ be a co-action of the CQG $(\mathcal{A}, \Delta)$. Then α induces a co-action on (V, E) preserving its quantum symmetry in Bichon's sense (see definition 2.5.2) iff $\alpha^{(2)}(C(E)) \subseteq C(E) \otimes \mathcal{A}$ and $\alpha^{(2)}|_{C(E)}$ is a co-action of $(\mathcal{A}, \Delta)$ on the algebra $C(E)$.*

Remark 2.5.4. *Let us consider the category $\mathcal{D}^{Bic}_{(V,E)}$ whose objects are compact quantum groups co-acting on (V, E) preserving its quantum symmetry in Bichon's sense and morphisms are quantum group homomorphisms intertwining such co-actions. From theorem 3.2 in [Bic03], it follows that the universal object in this category exists and called the* ***quantum automorphism group*** *of (V, E) in Bichon's sense. This automorphism group will be denoted as $S^{Bic}_{(V,E)}$ in this thesis.*

2.5.2 Banica's notion of quantum symmetry

Now we describe Banica's notion of quantum symmetry on a simple graph ([Ban05a]) and prove some results related to it.

Definition 2.5.5. *Let (V, E) be a simple graph. Then by a co-action α of a compact quantum group $(\mathcal{A}, \Delta)$ on (V, E) preserving its quantum symmetry in Banica's sense, we mean a co-action α on the algebra $C(V)$ such that the co-representation matrix of α commutes with the adjacency matrix of (V, E).*

Let us consider the category $\mathcal{D}^{Ban}_{(V,E)}$ whose objects are compact quantum groups co-acting on (V, E) and morphisms are quantum group homomorphisms intertwining such co-actions. The following result is due to Banica in [Ban05a].

Theorem 2.5.6. *Let $|V| = n$ and $Q = (q_{ij})_{i,j=1,..,n}$ be the co-representation matrix of the canonical co-action of S_n^+ on $C(V)$. The universal object in $\mathcal{D}^{Ban}_{(V,E)}$ exists and is given by $S_n^+/QW - WQ$ where W is the adjacency matrix of (V, E). This universal object is called* *the* **quantum automorphism group** *of* (V,E) *in Banica's sense and will be denoted as* $S^{Ban}_{(V,E)}$ *in this thesis.*

The above theorem holds true if we consider the matrix W to be any complex number valued matrix instead of just an adjacency matrix of a simple graph, which covers the case of weighted graphs and small metric spaces (see [Ban05a],[Ban05b]). If our simple graph is weighted with a weight function w, we will denote its quantum automorphism group as $S^{Ban}_{(V,E,w)}$.

Let (V,E,w) be a weighted simple graph with adjacency matrix W. Then we observe the following results:

Lemma 2.5.7. *Let α be a co-action of a CQG $(\mathcal{A},\Delta)$ on $C(V)$ with co-representation matrix $Q=(q_{ij})_{i,j\in V}$. Then the following conditions are equivalent:*

1. $QW=WQ$.
2. $\sum_{i,j\in V} W_{ij}q_{ki}q_{lj}=W_{kl}1$ *for all* $k,l\in V$.

Proof. Let us assume (1). We have

$$\begin{aligned}\sum_{i,j\in V} W_{ij}q_{ki}q_{lj} &= \sum_{j\in V}(\sum_{i\in V} q_{ki}W_{ij})q_{lj}\\ &= \sum_{j\in V}(\sum_{i\in V} W_{ki}q_{ij})q_{lj}\\ &= W_{kl}(\sum_{j\in V} q_{lj}) = W_{kl}1.\end{aligned}$$

Conversely, let us assume (2). Using antipode on the underlying Hopf * algebra of matrix elements of CQG $(\mathcal{A},\Delta)$ we observe that,

$$\sum_{i,j\in V} W_{ij}q_{jl}q_{ik}=W_{kl}1 \quad \text{for all} \quad k,l\in V.$$

Let us fix $i,j\in V$. Using above expression it follows that,

$$\begin{aligned}(QW)_{ij}=\sum_{k\in V} q_{ik}W_{kj} &= \sum_{k\in V}(\sum_{k',j'\in V} W_{k'j'}q_{j'j}q_{k'k})q_{ik}\\ &= \sum_{k\in V}\sum_{j'\in V} W_{ij'}q_{j'j}q_{ik}\\ &= \sum_{j'\in V} W_{ij'}q_{j'j}=(WQ)_{ij}.\end{aligned}$$

$\square$

For proving the next result we will use the technical lemma stated below.

Lemma 2.5.8. *Let P be an $n \times n$ matrix of positive operators on a Hilbert space H. Let $A \in M_n(\mathbb{C})$. Then P commutes with A if and only if P commutes with A_R and A_I, where A_R and A_I are in $M_n(\mathbb{R})$ such that $A = A_R + iA_I$.*

Theorem 2.5.9. *Let α be a co-action of a CQG $(\mathcal{A}, \Delta)$ on $C(V)$ with co-representation matrix $Q = (q_{ij})_{i,j \in V}$. Let us write $W = \sum_{c \in \mathbb{C}} W^c$ where $W^c_{ij} = 1$ iff $W_{ij} = c$ and $W^c_{ij} = 0$ otherwise. For $c \in \mathbb{C}$, we consider the linear subspace K^c of $L^2(V) \otimes L^2(V)$ defined by*

$$K^c = \textit{linear span}\{\chi_k \otimes \chi_l | W_{kl} = c; \ \ k, l \in V\}.$$

Then the following conditions are equivalent:

1. $QW = WQ$.
2. $\alpha^{(2)}(K^c) \subseteq K^c \otimes \mathcal{A}$ *for all* $c \in \mathbb{C}$.
3. $QW^c = W^c Q$ *for all* $c \in \mathbb{C}$.

Proof. We proceed through the following claims.

Claim:$(1) \implies (2)$.

We will follow the proof of theorem 3.11 in [GAH22]. Using lemma 2.5.8, without loss of generality, we can assume that the adjacency matrix W is a real valued matrix. Let us define $\overline{w} : V \times V \to \mathbb{C}$ by $\overline{w}(i, j) = W_{ij}$ where $i, j \in V$. Let $Image(\overline{w}) = \{s_1, s_2, .., s_r\}$ where $s_1 < s_2 < .. < s_r \in \mathbb{R}$. As the Haar state h is tracial on the algebra generated by the elements $\{q_{ij} | i, j \in V\}$ in $\mathcal{A}$, it follows that,

$$h(q_{ii'} q_{jj'} q_{ii'}) = h(q_{ii'} q_{jj'}) \quad \text{for all} \quad i, j, i', j' \in V.$$

Let us choose $(k, l) \in V \times V$ such that $\overline{w}(k, l) = s_r$. Using lemma 2.5.7 we observe that,

$$\sum_{i,j \in V} (W_{kl} - W_{ij}) h(q_{ki} q_{lj} q_{ki}) = h(\sum_{i,j \in V} (W_{kl} - W_{ij}) q_{ki} q_{lj}) = 0.$$

As h is faithful on the underlying dense Hopf * algebra of matrix elements of $(\mathcal{A}, \Delta)$ and $W_{kl} - W_{ij} \geq 0$ for all $i, j \in V$, it follows that,

$$\|q_{ki} q_{lj}\|^2 = \|q_{ki} q_{lj} q_{ki}\| = 0 \quad \text{whenever} \quad W_{ij} \neq s_r.$$

Using antipode κ on the underlying Hopf * algebra of matrix elements of $(\mathcal{A}, \Delta)$ we observe that,

$$q_{ik}q_{jl} = \kappa(q_{lj}q_{ki}) = \kappa((q_{ki}q_{lj})^*) = \kappa(0) = 0.$$

Hence we have proved our claim for $c = s_r$. Now we proceed by induction. Let us assume that for $c \geq s_t \in Image(\overline{w})$

$$\alpha^{(2)}(K^c) \subseteq K^c \otimes \mathcal{A}$$

which is equivalent to assuming that, for all $i, j, k, l \in V$,

$$q_{ki}q_{lj} = 0 \quad \text{when} \quad W_{ij} \geq s_t \text{ and } W_{kl} \neq W_{ij}.$$

We want to prove our assumption for $c \geq s_{t-1}$. It is enough to show that,

$$\alpha^{(2)}(K^c) \subseteq K^c \otimes \mathcal{A} \quad \text{for} \quad c = s_{t-1}. \tag{2.5.1}$$

Let us fix $(k, l) \in V \times V$ such that $W_{kl} = s_{t-1}$. Using induction hypothesis, we observe that,

$$\begin{aligned} \sum_{\substack{i,j \in V \\ W_{ij} < W_{kl}}} (W_{kl} - W_{ij}) h(q_{ki}q_{lj}q_{ki}) &= h(\sum_{\substack{i,j \in V \\ W_{ij} < W_{kl}}} (W_{kl} - W_{ij}) q_{ki}q_{lj}) \\ &= h(\sum_{i,j \in V} (W_{kl} - W_{ij}) q_{ki}q_{lj}) = 0. \end{aligned}$$

Therefore we observe that $q_{ki}q_{lj} = 0$ when $W_{ij} < W_{kl}$. Using induction hypothesis it further follows that,

$$q_{ki}q_{lj} = 0 \quad \text{when} \quad W_{ij} \neq s_{t-1}.$$

Hence 2.5.1 follows using similar arguments as in the case of $c = s_r$.

Claim:$(2) \implies (3)$.

It is clear from (2) that, $q_{ik}q_{jl} = 0$ for all $i, j, k, l \in V$ such that $W_{kl} \neq W_{ij}$.

Let us fix $i, j \in V$ and $c \in \mathbb{C}$ such that $W^c \neq 0$. We observe that,

$$\begin{aligned} (QW^c)_{ij} = \sum_{k \in V} q_{ik} W^c_{kj} &= \sum_{\substack{k \in V \\ W_{kj} = c}} q_{ik} \\ &= \sum_{\substack{k,l \in V \\ W_{kj} = c}} q_{ik}q_{lj} \end{aligned}$$

$$= \sum_{\substack{k,l \in V \\ W_{kj}=c=W_{il}}} q_{ik}q_{lj}$$

$$= \sum_{\substack{k,l \in V \\ W_{il}=c}} q_{ik}q_{lj}$$

$$= \sum_{\substack{l \in V \\ W_{il}=c}} q_{lj} = \sum_{l \in V} W^c_{il} q_{lj} = (W^c Q)_{ij}.$$

(3) $\implies$ (1).

$$QW = Q(\sum_{c \in \mathbb{C}} cW^c) = \sum_{c \in \mathbb{C}} cQW^c = \sum_{c \in \mathbb{C}} cW^c Q = (\sum_{c \in \mathbb{C}} cW^c)Q = WQ.$$

$\square$

Proposition 2.5.10. *Let α be a co-action of a CQG $(\mathcal{A}, \Delta)$ on a weighted simple graph (V, E, w) preserving its quantum symmetry in Banica's sense. We consider the source and target degree functions $f_s, f_t \in C(V)$ which are defined as follows:*

$$f_s(k) = \sum_{l \in V} W_{kl} \quad \textit{and} \quad f_t(k) = \sum_{l \in V} W_{lk} \quad \textit{for all} \ \ k \in V.$$

Then the following identities hold:

$$\alpha(f_s) = f_s \otimes 1 \quad \textit{and} \quad \alpha(f_t) = f_t \otimes 1.$$

Proof. Let us deal with only f_s as f_t can be dealt in identical manner. Let $Q = (q_{ij})_{i,j \in V}$ be the co-representation matrix of α. It follows that,

$$\begin{aligned} \alpha(f_s) = f_s \otimes 1 &\iff \sum_{k' \in V} f_s(k') q_{kk'} = f_s(k) \quad \text{for all} \quad k \in V \\ &\iff q_{kk'} = 0 \quad \text{whenever} \quad f_s(k) \neq f_s(k'). \end{aligned} \tag{2.5.2}$$

Let us choose $k, k' \in V$ such that $f_s(k) \neq f_s(k')$. We observe that,

$$\begin{aligned} f_s(k') q_{kk'} &= \sum_{i \in V} f_s(i) q_{ki} q_{kk'} \\ &= (\sum_{i,j \in V} W_{ij} q_{ki}) q_{kk'} \\ &= (\sum_{i,j \in V} W_{ki} q_{ij}) q_{kk'} \end{aligned}$$

$$= \sum_{i \in V} W_{ki} q_{kk'} = f_s(k) q_{kk'}.$$

As $f_s(k) \neq f_s(k')$, it follows that $q_{kk'} = 0$. From observation 2.5.2 the first identity in proposition 2.5.10 follows.

Now let $k, k' \in V$ be such that $f_t(k) \neq f_t(k')$. Using similar arguments as in previous case it follows that $f_t(k') q_{k'k} = f_t(k) q_{k'k}$, which implies $q_{k'k} = 0$ as $f_t(k) \neq f_t(k')$. □

We provide Banica's version of proposition 2.5.3 which is immidiate from theorem 2.5.9.

Proposition 2.5.11. *Let (V, E) be a simple graph and $\alpha : C(V) \to C(V) \otimes \mathcal{A}$ be a co-action of the CQG $(\mathcal{A}, \Delta)$ on $C(V)$. Then α preserves the quantum symmetry of (V, E) in Banica's sense if and only if $\alpha^{(2)}(L^2(E)) \subseteq L^2(E) \otimes \mathcal{A}$ where $L^2(E)$ is identified as a linear subspace of $L^2(V) \otimes L^2(V)$.*

We end this section with a small lemma which will be useful later during computation of examples.

Lemma 2.5.12. *Let (V, E) and (V, E') be two simple graphs without loops with adjacency matrices W and W' such that the following holds:*

$$W + W' = A$$

where A is a $|V| \times |V|$ real valued matrix such that $A_{ij} = 1 - \delta_{i,j}$ for all $i, j \in V$. Then we have

$$S^{Ban}_{(V,E)} \cong S^{Ban}_{(V,E')}$$

.

Proof. Let $\alpha : C(V) \to C(V) \otimes \mathcal{A}$ be a co-action of a CQG $(\mathcal{A}, \Delta)$ on $C(V)$ with co-representation matrix Q. As coefficients of the matrix Q satisfy **quantum permutation relations**, it is easy to see that

$$QW = WQ \iff Q(A - W) = (A - W)Q \iff QW' = W'Q.$$

From the above observation, the lemma follows using universality of $S^{Ban}_{(V,E)}$ and $S^{Ban}_{(V,E')}$. □

2.6 Free wreath product by quantum permutation groups

We recall the construction of **free wreath product by quantum permutation groups** formulated by Bichon in [Bic04]. If we consider any quantum subgroup of quantum permutation group, via same arguments in [Bic04] we can construct **free wreath product by subgroups of quantum permutation groups** (see [BB07a]).

Let $(\mathcal{B}, \Delta')$ be a quantum subgroup of S_n^+ where S_n^+ is the quantum permutation group on n elements. Let $(\mathcal{A}, \Delta)$ be another compact quantum group. We consider $\mathcal{A}^{*n}$ to be n times free product of the C* algebra $\mathcal{A}$ with the canonical inclusion maps $\nu_i : \mathcal{A} \to \mathcal{A}^{*n}$ where $i = 1, 2, .., n$. From proposition 2.4.1 it follows that $\mathcal{A}^{*n}$ has a natural co-product structure coming from $(\mathcal{A}, \Delta)$ making it a compact quantum group. We observe that there is a natural co-action $\alpha : \mathcal{A}^{*n} \to \mathcal{A}^{*n} \otimes \mathcal{B}$ of the CQG $(\mathcal{B}, \Delta')$ on the algebra $\mathcal{A}^{*n}$ which is given by,

$$\alpha(\nu_i(a)) = \sum_{j=1}^{n} \nu_j(a) \otimes x_{ji} \quad \text{where} \quad i = 1, 2, .., n \quad \text{and} \quad a \in \mathcal{A}. \tag{2.6.1}$$

Here $(x_{ij})_{i,j=1,..,n}$ is the matrix of canonical generators of $\mathcal{B}$ satisfying **quantum permutation relations**.

Definition 2.6.1. *The* **free wreath product of** $(\mathcal{A}, \Delta)$ **by** $(\mathcal{B}, \Delta')$ *is the quotient of the C* algebra $\mathcal{A}^{*n} * \mathcal{B}$ by a C* ideal generated by the elements:*

$$\nu_i(a)x_{ij} - x_{ij}\nu_i(a), \quad 1 \le i, j \le n, \quad a \in \mathcal{A}.$$

The free wreath product of $(\mathcal{A}, \Delta)$ by $(\mathcal{B}, \Delta')$ will be denoted by $\mathcal{A} *_w \mathcal{B}$.

We recall theorem 3.2 from [Bic04] which describes the co-product structure on $\mathcal{A} *_w \mathcal{B}$.

Theorem 2.6.2. *There is a natural co-product structure Δ_w on $\mathcal{A} *_w \mathcal{B}$ making it a compact quantum group. The co-product Δ_w satisfies:*

$$\Delta_w(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}, \quad \Delta_w(\nu_i(a)) = \sum_{k=1}^{n} \nu_i \otimes \nu_k(\Delta(a))(x_{ik} \otimes 1).$$

for all $i, j = 1, .., n$ and $a \in \mathcal{A}$.

As an immidiate application of the above construction in the theory of quantum symmetry in simple graphs we state theorem 4.2 from [Bic04]. (see also theorem 7.1 from [BB07a]) .

Theorem 2.6.3. *Let (V, E) be a finite connected simple graph without loops. Let us consider a simple graph (V^n, E^n) which is the disjoint union of n copies of (V, E). We have the following isomorphisms:*

$$S^{Bic}_{(V^n,E^n)} \cong S^{Bic}_{(V,E)} *_w S_n^+,$$
$$S^{Ban}_{(V^n,E^n)} \cong S^{Ban}_{(V,E)} *_w S_n^+$$

where the underlying co-action of S_n^+ is given in equation 2.6.1.

2.7 Setup and Notations

In this section we introduce some notations and conventions that we will use throughout the rest of this thesis. Let (V, E) be a directed graph with source and target maps $s : E \to V$ and $t : E \to V$. We further assume that there is no isolated vertex, that is, every vertex is either an initial or final vertex of some edge.

1. For $i, j \in V$ we denote the the subsets E^i, E_j and E_j^i of E by the following descriptions:

$$E_j^i := \{\tau \in E | s(\tau) = i \text{ and } t(\tau) = j\};$$
$$E^i := \{\tau \in E | s(\tau) = i\}; \qquad E_j := \{\tau \in E | t(\tau) = j\}.$$

2. If (V, E, j) is an undirected multigraph (see definition 2.1.7) then we will write $j(\tau) = \overline{\tau}$ for all $\tau \in E$. We also define a map $J : L^2(E) \to L^2(E)$ by

$$J(\chi_\tau) = \chi_{\overline{\tau}} \quad \text{where} \quad \tau \in E. \tag{2.7.1}$$

   It is clear that $\overline{\tau} = \tau$ when τ is a loop in (V, E, j).

3. As we are working generally with "directed" graphs, it is important to differentiate between initial and final vertex sets. Let us define $V^s \subseteq V$ and $V^t \subseteq V$ by

$$V^s = s(E) \quad \text{and} \quad V^t = t(E).$$

   As our graphs do not have any isolated vertex, it is clear that $V = V^s \cup V^t$.

4. There is a natural $C(V^s) - C(V^t)$ bimodule structure on $L^2(E)$ which is given by

$$\chi_i \cdot \chi_\tau = \delta_{i,s(\tau)} \chi_\tau \quad \text{and} \quad \chi_\tau \cdot \chi_j = \delta_{t(\tau),j} \chi_\tau \tag{2.7.2}$$

   where $i \in V^s, j \in V^t$ and $\tau \in E$. The Hilbert space $L^2(E)$ can also be treated as a $C(V) - C(V)$ bimodule with the same left and right module multiplication maps given by equations 2.7.2.

5. For $\tau \in E$, let p_τ denote the orthogonal projection onto a subspace generated by χ_τ in $L^2(E)$. We define two injective algebra maps $S : C(V^s) \to B(L^2(E))$ and $T : C(V^t) \to$

$B(L^2(E))$ by

$$S(\chi_v) = \sum_{\tau \in E^v} p_\tau \quad \text{and} \quad T(\chi_w) = \sum_{\tau \in E_w} p_\tau$$

for all v in V^s and w in V^t.

6. For $i, j \in V$ with $E^i_j \neq \phi$, let p_{ij} be the orthogonal projection onto a linear subspace in $L^2(E)$ generated by the elements $\{\chi_\tau | \tau \in E^i_j\}$. Let us define the following subalgebras in $B(L^2(E))$ by

$$M_{ij} := p_{ij} B(L^2(E)) p_{ij} \quad \text{and} \quad D_{ij} := p_{ij} D p_{ij}$$

where D is the algebra of diagonal operators spanned by the elements $\{p_\tau | \tau \in E\}$.

Chapter 3

Re-visiting quantum symmetry in simple graphs

Let us fix a simple graph (V, E) with adjacency matrix W, source and target maps $s : E \to V$ and $t : E \to V$. As an undirected simple graph is nothing but a "doubly directed" simple graph (see definition 2.1.6), it is enough to deal with the directed case. But we will indeed distinguish between "undirecteness" and "directedness" when we will be dealing with multigraphs in general.

3.1 Equivalent characterisations of quantum symmetries in a simple graph

Notation 3.1.1. *By a "bi-unitary" co-representation β of a compact quantum group $(\mathcal{A}, \Delta)$ we mean that β is a finite dimensional unitary co-representation such that the adjoint co-representation $\overline{\beta}$ is also unitary.*

Let us define $\xi_0 = \sum_{\tau \in E} \chi_\tau \in L^2(E)$. ξ_0 is the multiplicative identity when considered as an element in the algebra $C(E)$.

3.1.1 Banica's notion of quantum symmetry

Proposition 3.1.2. *Let $\alpha : C(V) \to C(V) \otimes \mathcal{A}$ be a co-action of a compact quantum group $(\mathcal{A}, \Delta)$ on $C(V)$. Then the following conditions are equivalent:*

1. *α preserves the quantum symmetry of (V, E) in Banica's sense.*
2. *There exists a bi-unitary co-representation $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ such that,*
   (a) *$\beta(\xi_0) = \xi_0 \otimes 1$.*

(b) $\alpha(\chi_i)\beta(\chi_\tau) = \beta(\chi_i.\chi_\tau)$ *and* $\beta(\chi_\tau)\alpha(\chi_i) = \beta(\chi_\tau.\chi_i)$ *for all* $i \in V$ *and* $\tau \in E$.

Proof. Let us consider a bi-unitary co-representation $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ with co-representation matrix $U = (u^\sigma_\tau)_{\sigma,\tau\in E}$.

We observe that,

$$\beta(\xi_0) = \xi_0 \otimes 1 \iff \sum_{\tau\in E} u^\sigma_\tau = 1 \quad \text{for all} \quad \sigma \in E. \tag{3.1.1}$$

Let $i \in V$ and $\tau \in E$. It follows that,

$$\begin{aligned}\alpha(\chi_i)\beta(\chi_\tau) &= (\sum_{i'\in V} \chi_{i'} \otimes q^{i'}_i)(\sum_{\sigma\in E} \chi_\sigma \otimes u^\sigma_\tau)\\ &= \sum_{\sigma\in E} \chi_\sigma \otimes q^{s(\sigma)}_i u^\sigma_\tau\end{aligned}$$

where $Q = (q^i_j)_{i,j\in V}$ is the co-representation matrix of the co-action α. Similarly, we have,

$$\beta(\chi_\tau)\alpha(\chi_i) = \sum_{\sigma\in E} \chi_\sigma \otimes u^\sigma_\tau q^{t(\sigma)}_i.$$

Hence we observe that, for all $\tau \in E$,

$$\alpha(\chi_i)\beta(\chi_\sigma) = \beta(\chi_i.\chi_\sigma) \iff q^{s(\sigma)}_i u^\sigma_\tau = \delta_{i,s(\tau)} u^\sigma_\tau \tag{3.1.2}$$

$$\text{and} \quad \beta(\chi_\tau)\alpha(\chi_i) = \beta(\chi_\tau.\chi_i) \iff u^\sigma_\tau q^{t(\sigma)}_i = \delta_{i,t(\tau)} u^\sigma_\tau \tag{3.1.3}$$

for all $\sigma \in E$.

Claim:$(1) \implies (2)$.

Let $(q^i_j)_{i,j\in V}$ be the co-representation matrix of α. From proposition 2.5.11 we have,

$$\alpha^{(2)}(L^2(E)) \subseteq L^2(E) \otimes \mathcal{A}.$$

By taking $\beta = \alpha^{(2)}|_{L^2(E)}$ it follows that $u^\sigma_\tau = q^{s(\sigma)}_{s(\tau)} q^{t(\sigma)}_{t(\tau)}$. As coefficients of the matrix $(q^i_j)_{i,j\in V}$ satisfy **quantum permutation relations**, using observations 3.1.1, 3.1.2 and 3.1.3, (2) follows.

Claim:$(2) \implies (1)$.

Let $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ be a bi-unitary co-representation satisfying conditions in (2). Using observations 3.1.1, 3.1.2, 3.1.3, it follows that,

$$u^\sigma_\tau = q^{s(\sigma)}_{s(\tau)} u^\sigma_\tau q^{t(\sigma)}_{t(\tau)} = q^{s(\sigma)}_{s(\tau)} (\sum_{\tau' \in E} u^\sigma_{\tau'}) q^{t(\sigma)}_{t(\tau)} = q^{s(\sigma)}_{s(\tau)} q^{t(\sigma)}_{t(\tau)}. \tag{3.1.4}$$

Let $\mathcal{B}$ be the Woronowicz C* subalgebra in $\mathcal{A}$ generated by the elements $\{q^i_j | i, j \in V\}$. As quantum permutation group is a quantum group of Kac type, the Haar measure h on $\mathcal{A}$, when restricted to $\mathcal{B}$, is tracial. Viewing $L^2(E)$ as a linear subspace of $C(V) \otimes C(V)$, we observe that, for $(k, l) \in E$,

$$\begin{aligned}
\sum_{\substack{i,j \in V \\ (i,j) \notin E}} h(q^i_k q^j_l q^i_k) &= \sum_{\substack{i,j \in V \\ (i,j) \notin E}} h(q^i_k q^j_l) \\
&= h(\sum_{i,j \in V} q^i_k q^j_l - \sum_{\substack{i,j \in V \\ (i,j) \in E}} q^i_k q^j_l) \\
&= h(1 - \sum_{(i,j) \in E} u^{(i,j)}_{(k,l)}) = 0 \quad \text{(from observation 3.1.1)}.
\end{aligned}$$

As h is faithful on the underlying Hopf * algebra of matrix elements of $\mathcal{A}$, it follows that,

$$\|q^i_k q^j_l\|^2 = \|q^i_k q^j_l q^i_k\| = 0$$

whenever $(i, j) \notin E$ and $(k, l) \in E$. Hence it follows that $\alpha^{(2)}(L^2(E)) \subseteq L^2(E) \otimes \mathcal{A}$ and using proposition 2.5.11 our claim is proved. □

3.1.2 Bichon's notion of quantum symmetry

From proposition 2.5.3 and similar arguments used above we give Bichon's version of proposition 3.1.2.

Proposition 3.1.3. *Let $\alpha : C(V) \to C(V) \otimes \mathcal{A}$ and $\beta : C(E) \to C(E) \otimes \mathcal{A}$ be two co-actions of a CQG $(\mathcal{A}, \Delta)$. Then the following conditions are equivalent:*

1. *(α, β) preserves quantum symmetry of (V, E) in Bichon's sense.*

2. *(α, β) respects the bi-module structure of (V, E), that is,*

$$\alpha(\chi_i)\beta(\chi_\tau) = \beta(\chi_i.\chi_\tau) \quad \textit{and} \quad \beta(\chi_\tau)\alpha(\chi_i) = \beta(\chi_\tau.\chi_i)$$

for all $i \in V$ and $\tau \in E$.

Remark 3.1.4. *In proposition 3.1.2 and proposition 3.1.3, if β satisfying (2) exists, then it is unique and is given by $\alpha^{(2)}|_{L^2(E)}$.*

The above propositions state the importance and effectiveness of the bi-module structure any simple graph has. The next two theorems give us the formula for capturing "permutation" of vertices in terms of "permutation" of edges which will be crucial for defining the notion of quantum symmetry in a multigraph.

3.2 Some useful observations

We will make some observations which we will be using several times through out this chapter. We will use the following characterization of $S(C(V^s))$ and $T(C(V^t))$ in $B(L^2(E))$ stated below:

Lemma 3.2.1. *Let $F \in B(L^2(E))$. Then,*

1. *$F \in S(C(V^s))$ if and only if the following holds:*

   For $\tau, \tau_1, \tau_2 \in E$, $F(\chi_\tau) = c_\tau \chi_\tau$ for some $c_\tau \in \mathbb{C}$ and $c_{\tau_1} = c_{\tau_2}$ whenever $s(\tau_1) = s(\tau_2)$.

2. *$F \in T(C(V^t))$ if and only if the following holds:*

   For $\tau, \tau_1, \tau_2 \in E$, $F(\chi_\tau) = c_\tau \chi_\tau$ for some $c_\tau \in \mathbb{C}$ and $c_{\tau_1} = c_{\tau_2}$ whenever $t(\tau_1) = t(\tau_2)$.

We will also need the another technical lemma for proceeding further.

Lemma 3.2.2. *Let $\{A_i | i = 1, 2, .., n\}$ be a finite set of operators on a Hilbert space and p , q be two projections.*

1. *If $\sum_{i=1}^n A_i A_i^* = p$, then $pA_i = A_i$.*

2. *If $\sum_{i=1}^n A_i^* A_i = q$, then $A_i q = A_i$.*

Proof. To prove (1), we observe that $A_i A_i^* \leq p$ for all i. It is enough to show that $\|(1-p)A_i\| = 0$ for all i.

$$\|(1-p)A_i\|^2 = \|(1-p)A_i A_i^*(1-p)\| \leq \|(1-p)p(1-p)\| = 0.$$

To prove (2), it is enough to observe that $qA_i^* = A_i^*$ which we get by replacing A_i with A_i^* in the first identity. □

3.3 Left equivariant co-representations on $L^2(E)$

Seeing $L^2(E)$ as a left $C(V^s)$ module we formulate an equivalent criterion for left equivariant bi-unitary co-representations on $L^2(E)$.

Theorem 3.3.1. *Let $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ be a bi-unitary co-representation of a CQG $(\mathcal{A}, \Delta)$. Let Ad_β be the co-action on $B(L^2(E))$ implemented by the unitary co-representation β (see definition 2.3.6). Then the following conditions are equivalent:*

1. *$Ad_\beta(S(C(V^s))) \subseteq S(C(V^s)) \otimes \mathcal{A}$.*
2. *There exists a co-action $\alpha_s : C(V^s) \to C(V^s) \otimes \mathcal{A}$ such that,*

$$\alpha_s(\chi_i)\beta(\chi_\tau) = \beta(\chi_i.\chi_\tau)$$

for all $i \in V^s$ and $\tau \in E$.

Proof. Let $U = (u^\sigma_\tau)_{\sigma,\tau \in E}$ be the co-representation matrix of β.

We make some observations first before proving the equivalence. Let us fix $k \in V^s$ and $\sigma_2 \in E$. We observe that,

$$\begin{aligned} Ad_\beta(S(\chi_k))(\chi_{\sigma_2} \otimes 1) &= Ad_\beta(\sum_{\tau \in E^k} p_\tau)(\chi_{\sigma_2} \otimes 1) \\ &= \beta(\sum_{\tau \in E^k} p_\tau \otimes 1)(\sum_{\tau' \in E} \chi_{\tau'} \otimes u^{\sigma_2 *}_{\tau'}) \\ &= \beta(\sum_{\tau \in E^k} \chi_\tau \otimes u^{\sigma_2 *}_\tau) \\ &= \sum_{\sigma_1 \in E} \chi_{\sigma_1} \otimes (\sum_{\tau \in E^k} u^{\sigma_1}_\tau u^{\sigma_2 *}_\tau). \end{aligned}$$

Applying lemma 3.2.1 we get that, for all $k \in V^s$ and $\sigma_1, \sigma_2 \in E$,

$$\begin{aligned} Ad_\beta(S(C(V^s))) \subseteq S(C(V^s)) \otimes \mathcal{A} \iff & \sum_{\tau \in E^k} u^{\sigma_1}_\tau u^{\sigma_2 *}_\tau = 0 \quad \text{if} \quad \sigma_1 \neq \sigma_2 \\ \text{and} \quad & \sum_{\tau \in E^k} u^{\sigma_1}_\tau u^{\sigma_1 *}_\tau = \sum_{\tau \in E^k} u^{\sigma_2}_\tau u^{\sigma_2 *}_\tau \quad \text{if} \quad s(\sigma_1) = s(\sigma_2). \end{aligned} \tag{3.3.1}$$

Let $\alpha_s : C(V^s) \to C(V^s) \otimes \mathcal{A}$ be a co-action on $C(V^s)$ with co-representation matrix $(q^i_j)_{i,j \in V^s}$. Let $i \in V^s$ and $\tau \in E$. As before we observe that,

$$\alpha_s(\chi_i)\beta(\chi_\tau) = \beta(\chi_i.\chi_\tau) \iff q^{s(\sigma)}_i u^\sigma_\tau = \delta_{i,s(\tau)} u^\sigma_\tau \tag{3.3.2}$$

for all $\sigma \in E$.

Claim: $(1) \implies (2)$.

From our assumption and observation 3.3.1 it follows that,

$$Ad_\beta(S(\chi_k))(\chi_{\sigma_2} \otimes 1) = \chi_{\sigma_2} \otimes (\sum_{\tau \in E^k} u_\tau^{\sigma_2} u_\tau^{\sigma_2 *}) \quad \text{for all} \quad k \in V^s, \sigma_2 \in E. \tag{3.3.3}$$

For $k \in V^s$ and $\sigma_2 \in E$, let us define

$$\sum_{\tau \in E^k} u_\tau^{\sigma_2} u_\tau^{\sigma_2 *} = q_k^{s(\sigma_2)}. \tag{3.3.4}$$

From equation 3.3.3 we further observe that,

$$\begin{aligned} Ad_\beta(S(\chi_k)) &= \sum_{\sigma \in E} p_\sigma \otimes q_k^{s(\sigma)} \\ &= \sum_{i \in V^s} (\sum_{\sigma \in E^i} p_\sigma) \otimes q_k^i \\ &= \sum_{i \in V^s} S(\chi_i) \otimes q_k^i. \end{aligned}$$

As $C(V^s) \cong S(C(V^s))$ as algebras and Ad_β is already a co-action on $S(C(V^s))$, we define a **quantum permutation** $\alpha_s : C(V^s) \to C(V^s) \otimes \mathcal{A}$ by the following expression:

$$\alpha_s(\chi_k) = \sum_{i \in V^s} \chi_i \otimes q_k^i \quad \text{for all} \quad k \in V^s.$$

Let us now fix $i \in V^s$ and $\sigma, \tau \in E$. From equation 3.3.4 and lemma 3.2.2 it follows that,

$$q_i^{s(\sigma)} u_\tau^\sigma = q_i^{s(\sigma)} (q_{s(\tau)}^{s(\sigma)} u_\tau^\sigma) = \delta_{i, s(\tau)} u_\tau^\sigma.$$

Using observation 3.3.2 we conclude that (2) follows.

Claim:$(2) \implies (1)$.

Let $(q_j^i)_{i,j \in V^s}$ and $(u_\tau^\sigma)_{\sigma, \tau \in E}$ be co-representation matrices of α_s and β.

Let $\sigma_1, \sigma_2 \in E$ and $k \in V^s$. As β is unitary, using observation 3.3.2 it follows that,

$$\sum_{\tau \in E^k} u_\tau^{\sigma_1} u_\tau^{\sigma_2 *} = q_k^{s(\sigma_1)} (\sum_{\tau \in E} u_\tau^{\sigma_1} u_\tau^{\sigma_2 *}) q_k^{s(\sigma_2)} = \delta_{\sigma_1, \sigma_2} q_k^{s(\sigma_1)} q_k^{s(\sigma_2)}.$$

Hence we get,

$$\sum_{\tau\in E^k} u_\tau^{\sigma_1} u_\tau^{\sigma_2 *} = 0 \quad \text{if} \quad \sigma_1 \neq \sigma_2$$
$$\text{and} \quad \sum_{\tau\in E^k} u_\tau^{\sigma_1} u_\tau^{\sigma_1 *} = \sum_{\tau\in E^k} u_\tau^{\sigma_2} u_\tau^{\sigma_2 *} \quad \text{if} \quad s(\sigma_1) = s(\sigma_2).$$

Therefore (1) follows from observation 3.3.1. □

3.4 Right equivariant co-representations on $L^2(E)$

Seeing $L^2(E)$ as a right $C(V^t)$ module we formulate a equivalent criterion for right equivariant bi-unitary co-representations on $L^2(E)$.

Theorem 3.4.1. *Let $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ be a bi-unitary co-representation of a CQG $(\mathcal{A}, \Delta)$. Let us consider the co-action $Ad_{\overline{\beta}}$ on $B(L^2(E))$ implemented by unitary co-representation $\overline{\beta}$. Then the following conditions are equivalent:*

1. *$Ad_{\overline{\beta}}(T(C(V^t))) \subseteq T(C(V^t)) \otimes \mathcal{A}$.*
2. *There exists a co-action $\alpha_t : C(V^t) \to C(V^t) \otimes \mathcal{A}$ such that,*

$$\beta(\chi_\tau)\alpha_t(\chi_j) = \beta(\chi_\tau.\chi_j)$$

for all $j \in V^t$ and $\tau \in E$.

Proof. The proof is done using similar arguments as in the proof of theorem 3.3.1. Let $U = (u_\tau^\sigma)_{\sigma,\tau\in E}$ be the co-representation matrix of β.

As before we make some observations first before proving the equivalence. Let us fix $l \in V^t$ and $\sigma_2 \in E$. We observe that,

$$\begin{aligned} Ad_{\overline{\beta}}(T(\chi_l))(\chi_{\sigma_2} \otimes 1) &= Ad_{\overline{\beta}}(\sum_{\tau\in E_l} p_\tau)(\chi_{\sigma_2} \otimes 1) \\ &= \overline{\beta}(\sum_{\tau\in E_l} p_\tau \otimes 1)(\sum_{\tau'\in E} \chi_{\tau'} \otimes u_{\tau'}^{\sigma_2}) \\ &= \overline{\beta}(\sum_{\tau\in E_l} \chi_\tau \otimes u_\tau^{\sigma_2}) \\ &= \sum_{\sigma_1\in E} \chi_{\sigma_1} \otimes (\sum_{\tau\in E_l} u_\tau^{\sigma_1 *} u_\tau^{\sigma_2}). \end{aligned}$$

Applying lemma 3.2.1 we get that, for all $l \in V^t$ and $\sigma_1, \sigma_2 \in E$,

$$Ad_{\overline{\beta}}(T(C(V^t))) \subseteq T(C(V^t)) \otimes \mathcal{A} \iff \sum_{\tau \in E_l} u_\tau^{\sigma_1 *} u_\tau^{\sigma_2} = 0 \quad \text{if} \quad \sigma_1 \neq \sigma_2 \in E$$
$$\text{and} \quad \sum_{\tau \in E_l} u_\tau^{\sigma_1 *} u_\tau^{\sigma_1} = \sum_{\tau \in E_l} u_\tau^{\sigma_2 *} u_\tau^{\sigma_2} \quad \text{if} \quad t(\sigma_1) = t(\sigma_2). \tag{3.4.1}$$

Let $\alpha_t : C(V^t) \to C(V^t) \otimes \mathcal{A}$ be a co-action on $C(V^t)$ with co-representation matrix $R = (r^i_j)_{i,j \in V^t}$. Let $j \in V^t$ and $\tau \in E$. As before we observe that,

$$\beta(\chi_\tau)\alpha_t(\chi_j) = \beta(\chi_\tau.\chi_j) \iff u_\tau^\sigma r_j^{t(\sigma)} = \delta_{t(\tau),j} u_\tau^\sigma \tag{3.4.2}$$

for all $\sigma \in E$.

Claim: $(1) \implies (2)$.

From our assumption and observation 3.4.1 it follows that,

$$Ad_{\overline{\beta}}(T(\chi_l))(\chi_{\sigma_2} \otimes 1) = \chi_{\sigma_2} \otimes (\sum_{\tau \in E_l} u_\tau^{\sigma_2 *} u_\tau^{\sigma_2}) \quad \text{for all} \quad l \in V^t, \sigma_2 \in E. \tag{3.4.3}$$

For $l \in V^t$ and $\sigma_2 \in E$, let us define

$$\sum_{\tau \in E_l} u_\tau^{\sigma_2 *} u_\tau^{\sigma_2} = r_l^{t(\sigma_2)}. \tag{3.4.4}$$

From equation 3.4.3 we further observe that,

$$\begin{aligned} Ad_{\overline{\beta}}(T(\chi_l)) &= \sum_{\sigma \in E} p_\sigma \otimes r_l^{t(\sigma)} \\ &= \sum_{j \in V^t} (\sum_{\sigma \in E_j} p_\sigma) \otimes r_l^j \\ &= \sum_{j \in V^t} T(\chi_j) \otimes r_l^j. \end{aligned}$$

As $C(V^t) \cong T(C(V^t))$ as algebras and $Ad_{\overline{\beta}}$ is already a co-action on $T(C(V^t))$, on V^t we define a **quantum permutation** $\alpha_t : C(V^t) \to C(V^t) \otimes \mathcal{A}$ by the following expression:

$$\alpha_t(\chi_l) = \sum_{j \in V^t} \chi_j \otimes r_l^j \quad \text{for all} \quad l \in V^t.$$

Let us fix $j \in V^t$ and $\sigma, \tau \in E$. From equation 3.4.4 and lemma 3.2.2 it follows that,

$$u_\tau^\sigma r_j^{t(\sigma)} = (u_\tau^\sigma r_{t(\tau)}^{t(\sigma)}) r_j^{t(\sigma)} = \delta_{t(\tau),j} u_\tau^\sigma.$$

Using observation 3.4.2 we conclude that (2) follows.

Claim:$(2) \implies (1)$.

Let $R = (r_j^i)_{i,j \in V^t}$ and $U = (u_\tau^\sigma)_{\sigma,\tau \in E}$ be co-representation matrices of α_t and β.

Let $\sigma_1, \sigma_2 \in E$ and $l \in V^t$. As $\overline{\beta}$ is unitary, using observation 3.4.2 it follows that,

$$\sum_{\tau \in E_l} u_\tau^{\sigma_1 *} u_\tau^{\sigma_2} = r_l^{t(\sigma_1)} (\sum_{\tau \in E} u_\tau^{\sigma_1 *} u_\tau^{\sigma_2}) r_l^{t(\sigma_2)} = \delta_{\sigma_1,\sigma_2} r_l^{t(\sigma_1)} r_l^{t(\sigma_2)}.$$

Hence we get,

$$\sum_{\tau \in E_l} u_\tau^{\sigma_1 *} u_\tau^{\sigma_2} = 0 \quad \text{if} \quad \sigma_1 \neq \sigma_2$$

$$\text{and} \quad \sum_{\tau \in E_l} u_\tau^{\sigma_1 *} u_\tau^{\sigma_1} = \sum_{\tau \in E_l} u_\tau^{\sigma_2 *} u_\tau^{\sigma_2} \quad \text{if} \quad t(\sigma_1) = t(\sigma_2).$$

Therefore (1) follows from observation 3.4.1. □

3.4.1 Induced permutations on V^s and V^t

It is clear that the co-actions α_s and α_t satisfying (2) in theorem 3.3.1 and theorem 3.4.1 are essentially unique as they are completely determined by the bi-unitary co-representation β. Given a bi-unitary co-representation β satisfying (1) in theorem 3.3.1 and theorem 3.4.1, we will refer α_s and α_t as **induced co-actions on** $C(V^s)$ **and** $C(V^t)$.

3.5 Induced permutations on $V^s \cap V^t$

3.5.1 Two graphs with isomorphic bimodule structure

It is not enough to only consider $C(V^s) - L^2(E) - C(V^t)$ bimodularity to capture the whole picture of quantum symmetry of (V, E). There does exist non-isomorphic graphs which have non-isomorphic quantum automorphism groups but isomorphic $C(V^s) - L^2(E) - C(V^t)$ bi-module structure. See the graphs in figure 3.1 for example, where the left one does not have any quantum symmetry (in Banica's sense) but the right one does have. Continuing our investigations further, we propose the following result:

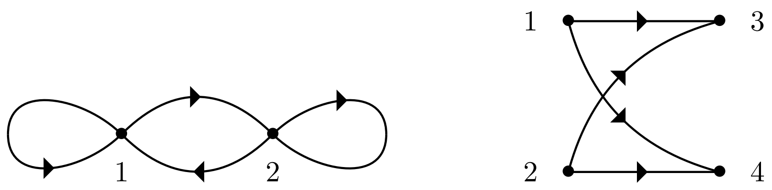

FIGURE 3.1: Two graphs with isomorphic $C(V^s) - L^2(E) - C(V^t)$ bimodule structure.

3.5.2 Right equivariance of α_s and left equivariance of α_t

Theorem 3.5.1. *Let $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ be a bi-unitary co-representation of a CQG $(\mathcal{A}, \Delta)$ such that the following conditions hold:*

1. $Ad_\beta(S(C(V^s))) \subseteq S(C(V^s)) \otimes \mathcal{A}$.
2. $Ad_{\overline{\beta}}(T(C(V^t))) \subseteq T(C(V^t)) \otimes \mathcal{A}$.

Furthermore, we also assume that the induced co-actions α_s and α_t (see subsection 3.4.1) both preserve $C(V^s \cap V^t)$, that is,

$$\alpha_s(C(V^s \cap V^t)) \subseteq C(V^s \cap V^t) \otimes \mathcal{A} \subseteq C(V^s) \otimes \mathcal{A},$$
$$\alpha_t(C(V^s \cap V^t)) \subseteq C(V^s \cap V^t) \otimes \mathcal{A} \subseteq C(V^t) \otimes \mathcal{A}.$$

Then the following conditions are equivalent:

1. $\alpha_s|_{C(V^s \cap V^t)} = \alpha_t|_{C(V^s \cap V^t)}$.
2. *For all $j \in V^s \cap V^t$ and $\tau \in E$,*
$$\beta(\chi_\tau)\alpha_s(\chi_j) = \beta(\chi_\tau . \chi_j).$$
3. *For all $i \in V^s \cap V^t$ and $\tau \in E$,*
$$\alpha_t(\chi_i)\beta(\chi_\tau) = \beta(\chi_i . \chi_\tau).$$

Proof. We define $E_{V^s}, E^{V^t} \subseteq E$ by

$$E_{V^s} = \{\tau \in E |\ t(\tau) \in V^s \cap V^t\},$$
$$E^{V^t} = \{\tau \in E |\ s(\tau) \in V^s \cap V^t\}.$$

We make some observations first. Let $(u_\tau^\sigma)_{\sigma,\tau\in E}$, $(q_j^l)_{l,j\in V^s}$ and $(r_i^k)_{k,i\in V^t}$ be co-representation matrices of β, α_s and α_t respectively. For $\tau \in E$ and $j \in V^s \cap V^t$ we observe that,

$$\begin{aligned}\beta(\chi_\tau)\alpha_s(j) &= (\sum_{\sigma\in E}\chi_\sigma \otimes u_\tau^\sigma)(\sum_{l\in V^s\cap V^t}\chi_l \otimes q_j^l)\\ &= \sum_{\sigma\in E_{V^s}}\chi_\sigma \otimes u_\tau^\sigma q_j^{t(\sigma)}.\end{aligned}$$

Hence for all $j \in V^s \cap V^t$ and $\tau \in E$,

$$\beta(\chi_\tau)\alpha_s(j) = \beta(\chi_\tau.\chi_j)$$

if and only if

$$\begin{aligned}&\beta(L^2(E_{V^s})) \subseteq L^2(E_{V^s}) \otimes \mathcal{A} \quad \text{and}\\ &u_\tau^\sigma q_j^{t(\sigma)} = \delta_{t(\tau),j}u_\tau^\sigma \quad \text{whenever} \quad \sigma \in E_{V^s}.\end{aligned} \tag{3.5.1}$$

Similarly it also follows that, for all $i \in V^s \cap V^t$ and $\tau \in E$,

$$\alpha_t(\chi_i)\beta(\chi_\tau) = \beta(\chi_i.\chi_\tau)$$

if and only if

$$\begin{aligned}&\beta(L^2(E^{V^t})) \subseteq L^2(E^{V^t}) \otimes \mathcal{A} \quad \text{and}\\ &r_i^{s(\sigma)}u_\tau^\sigma = \delta_{i,s(\tau)}u_\tau^\sigma \quad \text{whenever} \quad \sigma \in E^{V^t}.\end{aligned} \tag{3.5.2}$$

Now we proceed to prove our theorem.

Claim:$(1) \implies (2);(1) \implies (3)$.

As $\alpha_s|_{C(V^s\cap V^t)} = \alpha_t|_{C(V^s\cap V^t)}$, for $i \in V^s \cap V^t$ we have,

$$\begin{aligned}&q_i^k = r_i^k \quad \text{when} \quad k \in V^s \cap V^t \quad \text{and}\\ &q_i^k = 0 = r_i^l \quad \text{when} \quad k \in V^s \setminus V^t, l \in V^t \setminus V^s.\end{aligned}$$

From above expressions, theorem 3.4.1 and theorem 3.3.1 it follows that, for $\sigma, \tau \in E$,

$$\begin{aligned}&u_\tau^\sigma = u_\tau^\sigma r_{t(\tau)}^{t(\sigma)} = 0 \quad \text{whenever} \quad \sigma \notin E_{V^s} \quad \text{but} \quad \tau \in E_{V^s}\\ \text{and} \quad &u_\tau^\sigma = q_{s(\tau)}^{s(\sigma)}u_\tau^\sigma = 0 \quad \text{whenever} \quad \sigma \notin E^{V^t} \quad \text{but} \quad \tau \in E^{V^t}.\end{aligned}$$

Hence we have,

$$\beta(L^2(E_{V^s})) \subseteq L^2(E_{V^s}) \otimes \mathcal{A}$$
$$\text{and} \quad \beta(L^2(E^{V^t})) \subseteq L^2(E^{V^t}) \otimes \mathcal{A}.$$

We also observe that for $i, j \in V^s \cap V^t, \sigma_1 \in E_{V^s}$ and $\sigma_2 \in E^{V^t}$,

$$u_\tau^{\sigma_1} q_j^{t(\sigma_1)} = u_\tau^{\sigma_1} r_j^{t(\sigma_1)} = \delta_{j,t(\tau)} u_\tau^{\sigma_1}$$
$$\text{and} \quad r_i^{s(\sigma_2)} u_\tau^{\sigma_2} = q_i^{s(\sigma_2)} u_\tau^{\sigma_2} = \delta_{i,s(\tau)} u_\tau^{\sigma_2}.$$

As our choice of i, j, σ_1, σ_2 was arbitrary, from observations 3.5.1 and 3.5.2, (2) and (3) follow.

Claim:$(2) \implies (1)$.

Let $i, k \in V^s \cap V^t$ and $\sigma \in E$ be such that $t(\sigma) = k$. Using equations 3.4.4 and 3.5.1 we observe that,

$$r_i^k = \sum_{\tau \in E_i} u_\tau^{\sigma *} u_\tau^\sigma = \sum_{\tau \in E_i} u_\tau^{\sigma *} u_\tau^\sigma q_i^k = r_i^k q_i^k. \tag{3.5.3}$$

Hence it follows that,

$$r_i^k \leq q_i^k \quad \text{for all} \quad i, k \in V^s \cap V^t. \tag{3.5.4}$$

As coefficients of both matrices $(q_i^k)_{k,i \in V^s \cap V^t}$ and $(r_i^k)_{k,i \in V^s \cap V^t}$ satisfy **quantum permutation relations** it follows that, for $i \in V^s \cap V^t$,

$$1 = \sum_{k \in V^s \cap V^t} r_i^k \leq \sum_{k \in V^s \cap V^t} q_i^k = 1.$$

As $\{r_i^k \,|\, k \in V^s \cap V^t\}$ and $\{q_i^k \,|\, k \in V^s \cap V^t\}$ both are sets of mutually orthogonal projections, we have

$$q_i^k = r_i^k \quad \text{for all} \quad i, k \in V^s \cap V^t.$$

Therefore (1) follows.

Claim:$(3) \implies (1)$

Let $i, k \in V^s \cap V^t$ and $\sigma \in E$ be such that $s(\sigma) = k$. Using equations 3.3.4 and 3.5.2 we observe that,

$$q_i^k = \sum_{\tau \in E^i} u_\tau^\sigma u_\tau^{\sigma *} = r_i^k (\sum_{\tau \in E^i} u_\tau^\sigma u_\tau^{\sigma *}) = r_i^k q_i^k.$$

hence it follows that,

$$q_i^k \leq r_i^k \quad \text{for all} \quad i, k \in V^s \cap V^t.$$

Using similar arguments used in the previous case, (1) follows. □

3.6 Bi-unitarity and inversion in undirected graphs

Now we move to the case of undirected simple graphs, that is, where the adjacency matrix W is a symmetric matrix. All the results related to bi-unitarity can be replaced with unitarity alone becaue of the existence of inversion map in an undirected graph. We observe the following result which will be useful when we will be dealing with undirected multigraphs.

Theorem 3.6.1. *Let (V, E) be an undirected simple graph and $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ be a unitary co-representation of a CQG $(\mathcal{A}, \Delta)$ such that the following conditions hold:*

1. $Ad_\beta(S(C(V))) \subseteq S(C(V)) \otimes \mathcal{A}$.
2. $\beta(\xi_0) = \xi_0 \otimes 1_\mathcal{A}$ *where* $\xi_0 = \sum_{\tau \in E} \chi_\tau$.

Then the following conditions ((1) and (2)) are equivalent:

1. $\beta \circ J = (J \otimes id_\mathcal{A}) \circ \overline{\beta}$.
2. (a) $\overline{\beta}$ *is unitary.*
   (b) $Ad_{\overline{\beta}}(T(C(V))) \subseteq T(C(V)) \otimes \mathcal{A}$.
   (c) *The induced co-actions α_s and α_t coincide on $C(V)$.*

The map $J : L^2(E) \to L^2(E)$ is defined by equation 2.7.1.

Proof. Let $(u^\sigma_\tau)_{\sigma,\tau \in E}$ be the co-representation matrix of β. We make some observations first. For $\tau \in E$,

$$\beta \circ J(\chi_\tau) = \sum_{\sigma \in E} \chi_\sigma \otimes u^{\sigma}_{\overline{\tau}} \quad \text{(see notaion 2 in section 2.7)}$$

$$\text{and} \quad (J \otimes id_\mathcal{A}) \circ \overline{\beta}(\chi_\tau) = \sum_{\sigma \in E} J(\chi_\sigma) \otimes u^{\sigma*}_\tau = \sum_{\sigma \in E} \chi_\sigma \otimes u^{\overline{\sigma}*}_\tau .$$

Hence we have,

$$\beta \circ J = (J \otimes id_\mathcal{A}) \circ \overline{\beta} \iff u^{\overline{\sigma}}_{\overline{\tau}} = u^{\sigma*}_\tau \tag{3.6.1}$$

for all $\sigma, \tau \in E$.

Claim:$(1) \implies (2)$.

Let α_s be the induced co-action on $C(V)$ with co-representation matrix $(q^i_j)_{i,j\in V}$. Since β is unitary, using 3.6.1, we observe that,

$$\sum_{\tau\in E} u^{\sigma_1 *}_{\tau} u^{\sigma_2}_{\tau} = \sum_{\overline{\tau}\in E} u^{\overline{\sigma_1}}_{\overline{\tau}} u^{\overline{\sigma_2}*}_{\overline{\tau}} = \delta_{\overline{\sigma_1},\overline{\sigma_2}} = \delta_{\sigma_1,\sigma_2}$$
$$\text{and} \quad \sum_{\tau\in E} u^{\tau}_{\sigma_1} u^{\tau *}_{\sigma_2} = \sum_{\overline{\tau}\in E} u^{\overline{\tau}}_{\overline{\sigma_1}}{}^{*} u^{\overline{\tau}}_{\overline{\sigma_2}} = \delta_{\overline{\sigma_1},\overline{\sigma_2}} = \delta_{\sigma_1,\sigma_2}.$$

Hence $\overline{\beta}$ is also unitary. We further observe that, for $l \in V$,

$$\begin{aligned}\sum_{\tau\in E_l} u^{\sigma_1 *}_{\tau} u^{\sigma_2}_{\tau} = \sum_{\overline{\tau}\in E^l} u^{\overline{\sigma_1}}_{\overline{\tau}} u^{\overline{\sigma_2}*}_{\overline{\tau}} &= q_l^{s(\overline{\sigma_1})}(\sum_{\overline{\tau}\in E} u^{\overline{\sigma_1}}_{\overline{\tau}} u^{\overline{\sigma_2}*}_{\overline{\tau}}) q_l^{s(\overline{\sigma_1})} \\ &= \delta_{\overline{\sigma_1},\overline{\sigma_2}} q_l^{s(\overline{\sigma_1})} q_l^{s(\overline{\sigma_2})} \\ &= \delta_{\sigma_1,\sigma_2} q_l^{t(\sigma_1)} q_l^{t(\sigma_2)}.\end{aligned}$$

From observation 3.4.1 and equation 3.4.4 it follows that,

$$Ad_{\overline{\beta}}(T(C(V))) \subseteq T(C(V)) \otimes \mathcal{A} \quad \text{and} \quad \alpha_s = \alpha_t.$$

Claim:$(2) \implies (1)$.

Let $\alpha = \alpha_s = \alpha_t$. From our hypothesis and theorem 3.1.2 it follows that $\beta = \alpha^{(2)}|_{L^2(E)}$ where $L^2(E)$ is identified as a linear subspace of $C(V) \otimes C(V)$.

From the definition of $\alpha^{(2)}$, for $\tau \in E$ we have,

$$\alpha^{(2)}(\chi_\tau) = \sum_{\sigma\in E} \chi_\sigma \otimes q^{s(\sigma)}_{s(\tau)} q^{t(\sigma)}_{t(\tau)}$$
$$\text{and} \quad \overline{\alpha^{(2)}}(\chi_\tau) = \sum_{\sigma\in E} \chi_\sigma \otimes q^{t(\sigma)}_{t(\tau)} q^{s(\sigma)}_{s(\tau)}.$$

We further observe that,

$$\begin{aligned}\alpha^{(2)} \circ J(\chi_\tau) &= \sum_{\sigma\in E} \chi_\sigma \otimes q^{s(\sigma)}_{t(\tau)} q^{t(\sigma)}_{s(\tau)} \\ &= \sum_{\sigma\in E} J(\chi_\sigma) \otimes q^{t(\sigma)}_{t(\tau)} q^{s(\sigma)}_{s(\tau)} \\ &= (J \otimes id_{\mathcal{A}})\overline{\alpha^{(2)}}(\chi_\tau).\end{aligned}$$

Hence (1) follows. $\square$

3.7 Equivalent definitions of quantum symmetries in a simple graph

In the light of above discussions we give alternative definitions of quantum symmetry in a simple graph formulated in terms of bi-unitary maps.

Theorem 3.7.1. *Let (V, E) be a simple graph and $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ be a bi-unitary co-representation of a CQG $(\mathcal{A}, \Delta)$ on $L^2(E)$. The following conditions ((1) and (2)) are equivalent:*

1. *The bi-unitary co-representation β satisfies the following properties:*

    (a) $Ad_\beta(S(C(V^s))) \subseteq S(C(V^s)) \otimes \mathcal{A}$.

    (b) $Ad_{\overline{\beta}}(T(C(V^t))) \subseteq T(C(V^t)) \otimes \mathcal{A}$.

    (c) The induced co-actions α_s and α_t (see remark 3.4.1) both preserve $C(V^s \cap V^t)$ and agree on $C(V^s \cap V^t)$, that is,

    $$\alpha_s|_{C(V^s \cap V^t)} = \alpha_t|_{C(V^s \cap V^t)}.$$

    (d) β fixes the element $\xi_0 := \sum_{\tau \in E} \chi_\tau \in L^2(E)$, that is,

    $$\beta(\xi_0) = \xi_0 \otimes 1_{\mathcal{A}}.$$

2. *There exists a co-action $\alpha : C(V) \to C(V) \otimes \mathcal{A}$ which preserves quantum symmetry of (V, E) in Banica's sense and $\alpha^{(2)}|_{L^2(E)} = \beta$.*

Proof. **Claim:**$(1) \implies (2)$.
We define the required co-action $\alpha : C(V) \to C(V) \otimes \mathcal{A}$ by

$$\begin{aligned} \alpha(\chi_k) &= \alpha_s(\chi_k) \quad \text{if} \quad k \in V^s, \\ &= \alpha_t(\chi_k) \quad \text{if} \quad k \in V^t. \end{aligned}$$

The map α is well defined because of (c) of condition (1) and is a co-action as both α_s and α_t are co-actions on $C(V^s)$ and $C(V^t)$ respectively. From theorems 3.3.1, 3.4.1 and proposition 3.1.2 it follows that α preserves quantum symmetry of (V, E) in Banica's sense. The equation $\alpha^{(2)}|_{L^2(E)} = \beta$ follows from remark 3.1.4.
Claim: $(2) \implies (1)$.

Let (q^i_j) be the co-representation matrix of α. As α preserves quantum symmetry of (V, E) in Banica's sense, we have $q^i_k q^j_l = 0$ when $(i,j) \notin E$ and $(k,l) \in E$. For $i \notin V^s$, $k \in V^s$ and $(k,l) \in E$, we observe that,

$$q^i_k = \sum_{j \in V} q^i_k q^j_l = 0 \quad \text{as } (k,l) \in E \text{ and } (i,j) \notin E \text{ for all } j \in V.$$

Therefore it follows that $\alpha(C(V^s)) \subseteq C(V^s) \otimes \mathcal{A}$ and by similar arguments, $\alpha(C(V^t)) \subseteq C(V^t) \otimes \mathcal{A}$.

Let us define two co-actions $\alpha_s : C(V^s) \to C(V^s) \otimes \mathcal{A}$ and $\alpha_t : C(V^t) \to C(V^t) \otimes \mathcal{A}$ by $\alpha_s = \alpha|_{C(V^s)}$ and $\alpha_t = \alpha|_{C(V^t)}$. From proposition 3.1.2 it follows that,

$$\alpha_s(\chi_i)\beta(\chi_\tau) = \beta(\chi_i.\chi_\tau) \quad \text{and} \quad \beta(\chi_\tau)\alpha_t(\chi_j) = \beta(\chi_\tau.\chi_j) \tag{3.7.1}$$

for all $i \in V^s$, $j \in V^t$ and $\tau \in E$. Using theorems 3.3.1 and 3.4.1, we observe that (a) and (b) of (1) hold. The co-actions α_s and α_t become induced co-actions of β (see remark 3.4.1) which preserve and agree on $C(V^s \cap V^t)$ by their definitions.

□

As every co-action on $C(E)$ is also a bi-unitary co-representation on $L^2(E)$, using similar arguments in the above proof we give Bichon's version of theorem 3.7.1.

Theorem 3.7.2. *Let (V, E) be a simple graph and $\beta : C(E) \to C(E) \otimes \mathcal{A}$ be a co-action of a CQG $(\mathcal{A}, \Delta)$ on $C(E)$. The following conditions ((1) and (2)) are equivalent:*

1. *The co-action β satisfies the following properties:*
   - *(a) $Ad_\beta(S(C(V^s))) \subseteq S(C(V^s)) \otimes \mathcal{A}$.*
   - *(b) $Ad_\beta(T(C(V^t))) \subseteq T(C(V^t)) \otimes \mathcal{A}$.*
   - *(c) The induced co-actions α_s and α_t (see remark 3.4.1) preserve $C(V^s \cap V^t)$ and agree on $C(V^s \cap V^t)$, that is,*

$$\alpha_s|_{C(V^s \cap V^t)} = \alpha_t|_{C(V^s \cap V^t)}.$$

2. *There exists a co-action $\alpha : C(V) \to C(V) \otimes \mathcal{A}$ such that the pair (α, β) preserves quantum symmetry of (V, E) in Bichon's sense.*

Chapter 4

Quantum symmetry in directed and undirected multigraphs

4.1 Induced permutations on V^s and V^t from permutation of edges

We will be defining notions of quantum symmetry in a directed multigraph first. Let us fix a directed multigraph (V, E) with source and target maps $s : E \to V$ and $t : E \to V$. We will indeed use the machinery we have developed in the previous chapter for simple graphs. As we have done for simple graphs in the previous chapter, we will use the same technique to capture quantum permutations on V^s and V^t from a bi-unitary co-representation β where Ad_β preserves $C(V^s)$ and $Ad_{\overline{\beta}}$ preserves $C(V^t)$.

Theorem 4.1.1. *Let $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ be a bi-unitary co-representation of a CQG $(\mathcal{A}, \Delta)$. Let Ad_β be the co-action on $B(L^2(E))$ implemented by the unitary co-representation β. Then the following are equivalent:*

1. *$Ad_\beta(S(C(V^s))) \subseteq S(C(V^s)) \otimes \mathcal{A}$.*

2. *There exists a co-action $\alpha_s : C(V^s) \to C(V^s) \otimes \mathcal{A}$ such that,*

$$\alpha_s(\chi_i)\beta(\chi_\tau) = \beta(\chi_i.\chi_\tau)$$

for all $i \in V^s$ and $\tau \in E$.

Proof. The proof is using similar arguments as in the context of a simple graph (see proof of theorem 3.3.1). □

Theorem 4.1.2. *Let $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ be a bi-unitary co-representation of a CQG $(\mathcal{A}, \Delta)$. Let $Ad_{\overline{\beta}}$ be the co-action on $B(L^2(E))$ implemented by $\overline{\beta}$. Then the following are equivalent:*

1. $Ad_{\overline{\beta}}(T(C(V^t))) \subseteq T(C(V^t)) \otimes \mathcal{A}$.
2. *There exists a co-action $\alpha_t : C(V^t) \to C(V^t) \otimes \mathcal{A}$ such that,*

$$\beta(\chi_\tau)\alpha_t(\chi_j) = \beta(\chi_\tau.\chi_j)$$

for all $j \in V^t$ and $\tau \in E$.

Proof. The proof is using similar arguments as in the context of a simple graph (see proof of theorem 3.4.1). □

Remark 4.1.3. *As we have seen in the context of simple graph, the co-actions α_s and α_t in theorem 3.3.1 and theorem 3.4.1 are uniquely determined by β. Let $(q^i_k)_{i,k\in V^s}$, $(r^j_l)_{j,l\in V^t}$ and $(u^\tau_{\tau'})_{\tau,\tau'\in E}$ be co-representation matrices of α_s, α_t and β respectively. Then we have the following identities:*

$$\sum_{\tau\in E^k} u^{\sigma_1}_\tau u^{\sigma_2 *}_\tau = \delta_{\sigma_1,\sigma_2} q^{s(\sigma_1)}_k,$$
$$\sum_{\tau\in E_l} u^{\sigma_1 *}_\tau u^{\sigma_2}_\tau = \delta_{\sigma_1,\sigma_2} r^{t(\sigma_1)}_l$$

for all $k \in V^s, l \in V^t$ and $\sigma_1, \sigma_2 \in E$. For a bi-unitary co-representation β satisfying (1) in theorems 3.3.1 and 3.4.1, the co-actions α_s and α_t will be referred to as **induced co-actions on $C(V^s)$ and $C(V^t)$***.*

Theorem 4.1.4. *Let $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ be a bi-unitary co-representation such that*

1. $Ad_\beta(S(C(V^s))) \subseteq S(C(V^s)) \otimes \mathcal{A}$.
2. $Ad_{\overline{\beta}}(T(C(V^t))) \subseteq T(C(V^t)) \otimes \mathcal{A}$.

Furthermore, we also assume that the induced co-actions α_s and α_t both preserve $C(V^s \cap V^t)$, that is,

$$\alpha_s(C(V^s \cap V^t)) \subseteq C(V^s \cap V^t) \otimes \mathcal{A} \subseteq C(V^s) \otimes \mathcal{A},$$
$$\alpha_t(C(V^s \cap V^t)) \subseteq C(V^s \cap V^t) \otimes \mathcal{A} \subseteq C(V^t) \otimes \mathcal{A}.$$

Then the following conditions are equivalent:

1. $\alpha_s|_{C(V^s\cap V^t)} = \alpha_t|_{C(V^s\cap V^t)}$.

2. *For all* $j \in V^s \cap V^t$ *and* $\tau \in E$,

$$\beta(\chi_\tau)\alpha_s(\chi_j) = \beta(\chi_\tau.\chi_j).$$

3. *For all* $i \in V^s \cap V^t$ *and* $\tau \in E$,

$$\alpha_t(\chi_i)\beta(\chi_\tau) = \beta(\chi_i.\chi_\tau).$$

Proof. The proof is using same arguments as in the case of simple graph (see proof of theorem 3.5.1). □

4.2 Notions of quantum symmetry in a directed multigraph

Now we are in position to define our notions of quantum symmetry in a directed multigraph.

4.2.1 Main definitions

Definition 4.2.1. *A compact quantum group* $(\mathcal{A}, \Delta)$ *is said to co-act on* (V, E) *preserving its quantum symmetry in Bichon's sense if there exists a co-action* $\beta : C(E) \to C(E) \otimes \mathcal{A}$ *satisfying the following conditions:*

1. $Ad_\beta(S(C(V^s))) \subseteq S(C(V^s)) \otimes \mathcal{A}$.

2. $Ad_\beta(T(C(V^t))) \subseteq T(C(V^t)) \otimes \mathcal{A}$.

3. *The induced co-actions* α_s *and* α_t *both preserve* $C(V^s \cap V^t)$ *and agree on* $C(V^s \cap V^t)$, *that is,*

$$\alpha_s|_{C(V^s\cap V^t)} = \alpha_t|_{C(V^s\cap V^t)}.$$

Definition 4.2.2. *a compact quantum group* $(\mathcal{A}, \Delta)$ *is said to co-act on* (V, E) *preserving its quantum symmetry in Banica's sense if there exists a bi-unitary co-representation* $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ *satisfying the following conditions:*

1. $Ad_\beta(S(C(V^s))) \subseteq S(C(V^s)) \otimes \mathcal{A}$.

2. $Ad_{\overline{\beta}}(T(C(V^t))) \subseteq T(C(V^t)) \otimes \mathcal{A}$.

3. *The induced co-actions α_s and α_t both preserve $C(V^s \cap V^t)$ and agree on $C(V^s \cap V^t)$, that is,*

$$\alpha_s|_{C(V^s \cap V^t)} = \alpha_t|_{C(V^s \cap V^t)}.$$

4. *β fixes the element $\xi_0 := \sum_{\tau \in E} \chi_\tau \in L^2(E)$, that is,*

$$\beta(\xi_0) = \xi_0 \otimes 1_{\mathcal{A}}.$$

We will need additional conditions beyond those mentioned in definition 4.2.2 to capture the classical picture of multigraph automorphisms (see definition 2.1.11). Nevertheless, definition 4.2.2 provides the most general setup related to a multigraph, which we will use for several constructions in this thesis.

Definition 4.2.3. *A compact quantum group $(\mathcal{A}, \Delta)$ is said to co-act on (V, E) by preserving its quantum symmetry in our sense if there exists a bi-unitary co-representation $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ satisfying the following conditions:*

1. *$Ad_\beta(S(C(V^s))) \subseteq S(C(V^s)) \otimes \mathcal{A}$.*

2. *$Ad_{\overline{\beta}}(T(C(V^t))) \subseteq T(C(V^t)) \otimes \mathcal{A}$.*

3. *The induced co-actions α_s and α_t both preserve $C(V^s \cap V^t)$ and agree on $C(V^s \cap V^t)$, that is,*

$$\alpha_s|_{C(V^s \cap V^t)} = \alpha_t|_{C(V^s \cap V^t)}.$$

4. *β fixes the element $\xi_0 := \sum_{\tau \in E} \chi_\tau \in L^2(E)$, that is,*

$$\beta(\xi_0) = \xi_0 \otimes 1_{\mathcal{A}}.$$

5. *For $i, j, k, l \in V$ with $E^i_j \neq \phi$ and $E^k_l \neq \phi$, let us define $(Ad_\beta)^{ij}_{kl} : M_{ij} \to M_{kl} \otimes \mathcal{A}$ (similarly also $(Ad_{\overline{\beta}})^{ij}_{kl}$) by,*

$$(Ad_\beta)^{ij}_{kl}(T) = (p_{kl} \otimes 1) Ad_\beta(T)(p_{kl} \otimes 1); \quad T \in M_{ij}.$$

Then we assume that,

$$(Ad_\beta)^{ij}_{kl}(D_{ij}) \subseteq D_{kl} \otimes \mathcal{A}$$
$$\text{and} \quad (Ad_{\overline{\beta}})^{ij}_{kl}(D_{ij}) \subseteq D_{kl} \otimes \mathcal{A}.$$

Remark 4.2.4. *Let β be co-action of a compact quantum group $(\mathcal{A}, \Delta)$ on (V, E) preserving its quantum symmetry in Banica's sense, Bichon's sense or ours. From (1), (2) and (3) of our definitions it follows that the co-actions α_s and α_t induce a co-action $\alpha : C(V) \to C(V) \otimes \mathcal{A}$ by*

$$\begin{aligned} \alpha(\chi_k) &= \alpha_s(\chi_k) \quad \textit{if} \quad k \in V^s, \\ &= \alpha_t(\chi_k) \quad \textit{if} \quad k \in V^t. \end{aligned}$$

α is the required "permutation" of vertices derived from "permutation" of edges E and will be referred as **induced permutation on the set of vertices of** *(V, E).*

4.2.2 Left and right equivariance of the induced permutation α

Proposition 4.2.5. *Let β be a co-action of a CQG $(\mathcal{A}, \Delta)$ on (V, E) preserving its quantum symmetry in Banica's sense, Bichon's sense or ours. Let α be the induced co-action on $C(V)$. Then (α, β) respects $C(V^s) - L^2(E) - C(V^t)$ bi-module structure, that is,*

$$\begin{aligned} \alpha(\chi_i)\beta(\chi_\sigma) &= \beta(\chi_i.\chi_\sigma) \\ \textit{and} \quad \beta(\chi_\sigma)\alpha(\chi_j) &= \beta(\chi_\sigma.\chi_j). \end{aligned}$$

for all $i \in V^s, j \in V^t$ and $\sigma \in E$.

Proof. As $\alpha|_{(C(V^s)} = \alpha_s$ and $\alpha|_{C(V^t)} = \alpha_t$, from theorem 4.1.1 and theorem 4.1.2 the claim follows. Using similar arguments as in proof of theorem 3.1.2 we deduce the following identities:

$$q_i^{s(\sigma)} u_\tau^\sigma = \delta_{i,s(\tau)} u_\tau^\sigma \quad \text{and} \quad u_\tau^\sigma q_j^{t(\sigma)} = \delta_{t(\tau),j} u_\tau^\sigma \tag{4.2.1}$$

for all $\sigma \in E$ and $i \in V^s, j \in V^t$.

□

Instead of seeing $L^2(E)$ as a $C(V^s) - C(V^t)$ bimodule, we can also see it as $C(V) - C(V)$ bimodule by the same right and left module multiplication maps.

Proposition 4.2.6. *Let $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ be a bi-unitary co-representation and $\alpha : C(V) \to C(V) \otimes \mathcal{A}$ be a co-action of a CQG $(\mathcal{A}, \Delta)$. Then the following are equivalent:*

1. *$\alpha(\chi_i).\beta(\sigma) = \beta(\chi_i.\chi_\sigma)$ and $\beta(\chi_\sigma).\alpha(\chi_j) = \beta(\chi_\sigma.\chi_j)$ for all $i \in V^s, j \in V^t, \sigma \in E$.*
2. *$\alpha(\chi_i).\beta(\sigma) = \beta(\chi_i.\chi_\sigma)$ and $\beta(\chi_\sigma).\alpha(\chi_i) = \beta(\chi_\sigma.\chi_i)$ for all $i \in V, \sigma \in E$.*

Proof. Let $(u^\sigma_\tau)_{\sigma,\tau\in E}$ and $(q^i_j)_{i,j\in V}$ be the co-representation matrices of α and β.

$(1) \implies (2)$.

Let $\sigma \in E, i \in V^s$ and $j \in V^t$. As β is bi-unitary, we observe that,

$$q_i^{s(\sigma)} = q_i^{s(\sigma)}(\sum_{\tau\in E} u^\sigma_\tau u^{\sigma*}_\tau)q_i^{s(\sigma)} = \sum_{\tau\in E^i} u^\sigma_\tau u^{\sigma*}_\tau,$$
$$q_j^{t(\sigma)} = q_j^{t(\sigma)}(\sum_{\tau\in E} u^{\sigma*}_\tau u^{\sigma}_\tau)q_j^{t(\sigma)} = \sum_{\tau\in E_j} u^{\sigma*}_\tau u^{\sigma}_\tau.$$

We therefore have,

$$\sum_{i\in V^s} q_i^{s(\sigma)} = \sum_{\tau\in E} u^\sigma_\tau u^{\sigma*}_\tau = 1 \quad \text{and} \quad \sum_{j\in V^t} q_j^{t(\sigma)} = \sum_{\tau\in E} u^{\sigma*}_\tau u^{\sigma}_\tau = 1$$

Hence for $i \notin V^s$ and $j \notin V^t$ it follows that,

$$q_i^{s(\sigma)} = 0 \quad \text{and} \quad q_j^{t(\sigma)} = 0. \tag{4.2.2}$$

From our assumption and the observation made above, it follows that for all $i \in V$ and $\sigma, \tau \in E$,

$$\begin{aligned} q_i^{s(\sigma)} u^\sigma_\tau &= \delta_{i,s(\tau)} u^\sigma_\tau \quad \text{when} \quad i \in V^s; \\ &= 0 \\ &= \delta_{i,s(\tau)} u^\sigma_\tau \quad \text{when} \quad i \notin V^s. \end{aligned}$$

Via similar arguments we can prove the target version of above identites mentioned in equation 4.2.1. Hence (2) holds.

The converse $(2) \implies (1)$ is obvious. □

4.2.3 Some essential identities

In this subsection, we will prove some identites which we will be using implicitly and explicitly throughout this thesis. The fact, that these identities are readily available in the context of a simple graph, asserts that our descriptions of quantum symmetry in a multigraph are in fact correct generalizations.

Proposition 4.2.7. *Let β be a co-action of a CQG $(\mathcal{A}, \Delta)$ on (V, E) preserving its quantum symmetry in Banica's sense. Let $(u^\sigma_\tau)_{\sigma,\tau\in E}$ and $(q^i_j)_{i,j\in V}$ be the co-representation matrices of β and induced co-action α on $C(V)$. For $i, j, k, l \in V$ such that $E^i_j \neq \phi$ and $E^k_l \neq \phi$, we have the following:*

1. *For any* $\sigma \in E^i$, *we have* $\sum_{\tau \in E^k} u^\sigma_\tau = q^i_k$.

2. *For any* $\sigma \in E_j$, *we have* $\sum_{\tau \in E_l} u^\sigma_\tau = q^j_l$.

3. *For any* $\sigma \in E^i_j$, *we have* $\sum_{\tau \in E^k_l} u^\sigma_\tau = q^i_k q^j_l$.

As $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ *is bi-unitary co-representation, by using antipode on the underlying Hopf * algebra of matrix elements of* $(\mathcal{A}, \Delta)$, *it follows that the above identities are true if we consider sum in upper indices instead of lower indices.*

Proof. As β co-acts on (V, E) preserving its quantum symmetry, from (4) of definition 4.2.2 we have,

$$\sum_{\sigma \in E} u^\sigma_\tau = 1 = \sum_{\sigma \in E} u^\tau_\sigma \quad \text{for all} \quad \tau \in E.$$

For $\sigma \in E^i$, using equation 4.2.1 we observe that,

$$q^i_k = q^i_k (\sum_{\tau \in E} u^\sigma_\tau) = \sum_{\tau \in E^k} u^\sigma_\tau.$$

For $\sigma \in E_j$, using equation 4.2.1 we observe that,

$$q^j_l = (\sum_{\tau \in E} u^\sigma_\tau) q^j_l = \sum_{\tau \in E_l} u^\sigma_\tau.$$

For $\sigma \in E^i_j$, using equation 4.2.1 we note that,

$$q^i_k q^j_l = q^i_k (\sum_{\tau \in E} u^\sigma_\tau) q^j_l = \sum_{\tau \in E^k_l} u^\sigma_\tau.$$

□

Remark 4.2.8. *If* β *is a co-action on* (V, E) *preserving its quantum symmetry in Bichon's sense then in (3) of proposition 4.2.7, it follows that* q^i_k *and* q^j_l *commute with each other as* u^σ_τ*'s are already projections in* $\mathcal{A}$ *(*β *is a quantum permutation the edge set* E*).*

Proposition 4.2.9. *Let* β *be a co-action of a CQG* $(\mathcal{A}, \Delta)$ *on* (V, E) *preserving its quantum symmetry in Banica's sense. The following identity holds:*

$$QW = WQ$$

where $Q = (q^i_j)_{i,j \in V}$ *is the co-representation matrix of the induced co-action on* $C(V)$ *and* W *is the adjacency matrix of* (V, E).

Proof. Let $(u^\sigma_\tau)_{\sigma,\tau\in E}$ be the co-representation matrix of β.

Let us fix $i,j \in V$. If $i \notin V^s$ or $j \notin V^t$, using equation 4.2.2 it follows that,

$$(QW)^i_j = 0 = (WQ)^i_j.$$

Hence let us assume $i \in V^s$ and $j \in V^t$. For each $k \in V$ with $W^k_j \neq 0$ we fix an element τ_k in E^k_j. In a similar way, for each $k \in V$ with $W^i_k \neq 0$ we fix an element σ_k in E^i_k. We observe that,

$$\begin{aligned}
(QW)^i_j = \sum_{k\in V} q^i_k W^k_j &= \sum_{\substack{k\in V\\ W^k_j\neq 0}} W^k_j \big(\sum_{\sigma\in E^i} u^\sigma_{\tau_k}\big)\\
&= \sum_{\substack{k\in V\\ W^k_j\neq 0}} \big(\sum_{\substack{\sigma\in E^i\\ \tau\in E^k_j}} u^\sigma_\tau\big)\\
&= \sum_{\substack{\sigma\in E^i\\ \tau\in E_j}} u^\sigma_\tau\\
&= \sum_{\substack{k\in V\\ W^i_k\neq 0}} \big(\sum_{\substack{\sigma\in E^i_k\\ \tau\in E_j}} u^\sigma_\tau\big)\\
&= \sum_{\substack{k\in V\\ W^i_k\neq 0}} W^i_k \big(\sum_{\tau\in E_j} u^{\sigma_k}_\tau\big)\\
&= \sum_{k\in V} W^i_k q^k_j = (WQ)^i_j.
\end{aligned}$$

Hence our claim is proved.

□

Corollary 4.2.10. *Let β be a co-action of CQG $(\mathcal{A},\Delta)$ on (V,E) preserving its quantum symmetry in Banica's sense, Bichon's sense or ours. Let $i,j,k,l \in V$ be such that $|E^i_j| \neq |E^k_l|$. Then $q^i_k q^j_l = 0$ where $(q^i_j)_{i,j\in V}$ is the co-representation matrix of the induced co-action on $C(V)$.*

Proof. It is immidiate form proposition 4.2.9 and theorem 2.5.9. □

4.2.4 Complete orthogonality versus restricted orthogonality

We explore the consequences of condition (5) in the definition 4.2.3 which give us some "restricted orthogonality" among the edges once we have fixed two pairs of initial and final vertices.

We are saying "restricted" as we have considered $(Ad_\beta)^{ij}_{kl}$ and $(Ad_{\overline{\beta}})^{ij}_{kl}$ instead of Ad_β and $Ad_{\overline{\beta}}$. The next proposition shows that "full orthogonality" essentially leads to an equivalent description of co-action on (V, E) preserving its quantum symmetry in Bichon's sense.

Proposition 4.2.11. *Let β be a co-action of a CQG $(\mathcal{A}, \Delta)$ on a directed multigraph (V, E) preserving its quantum symmetry in Banica's sense (see definition 4.2.2). If*

$$Ad_\beta(D) \subseteq D \otimes \mathcal{A},$$

then β is a co-action on (V, E) which preserves its quantum symmetry in Bichon's sense. In particular, β is a quantum permutation on the edge set E.

Proof. We observe that, For $T \in B(L^2(E))$, $T \in D$ if and only if $\{\chi_\tau | \tau \in E\}$ is the complete set of eigenvectors of T.

Let $(u^\sigma_\tau)_{\sigma,\tau\in E}$ be the co-representation matrix of β. For $\tau, \sigma_2 \in E$, we note that,

$$\begin{aligned}
Ad_\beta(p_\tau)(\chi_{\sigma_2} \otimes 1) &= \beta(p_\tau \otimes 1)(\sum_{\tau' \in E} \chi_{\tau'} \otimes u^{\sigma_2 *}_{\tau'}) \\
&= \beta(\chi_\tau \otimes u^{\sigma_2 *}_\tau) \\
&= \sum_{\sigma_1 \in E} \chi_{\sigma_1} \otimes u^{\sigma_1}_\tau u^{\sigma_2 *}_\tau.
\end{aligned}$$

Using observation mentioned in the beginning we conclude that,

$$Ad_\beta(D) \subseteq D \otimes \mathcal{A} \iff u^{\sigma_1}_\tau u^{\sigma_2 *}_\tau = 0 \quad \text{when} \quad \sigma_1 \neq \sigma_2. \tag{4.2.3}$$

From (4) of definition 4.2.2 it follows that

$$\sum_{\sigma \in E} u^\sigma_\tau = 1 \quad \text{for all} \quad \tau \in E. \tag{4.2.4}$$

Using observation 4.2.3 and equation 4.2.4, for $\sigma_1, \tau \in E$ it follows that,

$$u^{\sigma_1}_\tau = u^{\sigma_1}_\tau (\sum_{\sigma_2 \in E} u^{\sigma_2 *}_\tau) = u^{\sigma_1}_\tau u^{\sigma_1 *}_\tau.$$

Using spectral calculus for normal operators, it follows that $u^{\sigma_1}_\tau$ is a projection. Combining the fact that $u^{\sigma_1}_\tau$'s are projections with observation 4.2.3 and equation 4.2.4 we conclude that the coefficients of the matrix $(u^{\sigma_1}_\tau)_{\sigma_1,\tau\in E}$ satisfy **quantum permutation relations** making β a co-action on $C(E)$ and therefore a co-action on (V, E) which preserve its quantum symmetry in Bichon's sense. $\square$

In light of the previous proposition, to capture some sort of orthogonality relations among edges, we resort to "restricted orthogonality" which serves as a middle ground between Bichon and Banica's notions of quantum symmetry. In the next section and the next chapter, we will see further justifications for our choice of "restricted orthogonality" relations.

Proposition 4.2.12. *Let β be a co-action of a CQG $(\mathcal{A}, \Delta)$ on (V, E) which preserves its quantum symmetry in Banica's sense. Let $i, j, k, l \in V$ be such that $E^i_j \neq \phi$ and $E^k_l \neq \phi$. Then the following are equivalent:*

1. $(Ad_\beta)^{kl}_{ij}(D_{kl}) \subseteq D_{ij} \otimes \mathcal{A}$ *and* $(Ad_{\overline{\beta}})^{kl}_{ij}(D_{kl}) \subseteq D_{ij} \otimes \mathcal{A}$.
2. *For all* $\sigma_1 \neq \sigma_2 \in E^i_j$ *and* $\tau \in E^k_l$,

$$u^{\sigma_1}_\tau u^{\sigma_2 *}_\tau = 0 \quad \textit{and} \quad u^{\sigma_1 *}_\tau u^{\sigma_2}_\tau = 0$$

where $(u^\sigma_\tau)_{\sigma, \tau \in E}$ *is the co-representation matrix of* β.

Proof. We obeserve that, for any $T \in M_{ij}$, $T \in D_{ij}$ if and only if $\{\chi_\sigma | \sigma \in E^i_j\}$ is a set of eigenvectors for T.

Let us fix $\tau \in E^k_l$ and $\sigma_2 \in E^i_j$. We observe that,

$$\begin{aligned}
(Ad_\beta)^{kl}_{ij}(p_\tau)(\chi_{\sigma_2} \otimes 1) &= (p_{ij} \otimes 1)\beta(p_\tau \otimes 1)(\sum_{\sigma \in E} \chi_\sigma \otimes u^{\sigma_2 *}_\sigma) \\
&= (p_{ij} \otimes 1)\beta(\chi_\tau \otimes u^{\sigma_2 *}_\tau) \\
&= (p_{ij} \otimes 1)(\sum_{\sigma_1 \in E} \chi_{\sigma_1} \otimes u^{\sigma_1}_\tau u^{\sigma_2 *}_\tau) \\
&= \sum_{\sigma_1 \in E^i_j} \chi_{\sigma_1} \otimes u^{\sigma_1}_\tau u^{\sigma_2 *}_\tau.
\end{aligned}$$

$$\text{Similarly,} \quad (Ad_{\overline{\beta}})^{kl}_{ij}(p_\tau)(\chi_{\sigma_2} \otimes 1) = \sum_{\sigma_1 \in E^i_j} \chi_{\sigma_1} \otimes u^{\sigma_1 *}_\tau u^{\sigma_2}_\tau.$$

From the observation mentioned in the beginning of the proof the equivalence follows. $\square$

4.3 Preservence and permutation of loops

In this section we show that any co-action on (V,E) preserving its quantum symmetry in our sense actually preserves the space of loops and is in fact is a quantum permutation on them.

Proposition 4.3.1. *Let β be a co-action of a CQG $(\mathcal{A},\Delta)$ on (V,E) which preserves its quantum symmetry in our sense. Then*

$$\beta(L^2(L)) \subseteq L^2(L) \otimes \mathcal{A}$$

where $L \subseteq E$ is the set of loops in (V,E). Moreover, $\beta|_{L^2(L)}$ is a **quantum permutation** *on L.*

Proof. Let $(u^\sigma_\tau)_{\sigma,\tau\in E}$ be the co-representation matrix of β. It is enough to show that $u^\sigma_\tau = 0$ when $\sigma \notin L$ and $\tau \in L$.

Let $i,j \in V$ be such that $E^i_j \neq \phi$ and $\tau \in L$. Let $s(\tau) = t(\tau) = k$ for some $k \in V$. From (3) of proposition 4.2.7 it follows that,

$$\sum_{\sigma\in E^i_j} u^\sigma_\tau = 0 \quad \text{when} \quad i \neq j.$$

Using proposition 4.2.12 we observe that,

$$u^{\sigma'*}_\tau u^{\sigma'}_\tau = u^{\sigma'*}_\tau \sum_{\sigma\in E^i_j} u^\sigma_\tau = 0$$

where $\sigma' \in E^i_j$ and $i \neq j$. As our choices of i and j were arbitrary, we have,

$$u^{\sigma'}_\tau = 0 \quad \text{where} \quad \sigma' \notin L, \tau \in L.$$

From proposition 5.3.10, it follows that u^σ_τ's are projections commuting with q^i_k where $\sigma \in E^i_i$ and $\tau \in E^k_k$. Using proposition 4.2.5 and proposition 4.2.12 we further observe that,

$$\begin{aligned} u^{\sigma_1}_\tau u^{\sigma_2}_\tau &= u^{\sigma_1}_\tau q^{s(\sigma_1)}_{s(\tau)} q^{s(\sigma_2)}_{s(\tau)} u^{\sigma_2}_\tau \\ &= \delta_{s(\sigma_1),s(\sigma_2)} u^{\sigma_1}_\tau u^{\sigma_2}_\tau \\ &= \delta_{\sigma_1,\sigma_2} u^{\sigma_1}_\tau. \end{aligned}$$

Hence $\beta|_{L^2(L)}$ is a **quantum permutation** on L. $\square$

Remark 4.3.2. *Proposition 4.3.1 also holds if we assume β to preserve quantum symmetry of (V, E) in Bichon's sense.*

4.4 Existence of Universal Objects:

4.4.1 The categories $\mathcal{C}^{Ban}_{(V,E)}$, $\mathcal{C}^{Bic}_{(V,E)}$ and $\mathcal{C}^{sym}_{(V,E)}$:

Definition 4.4.1. *Let β and β' be co-actions of two compact quantum groups $(\mathcal{A}, \Delta)$ and $(\mathcal{A}', \Delta')$ on (V, E) which preserve its quantum symmetry in Banica's sense. Then $\Phi : (\mathcal{A}, \Delta) \to (\mathcal{A}', \Delta')$, a quantum group homomorphism, is said to intertwin β and β' if the following diagram commutes:*

$$\begin{tikzcd} L^2(E) \arrow[r, "\beta'"] \arrow[d, "\beta"] & L^2(E) \otimes \mathcal{A}' \\ L^2(E) \otimes \mathcal{A} \arrow[ur, "id\otimes\Phi"] & \end{tikzcd}$$

Let us consider the category $\mathcal{C}^{Ban}_{(V,E)}$ whose objects are denoted by $(\mathcal{A}, \Delta_{\mathcal{A}}, \beta_{\mathcal{A}})$ where $(\mathcal{A}, \Delta_{\mathcal{A}})$ is a compact quantum group and $\beta_{\mathcal{A}}$ is a co-action of $(\mathcal{A}, \Delta_{\mathcal{A}})$ on (V, E) preserving its quantum symmetry in Banica's sense. Morphisms in this category are quantum group homomorphisms intertwining two such co-actions.

Similarly, we consider the categories $\mathcal{C}^{Bic}_{(V,E)}$ and $\mathcal{C}^{sym}_{(V,E)}$ whose objects are compact quantum groups co-acting on (V, E) preserving its quantum symmetry in Bichon's sense and ours respectively. Morphisms in these categories are quantum group homomorphisms intertwining two similar co-actions.

All the categories $\mathcal{C}^{Ban}_{(V,E)}$, $\mathcal{C}^{sym}_{(V,E)}$ and $\mathcal{C}^{Bic}_{(V,E)}$ are non-empty as $C(G^{aut}_{(V,E)})$ are in all of them where $G^{aut}_{(V,E)}$ is the group of classical automorphisms of (V, E).

4.4.2 Existence of universal object in $\mathcal{C}^{Ban}_{(V,E)}$ and $\mathcal{C}^{Bic}_{(V,E)}$

Before moving to the main theorems, let us start with the following definition.

Definition 4.4.2. *For a C* algebra $\mathcal{A}$ and a $\mathbb{C}$-linear map $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ we can always write,*

$$\beta(\chi_\tau) = \sum_{\sigma \in E} \chi_\sigma \otimes u^\sigma_\tau$$

where $T_\beta = (u^\sigma_\tau)_{\sigma,\tau \in E} \in M_m(\mathcal{A})$ and $m = |E|$. The matrix T_β will be referred to as the **transformation matrix** *of β. We will say that β is bi-unitary if T_β and $\overline{T_\beta} := (u^{\sigma *}_\tau)_{\sigma,\tau \in E}$ are both unitary in $M_m(\mathcal{A})$.*

Theorem 4.4.3. *For a directed multigraph (V, E), the category $\mathcal{C}^{Ban}_{(V,E)}$ admits a universal object.*

Proof. Let us consider the category $\mathcal{C}$ whose objects are $(\mathcal{A}, \beta_{\mathcal{A}})$ where $\mathcal{A}$ is a C* algebra and $\beta_{\mathcal{A}} : L^2(E) \to L^2(E) \otimes \mathcal{A}$ is a bi-unitary map satisfying conditions mentioned in definition 4.2.2, that is, coefficients of the transformation matrix of $\beta_{\mathcal{A}}$ satisfy the polynomial relations associated with conditions in definition 4.2.2. It should be noted that at the level of C* algebra, (1), (2) and (3) of definition 4.2.2 only induce C* homomorphisms $\alpha^{\mathcal{A}}_s$ and $\alpha^{\mathcal{A}}_t$ on $C(V^s)$ and $C(V^t)$ (in similar manner which we have done for a CQG) which preserve and agree on $C(V^s \cap V^t)$. Let $m = |E|$. By universality of the Wang algebra $A_u(m)$, there exists a C* algebra homomorphism $\Phi_{\mathcal{A}} : A_u(m) \to \mathcal{A}$ for each object $(\mathcal{A}, \beta_{\mathcal{A}})$ in $\mathcal{C}$. Let us define

$$\mathcal{I} = \cap_{(\mathcal{A},\beta_{\mathcal{A}})\in\mathcal{C}} Ker(\Phi_{\mathcal{A}}).$$

Let $\pi : A_u(m) \to {}^{A_u(m)}/_{\mathcal{I}}$ be the quotient map and we write $\pi(u^\sigma_\tau) = \overline{u^\sigma_\tau}$ where coefficients of the matrix $(u^\sigma_\tau)_{\sigma,\tau\in E}$ are the canonical generators of $A_u(m)$. It is clear that $\mathcal{Q} := {}^{A_u(m)}/_{\mathcal{I}}$ along with the bi-unitary map $\beta_{\mathcal{Q}} : L^2(E) \to L^2(E) \otimes \mathcal{Q}$ is a universal object in the category $\mathcal{C}$ where

$$\beta_{\mathcal{Q}}(\chi_\tau) = \sum_{\sigma\in E} \chi_\sigma \otimes \overline{u^\sigma_\tau}.$$

We will show that $\mathcal{Q}$ is in fact a CQG and therefore also universal in $\mathcal{C}^{Ban}_{(V,E)}$.

It suffices to define a co-product on $\mathcal{Q}$ such that $\beta_{\mathcal{Q}}$ becomes a bi-unitary co-representation. For $\sigma, \tau \in E$, let us define $U^\sigma_\tau \in \mathcal{Q} \otimes \mathcal{Q}$ by

$$U^\sigma_\tau := (\pi \otimes \pi)\Delta_m(u^\sigma_\tau) = \sum_{\tau'\in E} \overline{u^\sigma_{\tau'}} \otimes \overline{u^{\tau'}_\tau}$$

where Δ_m is the co-product on $A_u(m)$. Let us define a linear map $\beta : L^2(E) \to L^2(E) \otimes \mathcal{Q} \otimes \mathcal{Q}$ by

$$\beta(\chi_\tau) = \sum_{\sigma\in E} \chi_\sigma \otimes U^\sigma_\tau.$$

It further follows from routine computation that, $(\mathcal{Q} \otimes \mathcal{Q}, \beta) \in \mathcal{C}$. By universality of $\mathcal{Q}$, there exists a C* algebra homomorphism $\Delta_{Ban} : \mathcal{Q} \to \mathcal{Q} \otimes \mathcal{Q}$ such that,

$$\Delta_{Ban}(\overline{u^\sigma_\tau}) = \sum_{\tau'\in E} \overline{u^\sigma_{\tau'}} \otimes \overline{u^{\tau'}_\tau}.$$

The map Δ_{Ban} is co-associative as it is such on the set of generators. As $\Delta_{Ban} \circ \pi = (\pi \otimes \pi) \circ \Delta_m$, it follows that $\Delta_m(\mathcal{I}) \subseteq Ker(\pi \otimes \pi)$ making $\mathcal{I}$ a Woronowicz C* ideal in $A_u(m)$.

Therefore $\mathcal{Q}$ is a compact quantum group with the co-product Δ_{Ban}. The bi-unitary map $\beta_{\mathcal{Q}}$ is a co-representation on $L^2(E)$ as we clearly have

$$(id \otimes \Delta_{Ban}) \circ \beta_{\mathcal{Q}} = (\beta_{\mathcal{Q}} \otimes id) \circ \beta_{\mathcal{Q}}.$$

Therefore $(\mathcal{Q}, \Delta_{Ban}, \beta_Q)$ is universal in $\mathcal{C}^{Ban}_{(V,E)}$. □

Remark 4.4.4. *Let us denote the universal object in $\mathcal{C}^{Ban}_{(V,E)}$ by $Q^{Ban}_{(V,E)}$ and the respective co-product by Δ_{Ban}.*

Theorem 4.4.5. *For a directed multigraph (V, E), the category $\mathcal{C}^{Bic}_{(V,E)}$ admits a universal object.*

Proof. Let us denote $(\mathcal{A}, \Delta_{\mathcal{A}}, \beta_{\mathcal{A}})$ to be an object of $\mathcal{C}^{Bic}_{(V,E)}$ where $(\mathcal{A}, \Delta_{\mathcal{A}})$ is a CQG and $\beta_{\mathcal{A}} : C(E) \to C(E) \otimes \mathcal{A}$ is a co-action on (V, E) in the sense of definition 4.2.1. As any co-action $\beta : C(E) \to C(E) \otimes \mathcal{A}$ is also a bi-unitary co-representation on $L^2(E)$, by universality of $Q^{Ban}_{(V,E)}$, it follows that there exists a unique quantum group homomorphism $\Phi_{\mathcal{A}} : Q^{Ban}_{(V,E)} \to \mathcal{A}$ such that

$$\Phi_{\mathcal{A}}(u^\sigma_\tau) = v^\sigma_\tau \quad \text{for all} \quad \sigma, \tau \in E$$

where $(u^\sigma_\tau)_{\sigma,\tau\in E}$ is the matrix of canonical generators of $Q^{Ban}_{(V,E)}$ and $(v^\sigma_\tau)_{\sigma,\tau\in E}$ is the co-representation matrix of $\beta_{\mathcal{A}}$. Let $\mathcal{I}^{Bic}$ be a C* ideal in $Q^{Ban}_{(V,E)}$ generated by the following set of relations:

$$u^\sigma_\tau = u^{\sigma*}_\tau = u^{\sigma 2}_\tau \quad \text{for all} \quad \sigma, \tau \in E.$$

It is clear that $\mathcal{I}^{Bic} \subseteq Ker(\Phi_{\mathcal{A}})$ for all $(\mathcal{A}, \Delta_{\mathcal{A}}, \beta_{\mathcal{A}}) \in \mathcal{C}^{Bic}_{(V,E)}$. Let us denote $\mathcal{Q} = Q^{Ban}_{(V,E)} / \mathcal{I}^{Bic}$ and $\pi_{Bic} : Q^{Ban}_{(V,E)} \to \mathcal{Q}$ to be the natural quotient map. We write

$$\pi_{Bic}(u^\sigma_\tau) = \overline{u^\sigma_\tau} \quad \text{for all} \quad \sigma, \tau \in E.$$

If we show that $\mathcal{I}^{Bic}$ is a Woronowicz C* ideal in $Q^{Ban}_{(V,E)}$, then it follows that $\mathcal{Q}$ becomes a CQG with a natural co-action $\beta_{\mathcal{Q}} : C(E) \to C(E) \otimes \mathcal{Q}$ given by,

$$\beta_{\mathcal{Q}}(\chi_\tau) = \sum_{\sigma\in E} \chi_\sigma \otimes \overline{u^\sigma_\tau} \quad \text{where} \quad \tau \in E.$$

From definition of $\mathcal{I}^{Bic}$, it also follows that $(\mathcal{Q}, \Delta_{\mathcal{Q}}, \beta_{\mathcal{Q}}) \in \mathcal{C}^{Bic}_{(V,E)}$ and universal in $\mathcal{C}^{Bic}_{(V,E)}$ where the co-product $\Delta_{\mathcal{Q}}$ on $\mathcal{Q}$ is induced by Δ_{Ban} via the quotient map π_{Bic}.

Claim: The ideal $\mathcal{I}^{Bic}$ is a Woronowicz C* ideal in $Q^{Ban}_{(V,E)}$.

Let Δ_{Ban} be the co-product on $Q^{Ban}_{(V,E)}$ and $\sigma, \tau \in E$. We observe that,

$$\begin{aligned}
(\pi_{Bic} \otimes \pi_{Bic})\Delta_{Ban}(u^\sigma_\tau) &= \sum_{\tau' \in E} \overline{u^\sigma_{\tau'}} \otimes \overline{u^{\tau'}_\tau} \\
&= \sum_{\tau' \in E} \overline{u^{\sigma *}_{\tau'}} \otimes \overline{u^{\tau' *}_\tau} = (\pi_{Bic} \otimes \pi_{Bic})\Delta_{Ban}(u^{\sigma *}_\tau); \\
(\pi_{Bic} \otimes \pi_{Bic})\Delta_{Ban}(u^{\sigma^2}_\tau) &= \sum_{\tau_1, \tau_2 \in E} \overline{u^\sigma_{\tau_1}}\, \overline{u^\sigma_{\tau_2}} \otimes \overline{u^{\tau_1}_\tau}\, \overline{u^{\tau_2}_\tau} \\
&= \sum_{\tau_1 \in E} \overline{u^{\sigma 2}_{\tau_1}} \otimes \overline{u^{\tau_1 2}_\tau} \\
&= \sum_{\tau_1 \in E} \overline{u^\sigma_{\tau_1}} \otimes \overline{u^{\tau_1}_\tau} = (\pi_{Bic} \otimes \pi_{Bic})\Delta_{Ban}(u^\sigma_\tau).
\end{aligned}$$

Here we have used that the elements of the matrix $(\overline{u}^\sigma_\tau)_{\sigma,\tau \in E}$ satisfy **quantum permutation relations**. Therefore the claim follows and $(\mathcal{Q}, \Delta_\mathcal{Q}, \beta_\mathcal{Q})$ is the universal object in $\mathcal{C}^{Bic}_{(V,E)}$. □

Remark 4.4.6. *Let us denote the universal object in $\mathcal{C}^{Bic}_{(V,E)}$ by $Q^{Bic}_{(V,E)}$ and the respective co-product by Δ_{Bic}.*

4.4.3 Existence of universal object in $\mathcal{C}^{sym}_{(V,E)}$

We do not know whether $\mathcal{C}^{sym}_{(V,E)}$ admits a universal object or not for an arbitrary multigraph (V, E). However, it is worth mentioning that $\mathcal{C}^{sym}_{(V,E)}$ is strictly bigger than $\mathcal{C}^{Bic}_{(V,E)}$ for a large class of multigraphs. One of such classes consists of uniform multigraphs where Bichon's and Banica's notions of quantum symmetry differ for the underlying simple graph (see theorem 5.3.9 and example 5 in chapter 7). For such multigraphs, the **automorphism group of both source and target dependent quantum symmetries** $Q^{s,t}_{(V,E)}$ (see definition 6.4.3 in chapter 6) is a member of $\mathcal{C}^{sym}_{(V,E)}$ but **not** a member of $\mathcal{C}^{Bic}_{(V,E)}$.

4.5 Quantum automorphism groups of a directed multigraph

Definition 4.5.1. *Let (V, E) be a directed multigraph. The* **universal compact quantum group associated with** (V, E) *is the compact quantum group $(Q^{Ban}_{(V,E)}, \Delta_{Ban})$ where $Q^{Ban}_{(V,E)}$ is the universal C* algebra generated by elements of the matrix $(u^\sigma_\tau)_{\sigma,\tau \in E}$ satisfying the following relations:*

1. *The matrices* $U := (u^{\sigma}_{\tau})_{\sigma,\tau\in E}$ *and* $\overline{U} := (u^{\sigma *}_{\tau})_{\sigma,\tau\in E}$ *are both unitary, that is,*

$$\sum_{\tau\in E} u^{\sigma_1}_{\tau} u^{\sigma_2 *}_{\tau} = \delta_{\sigma_1,\sigma_2} 1 \quad \text{and} \quad \sum_{\tau\in E} u^{\tau *}_{\sigma_1} u^{\tau}_{\sigma_2} = \delta_{\sigma_1,\sigma_2} 1;$$
$$\sum_{\tau\in E} u^{\sigma_1 *}_{\tau} u^{\sigma_2}_{\tau} = \delta_{\sigma_1,\sigma_2} 1 \quad \text{and} \quad \sum_{\tau\in E} u^{\tau}_{\sigma_1} u^{\tau *}_{\sigma_2} = \delta_{\sigma_1,\sigma_2} 1$$

for all $\sigma_1, \sigma_2 \in E$.

2. $\sum_{\tau\in E} u^{\sigma}_{\tau} = 1$ *for all* $\sigma \in E$.

3. *Let* $k \in V^s$. *Then for all* $\sigma_1, \sigma_2 \in E$,

$$\sum_{\tau\in E^k} u^{\sigma_1}_{\tau} u^{\sigma_2 *}_{\tau} = 0 \quad \text{if} \quad \sigma_1 \neq \sigma_2;$$
$$\sum_{\tau\in E^k} u^{\sigma_1}_{\tau} u^{\sigma_1 *}_{\tau} = \sum_{\tau\in E^k} u^{\sigma_2}_{\tau} u^{\sigma_2 *}_{\tau} \quad \text{if} \quad s(\sigma_1) = s(\sigma_2).$$

4. *Let* $l \in V^t$. *Then for all* $\sigma_1, \sigma_2 \in E$,

$$\sum_{\tau\in E_l} u^{\sigma_1 *}_{\tau} u^{\sigma_2}_{\tau} = 0 \quad \text{if} \quad \sigma_1 \neq \sigma_2;$$
$$\sum_{\tau\in E_l} u^{\sigma_1 *}_{\tau} u^{\sigma_1}_{\tau} = \sum_{\tau\in E_l} u^{\sigma_2 *}_{\tau} u^{\sigma_2}_{\tau} \quad \text{if} \quad t(\sigma_1) = t(\sigma_2).$$

5. *Let* $i \in V^s \setminus V^t, j \in V^t \setminus V^s$ *and* $k \in V^s \cap V^t$. *Then for all* $\sigma_1 \in E^i, \sigma_2 \in E_j, \tau_1 \in E^k$ *and* $\tau_2 \in E_k$,

$$u^{\sigma_1}_{\tau_1} = 0 \quad \text{and} \quad u^{\sigma_2}_{\tau_2} = 0.$$

6. *Let* $i, k \in V^s \cap V^t$. *Then for all* $\sigma_1 \in E^i$ *and* $\sigma_2 \in E_i$,

$$\sum_{\tau\in E^k} u^{\sigma_1}_{\tau} u^{\sigma_1 *}_{\tau} = \sum_{\tau\in E_k} u^{\sigma_2 *}_{\tau} u^{\sigma_2}_{\tau}.$$

The co-product Δ_{Ban} *on* $Q^{Ban}_{(V,E)}$ *is given by,*

$$\Delta_{Ban}(u^{\sigma}_{\tau}) = \sum_{\tau'\in E} u^{\sigma}_{\tau'} \otimes u^{\tau'}_{\tau} \quad \text{for all} \quad \sigma, \tau \in E.$$

The canonical co-action $\beta_{Ban} : L^2(E) \to L^2(E) \otimes Q^{Ban}_{(V,E)}$ of $Q^{Ban}_{(V,E)}$ on (V, E) preserving its quantum symmetry in Banica's sense, is given by,

$$\beta_{Ban}(\chi_\tau) = \sum_{\sigma \in E} \chi_\sigma \otimes u^\sigma_\tau \quad \text{for all} \quad \tau \in E.$$

Definition 4.5.2. *Let (V, E) be a directed multigraph. The* **quantum automorphism group of** (V, E) **in Bichon's sense** *is the compact quantum group $(Q^{Bic}_{(V,E)}, \Delta_{Bic})$ where $Q^{Bic}_{(V,E)}$ is the universal C* algebra generated by the elements of the matrix $(u^\sigma_\tau)_{\sigma,\tau \in E}$ satisfying the following relations:*

1. $u^\sigma_\tau = u^{\sigma*}_\tau = u^{\sigma 2}_\tau$ *for all* $\sigma, \tau \in E$.
2. $\sum_{\tau \in E} u^\sigma_\tau = 1 = \sum_{\tau \in E} u^\tau_\sigma$ *for all* $\sigma \in E$.
3. *Let $k \in V^s$ and $l \in V^t$. Then for all $\sigma_1, \sigma_2 \in E$,*
$$\sum_{\tau \in E^k} u^{\sigma_1}_\tau = \sum_{\tau \in E^k} u^{\sigma_2}_\tau \quad \text{if} \quad s(\sigma_1) = s(\sigma_2),$$
$$\sum_{\tau \in E_l} u^{\sigma_1}_\tau = \sum_{\tau \in E_l} u^{\sigma_2}_\tau \quad \text{if} \quad t(\sigma_1) = t(\sigma_2).$$
4. *Let $i \in V^s \setminus V^t, j \in V^t \setminus V^s$ and $k \in V^s \cap V^t$. Then for all $\sigma_1 \in E^i, \sigma_2 \in E_j, \tau_1 \in E^k$ and $\tau_2 \in E_k$,*
$$u^{\sigma_1}_{\tau_1} = 0 \quad \text{and} \quad u^{\sigma_2}_{\tau_2} = 0.$$
5. *Let $i, k \in V^s \cap V^t$. Then for all $\sigma_1 \in E^i$ and $\sigma_2 \in E_i$*
$$\sum_{\tau \in E^k} u^{\sigma_1}_\tau = \sum_{\tau \in E_k} u^{\sigma_2}_\tau.$$

The co-product Δ_{Bic} on $Q^{Bic}_{(V,E)}$ is given by,

$$\Delta_{Bic}(u^\sigma_\tau) = \sum_{\tau' \in E} u^\sigma_{\tau'} \otimes u^{\tau'}_\tau \quad \text{for all} \quad \sigma, \tau \in E.$$

The canonical co-action $\beta_{Bic} : C(E) \to C(E) \otimes Q^{Bic}_{(V,E)}$ of $Q^{Bic}_{(V,E)}$ on (V, E) preserving its quantum symmetry in Bichon's sense is given by,

$$\beta_{Bic}(\chi_\tau) = \sum_{\sigma \in E} \chi_\sigma \otimes u^\sigma_\tau \quad \text{for all} \quad \tau \in E.$$

Remark 4.5.3. *It is clear that the universal commutative CQG in $\mathcal{C}^{Bic}_{(V,E)}$ is nothing but $C(G^{aut}_{(V,E)})$ where $G^{aut}_{(V,E)}$ is the group of classical automorphisms of a directed multigraph (V,E). Moreover, using proposition 5.3.10 one can conclude that $C(G^{aut}_{(V,E)})$ is also the universal commutative CQG in the category $\mathcal{C}^{sym}_{(V,E)}$.*

4.6 Quantum Symmetry in undirected multigraphs

4.6.1 Bi-unitarity and inversion in undirected multigraphs

In this section we will be developing notions of quantum symmetry in undirected multigraphs. Let us consider (V,E,j) to be an undirected multigraph which is a doubly directed multigraph with a chosen inversion map j (see definition 2.1.7). As our choice of j in a doubly directed multigraph might not be unique, the equivalence in theorem 3.6.1 fails to hold and becomes a one way implication.

Theorem 4.6.1. *Let (V,E,j) be an undirected multigraph and $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ be a unitary co-representation of a CQG $(\mathcal{A},\Delta)$ such that the following conditions hold:*

1. $Ad_\beta(S(C(V))) \subseteq S(C(V)) \otimes \mathcal{A}$.
2. $\beta \circ J = (J \otimes id_{\mathcal{A}}) \circ \overline{\beta}$.

Then the following statements are true:

1. $\overline{\beta}$ *is unitary.*
2. $Ad_{\overline{\beta}}(T(C(V))) \subseteq T(C(V)) \otimes \mathcal{A}$.
3. *The induced co-actions α_s and α_t on $C(V)$ coming from β and $\overline{\beta}$ coincide.*

The map $J : L^2(E) \to L^2(E)$ is defined by equation 2.7.1.

Proof. The proof is done using similar arguments as in proof of theorem 3.6.1. □

4.6.2 Notions of quantum symmetry in an undirected multigraph

We propose the following definitions.

Definition 4.6.2. *A compact quantum group $(\mathcal{A},\Delta)$ is said to co-act on an undirected multigraph (V,E,j) preserving its quantum symmetry in Bichon's sense if there exists a co-action $\beta : C(E) \to C(E) \otimes \mathcal{A}$ such that the following conditions hold:*

1. $Ad_\beta(S(C(V))) \subseteq S(C(V)) \otimes \mathcal{A}$.

2. $\beta \circ J = (J \otimes id_{\mathcal{A}}) \circ \beta$.

Definition 4.6.3. *A compact quantum group $(\mathcal{A}, \Delta)$ is said to co-act on an undirected multigraph (V, E, j) preserving its quantum symmetry in Banica's sense if there exists a unitary co-representation $\beta : L^2(E) \to \text{Ł}^2(E) \otimes \mathcal{A}$ such that the following conditions hold:*

1. $Ad_\beta(S(C(V))) \subseteq S(C(V)) \otimes \mathcal{A}$.

2. $\beta \circ J = (J \otimes id_{\mathcal{A}}) \circ \overline{\beta}$.

3. *β fixes the element $\xi_0 := \sum_{\tau \in E} \chi_\tau \in L^2(E)$, that is,*

$$\beta(\xi_0) = \xi_0 \otimes 1_{\mathcal{A}}.$$

Definition 4.6.4. *A compact quantum group $(\mathcal{A}, \Delta)$ is said to co-act on an undirected multigraph (V, E, j) preserving its quantum symmetry in our sense if there exists a unitary co-representation $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ such that the following conditions hold:*

1. $Ad_\beta(S(C(V))) \subseteq S(C(V)) \otimes \mathcal{A}$.

2. $\beta \circ J = (J \otimes id_{\mathcal{A}}) \circ \overline{\beta}$.

3. *β fixes the element $\xi_0 = \sum_{\tau \in E} \chi_\tau \in L^2(E)$, that is,*

$$\beta(\xi_0) = \xi_0 \otimes 1.$$

4. *For $i, j, k, l \in V$ with $E^i_j \neq \phi$ and $E^k_l \neq \phi$, we assume that,*

$$(Ad_\beta)^{ij}_{kl}(D_{ij}) \subseteq D_{kl} \otimes \mathcal{A}.$$

Using observation 3.6.1 which also holds for undirected multigraphs, we propose the following lemma.

Lemma 4.6.5. *Let β be a co-action of a CQG $(\mathcal{A}, \Delta)$ on (V, E, j) preserving its quantum symmetry in Banica's sense, Bichon's sense or ours. Let $(u^\sigma_\tau)_{\sigma,\tau \in E}$ be the co-representation matrix of β. Then we have*

$$u^\sigma_\tau = u^{\overline{\sigma}*}_{\overline{\tau}}$$

for all $\sigma, \tau \in E$.

We describe different **quantum automorphism groups** of an undirected multigraph.

Remark 4.6.6. *Let $\mathcal{C}^{Ban}_{(V,E,j)}, \mathcal{C}^{Bic}_{(V,E,j)}$ and $\mathcal{C}^{sym}_{(V,E,j)}$ be the respective categories of the co-actions mentioned in definitions 4.6.3, 4.6.2 and 4.6.4. The universal objects in $\mathcal{C}^{Ban}_{(V,E,j)}$ and $\mathcal{C}^{Bic}_{(V,E,j)}$ exist and are denoted by $Q^{Ban}_{(V,E,j)}$ and $Q^{Bic}_{(V,E,j)}$ respectively. Moreover, using lemma 4.6.5 it is easy to see that these quantum groups are the quantum automorphism groups of the "underlying doubly directed multigraph" (V,E) quotiented by the C* ideal generated by set of relations $\{u^{\sigma}_{\tau} = u^{\overline{\sigma}*}_{\overline{\tau}} | \sigma, \tau \in E\}$.*

Definition 4.6.7. *For an undirected multigraph (V,E,j), let us define the set of undirected edges E^u by $E^u = \{\{\tau,\overline{\tau}\}|\tau \in E\}$ where $\{.,.\}$ is an unordered pair of two elements.*

Proposition 4.6.8. *Let (V,E,j) be an undirected multigraph without any loops and β is a co-action of a CQG $(\mathcal{A},\Delta)$ on (V,E,j) preserving its quantum symmetry in Bichon's sense. Let us identify $C(E^u)$ as a subalgebra of $C(E)$ by,*

$$C(E^u) = \textit{linear span}\,\{(\chi_\tau + \chi_{\overline{\tau}})|\ \tau \in E\} \subseteq C(E)$$

Then the following holds:

$$\beta(C(E^u)) \subseteq C(E^u) \otimes \mathcal{A}.$$

Hence $\beta|_{C(E^u)}$ is a **quantum permutation** *on the elements of E^u.*

Proof. Using lemma 4.6.5 we observe that,

$$\begin{aligned}\beta(\chi_\tau + \chi_{\overline{\tau}}) &= \sum_{\sigma\in E} \chi_\sigma \otimes (u^{\sigma}_{\tau} + u^{\sigma}_{\overline{\tau}}) \\ &= \sum_{\{\sigma,\overline{\sigma}\}\in E^u} (\chi_\sigma + \chi_{\overline{\sigma}}) \otimes (u^{\sigma}_{\tau} + u^{\sigma}_{\overline{\tau}}).\end{aligned}$$

Therefore we have $\beta(C(E^u)) \subseteq C(E^u) \otimes \mathcal{A}$. As $C(E^u)$ is a subalgebra of $C(E)$, $\beta|_{C(E^u)}$ is a co-action on $C(E^u)$, a **quantum permutation** on the elements of E^u. □

Remark 4.6.9. *The assumption in proposition 4.6.8 that (V,E,j) does not have loops has been taken only for notational convenience. In proposition 4.3.1, we have shown that β is always a quantum permutation on the set of loops L whether β is a co-action on (V,E) in Bichon's sense or ours. If an undirected multigraph (V,E,j) has loops and β is a co-action on (V,E,j) preserving its quantum symmetry in Bichon's sense then β always permutes the elements of L and $E^u \setminus L$ (identifying L as a subset of E^u) separately.*

4.6.3 Underlying undirected multigraph of a directed multigraph

Let us consider the two multigraphs in figure 4.1 where the left one is undirected and the right one is directed. It is clear that any classical automorphism of the directed multigraph in the right can be realised as an automorphism of the underlying undirected multigraph in the left preserving the set of directed edges. This statement is true for any multigraph in general and

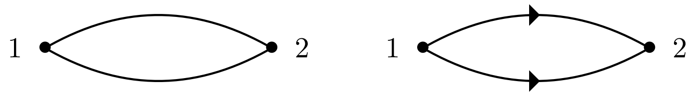

FIGURE 4.1: $G^{aut}_{(V,E)}$ for the left one is $\mathbb{Z}_2 \times \mathbb{Z}_2$ and for the right one is $\mathbb{Z}_2$.

can be observed in the quantum case also as we will see in this subsection. Before stating the main theorem, we need to properly define the notion of the **underlying undirected multigraph** of a **directed multigraph**.

The underlying undirected multigraph $(V, \mathcal{E}, j)$ consists of the same vertex set V with an edge set $\mathcal{E}$ consisting of edges coming from E with all directions removed (in our language, we make all edges doubly directed and identify them via an inversion map j). More precisely, We give the following definition.

Definition 4.6.10. *Let (V, E) be a directed multigraph. For each edge $\sigma \in E^k_l$ where $k \neq l \in V$, let us consider a new edge $\overline{\sigma}$ from l to k. We define $\mathcal{E} = \{\sigma, \overline{\sigma} | \sigma \in E \setminus L\} \cup L$ where L is the set of loops in (V, E). We say that $(V, \mathcal{E}, j)$ is the* **underlying undirected multigraph** *of (V, E) where $j : \mathcal{E} \to \mathcal{E}$ is given by*

$$j(\sigma) = \overline{\sigma}, \quad j(\overline{\sigma}) = \sigma \quad \textit{for all} \quad \sigma \in E \setminus L \quad \textit{and} \quad j|_L = id_L.$$

It is clear that $\mathcal{E} = E \cup j(E)$ and $L^2(\mathcal{E}) = L^2(E \setminus L) \oplus J(L^2(E \setminus L)) \oplus L^2(L)$ where L is the set of loops in (V, E).

Theorem 4.6.11. *Let (V, E) be a directed multigraph and $(V, \mathcal{E}, j)$ be the underlying undirected multigraph. If β is a co-action of a CQG $(\mathcal{A}, \Delta)$ on (V, E) preserving its quantum symmetry in our sense, then there exists a co-action β_u on $(V, \mathcal{E}, j)$ preserving its quantum symmetry in our sense such that,*

$$\beta_u(L^2(E)) \subseteq L^2(E) \otimes \mathcal{A}. \tag{4.6.1}$$

Conversely, if β_u is a co-action on $(V, \mathcal{E}, j)$ preserving its quantum symmetry in our sense and condition 4.6.1 holds, then $\beta = \beta_u|_{L^2(E)}$ preserves quantum symmetry of (V, E) in our sense.

Proof. Before proving the theorem let us make some observations.

1. If β is a co-action on (V, E) preserving its quantum symmetry in our sense then from proposition 4.3.1 it follows that,

$$\beta(L^2(L)) \subseteq L^2(L) \otimes \mathcal{A} \quad \text{and} \quad \beta|_{L^2(L)} = \overline{\beta}|_{L^2(L)}.$$

2. Let $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ a bi-unitary co-representation such that the following hold:

$$\beta(L^2(L)) \subseteq L^2(L) \otimes \mathcal{A} \quad \text{and} \quad \beta|_{L^2(L)} = \overline{\beta}|_{L^2(L)}. \tag{4.6.2}$$

We can always extend β to $\beta_u : L^2(\mathcal{E}) \to L^2(\mathcal{E}) \otimes \mathcal{A}$ by the following formula:

$$\begin{aligned} \beta_u(\chi_\sigma) &= \beta(\chi_\sigma) \quad \text{if} \quad \sigma \in E, \\ &= (J \otimes id)\overline{\beta}(\chi_{j(\sigma)}) \quad \text{if} \quad \sigma \in j(E). \end{aligned} \tag{4.6.3}$$

If $(v^\sigma_\tau)_{\sigma,\tau\in\mathcal{E}}$ and $(u^\sigma_\tau)_{\sigma,\tau\in E}$ are the co-representation matrices of β_u and β respectively, then it follows that,

$$v^\sigma_\tau = u^\sigma_\tau, \quad v^{j(\sigma)}_{j(\tau)} = u^{\sigma*}_\tau, \quad v^{\sigma_1}_{\overline{\tau_1}} = v^{\overline{\sigma_1}}_{\tau_1} = 0 \tag{4.6.4}$$

for all $\sigma, \tau \in E$ and $\sigma_1, \tau_1 \in E \setminus L$.

From observations 4.6.4 and 3.6.1 it is clear that β_u is unitary satisfying

$$\beta_u(L^2(E)) \subseteq L^2(E) \otimes \mathcal{A} \quad \text{and} \quad \beta_u \circ J = (J \otimes id) \circ \overline{\beta}_u. \tag{4.6.5}$$

3. Conversely, If $\beta_u : L^2(\mathcal{E}) \to L^2(\mathcal{E}) \otimes \mathcal{A}$ is a unitary co-representation such that conditions in 4.6.5 hold, then $\beta_u|_{L^2(E)}$ is a bi-unitary co-representation on $L^2(E)$ satisfying the relations in 4.6.2.

Now we proceed to prove the theorem. Let us fix β to be a bi-unitary co-representation on $L^2(E)$ satisfying the relations in 4.6.2 and β_u to be the extension of β on $L^2(\mathcal{E})$ satisfying conditions in 4.6.5. Let $\alpha : C(V) \to C(V) \otimes \mathcal{A}$ be a co-action on $C(V)$.

Claim 1: The following conditions are equivalent.

1. For all $i \in V$ and $\sigma \in \mathcal{E}$,

$$\alpha(\chi_i)\beta_u(\chi_\sigma) = \beta_u(\chi_i . \chi_\sigma).$$

.

2. For all $i \in V$ and $\sigma \in E$,

$$\alpha(\chi_i)\beta(\sigma) = \beta(\chi_i.\chi_\sigma) \quad \text{and} \quad \beta(\chi_\sigma)\alpha(\chi_i) = \beta(\chi_\sigma.\chi_i).$$

$(1) \implies (2)$.

Using theorem 4.6.1 and theorem 4.1.2, it follows that,

$$\beta_u(\chi_\sigma)\alpha(\chi_i) = \beta_u(\chi_\sigma.\chi_i)$$

for all $i \in V$ and $\sigma \in \mathcal{E}$. Using the fact that $\beta_u|_{L^2(E)} = \beta$, (2) follows.

$(2) \implies (1)$.

Let $(q^i_j)_{i,j\in V}$ be the co-representation matrix of α. Let $k \in V; \sigma, \tau \in E$ and $\sigma_1, \tau_1 \in E \setminus L$. Using identities in 4.2.1 and 4.6.4 we observe that,

$$q_k^{s(\sigma)} v^\sigma_\tau = q_k^{s(\sigma)} u^\sigma_\tau = \delta_{k,s(\tau)} u^\sigma_\tau = \delta_{k,s(\tau)} v^\sigma_\tau,$$
$$q_k^{s(\overline{\sigma_1})} v^{\overline{\sigma_1}}_{\overline{\tau_1}} = q_k^{t(\sigma_1)} u^{\sigma_1 *}_{\tau_1} = \delta_{k,t(\tau_1)} u^{\sigma_1 *}_{\tau_1} = \delta_{k,s(\overline{\tau_1})} v^{\overline{\sigma_1}}_{\overline{\tau_1}}.$$

Hence (1) follows.

Claim 2: Let $\xi_0' = \sum_{\tau\in\mathcal{E}} \chi_\tau \in L^2(\mathcal{E})$ and $\xi_0 = \sum_{\tau\in E} \chi_\tau \in L^2(E)$. Then

$$\beta_u(\xi_0') = \xi_0' \otimes 1 \quad \text{iff} \quad \beta(\xi_0) = \xi_0 \otimes 1.$$

We observe that,

$$\beta_u(\xi_0') = \xi_0' \otimes 1 \quad \text{iff} \quad \sum_{\tau\in\mathcal{E}} v^\sigma_\tau = 1 \quad \text{for all} \quad \sigma \in \mathcal{E}.$$
$$\beta(\xi_0) = \xi_0 \otimes 1 \quad \text{iff} \quad \sum_{\tau\in E} u^\sigma_\tau = 1 \quad \text{for all} \quad \sigma \in E.$$

To show claim 2, it is enough to observe that for $\sigma \in \mathcal{E}$,

$$\sum_{\tau\in\mathcal{E}} v^\sigma_\tau = \sum_{\tau\in E} u^\sigma_\tau \quad \text{if} \quad \sigma \in E$$
$$\text{and} \quad \sum_{\tau\in\mathcal{E}} v^\sigma_\tau = \sum_{\tau\in E} u^{j(\sigma)*}_\tau \quad \text{if} \quad \sigma \in j(E).$$

Claim 3: The following conditions (1) and (2) are equivalent:

1. $(Ad_{\beta_u})^{kl}_{ij}(D_{kl}) \subseteq D_{ij} \otimes \mathcal{A}$ for all $i, j, k, l \in V$ such that $\mathcal{E}^i_j$ and $\mathcal{E}^k_l$ are nonempty.

2. For all $i, j, k, l \in V$ such that E^i_j and E^k_l are nonempty we have,

   (a) $(Ad_\beta)^{kl}_{ij}(D_{kl}) \subseteq D_{ij} \otimes \mathcal{A}$,

   (b) $(Ad_{\overline{\beta}})^{kl}_{ij}(D_{kl}) \subseteq D_{ij} \otimes \mathcal{A}$.

From proposition 4.2.12 it follows that,

$$(1) \quad \text{holds} \quad \iff \quad v^{\sigma_1}_\tau v^{\sigma_2 *}_\tau = 0 \iff \quad v^{\sigma_1 *}_\tau v^{\sigma_2}_\tau = 0$$

where $\sigma, \sigma_2, \tau \in \mathcal{E}$ such that $\sigma_1 \neq \sigma_2$ and $(s(\sigma_1), t(\sigma_1)) = (s(\sigma_2), t(\sigma_2))$. As $v^{\sigma_1}_\tau = u^{\sigma_1}_\tau$ whenever $\sigma_1, \sigma_2, \tau \in E$, $(1) \implies (2)$ follows using proposition 4.2.12.

$(2) \implies (1)$.

Let us fix $\sigma_1 \neq \sigma_2 \in \mathcal{E}^i_j$ and $\tau \in \mathcal{E}^k_l$ where $i, j, k, l \in V$. We observe that,

$$\begin{aligned} v^{\sigma_1}_\tau v^{\sigma_2 *}_\tau &= u^{\sigma_1}_\tau u^{\sigma_2 *}_\tau = 0 \quad \text{if} \quad \sigma_1, \sigma_2, \tau \in E \\ &= u^{j(\sigma_1)*}_{j(\tau)} u^{j(\sigma_2)}_{j(\tau)} = 0 \quad \text{if} \quad \sigma_1, \sigma_2, \tau \in j(E). \end{aligned}$$

Hence (1) follows.

From claims 1, 2 and 3 the equivalence mentioned in the theorem 4.6.11 follows.

□

Remark 4.6.12. *Theorem 4.6.11 also holds true in the category $\mathcal{C}^{Bic}_{(V,E)}$. However, it does not hold in the category $\mathcal{C}^{Ban}_{(V,E)}$ as any member of $\mathcal{C}^{Ban}_{(V,E)}$ might not preserve the set of loops.*

Chapter 5

Further investigations into $\mathcal{C}^{sym}_{(V,E)}$ and $\mathcal{C}^{Bic}_{(V,E)}$

5.1 Decomposition of multigraphs into uniform multigraphs

Definition 5.1.1. *A multigraph (V, E) (directed or undirected with an inversion map j) is said to be* **uniform of degree m** *if $|E_l^k| = m$ for all $k, l \in V$ with $E_l^k \neq \phi$.*

Notation 5.1.2. *Any multigraph (V, E) (directed or undirected with an inversion map j) can be written as union of uniform multigraphs as follows. For a nonzero integer m we define $E_m \subseteq E$ and $V_m \subseteq V$ by,*

$$E_m = \{\tau \in E \mid \text{cardinality of the set } E_{t(\tau)}^{s(\tau)} = m\},$$
$$V_m = \{v \in V | v = s(\tau) \text{ or } v = t(\tau) \text{ for some } \tau \in E_m\}.$$

It is clear that, $E = \sqcup_m E_m$ and $V = \cup_m V_m$. For each nonzero integer m such that $E_m \neq \phi$, (V_m, E_m) is a uniform multigraph of degree m. By V_m^s and V_m^t, we will mean the sets of initial and final vertices of the uniform multigraph (V_m, E_m). We will write $(V, E) = \cup_m (V_m, E_m)$.

5.2 Co-actions on uniform components of a multigraph

Proposition 5.2.1. *Let $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ be a co-action of a CQG $(\mathcal{A}, \Delta)$ on (V, E) preserving its quantum symmetry in our sense. For any $m \in \mathbb{N}$ with $E_m \neq \phi$, it follows that*

$$\beta(L^2(E_m)) \subseteq L^2(E_m) \otimes \mathcal{A}.$$

Proof. Let $(u^\sigma_\tau)_{\sigma,\tau\in E}$ be the co-representation matrix of β. Let $i,j,k,l\in V$ be such that E^i_j and E^k_l are nonempty and $|E^i_j|\neq|E^k_l|$. It is enough to show that

$$u^\sigma_\tau=0 \quad \text{for all} \quad \sigma\in E^i_j, \tau\in E^k_l. \tag{5.2.1}$$

Using proposition 4.2.7 and proposition 4.2.12 we observe that,

$$\begin{aligned}\|u^\sigma_\tau\|^2 &= \|u^{\sigma*}_\tau u^\sigma_\tau\|\\ &= \|u^{\sigma*}_\tau(\sum_{\sigma_1\in E^i_j} u^{\sigma_1}_\tau)\|\\ &= \|u^{\sigma*}_\tau q^i_k q^j_l\| = 0 \qquad \text{(from corollary (4.2.10))}.\end{aligned}$$

$\square$

Now we prove the converse of the above proposition.

Proposition 5.2.2. *Let (V,E) be a multigraph (directed or undirected with an inversion map j) and $(\mathcal{A},\Delta)$ be a CQG. For every $m\in\mathbb{N}$ with $E_m\neq\phi$, let us consider $\beta_m : L^2(E_m)\to L^2(E_m)\otimes\mathcal{A}$ to be a bi-unitary co-representation. Let $\beta : L^2(E)\to L^2(E)\otimes\mathcal{A}$ be defined by*

$$\beta(\chi_\tau)=\beta_m(\chi_\tau) \quad \textit{where} \quad \tau\in E_m.$$

Then the following conditions are equivalent:

1. *β is a co-action on (V,E) preserving its quantum symmetry in our sense.*

2. *For every $m\in\mathbb{N}$ with $E_m\neq\phi$, β_m is a co-action on (V_m,E_m) preserving its quantum symmetry in our sense. Moreover, for all $m\neq n$,*

$$\alpha_m|_{C(V_m\cap V_n)}=\alpha_n|_{C(V_m\cap V_n)}$$

where α_m and α_n are induced quantum permutations on V_m and V_n respectively.

Proof. The proof is done by using theorem 4.1.1, theorem 4.1.2 and proposition 4.2.6 repeatedly. Throughout the proof, $(u^\sigma_\tau)_{\sigma,\tau\in E}$ will be the co-representation matrix of β. As $\beta_m=\beta|_{L^2(E_m)}$ for every $m\in\mathbb{N}$ such that $E_m\neq\phi$, it follows that $(u^\sigma_\tau)_{\sigma,\tau\in E_m}$ is the co-representation matrix of β_m.

Claim 1: $(1)\implies(2)$.

Let α be the induced co-action of β on $C(V)$ and $(q^i_k)_{i,k\in V}$ be the co-representation matrix of

α. Let $i \in V \setminus V_m$ and $k \in V_m$. Without loss of generality, we can assume that $i \in V_n^s$ and $k \in V_m^s$ where $m \neq n$. Using proposition 4.2.7 and proposition 5.2.1 we observe that,

$$q_k^i = \sum_{\sigma \in E^i} u_\tau^\sigma = 0$$

where $\tau \in E_m$ and $s(\tau) = k$. Hence it follows that,

$$\alpha(C(V_m)) \subseteq C(V_m) \otimes \mathcal{A}.$$

Let us define $\alpha_m : C(V_m) \to C(V_m) \otimes \mathcal{A}$ by $\alpha_m = \alpha|_{C(V_m)}$. It is clear that,

$$\alpha_m|_{C(V_m \cap V_n)} = \alpha_n|_{C(V_m \cap V_n)} \quad \text{when} \quad m \neq n.$$

As (α, β) respects the $C(V) - L^2(E) - C(V)$ bi-module structure, (α_m, β_m) respects the bimodularity $C(V_m) - L^2(E_m) - C(V_m)$ in (V_m, E_m) . Using proposition 4.2.6, theorem 4.1.1 and theorem 4.1.2 it follows that β_m satisfies (1), (2) and (3) of definition 4.2.3. To show that β_m satisfies (4) of definition 4.2.3, we observe that

$$1 = \sum_{\tau \in E} u_\tau^\sigma = \sum_{\tau \in E_m} u_\tau^\sigma. \quad \text{where} \quad \sigma \in E_m.$$

Similarly, To show β_m satisfies (5) of definition 4.2.3, it is easy to check that coefficients of the co-representation matrix of β_m satisfy the identities mentioned in proposition 4.2.12.

Claim : $(2) \implies (1)$.

As $\alpha_m|_{C(V_m \cap V_n)} = \alpha_n|_{C(V_m \cap V_n)}$ for all $m \neq n \in \mathbb{N}$, there exists a co-action $\alpha : C(V) \to C(V) \otimes \mathcal{A}$ such that,

$$\alpha(\chi_i) = \alpha_m(\chi_i) \quad \text{where} \quad i \in V_m.$$

Let $(q_k^i)_{i,k \in V}$ be the co-representation matrix of α. As we clearly have $\alpha(C(V_m \cap V_n)) \subseteq C(V_m \cap V_n) \otimes \mathcal{A}$ for all $m \neq n$, it follows that,

$$q_i^k = 0 \quad \text{where} \quad i \notin V_m \cap V_n \text{ and } k \in V_m \cap V_n. \tag{5.2.2}$$

We show that (α, β) respects the $C(V) - L^2(E) - C(V)$ bi-module structure. Let $\tau \in E_m$ and $i \in V_n$ for some nonzero integers m and n such that $m \neq n$. We observe that if $i \in V_m \cap V_n$ then

$$\alpha(\chi_i)\beta(\chi_\tau) = \alpha_m(\chi_i)\beta_m(\chi_\tau) = \beta_m(\chi_i.\chi_\tau) = \beta(\chi_i.\chi_\tau).$$

and if $i \in V_n \setminus V_m$, then using equation 5.2.2 we observe that,

$$\begin{aligned}\alpha(\chi_i)\beta(\chi_\tau) &= \alpha_n(\chi_i)\beta_m(\chi_\tau)\\ &= \Big(\sum_{\substack{k\in V_n\\ k\notin V_m\cap V_n}} \chi_k \otimes q^k_i\Big)\Big(\sum_{\sigma\in E_m} \chi_\sigma \otimes u^\sigma_\tau\Big)\\ &= 0 \quad (\text{as } \chi_k.\chi_\sigma = 0 \text{ for all } k \in V_n \setminus V_m)\\ &= \beta(\chi_i.\chi_\tau).\end{aligned}$$

Using similar arguments, it also follows that,

$$\beta(\chi_\tau)\alpha(\chi_i) = \beta(\chi_\tau.\chi_i) \quad \text{for all} \quad i \in V, \tau \in E.$$

Using proposition 4.2.6, theorem 4.1.1 and theorem 4.1.2 it follows that β saisfies (1), (2) and (3) of definition 4.2.3. To show that β satisfies (4) of definition 4.2.3 we note that, for $\sigma \in E$,

$$1 = \sum_{\tau\in E_m} u^\sigma_\tau = \sum_{\tau\in E} u^\sigma_\tau \quad \text{where } \sigma \in E_m \text{ for some } m.$$

Now we show that, β satisfies (5) of definition 4.2.3. Let $\sigma_1 \neq \sigma_2 \in E^i_j$ and $\tau \in E^k_l$ for some $i, j, k, l \in V$. If $|E^i_j| \neq |E^k_l|$, then from our hypothesis it follows that $u^{\sigma_1}_\tau = u^{\sigma_2}_\tau = 0$ and the identities mentioned in proposition 4.2.12 hold. If $|E^i_j| = |E^k_l| = m$ for some non zero integer m, identities mentioned in proposition 4.2.12 hold as we are given that β_m is a co-action on (V_m, E_m) and $(u^\sigma_\tau)_{\sigma,\tau\in E_m}$ is the co-representation matrix of β_m. □

Corollary 5.2.3. *It is clear that proposition 5.2.2 also holds true in the category $\mathcal{C}^{Bic}_{(V,E)}$. Let $(V, E) = \cup_m(V_m, E_m)$. Let us consider the free product of CQGs $Q^{Bic}_{(V_m,E_m)}$ and $\mathcal{I}$ to be a C* ideal in $*_m Q^{Bic}_{(V_m,E_m)}$ generated by the following relations:*

$$\begin{aligned}{}^m q^i_j &= {}^n q^i_j \quad \textit{for all} \quad i, j \in V_m \cap V_n\\ &= 0 \quad \textit{if} \quad i \in V_m \cap V_n, j \in V_m \setminus V_n\end{aligned}$$

for all nonzero integers m and n such that $E_m \neq \phi$, $E_n \neq \phi$ and $m \neq n$. Here $({}^m q^i_j)_{i,j\in V_m}$, $({}^n q^i_j)_{i,j\in V_n}$ are the co-representation matrices of induced co-actions α_m and α_n respectively.

Then $Q^{Bic}_{(V,E)}$ is given by,

$$Q^{Bic}_{(V,E)} = *_m Q^{Bic}_{(V_m,E_m)} / \mathcal{I}.$$

5.3 Co-actions on uniform multigraphs

In the last section we have seen that quantum symmetry of a multigraph is closely tied to quantum symmetries of its uniform components. In this section we will be investigating more into co-actions on uniform multigraphs.

5.3.1 New notations and a technical lemma

We introduce the following set of notations.

Notation 5.3.1. *Let (V, E) be a uniform multigraph of degree m. For each $k, l \in V$ such that $E^k_l \neq \phi$, let us consider a bijection $\mu_{kl} : \{1, .., m\} \to E^k_l$. This set of bijections $\{\mu_{kl} | E^k_l \neq \phi\}$ is said to be a* **representation of the multigraph** *(V, E). Once a representation is fixed, any $\tau \in E$ can be written as*

$$\tau = (k, l)r \quad \text{where} \quad s(\tau) = k,\ t(\tau) = l \ \text{and} \ 1 \leq r \leq m.$$

Furthermore, if (V, E) is undirected with an inversion map $j : E \to E$, then we will number the edges in a way such that,

$$j((k, l)r) = (l, k)r \quad \text{for all} \quad (k, l)r \in E.$$

For proceeding further we will be needing a technical lemma which is given below:

Lemma 5.3.2. *Let $\{A_i | i = 1, 2, .., n\}$ be a set of positive operators on a Hilbert space H such that $A_i A_j = 0$ when $i \neq j$. Let $T := \sum_{i=1}^n A_i$. For $i \in \{1, 2, .., n\}$, let p_i and P_T be range projections of A_i and T, that is, orthogonal projections onto the closures of ranges of A_i and T respectively. . Then the following identities are true:*

1. *$p_i p_j = 0 \quad \text{when} \quad i \neq j$.*
2. *$A_i = p_i T = T p_i \quad \text{for all} \quad i = 1, 2, .., n$.*
3. *$\sum_{i=1}^n p_i = P_T$.*

Proof. To prove (1), we will show that range of p_i is orthogonal to range of p_j whenever $i \neq j$. For $\xi, \eta \in H$, we observe that,

$$< A_i(\xi), A_j(\eta) >=< A_j A_i(\xi), \eta >= 0$$

whenever $i \neq j$. Hence (1) follows.

We note that,

$$p_iA_j = p_ip_jA_j = 0 \quad \text{and} \quad A_jp_i = (p_iA_j)^* = 0$$

whenever $i \neq j$. Using the relations mentioned above we observe that,

$$p_iT = p_i(\sum_{j=1}^{n} A_j) = p_iA_i = A_i,$$
$$Tp_i = (\sum_{j=1}^{n} A_j)p_i = A_ip_i = (p_iA_i)^* = A_i.$$

Hence (2) is proved.

To prove claim (3), it is enough to show that

$$\overline{Range(T)} = \oplus_{i=1}^{n}\overline{Range(A_i)}$$

where the direct sum is an orthogonal direct sum.

As for any $\xi \in H$, we have $A_i(\xi) = T(p_i(\xi))$, it follows that,

$$\overline{Range(A_i)} \subseteq \overline{Range(T)} \quad \text{and therefore} \quad \oplus_{i=1}^{n} \overline{Range(A_i)} \subseteq \overline{Range(T)}.$$

Now we prove the converse part. For any $\xi \in H$ we have,

$$T(\xi) = \sum_{i=1}^{n} A_i(\xi).$$

As $< A_i(\xi), A_j(\xi) >= 0$ for $i \neq j$, it follows that,

$$\overline{Range(T)} \subseteq \overline{\oplus_{i=1}^{n}Range(A_i)} = \oplus_{i=1}^{n}\overline{Range(A_i)}.$$

Therefore our claim is proved.

□

5.3.2 Nested quantum permutation relations

Notation 5.3.3. *For a compact quantum group $(\mathcal{A}, \Delta)$, let $\overline{\mathcal{A}}$ denote the universal enveloping Von Neumann algebra of $\mathcal{A}$. It is known that the co-product Δ extends to $\overline{\Delta}$ on $\overline{\mathcal{A}}$ as a normal homomorphism of Von Neumann algebras making $(\overline{\mathcal{A}}, \overline{\Delta})$ a Von Neumann algebraic quantum group.*

In this subsection, we fix β to be a co-action of a CQG $(\mathcal{A},\Delta)$ on a uniform multigraph (V,E) of degree m preserving its quantum symmetry in our sense. We fix a **representation of the multigraph** *of (V,E) (see notation 5.3.1). Let $(u^{(i,j)r}_{(k,l)s})_{(i,j)r,(k,l)s\in E}$ and $(q^i_j)_{i,j\in V}$ be the co-representation matrices of β and α respectively where α is the induced co-action on $C(V)$.*

Proposition 5.3.4. *Let i,j,k,l be in V such that E^i_j and E^k_l are nonempty. Then there exists a projection valued valued matrix $(p^{(i,j)r}_{(k,l)s})_{r,s=1,..,m}\in M_m(\mathbb{C})\otimes\overline{\mathcal{A}}$ such that the following holds:*

$$u^{(i,j)r}_{(k,l)s}u^{(i,j)r^*}_{(k,l)s}=p^{(i,j)r}_{(k,l)s}q^i_kq^j_lq^i_k.$$

Here $p^{(i,j)r}_{(k,l)s}$'s are the range projections of $u^{(i,j)r}_{(k,l)s}$ satisfying the following **quantum permutation** *like relations:*

1. *For r,r' and $s\in\{1,2,..,m\}$,* $p^{(i,j)r}_{(k,l)s}p^{(i,j)r'}_{(k,l)s}=\delta_{r,r'}p^{(i,j)r}_{(k,l)s}.$
2. *For r,s and $s'\in\{1,2,..,m\}$,* $p^{(i,j)r}_{(k,l)s}p^{(i,j)r}_{(k,l)s'}=\delta_{s,s'}p^{(i,j)r}_{(k,l)s}.$
3. $\sum_{s=1}^m p^{(i,j)r}_{(k,l)s}=\sum_{r=1}^m p^{(i,j)r}_{(k,l)s}=P_{q^i_kq^j_lq^i_k}$ *where $P_{q^i_kq^j_lq^i_k}$ is the range projection of $q^i_kq^j_lq^i_k$.*

Proof. Using proposition (4.2.7) and proposition (4.2.12) we observe that,

$$\begin{aligned}\sum_{s=1}^m u^{(i,j)r}_{(k,l)s}u^{(i,j)r^*}_{(k,l)s}&=(\sum_{s=1}^m u^{(i,j)r}_{(k,l)s})(\sum_{s=1}^m u^{(i,j)r}_{(k,l)s})^*\\&=q^i_kq^j_l(q^i_kq^j_l)^*=q^i_kq^j_lq^i_k.\end{aligned}$$

As $u^{(i,j)r}_{(k,l)s}u^{(i,j)r^*}_{(k,l)s}$'s are positive operators, using (2) of lemma 5.3.2 we conclude that

$$u^{(i,j)r}_{(k,l)s}u^{(i,j)r^*}_{(k,l)s}=p^{(i,j)r}_{(k,l)s}q^i_kq^j_lq^i_k$$

where $p^{(i,j)r}_{(k,l)s}$ is range projection of $u^{(i,j)r}_{(k,l)s}u^{(i,j)r^*}_{(k,l)s}$ which is same as the range projection of $u^{(i,j)r}_{(k,l)s}$. The **quantum permutation** like relations among $p^{(i,j)r}_{(k,l)s}$'s follow from the "orthogonality relations" mentioned in (1) and (3) of lemma 5.3.2.

□

Proposition 5.3.5. *Let i,j,k,l be in V such that E^i_j and E^k_l are nonempty. Then there exists a projection valued valued matrix $(\hat{p}^{(i,j)r}_{(k,l)s})_{r,s=1,..,m}\in M_m(\mathbb{C})\otimes\overline{\mathcal{A}}$ such that the following holds:*

$$u^{(i,j)r*}_{(k,l)s}u^{(i,j)r}_{(k,l)s}=\hat{p}^{(i,j)r}_{(k,l)s}q^j_lq^i_kq^j_l.$$

Here $\hat{p}^{(i,j)r}_{(k,l)s}$'s are the range projections of $u^{(i,j)r*}_{(k,l)s}$ satisfying the following **quantum permutation** *like relations:*

1. *For r, r' and $s \in \{1, 2, .., m\}$,* $\hat{p}^{(i,j)r}_{(k,l)s}\hat{p}^{(i,j)r'}_{(k,l)s} = \delta_{r,r'}\hat{p}^{(i,j)r}_{(k,l)s}$.
2. *For r, s and $s' \in \{1, 2, .., m\}$,* $\hat{p}^{(i,j)r}_{(k,l)s}\hat{p}^{(i,j)r}_{(k,l)s'} = \delta_{s,s'}\hat{p}^{(i,j)r}_{(k,l)s}$.
3. $\sum_{s=1}^m \hat{p}^{(i,j)r}_{(k,l)s} = \sum_{r=1}^m \hat{p}^{(i,j)r}_{(k,l)s} = P_{q^j_l q^i_k q^j_l}$ *where $P_{q^j_l q^i_k q^j_l}$ is the range projection of $q^j_l q^i_k q^j_l$.*

Proof. Using proposition 4.2.7 and proposition 4.2.12 we observe that,

$$\begin{aligned}\sum_{s=1}^m u^{(i,j)r*}_{(k,l)s} u^{(i,j)r}_{(k,l)s} &= (\sum_{s=1}^m u^{(i,j)r}_{(k,l)s})^*(\sum_{s=1}^m u^{(i,j)r}_{(k,l)s}) \\ &= (q^i_k q^j_l)^* q^i_k q^j_l = q^j_l q^i_k q^j_l.\end{aligned}$$

The claims follow from lemma 5.3.2 as it did for proposition 5.3.4. □

Corollary 5.3.6. *Let i, j, k, l, r, s be as in proposition 5.3.4 or proposition 5.3.5. Then we have the following commutation relations:*

1. $p^{(i,j)r}_{(k,l)s} q^i_k q^j_l q^i_k = q^i_k q^j_l q^i_k p^{(i,j)r}_{(k,l)s}$.
2. $\hat{p}^{(i,j)r}_{(k,l)s} q^j_l q^i_k q^j_l = q^j_l q^i_k q^j_l \hat{p}^{(i,j)r}_{(k,l)s}$.
3. $p^{(i,j)r}_{(k,l)s} u^{(i,j)r}_{(k,l)s} u^{(i,j)r*}_{(k,l)s} = u^{(i,j)r}_{(k,l)s} u^{(i,j)r*}_{(k,l)s} = u^{(i,j)r}_{(k,l)s} u^{(i,j)r*}_{(k,l)s} p^{(i,j)r}_{(k,l)s}$.
4. $\hat{p}^{(i,j)r}_{(k,l)s} u^{(i,j)r*}_{(k,l)s} u^{(i,j)r}_{(k,l)s} = u^{(i,j)r*}_{(k,l)s} u^{(i,j)r}_{(k,l)s} = u^{(i,j)r*}_{(k,l)s} u^{(i,j)r}_{(k,l)s} \hat{p}^{(i,j)r}_{(k,l)s}$.

Proof. (1) and (3) are immidiate from proposition 5.3.4, (2) and (4) are immidiate from proposition 5.3.5. □

Proposition 5.3.7. *Let $i, j, k, l \in V$ be such that E^i_j and E^k_l are nonempty and $r, s \in \{1, 2, .., m\}$. Then we have the following:*

$$u^{(i,j)r}_{(k,l)s} = p^{(i,j)r}_{(k,l)s} q^i_k q^j_l = q^i_k q^j_l \hat{p}^{(i,j)r}_{(k,l)s}$$

where $p^{(i,j)r}_{(k,l)s}$'s are described in proposition 5.3.4 and $\hat{p}^{(i,j)r}_{(k,l)s}$'s are described in proposition 5.3.5.

Proof. From proposition 4.2.7, proposition 5.3.4 and corollary 5.3.6 we observe that,

$$\begin{aligned}
&(u^{(i,j)r}_{(k,l)s} - p^{(i,j)r}_{(k,l)s} q^i_k q^j_l)(u^{(i,j)r}_{(k,l)s} - p^{(i,j)r}_{(k,l)s} q^i_k q^j_l)^* \\
=&u^{(i,j)r}_{(k,l)s} u^{(i,j)r^*}_{(k,l)s} - u^{(i,j)r}_{(k,l)s} q^j_l q^i_k p^{(i,j)r}_{(k,l)s} - p^{(i,j)r}_{(k,l)s} q^i_k q^j_l u^{(i,j)r^*}_{(k,l)s} + p^{(i,j)r}_{(k,l)s} q^i_k q^j_l q^j_l q^i_k p^{(i,j)r}_{(k,l)s} \\
=&u^{(i,j)r}_{(k,l)s} u^{(i,j)r^*}_{(k,l)s} - u^{(i,j)r}_{(k,l)s} (\sum_{r'=1}^{m} u^{(i,j)r'^*}_{(k,l)s}) p^{(i,j)r}_{(k,l)s} - p^{(i,j)r}_{(k,l)s} (\sum_{r'=1}^{m} u^{(i,j)r'}_{(k,l)s}) u^{(i,j)r^*}_{(k,l)s} \\
&+ u^{(i,j)r}_{(k,l)s} u^{(i,j)r^*}_{(k,l)s} p^{(i,j)r}_{(k,l)s} \\
=&u^{(i,j)r}_{(k,l)s} u^{(i,j)r^*}_{(k,l)s} - u^{(i,j)r}_{(k,l)s} u^{(i,j)r^*}_{(k,l)s} p^{(i,j)r}_{(k,l)s} - p^{(i,j)r}_{(k,l)s} u^{(i,j)r}_{(k,l)s} u^{(i,j)r^*}_{(k,l)s} + u^{(i,j)r}_{(k,l)s} u^{(i,j)r^*}_{(k,l)s} p^{(i,j)r}_{(k,l)s} \\
=&0.
\end{aligned}$$

We conclude that

$$u^{(i,j)r}_{(k,l)s} - p^{(i,j)r}_{(k,l)s} q^i_k q^j_l = 0 \quad \text{and hence} \quad u^{(i,j)r}_{(k,l)s} = p^{(i,j)r}_{(k,l)s} q^i_k q^j_l.$$

To prove the second equality, again using proposition 4.2.7, proposition 5.3.5 and corollary 5.3.6 we observe that,

$$\begin{aligned}
&(u^{(i,j)r}_{(k,l)s} - q^i_k q^j_l \hat{p}^{(i,j)r}_{(k,l)s})^*(u^{(i,j)r}_{(k,l)s} - q^i_k q^j_l \hat{p}^{(i,j)r}_{(k,l)s}) \\
=&u^{(i,j)r*}_{(k,l)s} u^{(i,j)r}_{(k,l)s} - u^{(i,j)r*}_{(k,l)s} q^i_k q^j_l \hat{p}^{(i,j)r}_{(k,l)s} - \hat{p}^{(i,j)r}_{(k,l)s} q^j_l q^i_k u^{(i,j)r}_{(k,l)s} + \hat{p}^{(i,j)r}_{(k,l)s} q^j_l q^i_k q^j_l \hat{p}^{(i,j)r}_{(k,l)s} \\
=&u^{(i,j)r*}_{(k,l)s} u^{(i,j)r}_{(k,l)s} - u^{(i,j)r*}_{(k,l)s} (\sum_{r'=1}^{m} u^{(i,j)r'}_{(k,l)s}) \hat{p}^{(i,j)r}_{(k,l)s} - \hat{p}^{(i,j)r}_{(k,l)s} (\sum_{r'=1}^{m} u^{(i,j)r'*}_{(k,l)s}) u^{(i,j)r}_{(k,l)s} \\
&+ u^{(i,j)r*}_{(k,l)s} u^{(i,j)r}_{(k,l)s} \hat{p}^{(i,j)r}_{(k,l)s} \\
=&u^{(i,j)r*}_{(k,l)s} u^{(i,j)r}_{(k,l)s} - u^{(i,j)r*}_{(k,l)s} u^{(i,j)r}_{(k,l)s} \hat{p}^{(i,j)r}_{(k,l)s} - \hat{p}^{(i,j)r}_{(k,l)s} u^{(i,j)r*}_{(k,l)s} u^{(i,j)r}_{(k,l)s} + u^{(i,j)r*}_{(k,l)s} u^{(i,j)r}_{(k,l)s} \hat{p}^{(i,j)r}_{(k,l)s} \\
=&0.
\end{aligned}$$

As before we conclude that,

$$u^{(i,j)r}_{(k,l)s} - q^i_k q^j_l \hat{p}^{(i,j)r}_{(k,l)s} = 0 \quad \text{and hence} \quad u^{(i,j)r}_{(k,l)s} = q^i_k q^j_l \hat{p}^{(i,j)r}_{(k,l)s}.$$

□

Remark 5.3.8. *If we consider β to be a co-action on (V, E) preserving its quantum symmetry in Bichon's sense, then it follows that $u^{(i,j)r}_{(k,l)s} = p^{(i,j)r}_{(k,l)s} = \hat{p}^{(i,j)r}_{(k,l)s}$ as $u^{(i,j)r}_{(k,l)s}$'s are already projections commuting with q^i_k and q^j_l.*

5.3.3 A necessary and sufficient condition for $\mathcal{C}^{sym}_{(V,E)} = \mathcal{C}^{Bic}_{(V,E)}$

In this section, we will see that our defined notions of quantum symmetry in the context of a multigraph are very closely connected to the quantum symmetry of the **underlying weighted graph**. (see definition 2.1.10).

Theorem 5.3.9. *Let (V, E) be a uniform multigraph (directed or undirected with an inversion map) of degree m. It follows that two categories $\mathcal{C}^{sym}_{(V,E)}$ and $\mathcal{C}^{Bic}_{(V,E)}$ coincide if and only if the categories $\mathcal{D}^{Ban}_{(V,\overline{E})}$ and $\mathcal{D}^{Bic}_{(V,\overline{E})}$ coincide where $(V,\overline{E})$ is the* ***underlying simple graph*** *of (V, E) (see definition 2.1.9).*

Proof. Let us assume that $\mathcal{C}^{sym}_{(V,E)} = \mathcal{C}^{Bic}_{(V,E)}$. Let $\alpha : C(V) \to C(V) \otimes \mathcal{A}$ be a co-action of a CQG $(\mathcal{A}, \Delta)$ such that the co-representation matrix $(q^i_j)_{i,j\in V}$ of α commutes with the adjacency matrix of $(V,\overline{E})$. Let us fix a **representation of the multigraph** (V, E) (see notation 5.3.1). We define a bi-unitary co-representation $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ by,

$$\beta(\chi_{(k,l)s}) = \sum_{(i,j)r\in E} \chi_{(i,j)r} \otimes \delta_{r,s} q^i_k q^j_l \quad \text{where} \quad (k, l)s \in E.$$

It is easy to see that β is in fact a co-action on (V, E) which preserves its quantum symmetry in our sense. From our assumption and remark 4.2.8 it follows that q^i_k commutes with q^j_l for all $(i, j),(k, l) \in \overline{E}$ making $\alpha^{(2)}$ (see notation 2.5.1) a co-action on $C(\overline{E})$. Hence α is a co-action on $(V,\overline{E})$ preserving its quantum symmetry in Bichon's sense. Therefore the categories $\mathcal{D}^{Ban}_{(V,\overline{E})}$ and $\mathcal{D}^{Bic}_{(V,\overline{E})}$ coincide.

Conversely, let us assume that $\mathcal{D}^{Ban}_{(V,\overline{E})} = \mathcal{D}^{Bic}_{(V,\overline{E})}$. Let us consider $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ to be a co-action of a CQG $(\mathcal{A}, \Delta)$ on (V, E) preserving its quantum symmetry in our sense. Let $(u^\sigma_\tau)_{\sigma,\tau\in E}$ and $(q^i_j)_{i,j\in V}$ be the co-representation matrices of β and the induced co-action α on $C(V)$. From proposition 4.2.9 and our assumption it follows that q^i_k commutes with q^j_l whenever E^i_j and E^k_l are nonempty. To show that β preserves quantum symmetry of (V, E) in Bichon's sense, it is enough to show that β is in fact a co-action on the algebra $C(E)$, that is, u^σ_τ's are projections in $\mathcal{A}$ satisfying **quantum permutation relations**. We proceed through the following claims.

Claim 1: $u^\tau_{\tau'}$'s are projections for all $\tau, \tau' \in E$.

Let $i, j, k, l \in V$ be such that E^i_j and E^k_l are nonempty. Let us fix $\tau \in E^i_j$ and $\tau' \in E^k_l$. Using proposition 4.2.7 and proposition 4.2.12 we observe the following relations:

$$\sum_{\sigma\in E^i_j} {u^\sigma_{\tau'}}^* u^\sigma_{\tau'} = (\sum_{\sigma\in E^i_j} u^\sigma_{\tau'})^*(\sum_{\sigma\in E^i_j} u^\sigma_{\tau'})$$

$$= q_l^j q_k^i q_l^j = q_l^j q_k^i \tag{5.3.1}$$

$$\sum_{\sigma \in E_j^i} u_{\tau'}^{\sigma} {u_{\tau'}^{\sigma}}^* = (\sum_{\sigma \in E_j^i} u_{\tau'}^{\sigma})(\sum_{\sigma \in E_j^i} u_{\tau'}^{\sigma})^*$$

$$= q_k^i q_l^j q_k^i = q_l^j q_k^i. \tag{5.3.2}$$

Using equations 5.3.1 and 5.3.2 it follows that,

$$\begin{aligned}
u_{\tau'}^{\tau} {u_{\tau'}^{\tau}}^* u_{\tau'}^{\tau} &= u_{\tau'}^{\tau} (\sum_{\sigma \in E_j^i} {u_{\tau'}^{\sigma}}^* u_{\tau'}^{\sigma}) \\
&= u_{\tau'}^{\tau} (q_l^j q_k^i) \\
&= u_{\tau'}^{\tau} (\sum_{\sigma \in E_j^i} {u_{\tau'}^{\sigma}}^*) = u_{\tau'}^{\tau} {u_{\tau'}^{\tau}}^*;
\end{aligned}$$

$$\begin{aligned}
\text{and similarly} \quad u_{\tau'}^{\tau} {u_{\tau'}^{\tau}}^* u_{\tau'}^{\tau} &= (q_l^j q_k^i) u_{\tau'}^{\tau} \\
&= (\sum_{\sigma \in E_j^i} u_{\tau'}^{\sigma *}) u_{\tau'}^{\tau} = u_{\tau'}^{\tau *} u_{\tau'}^{\tau}.
\end{aligned}$$

From the above relations we get

$$u_{\tau'}^{\tau} {u_{\tau'}^{\tau}}^* = {u_{\tau'}^{\tau}}^* u_{\tau'}^{\tau} = u_{\tau'}^{\tau} {u_{\tau'}^{\tau}}^* u_{\tau'}^{\tau}.$$

Using spectral calculus for normal operators, we conclude that $u_{\tau'}^{\tau}$ is a projection. Hence claim 1 follows.

Claim 2: Let $i, j, k, l \in V$ be such that E_j^i and E_l^k are nonempty. For every $\sigma \in E_j^i$ and $\tau \in E_l^k$, u_{τ}^{σ} commutes with q_k^i and q_l^j.

As u_{τ}^{σ}'s are projections (from claim 1) we observe that,

$$\begin{aligned}
u_{\tau}^{\sigma} q_k^i &= u_{\tau}^{\sigma *} q_k^i = u_{\tau}^{\sigma *} = u_{\tau}^{\sigma} = q_k^i u_{\tau}^{\sigma} \\
\text{and} \quad q_l^j u_{\tau}^{\sigma} &= q_l^j u_{\tau}^{\sigma *} = u_{\tau}^{\sigma *} = u_{\tau}^{\sigma} = u_{\tau}^{\sigma} q_l^j.
\end{aligned}$$

Hence claim 2 follows.

Claim 3: Let σ_1, σ_2 and τ be in E. Then we have the following orthogonality relation:

$$u_{\tau}^{\sigma_1} u_{\tau}^{\sigma_2} = \delta_{\sigma_1, \sigma_2} u_{\tau}^{\sigma_1}.$$

Let $\tau \in E^k_l, \sigma_1 \in E^i_j$ and $\sigma_2 \in E^{i'}_{j'}$ for some $i, j, i', j', k, l \in V$. As $u^{\sigma_1}_\tau$ and $u^{\sigma_2}_\tau$ are both projections, using claim 2 we observe that,

$$\begin{aligned}
u^{\sigma_1}_\tau u^{\sigma_2}_\tau &= q^i_k u^{\sigma_1}_\tau q^j_l q^{i'}_k u^{\sigma_2}_\tau q^{j'}_l \\
&= u^{\sigma_1}_\tau q^j_l q^i_k q^{i'}_k q^{j'}_l u^{\sigma_1}_\tau \\
&= u^{\sigma_1}_\tau \delta_{i,i'} \delta_{j,j'} u^{\sigma_2}_\tau \\
&= \delta_{\sigma_1,\sigma_2} u^{\sigma_1}_\tau \quad \text{(from proposition 4.2.12)}.
\end{aligned}$$

Hence claim 3 follows.

Using claims 1 and 3 and the fact that $\sum_{\tau \in E} u^\sigma_\tau = 1$ for all $\sigma \in E$, we conclude that β preserves the quantum symmetry of (V, E) in Bichon's sense.

□

The arguments used in proving converse part of the above theorem can be generalised beyond uniform multigraphs. Using similar arguments used in proving claim 1 and claim 2 we also can prove the following proposition.

Proposition 5.3.10. *Let β be a co-action of a CQG $(\mathcal{A}, \Delta)$ on a multigraph (V, E) (directed or undirected with an inversion map) preserving its quantum symmetry in our sense where (V, E) is not necessarily uniform. Let $(u^\sigma_\tau)_{\sigma,\tau \in E}$ and $(q^i_j)_{i,j \in V}$ be co-representation matrices of β and its induced permutation on the vertex set V. Let $i, j, k, l \in V$ be such that $E^i_j \neq \phi$ and $E^k_l \neq \phi$. If q^i_k and q^j_l commute with each other then for all $\sigma \in E^i_j, \tau \in E^k_l$,*

1. *u^σ_τ's are projections.*
2. *u^σ_τ commutes with q^i_k and q^j_l.*

Corollary 5.3.11. *Let (V, E) be a multigraph (directed or undirected with an inversion map) which is not necessarily uniform. Let $(V, \overline{E}, w)$ be its* **underlying weighted graph** *with weight function $w : \overline{E} \to \mathbb{C}$ given by,*

$$w((i,j)) = |E^i_j| \quad \text{where} \quad (i,j) \in \overline{E}.$$

If $(V, \overline{E}, w)$ does not have any quantum symmetry then

$$\mathcal{C}^{sym}_{(V,E)} = \mathcal{C}^{Bic}_{(V,E)}.$$

Proof. It is immidiate from proposition 5.3.10. □

5.3.4 Complete description of $Q^{Bic}_{(V,E)}$ for directed uniform multigraphs

Let us consider our multigraph (V,E) to be a directed uniform multigraph of degree m (see definition 5.1.1). In this subsection we will use notation 5.3.1 for denoting edges of (V,E). We will see that $Q^{Bic}_{(V,E)}$ turns out to be a free wreath product of the **quantum permutation group** S_m^+ by $S^{Bic}_{(V,\overline{E})}$, the **quantum automorphism group** of the simple graph $(V,\overline{E})$ in Bichon's sense. We introduce the following notations for dealing with free products.

Notation 5.3.12. *Let $n=|\overline{E}|$. Let us consider n times free product of the quantum permutation group S_m^+ which is itself a compact quantum group in its own right. We can write the canonical inclusion maps of the free product S_m^{+*n} as $\nu_{(i,j)}:S_m^+\to S_m^{+*n}$ where $(i,j)\in\overline{E}$. Let $(P^r_s)_{r,s=1,..,n}$ be the matrix of canonical generators of S_m^+ satisfying* **quantum permutation relations**. *We will write,*

$$P^{(i,j)r}_s=\nu_{(i,j)}(P^r_s)\quad \textit{where}\quad (i,j)\in\overline{E}\quad \textit{and}\quad r,s=1,2,..,m.$$

Now we state the main theorem.

Theorem 5.3.13. *Let (V,E) be a directed multigraph which is uniform of degree m. There is a natural co-action of $S^{Bic}_{(V,\overline{E})}$ on the algebra S_m^{+*n} which is given by*

$$\alpha(\nu_{(k,l)}(a))=\sum_{(i,j)\in\overline{E}}\nu_{(i,j)}(a)\otimes x^i_kx^j_l,\quad (k,l)\in\overline{E},a\in S_m^+. \tag{5.3.3}$$

where $(x^i_j)_{i,j\in V}$ is the matrix of canonical generators of $S^{Bic}_{(V,\overline{E})}$. Then it follows that, with respect to the co-action α,

$$Q^{Bic}_{(V,E)}\cong S_m^+*_wS^{Bic}_{(V,\overline{E})}.$$

Proof. As x^i_k and x^j_l commute with each other for $(i,j),(k,l)\in\overline{E}$, it is easy to check that α is a co-action of $S^{Bic}_{(V,\overline{E})}$ on the C* algebra S_m^{+*n}.

Let us start by showing that there is a co-action γ of $S_m^+*_wS^{Bic}_{(V,\overline{E})}$ on the multigraph (V,E) which preserves its quantum symmetry in Bichon's sense. Let us define $\gamma:C(E)\to C(E)\otimes(S_m^+*_wS^{Bic}_{(V,\overline{E})})$ by,

$$\gamma(\chi_{(k,l)s})=\sum_{\substack{(i,j)\in\overline{E}\\ r=1,..,m}}\chi_{(i,j)r}\otimes P^{(i,j)r}_sx^i_kx^j_l;\quad (k,l)\in\overline{E}\text{ and }s=1,2,..,m. \tag{5.3.4}$$

It is easy to check that γ preserves quantum symmetry of (V,E) in Bichon's sense. By the universality of $Q^{Bic}_{(V,E)}$ we have a quantum group homomorphism $\Phi:Q^{Bic}_{(V,E)}\to S_m^+*_wS^{Bic}_{(V,\overline{E})}$

satisfying

$$\Phi(u^{(i,j)r}_{(k,l)s}) = P^{(i,j)r}_{s} x^i_k x^j_l \quad \text{where} \quad (i,j),(k,l) \in \overline{E} \quad \text{and} \quad r,s = 1,2,..,m$$

where $u^{(i,j)r}_{(k,l)s}$'s are the canonical generators of $Q^{Bic}_{(V,E)}$. Let us denote $(q^i_j)_{i,j\in V}$ to be the co-representation matrix of the induced co-action of $Q^{Bic}_{(V,E)}$ on $C(V)$.

Now we construct the inverse of Φ to show that it is in fact an isomorphism of compact quantum groups.

From definition 4.5.2 it follows that $u^{(i,j)r}_{(k,l)s}$'s satisfy **quantum permutation relations**. For $(i,j) \in \overline{E}$ and $r,s \in \{1,2,..,m\}$ we define,

$$R^{(i,j)r}_s = \sum_{(k,l)\in\overline{E}} u^{(i,j)r}_{(k,l)s}.$$

Now we proceed through following claims.

Claim 1: Let $(i,j) \in \overline{E}$. The coefficients of the matrix $(R^{(i,j)r}_s)_{r,s=1,..,m}$ satisfy **quantum permutation relations**.

we observe the following relations:

$$R^{(i,j)r^2}_s = R^{(i,j)r}_s = R^{(i,j)r^*}_s$$

$$\text{and} \quad \sum_{r=1}^{m} R^{(i,j)r}_s = \sum_{\substack{r=1 \\ (k,l)\in\overline{E}}}^{m} u^{(i,j)r}_{(k,l)s} = \sum_{(k,l)\in\overline{E}} q^i_k q^j_l = 1 = \sum_{\substack{s=1 \\ (k,l)\in\overline{E}}}^{m} u^{(i,j)r}_{(k,l)s} = \sum_{s=1}^{m} R^{(i,j)r}_s.$$

Hence claim 1 follows.

Claim 2: For $(i,j),(k,l) \in \overline{E}$ and $r,s \in \{1,2,..,m\}$, we have the following relations:

$$u^{(i,j)r}_{(k,l)s} = R^{(i,j)r}_s q^i_k q^j_l \quad \text{and} \quad R^{(i,j)r}_s q^i_k q^j_l = q^i_k q^j_l R^{(i,j)r}_s.$$

We observe that,

$$\begin{aligned} R^{(i,j)r}_s q^i_k q^j_l &= \Big(\sum_{(k',l')\in\overline{E}} u^{(i,j)r}_{(k',l')s}\Big)\Big(\sum_{s'=1}^{m} u^{(i,j)r}_{(k,l)s'}\Big) \\ &= u^{(i,j)r}_{(k,l)s} \\ &= \Big(\sum_{s'=1}^{m} u^{(i,j)r}_{(k,l)s'}\Big)\Big(\sum_{(k',l')\in\overline{E}} u^{(i,j)r}_{(k',l')s}\Big) = q^i_k q^j_l R^{(i,j)r}_s. \end{aligned}$$

Hence claim 2 follows.

Claim 3: Let Δ_{Bic} denote the co-product on $Q^{Bic}_{(V,E)}$. The co-product identities in theorem 2.6.2 hold , that is,

$$\Delta_{Bic}(q^i_j) = \sum_{k \in V} q^i_k \otimes q^k_j$$

$$\text{and} \quad \Delta_{Bic}(R^{(i,j)r}_s) = \sum_{\substack{s'=1 \\ (k,l)\in\overline{E}}}^{m} (R^{(i,j)r}_{s'} \otimes R^{(k,l)s'}_s)(q^i_k q^j_l \otimes 1).$$

The first identity is immidiate. To prove the second one we observe that,

$$\begin{aligned}
\Delta_{Bic}(R^{(i,j)r}_s) &= \Delta_{Bic}\Big(\sum_{(k',l')\in\overline{E}} u^{(i,j)r}_{(k',l')s}\Big) \\
&= \sum_{(k',l')\in\overline{E}} \Big(\sum_{\substack{s'=1 \\ (k,l)\in\overline{E}}}^{m} u^{(i,j)r}_{(k,l)s'} \otimes u^{(k,l)s'}_{(k',l')s}\Big) \\
&= \sum_{\substack{s'=1 \\ (k,l)\in\overline{E}}}^{m} R^{(i,j)r}_{s'} q^i_k q^j_l \otimes \Big(\sum_{(k',l')\in\overline{E}} u^{(k,l)s'}_{(k',l')s}\Big) \\
&= \sum_{\substack{s'=1 \\ (k,l)\in\overline{E}}}^{m} R^{(i,j)r}_{s'} q^i_k q^j_l \otimes R^{(k,l)s'}_s .
\end{aligned}$$

Hence the second identity in claim 3 follows.

Using claim 1, claim 2, claim 3 and universality of free wreath product we get a surjective quantum group homomorphism $\Psi : S^+_m *_w S^{Bic}_{(V,\overline{E})} \to Q^{Bic}_{(V,E)}$ given by the following relations on generators:

$$\Psi(x^i_j) = q^i_j \quad \text{and} \quad \Psi(P^{(i',j')r}_s) = R^{(i',j')r}_s$$

where $i, j \in V, (i', j') \in \overline{E}$ and $r, s = 1, 2, .., m$.

It is clear that Φ and Ψ are inverses of each other as it is such on the set of generators. Hence theorem 5.3.13 is proved. □

5.3.5 Complete description of $Q^{Bic}_{(V,E,j)}$ for undirected uniform multigraphs

We want to prove a version of the theorem 5.3.13 for undirected multigraphs. In case of undirected multigraphs, we will see that it will turn out to be a quotient of free wreath product due to lemma 4.6.5.

Proposition 5.3.14. *Let (V, E, j) be an undirected multigraph which is uniform of degree m. Then the quantum automorphism group of (V, E, j) in Bichon's sense is given by*

$$Q^{Bic}_{(V,E,j)} = S^+_m *_w S^{Bic}_{(V,\overline{E})} / \mathcal{I}^u$$

where $\mathcal{I}^u$ is a C ideal generated by the following relations:*

$$P^{(i,j)r}_s = P^{(j,i)r}_s \quad \text{where} \quad (i,j) \in \overline{E}; \;\; r, s = 1, .., m.$$

It should be noted that the wreath product is with respect to the co-action 5.3.3.

Proof. Let (V, E) be the doubly directed multigraph associated with the undirected multigraph (V, E, j). From theorem 5.3.13, it follows that the quantum automorphism group $Q^{Bic}_{(V,E)}$ is given by,

$$Q^{Bic}_{(V,E)} = S^+_m *_w S^{Bic}_{(V,\overline{E})}.$$

Let us denote $(\mathcal{A}, \Delta_{\mathcal{A}}, \beta_{\mathcal{A}})$ to be an object in $\mathcal{C}^{Bic}_{(V,E,j)}$ where $(\mathcal{A}, \Delta_{\mathcal{A}})$ is a CQG co-acting on (V, E, j) preserving its quantum symmetry in Bichon's sense via the co-action $\beta_{\mathcal{A}} : C(E) \to C(E) \otimes \mathcal{A}$. As $\beta_{\mathcal{A}}$ is also a co-action on the underlying doubly directed multigraph (V, E), by universality of $Q^{Bic}_{(V,E)}$ it follows that there exists a unique quantum group homomorphism $\Phi_{\mathcal{A}} : Q^{Bic}_{(V,E)} \to \mathcal{A}$ such that

$$\Phi_{\mathcal{A}}(P^{(i,j)r}_s x^i_k x^j_l) = u^{(i,j)r}_{(k,l)s} = u^{(j,i)r}_{(l,k)s} = \Phi_{\mathcal{A}}(P^{(j,i)r}_s x^j_l x^i_k) \tag{5.3.5}$$

where $(u^{(i,j)r}_{(k,l)s})_{(i,j)r,(k,l)s \in E}$ is the co-representation matrix of $\beta_{\mathcal{A}}$. Summing over all $k, l \in V$ in equation 5.3.5 we get that,

$$\Phi_{\mathcal{A}}(P^{(i,j)r}_s) = \Phi_{\mathcal{A}}(P^{(j,i)r}_s) \quad \text{for all} \quad (i,j) \in \overline{E}.$$

Hence it follows that $I^u \subseteq Ker(\Phi_{\mathcal{A}})$ for all $(\mathcal{A}, \Delta_{\mathcal{A}}, \beta_{\mathcal{A}}) \in \mathcal{C}^{Bic}_{(V,E,j)}$. Let us consider the quotient algebra

$$\mathcal{Q} = S^+_m *_w S^{Bic}_{(V,\overline{E})} / I^u,$$

$\pi_u : S^+_m *_w S^{Bic}_{(V,\overline{E})} \to \mathcal{Q}$ to be the natural quotient map and we write $\pi_u(a) = \overline{a}$. To show that $\mathcal{Q}$ is itself a CQG, it is enough to show that I^u is a Woronowicz C* ideal in $\mathcal{Q}$. Let Δ_w be the co-product on $S_m^+ *_w S^{Bic}_{(V,\overline{E})}$. We observe that,

$$\begin{aligned}(\pi_u \otimes \pi_u)\Delta_w(P_s^{(i,j)r}) &= (\pi_u \otimes \pi_u) \sum_{\substack{s'=1 \\ (k,l)\in\overline{E}}}^{m} (P_{s'}^{(i,j)r} \otimes P_s^{(k,l)s'})(x_k^i x_l^j \otimes 1) \\ &= \sum_{\substack{s'=1 \\ (k,l)\in\overline{E}}}^{m} (\overline{P_{s'}^{(i,j)r}} \otimes \overline{P_s^{(k,l)s'}})(\overline{x_k^i x_l^j} \otimes 1) \\ &= \sum_{\substack{s'=1 \\ (k,l)\in\overline{E}}}^{m} (\overline{P_{s'}^{(j,i)r}} \otimes \overline{P_s^{(l,k)s'}})(\overline{x_l^j x_k^i} \otimes 1) \\ &= (\pi_u \otimes \pi_u)\Delta_w(P_s^{(j,i)r}).\end{aligned}$$

Therefore $\Delta_w(I^u) \subseteq Ker(\pi_u \otimes \pi_u)$ making I^u a Woronowicz C* ideal in $S_m^+ *_w S^{Bic}_{(V,\overline{E})}$. Hence $\mathcal{Q}$ is a CQG with the co-product $\Delta_{\mathcal{Q}}$ induced via π_u. Moreover there is a natural co-action $\beta_{\mathcal{Q}} : C(E) \to C(E) \otimes \mathcal{Q}$ of the CQG $(\mathcal{Q}, \Delta_{\mathcal{Q}})$ on $C(E)$ which is given by

$$\beta_{\mathcal{Q}}(\chi_{(k,l)s}) = \sum_{(i,j)r\in E} \chi_{(i,j)r} \otimes \overline{P_s^{(i,j)r} x_k^i x_l^j} \quad \text{for all} \quad (k,l)s \in E.$$

It is easy to see that $\beta_{\mathcal{Q}}$ preserves quantum symmetry of the undirected multigraph (V, E, j) in Bichon's sense. Therefore we conclude that $(\mathcal{Q}, \Delta_{\mathcal{Q}}, \beta_{\mathcal{Q}})$ is the universal object in $\mathcal{C}^{Bic}_{(V,E,j)}$. $\square$

Chapter 6

Source and target dependent co-actions on (V, E)

6.1 Source and target dependent automorphisms

We will look into some special classes of automorphisms in this chapter. It should be noted that these classes of automorphisms are more interesting for "directed" multigraphs than the "undirected" ones. Before going to the definitions, let us introduce an extension of notation 5.3.1 which we will use extensively in this chapter.

Notation 6.1.1. *Let (V, E) be a (directed or undirected with an inversion map j) multigraph. Let us write $(V, E) = \cup_m (V_m, E_m)$ where (V_m, E_m)'s are its uniform components (see notation 5.1.2). As each (V_m, E_m) is a uniform multigraph on its own, let us fix a representation for each uniform component (V_m, E_m) following the way described in notation 5.3.1. Now any $\sigma \in E$ can be written as,*

$$\sigma = (k, l)r \quad \textit{where} \quad s(\sigma) = k,\ t(\sigma) = l \ \textit{and}\ 1 \le r \le |E_l^k|.$$

From now on throughout the whole chapter, we will always be working with a fixed representation of (V, E).

We introduce another shorthand notation. Let m be a nonzero integer such that $E_m \neq \phi$. For $k, l \in V_m$, we write $k \to l$ in E_m if there exists $\sigma \in E_m$ with $s(\sigma) = k$, $t(\sigma) = l$.

Proposition 6.1.2. *Let $(V, E) = \cup_m (V_m, E_m)$ be a multigraph (directed or undirected). Let (f_V, f_E) be an automorphism of (V, E) in the sense of definition 2.1.11. Let m be a positive integer such that $E_m \neq \phi$. For each $k, l \in V_m$ such that $k \to l$ in E_m, there exists a permutation $\Gamma_{kl} \in S_m$ (classical permutation group on m objects) such that the following identity holds:*

$$f_E((k,l)r) = (f_V(k), f_V(l))\Gamma_{kl}(r) \quad \textit{for all} \quad 1 \leq r \leq m. \tag{6.1.1}$$

Proof. Let $k, l \in V$ be such that $E_l^k \neq \phi$. From definition 2.1.11 it follows that

$$f_E(E_l^k) \subseteq (E_{f_V(l)}^{f_V(k)}) \quad \text{and} \quad f_E^{-1}(E_{f_V(l)}^{f_V(k)}) \subseteq E_l^k. \tag{6.1.2}$$

As f_E and f_E^{-1} are one-one maps, from 6.1.2, it follows that for all $k, l \in V$ such that $E_l^k \neq \phi$,

$$|f(E_l^k)| = |E_{f_V(l)}^{f_V(k)}| = |E_l^k|. \tag{6.1.3}$$

From equation 6.1.3 we observe that, $f_E(E_m) \subseteq E_m$ and $f_V(V_m) \subseteq V_m$ for all m such that $E_m \neq \phi$. Hence the pair of maps (f_V, f_E) restricted to (V_m, E_m) is an automorphism of the uniform multigraph (V_m, E_m). Let us choose $k, l \in V_m$ such that $|E_l^k| = m$. We define a map $\Gamma_{kl} : \{1, .., m\} \to \{1, .., m\}$ by $\Gamma_{kl}(r) = s$ where $f_E((k,l)r) = (f_V(k), f_V(l))s$. It is easy to see that Γ_{kl} is a bijection and f_E has the desired form mentioned in the proposition 6.1.2.

□

Definition 6.1.3. *Let (V, E) be a finite quiver or a directed multigraph. A* **source dependent automorphism** *of (V, E) is an automorphism (f_V, f_E) of (V, E) where the map f_E can be written in the following form:*

$$f_E((k,l)r) = (f_V(k), f_V(l))\,{}^m\Gamma_k(r) \quad \textit{where} \quad (k,l)r \in E_m. \tag{6.1.4}$$

Here ${}^m\Gamma_k \in S_m$ and is a function of the initial vertex k and uniform component the edge lies in.

An automorphism (f_V, f_E) of the multigraph (V, E) is said to be **target dependent** *if f_E has the following form*

$$f_E((k,l)r) = (f_V(k), f_V(l))\,{}^m\Gamma_l(r) \quad \textit{where} \quad (k,l)r \in E_m. \tag{6.1.5}$$

Here ${}^m\Gamma_l \in S_m$ and is a function of the final vertex l and uniform component the edge lies in.

An automorphism (f_V, f_E) is said to be **both source and target dependent** *if f_E can be written in both forms given in equations 6.1.4 and 6.1.5.*

Let us denote the groups of all **source dependent**, **target dependent** *and* **both source and target dependent** *automorphisms by $G^s_{(V,E)}, G^t_{(V,E)}$ and $G^{s,t}_{(V,E)}$ respectively. It is clear* *that,*

$$G^{s,t}_{(V,E)} = G^s_{(V,E)} \cap G^t_{(V,E)}.$$

Remark 6.1.4. *Let (f_V, f_E) be an automorphism of the multigraph (V, E). Then from proposition 6.1.2 it follows that,*

1. *(f_V, f_E) is* **source dependent** *if for each m such that $E_m \neq \phi$, the following identity holds:*

$$\Gamma_{kl} = \Gamma_{kl'} \quad \text{for all} \quad k, l, l' \in V_m \quad \text{such that } k \to l \text{ and } k \to l' \text{ in } E_m.$$

2. *(f_V, f_E) is* **target dependent** *if for each m such that $E_m \neq \phi$, the following identity holds:*

$$\Gamma_{kl} = \Gamma_{k'l} \quad \text{for all} \quad k, k', l \in V_m \quad \text{such that } k \to l \text{ and } k' \to l \text{ in } E_m.$$

It should be noted that the classes of automorphisms in defintion 6.1.3 depend heavily on the representation chosen for the multigraph (V, E), that is, a source or target dependent automorphism of (V, E) might not remain source or target dependent once the representation of the multigraph is changed.

6.2 Source and target dependent quantum symmetries

6.2.1 Algebras of twisted digonal operators

Let (V, E) be a directed multigraph and a representation of (V, E) is fixed in the sense of notation 6.1.1. Let m be a nonzero integer such that $E_m \neq \phi$. For $i \in V^s_m, j \in V^t_m$ and $r = 1, 2, .., m$, let us define $\xi^m_{i,r}, \eta^m_{j,r} \in L^2(E)$ by

$$\xi^m_{i,r} = \sum_{j' \in V^t_m} \chi_{(i,j')r} \quad \text{and} \quad \eta^m_{j,r} = \sum_{i' \in V^s_m} \chi_{(i',j)r}.$$

In the above expressions we have summed over only those j''s in V^t_m and i''s in V^s_m where both $(i, j')r$ and $(i', j)r$ make sense as edges in E_m, that is, both $(i', j)r$ and $(i, j')r$ are in E_m. We will be using such shortened notations while expressing sums from now on. Let us define two algebras $X_m^s, X_m^t \subseteq B(L^2(E))$ by

$$X_m^s = \text{ linear span } \{|\xi_{i,r}^m\rangle\langle\xi_{i,r}^m| \,|i \in V_m^s,\ r = 1, 2, .., m\}$$
$$X_m^t = \text{ linear span } \{|\eta_{j,r}^m\rangle\langle\eta_{j,r}^m| \,|j \in V_m^t,\ r = 1, 2, .., m\}$$

The subalgebras X_m^s and X_m^t are called algebras of twisted diagonal operators. Here we have used **bra-ket** notations which are frequently used in quantum mechanics. We give a brief description here. Let H be a Hilbert space with inner product $<, >_H$. For $\xi, \eta \in H$, we define $\langle \xi \mid \eta \rangle \in \mathbb{C}$ and $|\xi\rangle\langle\eta| \in B(H)$ by

$$\langle \xi \mid \eta \rangle = < \xi, \eta >_H;$$
$$|\xi\rangle\langle \eta \mid \mu\rangle = < \eta, \mu >_H \xi \quad \text{where} \quad \mu \in H.$$

We will be using these notations throughout this chapter.

6.2.2 Main definitions

Definition 6.2.1. *Let (V, E) be a directed multigraph. Let β be a co-action of a CQG $(\mathcal{A}, \Delta)$ on (V, E) preserving its quantum symmetry in Banica's sense (see definition 4.2.2). β is said to preserve its source dependent quantum symmetry if the following holds:*

$$Ad_\beta(X_m^s) \subseteq X_m^s \otimes \mathcal{A} \quad \textit{for all} \quad m \ \textit{such that} \ E_m \neq \phi.$$

β is said to preserve its target dependent quantum symmetry if the following holds:

$$Ad_{\overline{\beta}}(X_m^t) \subseteq X_m^t \otimes \mathcal{A} \quad \textit{for all} \quad m \ \textit{such that} \ E_m \neq \phi.$$

β is said to preserve its both source and target dependent quantum symmetries if the following holds:

$$Ad_\beta(X_m^s) \subseteq X_m^s \otimes \mathcal{A} \ \textit{ and } \ Ad_{\overline{\beta}}(X_m^t) \subseteq X_m^t \otimes \mathcal{A}$$

for all m such that $E_m \neq \phi$.

Let us also define $\mathcal{C}^s_{(V,E)}$, $\mathcal{C}^t_{(V,E)}$ and $\mathcal{C}^{s,t}_{(V,E)}$ to be categories consisting of CQGs with co-actions preserving source dependent quantum symmetry, target dependent quantum symmetry and both source and target dependent quantum symmetry of (V, E) respectively. Morphisms in these categories are quantum group homomorphisms intertwining similar type co-actions.

6.2.3 Algebraic characterisations

In this subsection we formulate algebraic characterisations of source and target dependent quantum symmetries in terms of co-representation matrices.

The source dependent case:

Proposition 6.2.2. *Let β be a co-action of a CQG $(\mathcal{A}, \Delta)$ on a directed multigraph (V, E) which preserves its source dependent quantum symmetry. Let $(u^{\sigma}_{\tau})_{\sigma,\tau\in E}$ be the co-representation matrix of β. Then the following conditions hold:*

1. *$\beta(L^2(E_m)) \subseteq L^2(E_m) \otimes \mathcal{A}$ for all m such that $E_m \neq \phi$.*
2. *Let m be a nonzero integer such that $E_m \neq \phi$. The for any $(k,l)s, (k',l')s' \in E_m$,*

$$\sum_{j,j'\in V^t_m} u^{(k,l)s}_{(i,j)r} u^{(k',l')s'*}_{(i,j')r} = 0 \quad \textit{whenever} \quad (k,s) \neq (k',s')$$

and for all $l, l', l_1, l'_1 \in V^t_m$ such that $(k,l)s, (k,l')s, (k,l_1)s, (k,l'_1)s \in E_m$,

$$\sum_{j,j'\in V^t_m} u^{(k,l)s}_{(i,j)r} u^{(k,l')s*}_{(i,j')r} = \sum_{j,j'\in V^t_m} u^{(k,l_1)s}_{(i,j)r} u^{(k,l'_1)s*}_{(i,j')r}. \tag{6.2.1}$$

Proof. Let us fix a nonzero integer m such that $E_m \neq \phi$. Let $i \in V^s_m$ and $r \in \{1, 2, .., m\}$. We observe that,

$$\begin{aligned} Ad_\beta(|\xi^m_{i,r}\rangle\langle\xi^m_{i,r}|) &= |\beta(\xi^m_{i,r})\rangle\langle\beta(\xi^m_{i,r})| \\ &= \sum_{j,j'\in V^t_m} |\beta(\chi_{(i,j)r})\rangle\langle\beta(\chi_{(i,j')r})| \\ &= \sum_{\sigma,\tau\in E} |\chi_\sigma\rangle\langle\chi_\tau| \otimes (\sum_{j,j'\in V^t_m} u^{\sigma}_{(i,j)r} {u^{\tau}_{(i,j')r}}^*) \end{aligned} \tag{6.2.2}$$

As $Ad_\beta(X^s_m) \subseteq X^s_m \otimes \mathcal{A}$, from equation 6.2.2 we have the following relations:

1. Let $\sigma, \tau \in E$. If both σ and τ are not in E_m, then

$$\sum_{j,j'\in V^t_m} u^{\sigma}_{(i,j)r} {u^{\tau}_{(i,j')r}}^* = 0 \tag{6.2.3}$$

2. For any $(k,l)s$ and $(k',l')s' \in E_m$,

$$\sum_{j,j'\in V^t_m} u^{(k,l)s}_{(i,j)r} u^{(k',l')s'*}_{(i,j')r} = 0 \quad \text{whenever} \quad (k,s) \neq (k',s'). \tag{6.2.4}$$

3. For all $l,l',l_1,l_1'\in V_m^t$ such that $(k,l)s,(k,l')s,(k,l_1)s,(k,l_1')s\in E_m$,

$$\sum_{j,j'\in V_m^t} u_{(i,j)r}^{(k,l)s} u_{(i,j')r}^{(k,l')s^*} = \sum_{j,j'\in V_m^t} u_{(i,j)r}^{(k,l_1)s} u_{(i,j')r}^{(k,l_1')s^*}. \tag{6.2.5}$$

We note that (2) of proposition 6.2.2 follows from equation 6.2.4 and equation 6.2.5. Now to prove (1) it is enough to show that $u^{\sigma}_{(i,j)r}=0$ whenever $\sigma\notin E_m$ and $(i,j)r\in E_m$.

Putting $\sigma=\tau\in E\setminus E_m$ in equation 6.2.3 and using equation 4.2.1 we observe that,

$$\begin{aligned}0=\sum_{j,j'\in V_m^t} u^{\sigma}_{(i,j)r}{u^{\sigma}_{(i,j')r}}^* &= \sum_{j,j'\in V_m^t} u^{\sigma}_{(i,j)r} q_j^{t(\sigma)} q_{j'}^{t(\sigma)} {u^{\sigma}_{(i,j')r}}^* \\ &= \sum_{j\in V_m^t} u^{\sigma}_{(i,j)r}{u^{\sigma}_{(i,j)r}}^*.\end{aligned}$$

As $u^{\sigma}_{(i,j)r}{u^{\sigma}_{(i,j)r}}^*$'s are positive operators, we conclude that,

$$u^{\sigma}_{(i,j)r}=0.$$

Hence (1) follows.

□

The target dependent case:

Proposition 6.2.3. *Let β be a co-action of a CQG $(\mathcal{A},\Delta)$ on a directed multigraph (V,E) preserving its target dependent quantum symmetry. Let $(u^{\sigma}_{\tau})_{\sigma,\tau\in E}$ be the co-representation matrix of β. Then the following conditions hold:*

1. *$\beta(L^2(E_m))\subseteq L^2(E_m)\otimes\mathcal{A}$ for all m such that $E_m\neq\phi$.*

2. *Let m be a nonzero integer such that $E_m\neq\phi$. The for any $(k,l)s,(k',l')s'\in E_m$,*

$$\sum_{i,i'\in V_m^t} u_{(i,j)r}^{(k,l)s^*} u_{(i',j)r}^{(k',l')s'} = 0 \quad \textit{whenever} \quad (l,s)\neq(l',s') \tag{6.2.6}$$

and for all $k,k',k_1,k_1'\in V_m^s$ such that $(k,l)s,(k',l)s,(k_1,l)s,(k_1',l)s\in E_m$,

$$\sum_{i,i'\in V_m^s} u_{(i,j)r}^{(k,l)s^*} u_{(i',j)r}^{(k',l)s} = \sum_{i,i'\in V_m^s} u_{(i,j)r}^{(k_1,l)s^*} u_{(i',j)r}^{(k_1',l)s}. \tag{6.2.7}$$

Proof. The proof is very similar to the proof of proposition 6.2.2. Let us fix a nonzero integer m such that $E_m \neq \phi$. Let $j \in V_m^t$ and $r \in \{1, 2, .., m\}$. We observe that,

$$\begin{aligned} Ad_{\overline{\beta}}(|\eta_{j,r}^m\rangle\langle\eta_{j,r}^m|) &= |\overline{\beta}(\eta_{j,r}^m)\rangle\langle\overline{\beta}(\eta_{j,r}^m)| \\ &= \sum_{i,i' \in V_m^s} |\overline{\beta}(\chi_{(i,j)r})\rangle\langle\overline{\beta}(\chi_{(i',j)r})| \\ &= \sum_{\sigma,\tau \in E} |\chi_\sigma\rangle\langle\chi_\tau| \otimes (\sum_{i,i' \in V_m^s} {u_{(i,j)r}^{\sigma}}^* u_{(i',j)r}^{\tau}). \end{aligned} \tag{6.2.8}$$

As $Ad_{\overline{\beta}}(X_m^t) \subseteq X_m^t \otimes \mathcal{A}$, from equation 6.2.8 we have the following relations:

1. Let $\sigma, \tau \in E$. If both σ and τ are not in E_m, then
$$\sum_{i,i' \in V_m^s} {u_{(i,j)r}^{\sigma}}^* u_{(i',j)r}^{\tau} = 0. \tag{6.2.9}$$

2. For any $(k,l)s$ and $(k',l')s' \in E_m$,
$$\sum_{i,i' \in V_m^s} u_{(i,j)r}^{(k,l)s^*} u_{(i',j)r}^{(k',l')s'} = 0 \quad \text{whenever} \quad (l,s) \neq (l',s'). \tag{6.2.10}$$

3. For all $k, k', k_1, k_1' \in V_m^s$ such that $(k,l)s, (k',l)s, (k_1,l)s, (k_1',l)s \in E_m$,
$$\sum_{i,i' \in V_m^s} u_{(i,j)r}^{(k,l)s^*} u_{(i',j)r}^{(k',l)s} = \sum_{i,i' \in V_m^s} u_{(i,j)r}^{(k_1,l)s^*} u_{(i',j)r}^{(k_1',l)s}. \tag{6.2.11}$$

We note that (2) of proposition 6.2.3 follows from equation 6.2.10 and equation 6.2.11. Now to prove (1) it is enough to show that $u_{(i,j)r}^{\sigma} = 0$ whenever $\sigma \notin E_m$ and $(i,j)r \in E_m$.

Putting $\sigma = \tau \in E \setminus E_m$ in equation 6.2.9 and equation 4.2.1 we observe that,

$$\begin{aligned} 0 = \sum_{i,i' \in V_m^s} {u_{(i,j)r}^{\sigma}}^* u_{(i',j)r}^{\sigma} &= \sum_{i,i' \in V_m^s} {u_{(i,j)r}^{\sigma}}^* q_i^{s(\sigma)} q_{i'}^{s(\sigma)} u_{(i',j)r}^{\sigma} \\ &= \sum_{i \in V_m^s} {u_{(i,j)r}^{\sigma}}^* u_{(i,j)r}^{\sigma}. \end{aligned}$$

As ${u_{(i,j)r}^{\sigma}}^* u_{(i,j)r}^{\sigma}$'s are positive operators, we conclude that,

$$u_{(i,j)r}^{\sigma} = 0.$$

Therefore (1) follows.

$\square$

Remark 6.2.4. *In proposition 6.2.2, conditions 1 and 2 can be taken as a characterization for co-actions on (V, E) which preserve its source dependent quantum symmetry. It is easy to check that any co-action on (V, E) preserving its quantum symmetry in Banica's sense satisfying 1 and 2 of proposition 6.2.2 also satisfies identites 6.2.3, 6.2.5 and 6.2.4 and hence a co-action on (V, E) which preserves its source dependent quantum symmetry.*

In a similar way it can be argued that the any co-action on (V, E) preserving its quantum symmetry in Banica's sense also preserves its target dependent quantum symmetry if it satisfies conditions 1 and 2 in proposition 6.2.3.

6.2.4 More familiar forms and partial wreath product relations

The algebraic characterisations we got in the previous subsection will be simplified further to get more familiar forms resembling to the classical picture of source and target dependent symmetries. We have also been able to recover **wreath product relations** (similar to described in subsection 5.3.4) with respect to permutations of V^s or V^t (depending on whether source dependent or target dependent quantum symmetry is preserved). Let us start by following notations and observations.

Notation 6.2.5. *We make some observations and introduce two new quantum permutation matrices which will be crucial for the upcoming results later on.*

1. *Let β be a co-action of a CQG $(\mathcal{A}, \Delta)$ on (V, E) preserving its source dependent quantum symmetry and $(u^{\sigma}_{\tau})_{\sigma,\tau\in E}$ be the co-representation matrix of β. Let m be a nonzero integer such that $E_m \neq \phi$. By putting $l' = l$ and $l_1 = l'_1$ in equation 6.2.1 we observe that for all $l, l_1 \in V^t_m$,*
$$\sum_{j\in V^t_m} u^{(k,l)s}_{(i,j)r} u^{(k,l)s^*}_{(i,j)r} = \sum_{j\in V^t_m} u^{(k,l_1)s}_{(i,j)r} u^{(k,l_1)s^*}_{(i,j)r} =: {}^m\theta^{ks}_{ir}.$$
Where ${}^m\theta^{ks}_{ir}$ is an element in $\mathcal{A}$ depending on m-th uniform component, the vertices $i, k \in V^s_m$ and $r, s \in \{1, 2, .., m\}$. As Ad_β preserves X^s_m and is a C algebra homomorphism, it follows that Ad_β is in fact a quantum permutation on the set of operators $\{|\xi^m_{i,r}\rangle\langle\xi^m_{i,r}|\,|i \in V^s_m;\ r = 1, .., m\}$. Hence the elements of the matrix $({}^m\theta^{ks}_{ir})_{(ks),(ir)}$ satisfy* **quantum permutation relations**.

2. *Now let us consider β to be a co-action of a CQG $(\mathcal{A}, \Delta)$ on (V, E) preserving its target dependent quantum symmetry. Let us fix an integer m such that $E_m \neq \phi$. By similar*

argument used in previous paragraph, for $j, l \in V_m^t$ and $r, s \in \{1, 2, .., m\}$ we can define

$$^{m}\hat{\theta}_{jr}^{ls} := \sum_{i \in V_m^s} u_{(i,j)r}^{(k,l)s^*} u_{(i,j)r}^{(k,l)s}.$$

As $Ad_{\overline{\beta}}$ preserves X_m^t and is a C algebra homomorphism on X_m^t, it follows that $Ad_{\overline{\beta}}$ is a quantum permutation on the set of operators $\{|\eta_{j,r}^m\rangle\langle\eta_{j,r}^m| | j \in V_m^t; r = 1, .., m\}$. Therefore the elements of the matrix $(^{m}\hat{\theta}_{jr}^{ls})_{(ls),(jr)}$ satisfy* **quantum permutation relations**.

Before proceeding further, let us make some observations about the elements $^{m}\theta_{ir}^{ks}$'s and $^{m}\hat{\theta}_{jr}^{ls}$'s which will be used for our constructions later.

Remark 6.2.6. *Let β be a co-action on a directed multigraph (V, E) which preserves its source dependent quantum symmetry. Let m be a nonzero integer such that $E_m \neq \phi$. We have the following identities:*

1. $^{m}\theta_{ir}^{ks} q_i^k = {}^{m}\theta_{ir}^{ks} = q_i^k \, {}^{m}\theta_{ir}^{ks}$ *for all* $k, i \in V_m^s$; $s, r \in \{1, .., m\}$.
2. *Let us choose $k, i \in V_m^s$ and $s \in \{1, .., m\}$. Then it follows that*

$$\sum_{r=1}^{m} {}^{m}\theta_{ir}^{ks} = \sum_{\substack{r=1 \\ j \in V_m^t}}^{m} u_{(i,j)r}^{(k,l)s} u_{(i,j)r}^{(k,l)s^*} = q_i^k.$$

In the above computation we have used the fact that $u_\tau^\sigma = 0$ when $\sigma \in E_m$ and $\tau \in E_n$ with $m \neq n$.

Now let us consider β to be a co-action on (V, E) which preserves its target dependent quantum symmetry. Let m be a nonzero integer such that $E_m \neq \phi$. Then we have the following identities:

1. $^{m}\hat{\theta}_{jr}^{ls} q_j^l = {}^{m}\hat{\theta}_{jr}^{ls} = q_j^l \, {}^{m}\hat{\theta}_{jr}^{ls}$ *for all* $l, j \in V_m^t$; $s, r \in \{1, .., m\}$.
2. *Let us choose $l, j \in V_m^t$ and $s \in \{1, .., m\}$. Then it follows that,*

$$\sum_{r=1}^{m} {}^{m}\hat{\theta}_{jr}^{ls} = \sum_{\substack{r=1 \\ i \in V_m^s}}^{m} u_{(i,j)r}^{(k,l)s^*} u_{(i,j)r}^{(k,l)s} = q_j^l.$$

Co-actions preserving source dependent quantum symmetry:

Theorem 6.2.7. *Let β be a co-action of a CQG $(\mathcal{A}, \Delta)$ on (V, E) which preserves its source dependent quantum symmetry. Let $(u_\tau^\sigma)_{\sigma,\tau \in E}$ and $(q_j^i)_{i,j \in V}$ be the co-representation matrices* *of β and its induced co-action on $C(V)$. Let m be a nonzero integer such that $E_m \neq \phi$. Then for any $(i,j)r, (k,l)s \in E_m$,*

$$u^{(k,l)s}_{(i,j)r} = {}^m\theta^{ks}_{ir} q^l_j$$

where ${}^m\theta^{ks}_{ir}$'s are described in notation 6.2.5.

Proof. Let us define $T^{(k,l)s}_{ir} = \sum_{j \in V^t_m} u^{(k,l)s}_{(i,j)r}$. We observe that,

$$\begin{aligned} T^{(k,l)s}_{ir} T^{(k,l)s^*}_{ir} &= \sum_{j,j' \in V^t_m} u^{(k,l)s}_{(i,j)r} u^{(k,l)s^*}_{(i,j')r} \\ &= \sum_{j \in V^t_m} u^{(k,l)s}_{(i,j)r} u^{(k,l)s^*}_{(i,j)r} = {}^m\theta^{ks}_{ir}. \end{aligned}$$

As ${}^m\theta^{ks}_{ir}$ is a projection, it follows that ${}^m\theta^{ks}_{ir}$ is the range projection of $T^{(k,l)s}_{ir}$. It further follows that,

$$\begin{aligned} (T^{(k,l)s^*}_{ir} T^{(k,l)s}_{ir})^2 &= T^{(k,l)s^*}_{ir} T^{(k,l)s}_{ir} T^{(k,l)s^*}_{ir} T^{(k,l)s}_{ir} \\ &= T^{(k,l)s^*}_{ir} \, {}^m\theta^{ks}_{ir} \; T^{(k,l)s}_{ir} \\ &= T^{(k,l)s^*}_{ir} T^{(k,l)s}_{ir}. \end{aligned}$$

Hence $T^{(k,l)s}_{ir}$'s are partial isometries. Let h be the Haar funtional of the CQG $(\mathcal{A}, \Delta)$. We observe that,

$$\begin{aligned} &\sum_{\substack{r=1 \\ j \in V^t_m}}^{m} h(u^{(k,l)s}_{(i,j)r} - {}^m\theta^{ks}_{ir} q^l_j)(u^{(k,l)s}_{(i,j)r} - {}^m\theta^{ks}_{ir} q^l_j)^* \\ =& h(\sum_{\substack{r=1 \\ j \in V^t_m}}^{m} u^{(k,l)s}_{(i,j)r} u^{(k,l)s^*}_{(i,j)r}) - h(\sum_{\substack{r=1 \\ j \in V^t_m}}^{m} u^{(k,l)s}_{(i,j)r} q^l_j \, {}^m\theta^{ks}_{ir}) - h(\sum_{\substack{r=1 \\ j \in V^t_m}}^{m} {}^m\theta^{ks}_{ir} q^l_j u^{(k,l)s^*}_{(i,j)r}) + h(\sum_{\substack{r=1 \\ j \in V^t_m}}^{m} {}^m\theta^{ks}_{ir} q^l_j \, {}^m\theta^{ks}_{ir}) \\ =& h(q^k_i) - h(\sum_{r=1}^{m} (\sum_{j \in V^t_m} u^{(k,l)s}_{(i,j)r}) \, {}^m\theta^{ks}_{ir}) - h(\sum_{r=1}^{m} {}^m\theta^{ks}_{ir} (\sum_{j \in V^t_m} u^{(k,l)s^*}_{(i,j)r})) + h(\sum_{r=1}^{m} {}^m\theta^{ks}_{ir} (\sum_{j \in V^t_m} q^l_j) \, {}^m\theta^{ks}_{ir}) \\ =& h(q^k_i) - h(\sum_{r=1}^{m} T^{(k,l)s}_{ir} \, {}^m\theta^{ks}_{ir}) - h(\sum_{r=1}^{m} {}^m\theta^{ks}_{ir} T^{(k,l)s^*}_{ir}) + h(\sum_{r=1}^{m} {}^m\theta^{ks}_{ir}) \\ =& h(q^k_i) - h(\sum_{r=1}^{m} {}^m\theta^{ks}_{ir} T^{(k,l)s}_{ir}) - h(\sum_{r=1}^{m} T^{(k,l)s^*}_{ir} \, {}^m\theta^{ks}_{ir}) + h(q^k_i) \\ =& h(q^k_i) - h(\sum_{r=1}^{m} T^{(k,l)s}_{ir}) - h(\sum_{r=1}^{m} T^{(k,l)s^*}_{ir}) + h(q^k_i) \\ =& h(q^k_i) - h(q^k_i) - h(q^k_i) + h(q^k_i) = 0 \end{aligned}$$

As h is a positive linear functional and is faithful on the underlying Hopf* algebra of matrix elements of the CQG $(\mathcal{A},\Delta)$, it follows that each of the summand of the above expression is zero. We conclude that,

$$(u^{(k,l)s}_{(i,j)r} - {}^m\theta^{ks}_{ir} q^l_j)(u^{(k,l)s}_{(i,j)r} - {}^m\theta^{ks}_{ir} q^l_j)^* = 0 \quad \text{for all} \quad (i,j)r,(k,l)s \in E_m.$$

Therefore we have,

$$u^{(k,l)s}_{(i,j)r} = {}^m\theta^{ks}_{ir} q^l_j \quad \text{for all} \quad (i,j)r,(k,l)s \in E_m.$$

□

The form that we have got in proposition 6.2.7 can be improved further to give a form resembling more to classical case.

Proposition 6.2.8. *Let β be a co-action of the CQG $(\mathcal{A},\Delta)$ on (V,E) which preserves its source dependent quantum symmetry. Let $(u^\sigma_\tau)_{\sigma,\tau\in E}$ and $(q^i_j)_{i,j\in V}$ be the co-representation matrices of β and its induced permutation on vertices respectively. Let m be a nonzero integer such that $E_m \neq \phi$. Then for $(i,j)r,(k,l)s \in E_m$, the following identities hold:*

$$u^{(k,l)s}_{(i,j)r} = {}^m\gamma^{ks}_r \, q^k_i q^l_j \quad \textit{and} \quad \Delta({}^m\gamma^{ks}_r) = \sum_{\substack{s'=1\\ k'\in V^s_m}}^{m} {}^m\gamma^{ks}_{s'} q^k_{k'} \otimes {}^m\gamma^{k's'}_r$$

where

$${}^m\gamma^{ks}_r := \sum_{i\in V^s_m} {}^m\theta^{ks}_{ir}.$$

Proof. The first identity is clear from theorem 6.2.7, definition of ${}^m\gamma^{ks}_r$ and the fact that,

$${}^m\theta^{ks}_{ir} q^k_i = {}^m\theta^{ks}_{ir} = q^k_i \; {}^m\theta^{ks}_{ir} \quad \text{for all} \quad k,i \in V^s_m;\; r,s = 1,..,m.$$

As Ad_β is a quantum permutation on the set of generators of X^s_m, it follows that,

$$\Delta({}^m\theta^{ks}_{ir}) = \sum_{\substack{s'=1\\ k'\in V^s_m}}^{m} {}^m\theta^{ks}_{k's'} \otimes {}^m\theta^{k's'}_{ir} \quad \text{for all} \quad k,i \in V^s_m;\; r,s = 1,..,m.$$

From the above identity and definition of ${}^m\gamma^{ks}_r$, it follows that,

$$\Delta({}^m\gamma^{ks}_r) = \Delta(\sum_{i\in V^s_m} {}^m\theta^{ks}_{ir})$$

$$= \sum_{\substack{s'=1\\ k'\in V_m^s}}^{m} {}^m\theta_{k's'}^{ks} \otimes (\sum_{i\in V_m^s} {}^m\theta_{ir}^{k's'})$$
$$= \sum_{\substack{s'=1\\ k'\in V_m^s}}^{m} {}^m\gamma_{s'}^{ks} q_{k'}^{k} \otimes {}^m\gamma_r^{k's'}.$$

Hence the second identity follows.

□

Co-actions preserving target depenedent quantum symmetry:

Theorem 6.2.9. *Let β be a co-action of a CQG $(\mathcal{A},\Delta)$ on (V,E) which preserves its target dependent quantum symmetry. Let $(u_\tau^\sigma)_{\sigma,\tau\in E}$ and $(q_j^i)_{i,j\in V}$ be the co-representation matrices of β and its induced co-action on $C(V)$. Let m be a nonzero integer such that $E_m \neq \phi$. Then for any $(i,j)r,(k,l)s \in E_m$,*

$$u_{(i,j)r}^{(k,l)s} = q_i^k \, {}^m\hat{\theta}_{jr}^{ls}$$

where ${}^m\hat{\theta}_{jr}^{ls}$'s are described in notation 6.2.5.

Proof. Let us define $S_{jr}^{(k,l)s} = \sum_{i\in V_m^s} u_{(i,j)r}^{(k,l)s}$. We observe that,

$$S_{jr}^{(k,l)s^*} S_{jr}^{(k,l)s} = \sum_{i,i'\in V_m^s} u_{(i,j)r}^{(k,l)s^*} u_{(i',j)r}^{(k,l)s}$$
$$= \sum_{i\in V_m^s} u_{(i,j)r}^{(k,l)s^*} u_{(i,j)r}^{(k,l)s} = {}^m\hat{\theta}_{jr}^{ls}.$$

As ${}^m\hat{\theta}_{jr}^{ls}$ is a projection, it follows that ${}^m\hat{\theta}_{jr}^{ls}$ is the range projection of $S_{jr}^{(k,l)s^*}$. It further follows that $S_{jr}^{(k,l)s}$'s are partial isometries. Let h be the Haar funtional of the CQG $(\mathcal{A},\Delta)$. We observe that,

$$\sum_{\substack{r=1\\ i\in V_m^s}}^{m} h(u_{(i,j)r}^{(k,l)s} - q_i^k \, {}^m\hat{\theta}_{jr}^{ls})^*(u_{(i,j)r}^{(k,l)s} - q_i^k \, {}^m\hat{\theta}_{jr}^{ls})$$
$$= h(\sum_{\substack{r=1\\ i\in V_m^s}}^{m} u_{(i,j)r}^{(k,l)s^*} u_{(i,j)r}^{(k,l)s}) - h(\sum_{\substack{r=1\\ i\in V_m^s}}^{m} u_{(i,j)r}^{(k,l)s^*} q_i^k \, {}^m\hat{\theta}_{jr}^{ls}) - h(\sum_{\substack{r=1\\ i\in V_m^s}}^{m} {}^m\hat{\theta}_{jr}^{ls} q_i^k u_{(i,j)r}^{(k,l)s}) + h(\sum_{\substack{r=1\\ i\in V_m^s}}^{m} {}^m\hat{\theta}_{jr}^{ls} q_i^k \, {}^m\hat{\theta}_{jr}^{ls})$$
$$= h(q_j^l) - h(\sum_{r=1}^{m}(\sum_{i\in V_m^s} u_{(i,j)r}^{(k,l)s^*})\, {}^m\hat{\theta}_{jr}^{ls}) - h(\sum_{r=1}^{m} {}^m\hat{\theta}_{jr}^{ls}(\sum_{i\in V_m^s} u_{(i,j)r}^{(k,l)s})) + h(\sum_{r=1}^{m} {}^m\hat{\theta}_{jr}^{ls}(\sum_{i\in V_m^s} q_i^k)\, {}^m\hat{\theta}_{jr}^{ls})$$
$$= h(q_j^l) - h(\sum_{r=1}^{m} S_{jr}^{(k,l)s^*}\, {}^m\hat{\theta}_{jr}^{ls}) - h(\sum_{r=1}^{m} {}^m\hat{\theta}_{jr}^{ls} S_{jr}^{(k,l)s}) + h(\sum_{r=1}^{m} {}^m\hat{\theta}_{jr}^{ls})$$

$$=h(q_j^l) - h(\sum_{r=1}^{m} {}^m\hat{\theta}_{jr}^{ls} S_{jr}^{(k,l)s^*}) - h(\sum_{r=1}^{m} S_{jr}^{(k,l)s} \, {}^m\hat{\theta}_{jr}^{ls}) + h(q_j^l)$$
$$=h(q_j^l) - h(\sum_{r=1}^{m} S_{jr}^{(k,l)s^*}) - h(\sum_{r=1}^{m} S_{jr}^{(k,l)s}) + h(q_j^l)$$
$$=h(q_j^l) - h(q_j^l) - h(q_j^l) + h(q_j^l) = 0.$$

As h is a positive linear functional and is faithful on the underlying Hopf* algebra of matrix elements of the CQG $(\mathcal{A}, \Delta)$, it follows that each of the summand of the above expression is zero. We conclude that,

$$(u_{(i,j)r}^{(k,l)s} - q_i^k \, {}^m\hat{\theta}_{jr}^{ls})^*(u_{(i,j)r}^{(k,l)s} - q_i^k \, {}^m\hat{\theta}_{jr}^{ls}) = 0 \quad \text{for all} \quad (i,j)r, (k,l)s \in E_m.$$

Therefore we have,

$$u_{(i,j)r}^{(k,l)s} = q_i^k \, {}^m\hat{\theta}_{jr}^{ls} \quad \text{for all} \quad (i,j)r, (k,l)s \in E_m.$$

□

As we have done for co-actions preserving source-dependent quantum symmetry, same can be argued for the target dependent case.

Proposition 6.2.10. *Let β be a co-action of the CQG $(\mathcal{A}, \Delta)$ on (V, E) preserving its target dependent quantum symmetry. Let $(u_\tau^\sigma)_{\sigma,\tau\in E}$ and $(q_j^i)_{i,j\in V}$ be the co-representation matrices of β and its induced permutation on the vertices. Let m be a nonzero integer such that $E_m \neq \phi$. Then for $(i,j)r, (k,l)s \in E_m$, the following identities hold:*

$$u_{(i,j)r}^{(k,l)s} = q_i^k q_j^l \, {}^m\nu_r^{ls} \quad \textit{and} \quad \Delta({}^m\nu_r^{ls}) = \sum_{\substack{s'=1 \\ l'\in V_m^t}}^{m} {}^m\nu_{s'}^{ls} q_{l'}^l \otimes {}^m\nu_r^{l's'}$$

where

$${}^m\nu_r^{ls} = \sum_{j\in V_m^t} {}^m\hat{\theta}_{jr}^{ls}.$$

Proof. The proof is similar to the proof of proposition 6.2.8. The first identity is clear from theorem 6.2.9, definition of ${}^m\nu_r^{ls}$ and the fact that,

$${}^m\hat{\theta}_{jr}^{ls} q_j^l = {}^m\hat{\theta}_{jr}^{ls} = q_j^l \, {}^m\hat{\theta}_{jr}^{ls} \quad \text{for all} \quad l,j \in V_m^t;\ r,s = 1,..,m.$$

As $Ad_{\overline{\beta}}$ is a quantum permutation on the set of generators of X^t_m, it follows that,

$$\Delta({}^m\hat{\theta}^{ls}_{jr}) = \sum_{\substack{s'=1 \\ l' \in V^t_m}}^{m} {}^m\hat{\theta}^{ls}_{l's'} \otimes {}^m\hat{\theta}^{l's'}_{jr} \quad \text{for all} \quad l, j \in V^t_m;\ r, s = 1, .., m.$$

From the above identity and the definition of ${}^m\nu^{ls}_r$, it follows that,

$$\begin{aligned} \Delta({}^m\nu^{ls}_r) &= \Delta(\sum_{j \in V^t_m} {}^m\hat{\theta}^{ls}_{jr}) \\ &= \sum_{\substack{s'=1 \\ l' \in V^t_m}}^{m} {}^m\hat{\theta}^{ls}_{l's'} \otimes (\sum_{j \in V^t_m} {}^m\hat{\theta}^{l's'}_{jr}) \\ &= \sum_{\substack{s'=1 \\ l' \in V^t_m}}^{m} {}^m\nu^{ls}_{s'} q^l_{l'} \otimes {}^m\nu^{l's'}_r \end{aligned}$$

Hence the second identity follows. □

Co-actions preserving both source and target dependent quantum symmetries:

Proposition 6.2.11. *Let $(V, E) = \cup_m (V_m, E_m)$. Let β be a co-action of a CQG $(\mathcal{A}, \Delta)$ on (V, E) preserving its source dependent quantum symmetry. The co-action β also preserves its target dependent quantum symmetry if and only if for each m such that $E_m \neq \phi$, the following conditions hold:*

1. *${}^m\gamma^{ks}_r = {}^m\gamma^{k's}_r$ for all $s, r = 1, .., m$ and $k, k' \in V^s_m$ such that $k \to l$ and $k' \to l$ in E_m for some $l \in V_m$.*
2. *${}^m\gamma^{ks}_r q^l_j = q^l_j\, {}^m\gamma^{ks}_r$ for all $j, l \in V^t_m$ and $k \in V^s_m$ such that $k \to l$ in E_m.*

Proof. Let us assume that β preserves target dependent quantum symmetry of (V, E). Then from proposition 6.2.8 and proposition 6.2.10 it follows that, for all $(i, j)r, (k, l)s \in E_m$,

$${}^m\gamma^{ks}_r q^k_i q^l_j = q^k_i q^l_j\, {}^m\nu^{ls}_r.$$

Summing over $i \in V^s_m$ and $j \in V^t_m$ we observe that,

$${}^m\gamma^{ks}_r = {}^m\nu^{ls}_r \quad \text{for all} \quad k, l \in V_m \quad \text{such that} \quad k \to l \quad \text{in } E_m \text{ and} \quad s, r = 1, .., m. \tag{6.2.12}$$

(1) and (2) in proposition 6.2.11 follows from the above observation.

Now let us assume that β is a co-action on (V,E) preserving its source dependent quantum symmetry and satisfying (1) and (2) of proposition 6.2.11. It is enough to show that the coefficients of the co-representation matrix $(u^{\sigma}_{\tau})_{\sigma,\tau\in E}$ satisfy the identities 6.2.6 and 6.2.7 in proposition 6.2.3. Let $j\in V^t_m, r\in\{1,..,m\}$ and $(k,l)s,(k',l')s'\in E_m$. We observe that,

$$\begin{aligned}\sum_{i,i'\in V^s_m} u^{(k,l)s*}_{(i,j)r}u^{(k',l')s'}_{(i',j)r} &= \sum_{i,i'\in V^s_m} q^l_j q^k_i\, {}^m\gamma^{ks}_r\, {}^m\gamma^{k's'}_r q^{k'}_{i'} q^{l'}_j \\ &= {}^m\gamma^{ks}_r q^l_j\Big(\sum_{i,i'\in V^s_m} q^k_i q^{k'}_{i'}\Big)q^{l'}_j\, {}^m\gamma^{k's'}_r \\ &= \delta_{l,l'} q^l_j\, {}^m\gamma^{ks}_r\, {}^m\gamma^{k's'}_r \\ &= \delta_{l,l'} q^l_j\, {}^m\gamma^{ks}_r\, {}^m\gamma^{ks'}_r \\ &= \delta_{l,l'}\delta_{s,s'}\, {}^m\gamma^{ks}_r q^l_j. \end{aligned} \tag{6.2.13}$$

From the above relation it is clear that

$$\sum_{i,i'\in V^s_m} u^{(k,l)s*}_{(i,j)r}u^{(k',l')s'}_{(i',j)r}=0 \quad \text{whenever} \quad (l,s)\neq(l',s').$$

Moreover, as ${}^m\gamma^{ks}_r={}^m\gamma^{k_1s}_r$ for any $k,k_1\in V^s_m$ and $l\in V^t_m$ such that $k\to l$ and $k_1\to l$ in E_m, it further follows from equation 6.2.13 that,

$$\sum_{i,i'\in V^s_m} u^{(k,l)s*}_{(i,j)r}u^{(k',l)s}_{(i',j)r} = {}^m\gamma^{ks}_r q^l_j = {}^m\gamma^{k_1s}_r q^l_j = \sum_{i,i'\in V^s_m} u^{(k_1,l)s*}_{(i,j)r}u^{(k_1',l)s}_{(i',j)r}$$

where $(k,l)s$, $(k',l)s$, $(k_1,l)s$, $(k_1',l)s\in E_m$, $j\in V^t_m$ and $r\in\{1,..,m\}$. Hence equation 6.2.7 follows. $\square$

In the next proposition we will show that $\mathcal{C}^{s,t}_{(V,E)}$ is a subcategory $\mathcal{C}^{sym}_{(V,E)}$. It is in fact a subcategory of $\mathcal{C}^{sym}_{(V,E)}$ which is **non-Bichon** in the sense that any co-action β on (V,E) preserving both of its source and target dependent quantum symmetries do not necessarily need to be a **quantum permutation** on the edge set E. In later sections, we prove that $\mathcal{C}^{s,t}_{(V,E)}$ admits a universal object which is the **automorphism group of source and target dependent quantum symmetries of** (V,E) (see definition 6.4.3).

Proposition 6.2.12. *Let β be a co-action of a CQG $(\mathcal{A},\Delta)$ on (V,E) which preserves both of its source and target dependent quantum symmetries. Then β is a co-action on (V,E) which preserves its quantum symmetry in our sense (see definition 4.2.3).*

Proof. Let $(u^{\sigma}_{\tau})_{\sigma,\tau\in E}$ be the co-representation matrix of β. It is enough to show that the elements u^{σ}_{τ}'s satisfy the relations mentioned in (2) of proposition 4.2.12. As we already have

$$\beta(L^2(E_m)) \subseteq L^2(E_m)\otimes\mathcal{A}$$

for all uniform components (V_m,E_m), it is enough to check the relations mentioned in (2) of proposition 4.2.12 in uniform components. Let us fix nonzero integer m such that $E_m\neq\phi$.

From proposition 6.2.8 and proposition 6.2.10 it follows that

$$u^{(k,l)s}_{(i,j)r} = {}^{m}\gamma^{ks}_{r}q^k_i q^l_j = q^k_i q^l_j\,{}^{m}\nu^{ls}_{r} \quad \text{for all} \quad (i,j)r,(k,l)s\in E_m.$$

It further follows that,

$$\begin{aligned}
u^{(k,l)s}_{(i,j)r}u^{(k,l)s'*}_{(i,j)r} &= (q^k_i q^l_j\,{}^{m}\nu^{ls}_{r})({}^{m}\nu^{ls'}_{r}q^l_j q^k_i) = 0 \quad \text{if} \quad s\neq s';\\
u^{(k,l)s*}_{(i,j)r}u^{(k,l)s'}_{(i,j)r} &= (q^l_j q^k_i\,{}^{m}\gamma^{ks}_{r})({}^{m}\gamma^{ks'}_{r}q^k_i q^l_j) = 0 \quad \text{if} \quad s\neq s'.
\end{aligned}$$

Hence coefficients of the matrix $(u^{\sigma}_{\tau})_{\sigma,\tau\in E}$ satisfy (2) of proposition 4.2.12 making β a co-action on (V,E) preserving its quantum symmetry in our sense. □

Final remark:

The propositions 6.2.8, 6.2.10 and 6.2.11 can be taken as characterisations of co-actions on (V,E) preserving its source dependent, target dependent and both source and target dependent quantum symmetries respectively. More precisely, starting with a co-action on (V,E) in Banica's sense, if the coefficients of co-representation matrix are one of the prescribed forms mentioned in those propositions, then we can conclude that the co-action preserves the respective quantum symmetry.

In proposition 6.2.8, the coefficients of the quantum permutation matrix $({}^{m}\gamma^{ks}_{r})_{s,r=1,..,m}$ commute with q^k_i for all $i\in V^s_m$ giving us **free wreath product relations** with respect to induced permutation on V^s_m. Similarly, in proposition 6.2.10 the entries of the quantum permutation matrix ${}^{m}\nu^{ls}_{r}$ commute with q^l_j for all $j\in V^t_m$ giving us **free wreath product relations** with respect to the induced permutation on V^t_m.

6.3 Existence of universal objects

Theorem 6.3.1. *For a directed multigraph (V,E), the categories $\mathcal{C}^s_{(V,E)}$, $\mathcal{C}^t_{(V,E)}$ and $\mathcal{C}^{s,t}_{(V,E)}$ (see definition 6.2.1) admit universal objects.*

Proof. Let us start by showing that $\mathcal{C}^s_{(V,E)}$ admits a universal object. Let $(\mathcal{A}, \Delta_{\mathcal{A}}, \beta_{\mathcal{A}}) \in \mathcal{C}^s_{(V,E)}$ where $(\mathcal{A}, \Delta_A)$ is a CQG and $\beta_{\mathcal{A}}$ is co-action of $(\mathcal{A}, \Delta_{\mathcal{A}})$ on (V, E) preserving its source dependent quantum symmetry. By universality of $Q^{Ban}_{(V,E)}$, there exists a unique quantum group homomorphism $\Phi_{\mathcal{A}} : Q^{Ban}_{(V,E)} \to \mathcal{A}$ such that

$$\Phi_{\mathcal{A}}(u^{\sigma}_{\tau}) = v^{\sigma}_{\tau} \quad \text{for all} \quad \sigma, \tau \in E$$

where $(u^{\sigma}_{\tau})_{\sigma,\tau\in E}$ is the matrix of canonical generators of $Q^{Ban}_{(V,E)}$ and $(v^{\sigma}_{\tau})_{\sigma,\tau\in E}$ is the co-representation matrix of $\beta_{\mathcal{A}}$.

Let $\mathcal{I}^s$ be a C* ideal in $Q^{Ban}_{(V,E)}$ generated by the following relations:

1. $u^{\sigma}_{\tau} = 0$ for all $\sigma \in E_m, \tau \in E_n$ where m and n are two non zero integers such that $m \neq n$.

2. Let m be a nonzero integer such that $E_m \neq \phi$. The for any $(k, l)s, (k', l')s' \in E_m$, $i \in V^s_m$ and $r \in \{1, .., m\}$,

$$\sum_{j,j' \in V^t_m} u^{(k,l)s}_{(i,j)r} u^{(k',l')s'*}_{(i,j')r} = 0 \quad \text{whenever} \quad (k, s) \neq (k', s')$$

and for all $l, l', l_1, l'_1 \in V^t_m$ such that $(k, l)s, (k, l')s, (k, l_1)s, (k, l'_1)s \in E_m$, $i \in V^s_m$ and $r \in \{1, .., m\}$,

$$\sum_{j,j' \in V^t_m} u^{(k,l)s}_{(i,j)r} u^{(k,l')s*}_{(i,j')r} = \sum_{j,j' \in V^t_m} u^{(k,l_1)s}_{(i,j)r} u^{(k,l'_1)s*}_{(i,j')r}.$$

From remark 6.2.4 it is clear that $\mathcal{I}^s \subseteq Ker(\Phi_{\mathcal{A}})$ for all $(\mathcal{A}, \Delta_{\mathcal{A}}, \beta_{\mathcal{A}}) \in \mathcal{C}^s_{(V,E)}$. Let us consider $\mathcal{Q}^s = Q^{Ban}_{(V,E)}/\mathcal{I}^s$ and $\pi_s : Q^{Ban}_{(V,E)} \to \mathcal{Q}^s$ to be the natural quotient map. We write

$$\pi_s(u^{\sigma}_{\tau}) = v^{\sigma}_{\tau} \quad \text{for all} \quad \sigma, \tau \in E.$$

If we show that $\mathcal{I}^s$ is a Woronowicz C* ideal in $Q^{Ban}_{(V,E)}$, it will follow that $\mathcal{Q}^s$ is a CQG with the co-product $\Delta_{\mathcal{Q}}$ induced by Δ_{Ban} via the quotient map π_s. Moreover, there is also a natural co-action $\beta_{\mathcal{Q}^s}$ of the CQG $(\mathcal{Q}^s, \Delta_{\mathcal{Q}^s})$ on (V, E) which preserves its source dependent quantum symmetry. The map $\beta_{\mathcal{Q}^s} : L^2(E) \to L^2(E) \otimes \mathcal{Q}^s$ is given by,

$$\beta_{\mathcal{Q}^s}(\chi_\tau) = \sum_{\sigma\in E} \chi_\sigma \otimes v^{\sigma}_{\tau} \quad \text{where} \quad \tau \in E.$$

Therefore it follows that $(\mathcal{Q}^s, \Delta_{\mathcal{Q}^s}, \beta_{\mathcal{Q}^s})$ is universal in $\mathcal{C}^s_{(V,E)}$. To prove that $\mathcal{I}^s$ is a Woronowicz C* ideal, we observe the following relations:

1. Let $\sigma \in E_m$ and $\tau \in E_n$ such that $m \neq n$. Then

$$(\pi_s \otimes \pi_s)\Delta_{Ban}(u^{\sigma}_{\tau}) = \sum_{\tau' \in E} v^{\sigma}_{\tau'} \otimes v^{\tau'}_{\tau} = 0$$

as $v^{\tau'}_{\tau} = 0$ when $\tau' \in E_m$ and $v^{\sigma}_{\tau'} = 0$ when $\tau' \in E_n$.

2. Let m be a nonzero integer such that $E_m \neq \phi$. Let $(k,l)s, (k',l')s' \in E_m$ be such that $(k,s) \neq (k',s')$. Then we observe that,

$$\begin{aligned}
&(\pi_s \otimes \pi_s)\Delta_{Ban}\Big(\sum_{j,j' \in V^t_m} u^{(k,l)s}_{(i,j)r} u^{(k',l')s'^*}_{(i,j')r}\Big) \\
&= \sum_{\substack{(k_1,l_1)s_1,\\ (k_2,l_2)s_2 \in E_m}} v^{(k,l)s}_{(k_1,l_1)s_1} v^{(k',l')s'^*}_{(k_2,l_2)s_2} \otimes \Big(\sum_{j,j' \in V^t_m} v^{(k_1,l_1)s_1}_{(i,j)r} v^{(k_2,l_2)s_2^*}_{(i,j')r}\Big) \\
&= \sum_{\substack{(k_1,l_1)s_1,\\ (k_1,l_2)s_1 \in E_m}} v^{(k,l)s}_{(k_1,l_1)s_1} v^{(k',l')s'^*}_{(k_1,l_2)s_1} \otimes \Big(\sum_{j,j' \in V^t_m} v^{(k_1,l_1)s_1}_{(i,j)r} v^{(k_1,l_2)s_1^*}_{(i,j')r}\Big) \\
&= \sum_{\substack{s_1=1\\ k_1 \in V^s_m}}^{m} \Big(\sum_{l_1,l_2 \in V^t_m} v^{(k,l)s}_{(k_1,l_1)s_1} v^{(k',l')s'^*}_{(k_1,l_2)s_1}\Big) \otimes {}^m\theta^{k_1 s_1}_{ir} = 0 \quad \text{as } (k,s) \neq (k',s').
\end{aligned}$$

The quantity ${}^m\theta^{k_1 s_1}_{ir}$ in the above expression is independent of $l_1, l_2 \in V^t_m$. Now let us choose $l, l', l_1, l_1' \in V^t_m$ such that $(k,l)s, (k,l')s, (k,l_1)s, (k,l_1')s \in E_m$, then it follows that,

$$\begin{aligned}
&(\pi_s \otimes \pi_s)\Delta_{Ban}\Big(\sum_{j,j' \in V^t_m} u^{(k,l)s}_{(i,j)r} u^{(k,l')s^*}_{(i,j')r}\Big) \\
&= \sum_{\substack{(i_1,j_1)r_1,\\ (i_2,j_2)r_2 \in E_m}} v^{(k,l)s}_{(i_1,j_1)r_1} v^{(k,l')s^*}_{(i_2,j_2)r_2} \otimes \Big(\sum_{j,j' \in V^t_m} v^{(i_1,j_1)r_1}_{(i,j)r} v^{(i_2,j_2)r_2^*}_{(i,j')r}\Big) \\
&= \sum_{\substack{(i_1,j_1)r_1,\\ (i_1,j_2)r_1 \in E_m}} v^{(k,l)s}_{(i_1,j_1)r_1} v^{(k,l')s^*}_{(i_1,j_2)r_1} \otimes \Big(\sum_{j,j' \in V^t_m} v^{(i_1,j_1)r_1}_{(i,j)r} v^{(i_1,j_2)r_1^*}_{(i,j')r}\Big) \\
&= \sum_{\substack{r_1=1\\ i_1 \in V^s_m}}^{m} \Big(\sum_{j_1,j_2 \in V^t_m} v^{(k,l)s}_{(i_1,j_1)r_1} v^{(k,l')s^*}_{(i_1,j_2)r_1}\Big) \otimes {}^m\theta^{i_1 r_1}_{ir} \\
&= \sum_{\substack{r_1=1\\ i_1 \in V^s_m}}^{m} {}^m\theta^{ks}_{i_1 r_1} \otimes {}^m\theta^{i_1 r_1}_{ir}.
\end{aligned}$$

As the above expression is independent of l and l', we observe that,

$$(\pi_s\otimes\pi_s)\Delta_{Ban}(\sum_{j,j'\in V_m^t} u_{(i,j)r}^{(k,l)s}u_{(i,j')r}^{(k,l')s^*})=(\pi_s\otimes\pi_s)\Delta_{Ban}(\sum_{j,j'\in V_m^t} u_{(i,j)r}^{(k,l_1)s}u_{(i,j')r}^{(k,l_1')s^*})$$

Hence $\mathcal{I}^s$ is a Woronowicz C*ideal in $Q_{(V,E)}^{Ban}$ making $Q_{(V,E)}^{Ban}/\mathcal{I}^s$ universal in $\mathcal{C}_{(V,E)}^s$.

We will use similar arguments to show that $\mathcal{C}_{(V,E)}^t$ also admits a universal object. Let $\mathcal{I}^t$ be a C* ideal in $Q_{(V,E)}^{Ban}$ generated by the following relations,

1. $u_\tau^\sigma=0$ for all $\sigma\in E_m,\tau\in E_n$ where m and n are two non zero integers such that $m\neq n$.

2. Let m be a nonzero integer such that $E_m\neq\phi$. The for any $(k,l)s,(k',l')s'\in E_m$, $j\in V_m^t$ and $r\in\{1,..,m\}$,

$$\sum_{i,i'\in V_m^s} u_{(i,j)r}^{(k,l)s^*}u_{(i',j)r}^{(k',l')s'}=0\quad\text{whenever}\quad(l,s)\neq(l',s')$$

and for all $k,k',k_1,k_1'\in V_m^s$ such that $(k,l)s,(k',l)s,(k_1,l)s,(k_1',l)s\in E_m$, $j\in V_m^t$ and $r\in\{1,..,m\}$,

$$\sum_{i,i'\in V_m^s} u_{(i,j)r}^{(k,l)s^*}u_{(i,j')r}^{(k',l)s}=\sum_{i,i'\in V_m^s} u_{(i,j)r}^{(k_1,l)s^*}u_{(i',j)r}^{(k_1',l)s}.$$

As in the previous case, it is enough to observe that $\mathcal{I}^t$ is a Woronowicz C* ideal in $Q_{(V,E)}^{Ban}$. Let $\pi_t:Q_{(V,E)}^{Ban}\to Q_{(V,E)}^{Ban}/\mathcal{I}^t$ be the natural quotient map and we write $\pi_t(u_\tau^\sigma)=v_\tau^\sigma$ for all $\sigma,\tau\in E$. Let m be a nonzero integer such that $E_m\neq\phi$. Let $(k,l)s,(k',l')s'\in E_m$ be such that $(l,s)\neq(l',s')$. Then we observe that,

$$\begin{aligned}
&(\pi_t\otimes\pi_t)\Delta_{Ban}(\sum_{i,i'\in V_m^s} u_{(i,j)r}^{(k,l)s^*}u_{(i',j)r}^{(k',l')s'})\\
=&\sum_{\substack{(k_1,l_1)s_1,\\(k_2,l_2)s_2\in E_m}} v_{(k_1,l_1)s_1}^{(k,l)s^*}v_{(k_2,l_2)s_2}^{(k',l')s'}\otimes(\sum_{i,i'\in V_m^s} v_{(i,j)r}^{(k_1,l_1)s_1^*}v_{(i',j)r}^{(k_2,l_2)s_2})\\
=&\sum_{\substack{(k_1,l_1)s_1,\\(k_2,l_1)s_1\in E_m}} v_{(k_1,l_1)s_1}^{(k,l)s^*}v_{(k_2,l_1)s_1}^{(k',l')s'}\otimes(\sum_{i,i'\in V_m^s} v_{(i,j)r}^{(k_1,l_1)s_1^*}v_{(i',j)r}^{(k_2,l_1)s_1})\\
=&\sum_{\substack{s_1=1\\l_1\in V_m^t}}^{m}(\sum_{k_1,k_2\in V_m^s} v_{(k_1,l_1)s_1}^{(k,l)s^*}v_{(k_2,l_1)s_1}^{(k',l')s'})\otimes{}^m\hat{\theta}_{jr}^{l_1s_1}=0\quad\text{as }(l,s)\neq(l',s').
\end{aligned}$$

The quantity ${}^m\hat{\theta}^{l_1 s_1}_{jr}$ is independent of k_1 and k_2. Now let $k,k',k_1,k_1' \in V^s_m$ be such that $(k,l)s,(k',l)s,(k_1,l)s,(k_1',l)s \in E_m$. As before, we observe that,

$$\begin{aligned}
&(\pi_t \otimes \pi_t)\Delta_{Ban}\Big(\sum_{i,i' \in V^s_m} u^{(k,l)s^*}_{(i,j)r} u^{(k',l)s}_{(i',j)r}\Big) \\
&= \sum_{\substack{(i_1,j_1)r_1,\\ (i_2,j_2)r_2 \in E_m}} v^{(k,l)s^*}_{(i_1,j_1)r_1} v^{(k',l)s}_{(i_2,j_2)r_2} \otimes \Big(\sum_{i,i' \in V^s_m} v^{(i_1,j_1)r_1^*}_{(i,j)r} v^{(i_2,j_2)r_2}_{(i',j)r}\Big) \\
&= \sum_{\substack{(i_1,j_1)r_1,\\ (i_2,j_1)r_1 \in E_m}} v^{(k,l)s^*}_{(i_1,j_1)r_1} v^{(k',l)s}_{(i_2,j_1)r_1} \otimes \Big(\sum_{i,i' \in V^s_m} v^{(i_1,j_1)r_1^*}_{(i,j)r} v^{(i_2,j_1)r_1}_{(i',j)r}\Big) \\
&= \sum_{\substack{r_1=1\\ j_1 \in V^t_m}}^{m} \Big(\sum_{i_1,i_2 \in V^s_m} v^{(k,l)s}_{(i_1,j_1)r_1} v^{(k',l)s^*}_{(i_2,j_1)r_1}\Big) \otimes {}^m\hat{\theta}^{j_1 r_1}_{jr} \\
&= \sum_{\substack{r_1=1\\ j_1 \in V^t_m}}^{m} {}^m\hat{\theta}^{ls}_{j_1 r_1} \otimes {}^m\hat{\theta}^{j_1 r_1}_{jr}.
\end{aligned}$$

As the above expression is independent of k and k' it follows that for all $k,k',k_1,k_1' \in V^s_m$,

$$(\pi_t \otimes \pi_t)\Delta_{Ban}\Big(\sum_{i,i' \in V^s_m} u^{(k,l)s^*}_{(i,j)r} u^{(k',l)s}_{(i',j)r}\Big) = (\pi_t \otimes \pi_t)\Delta_{Ban}\Big(\sum_{i,i' \in V^s_m} u^{(k_1,l)s^*}_{(i,j)r} u^{(k_1',l)s}_{(i',j)r}\Big)$$

Hence $\mathcal{I}^t$ is a Woronowicz C* ideal in $Q^{Ban}_{(V,E)}$ making $Q^{Ban}_{(V,E)}/\mathcal{I}^t$ universal in $\mathcal{C}^t_{(V,E)}$.

Now for the last part, to show that the category $\mathcal{C}^{s,t}_{(V,E)}$ admits a universal object, we define a C* ideal in $\mathcal{I}^{s,t}$ in $Q^{Ban}_{(V,E)}$ which is generated by the elements of the set $\mathcal{I}^s \cup \mathcal{I}^t$. From the previous two cases it is clear $\mathcal{I}^{s,t}$ is a Woronowicz C* ideal in $Q^{Ban}_{(V,E)}$ making $Q^{Ban}_{(V,E)}/\mathcal{I}^{s,t}$ universal in $\mathcal{C}^{s,t}_{(V,E)}$. □

Remark 6.3.2. *Let us call the universal objects in $\mathcal{C}^s_{(V,E)}$, $\mathcal{C}^t_{(V,E)}$ and $\mathcal{C}^{s,t}_{(V,E)}$ by $Q^s_{(V,E)}$, $Q^t_{(V,E)}$ and $Q^{s,t}_{(V,E)}$ respectively with respective co-products Δ_s, Δ_t and $\Delta_{s,t}$.*

6.4 Automorphism groups of source and target dependent quantum symmetries

The universal objects $Q^s_{(V,E)}$, $Q^t_{(V,E)}$ and $Q^{s,t}_{(V,E)}$ are the automorphism groups of source and target dependent quantum symmetries of (V,E). In light of the final remark mentioned in the end of subsection 6.2.4 we propose the following set of definitions.

Definition 6.4.1. *Let $(V,E) = \cup_m (V_m,E_m)$ be a directed multigraph where (V_m,E_m) is the uniform component with degree m. The* **automorphism group of source dependent**

***quantum symmetry** of (V,E) is the universal C* algebra $Q^s_{(V,E)}$ generated by the elements of the set*

$$\cup_m\{{}^m\gamma^{ks}_r\,|k\in V^s_m, r,s=1,..,m\}\cup\{q^i_j|i,j\in V\}$$

such that the following conditions hold:

1. *The matrix $Q=(q^i_j)_{i,j\in V}$ is a **quantum permutation matrix** satisfying*

$$QW=WQ$$

   where W is the adjacency matrix of (V,E).

2. *For a nonzero m such that $E_m\neq\phi$ the matrix $({}^m\gamma^{ks}_r)_{s,r=1,..,m}$ is a **quantum permutation matrix** satisfying*

$$q^k_i({}^m\gamma^{ks}_r)={}^m\gamma^{ks}_r q^k_i \quad \textit{for all} \quad k,i\in V^s_m;\ r,s=1,..,m.$$

The co-product Δ_s on $Q^s_{(V,E)}$ is given by

$$\Delta_s(q^i_j)=\sum_{k\in V}q^i_k\otimes q^k_j \quad \textit{and} \quad \Delta_s({}^m\gamma^{ks}_r)=\sum_{\substack{s'=1\\ k'\in V^s_m}}^{m}{}^m\gamma^{ks}_{s'}q^k_{k'}\otimes{}^m\gamma^{k's'}_r.$$

*The canonical co-action β_s of $Q^s_{(V,E)}$ on (V,E) which preserves its **source dependent quantum symmetry** is given by,*

$$\beta_s(\chi_{(i,j)r})=\sum_{(k,l)s\in E_m}\chi_{(k,l)s}\otimes{}^m\gamma^{ks}_r q^k_i q^l_j \quad \textit{where} \quad (i,j)r\in E_m.$$

Definition 6.4.2. *Let $(V,E)=\cup_m(V_m,E_m)$ be a directed multigraph where (V_m,E_m) is the uniform component with degree m. The **automorphism group of target dependent quantum symmetry** of (V,E) is the universal C* algebra $Q^t_{(V,E)}$ generated by the elements of the set*

$$\cup_m\{{}^m\nu^{ls}_r\,|l\in V^t_m, r,s=1,..,m\}\cup\{q^i_j|i,j\in V\}$$

such that the following conditions hold:

1. *The matrix $Q=(q^i_j)_{i,j\in V}$ is a **quantum permutation matrix** satisfying*

$$QW=WQ$$

where W is the adjacency matrix of (V, E).

2. *For a nonzero integer m such that $E_m \neq \phi$ the matrix $({}^m\nu_r^{ls})_{s,r=1,..,m}$ is a* **quantum permutation matrix** *satisfying*

$$ {}^m\nu_r^{ls} q_j^l = q_j^l \, {}^m\nu_r^{ls} \quad \text{for all} \quad l, j \in V_m^t; \; s, r = 1, .., m. $$

The co-product Δ_t on $Q_{(V,E)}^t$ is given by,

$$ \Delta_t(q_j^i) = \sum_{k \in V} q_k^i \otimes q_j^k \quad \text{and} \quad \Delta_t({}^m\nu_r^{ls}) = \sum_{\substack{s'=1 \\ l' \in V_m^t}}^{m} {}^m\nu_{s'}^{ls} q_{l'}^l \otimes {}^m\nu_r^{l's'}. $$

The canonical co-action β_t of $Q_{(V,E)}^t$ on (V, E) which preserves its target dependent quantum symmetry is given by,

$$ \beta_t(\chi_{(i,j)r}) = \sum_{(k,l)s \in E_m} \chi_{(k,l)s} \otimes q_i^k q_j^l \; {}^m\nu_r^{ls} \quad \text{where} \quad (i, j)r \in E_m. $$

Definition 6.4.3. *Let $(V, E) = \cup_m (V_m.E_m)$ be a directed multigraph where (V_m, E_m) is the uniform component of degree m. The* **automorphism group of source and target dependent quantum symmetries** *is the universal C* algebra $Q_{(V,E)}^{s,t}$ generated by the elements of the set*

$$ \cup_m \{ {}^m\gamma_r^{ks} | k \in V_m^s; r, s = 1, .., m \} \cup \{ q_j^i | i, j \in V \} $$

such that the following conditions hold:

1. *The matrix $Q = (q_j^i)_{i,j \in V}$ is a* **quantum permutation matrix** *satisfying*

$$ QW = WQ $$

   where W is the adjacency matrix of (V, E).

2. *For a nonzero m such that $E_m \neq \phi$ the matrix $({}^m\gamma_r^{ks})_{s,r=1,..,m}$ is a* **quantum permutation matrix***. Let $k_1, k_2 \in V_m^s$ and $l \in V_m^t$ be such that $k_1 \to l$ and $k_2 \to l$ in E_m (see notation 6.1.1). Then we assume that,*

$$ {}^m\gamma_r^{k_1 s} = {}^m\gamma_r^{k_2 s} \quad \text{for all} \quad s, r = 1, .., m; $$

$$ {}^m\gamma_r^{k_1 s} q_i^{k_1} = q_i^{k_1} \, {}^m\gamma_r^{k_1 s} \quad \text{and} \quad {}^m\gamma_r^{k_1 s} q_j^l = q_j^l \, {}^m\gamma_r^{k_1 s} $$

for all $i \in V_m^s$, $j \in V_m^t$ and $s, r \in \{1, .., m\}$.

The co-product $\Delta_{s,t}$ on $Q^{s,t}_{(V,E)}$ is given by,

$$\Delta_{s,t}(q^i_j) = \sum_{k \in V} q^i_k \otimes q^k_j \quad \text{and} \quad \Delta_{s,t}({}^m\gamma^{ks}_r) = \sum_{\substack{s'=1 \\ k' \in V^s_m}}^{m} {}^m\gamma^{ks}_{s'} q^k_{k'} \otimes {}^m\gamma^{k's'}_r$$

The canonical co-action $\beta_{s,t}$ of $Q^{s,t}_{(V,E)}$ on (V, E) which preserves both of its source and target dependent quantum symmetries is given by,

$$\beta_{s,t}(\chi_{(i,j)r}) = \sum_{(k,l)s \in E_m} \chi_{(k,l)s} \otimes q^k_i q^l_j \ {}^m\gamma^{ks}_r \quad \text{where} \quad (i, j)r \in E_m.$$

Remark 6.4.4. *It is clear from the definitions themselves that the universal commutative CQG's in the categories $\mathcal{C}^s_{(V,E)}$, $\mathcal{C}^t_{(V,E)}$ and $\mathcal{C}^{s,t}_{(V,E)}$ are $C(G^s_{(V,E)})$, $C(G^t_{(V,E)})$ and $C(G^{s,t}_{(V,E)})$ respectively.*

We also want to mention that despite having wreath product relations with respect to induced permutations on V^s and V^t, in general $Q^{s,t}_{(V,E)}$ is not a quantum subgroup of $Q^{Bic}_{(V,E)}$ as q^k_i and q^l_j do not need to commute whenever E^k_l and E^i_j are nonempty.

6.5 Source and target dependent quantum symmetries of an undirected multigraph

We define the notion of source and target dependent quantum symmetries in an undirected mutigraph in the same way as we did in case of a directed graph. For an undirected multigraph (V, E, j) all the categories $\mathcal{C}^s_{(V,E,j)}$, $\mathcal{C}^t_{(V,E,j)}$ and $\mathcal{C}^{s,t}_{(V,E,j)}$ coincide as it should, because there is no inherent meaning of source and target maps in an undirected multigraph. We will use the same notation described in section 6.2.1. Let us start with the following proposition.

Proposition 6.5.1. *Let $(V, E, j) = \cup_m (V_m, E_m, j)$ be an undirected multigraph where (V_m, E_m, j)'s are its unifrom components. Let β be a co-action of a CQG $(\mathcal{A}, \Delta)$ on (V, E, j) preserving its quantum symmetry in Banica's sense. Then the following conditions are equivalent:*

1. *$Ad_\beta(X^s_m) \subseteq X^s_m \otimes \mathcal{A}$ for each nonzero integer m such that $E_m \neq \phi$.*

2. *$Ad_{\overline{\beta}}(X^t_m) \subseteq X^t_m \otimes \mathcal{A}$ for each nonzero integer m such that $E_m \neq \phi$.*

Proof. Let $(u^\sigma_\tau)_{\sigma,\tau \in E}$ be the co-representation matrix of β. Let m be a nonzero integer such that $E_m \neq \phi$. Let $i \in V_m$, $r \in \{1, .., m\}$, $(k, l)s$ and $(k', l')s' \in E_m$. Using lemma 4.6.5 we observe that,

$$\sum_{j,j' \in V_m} u^{(k,l)s}_{(i,j)r} u^{(k',l')s'*}_{(i,j')r} = \sum_{j,j' \in V_m} u^{(l,k)s^*}_{(j,i)r} u^{(l',k')s'}_{(j',i)r}.$$

The equivalence in proposition 6.5.1 follows from remark 6.2.4 and the above observation.

□

Definition 6.5.2. *Let $(V, E.j)$ be an undirected multigraph and β be a co-action of a CQG $(\mathcal{A}, \Delta)$ on (V, E, j) preserving its quantum symmetry in Banica's sense. Then β is said to preserve its source and target dependent quantum symmetries if the following condition holds:*

$$Ad_\beta(X^s_m) \subseteq X^s_m \otimes \mathcal{A}$$

for all nonzero integer m such that $E_m \neq \phi$.

Before moving to the main result of this section we recall the defintition of a path in a multigraph.

Definition 6.5.3. *Let (V, E) be a multigraph (directed or undirected with an inversion map j). Let $k, k' \in V$. A path from k to k' is a finite sequence of edges $\{\sigma_1, \sigma_2, ..., \sigma_n\}$ such that*

$$s(\sigma_{i+1}) = t(\sigma_i) \quad \text{for all} \quad i = 1, .., n-1;$$
$$s(\sigma_1) = k \quad \text{and} \quad t(\sigma_n) = k'.$$

Theorem 6.5.4. *Let $(V, E, j) = \cup_m (V_m, E_m, j)$ be an undirected multigraph and β be a co-action of a CQG $(\mathcal{A}, \Delta)$ on (V, E, j) preserving its source and target dependent quantum symmetries. Let m be a nonzero integer such that $E_m \neq \phi$ and $({}^m\gamma^{ks}_r)_{s,r=1,..,m}$ be the* **quantum permutation matrices** *defined in proposition 6.2.8. Then we have the following:*

$${}^m\gamma^{ks}_r = {}^m\gamma^{k's}_r \quad \text{for all} \quad s, r = 1, .., m \quad \text{and} \quad k, k' \in V_m$$

such that there is a path from k to k' in (V_m, E_m, j).

Proof. Let us fix a nonzero integer m such that $E_m \neq \phi$. Let $k, l \in V_m$ be such that $k \to l$ in E_m (see notation 6.1.1). As our multigraph is undirected, we also have $l \to k$ in E_m. To prove our theorem, it is enough to show that

$${}^m\gamma^{ks}_r = {}^m\gamma^{ls}_r \quad \text{for all} \quad s, r = 1, .., m.$$

Let $i, j, k, l \in V_m$ be such that $i \to j$ and $k \to l$ in E_m. From lemma 4.6.5 and proposition 6.2.8 we observe that,

$$ {}^m\gamma_r^{ks} q_i^k q_j^l = ({}^m\gamma_r^{ls} q_j^l q_i^k)^* = q_i^k q_j^l \, {}^m\gamma_r^{ls}. \tag{6.5.1} $$

Using equation 6.5.1 we further observe that,

$$ \begin{aligned} {}^m\gamma_r^{ks} = \sum_{i,j \in V_m} {}^m\gamma_r^{ks} q_i^k q_j^l &= \sum_{\substack{i,j \in V_m \\ i \to j}} {}^m\gamma_r^{ks} q_i^k q_j^l \\ &= \sum_{\substack{i,j \in V_m \\ i \to j}} q_i^k q_j^l \, {}^m\gamma_r^{ls} = \sum_{i,j \in V_m} q_i^k q_j^l \, {}^m\gamma_r^{ls} = {}^m\gamma_r^{ls}. \end{aligned} $$

Hence the theorem is proved. □

We propose the following definition of automorphism group of source and target dependent quantum symmetries of an undirected multigraph (V, E, j).

Definition 6.5.5. *Let $(V, E, j) = \cup_m (V_m, E_m, j)$ be an undirected multigraph. The* **automorphism group of source and target dependent quantum symmetries** *of (V, E, j) is the universal C* algebra $Q^{s,t}_{(V,E,j)}$ generated by the following set of elements*

$$ \cup_m \{ {}^m\gamma_r^{ks} | k \in V_m; r, s = 1, .., m \} \cup \{ q_j^i | i, j \in V \} $$

such that the following conditions hold:

1. *The matrix $Q = (q_j^i)_{i,j \in V}$ is a* **quantum permutation matrix** *satisfying*

$$ QW = WQ $$

   where W is the adjacency matrix of (V, E, j).

2. *For a nonzero m such that $E_m \neq \phi$ the matrix $({}^m\gamma_r^{ks})_{s,r=1,..,m}$ is a* **quantum permutation matrix**. *Let us consider $k_1, k_2 \in V_m$ such that there is a path in E_m between k_1 and k_2 (see definition 6.5.3). Then we assume that,*

$$ {}^m\gamma_r^{k_1 s} = {}^m\gamma_r^{k_2 s} \quad \text{and} \quad {}^m\gamma_r^{k_1 s} q_i^{k_1} = q_i^{k_1} \, {}^m\gamma_r^{k_1 s} $$

   for all $i \in V_m$ and $s, r \in \{1, .., m\}$.

The co-product $\Delta_{s,t}$ *on* $Q^{s,t}_{(V,E,j)}$ *is given by,*

$$\Delta_{s,t}(q^i_j) = \sum_{k \in V} q^i_k \otimes q^k_j \quad \textit{and} \quad \Delta_{s,t}({}^m\gamma^{ks}_r) = \sum_{\substack{s'=1 \\ k' \in V_m}}^{m} {}^m\gamma^{ks}_{s'} q^k_{k'} \otimes {}^m\gamma^{k's'}_r.$$

The canonical co-action $\beta_{s,t}$ *of* $Q^{s,t}_{(V,E,j)}$ *on* (V, E, j) *which preserves both of its source and target dependent quantum symmetries is given by,*

$$\beta_{s,t}(\chi_{(i,j)r}) = \sum_{(k,l)s \in E_m} \chi_{(k,l)s} \otimes q^k_i q^l_j \ {}^m\gamma^{ks}_r \quad \textit{where} \quad (i,j)r \in E_m.$$

Chapter 7

Examples and applications

7.1 A summary of what has been done

Before going to examples, we start this chapter with a diagrammatic summary of all different categories of quantum symmetry preserving co-actions on a directed multigraph (V, E). In figure 7.1, the arrows denote inclusion functors between two categories where the former is always a "full" subcategory of the latter. For an "undirected" multigraph (V, E, j), (see definition 2.1.7), the categories $\mathcal{C}^s_{(V,E,j)}$, $\mathcal{C}^t_{(V,E,j)}$ and $\mathcal{C}^{s,t}_{(V,E,j)}$ coincide (see proposition 6.5.1). We conclude this

$$
\begin{tikzcd}
 & & \mathcal{C}^{Ban}_{(V,E)} & \\
\mathcal{C}^s_{(V,E)} \arrow[urr] & \mathcal{C}^{s,t}_{(V,E)} \arrow[l] \arrow[ur] \arrow[rr] \arrow[dr] & & \mathcal{C}^t_{(V,E)} \arrow[ul] \\
 & & \mathcal{C}^{sym}_{(V,E)} \arrow[uu] & \\
 & & \mathcal{C}^{Bic}_{(V,E)} \arrow[u] &
\end{tikzcd}
$$

FIGURE 7.1: Inclusions of different categories for a directed multigraph (V, E)

section with a brief summary of similarities and differences of these different categories of quantum symmetry preserving co-actions in form of a table.

Category name	**Existence of universal object**	**classical variant of the universal object (if it exists)**	**preservence of uniform components of (V,E)**
$\mathcal{C}^{Ban}_{(V,E)}$	Yes (theorem 4.4.3)	strictly bigger than $G^{aut}_{(V,E)}$(see example 1)	No
$\mathcal{C}^{sym}_{(V,E)}$	Yes for a class of multigraphs (theorems 5.3.9 and 4.4.5), in general, not known.	$G^{aut}_{(V,E)}$ (definition 2.1.11)	Yes (propositions 5.2.1 and 5.2.2)
$\mathcal{C}^{Bic}_{(V,E)}$	Yes (theorem 4.4.5)	$G^{aut}_{(V,E)}$	Yes (propositions 5.2.1 and 5.2.2)
$\mathcal{C}^{s}_{(V,E)}$	Yes (theorem 6.3.1)	$G^{s}_{(V,E)}$ (definition 6.1.3)	Yes (proposition 6.2.2)
$\mathcal{C}^{t}_{(V,E)}$	Yes (theorem 6.3.1)	$G^{t}_{(V,E)}$ (definition 6.1.3)	Yes (proposition 6.2.3)
$\mathcal{C}^{s,t}_{(V,E)}$	Yes (theorem 6.3.1)	$G^{s,t}_{(V,E)}$ (definition 6.1.3)	Yes (proposition 6.2.2 or 6.2.3)

7.2 Examples and Computations

In this section we compute various quantum automorphism groups for a few selected multigraphs.

7.2.1 Example 1:

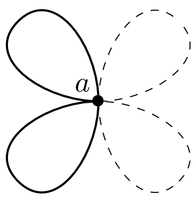

FIGURE 7.2: A multigraph with n loops on a single vertex

Let us consider the multigraph in figure 7.2 where the vertex set has a single element a and edge set E has n number of loops. The multigraph can be regarded both as a directed multigraph (V, E) or an undirected multigraph (V, E, j) where the inversion map $j = id_E$. The universal CQG associated with (V, E), $Q^{Ban}_{(V,E)}$ is generated as a universal C* algebra by the elements of the matrix $(u^{\sigma}_{\tau})_{\sigma,\tau\in E}$ satisfying the following relations:

$$\sum_{\tau\in E} u^{\sigma_1}_{\tau} u^{\sigma_2 *}_{\tau} = \delta_{\sigma_1,\sigma_2} 1, \quad \sum_{\tau\in E} u^{\sigma_1 *}_{\tau} u^{\sigma_2}_{\tau} = \delta_{\sigma_1,\sigma_2} 1,$$
$$\sum_{\tau\in E} u^{\tau}_{\sigma_1} u^{\tau *}_{\sigma_2} = \delta_{\sigma_1,\sigma_2} 1, \quad \sum_{\tau\in E} u^{\tau *}_{\sigma_1} u^{\tau}_{\sigma_2} = \delta_{\sigma_1,\sigma_2} 1,$$
$$\text{and} \quad \sum_{\tau\in E} u^{\sigma_1}_{\tau} = 1$$

where $\sigma_1, \sigma_2 \in E$.

The universal C* algebra associated to (V, E, j) is given by

$$Q^{Ban}_{(V,E,j)} = Q^{Ban}_{(V,E)} \Big/ \langle u^{\sigma}_{\tau} - u^{\sigma *}_{\tau} | \sigma, \tau \in E \rangle \cdot$$

From theorem 5.3.9, it follows that the category $\mathcal{C}^{sym}_{(V,E)}$ admits universal object which is $Q^{Bic}_{(V,E)}$. Moreover it follows that $Q^{Bic}_{(V,E,j)} = Q^{Bic}_{(V,E)} = S^+_4$ where S^+_n is the **quantum permutation group** on n elements. As the vertex set has only one element, it also follows that the automorphism groups of source and target dependent quantum symmetries are

$$Q^{s}_{(V,E)} = Q^{t}_{(V,E)} = Q^{s,t}_{(V,E)} = Q^{s,t}_{(V,E,j)} = S^+_n.$$

7.2.2 Example 2:

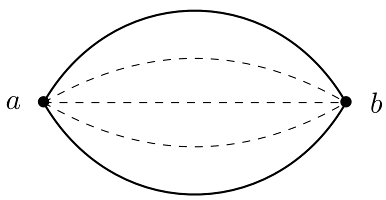

FIGURE 7.3: An undirected multigraph with two vertices and n edges among them.

Let us consider the multigraph (V, E, j) in figure 7.3 where there are n number of "undirected" edges between two vertices. According to definition 2.1.7, the edge set E consists of $2n$ elements with an inversion map j identifying two oppositely directed edges producing an undirected edge. Let us fix a representation for (V, E, j) (see definition 6.1.1). Using theorem

5.3.9 it follows that $\mathcal{C}^{sym}_{(V,E,j)}$ admits universal object which is $Q^{Bic}_{(V,E,j)}$. The quantum automorphism group of the underlying simple graph $S^{Ban}_{(V,\overline{E})}$ is S_2^+ and is generated by the coefficients of the following matrix:

$$\begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

where p is a projection. The quantum automorphism group $Q^{Bic}_{(V,E,j)}$ is the universal C* algebra generated by p and the coefficients of a **quantum permutation matrix** $(u^{(i,j)r}_{(k,l)s})_{(i,j)r,(k,l)s\in E}$ satisfying the following identities:

$$\sum_{s=1}^{n} u^{(a,b)r}_{(a,b)s} = p \quad \text{and} \quad \sum_{s=1}^{n} u^{(a,b)r}_{(b,a)s} = (1-p) \quad \text{for all} \quad r \in \{1, 2, .., n\}.$$

By defining $v^r_s = u^{(a,b)r}_{(a,b)s} + u^{(a,b)r}_{(b,a)s}$ we observe that,

1. Coefficients of the matrix $(v^r_s)_{r,s=1,..,n}$ satisfy **quantum permutation relations**.
2. $u^{(a,b)r}_{(a,b)s} = v^r_s p = p v^r_s = u^{(b,a)r}_{(b,a)s}$ and $u^{(a,b)r}_{(b,a)s} = (1-p)v^r_s = v^r_s(1-p) = u^{(b,a)r}_{(a,b)s}$ for all $r, s = 1, .., n$.

Therefore it follows that $Q^{Bic}_{(V,E,j)} = S_n^+ \otimes S_2^+$. The automorphism group of source and target dependent quantum symmetries $Q^{s,t}_{(V,E,j)}$ is also $S_n^+ \otimes S_2^+$.

7.2.3 Example 3:

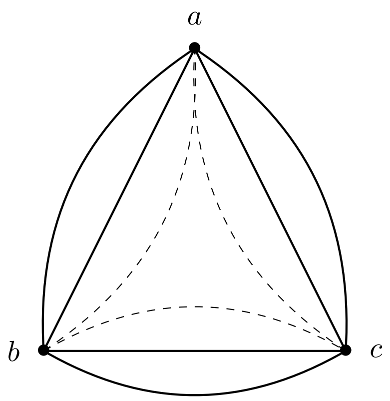

FIGURE 7.4: An undirected graph of a triangle with n edges between two vertices.

We consider the undirected multigraph (V, E, j) in figure 7.4 where there are n number of "undirected" edges between the pairs of vertices $\{a, b\}$, $\{b, c\}$ and $\{c, a\}$. We fix a representation of (V, E, j) (see notation 5.3.1). From theorem 5.3.9, it follows that $\mathcal{C}^{sym}_{(V,E,j)}$ admits a universal object which is $Q^{Bic}_{(V,E,j)}$. It is also clear that $S^{Bic}_{(V,\overline{E})} = S_3^+$ where $(V, \overline{E})$ is the underlying simple graph. Let $(u^{(i,j)r}_{(k,l)s})_{(i,j)r,(k,l)s\in E}$ be the matrix of canonical generators of $Q^{Bic}_{(V,E,j)}$. For $(i,j)\in\overline{E}$ and $r,s\in\{1,2,..,n\}$, let us define,

$$P^{\{i,j\}r}_s=\sum_{(k,l)\in\overline{E}}u^{(i,j)r}_{(k,l)s}$$

where $\{i,j\}$ is an unordered pair of two vertices. The quantity $P^{\{i,j\}r}_s$ is well defined because of lemma 4.6.5. It follows that for all $(i,j)\in\overline{E}$ and $k,l\in V$, $P^{\{i,j\}r}_s$ commutes with q^i_k and q^j_l. Fixing a $k\in V$ and $(i,j)\in\overline{E}$, we observe that $P^{\{i,j\}r}_s$ commutes with $q^{i'}_k$ for all $i'\in V$ as there are only three points in the vertex set. The quantum automorphism group $Q^{Bic}_{(V,E,j)}$ is generated by the following set of generators:

$$\cup_{i\neq j\in V}\{P^{\{i,j\}r}_s|r,s=1,2,..,n\}\cup\{q^k_l|k,l\in V\}$$

such that the following conditions hold:

1. The matrix $(q^k_l)_{k,l\in V}$ is a **quantum permutation matrix**.
2. For each $i,j\in V$ such that $i\neq j$, the matrix $(P^{\{i,j\}r}_s)_{r,s=1,2,..,n}$ is a **quantum permutation matrix**.
3. $P^{\{i,j\}r}_s$ commutes with q^k_l for all $k,l\in V;(i,j)\in\overline{E};r,s=1,2,..,n$.

It is clear that as an algebra $Q^{Bic}_{(V,E,j)}$ is $(S^+_n*S^+_n*S^+_n)\otimes S^+_3$. Moreover the co-product Δ_{Bic} on $Q^{Bic}_{(V,E,j)}$ is given by

$$\Delta_{Bic}(q^k_l)=\sum_{k'\in V}q^k_{k'}\otimes q^{k'}_l\quad\text{and}\quad\Delta_{Bic}(P^{\{i,j\}r}_s)=\sum_{\substack{s'=1\\k\neq l\in V}}^{n}(P^{\{i,j\}r}_{s'}\otimes P^{\{k,l\}s'}_s)(q^i_kq^j_l\otimes 1).$$

The automorphism group of source and target dependent quantum symmetries is $Q^{s,t}_{(V,E,j)}=S^+_n\otimes S^+_3$.

7.2.4 Example 4

FIGURE 7.5: Disjoint union of two undirected multigraphs

Let us consider the undirected multigraph (V, E, j) shown in figure 7.5 where there n number of "undirected" edges between the pairs of vertices $\{a, b\}$ and $\{c, d\}$. We fix a representation for (V, E, j) (see notation 5.3.1). Let $Q = (q^i_j)_{i,j\in V}$ be the matrix of canonical generators of $S^{Ban}_{(V,\overline{E})}$ where $(V, \overline{E})$ is the underlying simple graph. From theorem 7.1 of [BB07a] it follows that the matrix Q is of the form,

$$\begin{bmatrix} pt & (1-p)t & p(1-t) & (1-p)(1-t) \\ (1-p)t & pt & (1-p)(1-t) & p(1-t) \\ q(1-t) & (1-q)(1-t) & qt & (1-q)t \\ (1-q)(1-t) & q(1-t) & (1-q)t & qt \end{bmatrix} \tag{7.2.1}$$

where p, q, t are projections satisfying $pt = tp$ and $qt = tq$. It is easy to see that q^i_k commutes with q^j_l for all $(i, j),(k, l) \in \overline{E}$. Hence we have $S^{Ban}_{(V,\overline{E})} \cong S^{Bic}_{(V,\overline{E})}$. Furthermore from theorem 5.3.9, it follows that $\mathcal{C}^{sym}_{(V,E,j)}$ admits a universal object which is $Q^{Bic}_{(V,E,j)}$. The quantum automorphism group $Q^{Bic}_{(V,E,j)}$ is the universal C* algebra generated by the following set of projections:

$$\cup_{(i,j)\in\overline{E}}\{P^{\{i,j\}r}_s | r, s = 1, 2, .., n\} \cup \{p, q, t | pt = tp; \;\; qt = tq\}$$

such that the following conditions hold:

1. For all $(i, j) \in \overline{E}$ the matrix $(P^{\{i,j\}r}_s)_{r,s=1,2,..,n}$ is a **quantum permutation matrix**.
2. $P^{\{a,b\}r}_s$ commutes with p and t and $P^{\{c,d\}r}_s$ commutes with q and t for all $r, s = 1, 2, .., n$.

The co-product Δ_{Bic} on the set of generators is given by

$$\begin{aligned}
\Delta_{Bic}(p) &= pt \otimes p + (1-p)t \otimes (1-p) + p(1-t) \otimes q + (1-p)(1-t) \otimes (1-q),\\
\Delta_{Bic}(q) &= qt \otimes q + (1-q)t \otimes (1-q) + q(1-t) \otimes p + (1-q)(1-t) \otimes (1-p),\\
\Delta_{Bic}(t) &= t \otimes t + (1-t) \otimes (1-t),\\
\Delta_{Bic}(P^{\{a,b\}r}_s) &= \sum_{s'=1}^{n} (tP^{\{a,b\}r}_{s'} \otimes P^{\{a,b\}s'}_s) + ((1-t)P^{\{a,b\}r}_{s'} \otimes P^{\{c,d\}s'}_s),\\
\text{and} \quad \Delta_{Bic}(P^{\{c,d\}r}_s) &= \sum_{s'=1}^{n} (tP^{\{c,d\}r}_{s'} \otimes P^{\{c,d\}s'}_s) + ((1-t)P^{\{c,d\}r}_{s'} \otimes P^{\{a,b\}s'}_s).
\end{aligned}$$

As (V, E) has two connected components corresponding to two unordered pairs of vertices $\{a, b\}$ and $\{c, d\}$, from definition 6.5.5 it follows that $Q^{s,t}_{(V,E,j)} \cong Q^{Bic}_{(V,E,j)}$.

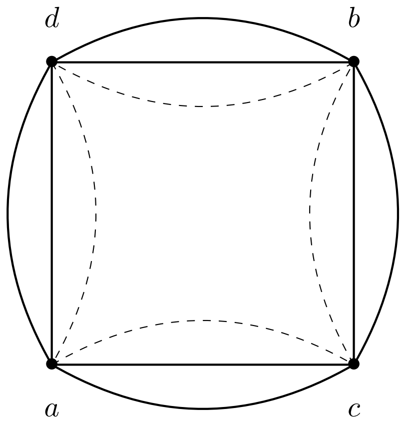

FIGURE 7.6: Graph of a square with n edges between two vertices.

7.2.5 Example 5

Let us consider the undirected multigraph (V, E, j) mentioned in figure 7.6 where there are n number of "undirected" edges between the pairs of vertices $\{a, c\}$, $\{c, b\}$, $\{b, d\}$ and $\{d, a\}$. We fix a representation for (V, E, j) (see notation 5.3.1). Let $(V, \overline{E})$ be the underlying simple graph. Using lemma 2.5.12 for underlying simple graphs of the multigraphs mentioned in figure 7.5 and 7.6, it follows that $S^{Ban}_{(V,\overline{E})}$ is generated by the entries of the matrix 7.2.1 where p, q, t are projections such that $pt = tp$ and $qt = tq$. As p and q are free, it follows that

$$q^a_d q^c_b = (1-p)(1-q)(1-t) \neq (1-q)(1-p)(1-t) = q^c_b q^a_d.$$

Therefore $S^{Ban}_{(V,\overline{E})}$ and $S^{Bic}_{(V,\overline{E})}$ are not isomorphic as compact quantum groups. From theorem 5.3.9, it follows that $\mathcal{C}^{sym}_{(V,E,j)}$ is strictly bigger than $\mathcal{C}^{Bic}_{(V,E,j)}$. Though we do not know whether the category $\mathcal{C}^{sym}_{(V,E,j)}$ admits universal object or not, in chapter 6, we have constructed a subcategory $\mathcal{C}^{s,t}_{(V,E,j)}$ of $\mathcal{C}^{sym}_{(V,E,j)}$ which does admit a universal object namely $Q^{s,t}_{(V,E,j)}$. The compact quantum group $Q^{s,t}_{(V,E,j)}$ is a **non-Bichon** type quantum subgroup of $Q^{Ban}_{(V,E,j)}$. The **automorphism group of source and target dependent quantum symmetries** $Q^{s,t}_{(V,E,j)}$ is the universal C* algebra generated by the following set of elements:

$$\{\gamma^r_s | r, s = 1, 2, .., n\} \cup \{p, q, t | pt = tp;\ \ qt = tq\}$$

where γ^r_s, p, q, t all are projections satisfying the following conditions:

1. $(\gamma^r_s)_{r,s=1,2,..,n}$ is a **quantum permutation matrix**.

2. $\gamma^r_s p = p\gamma^r_s, \gamma^r_s q = q\gamma^r_s$ and $t\gamma^r_s = \gamma^r_s t$ for all $r, s = 1, 2, .., n$.

The co-product $\Delta_{s,t}$ is given by

$$\Delta_{s,t}(p) = pt \otimes p + (1-p)t \otimes (1-p) + p(1-t) \otimes q + (1-p)(1-t) \otimes (1-q),$$
$$\Delta_{s,t}(q) = qt \otimes q + (1-q)t \otimes (1-q) + q(1-t) \otimes p + (1-q)(1-t) \otimes (1-p),$$
$$\Delta_{s,t}(t) = t \otimes t + (1-t) \otimes (1-t)$$
$$\text{and} \quad \Delta_{s,t}(\gamma_s^r) = \sum_{s'=1}^{n} \gamma_{s'}^r \otimes \gamma_s^{s'}.$$

We finish this section with the following question:

Question:

It is interesting to ask whether $\mathcal{C}^{sym}_{(V,E,j)}$ admits a universal object or not in the context of example 5. If the answer is negative, then it is also worth investigating whether $\mathcal{C}^{s,t}_{(V,E,j)}$ is the "largest" subcategory of $\mathcal{C}^{sym}_{(V,E,j)}$ which admits a universal object of **non-Bichon** type.

7.3 Applications:

7.3.1 Quantum symmetry of Graph C* algebras

In the context of quantum symmetry, it is interesting to study the graph C* algebras as they are mostly infinite dimensional although the function algebras associated with graphs are not. In this subsection, we will see that our notions of quantum symmetry in multigraphs lift to the level of graph C* algebras, We recall the definition of a graph C* algebra associated with a directed multigraph (V, E). For more details, see [Rae05], [BEVW22], [PR06] and references within.

Definition 7.3.1. *For a finite directed multigraph $\Gamma = (V, E)$ the graph C* algebra $C^*(\Gamma)$ is the universal C*algebra generated by a set of partial isometries $\{s_\tau | \tau \in E\}$ and a set of mutually orthogonal projections $\{p_i | i \in V\}$ satisfying the following relations among them:*

1. *$s_\tau^* s_\tau = p_{t(\tau)}$ for all $\tau \in E$ where $t : E \to V$ is the target map of Γ.*
2. *$\sum_{\tau \in E^i} s_\tau s_\tau^* = p_i$ for all $i \in V^s$ where V^s is the set of initial vertices in Γ.*

We have the following properties of graph C* algebras (subsection 2.1 of [PR06]).

1. $\sum_{i \in V} p_i = 1$ in $C^*(\Gamma)$.
2. For any $i \in V^s$, $\{s_\tau s_\tau^* | \tau \in E^i\}$ is a set of mutually orthogonal projections and $s_{\tau_1}^* s_{\tau_2} = 0$ for all $\tau_1 \neq \tau_2 \in E$.

We will be generalising the main result in [SW18] in our framework of quantum symmetry in multigraphs using similar arguments.

Theorem 7.3.2. *Let $\Gamma = (V, E)$ be a directed multigraph and β be a co-action of a CQG $(\mathcal{A}, \Delta)$ on (V, E) preserving its quantum symmetry in Banica's sense (see definition 4.2.2). Then β induces a co-action $\beta' : C^*(\Gamma) \to C^*(\Gamma) \otimes \mathcal{A}$ satisfying,*

$$\beta'(p_i) = \sum_{k \in V} p_k \otimes q_i^k$$
$$\beta'(s_\tau) = \sum_{\sigma \in E} s_\sigma \otimes u_\tau^\sigma$$

where $(u_\tau^\sigma)_{\sigma,\tau \in E}$ and $(q_i^k)_{k,i \in V}$ are the co-representation matrices of β and its induced co-action α on $C(V)$.

Proof. For $\tau \in E, i \in V$, let us define $S_\tau, P_v \in C^*(\Gamma) \otimes \mathcal{A}$ by

$$S_\tau = \sum_{\sigma \in E} s_\sigma \otimes u_\tau^\sigma,$$
$$P_i = \sum_{k \in V} p_k \otimes q_i^k.$$

For $i, j \in V$, we observe that,

$$P_i P_j = \sum_{k \in V} p_k \otimes q_i^k q_j^k = \delta_{i,j} \sum_{k \in V} p_k \otimes q_i^k = \delta_{i,j} P_i.$$

Hence $\{P_i | i \in V\}$ is a set of mutually orthogonal projections in $C^*(\Gamma) \otimes \mathcal{A}$. Using the properties of $C^*(\Gamma)$ mentioned in definition 7.3.1 we observe that,

$$\begin{aligned}
\text{for} \quad \tau \in E, \quad S_\tau^* S_\tau &= \sum_{\sigma_1, \sigma_2 \in E} s_{\sigma_1}^* s_{\sigma_2} \otimes u_\tau^{\sigma_1 *} u_\tau^{\sigma_2} \\
&= \sum_{\sigma \in E} s_\sigma^* s_\sigma \otimes u_\tau^{\sigma *} u_\tau^\sigma \\
&= \sum_{\sigma \in E} p_{t(\sigma)} \otimes u_\tau^{\sigma *} u_\tau^\sigma \\
&= \sum_{k \in V^t} p_k \otimes \sum_{\sigma \in E_k} u_\tau^{\sigma *} u_\tau^\sigma \\
&= \sum_{k \in V} p_k \otimes q_{t(\tau)}^k = P_{t(\tau)}
\end{aligned}$$

$$\text{and for} \quad i \in V^s, \quad \sum_{\tau \in E^i} S_\tau S_\tau^* = \sum_{\sigma_1, \sigma_2 \in E} s_{\sigma_1} s_{\sigma_2}^* \otimes \sum_{\tau \in E^i} u_\tau^{\sigma_1} u_\tau^{\sigma_2 *}$$

$$= \sum_{\sigma_1,\sigma_2 \in E} s_{\sigma_1} s^*_{\sigma_2} \otimes \delta_{\sigma_1,\sigma_2} q_i^{s(\sigma_1)}$$
$$= \sum_{k \in V^s} (\sum_{\sigma \in E^k} s_\sigma s^*_\sigma) \otimes q_i^k$$
$$= \sum_{k \in V} p_k \otimes q_i^k = P_i.$$

By universality of $C^*(\Gamma)$, there exists a C* homomorphism $\beta' : C^*(\Gamma) \to C^*(\Gamma) \otimes \mathcal{A}$ such that,

$$\beta'(s_\tau) = S_\tau \quad \text{and} \quad \beta'(p_i) = P_i$$

for all $\tau \in E$ and $i \in V$. It only remains to show that β' is in fact a co-action of $(\mathcal{A}, \Delta)$ on $C^*(\Gamma)$. The co-product identity holds as it is easy to check that on the set of generators of $C^*(\Gamma)$. Let us define

$$\mathcal{S} = \text{linear span } \beta'(C^*(\Gamma))(1 \otimes \mathcal{A}) \subseteq C^*(\Gamma) \otimes \mathcal{A}.$$

To conclude that β' is a co-action of $(\mathcal{A}, \Delta)$ it is enough to show that $\mathcal{S}$ is norm-dense in $C^*(\Gamma) \otimes \mathcal{A}$. We proceed through following claims:

Claim 1: $p_i \otimes 1, s_\tau \otimes 1, s^*_\tau \otimes 1 \in \mathcal{S}$ for all $i \in V, \tau \in E$.

Let $i \in V, \tau \in E$. We observe that,

$$\sum_{j \in V} \beta'(p_j)(1 \otimes q^i_j) = \sum_{l \in V} p_l \otimes (\sum_{j \in V} q^l_j q^i_j) = p_i \otimes \sum_{j \in V} q^i_j = p_i \otimes 1,$$
$$\sum_{\sigma \in E} \beta'(s_\sigma)(1 \otimes u^{\tau *}_\sigma) = \sum_{\sigma' \in E} s_{\sigma'} \otimes (\sum_{\sigma \in E} u^{\sigma'}_\sigma u^{\tau *}_\sigma) = s_\tau \otimes 1,$$
$$\sum_{\sigma \in E} \beta'(s^*_\sigma)(1 \otimes u^\tau_\sigma) = \sum_{\sigma' \in E} s^*_{\sigma'} \otimes (\sum_{\sigma \in E} u^{\sigma' *}_\sigma u^\tau_\sigma) = s^*_\tau \otimes 1.$$

In the above computation we have used the fact that β and $\overline{\beta}$ both are unitary co-representations on $L^2(E)$. As all the elements mentioned in the left are in $\mathcal{S}$, claim 1 follows.

Claim 2: If $x \otimes 1, y \otimes 1 \in \mathcal{S}$, then $xy \otimes 1 \in \mathcal{S}$.

Let us assume that,

$$x \otimes 1 = \sum_{i=1}^n \beta'(e_i)(1 \otimes f_i) \quad \text{and} \quad y \otimes 1 = \sum_{j=1}^m \beta'(g_j)(1 \otimes h_j)$$

where $e_i, g_j \in C^*(\Gamma)$ and $f_i, h_j \in \mathcal{A}$ for all i, j. We observe that,

$$\begin{aligned} xy \otimes 1 &= \sum_{i=1}^{n} \beta'(e_i)(1 \otimes f_i)(y \otimes 1) \\ &= \sum_{i=1}^{n} \beta'(e_i)(y \otimes 1)(1 \otimes f_i) \\ &= \sum_{i,j} \beta'(e_i)\beta'(g_j)(1 \otimes g_j)(1 \otimes f_i) \\ &= \sum_{i,j} \beta'(e_i g_j)(1 \otimes g_j f_i) \quad \in \mathcal{S}. \end{aligned}$$

Hence claim 2 follows.

From claim 1 and claim 2 it is clear that,

$$C^*(\Gamma) \otimes 1 \subseteq \text{norm closure of } \mathcal{S}.$$

As for any $T \in \mathcal{S}$ and $x \in \mathcal{A}$, $T(1 \otimes x)$ is also in $\mathcal{S}$, we conclude that,

$$C^*(\Gamma) \otimes \mathcal{A} \subseteq \text{norm closure of } \mathcal{S}.$$

Hence our theorem is proved. □

7.3.2 Co-actions on C* correspondences

In a private communication with Jyotishman Bhowmick ([Bho22]), he pointed out that our framework of quantum symmetry in multigraphs also fits into the framework of co-actions on C* correspondences. We start with the following definitions. For more details, see [Kat04], [KQR15], [BJ] and references within.

Definition 7.3.3. *Let B be a unital C* algebra. A C* correspondence over B is a pair (X, ϕ) where X is a right Hilbert B-module and $\phi : B \to \mathcal{L}_B(X)$ is a C* algebra homomorphism. Moreover, (X, ϕ) is said to be "non-degenerate" if linear span of $\phi(B)X$ is norm-dense in X.*

Definition 7.3.4. *A co-action of a CQG $(\mathcal{A}, \Delta)$ on a C* correspondence (X, ϕ) over a unital C* algebra B is a pair of maps (β, α) where $\alpha : B \to B \otimes \mathcal{A}$ is a co-action of $(\mathcal{A}, \Delta)$ on the C* algebra B and $\beta : X \to X \otimes \mathcal{A}$ is a $\mathbb{C}$-linear map satisfying the following:*

1. $(\beta \otimes id) \circ \beta = (id \otimes \Delta) \circ \beta$.

2. *Linear span of $\beta(X)(1 \otimes \mathcal{A})$ is norm-dense in $X \otimes \mathcal{A}$.*

3. $\beta(\phi(b)\xi) = (\phi \otimes id_{\mathcal{A}})(\alpha(b))\beta(\xi)$ *for all* $b \in B, \xi \in X$.

4. $< \beta(\xi), \beta(\eta) >_{B\otimes\mathcal{A}} = \alpha(< \xi, \eta >_B)$ *where* $<,>_B$ *is the* B*-valued inner product on* X *and* $<,>_{B\otimes\mathcal{A}}$ *is defined by*

$$< \xi \otimes a, \eta \otimes b >_{B\otimes\mathcal{A}} = < \xi, \eta >_B \otimes a^* b$$

for all $\xi \otimes a, \eta \otimes b \in X \otimes \mathcal{A}$.

Remark 7.3.5. *In [Kat04], Katsura showed that (3) and (4) in definition 7.3.4 together imply,*

$$\beta(\xi.b) = \beta(\xi)\alpha(b) \quad \textit{for all} \quad \xi \in X, b \in B.$$

For a finite directed multigraph (V, E), $L^2(E)$ is a right $C(V)$-module where the right module structure and $C(V)$-valued inner product are given by,

$$\chi_\tau . \chi_j = \delta_{t(\tau),j}\chi_\tau \quad \text{and} \quad < \chi_\sigma, \chi_\tau >_{C(V)} = \delta_{\sigma,\tau}\chi_{t(\sigma)}$$

for all $\sigma, \tau \in E$ and $j \in V$.

The pair $(L^2(E), \phi)$ is a C* correspondence over $C(V)$ where the C* algebra homomorphism $\phi : C(V) \to \mathcal{L}_{C(V)}(L^2(E))$ is given by,

$$\phi(f)(\chi_\sigma) = f(s(\sigma))\chi_\sigma \quad \text{for all} \quad f \in C(V), \sigma \in E.$$

Proposition 7.3.6. *Let* β *be a co-action of a CQG* $(\mathcal{A}, \Delta)$ *on a finite directed multigraph* (V, E) *preserving its quantum symmetry in Banica's sense (see definition 4.2.2). Let* α *be the induced co-action on* $C(V)$*. Then* (β, α) *is a co-action on the C* correspondence* $(L^2(E), \phi)$ *in the sense of definition 7.3.4.*

Proof. As $\beta : L^2(E) \to L^2(E) \otimes \mathcal{A}$ is a bi-unitary co-representation on the Hilbert space $L^2(E)$ it is easy to see that β satisfies (1) and (2) of definition 7.3.4. (3) of definition 7.3.4 follows from propositions 4.2.5, 4.2.6 and the following observation:

$$\phi(\chi_i).\chi_\tau = \delta_{i,s(\tau)}\chi_\tau = \chi_i.\chi_\tau \quad \text{for all} \quad \tau \in E, i \in V.$$

To show that β satisfies (4), for $\tau_1, \tau_2 \in E$, we observe that

$$\begin{aligned}
< \beta(\chi_{\tau_1}), \beta(\chi_{\tau_2}) >_{C(V)\otimes\mathcal{A}} &=< \sum_{\sigma_1\in E} \chi_{\sigma_1} \otimes u^{\sigma_1}_{\tau_1}, \sum_{\sigma_2\in E} \chi_{\sigma_2} \otimes u^{\sigma_2}_{\tau_2} >_{C(V)\otimes\mathcal{A}} \\
&= \sum_{\sigma_1,\sigma_2\in E} < \chi_{\sigma_1}, \chi_{\sigma_2} >_{C(V)} \otimes\ u^{\sigma_1 *}_{\tau_1} u^{\sigma_2}_{\tau_2} \\
&= \sum_{\sigma\in E} \chi_{t(\sigma)} \otimes u^{\sigma *}_{\tau_1} u^{\sigma}_{\tau_2} \\
&= \sum_{j\in V^t} \chi_j \otimes (\sum_{\sigma\in E_j} u^{\sigma *}_{\tau_1} u^{\sigma}_{\tau_2}).
\end{aligned}$$

Using remark 4.1.3 and antipode on the underlying Hopf * algebra of matrix elements of $(\mathcal{A}, \Delta)$ we get,

$$\begin{aligned}
< \beta(\chi_{\tau_1}), \beta(\chi_{\tau_2}) >_{C(V)\otimes\mathcal{A}} &= \sum_{j\in V^t} \chi_j \otimes \delta_{\tau_1,\tau_2} q^{j}_{t(\tau_1)} \\
&= \delta_{\tau_1,\tau_2}(\sum_{j\in V} \chi_j \otimes q^{j}_{t(\tau_1)}) = \alpha(< \chi_{\tau_1}, \chi_{\tau_2} >_{C(V)})
\end{aligned}$$

As $\{\chi_\tau | \tau \in E\}$ linearly spans $L^2(E)$, our proposition is proved. $\square$

Remark 7.3.7. *In [BJ] we see another approach to quantum symmetry in simple graphs where one considers a restricted category of CQGs co-acting equivariantly on the C* correspondence coming naturally from a simple graph. It is interesting to compare this approach to our approaches of quantum symmetry and see whether the idea can be extended into the realm of multigraphs.*

7.3.3 Quantum symmetry on Potts Model

Potts model is one of the fundamental models in statistical mechanics. We have given already a brief description in section 2.2 of chapter 2. In [GAH22] we have been able to define a notion of quantum symmetry in q-state Potts model where the underlying graphs are simple graphs without loops. Using the machinery developed there we have shown that in some toy models, how slight fluctuations of Hamiltonian can lead to drastic changes in quantum symmetry making a possible case of phase transition in the system. In chapter 8 we have extended our treatment in the context of multigraphs.

Chapter 8

Quantum symmetry in q-state Potts model

In this chapter we will be describing the notion of quantum symmetry in a q-state Potts model and demonstrate the importance of quantum symmetry in the theory of phase transitions in some simple physical models through examples. Though the original work done by us in [GAH22] were concerned with only undirected simple graphs, similar computations can be carried out in the context of undirected multigraphs as we will see in this chapter. Physically, a multigraph can correspond to lattices of atoms with double or triple bonds. We will see later through examples that slight change in energy in one of the bonds can effect the quantum symmetry in the system drastically. We will also be using a simpler version of Potts model for simpler mathematical treatment.

8.1 Hamiltonian on an undirected multigraph

Let (V, E, j) be an undirected multigraph with no loops. A q-state Potts model ($q \in \mathbb{N}$ and $q \geq 2$) on (V, E, j) consists of a set of configurations Ω_P and a Hamiltonian $H_P : \Omega_P \to \mathbb{C}$ defined as follows:

Definition 8.1.1. *A configuration ω for a q-state Potts Model on (V, E, j) is a function from V to a finite set X_q consisting of q number of elements. The Hamiltonian H_P is defined to be:*

$$H_P(\omega) := \sum_{\tau \in E} J_\tau \delta_{\omega(s(\tau)),\omega(t(\tau))} \qquad \text{for all} \quad \omega \in \Omega_P \tag{8.1.1}$$

where $J_\tau \in \mathbb{C}$ and $J_\tau = J_{\overline{\tau}}$ for all $\tau \in E$. The expression $\delta_{\omega(k),\omega(l)}$ is equal to 1 if $\omega(k) = \omega(l)$ and is 0 otherwise.

A word of caution

One should not confuse the linear map $J : L^2(E) \to L^2(E)$ coming from the inversion map in an undirected multigraph with the set of parameters $\{J_\tau | \tau \in E\}$ of the Hamiltonian in definition 8.1.1 as they are completely unrelated. There will not be any notational confusion as the linear map $J : L^2(E) \to L^2(E)$ itself will not be used in this chapter explicitly anywhere.

We end this section with the following remark.

Remark 8.1.2. *For any $k, l \in V$ let us define $A_l^k = \sum_{\tau \in E_l^k} J_\tau$ if $E_l^k \neq \phi$ and $A_l^k = 0$ otherwise. From equation 8.1.1 for any configuration $\omega \in \Omega_P$ we have,*

$$H_P(\omega) = \sum_{\tau \in E} J_\tau \delta_{\omega(s(\tau)),\omega(t(\tau))} = \sum_{\substack{k,l \in V \\ E_l^k \neq \phi}} \sum_{\tau \in E_l^k} J_\tau \delta_{\omega(k),\omega(l)} = \sum_{k,l \in V} A_l^k \delta_{\omega(k),\omega(l)}. \tag{8.1.2}$$

The matrix $(A_l^k)_{k,l \in V}$ is a symmetric matrix as $J_\tau = J_{\overline{\tau}}$ for all $\tau \in E$.

Now we discuss the notion of quantum symmetry in Potts model.

8.2 Hamiltonian as a bilinear form

Let (V, E, j) be an undirected multigraph without loops with a specified Hamiltonian H_P. Let us consider α to be a co-action of a compact quantum group $(\mathcal{A}, \Delta)$ on $C(V)$. We want to describe what it means for α to preserve the Hamiltonian H_P. Such a co-action can be described to preserve the quantum symmetry of the q-state Potts model on (V, E, j). For our purpose, it is convenient to see a configuration ω as an element of $C(V) \otimes C^*(\mathbb{Z}_q)$ such that,

$$\omega(k) = \chi_{g_k} \qquad \text{for some} \quad g_k \in \mathbb{Z}_q.$$

Let $\tau : C^*(\mathbb{Z}_q) \to \mathbb{C}$ be a linear functional defined by $\tau(f) = f(e)$, where e is the identity of the cyclic group $\mathbb{Z}_q$. Let us define a bilinear form $<,>_{H_P}$ on $C(V) \otimes C^*(\mathbb{Z}_q)$ by

$$< f, h >_{H_P} = \sum_{k,l \in V} A_l^k \tau(f(k)^* * h(l))$$

where f and h are arbitrary elements in $C(V) \otimes C^*(\mathbb{Z}_q)$ and $f(k)^*(g) = \overline{f(k)(g^{-1})}$. We observe that,

$$\begin{aligned}
< f,h >_{H_P} &= \sum_{k,l\in V} A_l^k \tau(f(k)^* * h(l)) \\
&= \sum_{k,l\in V} A_l^k \tau\big((\sum_{g_1\in\mathbb{Z}_q} \overline{f(k)(g_1)}\chi_{g_1}^*) * (\sum_{g_2\in\mathbb{Z}_q} h(l)(g_2)\chi_{g_2})\big) \\
&= \sum_{\substack{k,l\in V\\ g_1,g_2\in\mathbb{Z}_q}} A_l^k \overline{f(k)(g_1)} h(l)(g_2)\tau\big(\chi_{g_1^{-1}g_2}\big) \\
&= \sum_{\substack{k,l\in V\\ g\in\mathbb{Z}_q}} A_l^k \overline{f(k)(g)} h(l)(g) \qquad \text{(as } \tau(\chi_g) = 1 \text{ iff } g = e)
\end{aligned}$$

Let $\omega \in \Omega_P$. We observe that,

$$\begin{aligned}
< \omega,\omega >_{H_P} &= \sum_{\substack{k,l\in V\\ g\in\mathbb{Z}_q}} A_l^k \overline{\omega(k)(g)}\omega(l)(g) \\
&= \sum_{k,l\in V} A_l^k (\sum_{g\in\mathbb{Z}_q} \overline{\omega(k)(g)}\omega(l)(g)) \\
&= \sum_{k,l\in V} A_l^k \delta_{g_k,g_l} \qquad \text{(as } \omega(k)(g) = \chi_{g_k}) \\
&= H_P(\omega). \qquad (8.2.1)
\end{aligned}$$

$<,>_{H_P}$ induces an $\mathcal{A}$ valued bilinear form $<,>_{H_P\otimes\mathcal{A}}$ on $C(V)\otimes C^*(\mathbb{Z}_q)\otimes\mathcal{A}$ given by

$$< f\otimes a, h\otimes b >_{H_P\otimes\mathcal{A}} := < f,h >_{H_P} a^* b.$$

Let us define $\alpha' : C(V)\otimes C^*(\mathbb{Z}_q) \to C(V)\otimes C^*(\mathbb{Z}_q)\otimes\mathcal{A}$ given by,

$$\alpha' = (id\otimes\sigma_{23})(\alpha\otimes id)$$

where σ_{23} is the standard flip between 2nd and 3rd coordinates. It is easy to see that α' is a co-action on the algebra $C(V)\otimes C^*(\mathbb{Z}_q)$.

8.3 Notion of quantum symmetry in Potts model

8.3.1 Co-action preserving the Hamiltonian

Definition 8.3.1. *Let (V, E, j) be an undirected multigraph without loops and there is a specified Hamiltonian H_P on it. Let α be a co-action of a CQG $(\mathcal{A}, \Delta)$ on $C(V)$. The co-action α is said to preserve H_P if the following holds:*

$$< \omega, \omega >_{H_P} = < \alpha'(\omega), \alpha'(\omega) >_{H_P \otimes \mathcal{A}} \quad \text{for all} \quad \omega \in \Omega_P \tag{8.3.1}$$

where $<, >_{H_P}$ and α' are defined in section 8.2.

8.3.2 Algebraic characterisations

We introduce some notations for our convenience.

Notation 8.3.2. *Let α be a co-action of a CQG $(\mathcal{A}, \Delta)$ on $C(V)$ with co-representation matrix $(q_l^k)_{k,l \in V}$. For $\beta, \gamma \in V$, we define,*

$$S_{\beta\gamma} := \sum_{k,l \in V} A_l^k q_\beta^k q_\gamma^l.$$

Let $f \in C(V)$ be defined by,

$$f(\beta) = \sum_{l \in V} A_l^\beta = \sum_{k \in V} A_\beta^k \quad \text{for all} \quad \beta \in V.$$

By evaluating right hand side of equation (8.3.1) we get,

$$\begin{aligned}
< \alpha'(\omega), \alpha'(\omega) >_{H_P \otimes \mathcal{A}} &= < \sum_{k \in V} \alpha'(\chi_k \otimes \omega(k)), \sum_{l \in V} \alpha'(\chi_l \otimes \omega(l)) >_{H_P \otimes \mathcal{A}} \\
&= \sum_{k',l',k,l \in V} < \chi_{k'} \otimes \omega(k), \chi_{l'} \otimes \omega(l) >_{H_P} q_k^{k'} q_l^{l'} \\
&= \sum_{\substack{k',l',k,l \in V \\ g \in \mathbb{Z}_q}} A_{l'}^{k'} \overline{\omega(k)(g)} \omega(l)(g) q_k^{k'} q_l^{l'} \\
&= \sum_{\substack{k,l \in V \\ g \in \mathbb{Z}_q}} \overline{\omega(k)(g)} \omega(l)(g) S_{kl}.
\end{aligned} \tag{8.3.2}$$

Remark 8.3.3. *From equations 8.2.1 and 8.3.2 it follows that α preserves the Hamiltonian of q-state Potts model on (V, E, j) iff the following holds:*

$$\sum_{\substack{k,l\in V\\ g\in\mathbb{Z}_q}} \overline{\omega(k)(g)}\omega(l)(g)A_l^k 1 = \sum_{\substack{k,l\in V\\ g\in\mathbb{Z}_q}} \overline{\omega(k)(g)}\omega(l)(g)S_{kl} \qquad \text{for all} \quad \omega\in\Omega_P. \tag{8.3.3}$$

8.4 Some useful observations

Let us fix an undirected multigraph (V, E, j) without loops and Hamiltonian H_P on (V, E, j). Let us make some observations which will be crucial in the next section for proving the main theorem of this chapter.

Proposition 8.4.1. *Let α be a co-action of a CQG $(\mathcal{A}, \Delta)$ on $C(V)$. If the co-representation matrix Q corresponding to α commutes with the matrix $(A_l^k)_{k,l\in V}$, then α preserves the Hamiltonian H_P in the sense of definition 8.3.1.*

Proof. Let $k, l \in V$. We observe that,

$$\begin{aligned} S_{kl} = \sum_{k',l'\in V} A_{l'}^{k'} q_k^{k'} q_l^{l'} &= \sum_{k'\in V} q_k^{k'} (\sum_{l'\in V} A_{l'}^{k'} q_l^{l'}) \\ &= \sum_{k'\in V}\sum_{l'\in V} A_l^{l'} q_k^{k'} q_{l'}^{k'} \\ &= \sum_{k'\in V} A_l^k q_k^{k'} = A_l^k 1. \end{aligned}$$

Hence from equation 8.3.3, the result follows. ∎

Proposition 8.4.2. *Let α be the co-action of a CQG $(\mathcal{A}, \Delta)$ on $C(V)$ and h be the Haar functional on $(\mathcal{A}, \Delta)$. If $h(S_{\beta\gamma}) = A_\gamma^\beta$ for all $\beta, \gamma \in V$, then the co-representation matrix Q of α commutes with the matrix $A = (A_l^k)_{k,l\in V}$.*

Proof. Let $\beta, \gamma \in V$. Then we observe that,

$$\begin{aligned} \Delta(S_{\beta\gamma}) &= \Delta(\sum_{k,l\in V} A_l^k q_\beta^k q_\gamma^l) \\ &= \sum_{k,l\in V} A_l^k \Delta(q_\beta^k)\Delta(q_\gamma^l) \\ &= \sum_{k,l\in V} A_l^k (\sum_{k'\in V} q_{k'}^k \otimes q_\beta^{k'})(\sum_{l'\in V} q_{l'}^l \otimes q_\gamma^{l'}) \\ &= \sum_{k,l,k',l'\in V} A_l^k (q_{k'}^k q_{l'}^l \otimes q_\beta^{k'} q_\gamma^{l'}). \end{aligned} \tag{8.4.1}$$

As h is the Haar functional, we have,

$$(h \otimes id)\Delta(a) = h(a)1 \quad \text{for all} \quad a \in \mathcal{A}. \tag{8.4.2}$$

From equation (8.4.1) we get,

$$\begin{aligned}
(h \otimes id)\Delta(S_{\beta\gamma}) &= \sum_{k,l,k',l' \in V} A^k_l h(q^k_{k'} q^l_{l'}) q^{k'}_\beta q^{l'}_\gamma \\
&= \sum_{k',l' \in V} (h(\sum_{k,l \in V} A^k_l q^k_{k'} q^l_{l'})) q^{k'}_\beta q^{l'}_\gamma \\
&= \sum_{k',l' \in V} q^{k'}_\beta q^{l'}_\gamma h(S_{k'l'}) \\
&= \sum_{k',l' \in V} q^{k'}_\beta q^{l'}_\gamma A^{k'}_{l'} = S_{\beta\gamma}.
\end{aligned}$$

As h is the Haar funtional on $(\mathcal{A}, \Delta)$, from our hypothesis it follows that

$$S_{\beta\gamma} = A^\beta_\gamma 1 \quad \text{for all} \quad \beta, \gamma \in V. \tag{8.4.3}$$

Finally we observe,

$$\begin{aligned}
(QA)^i_j = \sum_{k \in V} q^i_k A^k_j &= \sum_{k \in V} q^i_k (\sum_{k',l \in V} A^{k'}_l q^{k'}_k q^l_j) \quad \text{(from (8.4.3))} \\
&= \sum_{k,l \in V} A^i_l q^i_k q^l_j \\
&= \sum_{l \in V} A^i_l q^l_j = (AQ)^i_j.
\end{aligned}$$

Hence we get that Q and A commutes. □

From proposition (8.4.1) and proposition (8.4.2), we get the following result:

Theorem 8.4.3. *Let $(\mathcal{A}, \Delta)$ be a compact quantum group co-acting on $C(V)$. The co-representation matrix Q commutes with the matrix $(A^k_l)_{k,l \in V}$ if and only if $S_{\beta\gamma} = A^\beta_\gamma 1$ for all $\beta, \gamma \in V$.*

8.5 Preservence of Hamiltionian and weighted symmetry

Let α be a quantum permutation of the vertex set V. We have seen in proposition 8.4.1 that if the co-representation matrix Q commutes with $(A^k_l)_{k,l \in V}$ then α preserves the Hamiltonian

H_p on (V, E, j). It turns out that the converse is also true. To show that, we will need the following lemma.

Lemma 8.5.1. *Let α be a co-action of a CQG $(\mathcal{A}, \Delta)$ on $C(V)$. If α preserves the Hamiltonian H_P on (V, E, j), then $\alpha(f) = f \otimes 1$ where f is described in notation 8.3.2.*

Proof. To show that $\alpha(f) = f \otimes 1$, it is enough to show

$$\sum_{k \in V} f(k) q_k^\beta = f(\beta) 1 \tag{8.5.1}$$

for all $\beta \in V$.

Let us fix β in V and $g_0 \in \mathbb{Z}_q$ such that $g_0 \neq e$. We define $\omega : V \to C^*(\mathbb{Z}_q)$ by $\omega(\beta) = \chi_{g_0}$ and $\omega(k) = \chi_e$ for $k \neq \beta$. For ω, we evaluate right hand side of equation (8.3.3) as follows:

$$\begin{aligned}
\sum_{\substack{k,l \in V \\ g \in \mathbb{Z}_q}} \overline{\omega(k)(g)} \omega(l)(g) S_{kl} &= \sum_{\substack{k \in V \\ g \in \mathbb{Z}_q}} \overline{\omega(k)(g)} \omega(\beta)(g) S_{k\beta} + \sum_{\substack{l \in V \\ g \in \mathbb{Z}_q}} \overline{\omega(\beta)(g)} \omega(l)(g) S_{\beta l} \\
&\quad + \sum_{\substack{k,l \neq \beta \\ g \in \mathbb{Z}_q}} \overline{\omega(k)(g)} \omega(l)(g) S_{kl} \\
&= \sum_{k,l \neq \beta} S_{kl} \\
&= \sum_{k,l \neq \beta} \sum_{k',l' \in V} A_{l'}^{k'} q_k^{k'} q_l^{l'} \\
&= \sum_{k',l' \in V} A_{l'}^{k'} (1 - q_\beta^{k'})(1 - q_\beta^{l'}) \\
&= \sum_{k',l' \in V} A_{l'}^{k'} (1 - q_\beta^{k'} - q_\beta^{l'}) \\
&= \sum_{k' \in V} f(k') - 2 \sum_{l' \in V} f(l') q_\beta^{l'}.
\end{aligned}$$

For ω, evaluating left hand side of equation (8.3.3) we get,

$$\begin{aligned}
\sum_{\substack{k,l \in V \\ g \in \mathbb{Z}_q}} A_l^k \overline{\omega(k)(g)} \omega(l)(g) &= \sum_{\substack{k \in V \\ g \in \mathbb{Z}_q}} A_\beta^k \overline{\omega(k)(g)} \omega(\beta)(g) + \sum_{\substack{l \in V \\ g \in \mathbb{Z}_q}} A_l^\beta \overline{\omega(\beta)(g)} \omega(l)(g) \\
&\quad + \sum_{\substack{k,l \neq \beta \\ g \in \mathbb{Z}_q}} A_l^k \overline{\omega(k)(g)} \omega(l)(g) \\
&= \sum_{k,l \neq \beta} A_l^k \\
&= \sum_{l \neq \beta} (f(l) - A_l^\beta)
\end{aligned}$$

$$= \sum_{l \in V} f(l) - 2f(\beta).$$

By our hypothesis and remark (8.3.3) we know that equation (8.3.3) holds. Hence, we conclude that equation (8.5.1) is true and our proposition is proved. □

Now we are in a position to prove the main result of this chapter.

Theorem 8.5.2. *Let α be a co-action of a CQG $(\mathcal{A}, \Delta)$ on $C(V)$. If α preserves the Hamiltonian H_P on (V, E, j), then the co-representation matrix Q of α commutes with the matrix $A = (A_l^k)_{k,l \in V}$.*

Proof. We fix $\beta, \gamma \in V$ such that $\beta \neq \gamma$ and $g_0 \in \mathbb{Z}_q$ which is not e. We define a configuration $\omega : V \to C^*(\mathbb{Z}_q)$ by

$$\omega(\beta) = \chi_{g_0},$$
$$\omega(\gamma) = \chi_{g_0},$$
$$\omega(k) = \chi_e \quad \text{for} \quad k \neq \beta, \gamma.$$

.

Evaluating right hand side of equation (8.3.3) for ω and using lemma 8.5.1 we get,

$$\begin{aligned}
\sum_{\substack{k,l \in V \\ g \in \mathbb{Z}_q}} \overline{\omega(k)(g)}\omega(l)(g)S_{kl} &= \sum_{\substack{l \neq \beta,\gamma \\ g \in \mathbb{Z}_q}} \overline{\omega(\beta)(g)}\omega(l)(g)S_{\beta l} + \sum_{\substack{l \neq \beta,\gamma \\ g \in \mathbb{Z}_q}} \overline{\omega(\gamma)(g)}\omega(l)(g)S_{\gamma l} \\
&\quad + \sum_{\substack{k \neq \beta,\gamma \\ g \in \mathbb{Z}_q}} \overline{\omega(k)(g)}\omega(\beta)(g)S_{k\beta} + \sum_{\substack{k \neq \beta,\gamma \\ g \in \mathbb{Z}_q}} \overline{\omega(k)(g)}\omega(\gamma)(g)S_{k\gamma} \\
&\quad + \sum_{\substack{k \neq \beta,\gamma \\ l \neq \beta,\gamma \\ g \in \mathbb{Z}_q}} \overline{\omega(k)(g)}\omega(l)(g)S_{kl} + S_{\beta\gamma} + S_{\gamma\beta} \\
&= \sum_{\substack{k \neq \beta,\gamma \\ l \neq \beta,\gamma}} S_{kl} + S_{\beta\gamma} + S_{\gamma\beta} \\
&= \sum_{k',l' \in V} A_{l'}^{k'} \Big(\sum_{\substack{k \neq \beta,\gamma \\ l \neq \beta,\gamma}} q_k^{k'} q_l^{l'} \Big) + S_{\beta\gamma} + S_{\gamma\beta} \\
&= \sum_{k',l' \in V} A_{l'}^{k'} (1 - q_\beta^{k'} - q_\gamma^{k'})(1 - q_\beta^{l'} - q_\gamma^{l'}) + S_{\beta\gamma} + S_{\gamma\beta} \\
&= \sum_{k',l' \in V} A_{l'}^{k'} (1 - q_\beta^{k'} - q_\gamma^{k'} - q_\beta^{l'} + q_\gamma^{k'} q_\beta^{l'} - q_\gamma^{l'} + q_\beta^{k'} q_\gamma^{l'}) \\
&\quad + S_{\beta\gamma} + S_{\gamma\beta}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k' \in V} f(k') - f(\beta)1 - f(\gamma)1 - f(\beta)1 + S_{\gamma\beta} - f(\gamma)1 \\
&\quad + S_{\beta\gamma} + S_{\beta\gamma} + S_{\gamma\beta} \\
&= \sum_{k' \in V} f(k') - 2f(\beta)1 - 2f(\gamma)1 + 2S_{\beta\gamma} + 2S_{\gamma\beta}. \qquad (8.5.2)
\end{aligned}
$$

By evaluating left hand side of equation (8.3.3) for ω we get,

$$
\begin{aligned}
\sum_{\substack{k,l \in V \\ g \in \mathbb{Z}_q}} \overline{\omega(k)(g)}\omega(l)(g)A_l^k &= \sum_{\substack{l \neq \beta,\gamma \\ g \in \mathbb{Z}_q}} \overline{\omega(\beta)(g)}\omega(l)(g)A_l^\beta + \sum_{\substack{l \neq \beta,\gamma \\ g \in \mathbb{Z}_q}} \overline{\omega(\gamma)(g)}\omega(l)(g)A_l^\gamma \\
&\quad + \sum_{\substack{k \neq \beta,\gamma \\ g \in \mathbb{Z}_q}} \overline{\omega(k)(g)}\omega(\beta)(g)A_\beta^k + \sum_{\substack{k \neq \beta,\gamma \\ g \in \mathbb{Z}_q}} \overline{\omega(k)(g)}\omega(\gamma)(g)A_\gamma^k \\
&\quad + \sum_{\substack{k \neq \beta,\gamma \\ l \neq \beta,\gamma \\ g \in \mathbb{Z}_q}} \overline{\omega(k)(g)}\omega(l)(g)A_l^k + A_\gamma^\beta + A_\beta^\gamma \\
&= \sum_{\substack{k \neq \beta,\gamma \\ l \neq \beta,\gamma}} A_l^k + A_\gamma^\beta + A_\beta^\gamma \\
&= \sum_{k \neq \beta,\gamma} (f(k) - A_\beta^k - A_\gamma^k) + A_\gamma^\beta + A_\beta^\gamma \\
&= \sum_{k \neq \beta,\gamma} f(k) - \sum_{k \neq \beta,\gamma} A_\beta^k - \sum_{k \neq \beta,\gamma} A_\gamma^k + A_\gamma^\beta + A_\beta^\gamma \\
&= \sum_{k \in V} f(k) - f(\beta) - f(\gamma) - f(\beta) + A_\beta^\gamma - f(\gamma) + A_\gamma^\beta \\
&\quad + A_\gamma^\beta + A_\beta^\gamma \\
&= \sum_{k \in V} f(k) - 2f(\beta) - 2f(\gamma) + 2A_\gamma^\beta + 2A_\beta^\gamma. \qquad (8.5.3)
\end{aligned}
$$

From our hypothesis and remark (8.3.3) we know that equation (8.3.3) holds. Hence, from equations (8.5.2) and (8.5.3) we get,

$$
S_{\beta\gamma} + S_{\gamma\beta} = (A_\gamma^\beta + A_\beta^\gamma)1
$$

which implies $\quad h(S_{\beta\gamma}) = A_\gamma^\beta.$

as h is tracial on the algebra generated by the coefficients of the matrix $(q_l^k)_{k,l \in V}$ inside $\mathcal{A}$. Since our choice of β, γ was arbitrary, from proposition 8.4.2 the theorem follows.

□

Theorem 8.5.3. *There exists a unique universal object in the category of compact quantum groups co-acting on (V, E, j) preserving the Hamiltonian of q-state Potts model.*

Proof. From lemma (8.4.1) and theorem (8.5.2), it follows that a compact quantum group $(\mathcal{A}, \Delta)$ co-acts on (V, E, j) via preserving the Hamiltonian iff the co-representation matrix Q of the co-action $\alpha : C(V) \to C(V) \otimes \mathcal{A}$ commutes with the matrix $(A_l^k)_{k,l \in V}$. Hence, from theorem (2.5.6) our claim follows. □

Let us call this unique universal object **Quantum symmetry group of Potts model** on (V, E, j).

8.6 Phase transition in some simple models

In this section, we look at some simple examples of Potts models where a slight fluctuation in the Hamiltonian can destroy the quantum symmetry present in system and turn it into a classical one or the opposite. This abrupt change in symmetry can indicate towards a phase transition in the system.

8.6.1 Example 1

We start with the graph of a cube as shown in figure (8.1). Let the set of vertices be $V = \{1, 2, 3, 4, 1', 2', 3', 4'\}$ and the edges E are as shown in the picture. To specify a Hamiltonian on (V, E) it is enough to specify the matrix $(A_l^k)_{k,l \in V}$ (see remark 8.1.2).

Let $\lambda \in \mathbb{C}$. We consider the Hamiltonian H_λ given by,

$$H_\lambda(\omega) = \sum_{k,l \in V} A_l^k \delta_{\omega(k),\omega(l)} \quad \forall\ \omega \in \Omega_P \tag{8.6.1}$$

where the matrix $(A_l^k)_{k,l \in V}$ is given by,

$$\begin{aligned} A_l^k &= 1 \quad \text{if} \quad (k, l) \neq (4, 4'), (4', 4), \\ A_{4'}^4 &= A_4^{4'} = \lambda. \end{aligned}$$

Remark 8.6.1. *When $\lambda = 1$, The quantum symmetry group of Potts Model is the quantum automoprhism group of the graph (V, E) in Banica's sense which is a non-classical compact quantum group. See for instance ([BBC07b], [BBC07a]).*

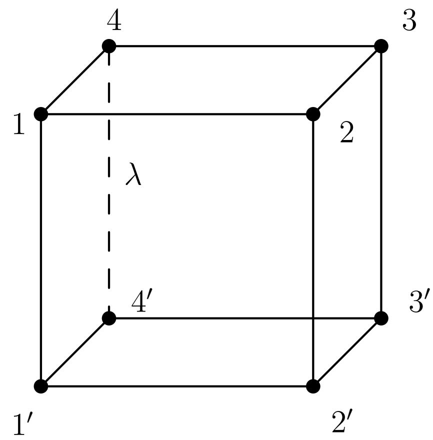

FIGURE 8.1: The graph of a cube with a specified Hamiltonian on it.

Lemma 8.6.2. *When λ is not 1, the quantum symmetry group of Potts model on (V, E) becomes commutative.*

Proof. Let λ be a complex number which is not 1. Let α be a co-action of a compact quantum group $(\mathcal{A}, \Delta)$ on $C(V)$ preserving the Hamiltonian H_λ. From theorem 8.5.2, the co-representation matrix Q of α commutes with $(A^k_l)_{k,l \in V}$.

From proposition 2.5.10 it follows that $q^k_4 = q^4_l = q^k_{4'} = q^{4'}_l = 0$ when $k \neq 4, 4'$ and $l \neq 4, 4'$. Hence it follows that

$$q^4_{4'} = q^{4'}_4 \quad \text{and} \quad q^4_4 = q^{4'}_{4'}$$

Using $QA = AQ$, we get the following commutation relations:

$$\begin{aligned} q^4_4 =& (AQ)^1_4 = (QA)^1_4 = q^1_1 + q^1_3 \\ 0 =& (AQ)^2_4 = (QA)^2_4 = q^2_1 + q^2_3 \\ q^{4'}_4 =& (AQ)^{1'}_4 = (QA)^{1'}_4 = q^{1'}_1 + q^{1'}_3 \\ 0 =& (AQ)^{2'}_4 = (QA)^{2'}_4 = q^{2'}_1 + q^{2'}_3 \\ q^{4'}_{4'} =& (AQ)^{1'}_{4'} = (QA)^{1'}_{4'} = q^{1'}_{1'} + q^{1'}_{3'} \\ 0 =& (AQ)^{2'}_{4'} = (QA)^{2'}_{4'} = q^{2'}_{1'} + q^{2'}_{3'} \end{aligned} \tag{8.6.2}$$

From equations (8.6.2) we note that $q_1^2 + q_3^2 = 0$, which implies $q_1^2 = 0$ and $q_3^2 = 0$. Similarly $q_1^{2'} = q_3^{2'} = q_{1'}^2 = q_{3'}^2 = q_{1'}^{2'} = q_{3'}^{2'} = 0$. The co-representation matrix Q becomes,

$$\begin{bmatrix} q_1^1 & 0 & q_3^1 & 0 & q_{1'}^1 & 0 & q_{3'}^1 & 0 \\ 0 & q_2^2 & 0 & 0 & 0 & q_{2'}^2 & 0 & 0 \\ q_1^3 & 0 & q_3^3 & 0 & q_{1'}^3 & 0 & q_{3'}^3 & 0 \\ 0 & 0 & 0 & q_4^4 & 0 & 0 & 0 & q_{4'}^4 \\ q_1^{1'} & 0 & q_3^{1'} & 0 & q_{1'}^{1'} & 0 & q_{3'}^{1'} & 0 \\ 0 & q_2^{2'} & 0 & 0 & 0 & q_{2'}^{2'} & 0 & 0 \\ q_1^{3'} & 0 & q_3^{3'} & 0 & q_{1'}^{3'} & 0 & q_{3'}^{3'} & 0 \\ 0 & 0 & 0 & q_4^{4'} & 0 & 0 & 0 & q_{4'}^{4'} \end{bmatrix}$$

By equating 1st row of (AQ) and (QA) we get,

$$\begin{aligned} q_1^{1'} &= (AQ)_1^1 = (QA)_1^1 = q_{1'}^1 \\ q_2^2 &= (AQ)_2^1 = (QA)_2^1 = q_1^1 + q_3^1 = q_4^4 \\ q_3^{1'} &= (AQ)_3^1 = (QA)_3^1 = q_{3'}^1 \\ q_{1'}^{1'} &= (AQ)_{1'}^1 = (QA)_{1'}^1 = q_1^1 \\ q_{2'}^2 &= (AQ)_{2'}^1 = (QA)_{2'}^1 = q_{1'}^1 + q_{3'}^1 = q_{4'}^4 \\ q_{3'}^{1'} &= (AQ)_{3'}^1 = (QA)_{3'}^1 = q_3^1 \end{aligned} \tag{8.6.3}$$

Finally, from 8.6.3 and 8.6.2 we observe that,

$$q_j^{i'} = q_{j'}^i \quad \text{and} \quad q_{j'}^{i'} = q_j^i \quad \text{for all} \quad i, j \in \{1, 2, 3, 4\}. \tag{8.6.4}$$

These are enough relations to conclude that the entries of Q commute with each other. Hence the quantum symmetry group of Potts model on (V, E) is commutative. □

8.6.2 Example 2

We observe a phenomena similar to that we observed in example 1 but in the context of multigraphs. Let us consider the multigraph (V, E, j) given in figure 8.2. The vertex set V consists of 5 elements and edge set E has 10 directed edges where oppositely directed edges are identified via the inversion map j to produce an undirected edge. Let us denote the edges emitting from 1 by $\{\sigma_2, \sigma_2', \sigma_3, \sigma_4, \sigma_5\}$ where σ_2 and σ_2' are edges from 1 to 2 and $\sigma_3, \sigma_4, \sigma_5$ are edges from 1 to $3, 4$ and 5 respectively. Let us consider $\lambda_2, \lambda_2' \in \mathbb{C}$. We consider a Hamiltonian

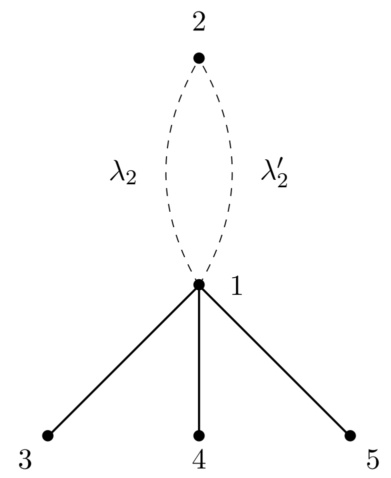

FIGURE 8.2: The multigraph in example 2 with the corresponding Hamiltonian.

H_P on (V, E, j) by

$$H_P(\omega) := \sum_{\tau \in E} J_\tau \delta_{\omega(s(\tau)),\omega(t(\tau))} \qquad \text{for all} \quad \omega \in \Omega_P$$

where $J_{\sigma_2} = \lambda_2 = J_{\overline{\sigma_2}}$, $J_{\sigma_2'} = \lambda_2' = J_{\overline{\sigma_2'}}$ and $J_\tau = 1$ otherwise. Using theorem 8.5.2 it is easy to see that when $\lambda_2 + \lambda_2' = 1$, the quantum symmetry group of Potts model on (V, E, j) is S_4^+ which signifies that the system has quantum symmetry. Any slight fluctuation in the parameters λ_1 and λ_2 such that $\lambda_1 + \lambda_2 \neq 1$, the quantum symmetry group for Potts model becomes S_3^+ and the system loses its quantum symmetry completely.

8.6.3 Example 3

We look at an example of Potts model where slight fluctuation of Hamiltonian changes the quantum symmetry of the system but does not affect its classical symmetry.

We consider the graph (V, E) shown in figure (8.3). The vertex set $V = \{1, 2, 3, ..., 8\}$ has 8 elements and the edge set is as shown in the figure. To specify a Hamiltonian on (V, E) we simply specify the matrix $(A_l^k)_{k,l \in V}$. Let us consider $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $|\lambda_1| \leq 1$ and $|\lambda_2| \leq 1$. We consider the following Hamiltonian H_P on (V, E) given by

$$H_P(\omega) = \sum_{k,l \in V} A_l^k \delta_{\omega(k),\omega(l)} \quad \forall \quad \omega \in \Omega_P$$

where $(A_l^k)_{k,l \in V}$ is a symmetric matrix and $A_7^1 = A_8^1 = A_5^2 = A_6^2 = A_8^7 = \lambda_1$, $A_6^5 = \lambda_2$ and $A_l^k = 1$ otherwise. The underlying simple graph (V, E) (that is, when $\lambda_1 = \lambda_2 = 1$) does not have any quantum symmetry (see section (4.4) of [Sch20b]). In light of theorem 2.5.9, we observe that,

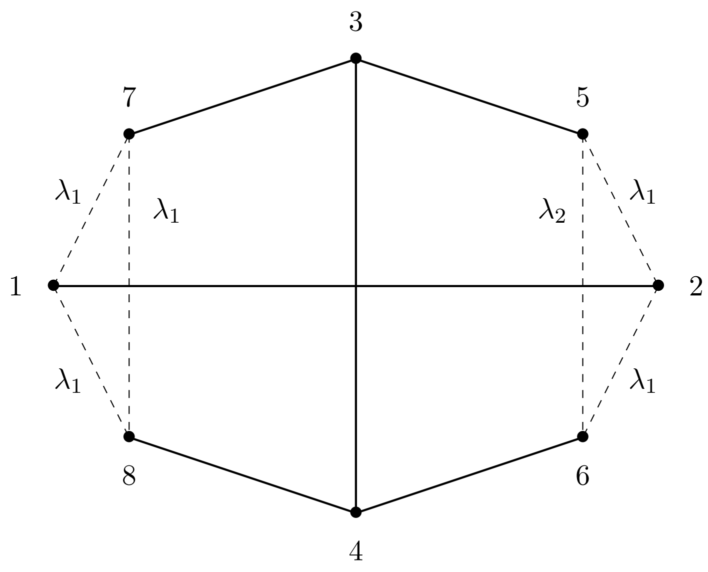

FIGURE 8.3: The graph in example 3 with the corresponding Hamiltonian.

Remark 8.6.3. *When $\lambda_1 \neq 0$ and $\lambda_1 = \lambda_2$, the quantum symmetry group of Potts model on (V, E) is $\mathbb{C}(\mathbb{Z}_2) \otimes \mathbb{C}(\mathbb{Z}_2)$.*

Lemma 8.6.4. *When $\lambda_1 = 0$ and $\lambda_1 \neq \lambda_2$, the quantum symmetry group of Potts model on (V, E) becomes $\mathbb{C}(\mathbb{Z}_2) * \mathbb{C}(\mathbb{Z}_2)$.*

Proof. Let $(\mathcal{A}, \Delta)$ be the quantum symmetry group for Potts model on (V, E) co-acting on $C(V)$ via α preserving the Hamiltonian H_P. As before, from theorem 8.5.2 it follows that the co-representation matrix Q of α commutes with the matrix $A = (A^k_l)_{k,l \in V}$.

From proposition (2.5.10) it follows that,

$$\begin{aligned} q^3_l = q^4_l &= 0 \quad \text{for} \quad l \neq 3, 4, \\ q^1_l = q^2_l &= 0 \quad \text{for} \quad l \in \{3, 4, 5, 6\}, \\ q^7_l = q^8_l &= 0 \quad \text{for} \quad l \in \{3, 4, 5, 6\}. \end{aligned} \tag{8.6.5}$$

We observe that,

$$\begin{aligned} q^3_4 = (AQ)^7_4 &= (QA)^7_4 = q^7_8 \\ q^3_4 = (AQ)^5_4 &= (QA)^5_4 = q^5_6 \\ 0 = (AQ)^1_3 &= (QA)^1_3 = q^1_7 \\ 0 = (AQ)^1_4 &= (QA)^1_4 = q^1_8 \\ 0 = (AQ)^2_3 &= (QA)^2_3 = q^2_7 \\ 0 = (AQ)^2_4 &= (QA)^2_4 = q^2_8. \end{aligned} \tag{8.6.6}$$

In light of equations (8.6.5) and (8.6.6), the co-representation matrix Q becomes,

$$\begin{bmatrix} 1-p & p & 0 & 0 & 0 & 0 & 0 & 0 \\ p & 1-p & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-q & q & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 1-q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-q & q & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 1-q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-q & q \\ 0 & 0 & 0 & 0 & 0 & 0 & q & 1-q \end{bmatrix}$$

where two projections p and q are free. Hence we conclude that the quantum symmetry group of Potts model on (V, E) is $\mathbb{C}(\mathbb{Z}_2) * \mathbb{C}(\mathbb{Z}_2)$. □

Remark 8.6.5. *For $\lambda_1 = 0$ and $\lambda_1 \neq \lambda_2$, the quantum symmetry group for Potts model is $\mathbb{C}(\mathbb{Z}_2) * \mathbb{C}(\mathbb{Z}_2)$ which is indeed a non-classical comapct quantum group. On the other hand, the classical symmetry group for Potts model is $\mathbb{C}(\mathbb{Z}_2) \otimes \mathbb{C}(\mathbb{Z}_2)$, which is same as the case when $\lambda_1 \neq 0$ and $\lambda_1 = \lambda_2$. Hence we observe that slight changing the parameters λ_1 and λ_2 in certain manner keeps the classical symmetry same but affects quantum symmetry drastically.*

We end this thesis with the following question:

Question

For an arbitrary directed multigraph (V, E), does the category $\mathcal{C}^{sym}_{(V,E)}$ always admit a universal object? If the answer is negative, then are the categories $\mathcal{C}^{s,t}_{(V,E)}$'s, for different representations of (V, E), the largest subcategories of **non-Bichon** type in $\mathcal{C}^{sym}_{(V,E)}$ where universal objects exist and are algebraically describable?

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