Commutant Lifting, Interpolation And Toeplitz Operators In Several Variables

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Dedicated to my Mother, Father, Brother and wife.

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Contents

Notations & Abbreviations

$\mathbb{N}$	Set of all Natural numbers.
$\mathbb{Z}_+$	$\mathbb{N} \cup \{0\}$.
$\mathbb{N}^n$	$\{\boldsymbol{k} = (k_1, \ldots, k_n) : k_i \in \mathbb{N}, i = 1, \ldots, n\}$.
$\mathbb{Z}_+^n$	$\{\boldsymbol{t} = (t_1, \ldots, t_n) : t_i \in \mathbb{Z}_+, i = 1, \ldots, n\}$.
$\boldsymbol{z}$	$(z_1, \ldots, z_n) \in \mathbb{C}^n$.
$\boldsymbol{z}^{\boldsymbol{k}}$	$z_1^{k_1} \ldots z_n^{k_n}$.
$\lvert\boldsymbol{k}\rvert$	$k_1 + \ldots + k_n$.
$(T_1, \ldots, T_n)$	n-tuple of commuting operators on Hilbert spaces.
$T^{\boldsymbol{k}}$	$T_1^{k_1} \ldots T_n^{k_n}$.
$\mathbb{D}^n$	$\{\boldsymbol{z} : \lvert z_i\rvert < 1, i = 1, \ldots, n\}$.
$\mathbb{B}^n$	$\{\boldsymbol{z} : \sum_{i=1}^n \lvert z_i\rvert^2 < 1\}$.
$\mathcal{E}, \mathcal{E}_*$	Hilbert spaces.
$\mathcal{O}(\Omega, \mathcal{E})$	The set of all holomorphic functions on $\Omega \subseteq \mathbb{C}^n$ to $\mathcal{E}$.
$\mathcal{O}(\mathbb{B}^n, \mathcal{B}(\mathcal{E}, \mathcal{E}_*))$	The set of all $\mathcal{B}(\mathcal{E}, \mathcal{E}_*)$-valued holomorphic functions on $\mathbb{B}^n$.
$A(\mathbb{B}^n)$	Ball algebra.
$H^\infty(\mathbb{D}^n)$	The set of all bounded analytic functions on $\mathbb{D}^n$.

Introduction

The purpose of this thesis is to examine some classical one variable Hilbert function space theoretic results in the context of several complex variables and commuting tuples of bounded linear operators on Hilbert spaces. More specifically, we will be interested in the classical Sarason's commutant lifting theorem on $\mathbb{D}$, where

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

the open unit disc in $\mathbb{C}$. A significant part of our discussion in this thesis will revolve around the commutant lifting theorem in two different contexts, as well as its following applications of independent importance. Another important object of study will be Toeplitz operators on the polydisc $\mathbb{D}^n$, $n \geq 1$.

It is worth noting that the operator theory, in terms of complexity and known as well as unknown, is different for commuting tuples of contractions and commuting tuples of row contractions, just like the theory of analytic functions differs from the open unit ball to the open unit polydisc. From this perspective, we talk about the commutant lifting theorem in the context of the open unit ball and the polydisc. As we will see in this thesis, the latter scenario seems to be more interesting and challenging.

The main contributions of this thesis are:

1. *Partially isometric Toeplitz operators on the polydisc:* We prove that a Toeplitz operator T_φ, $\varphi \in L^\infty(\mathbb{T}^n)$, is a partial isometry if and only if there exist inner functions $\varphi_1, \varphi_2 \in H^\infty(\mathbb{D}^n)$ such that φ_1 and φ_2 depend on different variables and $\varphi = \bar{\varphi}_1 \varphi_2$. In particular, for $n = 1$, along with new proof, this recovers a classical theorem of Brown and Douglas. We also prove that a partially isometric Toeplitz operator is hyponormal if and only if the corresponding symbol is an inner function in $H^\infty(\mathbb{D}^n)$. Moreover, partially isometric Toeplitz operators are always power partial isometry (following Halmos and Wallen), and hence, up to unitary equivalence, a partially isometric Toeplitz operator with a symbol in $L^\infty(\mathbb{T}^n)$, $n > 1$, is either a shift, or a co-shift, or a direct sum of truncated shifts. Along the way, we prove that T_φ is a shift whenever φ is inner in $H^\infty(\mathbb{D}^n)$.

2. *Commutant lifting and Nevanlinna-Pick interpolation on the polydisc:* The fundamental theorem on commutant lifting due to Sarason does not carry over to the

setting of the polydisc. This chapter presents two classifications of commutant lifting in several variables. The first classification links the lifting problem to the contractivity of certain linear functionals. The second one transforms it into non negative real numbers. We also solve the Nevanlinna-Pick interpolation problem for bounded analytic functions on the polydisc. Commutant lifting and interpolation on the polydisc solve two well-known problems in Hilbert function space theory.

3. *Perturbations of analytic functions on the polydisc:* In the context of Schur functions on $\mathbb{D}^n$, we solve a perturbation problem.

4. *Commutant lifting and Nevanlinna-Pick interpolation on the ball:* We prove a commutant lifting theorem and a Nevanlinna-Pick type interpolation result in the setting of multipliers from vector-valued Drury-Arveson space to a large class of vector-valued reproducing kernel Hilbert spaces over the unit ball. The special case of reproducing kernel Hilbert spaces includes all natural examples of Hilbert spaces like Hardy space, Bergman space and weighted Bergman spaces over the unit ball.

Let us now elaborate on the preceding content chapter by chapter.

Chapter 1: Partially isometric Toeplitz operators on the polydisc.

Toeplitz operators are one of the most useful and prevalent objects in matrix theory, operator theory, operator algebras, and its related fields. For instance, Toeplitz operators provide some of the most important links between index theory, C^*-algebras, function theory, and non-commutative geometry. See the monograph by Higson and Roe [62] for a thorough presentation of these connections, and consult the paper by Axler [14] for a rapid introduction to Toeplitz operators.

Evidently, a lot of work has been done in the development of one variable Toeplitz operators, and it is still a subject of very active research, with an ever-increasing list of connections and applications. But on the other hand, many questions remain to be settled in the several variables case, and more specifically in the open unit polydisc case (however, see [42, 43, 56, 72, 99]). The difficulty lies in the obvious fact that the standard (and classical) single variable tools are either unavailable or not well developed in the setting of polydisc. Evidently, advances in Toeplitz operators on the polydisc have frequently resulted in a number of new tools and techniques in operator theory, operator algebras, and related fields.

Our objective of this chapter is to address the following basic question: Characterize partially isometric Toeplitz operators on $H^2(\mathbb{D}^n)$, where $H^2(\mathbb{D}^n)$ denotes the Hardy space over the unit polydisc $\mathbb{D}^n$. Recall that a partial isometry [58] is a bounded linear operator whose restriction to the orthogonal complement of its null space is an isometry.

Before we answer the above question, we first recall that $H^2(\mathbb{D}^n)$ is the Hilbert space of all analytic functions f on $\mathbb{D}^n$ such that

$$\|f\| := \Big(\sup_{0\le r<1} \int_{\mathbb{T}^n} |f(rz_1,\dots,rz_n)|^2 d\mu(z)\Big)^{\frac{1}{2}} < \infty,$$

where $d\mu$ is the normalized Lebesgue measure on the n-torus $\mathbb{T}^n$, and $z = (z_1,\dots,z_n)$. We denote by $L^2(\mathbb{T}^n)$ the Hilbert space $L^2(\mathbb{T}^n, d\mu)$. From the radial limits of square summable analytic functions point of view [87], one can identify $H^2(\mathbb{D}^n)$ with the closed subspace $H^2(\mathbb{T}^n)$ of $L^2(\mathbb{T}^n)$. Let $L^\infty(\mathbb{T}^n)$ denote the standard C^*-algebra of $\mathbb{C}$-valued essentially bounded Lebesgue measurable functions on $\mathbb{T}^n$. The *Toeplitz operator* T_φ with symbol $\varphi \in L^\infty(\mathbb{T}^n)$ is defined by

$$T_\varphi f = P_{H^2(\mathbb{D}^n)}(\varphi f) \qquad (f \in H^2(\mathbb{D}^n)),$$

where $P_{H^2(\mathbb{D}^n)}$ denotes the orthogonal projection from $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{D}^n)$. Also recall that

$$H^\infty(\mathbb{D}^n) = L^\infty(\mathbb{T}^n) \cap H^2(\mathbb{D}^n),$$

where $H^\infty(\mathbb{D}^n)$ denotes the Banach algebra of all bounded analytic functions on $\mathbb{D}^n$. A function $\varphi \in H^\infty(\mathbb{D}^n)$ is called *inner* if φ is unimodular on $\mathbb{T}^n$.

The answer to the above question is contained in the following theorem:

Theorem 0.0.1. *Let φ be a nonzero function in $L^\infty(\mathbb{T}^n)$. Then T_φ is a partial isometry if and only if there exist inner functions $\varphi_1, \varphi_2 \in H^\infty(\mathbb{D}^n)$ such that φ_1 and φ_2 depend on different variables and*

$$T_\varphi = T_{\varphi_1}^* T_{\varphi_2}.$$

In particular, if $n = 1$, then the only nonzero Toeplitz operators that are partial isometries are those of the form T_φ and T_φ^*, where $\varphi \in H^\infty(\mathbb{D})$ is an inner function. This was proved by Brown and Douglas in [27]. Actually, as we will see soon in this case that T_φ is not only an isometry but a shift.

A key ingredient in the proof of the Brown and Douglas theorem is the classical Beurling theorem [23]. Recall that the Beurling theorem connects inner functions in $H^\infty(\mathbb{D})$ with shift invariant subspaces of $H^2(\mathbb{D})$. However, in the present case of higher dimensions, this approach does not work, as is well known, Beurling type classification does not hold for shift invariant subspaces of $H^2(\mathbb{D}^n)$, $n > 1$. Here, we exploit more analytic and geometric structures of $H^2(\mathbb{D}^n)$ and $L^2(\mathbb{T}^n)$ to achieve the main goal.

Along the way, we prove some basic properties of Toeplitz operators on the polydisc. Some of these observations are perhaps known (if not readily available in the literature) to experts, but they are necessary for our purposes here. We also remark that our proof of

$$\|T_\varphi\| = \|\varphi\|_\infty \qquad (\varphi \in L^\infty(\mathbb{T}^n)), \tag{0.0.1}$$

seems to be different even in the case of $n = 1$, as it avoids the standard techniques of the spectral radius formula (see Brown and Halmos [26, page 99] and the monographs [45, 68, 78]).

Moreover, we prove the following result, which connects inner functions with shift operators, and is also of independent interest: *If $\varphi \in H^\infty(\mathbb{D}^n)$ is a nonconstant inner function, then M_φ is a shift.*

Here, and in what follows, M_φ denotes the analytic Toeplitz operator T_φ whenever $\varphi \in H^\infty(\mathbb{D}^n)$. In this case, M_φ is simply the standard multiplication operator on $H^2(\mathbb{D}^n)$, that is, $M_\varphi f = \varphi f$ for all $f \in H^2(\mathbb{D}^n)$.

As a first application to Theorem 0.0.1, we classify partially isometric hyponormal Toeplitz operators. Recall that a bounded linear operator T on some Hilbert space is called hyponormal if

$$T^*T - TT^* \geq 0.$$

We prove the following: If T_φ, $\varphi \in L^\infty(\mathbb{T}^n)$, is a partial isometry, then T_φ is hyponormal if and only if φ is an inner function in $H^\infty(\mathbb{D}^n)$.

Secondly, following the Halmos and Wallen [59] notion of power partial isometries (also see an Huef, Raeburn and Tolich [63]), we prove that partially isometric Toeplitz operators are always power partial isometry. We further exploit the Halmos and Wallen models of power partial isometries, and obtain a connection between partially isometric Toeplitz operators, shifts, co-shifts, and direct sums of truncated shifts.

Finally, collecting all these results together, from an operator theoretic point of view, we obtain the following refinement of Theorem 0.0.1: *Suppose T_φ, $\varphi \in L^\infty(\mathbb{T}^n)$, is partially isometric. Then, up to unitary equivalence, T_φ is either a shift, or a co-shift, or a direct sum of truncated shifts.*

We stress that the latter possibility is only restricted to the $n > 1$ case.

Chapter 2: Commutant lifting and Nevanlinna-Pick interpolation on the polydisc.

Sarason's commutant lifting theorem [91] is fundamental, with significant applications to virtually every aspect of Hilbert function space theory. One of them is the Nevanlinna-Pick interpolation theorem on $\mathbb{D}$, which we will quickly review before moving on to the lifting theorem. Given m distinct points $\mathcal{Z} = \{z_1, \ldots, z_m\} \subset \mathbb{D}$ (interpolation nodes) and m scalars $\mathcal{W} = \{w_1, \ldots, w_m\} \subset \mathbb{D}$ (target data), there exists $\varphi \in H^\infty(\mathbb{D})$ (interpolating function) such that

$$\|\varphi\|_\infty := \sup_{z \in \mathbb{D}} |\varphi(z)| \leq 1$$

and

$$\varphi(z_i) = w_i,$$

for all $i = 1, \ldots, m$, if and only if the $m \times m$ *Pick matrix* $\mathfrak{P}_{\mathcal{Z},\mathcal{W}}$ is positive semi-definite, where

$$\mathfrak{P}_{\mathcal{Z},\mathcal{W}} := \Big(\frac{1 - w_i \bar{w}_j}{1 - z_i \bar{z}_j}\Big)_{i,j=1}^m.$$

This was proved by G. Pick [85] more than a century ago. R. Nevanlinna [77] independently solved the same problem at a very similar time. The methods of Pick and Nevanlinna are different, interesting on their own, and still relevant. For instance, Pick focused on interpolation on the upper half-plane, whereas the Schur algorithm (see I. Schur [94, 95]) served as the driving force behind Nevanlinna's strategy [54, 80].

After four decades of Pick's paper, D. Sarason [91] provided a solid Hilbert function space theoretical foundation for Nevanlinna and Pick's analytic and algebraic methods for the solution of the interpolation problem. Sarason's elegant result, known as the *commutant lifting theorem*, represents the commutant of model operators in terms of nicer operators (say Toeplitz operators) without changing the norms. To be more specific, let us identify the class of functions of interest. We denote the closed unit ball of $H^\infty(\mathbb{D}^n)$ by

$$\mathcal{S}(\mathbb{D}^n) = \{\varphi \in H^\infty(\mathbb{D}^n) : \|\varphi\|_\infty \leq 1\}.$$

The members of $\mathcal{S}(\mathbb{D}^n)$ are known as *Schur functions*. Recall that the *analytic Toeplitz operator* T_φ on $H^2(\mathbb{T}^n)$, $\varphi \in H^\infty(\mathbb{D}^n)$, is defined by

$$T_\varphi f = \varphi f,$$

for all $f \in H^2(\mathbb{T}^n)$. In particular, for $\varphi = z_i$, we get T_{z_i}, the multiplication operator by coordinate function z_i on $H^2(\mathbb{T}^n)$, $i = 1, \ldots, n$. The following equality describes how the commutant of $\{T_{z_i}\}_{i=1}^n$ connects the Banach algebra $H^\infty(\mathbb{D}^n)$ to $\mathcal{B}(H^2(\mathbb{T}^n))$:

$$\{T_{z_1}, \ldots, T_{z_n}\}' = \{T_\varphi : \varphi \in H^\infty(\mathbb{D}^n)\}.$$

Moreover, we know that (see (0.0.1))

$$\|T_\varphi\| = \|\varphi\|_\infty \qquad (\varphi \in H^\infty(\mathbb{D}^n)).$$

We now return to the classical case where $n = 1$. Let $\mathcal{Q}$ be a T_z^*-invariant closed subspace of $H^2(\mathbb{T})$, and let X be a bounded linear operator on $\mathcal{Q}$ (in short, $X \in \mathcal{B}(\mathcal{Q})$). Sarason's commutant lifting theorem states the following: Suppose X commutes with the *model operator* $P_{\mathcal{Q}} T_z|_{\mathcal{Q}} \in \mathcal{B}(\mathcal{Q})$, that is

$$X(P_{\mathcal{Q}} T_z|_{\mathcal{Q}}) = (P_{\mathcal{Q}} T_z|_{\mathcal{Q}})X.$$

Then there exists $\varphi \in H^\infty(\mathbb{D})$ such that

$$X = P_{\mathcal{Q}} T_\varphi|_{\mathcal{Q}},$$

and

$$\|X\| = \|\varphi\|_\infty.$$

Here (and in what follows) $P_\mathcal{Q}$ denotes the orthogonal projection from $H^2(\mathbb{T})$ onto $\mathcal{Q}$. In other words, along with $\|X\| = \|T_\varphi\|$, the following diagram commutes:

$$\begin{tikzcd}
H^2(\mathbb{T}) \arrow[r, "T_\varphi"] & H^2(\mathbb{T}) \arrow[d, "P_\mathcal{Q}"] \\
\mathcal{Q} \arrow[u, "i_\mathcal{Q}"] \arrow[r, "X"'] & \mathcal{Q}
\end{tikzcd}$$

where $i_\mathcal{Q} : \mathcal{Q} \hookrightarrow H^2(\mathbb{T})$ denotes the inclusion map. The Nevanlinna-Pick interpolation theorem then easily follows from this applied to zero-based finite-dimensional T_z^*-invariant subspaces of $H^2(\mathbb{T})$. The most important aspect of Sarason's lifting theorem, however, is the lifting of the commutant of model operators to the commutant of T_z keeping the norms the same.

We remind the reader that Sarason's commutant lifting theorem has a stellar reputation in its application to the classical operator and function theoretic results like the Carathéodory-Fejér interpolation problem, Nehari interpolation problem, von Neumann inequality, isometric dilations, and the Ando dilation, just to name a few. The expanded list easily includes control theory and electrical engineering [51, 61]. When dealing with several variables, however, each analogue question poses a unique set of challenges and frequently offers less opportunity for a comprehensive theory (however, see [8, 20, 21, 39, 44, 55]). In fact, it is known that Sarason's commutant lifting theorem does not hold true in general in the setting of $\mathbb{D}^n$. Understanding the obstacle of commutant lifting over $\mathbb{D}^n$ is thus one of the most important problems in Hilbert function space theory.

In this chapter, we solve the commutant lifting problem on $H^2(\mathbb{T}^n)$, $n \geq 1$. That is, given a closed subspace $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ that is invariant under $T_{z_i}^*$, $i = 1, \ldots, n$, we classify contractions $X \in \mathcal{B}(\mathcal{Q})$ satisfying the condition that

$$X(P_\mathcal{Q} T_{z_i}|_\mathcal{Q}) = (P_\mathcal{Q} T_{z_i}|_\mathcal{Q})X \qquad (i = 1, \ldots, n),$$

so that the following diagram commutes

$$\begin{tikzcd}
H^2(\mathbb{T}^n) \arrow[r, dotted, "T_\varphi"] & H^2(\mathbb{T}^n) \arrow[d, "P_\mathcal{Q}"] \\
\mathcal{Q} \arrow[u, "i_\mathcal{Q}"] \arrow[r, "X"'] & \mathcal{Q}
\end{tikzcd}$$

for some $\varphi \in \mathcal{S}(\mathbb{D}^n)$. Several attempts have been made to solve this problem, but they appear to be quite abstract and only applicable to a smaller class of operators (or functions). The most notable one is perhaps the work of Ball, Li, Timotin, and Trent [20]. The class of functions considered in [20] is the so-called Schur-Agler class functions. This class is significantly smaller than even the polydisc algebra when $n > 2$, and it is the same as the Schur class when $n = 2$. Even in the $n = 2$ case, however, the existing results are abstract. In the context of interpolation for the $n = 2$ case, we refer the reader to the seminal papers by Agler [5, 6] (also, see the discussion following Theorem 0.0.5).

Our approach and solution to the commutant lifting problem are both concrete and function-theoretic. As part of the application, we moreover solve the interpolation problem for Schur functions on $\mathbb{D}^n$. In the context of Schur functions on $\mathbb{D}^n$, we also solve a perturbation problem. Like our commutant lifting theorem, all results are concrete and quantify the complexity of the problem by nonnegative real numbers.

Now we provide a more thorough summary of this chapter's key contribution. Unless otherwise specified, we will always assume that $n \geq 1$ is a natural number. Given a Hilbert space $\mathcal{H}$, set

$$\mathcal{B}_1(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \|T\| \leq 1\}.$$

Given a nonempty subset $S \subseteq H^2(\mathbb{T}^n)$, we define the conjugate space S^{conj} as

$$S^{conj} = \{\bar{f} : f \in S\}.$$

Let $\mathcal{S} \subseteq H^2(\mathbb{T}^n)$ be a closed subspace. We say that $\mathcal{S}$ is a *shift invariant subspace* (or submodule) if

$$z_i\mathcal{S} \subseteq \mathcal{S},$$

for all $i = 1, \ldots, n$. We say that $\mathcal{S}$ is a *backward shift invariant subspace* (or quotient module) if $\mathcal{S}^\perp$ is a shift invariant subspace, or equivalently,

$$T^*_{z_i}\mathcal{S} \subseteq \mathcal{S},$$

for all $i = 1, \ldots, n$. Given a backward shift invariant subspace $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$, we define the *model operator* S_{z_i}, for each $i = 1, \ldots, n$, by

$$S_{z_i} = P_{\mathcal{Q}} T_{z_i}|_{\mathcal{Q}}.$$

Now we define lifting on backward shift invariant subspaces.

Definition 0.0.2. *Let $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ be a backward shift invariant subspace, $X \in \mathcal{B}_1(\mathcal{Q})$, and suppose $XS_{z_i} = S_{z_i}X$ for all $i = 1, \ldots, n$. If there exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that*

$$X = P_{\mathcal{Q}} T_{\varphi}|_{\mathcal{Q}},$$

then X is said to have a lift, or to be liftable.

We need to familiarise ourselves with a few more additional concepts. First, we define the closed subspace of "mixed functions" of $L^2(\mathbb{T}^n)$ as

$$\mathcal{M}_n = L^2(\mathbb{T}^n) \ominus (H^2(\mathbb{T}^n)^{conj} + H^2(\mathbb{T}^n)).$$

This space has a significant role to perform in the entire paper. It is crucial to observe that $\mathcal{M}_n \cap H^2(\mathbb{T}^n) = \{0\}$, and

$$\mathcal{M}_1 = \{0\}.$$

Let $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ be a backward shift invariant subspace. Set

$$\mathcal{M}_\mathcal{Q} = \mathcal{Q}^{conj} \dotplus (\mathcal{M}_n \dotplus H_0^2(\mathbb{T}^n)), \tag{0.0.2}$$

where

$$H_0^2(\mathbb{T}^n) = H^2(\mathbb{T}^n) \ominus \{1\},$$

the closed subspace of $H^2(\mathbb{T}^n)$ of functions vanishing at the origin. Note that $\dotplus$ signifies the skew sum of Banach spaces. In what follows, we treat $\mathcal{M}_\mathcal{Q}$ as a subspace of the classical Banach space $L^1(\mathbb{T}^n)$, and denote it by $(\mathcal{M}_\mathcal{Q}, \|\cdot\|_1)$. In other words

$$(\mathcal{M}_\mathcal{Q}, \|\cdot\|_1) \subset (L^1(\mathbb{T}^n), \|\cdot\|_1).$$

Let $X \in \mathcal{B}(\mathcal{Q})$, and suppose

$$\psi = X(P_\mathcal{Q} 1). \tag{0.0.3}$$

Define a functional $X_\mathcal{Q} : (\mathcal{M}_\mathcal{Q}, \|\cdot\|_1) \longrightarrow \mathbb{C}$ by

$$X_\mathcal{Q} f = \int_{\mathbb{T}^n} \psi f \, d\mu \qquad (f \in \mathcal{M}_\mathcal{Q}).$$

Recall that $d\mu$ is the normalized Lebesgue measure on $\mathbb{T}^n$. Finally, set

$$\tilde{\mathcal{M}}_{\mathcal{Q},X} = (\mathcal{Q}^{conj} \ominus \{\bar{\psi}\}) \dotplus (\mathcal{M}_n \dotplus H_0^2(\mathbb{T}^n)),$$

and again treat it as a subspace of $L^1(\mathbb{T}^n)$:

$$(\tilde{\mathcal{M}}_{\mathcal{Q},X}, \|\cdot\|_1) \subset (L^1(\mathbb{T}^n), \|\cdot\|_1).$$

Now that we have these notations, we can say how the lifting of commutants in higher dimensions is classified:

Theorem 0.0.3. *Let $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ be a backward shift invariant subspace and let $X \in \mathcal{B}(\mathcal{Q})$ be a contraction. Suppose $XS_{z_i} = S_{z_i}X$ for all $i = 1, \ldots, n$. The following conditions are equivalent:*

1. *X admits a lift.*

2. *$X_{\mathcal{Q}} : (\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1) \longrightarrow \mathbb{C}$ is a contractive functional, where*

$$X_{\mathcal{Q}} f = \int_{\mathbb{T}^n} \psi f \, d\mu \qquad (f \in \mathcal{M}_{\mathcal{Q}}).$$

3. *$dist_{L^1(\mathbb{T}^n)}\left(\frac{\bar{\psi}}{\|\psi\|_2^2}, \tilde{\mathcal{M}}_{\mathcal{Q},X}\right) \geq 1$.*

This solves the long-standing commutant lifting problem for $H^2(\mathbb{T}^n)$, $n > 1$. We believe that the technique used to prove our lifting theorem is interesting on its own.

Now we will explain the solution to the interpolation problem, which also resolves the long-standing question on interpolation with Schur functions as interpolating functions on $\mathbb{D}^n$, $n > 1$. We will start by laying the groundwork. Recall that $H^2(\mathbb{T}^n)$ is a reproducing kernel Hilbert space corresponding to the *Szegö kernel* $\mathbb{S} : \mathbb{D}^n \times \mathbb{D}^n \to \mathbb{C}$ (see the monograph [84] for more details), where

$$\mathbb{S}(z, w) = \prod_{i=1}^{n} \frac{1}{1 - z_i \bar{w}_i} \qquad (z, w \in \mathbb{D}^n).$$

For each $w \in \mathbb{D}^n$, define $\mathbb{S}(\cdot, w) : \mathbb{D}^n \to \mathbb{C}$ by $(\mathbb{S}(\cdot, w))(z) = \mathbb{S}(z, w)$ for all $z \in \mathbb{D}^n$. In view of the standard reproducing kernel property, it follows that $\{\mathbb{S}(\cdot, w) : w \in \mathbb{D}^n\} \subseteq H^2(\mathbb{T}^n)$ is a set of linearly independent functions, and

$$\mathbb{S}(z, w) = \langle \mathbb{S}(\cdot, w), \mathbb{S}(\cdot, z) \rangle_{H^2(\mathbb{T}^n)},$$

for all $z, w \in \mathbb{D}^n$. Given a set of distinct points $\mathcal{Z} = \{z_1, \ldots, z_m\} \subset \mathbb{D}^n$, we define an m-dimensional subspace of $H^2(\mathbb{T}^n)$ as

$$\mathcal{Q}_{\mathcal{Z}} = \text{span}\{\mathbb{S}(\cdot, z_j) : j = 1, \ldots, m\}.$$

It follows that $\mathcal{Q}_{\mathcal{Z}}$ is a backward shift invariant subspace of $H^2(\mathbb{T}^n)$. Define

$$\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}} = \mathcal{Q}_{\mathcal{Z}}^{conj} \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)).$$

In addition, given a set of scalars $\{w_i\}_{i=1}^m \subset \mathbb{D}$, define $X_{\mathcal{Z},\mathcal{W}} \in \mathcal{B}(\mathcal{Q}_{\mathcal{Z}})$ by

$$X_{\mathcal{Z},\mathcal{W}}^* \mathbb{S}(\cdot, z_j) = \bar{w}_j \mathbb{S}(\cdot, z_j) \qquad (j = 1, \ldots, m).$$

The fact that $X_{\mathcal{Z},\mathcal{W}}$ on $\mathcal{Q}_{\mathcal{Z}}$ is a natural operator and that it meets the crucial condition that $X_{\mathcal{Z},\mathcal{W}} S_{z_i} = S_{z_i} X_{\mathcal{Z},\mathcal{W}}$, $i = 1, \ldots, m$, is noteworthy.

Here is a summary of our main interpolation results:

Theorem 0.0.4. *Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{D}^n$ be m distinct points, and let $\mathcal{W} = \{w_i\}_{i=1}^m \subset \mathbb{D}$ be m scalars. The following conditions are equivalent:*

1. *There exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that $\varphi(z_i) = w_i$ for all $i = 1, \ldots, m$.*

2. *$M_{\mathcal{Z},\mathcal{W}} : (\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}, \|\cdot\|_1) \to \mathbb{C}$ is a contraction, where*

$$M_{\mathcal{Z},\mathcal{W}} f = \int_{\mathbb{T}^n} \psi_{\mathcal{Z},\mathcal{W}} f \, d\mu,$$

for all $f \in \mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}$, and

$$\psi_{\mathcal{Z},\mathcal{W}} = \sum_{i=1}^{m} c_i \mathbb{S}(\cdot, z_i),$$

and the scalar coefficients $\{c_i\}_{i=1}^m$ are given by

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} \mathbb{S}(z_1,z_1) & \mathbb{S}(z_1,z_2) & \cdots & \mathbb{S}(z_1,z_m) \\ \mathbb{S}(z_2,z_1) & \mathbb{S}(z_2,z_2) & \cdots & \mathbb{S}(z_2,z_m) \\ \vdots & \ddots & \ddots & \vdots \\ \mathbb{S}(z_m,z_1) & \mathbb{S}(z_m,z_2) & \cdots & \mathbb{S}(z_m,z_m) \end{bmatrix}^{-1} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}.$$

3. *Let $\psi := X_{\mathcal{Z},\mathcal{W}}(P_{\mathcal{Q}_{\mathcal{Z}}} 1)$, and suppose*

$$\tilde{\mathcal{M}}_{\mathcal{Z},\mathcal{W}} := (\mathcal{Q}_{\mathcal{Z}}^{conj} \ominus \{\bar{\psi}\}) \dotplus (\mathcal{M}_n \dotplus H_0^2(\mathbb{T}^n)).$$

Then

$$dist_{L^1(\mathbb{T}^n)}\Big(\frac{\bar{\psi}}{\|\psi\|_2^2}, \tilde{\mathcal{M}}_{\mathcal{Q}_{\mathcal{Z}}}\Big) \geq 1.$$

Note that the matrix in part (2) of the above theorem is the inverse of the Gram matrix

$$\Big(\mathbb{S}(z_i, z_j)\Big)_{i,j=1}^m,$$

corresponding to the m Szegö kernel functions $\{\mathbb{S}(\cdot, z_i)\}_{i=1}^m$. Also, observe that part (3) provides a useful quantitative criterion to check interpolation on the polydisc. Indeed, as we will see, the quantitative criterion yields examples of interpolation on $\mathbb{D}^n$, $n \geq 2$. Notable is the fact that interpolating functions in this case are polynomials.

It is noteworthy that the answer to natural questions, as in Theorems 0.0.3, 0.0.6, and 0.0.4, has a connection to the set of nonnegative real numbers. This is a common and classical occurrence. The classical Nehari theorem [79], for example, establishes a direct link with such a distance function. Another instance is the celebrated Adamyan-Arov-Krein formulae [2, 3, 4].

We also recover Sarason's lifting theorem as an application to Theorem 0.0.3, resulting in yet another proof of the classical lifting theorem:

Theorem 0.0.5. *Let $\mathcal{Q} \subseteq H^2(\mathbb{T})$ be a backward shift invariant subspace, and let $X \in \mathcal{B}_1(\mathcal{Q})$. If*

$$XS_z = S_z X,$$

then X is liftable.

In the proof of the above theorem, $\mathcal{M}_{\mathcal{Q}}$ (defined as in (0.0.2)) admits a more compact form, namely

$$\mathcal{M}_{\mathcal{Q}} = \overline{\varphi}(zH^2(\mathbb{T})),$$

where $\varphi \in H^\infty(\mathbb{D})$ is an inner function (that is, $|\varphi| = 1$ on $\mathbb{T}$ a.e.) and $\mathcal{Q} = (\varphi H^2(\mathbb{T}))^\perp$. Moreover, we employ all the standard one variable types of machinery like the Beurling theorem, inner-outer factorizations [23, 54], etc. On the one hand, this is to be expected, given that Sarason uses similar tools for his lifting theorem. This, on the other hand, explains both the challenges associated with the commutant lifting theorem and the potential for extensions of relevant function theoretic results on the polydisc.

Chapter 3: Perturbations of analytic functions on the polydisc.

In this chapter, we solve a perturbation problem: Given a nonzero function $f \in H^2(\mathbb{T}^n)$, does there exist $g \in H^2(\mathbb{T}^n)$ such that

$$f + g \in \mathcal{S}(\mathbb{D}^n)?$$

Of course, to avoid triviality (that $g = -f$, for instance), we assume that $g \in \{f\}^\perp$. Set

$$\mathcal{L}_n = \mathcal{M}_n \oplus H^2_0(\mathbb{T}^n),$$

and treat it as a subspace of $L^1(\mathbb{T}^n)$. We present a complete solution to this problem as follows:

Theorem 0.0.6. *Let $f \in H^2(\mathbb{T}^n)$ be a nonzero function. Then there exists $g \in \{f\}^\perp$ such that*

$$f + g \in \mathcal{S}(\mathbb{D}^n),$$

if and only if

$$dist_{L^1(\mathbb{T}^n)}\Big(\frac{\bar{f}}{\|f\|_2^2}, \mathcal{L}_n\Big) \geq 1.$$

We will offer nontrivial examples to demonstrate the aforementioned result.

Chapter 4: Commutant lifting and Nevanlinna-Pick interpolation on the ball:

In this chapter, we make a contribution to a commutant lifting theorem and a version of Nevanlinna-Pick interpolation in the setting of the open unit ball $\mathbb{B}^n$, where

$$\mathbb{B}^n = \{\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{i=1}^n |z_i|^2 < 1\}.$$

To be more precise, let $m \geq 1$ and let $\mathcal{H}_m$ denote the reproducing kernel Hilbert space corresponding to the kernel k_m on $\mathbb{B}^n$, where

$$k_m(\boldsymbol{z}, \boldsymbol{w}) = (1 - \sum_{i=1}^n z_i \bar{w}_i)^{-m} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

Recall that $\mathcal{H}_m$ is the Drury-Arveson space (popularly denoted by H^2_n), the Hardy space, the Bergman space and the weighted Bergman space over $\mathbb{B}^n$ for $m = 1$, $m = n$, $m = n + 1$ and $m > n + 1$, respectively.

Our main results, restricted to $\mathcal{H}_m$, $m > 1$, can now be formulated as follows:

Commutant lifting theorem: Suppose $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are joint $(M_{z_1}, \ldots, M_{z_n})$ co-invariant subspaces of $H^2_n (= \mathcal{H}_1)$ and $\mathcal{H}_m$, respectively. Let $X \in \mathcal{B}(\mathcal{Q}_1, \mathcal{Q}_2)$ and $\|X\| \leq 1$. If

$$X(P_{\mathcal{Q}_1} M_{z_i}|_{\mathcal{Q}_1}) = (P_{\mathcal{Q}_2} M_{z_i}|_{\mathcal{Q}_2})X,$$

for all $i = 1, \ldots, n$, then there exists a holomorphic function $\varphi : \mathbb{B}^n \to \mathbb{C}$ such that the multiplication operator $M_\varphi \in \mathcal{B}(H^2_n, \mathcal{H}_m)$, $\|M_\varphi\| \leq 1$ (that is, φ is a *contractive multiplier*), and

$$X = P_{\mathcal{Q}_2} M_\varphi|_{\mathcal{Q}_1}.$$

Thus, we have the following commutative diagram:

$$\begin{tikzcd} H^2_n \arrow[r, dotted, "M_\varphi"] \arrow[d, "P_{\mathcal{Q}_1}"'] & \mathcal{H}_m \arrow[d, "P_{\mathcal{Q}_2}"] \\ \mathcal{Q}_1 \arrow[r, "X"'] & \mathcal{Q}_2 \end{tikzcd}$$

Given a closed subspace $\mathcal{S}$ of a Hilbert space $\mathcal{H}$ we denote by $P_{\mathcal{S}}$ the orthogonal projection of $\mathcal{S}$ on $\mathcal{H}$.

Nevanlinna–Pick interpolation theorem: Given distinct p points

$$\{\boldsymbol{z}_i\}_{i=1}^p \subseteq \mathbb{B}^n,$$

and n points

$$\{w_i\}_{i=1}^p \subseteq \mathbb{D},$$

there exists a contractive multiplier φ such that

$$\varphi(\boldsymbol{z}_i) = w_i,$$

for all $i = 1, \ldots, p$ if and only if the matrix

$$\left[\tfrac{1}{(1-\langle \boldsymbol{z}_i, \boldsymbol{z}_j \rangle)^m} - \tfrac{w_i \bar{w}_j}{1 - \langle \boldsymbol{z}_i, \boldsymbol{z}_j \rangle}\right]_{i,j=1}^p,$$

is positive semi-definite. Here $\langle \boldsymbol{z}, \boldsymbol{w} \rangle$, denotes the Euclidean inner product of $\boldsymbol{z}$ and $\boldsymbol{w}$ in $\mathbb{C}^n$.

We make strong use of the commutant lifting theorem in the setting of Drury-Arveson space and a refined factorization result concerning multipliers between Drury-Arveson space and a large class of analytic reproducing kernel Hilbert space over $\mathbb{B}^n$.

We point out that the above interpolation theorem, in the setting of normalized complete Pick kernel, is due to Aleman, Hartz, McCarthy and Richter [9]. Their proof relies on Leech's theorem (or Toeplitz corona theorem). From this point of view, in this paper we prove that the interpolation theorem is a consequence of the commutant lifting theorem. Furthermore, our interpolation result holds for operator-valued multipliers.

Note that there are also free noncommutative versions of interpolation theory (cf. [17]).

Chapter 1

Partially Isometric Toeplitz Operators On The Polydisc

1.1 Introduction

Our objective of this chapter is to address the following basic question: Characterize partially isometric Toeplitz operators on $H^2(\mathbb{D}^n)$, where $H^2(\mathbb{D}^n)$ denotes the Hardy space over the unit polydisc $\mathbb{D}^n$. Recall that a partial isometry [58] is a bounded linear operator whose restriction to the orthogonal complement of its null space is an isometry.

The answer to the above question is contained in the following theorem:

Theorem 1.1.1. *Let φ be a nonzero function in $L^\infty(\mathbb{T}^n)$. Then T_φ is a partial isometry if and only if there exist inner functions $\varphi_1, \varphi_2 \in H^\infty(\mathbb{D}^n)$ such that φ_1 and φ_2 depend on different variables and*

$$T_\varphi = T^*_{\varphi_1} T_{\varphi_2}.$$

In particular, if $n = 1$, then the only nonzero Toeplitz operators that are partial isometries are those of the form T_φ and T^*_φ, where $\varphi \in H^\infty(\mathbb{D})$ is an inner function. This was proved by Brown and Douglas in [27]. Actually, as we will see soon in this case that T_φ is not only an isometry but a shift.

Section 1.3 contains the proof of the above theorem. Along the way to the proof of Theorem 1.1.1, in Section 1.2 we prove some basic properties of Toeplitz operators on the polydisc. Some of these observations are perhaps known (if not readily available in the literature) to experts, but they are necessary for our purposes here. We also remark that the proof of $\|T_\varphi\| = \|\varphi\|_\infty$, $\varphi \in L^\infty(\mathbb{T}^n)$, in Proposition 1.2.2 seems to be different even in the case of $n = 1$, as it avoids the standard techniques of the spectral radius formula (see Brown and Halmos [26, page 99] and the monographs [45, 68, 78]).

Moreover, in Section 1.4, we prove the following result, which connects inner functions with shift operators, and is also of independent interest: *If $\varphi \in H^\infty(\mathbb{D}^n)$ is a nonconstant inner function, then M_φ is a shift.*

Here, and in what follows, M_φ denotes the analytic Toeplitz operator T_φ whenever $\varphi \in H^\infty(\mathbb{D}^n)$. In this case, M_φ is simply the standard multiplication operator on $H^2(\mathbb{D}^n)$, that is, $M_\varphi f = \varphi f$ for all $f \in H^2(\mathbb{D}^n)$.

In Section 1.5, as a first application to Theorem 1.1.1, we classify partially isometric hyponormal Toeplitz operators. Recall that a bounded linear operator T on some Hilbert space is called hyponormal if $T^*T - TT^* \geq 0$. In Corollary 1.5.1, we prove the following: If T_φ, $\varphi \in L^\infty(\mathbb{T}^n)$, is a partial isometry, then T_φ is hyponormal if and only if φ is an inner function in $H^\infty(\mathbb{D}^n)$.

Secondly, following the Halmos and Wallen [59] notion of power partial isometries (also see an Huef, Raeburn and Tolich [63]), in Corollary 1.5.2 we prove that partially isometric Toeplitz operators are always power partial isometry. In Theorem 1.5.3, we further exploit the Halmos and Wallen models of power partial isometries, and obtain a connection between partially isometric Toeplitz operators, shifts, co-shifts, and direct sums of truncated shifts.

Finally, collecting all these results together, from an operator theoretic point of view, we obtain the following refinement of Theorem 1.1.1:

Suppose T_φ, $\varphi \in L^\infty(\mathbb{T}^n)$, is partially isometric. Then, up to unitary equivalence, T_φ is either a shift, or a co-shift, or a direct sum of truncated shifts.

We stress that the latter possibility is only restricted to the $n > 1$ case.

1.2 Preparatory results

In this section, we develop the necessary tools leading to the proof of Theorem 1.1.1. In this respect, we again remark that in what follows, we will often identify (via radial limits) $H^2(\mathbb{D}^n)$ with $H^2(\mathbb{T}^n)$ without further explanation. Given $\varphi \in L^\infty(\mathbb{T}^n)$, we denote by L_φ the *Laurent operator* on $L^2(\mathbb{T}^n)$, that is, $L_\varphi f = \varphi f$ for all $f \in L^2(\mathbb{T}^n)$. Note that

$$\|L_\varphi\|_{\mathcal{B}(L^2(\mathbb{T}^n))} = \|\varphi\|_\infty,$$

where $\|\varphi\|_\infty$ denotes the essential supremum norm of φ. The Toeplitz operator T_φ with symbol $\varphi \in L^\infty(\mathbb{T}^n)$ is given by

$$T_\varphi = P_{H^2(\mathbb{D}^n)} L_\varphi|_{H^2(\mathbb{D}^n)}.$$

Clearly, $T_\varphi \in \mathcal{B}(H^2(\mathbb{D}^n))$. Also note that a function $f = \sum\limits_{k \in \mathbb{Z}^n} a_k z^k \in L^2(\mathbb{T}^n)$ is in $H^2(\mathbb{D}^n)$ if and only if $a_k = 0$ whenever at least one of the k_j, $j = 1, \ldots, n$, in $k = (k_1, \ldots, k_n)$ is negative. Recall that μ is the normalized Lebesgue measure on $\mathbb{T}^n$.

We start with several variables analogue of brother Riesz theorem. We denote the set of zeros of a scalar-valued function f by $\mathcal{Z}(f)$.

Lemma 1.2.1. *If $f \in H^2(\mathbb{D}^n)$ is nonzero, then $\mu(\mathcal{Z}(f)) = 0$.*

Proof. Let m denote the normalized Lebesgue measure on $\mathbb{T}$. Suppose f is a nonzero function in $H^2(\mathbb{D}^2)$. For w_1 and w_2 in $\mathbb{T}$ a.e., we define the slice functions f_{w_1} and f_{w_2} by $f_{w_1}(z) = f(w_1, z)$ and $f_{w_2}(z) = (z, w_2)$ for all $z \in \mathbb{T}$. Set

$$\mathcal{Z} = \{w_2 \in \mathbb{T} : f_{w_2} \equiv 0\}.$$

Note that $\mathcal{Z} \subseteq \mathcal{Z}(f_{w_1})$ for all $w_1 \in \mathbb{T}$. If $m(\mathcal{Z}) > 0$, then the classical brother Riesz theorem implies that f is identically zero. Therefore, $m(\mathcal{Z}) = 0$, where m is the normalized Lebesgue measure on $\mathbb{T}$. Evidently

$$m(\mathcal{Z}(f_{w_2})) = \begin{cases} 1 & \text{if } w_2 \in \mathcal{Z} \\ 0 & \text{if } w_2 \in \mathcal{Z}^c, \end{cases}$$

and hence $w_2 \mapsto m(\mathcal{Z}(f_{w_2}))$ is a measurable function. By the Tonelli and Fubini theorem, we see that

$$\begin{aligned} (m \times m)(\mathcal{Z}(f)) &= \int_{\mathbb{T}} m(\mathcal{Z}(f_{z_2}))\, dm(z_2) \\ &= \int_{\mathcal{Z}} m(\mathcal{Z}(f_{z_2}))\, dm(z_2) + \int_{\mathcal{Z}^c} m(\mathcal{Z}(f_{z_2}))\, dm(z_2) \\ &= 0. \end{aligned}$$

The rest of the proof now follows easily by the induction on n. $\square$

We refer to Rudin [87, Theorem 3.3.5] for a different proof of the above lemma (even in the context of functions in the Nevanlinna class). Also, see [101] for the same for functions in $H^\infty(\mathbb{D}^n)$. However, the present proof is direct and avoids the use of heavy machinery from function theory.

We now prove that $\|T_\varphi\|_{\mathcal{B}(H^2(\mathbb{D}^n))} = \|\varphi\|_\infty$. As we have pointed out already in the introductory section above, this may be known to experts. However, even when $n = 1$, the present proof seems to be direct as it avoids the standard techniques of the spectral radius formula. For instance, see the classic monograph [45, Corollary 7.8] and the recent monograph [68, Corollary 3.3.2].

Proposition 1.2.2. *$\|T_\varphi\| = \|\varphi\|_\infty$ for all $\varphi \in L^\infty(\mathbb{T}^n)$.*

Proof. Let $\mathcal{L}$ denote the set of Laurent polynomials in n variables. We compute

$$\begin{aligned} \|T_\varphi\| &= \sup\{|\langle \varphi f, g\rangle| : f, g \in H^2(\mathbb{D}^n), \|f\|, \|g\| \le 1\} \\ &= \sup\{|\langle \varphi f, g\rangle| : f, g \in \mathbb{C}[z_1, \dots, z_n], \|f\|, \|g\| \le 1\} \quad \text{(by density of polynomials)} \\ &= \sup\{|\langle \varphi f, g\rangle| : f, g \in \mathcal{L}, \|f\|, \|g\| \le 1\} \\ &= \|L_\varphi\| \\ &= \|\varphi\|_\infty. \end{aligned}$$

Note the third equality follows because any Laurent polynomial can be multiplied by a monomial to put it into polynomials. This completes the proof of the proposition. □

The above elegant proof is due to Professor Greg Knese and replaces our original proof, which was longer and technical.

Before proceeding to the proof of the main theorem, we conclude this section with a result concerning unimodular functions in $L^\infty(\mathbb{T}^n)$.

Corollary 1.2.3. *Suppose φ is a nonzero function in $L^\infty(\mathbb{T}^n)$. If $\|T_\varphi f\| = \|\varphi\|_\infty \|f\|$ for some nonzero $f \in H^2(\mathbb{D}^n)$, then $\frac{1}{\|\varphi\|_\infty}\varphi$ is unimodular in $L^\infty(\mathbb{T}^n)$.*

Proof. In view of Proposition 1.2.2, without loss of generality we may assume that $\|T_\varphi\| = 1$. Then

$$\int_{\mathbb{T}^n} |\varphi(z)|^2 |f(z)|^2 d\mu(z) = \int_{\mathbb{T}^n} |f(z)|^2 d\mu(z).$$

By Lemma 1.2.1, $|\varphi(z)| = 1$ for all $z \in \mathbb{T}^n$ a.e. and the result follows. □

In particular, if T_φ, $\varphi \in L^\infty(\mathbb{T}^n)$, is a partial isometry, then φ is unimodular.

1.3 Proof of Theorem 1.1.1

In this section, without explicitly mentioning it in each instance, we always assume that T_φ, $\varphi \in L^\infty(\mathbb{T}^n)$, is partially isometric. Also, we frequently make use of the identification $H^2(\mathbb{D}^n) \cong H^2(\mathbb{T}^n)$ without mentioning it (see Section 1.2).

For simplicity we denote by $\mathcal{R}(T)$ the range of a bounded linear operator T. Clearly, $\mathcal{R}(T_\varphi)$ is a closed subspace of $H^2(\mathbb{D}^n)$.

Lemma 1.3.1. *$\mathcal{R}(T_\varphi)$ is invariant under M_{z_i}, $i = 1, \ldots, n$.*

Proof. Note that, since $\|T_\varphi\| = 1$, we have $\|\varphi\|_\infty = 1$. Suppose $f \in \mathcal{R}(T_\varphi)$. By Corollary 1.2.3, it follows that φ is unimodular, and hence $\|L_{\bar{\varphi}} f\| = \|f\|$. Since T_φ^* is an isometry on $\mathcal{R}(T_\varphi)$, we have

$$\|f\| = \|T_\varphi^* f\| \leq \|L_{\bar{\varphi}} f\| = \|\bar{\varphi} f\| = \|f\|.$$

Therefore, $\|P_{H^2(\mathbb{D}^n)}(\bar{\varphi} f)\| = \|\bar{\varphi} f\|$, that is, $P_{H^2(\mathbb{D}^n)}(\bar{\varphi} f) = \bar{\varphi} f$. This implies that

$$\bar{\varphi} f \in H^2(\mathbb{D}^n), \tag{1.3.1}$$

and hence $z_i \bar{\varphi} f \in H^2(\mathbb{D}^n)$ for all $i = 1, \ldots, n$. Then

$$T_\varphi T_\varphi^*(z_i f) = T_\varphi(\bar{\varphi} z_i f) = P_{H^2(\mathbb{D}^n)}(|\varphi|^2 z_i f) = P_{H^2(\mathbb{D}^n)}(z_i f) = z_i f,$$

implies that $z_i f \in \mathcal{R}(T_\varphi)$ for all $i = 1, \ldots, n$. This completes the proof. □

In what follows, if $i \in \{1, \ldots, n\}$ and k_i is a negative integer, then we write $z_i^{k_i} = \bar{z}_i^{-k_i}$.

Lemma 1.3.2. *For each $i = 1, \ldots, n$, the function φ cannot depend on both z_i and $\bar{z}_i$ variables at a time.*

Proof. We shall prove this by contradiction. Assume without loss of generality that φ depends on both z_1 and $\bar{z}_1$. Then

$$\varphi = \sum_{k=1}^{\infty} \bar{z}_1^k \varphi_{-k} \oplus \sum_{k=0}^{\infty} z_1^k \varphi_k,$$

and $\varphi_{-k_0} \neq 0$ for some $k_0 \neq 0$. Here $\varphi_k \in L^2(\mathbb{T}^{n-1})$, $k \in \mathbb{Z}$, is a function of $\{z_i, \bar{z}_j : i, j = 2, \ldots, n\}$. There exist non-negative integers $k_2, \ldots, k_n$, and $l_2, \ldots, l_n$ such that the coefficient of $\bar{z}_2^{k_2} \cdots \bar{z}_n^{k_n} z_2^{l_2} \cdots z_n^{l_n}$ in the expansion of the Fourier series of φ_{-k_0} is nonzero. Set

$$Z_{kl} := z_2^{k_2} \cdots z_n^{k_n} z_2^{l_2} \cdots z_n^{l_n},$$

and

$$f := T_\varphi(z_1^{k_0} Z_{kl}) - z_1 T_\varphi(z_1^{k_0 - 1} Z_{kl}).$$

Note that f is a nonzero function in $H^2(\mathbb{D}^n)$, and f does not depend on z_1. Since $T_\varphi(z_1^{k_0-1} Z_{kl}) \in \mathcal{R}(T_\varphi)$, Lemma 1.3.1 implies that $f \in \mathcal{R}(T_\varphi)$. In particular, by (1.3.1), $\bar{\varphi} f \in H^2(\mathbb{D}^n)$. On the other hand, since

$$\bar{\varphi} f = \sum_{k=1}^{\infty} z_1^k (f \bar{\varphi}_{-k}) \oplus \sum_{k=0}^{\infty} \bar{z}_1^k (f \bar{\varphi}_k),$$

it follows that $f \bar{\varphi}_k = 0$ for all $k > 0$. Since $\boldsymbol{m}(\{z \in \mathbb{T}^n : f(z) = 0\}) = 0$, we have $\bar{\varphi}_k = 0$ for all $k > 0$. This yields

$$\varphi = \sum_{k=0}^{\infty} \bar{z}_1^k \varphi_{-k},$$

and hence φ depends on $\bar{z}_1$ and does not depend on z_1. This is a contradiction. $\square$

We are now ready for the proof of Theorem 1.1.1.

Proof of Theorem 1.1.1. Suppose T_φ is a partial isometry. In view of Lemma 1.3.2, there exists a (possibly empty) subset C of $\{z_1, \ldots, z_n\}$ such that φ is analytic in z_i for all $z_i \in A := C^c$, and co-analytic in z_j for all $z_j \in C$. Let $A = \{z_{i_1}, \ldots, z_{i_p}\}$ and $C = \{z_{j_1}, \ldots, z_{j_q}\}$. Then $p + q = n$, and

$$\varphi = \sum_{k \in \mathbb{Z}_+^q} \bar{z}_C^k \varphi_{A,k},$$

where $\varphi_{A,k} \in H^2(\mathbb{D}^p)$ is a function of $\{z_{i_1}, \ldots, z_{i_p}\}$, $\bar{z}_C^k = \bar{z}_{j_1}^{k_1} \cdots \bar{z}_{j_q}^{k_q}$, and $k = (k_1, \ldots, k_q) \in \mathbb{Z}_+^q$. Note that

$$\varphi_{A,l} \in \mathcal{R}(T_\varphi) \qquad (l \in \mathbb{Z}_+^q).$$

Indeed, $\varphi_{A,0} = T_\varphi 1 \in \mathcal{R}(T_\varphi)$. Moreover, for each $l \in \mathbb{Z}_+^q \setminus \{0\}$, we have

$$T_\varphi z^l = P_{H^2(\mathbb{D}^n)}\Big(\sum_{k\in\mathbb{Z}_+^q} z_C^{l-k}\varphi_{A,k}\Big),$$

that is

$$T_\varphi z^l = \sum_{l-k\geq 0} z_C^{l-k}\varphi_{A,k}.$$

Here $l-k \geq 0$ means that $l_i - k_i \geq 0$ for all $i = 1, \ldots, q$. Thus the claim follows by induction. By (1.3.1), we have $\bar\varphi\varphi_{A,l} \in H^2(\mathbb{D}^n)$, $l \in \mathbb{Z}_+^q$. Therefore

$$\bar\varphi\varphi_{A,l} = \sum_{k\in\mathbb{Z}_+^q} z_C^k \overline{\varphi_{A,k}}\varphi_{A,l} \in H^2(\mathbb{D}^n) \qquad (l \in \mathbb{Z}_+^q).$$

Consequently, $\overline{\varphi}_{A,k}\varphi_{A,l} \in H^2(\mathbb{D}^p)$ for all k and l, and hence, in particular, we have

$$\overline{\varphi}_{A,l}\varphi_{A,l} \in H^2(\mathbb{D}^p) \qquad (l \in \mathbb{Z}_+^q).$$

This immediately implies that $\overline{\varphi}_{A,l}\varphi_{A,l}$ is a constant function, and hence $\varphi_{A,l} = \alpha_l\psi_l$ for some inner function $\psi_l \in H^\infty(\mathbb{D}^p)$ and scalar α_l such that $|\alpha_l| \leq 1$, $l \in \mathbb{Z}_+^q$. Assume without loss of generality that $\varphi_{A,0} \neq 0$. Now by the fact that $\overline{\varphi}_{A,0}\varphi_{A,k}$ and $\overline{\varphi}_{A,k}\varphi_{A,0}$ are in $H^2(\mathbb{D}^p)$, we have $\varphi_{A,k} = \beta_k\psi_0$, $k \in \mathbb{Z}_+^q$. Therefore

$$\varphi = \Big(\sum_{k\in\mathbb{Z}_+^q}\beta_k \bar z_C^k\Big)\psi_0 = \bar\varphi_1\varphi_2,$$

where $\varphi_1 = \sum_{k\in\mathbb{Z}_+^q}\bar\beta_k z_C^k$ and $\varphi_2 = \psi_0$.

We now turn to the converse part. First we have clearly

$$T_{\varphi_1}T_{\varphi_2} = T_{\varphi_2}T_{\varphi_1}. \tag{1.3.2}$$

We also claim that

$$T_{\varphi_1}T_{\varphi_2}^* = T_{\varphi_2}^*T_{\varphi_1}. \tag{1.3.3}$$

This holds trivially when one of the functions φ_1 or φ_2 is constant. We continue with the above notation, and assume that both A and C are nonempty subsets of $\{z_1, \ldots, z_n\}$. First we observe that φ_1 and φ_2 depends only on $\{z_{i_1}, \ldots, z_{i_p}\}$ and $\{z_{j_1}, \ldots, z_{j_q}\}$, respectively. Consider a monomial $z^k \in \mathbb{C}[z_1, \ldots, z_n]$. Suppose $k = (k_1, \ldots, k_n)$, and write

$$z^k = z_C^{k_c} z_A^{k_a},$$

where $k_c = (k_{j_1}, \ldots, k_{j_q}) \in \mathbb{Z}_+^q$, and $k_a \in \mathbb{Z}_+^p$ is the ordered p tuple made out of $\{k_i\}_{i=1}^n \setminus \{k_{j_t}\}_{t=1}^q$. Since the analytic function φ_2 depends only on $z_{j_s} \in C$, $s = 1, \ldots, p$, it is clear that

$$\bar\varphi_2 z_C^{k_c} = \varphi_a + \varphi_c,$$

where φ_a depends only on $\{z_{j_s}\}_{s=1}^p$ (and hence it is an analytic function) and $\varphi_c \in L^2(\mathbb{T}^q) \ominus H^2(\mathbb{D}^q)$ is a function of $\{z_{j_t}, \bar{z}_{j_t}\}_{t=1}^q$. Note that the latter property ensures that $\varphi_c(0) = 0$. Then, on one hand, we have

$$T_{\varphi_2}^* T_{\varphi_1} z^k = P_{H^2(\mathbb{D}^n)}(\bar{\varphi}_2 \varphi_1 z^k) = P_{H^2(\mathbb{D}^n)}\Big((\varphi_a + \varphi_c)\varphi_1 z_A^{k_a}\Big) = \varphi_a \varphi_1 z_A^{k_a},$$

and on the other hand that

$$T_{\varphi_1} T_{\varphi_2}^* z^k = \varphi_1 P_{H^2(\mathbb{D}^n)}(\bar{\varphi}_2 z^k) = \varphi_1 P_{H^2(\mathbb{D}^n)}\Big((\varphi_a + \varphi_c) z_A^{k_a}\Big) = \varphi_1 \varphi_a z_A^{k_a}.$$

Consequently, $T_{\varphi_2}^* T_{\varphi_1} z^k = T_{\varphi_1} T_{\varphi_2}^* z^k$ for all $k \in \mathbb{Z}_+^n$, which proves our claim. Now suppose that $T_\varphi = T_{\varphi_1}^* T_{\varphi_2}$, where φ_1 and φ_2 depends on different variables. Using (1.3.2) and (1.3.3), we obtain

$$T_\varphi T_\varphi^* = T_{\varphi_1}^* T_{\varphi_2} T_{\varphi_2}^* T_{\varphi_1} = (T_{\varphi_1}^* T_{\varphi_1})(T_{\varphi_2} T_{\varphi_2}^*) = P_{\mathcal{R}(T_{\varphi_2})}, \tag{1.3.4}$$

which implies that T_φ is a partial isometry. □

We remark that the commutativity and doubly commutativity of T_{φ_1} and T_{φ_2} in (1.3.2) and (1.3.3) will be useful in the particular applications to Theorem 1.1.1 in the final section.

1.4 Inner functions and shifts

In this short section, we pause to prove an auxiliary result that is both a necessary tool for our final refinement of partial isometric Toeplitz operators and a subject of independent interest with its own applications.

Let $\varphi \in H^\infty(\mathbb{D}^n)$, and suppose the multiplication operator M_φ is an isometry on $H^2(\mathbb{D}^n)$. Then

$$\|\varphi\|_\infty = \|M_\varphi\|_{\mathcal{B}(H^2(\mathbb{D}^n))} = 1,$$

and hence Corollary 1.2.3 implies that φ is a unimodular function in $H^\infty(\mathbb{D}^n)$, that is, φ is an inner function. Now we prove that a nonconstant inner function always defines a shift (and not only isometry). Recall that an operator $V \in \mathcal{B}(\mathcal{H})$ is said to be a *shift* if V is an isometry and $V^{*m} \to 0$ as $m \to \infty$ in the strong operator topology.

Recall that a closed subspace $\mathcal{S} \subseteq H^2(\mathbb{D}^n)$ is of *Beurling type* if there exists an inner function $\theta \in H^\infty(\mathbb{D}^n)$ such that $\mathcal{S} = \theta H^2(\mathbb{D}^n)$. It is also known that (cf. [71, Corollary 6.3] and [67]) a closed subspace $\mathcal{S} \subseteq H^2(\mathbb{D}^n)$, $n > 1$, is of Beurling type if and only if $R_i^* R_j = R_j R_i^*$ for all $1 \leq i < j \leq n$, where $R_p = M_{z_p}|_{\mathcal{S}} \in \mathcal{B}(\mathcal{S})$ is the restriction operator and $p = 1, \ldots, n$. Note that

$$R_i^* R_j = P_{\mathcal{S}} M_{z_i}^* M_{z_j}|_{\mathcal{S}} \text{ and } R_j R_i^* = M_{z_j} P_{\mathcal{S}} M_{z_i}^*|_{\mathcal{S}}, \tag{1.4.1}$$

for all $i, j = 1, \ldots, n$.

Theorem 1.4.1. *If $\varphi \in H^\infty(\mathbb{D}^n)$ is a nonconstant inner function, then M_φ is a shift.*

Proof. It is well known (as well as easy to see) that M_φ is an isometry. Following the classical von Neumann and Wold decomposition for isometries, we only need to prove that

$$\mathcal{H}_u := \bigcap_{m=0}^{\infty} \varphi^m H^2(\mathbb{D}^n) = \{0\}.$$

Assuming the contrary, suppose that $\mathcal{H}_u \neq \{0\}$. We claim that $\mathcal{H}_u$ is of Beurling type. Since the $n = 1$ case is obvious, we assume that $n > 1$. As $\varphi^p H^2(\mathbb{D}^n) \subseteq \varphi^q H^2(\mathbb{D}^n)$ for all $p \geq q$, we have

$$P_{\mathcal{H}_u} = SOT - \lim_{m\to\infty} P_{\varphi^m H^2(\mathbb{D}^n)}.$$

Since $\varphi^m H^2(\mathbb{D}^n)$, $m \geq 1$, is a Beurling type invariant subspace, in view of (1.4.1), it follows that

$$P_{\mathcal{H}_u} M_{z_i}^* M_{z_j} h = M_{z_j} P_{\mathcal{H}_u} M_{z_i}^* h,$$

for all $h \in \mathcal{H}_u$. Then (1.4.1) again implies that $\mathcal{H}_u$ is of Beurling type. Therefore, there exists an inner function $\theta \in H^\infty(\mathbb{D}^n)$ such that $\mathcal{H}_u = \theta H^2(\mathbb{D}^n)$ (note that the $n = 1$ case directly follows from Beurling). Then, for each $m \geq 1$, there exists an inner function $\psi_m \in H^\infty(\mathbb{D}^n)$ such that $\theta = \varphi^m \psi_m$ (for instance, see (1.5.1)). Since φ is a nonconstant inner function, by the maximum modulus principle [97, §2, Theorem 6], we have $|\varphi(z)| < 1$ for all $z \in \mathbb{D}^n$. For each fixed $z_0 \in \mathbb{D}^n$, it follows that

$$|\theta(z_0)| = |\varphi(z_0)|^m |\psi_m(z_0)| \leq |\varphi(z_0)|^m \to 0 \text{ as } m \to \infty,$$

and hence $\theta \equiv 0$. This contradiction shows that $\mathcal{H}_u = \{0\}$. $\square$

In fact, the above argument yields something more: Suppose $\{\mathcal{S}_m\}_{m\geq 1}$ be a sequence of Beurling type invariant subspaces of $H^2(\mathbb{D}^n)$. Then $\bigcap_{m=1}^\infty \mathcal{S}_m$ is also a Beurling type invariant subspace. Indeed, we let $\mathcal{H}_m = \bigcap_{i=1}^m \mathcal{S}_m$. Then $\{\mathcal{H}_m\}_{m\geq 1}$ forms a decreasing sequence of Beurling type invariant subspaces, and hence

$$P_{\bigcap_{m=1}^\infty \mathcal{S}_m} = P_{\bigcap_{m=1}^\infty \mathcal{H}_m} = SOT - \lim_{m\to\infty} P_{\mathcal{H}_m}.$$

The rest of the proof is then much as before.

We also wish to point out that Theorem 1.4.1 can be proved by using (analytic) reproducing kernel Hilbert space techniques. We believe that the algebraic tools described above might be useful in other settings.

1.5 Applications and further refinements

We begin with partially isometric Toeplitz operators that are hyponormal. A bounded linear operator T acting on a Hilbert space is called *hyponormal* if $[T^*, T] \geq 0$, where

$$[T^*, T] = T^*T - TT^*,$$

is the self commutator of T.

Now suppose T_φ, $\varphi \in L^\infty(\mathbb{T}^n)$, is a partial isometry. If $\varphi \in H^\infty(\mathbb{D}^n)$ is inner, then T_φ is an isometry and hence is hyponormal. For the converse direction, we note by Theorem 1.1.1 that $T_\varphi = T_{\varphi_1}^* T_{\varphi_2}$ for some inner functions φ_1 and φ_2 in $H^\infty(\mathbb{D}^n)$ which depends on different variables. If φ_1 is a constant function, then $T_\varphi = T_{\varphi_2} = M_{\varphi_2}$ is an isometry, and hence T_φ is hyponormal. If φ_2 is a constant function, then $T_\varphi = T_{\varphi_1}^* = M_{\varphi_1}^*$ is a co-isometry, and hence T_φ cannot be hyponormal. Suppose both φ_1 and φ_2 are nonconstant functions. Now (1.3.2) and (1.3.3) imply that

$$T_\varphi^* T_\varphi = T_{\varphi_2}^* T_{\varphi_1} T_{\varphi_1}^* T_{\varphi_2} = (T_{\varphi_2}^* T_{\varphi_2})(T_{\varphi_1} T_{\varphi_1}^*) = T_{\varphi_1} T_{\varphi_1}^*.$$

Then, by (1.3.4) we see that $[T_\varphi^*, T_\varphi] \geq 0$ implies $T_{\varphi_2} T_{\varphi_2}^* \leq T_{\varphi_1} T_{\varphi_1}^*$. By noting that φ_1 and φ_2 are analytic functions, we see

$$M_{\varphi_2} M_{\varphi_2}^* \leq M_{\varphi_1} M_{\varphi_1}^*,$$

which, by the Douglas range inclusion theorem, is equivalent to $M_{\varphi_2} = M_{\varphi_1} X$ for some $X \in \mathcal{B}(H^2(\mathbb{D}^n))$. Observe that

$$M_{\varphi_1} M_{z_i} X = M_{z_i} M_{\varphi_1} X = M_{z_i} M_{\varphi_2} = M_{\varphi_2} M_{z_i} = M_{\varphi_1} X M_{z_i}, \tag{1.5.1}$$

implies that $M_{z_i} X = X M_{z_i}$ for all $i = 1, \ldots, n$, and hence $X = M_\psi$ for some $\psi \in H^\infty(\mathbb{D}^n)$. Hence, we conclude that $\varphi_2 = \varphi_1 \psi$. Since φ_1 and φ_2 are inner functions, $\psi \in H^\infty(\mathbb{D}^n)$ is inner. Moreover, since φ_1 and φ_2 are nonconstant functions and depend on different variables, by comparing the Fourier coefficients, one sees that the equality $\varphi_2 = \psi \varphi_1$ cannot be solved for $\psi \in H^\infty(\mathbb{D}^n)$. We have therefore shown the following result:

Corollary 1.5.1. *Let T_φ, $\varphi \in L^\infty(\mathbb{T}^n)$, be a partial isometry. Then T_φ is hyponormal if and only if φ is an inner function in $H^\infty(\mathbb{D}^n)$.*

Therefore, in view of Theorem 1.4.1, T_φ is hyponormal if and only if (up to unitary equivalence) T_φ is a shift.

We recall [59, Halmos and Wallen] that a bounded linear operator T acting on some Hilbert space is called a *power partial isometry* if T^m is partially isometric for all $m \geq 1$. Clearly, Theorem 1.1.1 and the equalities in (1.3.2) and (1.3.3) imply the following statement:

Corollary 1.5.2. *Partially isometric Toeplitz operators are power partial isometry.*

We also recall from Halmos and Wallen [59] (also see [63]) that every power partial isometry is a direct sum whose summands are unitary operators, shifts, co-shifts, and truncated shifts. Recall that a *truncated shift* S of *index* p, $p \in \mathbb{N}$, on some Hilbert space $\mathcal{H}$ is an operator of the form

$$S = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ I_{\mathcal{H}_0} & 0 & 0 & \cdots & 0 & 0 \\ 0 & I_{\mathcal{H}_0} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & I_{\mathcal{H}_0} & 0 \end{bmatrix}_{p\times p},$$

where $\mathcal{H}_0$ is a Hilbert space, and $\mathcal{H} = \underbrace{\mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_0}_{p}$.

We prove that, up to unitary equivalence, a partial isometric T_φ is simply direct sum of truncated shifts, or a shift, or a co-shift (that is, adjoint of a shift). The proof is essentially contained in Theorem 1.4.1 and the Halmos and Wallen models of power partial isometries.

Theorem 1.5.3. *Up to unitary equivalence, a partially isometric Toeplitz operator is either a shift, or a co-shift, or a direct sum of truncated shifts.*

Proof. Suppose T_φ, $\varphi \in L^\infty(\mathbb{T}^n)$, is a partial isometry. By Theorem 1.1.1, $T_\varphi = T^*_{\varphi_1} T_{\varphi_2}$, where φ_1 and φ_2 are inner functions in $H^\infty(\mathbb{D}^n)$ and depends on different variables. Moreover, by Corollary 1.5.2, T_φ is a power partial isometry. If φ_1 is a constant function, then T_φ is a shift, and if φ_2 is a constant function, then T_φ is a co-shift. Now let both φ_1 and φ_2 are nonconstant functions. Following the construction of Halmos and Wallen [59, page 660] (also see [63]), we set $E_m = T_\varphi^{*m} T_\varphi^m$ and $F_m = T_\varphi^m T_\varphi^{*m}$ for the initial and final projections of the partial isometry T_φ^m, $m \geq 1$. By (1.3.2) and (1.3.3) it follows that $E_m = T^m_{\varphi_1} T^{*m}_{\varphi_1}$ and $F_m = T^m_{\varphi_2} T^{*m}_{\varphi_2}$, and hence

$$\mathcal{R}(E_m) = \varphi_1^m H^2(\mathbb{D}^n) \quad \text{and} \quad \mathcal{R}(F_m) = \varphi_2^m H^2(\mathbb{D}^n),$$

for all $m \geq 1$. Then, by Theorem 1.4.1, we have

$$\bigcap_{m\geq 0} \mathcal{R}(E_m) = \bigcap_{m\geq 0} \varphi_1^m H^2(\mathbb{D}^n) = \{0\},$$

and similarly $\bigcap_{m\geq 0} \mathcal{R}(F_m) = \{0\}$. Therefore, the unitary part, the shift part, and the co-shift part of the Halmos and Wallen model of T_φ are trivial (see [59, page 661] or [63]). Hence in this case, T_φ is a direct sum of truncated shifts. □

Clearly, Corollary 1.5.1 immediately follows from the above result as well. Also, note that the Halmos and Wallen models of power partial isometries played an important role in the proof of the above theorem. We refer [63, 25, 52] for a more recent viewpoint of power partial isometries.

Finally, summarizing our results from an operator theoretic point of view, we conclude the following: Let T_φ, $\varphi \in L^\infty(\mathbb{T}^n)$, be a partially isometric Toeplitz operator. Then the following hold:

1. If $n = 1$, then T_φ is either an isometry, or a coisometry. This is due to Brown and Douglas. And, in view of Theorem 1.4.1, T_φ is either a shift, or a co-shift.

2. If $n > 1$, then, up to unitary equivalence, T_φ is either a shift, or a co-shift, or a direct sum of truncated shifts.

Chapter 2

Commutant Lifting And Interpolation On The Polydisc

2.1 Introduction

In this chapter, we solve the commutant lifting problem on $H^2(\mathbb{T}^n)$ and the Nevanlinna-Pick interpolation problem for bounded analytic functions on $\mathbb{D}^n$. The commutant lifting problem on $\mathbb{D}^n$ refers to the following problem: Given a closed subspace $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ that is invariant under $T_{z_i}^*$, $i = 1, \ldots, n$, we classify contractions $X \in \mathcal{B}(\mathcal{Q})$ satisfying the condition that

$$X(P_{\mathcal{Q}} T_{z_i}|_{\mathcal{Q}}) = (P_{\mathcal{Q}} T_{z_i}|_{\mathcal{Q}})X \qquad (i = 1, \ldots, n),$$

so that the following diagram commutes

$$\begin{tikzcd}
H^2(\mathbb{T}^n) \arrow[r, dotted, "T_\varphi"] & H^2(\mathbb{T}^n) \arrow[d, "P_{\mathcal{Q}}"] \\
\mathcal{Q} \arrow[u, "i_{\mathcal{Q}}"] \arrow[r, "X"'] & \mathcal{Q}
\end{tikzcd}$$

for some $\varphi \in \mathcal{S}(\mathbb{D}^n)$. Recall that

$$\mathcal{S}(\mathbb{D}^n) = \{\varphi \in H^\infty(\mathbb{D}^n) : \|\varphi\|_\infty \leq 1\},$$

and the members of $\mathcal{S}(\mathbb{D}^n)$ are known as *Schur functions*. We apply the solution to the above problem for solution to the Nevanlinna-Pick interpolation problem on $\mathbb{D}^n$.

Let us point out some facts and thoughts regarding the commutant lifting and interpolation problems, as well as the context of our approach. In 1968, Sz.-Nagy and Foiaş [70] generalized the Sarason lifting theorem to vector-valued Hardy spaces. In subsequent papers, many researchers presented a variety of alternative proofs of independent interest (cf. [12, 46, 90]). However, the dilation theory (pioneered by Halmos [57] and advanced by Sz.-Nagy [69]) is the primary technique employed in all of these papers which is powerful enough to negate the heavy use of function theoretic tools. For different versions of the commutant lifting theorem and its applications, we refer to Bercovici, Foiaş and Tannenbaum [22], and the monographs by Nikolski [82], Sz.-Nagy and Foiaş [100], and Foiaş and Frazho [53] (also see Nikolski and Volberg [83] and Seip [96]).

In several variables, the earlier approach to the lifting theorem also appears to be dilation theoretic or under the assumption of von Neumann inequality, where dilation theory and von Neumann inequality for commuting contractions are complex subjects in and of themselves.

On the other hand, if the solution to the interpolation problem on $\mathbb{D}^n$, $n \geq 1$, is sought in terms of the Pick matrix's positive semi-definiteness, then the interpolation problem becomes equivalent to the commutant lifting theorem on finite-dimensional zero-based subspaces (cf. Proposition 2.9.5). Consequently, in one variable, thanks to Sarason, the commutant lifting property, the Pick positivity, and the solution to the interpolation problem appear to be inextricably linked. In higher variables, however, because the commutant lifting property is rather erratic (cf. Section 2.3), it is perhaps necessary to disencumber the positivity of the Pick matrix from the interpolation problem. In some ways, these observations seek a different perspective on the several variables interpolation problem, one that is not as similar to the classical case of positivity of the Pick matrix (nor even positivity of a family of Pick matrices as in [1, 31, 44, 60]). As a consequence, we approach the problem from a completely different angle: more along the function theoretic path pioneered by Sarason. The difficulty here, of course, is dealing with the sensitivity of several complex variables as well as the lack of all standard one variable tools.

Finally, a few words about this chapter's methodology. We heavily use the duality of classical Banach spaces, namely

$$(L^1(\mathbb{T}^n))^* \cong L^\infty(\mathbb{T}^n).$$

Other common tools used in this chapter include the classical Hahn-Banach theorem, the geometry of Banach spaces, and the Hilbert function space theory.

The remainder of the chapter is structured as follows. Section 2.2 introduces some preliminary concepts. Section 2.3 outlines explicit examples of non-liftable maps. Section 2.5 presents the first classification of the interpolation on $\mathbb{D}^n$. A quantitative classification for interpolation is presented in Section 2.6. In the same section, by using the quantitative classification, we provide examples of interpolation on $\mathbb{D}^n$, $n \geq 2$. The commutant lifting theorem on $\mathbb{D}^n$ is tested in Section 2.7 with some concrete examples. As an application to our main commutant lifting theorem, Section 2.8 provides new proof for the classical lifting theorem. In Section 2.9 we make some general observations such as the Carathéodory-Fejér interpolation problem, weak interpolation, and decomposing a polynomial as a sum of bounded analytic functions. Section 2.10 concludes with some closing remarks and thoughts on some other known results.

The chapter contains an abundance of examples and counterexamples, as well as numerous auxiliary results of independent interest in both one and several variables.

2.2 Preliminaries

In this section, we will introduce some necessary Hilbert function space theoretic preliminaries. These include Hardy space, submodules, quotient modules, and a formal definition of lifting. We begin by looking at the Hardy space. We again remind the reader that throughout the chapter, n will denote a natural number, and (unless otherwise stated) we always assume that $n \geq 1$.

We denote as usual by $L^2(\mathbb{T}^n)$ the space of square-integrable functions on $\mathbb{T}^n$. Recall that $\mathbb{T}^n$ is the Šhilov boundary of $\mathbb{D}^n$. The *Hardy space* $H^2(\mathbb{T}^n)$ is the closed subspace of $L^2(\mathbb{T}^n)$ consisting of those functions whose Fourier coefficients vanish off $\mathbb{Z}_+^n$. More specifically, consider $f \in L^2(\mathbb{T}^n)$ with Fourier series representation

$$f = \sum_{k \in \mathbb{Z}^n} a_k z^k \qquad (z \in \mathbb{T}^n),$$

where $z^k = z_1^{k_1} \cdots z_n^{k_n}$ for all $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$. Then $f \in H^2(\mathbb{T}^n)$ if and only if $a_k = 0$ whenever at least one of the k_j, $j = 1, \ldots, n$, in $k = (k_1, \ldots, k_n)$ is negative. The usage of radial limits is another neat way to represent the Hardy space (see Rudin [87]). In other words, we will identify $H^2(\mathbb{T}^n)$ with $H^2(\mathbb{D}^n)$, the Hilbert space analytic functions $f \in \mathcal{O}(\mathbb{D}^n)$ such that

$$\|f\|_2 := \Big(\sup_{0<r<1} \int_{\mathbb{T}^n} |f(rz)|^2 d\mu(z) \Big)^{\frac{1}{2}} < \infty, \tag{2.2.1}$$

where $d\mu$ denotes the normalized Lebesgue measure on $\mathbb{T}^n$, and $rz = (rz_1, \ldots, rz_n)$. The identification is canonical, that is, given $f \in H^2(\mathbb{D}^n)$, the radial limit

$$\tilde{f}(z) := \lim_{r \to 1^-} f(rz),$$

exists for almost every $z \in \mathbb{T}^n$, and $\tilde{f} \in H^2(\mathbb{T}^n)$, and vice-versa. In what follows (and unless otherwise stated) we will not distinguish between $f \in \mathcal{O}(\mathbb{D}^n)$ satisfying (2.2.1) and its radial limit representation $\tilde{f} \in H^2(\mathbb{T}^n)$. Therefore, we will not distinguish between $H^2(\mathbb{T}^n)$ and $H^2(\mathbb{D}^n)$ and will use the same notation $H^2(\mathbb{T}^n)$ for both.

It is frequently useful to represent $H^2(\mathbb{T}^n)$ as the Hilbert space of square-summable analytic functions on $\mathbb{D}^n$, that is

$$H^2(\mathbb{T}^n) = \left\{ \sum_{k\in\mathbb{Z}_+^n} a_k z^k \in \mathcal{O}(\mathbb{D}^n) : \sum_{k\in\mathbb{Z}_+^n} |a_k|^2 < \infty \right\}.$$

The Hardy space $H^2(\mathbb{T}^n)$ is equipped with the tuple of multiplication operators by coordinate functions $\{z_1, \ldots, z_n\}$, which we denote by $(T_{z_1}, \ldots, T_{z_n})$. Therefore, by definition, we have

$$(T_{z_i} f)(w) = w_i f(w),$$

for all $f \in H^2(\mathbb{T}^n), w \in \mathbb{D}^n$, and $i = 1, \ldots, n$. It is easy to see that $(T_{z_1}, \ldots, T_{z_n})$ is an n-tuple of commuting isometries, that is

$$T_{z_i}^* T_{z_i} = I_{H^2(\mathbb{T}^n)}, \text{ and } T_{z_i} T_{z_j} = T_{z_j} T_{z_i},$$

for all $i, j = 1, \ldots, n$. We will also need to use the doubly commutativity property

$$T_{z_i}^* T_{z_j} = T_{z_j} T_{z_i}^* \qquad (i \neq j).$$

From the analytic function space perspective, recall that $H^2(\mathbb{T}^n)$ is a reproducing kernel Hilbert space corresponding to the *Szegö kernel* $\mathbb{S}$ on $\mathbb{D}^n$, where

$$\mathbb{S}(z, w) = \prod_{i=1}^{n} \frac{1}{1 - z_i \bar{w}_i} \qquad (z, w \in \mathbb{D}^n).$$

For each $w \in \mathbb{D}^n$, the *kernel function* $\mathbb{S}(\cdot, w) : \mathbb{D}^n \to \mathbb{C}$ defined by

$$(\mathbb{S}(\cdot, w))(z) = \mathbb{S}(z, w) \qquad (z \in \mathbb{D}^n),$$

generates the joint eigenspace of the backward shifts, that is

$$\bigcap_{i=1}^{n} \ker(T_{z_i} - w_i I_{H^2(\mathbb{T}^n)})^* = \mathbb{C}\mathbb{S}(\cdot, w). \tag{2.2.2}$$

The above equality essentially follows from the fact that

$$T_{z_i}^* \mathbb{S}(\cdot, w) = \bar{w}_i \mathbb{S}(\cdot, w), \tag{2.2.3}$$

for all $w \in \mathbb{D}^n$ and $i = 1, \ldots, n$, and

$$\sum_{k\in\{0,1\}^n} (-1)^{|k|} T_z^k T_z^{*k} = P_{\mathbb{C}},$$

where $P_{\mathbb{C}}$ is the orthogonal projection onto the space of constant functions, and $T_z^k = T_{z_1}^{k_1} \cdots T_{z_n}^{k_n}$ for all $k \in \{0,1\}^n \subset \mathbb{Z}_+^n$. Moreover, the set of kernel functions $\{\mathbb{S}(\cdot, w) : w \in$

$\mathbb{D}^n\}$ forms a total set in $H^2(\mathbb{T}^n)$ and satisfy the *reproducing property*

$$f(w) = \left\langle f, \mathbb{S}(\cdot, w) \right\rangle_{H^2(\mathbb{T}^n)}, \tag{2.2.4}$$

for all $f \in H^2(\mathbb{T}^n)$ and $w \in \mathbb{D}^n$.

Recall from Section 2.1 that a closed subspace $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ is called a *quotient module* if $T_{z_i}^* \mathcal{Q} \subseteq \mathcal{Q}$ for all $i = 1, \ldots, n$. A closed subspace $\mathcal{S} \subseteq H^2(\mathbb{T}^n)$ is called a *submodule* if $z_i \mathcal{S} \subseteq \mathcal{S}$ for all $i = 1, \ldots, n$. Equivalently, $\mathcal{S}^\perp \cong H^2(\mathbb{T}^n)/\mathcal{S}$ is a quotient module. In summary, we have the following identifications:

$$\{\text{submodules}\} \longleftrightarrow \{\text{shift invariant subspaces}\},$$

and

$$\{\text{quotient modules}\} \longleftrightarrow \{\text{backward shift invariant subspaces}\}.$$

The classical *Laurent operator* L_φ with symbol $\varphi \in L^\infty(\mathbb{T}^n)$ is the bounded linear operator on $L^2(\mathbb{T}^n)$ defined by

$$L_\varphi f = \varphi f,$$

for all $f \in L^2(\mathbb{T}^n)$. The corresponding *Toeplitz operator* is the compression of L_φ to $H^2(\mathbb{T}^n)$, that is

$$T_\varphi f = P_{H^2(\mathbb{T}^n)}(\varphi f),$$

for all $f \in H^2(\mathbb{T}^n)$. As usual, $P_{H^2(\mathbb{T}^n)}$ denotes the orthogonal projection from $L^2(\mathbb{T}^n)$ onto $H^2(\mathbb{T}^n)$. Recall that (see [41])

$$\|T_\varphi\|_{\mathcal{B}(H^2(\mathbb{T}^n))} = \|L_\varphi\|_{\mathcal{B}(L^2(\mathbb{T}^n))} = \|\varphi\|_\infty, \tag{2.2.5}$$

for all $\varphi \in L^\infty(\mathbb{T}^n)$. It is useful to point out that the Toeplitz operator with analytic symbol $\varphi \in H^\infty(\mathbb{D}^n)$ is given by

$$T_\varphi = L_\varphi|_{H^2(\mathbb{T}^n)}.$$

This follows from the general fact that if $\mathcal{S}$ is a submodule of $H^2(\mathbb{T}^n)$, then $\varphi\mathcal{S} \subseteq \mathcal{S}$ for all $\varphi \in H^\infty(\mathbb{D}^n)$. Finally, given a quotient module $\mathcal{Q}$ of $H^2(\mathbb{T}^n)$ and an analytic symbol $\varphi \in H^\infty(\mathbb{D}^n)$, we define the compression operator S_φ on $\mathcal{Q}$ by

$$S_\varphi = P_\mathcal{Q} T_\varphi|_\mathcal{Q}.$$

In particular, for each $i = 1, \ldots, n$, we have the compression of T_{z_i} on $\mathcal{Q}$ as

$$S_{z_i} = P_\mathcal{Q} T_{z_i}|_\mathcal{Q}.$$

Clearly, $S_\varphi S_{z_i} = S_{z_i} S_\varphi$ for all $i = 1, \ldots, n$. From this point of view, we also call that S_φ a module map. In general:

Definition 2.2.1. *Let $\mathcal{Q}$ be a quotient module of $H^2(\mathbb{T}^n)$. An operator $X \in \mathcal{B}(\mathcal{Q})$ is said to be a module map if*

$$XS_{z_i} = S_{z_i}X \qquad (i = 1, \dots, n).$$

Another common name for module maps is *truncated Toeplitz operators* (even with $L^\infty(\mathbb{T}^n)$-symbols). See Sarason [88] and also the classic by Brown and Halmos [26].

We conclude this section with the definition of the central concept of this chapter. Given a Hilbert space $\mathcal{H}$, recall again that

$$\mathcal{B}_1(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \|T\| \leq 1\}.$$

Definition 2.2.2. *Let $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ be a quotient module and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map. If there is a $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that*

$$X = S_\varphi,$$

then we say that X has a lift, or X is liftable, or X admits a lift. We also say that φ is a lift of X.

In the case of $n = 1$, Sarason's result states that contractive module maps are always liftable. In the following section, we demonstrate that such a statement is no longer true whenever $n > 1$.

2.3 Homogeneous quotient modules

The purpose of this section is to outline explicit and basic examples of non-liftable module maps on quotient modules of $H^2(\mathbb{T}^n)$, $n > 1$. Our quotient modules are as simple as homogenous quotient modules and the module maps are compressions of homogeneous polynomials. We begin with a (probably known) classification of inner polynomials on $\mathbb{D}^n$. A function $\varphi \in H^\infty(\mathbb{D}^n)$ is called *inner* if φ is unimodular a.e. on $\mathbb{T}^n$ (in the sense of radial limits).

Lemma 2.3.1. *Let p be a nonzero polynomial in $\mathbb{C}[z_1, \dots, z_n]$. Then p is inner if and only if*

$$p = \textit{unimodular constant} \times \textit{monomial}.$$

Proof. We assume $n > 1$ because the $n = 1$ case is simpler and follows the same line of proof as the $n > 1$ case. By definition, p is inner if and only if $|p| = 1$ on $\mathbb{T}^n$. The sufficient part is now trivial. For the reverse direction, assume that p is inner. If p is a constant multiple of a monomial, then passing to the boundary value, the assertion will follow immediately. Therefore, assume that p has more than one term. There exists

$N_1 \in \mathbb{N}$ such that

$$p = \sum_{j=0}^{N_1} z_1^j p_j,$$

where $p_j \in \mathbb{C}[z_2, \ldots, z_n]$ for all $j = 0, 1, \ldots, N_1$, and

$$p_{N_1} \neq 0.$$

Here we are assuming without loss of generality that p has a monomial term with z_1 as a factor (otherwise, we pass to the same but with respect to z_2 and so on). Since p is inner, on $\mathbb{T}^n$, we have

$$\begin{aligned} 1 &= p\bar{p} \\ &= \bar{z}_1^{N_1}(p_0 \bar{p}_{N_1} \oplus \cdots). \end{aligned}$$

This implies

$$p_0 \bar{p}_{N_1} = 0,$$

and hence $p_0 = 0$. Continuing exactly in the same way, we obtain that

$$p = z_1^{N_1} p_{N_1},$$

for some $p_{N_1} \in \mathbb{C}[z_2, \ldots, z_n]$. Applying the above recipe to p_{N_1}, we get $p_{N_1} = z_2^{N_2} p_{N_2}$ for some $N_2 \in \mathbb{Z}_+$ and $p_{N_2} \in \mathbb{C}[z_3, \ldots, , z_n]$. Hence

$$p = z_1^{N_1} z_2^{N_2} p_{N_2}.$$

Therefore, applying this method repeatedly, we finally deduce that p is a unimodular constant multiple of some monomial. $\square$

Now we turn to the construction of the quotient modules of interest. As is well known and also evident from the definition of the Hardy space, polynomials are dense in $H^2(\mathbb{T}^n)$, that is

$$H^2(\mathbb{T}^n) = \overline{\mathbb{C}[z_1, \ldots, z_n]}^{L^2(\mathbb{T}^n)}.$$

Therefore, the standard grading on $\mathbb{C}[z_1, \ldots, z_n]$ induces a graded structure on $H^2(\mathbb{T}^n)$. We are essentially going to exploit this simple property in our construction of module maps. For each $t \in \mathbb{Z}_+$, denote by $H_t \subseteq \mathbb{C}[z_1, \ldots, z_n]$ the complex vector space of homogeneous polynomials of degree t. We have the vector space direct sum

$$\mathbb{C}[z_1, \ldots, z_n] = \bigoplus_{t \in \mathbb{Z}_+} H_t.$$

We consider from now on the finite-dimensional subspace $\mathcal{H}_t$ as a closed subspace of $H^2(\mathbb{T}^n)$. Also, for each $m \in \mathbb{N}$, set

$$\mathcal{Q}_m = \bigoplus_{t=0}^{m} H_t.$$

Since $T^*_{z_i}\mathcal{Q}_m \subseteq \mathcal{Q}_m$ for all $m \geq 1$, it follows that $\mathcal{Q}_m \subseteq \mathbb{C}[z_1, \dots, z_n]$ is a finite-dimensional quotient module of $H^2(\mathbb{T}^n)$, and $\deg f \leq m$ for all $f \in \mathcal{Q}_m$. Fix $m \in \mathbb{N}$ and fix a homogeneous polynomial of degree m as

$$p = \sum_{|k|=m} a_k z^k \in H_m.$$

Suppose that $\|p\|_2 = 1$. By the definition of the norm on $H^2(\mathbb{T}^n)$, we have

$$\sum_{|k|=m} |a_k|^2 = 1.$$

We aim to investigate the lifting of the module map

$$S_p = P_{\mathcal{Q}_m} T_p|_{\mathcal{Q}_m}.$$

By $S_p f = P_{\mathcal{Q}_m}(pf)$, $f \in \mathcal{Q}_m$, we have on one hand $S_p 1 = p$, and on the other hand

$$S_p f = 0,$$

for all $f \in \mathcal{Q}_m$ such that $f(0) = 0$. Therefore, $\ker S_p = \mathcal{Q}_m \ominus \mathbb{C}$ or, equivalently

$$\ker S_p = \bigoplus_{t=1}^{m} H_t.$$

This allows us to conclude that

$$\|S_p\| = 1. \tag{2.3.1}$$

We recall in passing that $\|T_\varphi\|_{\mathcal{B}(H^2(\mathbb{T}^n))} = \|\varphi\|_\infty$ for all $\varphi \in L^\infty(\mathbb{T}^n)$ (see (2.2.5)).

Theorem 2.3.2. *S_p admits a lift if and only if p is a unimodular constant multiple of a monomial.*

Proof. Suppose S_p is liftable. There exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that $S_p = S_\varphi$. Then

$$S_p = S_\varphi = P_{\mathcal{Q}_m} T_\varphi|_{\mathcal{Q}_m},$$

and

$$\|\varphi\|_\infty \leq 1.$$

Note that $1 \in \mathcal{Q}_m$. Since $S_p 1 = p$, it is clear that $P_{\mathcal{Q}_m}\varphi = p$, and hence there exists $\psi \in \mathcal{Q}_m^{\perp}$ such that

$$\varphi = p \oplus \psi \in \mathcal{Q}_m \oplus \mathcal{Q}_m^{\perp}.$$

It is a well known general fact that $\|\varphi\|_2 \leq \|\varphi\|_\infty$. Indeed

$$\begin{aligned}\|\varphi\|_2 &= \|T_\varphi 1\|_2 \\ &\leq \|T_\varphi\|_{\mathcal{B}(H^2(\mathbb{T}^n))}\|1\|_2 \\ &= \|\varphi\|_\infty.\end{aligned}$$

Now that $\|p\|_2 = 1$, we compute

$$\begin{aligned}1 + \|\psi\|_2^2 &= \|\varphi\|_2^2 \\ &\leq \|\varphi\|_\infty^2 \\ &\leq 1,\end{aligned}$$

which implies $\psi = 0$. Therefore

$$\varphi = p \in \mathcal{Q}_m.$$

By using the same computation (or the standard norm equality) as above, we have

$$1 = \|p\|_2 \leq \|p\|_\infty = \|\varphi\|_\infty = 1,$$

which implies that $\|p\|_\infty = 1$. This combined with

$$\|T_p 1\|_2 = \|p\|_2 = 1,$$

imply that the Toeplitz operator T_p is norm attaining. Consequently [41, Corollary 2.3], p is inner (as $p \in H^\infty(\mathbb{D}^n)$). Then by Lemma 2.3.1 we conclude that p is a unimodular constant multiple of a monomial. The converse is obvious. □

The following corollary is now straight. Here we need to assume that $n > 1$.

Corollary 2.3.3. *Suppose $n > 1$. Let p be a homogeneous polynomial of degree m and assume that $\|p\|_2 = 1$. Suppose*

$$p = \sum_{|k|=m} a_k z^k \in H_m.$$

If $a_k, a_{\boldsymbol{\lambda}} \neq 0$ for some $k, \boldsymbol{\lambda} \in \mathbb{Z}_+^n$, then S_p on $\mathcal{Q}_m$ is not liftable.

The following fact was used to prove the above theorem [41, Corollary 2.3.]: For $\varphi \in H^\infty(\mathbb{T}^n)$ with $\|\varphi\|_\infty = 1$, if the Toeplitz operator T_φ is norm attaining, then the symbol φ is inner. In this context, it is worth noting that the lift of a commutant is highly nonunique, and the issue of uniqueness of Sarason's commutant lifting theorem is inextricably linked to the norm attainment property [91, Section 5].

Now we consider a simple class of quotient modules where all module maps admit lifting. Our idea is fairly elementary: embed one Hardy space into another Hardy space. Fix a natural number m such that $1 < m < n$. Define

$$\mathcal{S} = \sum_{j=1}^{m} z_j H^2(\mathbb{T}^n).$$

Then $\mathcal{S}$ is a closed subspace and hence a submodule of $H^2(\mathbb{T}^n)$ [93]. Our interest is in the corresponding quotient module $\mathcal{Q}$, that is

$$\mathcal{Q} := \Big(\sum_{j=1}^{m} z_j H^2(\mathbb{T}^n) \Big)^{\perp}.$$

A simple calculation reveals that

$$\mathcal{Q} = \mathbb{C} \otimes H^2(\mathbb{T}^{n-m}),$$

that is, $\mathcal{Q}$ is simply the space of functions on $H^2(\mathbb{T}^n)$ that does not depend on the variables $\{z_1, \ldots, z_m\}$ (again, see [93]). Because $S_{z_i} = P_{\mathcal{Q}} T_{z_i}|_{\mathcal{Q}}$, we have

$$S_{z_i} = \begin{cases} 0 & \text{if } i = 1, \ldots, m \\ T_{z_i} & \text{if } i = m+1, \ldots, n. \end{cases}$$

Suppose $X \in \mathcal{B}(\mathcal{Q})$. Then, by a routine argument, X is a module map, that is

$$X P_{\mathcal{Q}} M_{z_i}|_{\mathcal{Q}} = P_{\mathcal{Q}} M_{z_i}|_{\mathcal{Q}} X,$$

for all $i = 1, \ldots, n$, if and only if there exists $\varphi \in H^\infty(\mathbb{D}^n)$ such that φ does not depend on the variables $\{z_1, \ldots, z_m\}$ and

$$X = T_\varphi.$$

This immediately implies the following result: Let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map. Then $X = T_\varphi$ for some $\varphi \in \mathcal{S}(\mathbb{D}^n)$. In particular, X lifts to T_φ itself.

In Section 2.7, we will show examples of module maps on nonhomogeneous quotient modules that cannot be lifted.

2.4 Classifications of commutant lifting

Given the examples in the preceding section, it is clear that a module map on a quotient module of $H^2(\mathbb{T}^n)$, $n \geq 2$, may not admit a lift in general. In this section, we classify liftable module maps defined on quotient modules of $H^2(\mathbb{T}^n)$, $n \geq 1$. We begin with the well known duality of classical Banach spaces. Recall that $L^1(\mathbb{T}^n)$ is a Banach space predual of $L^\infty(\mathbb{T}^n)$. More specifically, we have

$$(L^1(\mathbb{T}^n))^* \cong L^\infty(\mathbb{T}^n),$$

via the isometrically isomorphic map $\chi : L^\infty(\mathbb{T}^n) \to (L^1(\mathbb{T}^n))^*$ defined by $\varphi \in L^\infty(\mathbb{T}^n) \mapsto \chi_\varphi \in (L^1(\mathbb{T}^n))^*$, where for each $\varphi \in L^\infty(\mathbb{T}^n)$, $\chi_\varphi \in (L^1(\mathbb{T}^n))^*$ is defined by

$$\chi_\varphi f = \int_{\mathbb{T}^n} \varphi f \, d\mu, \tag{2.4.1}$$

for all $f \in L^1(\mathbb{T}^n)$. Moreover, we have the isometric property

$$\|\chi_\varphi\| = \|\varphi\|_\infty,$$

for all $\varphi \in L^\infty(\mathbb{T}^n)$. For a nonempty $X \subseteq L^2(\mathbb{T}^n)$, we define

$$X^{conj} = \{\bar{f} : f \in X\}.$$

We also define the subspace of "mixed functions" of $L^2(\mathbb{T}^n)$ as

$$\mathcal{M}_n = L^2(\mathbb{T}^n) \ominus (H^2(\mathbb{T}^n)^{conj} + H^2(\mathbb{T}^n)).$$

This is the closed subspace of $L^2(\mathbb{T}^n)$ generated by monomials that are neither analytic nor coanalytic. Let $I_n = \{1, \dots, n\}$. Given $A \subseteq I_n$, we set

$$|A| = \#A,$$

the cardinality of A. The following easy-to-see equality explains the terminology of "mixed functions":

$$\mathcal{M}_n = \overline{\operatorname{span}}\{z_A^{k_A} \bar{z}_B^{k_B} : A, B \subseteq I_n, A \cap B = \emptyset, A, B \neq \emptyset, k_A \in \mathbb{Z}_+^{|A|}, k_B \in \mathbb{Z}_+^{|B|}\}, \tag{2.4.2}$$

where for a nonempty subset $A = \{i_1, \dots, i_m\} \subsetneqq I_n$ and $k_A = (k_1, \dots, k_m) \in \mathbb{Z}_+^{|A|}$, we define the monomial

$$z_A^{k_A} := z_{i_1}^{k_1} \cdots z_{i_m}^{k_m}.$$

Note that $\mathcal{M}_n$ is self-adjoint, that is

$$\mathcal{M}_n^{conj} = \mathcal{M}_n. \tag{2.4.3}$$

It is also crucial to observe that if $n = 1$, then $\mathcal{M}_n$ is trivial:

$$\mathcal{M}_1 = \{0\}. \tag{2.4.4}$$

Given a quotient module $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$, as per our convention, we have $\mathcal{Q}^{conj} := \{\bar{f} : f \in \mathcal{Q}\}$, and hence $\mathcal{Q}^{conj}$ is a closed subspace of $L^2(\mathbb{T}^n)$ and

$$\mathcal{Q}^{conj} \perp H_0^2(\mathbb{T}^n),$$

where $H_0^2(\mathbb{T}^n) = H^2(\mathbb{T}^n) \ominus \{1\}$. It is easy to check (for instance, by using $\mathbb{S}(\cdot, 0) \equiv 1$) that

$$H_0^2(\mathbb{T}^n) = \{f \in H^2(\mathbb{T}^n) : f(0) = 0\},$$

the closed subspace of $H^2(\mathbb{T}^n)$ of functions vanishing at the origin. Finally, given a quotient module $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$, we set

$$\mathcal{M}_{\mathcal{Q}} = \mathcal{Q}^{conj} \dotplus (\mathcal{M}_n \dotplus H_0^2(\mathbb{T}^n)).$$

The skew sums in the above definition are in fact Hilbert space orthogonal direct sums in $L^2(\mathbb{T}^n)$. However, in what follows, we will represent $\mathcal{M}_{\mathcal{Q}}$ as a linear subspace of the Banach space $L^1(\mathbb{T}^n)$, and denote it by

$$(\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1).$$

We are now ready for our first lifting theorem.

Theorem 2.4.1. *Let $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ be a quotient module, let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map, and suppose $\psi = X(P_{\mathcal{Q}}1)$. Define $X_{\mathcal{Q}} : (\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1) \longrightarrow \mathbb{C}$ by*

$$X_{\mathcal{Q}} f = \int_{\mathbb{T}^n} \psi f \, d\mu,$$

for all $f \in \mathcal{M}_{\mathcal{Q}}$. Then X is liftable if and only if $X_{\mathcal{Q}}$ is a contractive functional on $(\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1)$.

Proof. Let $\varphi \in \mathcal{S}(\mathbb{D}^n)$ be a lift of X. Then $X = S_\varphi$, where, by definition, $S_\varphi = P_{\mathcal{Q}} T_\varphi|_{\mathcal{Q}}$. Since $\varphi \in \mathcal{S}(\mathbb{D}^n)$ (that is, $\|\varphi\|_\infty \leq 1$), it follows that the functional $\chi_\varphi : L^1(\mathbb{T}^n) \to \mathbb{C}$ defined by

$$\chi_\varphi(f) = \int_{\mathbb{T}^n} f \varphi \, d\mu,$$

for all $f \in L^1(\mathbb{T}^n)$, is a contraction (see (2.4.1)). In view of the fact that $\varphi \mathcal{Q}^\perp \subseteq \mathcal{Q}^\perp$ (as submodules are invariant under $H^\infty(\mathbb{D}^n)$), we have $P_{\mathcal{Q}} T_\varphi P_{\mathcal{Q}} = P_{\mathcal{Q}} T_\varphi$, and hence

$$\begin{aligned} S_\varphi P_{\mathcal{Q}} 1 &= P_{\mathcal{Q}} T_\varphi|_{\mathcal{Q}} P_{\mathcal{Q}} 1 \\ &= P_{\mathcal{Q}} T_\varphi 1 \\ &= P_{\mathcal{Q}} \varphi. \end{aligned}$$

Also, $X = S_\varphi$ implies $\psi = S_\varphi P_{\mathcal{Q}} 1$. This combined with $S_\varphi P_{\mathcal{Q}} 1 = P_{\mathcal{Q}} \varphi$ yields

$$\psi = P_{\mathcal{Q}} \varphi.$$

We now prove that $X_{\mathcal{Q}}$ on $(\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1)$ is a contractive functional. First we consider $X_{\mathcal{Q}}$ on $\mathcal{Q}^{conj} \subseteq \mathcal{M}_{\mathcal{Q}}$. Let $\bar{h} \in \mathcal{Q}^{conj}$. Then $h \in \mathcal{Q}$ or, equivalently, $P_{\mathcal{Q}}h = h$, and we have

$$\begin{aligned}\int_{\mathbb{T}^n} \varphi \bar{h}\, d\mu &= \langle \varphi, h\rangle_{H^2(\mathbb{T}^n)} \\ &= \langle \varphi, P_{\mathcal{Q}}h\rangle_{H^2(\mathbb{T}^n)} \\ &= \langle P_{\mathcal{Q}}\varphi, h\rangle_{H^2(\mathbb{T}^n)} \\ &= \langle \psi, h\rangle_{H^2(\mathbb{T}^n)}.\end{aligned}$$

Thus we conclude that

$$\int_{\mathbb{T}^n} \psi \bar{h}\, d\mu = \int_{\mathbb{T}^n} \varphi \bar{h}\, d\mu,$$

for all $\bar{h} \in \mathcal{Q}^{conj}$, equivalently

$$X_{\mathcal{Q}} = \chi_\varphi \text{ on } \mathcal{Q}^{conj}.$$

Next, we consider $X_{\mathcal{Q}}$ on $\mathcal{M}_n$. Since

$$\mathcal{M}_n \subseteq L^2(\mathbb{T}^n) \ominus (H^2(\mathbb{T}^n) + H^2(\mathbb{T}^n)^{conj}),$$

functions in $\mathcal{M}_n$ do not have an analytic part. Moreover, since $\mathcal{M}_n$ is self-adjoint (see (2.4.3)), we have

$$P_{\mathcal{Q}}\mathcal{M}_n^{conj} = P_{\mathcal{Q}}\mathcal{M}_n = \{0\}.$$

By using the identity $\psi = P_{\mathcal{Q}}\varphi$ and following the computation as in the previous case, for each $h \in \mathcal{M}_n$, we have

$$\begin{aligned}\int_{\mathbb{T}^n} \psi h\, d\mu &= \langle \psi, \bar{h}\rangle_{L^2(\mathbb{T}^n)} \\ &= \langle P_{\mathcal{Q}}\varphi, \bar{h}\rangle_{L^2(\mathbb{T}^n)} \\ &= \langle \varphi, P_{\mathcal{Q}}\bar{h}\rangle_{H^2(\mathbb{T}^n)} \\ &= 0,\end{aligned}$$

as $P_{\mathcal{Q}}h = P_{\mathcal{Q}}\bar{h} = 0$. This proves that

$$X_{\mathcal{Q}} = \chi_\varphi = 0 \text{ on } \mathcal{M}_n.$$

Finally, if $h \in H_0^2(\mathbb{T}^n)$, then $h(0) = 0$, and hence (as $\psi \in \mathcal{Q} \subseteq H^2(\mathbb{T}^n)$)

$$\langle \bar{h}, \psi\rangle_{L^2(\mathbb{T}^n)} = 0.$$

Therefore, again

$$\begin{aligned}\int_{T^n} \psi h\, d\mu &= \langle \psi, \bar{h}\rangle_{L^2(\mathbb{T}^n)} \\ &= 0,\end{aligned}$$

as $\psi \in \mathcal{Q} \subset H^2(\mathbb{T}^n)$. This implies, again, that

$$X_{\mathcal{Q}} = \chi_\varphi = 0 \text{ on } H_0^2(\mathbb{T}^n).$$

Thus we conclude that $X_{\mathcal{Q}} = \chi_\varphi$ on $\mathcal{M}_{\mathcal{Q}}$. On the other hand, $\chi_\varphi : L^1(\mathbb{T}^n) \to \mathbb{C}$ is a contraction. In particular, $\chi_\varphi|_{\mathcal{M}_{\mathcal{Q}}}$ is a contraction, which proves our claim that $X_{\mathcal{Q}} : \mathcal{M}_{\mathcal{Q}} \to \mathbb{C}$ is contractive.

For the converse direction, assume that $X_{\mathcal{Q}} : (\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1) \to \mathbb{C}$ is a contraction. By the Hahn-Banach theorem, there is a linear functional $\tilde{X}_{\mathcal{Q}} : L^1(\mathbb{T}^n) \longrightarrow \mathbb{C}$ such that

$$\tilde{X}_{\mathcal{Q}}|_{\mathcal{M}_{\mathcal{Q}}} = X_{\mathcal{Q}},$$

and

$$\|\tilde{X}_{\mathcal{Q}}\| = \|X_{\mathcal{Q}}\| \leq 1.$$

By the duality $(L^1(\mathbb{T}^n))^* \cong L^\infty(\mathbb{T}^n)$, as outlined in (2.4.1), there exists $\varphi \in L^\infty(\mathbb{T}^n)$ such that

$$\chi_\varphi = \tilde{X}_{\mathcal{Q}},$$

and

$$\|\varphi\|_\infty \leq 1.$$

In particular, $\chi_\varphi|_{\mathcal{M}_{\mathcal{Q}}} = \tilde{X}_{\mathcal{Q}}|_{\mathcal{M}_{\mathcal{Q}}} = X_{\mathcal{Q}}$. Since

$$\chi_\varphi h = \int_{\mathbb{T}^n} \varphi h \, d\mu,$$

for all $h \in \mathcal{M}_{\mathcal{Q}}$, it follows that

$$\int_{\mathbb{T}^n} \varphi h \, d\mu = \int_{\mathbb{T}^n} \psi h \, d\mu, \tag{2.4.5}$$

for all $h \in \mathcal{M}_{\mathcal{Q}}$. We consider a typical monomial f from $\mathcal{M}_n \dotplus H_0^2(\mathbb{T}^n)$. In other words, we let

$$f = z^k,$$

for some $k \in \mathbb{N}^n$, or let

$$f = z_A^{k_A} \bar{z}_B^{k_B},$$

for some $k_A \in \mathbb{Z}_+^{|A|}$ and $k_B \in \mathbb{Z}_+^{|B|}$, where $A, B \subseteq \{1, \dots, n\}$, $A \cap B = \emptyset$, and $A, B \neq \emptyset$ (see the definition of $\mathcal{M}_n$ in (2.4.2)). As $\psi = X(P_{\mathcal{Q}}1) \in \mathcal{Q} \subseteq \mathrm{Hol}(\mathbb{D}^n)$, it follows that $\langle \psi, \bar{f} \rangle_{L^2(\mathbb{T}^n)} = 0$ and hence

$$\int_{\mathbb{T}^n} \psi f \, d\mu = 0.$$

Consequently, the identity in (2.4.5) yields

$$\int_{\mathbb{T}^n} \varphi z^k \, d\mu = 0,$$

for all $k \in \mathbb{N}^n$, as well as

$$\int_{\mathbb{T}^n} \varphi z_A^{k_A} \bar{z}_B^{k_B} \, d\mu = 0,$$

for all $k_A \in \mathbb{Z}_+^{|A|}$ and $k_B \in \mathbb{Z}_+^{|B|}$, where $A, B \subseteq \{1, \ldots, n\}$, $A \cap B = \emptyset$, and $A, B \neq \emptyset$. This implies φ is analytic, and hence $\varphi \in \mathcal{S}(\mathbb{D}^n)$. To complete the proof, it remains to show that $X = S_\varphi$. Note, by (2.4.5) again, we have that

$$\int_{\mathbb{T}^n} \psi \bar{h} \, d\mu = \int_{\mathbb{T}^n} \varphi \bar{h} \, d\mu,$$

for all $\bar{h} \in \mathcal{Q}^{conj}$. Equivalently, for each $\bar{h} \in \mathcal{Q}^{conj}$, we have

$$\langle \varphi, h \rangle_{L^2(\mathbb{T})} = \langle \psi, h \rangle_{L^2(\mathbb{T}^n)},$$

and hence

$$\langle P_\mathcal{Q} \varphi, h \rangle_{H^2(\mathbb{T}^n)} = \langle \psi, h \rangle_{H^2(\mathbb{T}^n)},$$

from which we conclude that

$$P_\mathcal{Q} \varphi = \psi.$$

As before, we write $\varphi \in \mathcal{S}(\mathbb{D}^n) \subseteq H^2(\mathbb{T}^n)$ with respect to $\mathcal{Q} \oplus \mathcal{Q}^\perp = H^2(\mathbb{T}^n)$ as

$$\varphi = \psi \oplus \rho \in \mathcal{Q} \oplus \mathcal{Q}^\perp.$$

Since $P_\mathcal{Q} \varphi = P_\mathcal{Q} T_\varphi P_\mathcal{Q} 1 = S_\varphi(P_\mathcal{Q} 1)$ and $P_\mathcal{Q} \varphi = \psi$, we have

$$\psi = S_\varphi(P_\mathcal{Q} 1).$$

This combined with $\psi = X(P_\mathcal{Q} 1)$ yields

$$S_\varphi(P_\mathcal{Q} 1) = X(P_\mathcal{Q} 1).$$

Finally, let us fix $k \in \mathbb{Z}_+^n$ and observe

$$\begin{aligned} P_\mathcal{Q} z^k &= P_\mathcal{Q} z^k (P_\mathcal{Q} 1) \\ &= S_z^k (P_\mathcal{Q} 1). \end{aligned}$$

Therefore, $S_\varphi S_z^k = S_z^k S_\varphi$ implies

$$\begin{aligned} S_\varphi(P_\mathcal{Q} z^k) &= S_\varphi S_z^k (P_\mathcal{Q} 1) \\ &= S_z^k S_\varphi (P_\mathcal{Q} 1) \\ &= S_z^k X (P_\mathcal{Q} 1) \\ &= X S_z^k (P_\mathcal{Q} 1) \\ &= X(P_\mathcal{Q} z^k). \end{aligned}$$

Then, in view of the fact that $\mathcal{Q} = \overline{\text{span}}\{P_{\mathcal{Q}} z^k : k \in \mathbb{Z}_+^n\}$, the equality $X = S_\varphi$ is immediate. This completes the proof of the theorem. $\square$

The proof of the above theorem says more than what it states. In fact, we have the identity

$$X_{\mathcal{Q}}|_{\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)} \equiv 0,$$

and hence

$$\ker X_{\mathcal{Q}} \supseteq \mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n).$$

In other words, $\mathcal{Q}^{conj}$ is the supporting space of the functional $X_{\mathcal{Q}}$. Another way to put it is that there is a contractive extension of $X_{\mathcal{Q}}|_{\mathcal{Q}^{conj}}$ to the entire $\mathcal{M}_{\mathcal{Q}}$ that vanishes on the completely analytic and completely co-analytic parts.

Remark 2.4.1. *It is clear from the construction that the subspace $(\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1)$ is independent of X.*

Our second lifting theorem is a consequence of the first, and it appears to be in a more compact form. Given a quotient module $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ and a module map $X \in \mathcal{B}(\mathcal{Q})$, we define a subspace of $L^1(\mathbb{T}^n)$ as

$$\tilde{\mathcal{M}}_{\mathcal{Q},X} = (\mathcal{Q}^{conj} \ominus \{\bar{\psi}\}) \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)).$$

Keep in mind, in contrast to Remark 2.4.1, that $(\tilde{\mathcal{M}}_{\mathcal{Q},X}, \|\cdot\|_1)$ is dependent on X.

Theorem 2.4.2. *Let $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ be a quotient module, let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map, and suppose $\psi = X(P_{\mathcal{Q}}1)$. Then X is liftable if and only if*

$$dist_{L^1(\mathbb{T}^n)}\Big(\frac{\bar{\psi}}{\|\psi\|_2^2}, \tilde{\mathcal{M}}_{\mathcal{Q},X}\Big) \geq 1.$$

Proof. In view of $\bar{\psi} \in \mathcal{Q}^{conj}$, first we observe that

$$\mathcal{M}_{\mathcal{Q}} = \mathbb{C}\bar{\psi} \dot{+} \tilde{\mathcal{M}}_{\mathcal{Q},X}.$$

Suppose X is liftable. By Theorem 2.4.1, we have

$$\Big|\int_{\mathbb{T}^n} \psi f \, d\mu\Big| \leq \|f\|_1 \qquad (f \in \mathcal{M}_{\mathcal{Q}}). \tag{2.4.6}$$

Pick $g \in \mathcal{M}_{\mathcal{Q}}$. There exists a scalar c and a function $\tilde{g} \in \tilde{\mathcal{M}}_{\mathcal{Q},X}$ such that

$$g = c\bar{\psi} + \tilde{g}.$$

We compute

$$\begin{aligned}\int_{\mathbb{T}^n} \psi(c\bar{\psi} + \tilde{g})\, d\mu &= c\int_{\mathbb{T}^n} \psi\bar{\psi}\, d\mu + \int_{\mathbb{T}^n} \psi\tilde{g}\, d\mu \\ &= c\|\psi\|_2^2 + \langle \psi, \bar{\tilde{g}}\rangle \\ &= c\|\psi\|_2^2,\end{aligned}$$

as

$$\langle \psi, \bar{\tilde{g}}\rangle = 0,$$

which follows from the definition of $\tilde{g}$ and the fact that ψ is analytic. Now (2.4.6) implies

$$\Big| \int_{\mathbb{T}^n} \psi(c\bar{\psi} + \tilde{g})\, d\mu \Big| \leq \|c\bar{\psi} + \tilde{g}\|_1,$$

and hence

$$|c|\|\psi\|_2^2 \leq \|c\bar{\psi} + \tilde{g}\|_1,$$

or equivalently

$$\Big\| \frac{\bar{\psi}}{\|\psi\|_2^2} + \tilde{g} \Big\|_1 \geq 1,$$

for all $\tilde{g} \in \tilde{\mathcal{M}}_{\mathcal{Q},X}$, and completes the proof of the forward direction. To prove the reverse direction, let the above inequality holds for all $\tilde{g} \in \tilde{\mathcal{M}}_{\mathcal{Q},X}$. Equivalently

$$\|\psi\|_2^2 \leq \|\bar{\psi} + \tilde{g}\|_1 \qquad (\tilde{g} \in \tilde{\mathcal{M}}_{\mathcal{Q},X}).$$

Fix $f \in \mathcal{M}_{\mathcal{Q}}$, and write $f = c\bar{\psi} + \tilde{f}$ for some scalar c and some function $\tilde{f} \in \tilde{\mathcal{M}}_{\mathcal{Q},X}$. Following the proof of the forward direction, we have

$$\begin{aligned}c\|\psi\|_2^2 &= \int_{\mathbb{T}^n} \psi(c\bar{\psi} + \tilde{f})\, d\mu \\ &= c\int_{\mathbb{T}^n} \psi f\, d\mu,\end{aligned}$$

which leads to (2.4.6). Theorem 2.4.1 now completes the proof of the theorem. □

Combining Theorem 2.4.1 and Theorem 2.4.2, we have the following:

Theorem 2.4.3. *Let $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ be a quotient module, and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map. Set*

$$\psi = X(P_{\mathcal{Q}}1),$$

and suppose

$$\mathcal{M}_{\mathcal{Q}} = \mathcal{Q}^{conj} \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)),$$

and

$$\tilde{\mathcal{M}}_{\mathcal{Q},X} = (\mathcal{Q}^{conj} \ominus \{\bar{\psi}\}) \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)).$$

Then the following conditions are equivalent:

1. *X is liftable.*
2. *$X_{\mathcal{Q}} : (\mathcal{M}_{\mathcal{Q}}, \|\cdot\|_1) \longrightarrow \mathbb{C}$ is a contractive functional, where*
$$X_{\mathcal{Q}} f = \int_{\mathbb{T}^n} \psi f \, d\mu \qquad (f \in \mathcal{M}_{\mathcal{Q}}).$$
3. *$dist_{L^1(\mathbb{T}^n)}\left(\frac{\bar{\psi}}{\|\psi\|_2^2}, \tilde{\mathcal{M}}_{\mathcal{Q},X}\right) \geq 1$.*

The techniques involved in the association of the existence of commutant lifting with the distance formula are far-reaching. In the following section, we will apply some of the concepts introduced here to solve a perturbation problem.

2.5 Interpolation

The goal of this section is to provide a solution to the interpolation problem. As previously mentioned, Sarason's commuting lifting theorem recovers the Nevanlinna-Pick interpolation with an elegant proof. However, Sarason only needed to use his lifting theorem for some special finite-dimensional quotient modules. These quotient modules are generated by finitely many kernel functions.

First, we prove that Sarason-type quotient modules (we call them zero-based quotient modules) in several variables always admit lifting to $H^\infty(\mathbb{D}^n)$-functions (we call it weak lifting).

Definition 2.5.1. *Let $\mathcal{Q} \subseteq H^2(\mathbb{D}^n)$ be a quotient module, and let $X \in \mathcal{B}(\mathcal{Q})$. Suppose $XS_{z_i} = S_{z_i}X$ for all $i = 1, \ldots, n$. We say that X admits a weak lift or X is weakly liftable if there exists $\varphi \in H^\infty(\mathbb{D}^n)$ such that*
$$X = S_\varphi.$$

To put it another way, a weak lifting is a lifting that lacks control over the norm. Given a set $\mathcal{Z} \subseteq \mathbb{D}^n$, define
$$\mathcal{Q}_{\mathcal{Z}} = \overline{\text{span}}\{\mathbb{S}(\cdot, w) : w \in \mathcal{Z}\}.$$

Definition 2.5.2. *A quotient module $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ is said to be zero-based if there exists $\mathcal{Z} \subseteq \mathbb{D}^n$ such that $\mathcal{Q} = \mathcal{Q}_{\mathcal{Z}}$.*

For a zero-based quotient module $\mathcal{Q}_{\mathcal{Z}}$, by using the reproducing property (2.2.4), we have the following representation of the corresponding submodule (hence the name zero-based)
$$\mathcal{Q}_{\mathcal{Z}}^\perp = \{f \in H^2(\mathbb{T}^n) : f(w) = 0 \text{ for all } w \in \mathcal{Z}\}.$$

Since $\{\mathbb{S}(\cdot, w) : w \in \mathcal{Z}\}$ is a set of linearly independent vectors, a zero-based quotient module $\mathcal{Q}_{\mathcal{Z}}$ is finite-dimensional if and only if

$$\#\mathcal{Z} = \dim \mathcal{Q}_{\mathcal{Z}} < \infty.$$

For each $j \in \{1, \ldots, n\}$, denote by $\pi_j : \mathbb{C}^n \longrightarrow \mathbb{C}$ the projection map onto the j-th coordinate. In particular, $z \in \mathbb{C}^n$ can be expressed as

$$z = (\pi_1(z), \ldots, \pi_n(z)).$$

The following easy-to-see lemma will be useful in what follows.

Lemma 2.5.3. *Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{D}^n$ be a set of distinct points, and let $X \in \mathcal{B}(\mathcal{Q}_{\mathcal{Z}})$. Then X is module map if and only if there exists $\{w_i\}_{i=1}^m \subset \mathbb{C}$ such that*

$$X^* \mathbb{S}(\cdot, z_j) = w_j \mathbb{S}(\cdot, z_j),$$

for all $j = 1, \ldots, m$.

Proof. Let $X \in \mathcal{B}(\mathcal{Q}_{\mathcal{Z}})$ and suppose $X S_{z_i} = S_{z_i} X$ for all $i = 1, \ldots, n$. Since $X^* S_{z_i}^* = S_{z_i}^* X^*$, using the fact that $\mathcal{Q}_{\mathcal{Z}}$ is a quotient module, we find

$$T_{z_i}^*|_{\mathcal{Q}_{\mathcal{Z}}} X^* = X^* T_{z_i}^*|_{\mathcal{Q}_{\mathcal{Z}}},$$

for all $i = 1, \ldots, n$. In view of (2.2.3), we compute

$$\begin{aligned}(T_{z_i}^*|_{\mathcal{Q}_{\mathcal{Z}}} X^*)\mathbb{S}(\cdot, z_j) &= (X^* T_{z_i}^*|_{\mathcal{Q}_{\mathcal{Z}}})\mathbb{S}(\cdot, z_j) \\ &= X^* T_{z_i}^* \mathbb{S}(\cdot, z_j) \\ &= \overline{\pi_i(z_j)} X^* \mathbb{S}(\cdot, z_j).\end{aligned}$$

Since $(T_{z_i}^*|_{\mathcal{Q}_{\mathcal{Z}}} X^*)\mathbb{S}(\cdot, z_j) = T_{z_i}^*(X^* \mathbb{S}(\cdot, z_j))$, it follows that

$$T_{z_i}^*(X^* \mathbb{S}(\cdot, z_j)) = \overline{\pi_i(z_j)}(X^* \mathbb{S}(\cdot, z_j)),$$

for all $i = 1, \ldots, n$, and $j = 1, \ldots, m$. Equivalently

$$X^* \mathbb{S}(\cdot, z_j) \in \bigcap_{i=1}^n \ker(T_{z_i} - \pi_i(z_j) I_{H^2(\mathbb{T}^n)})^*,$$

for all $j = 1, \ldots, m$. Now, in view of the joint eigenspace property (2.2.2), the right side of the above is $\mathbb{C}\mathbb{S}(\cdot, z_j)$, and hence, there exists a scalar w_j such that

$$X^* \mathbb{S}(\cdot, z_j) = w_j \mathbb{S}(\cdot, z_j),$$

for all $j = 1, \ldots, m$. The converse direction is easy and follows again from (2.2.3) and the definition of $\mathcal{Q}_{\mathcal{Z}}$. $\square$

The proposition that follows is very crucial and will be used in what follows.

Proposition 2.5.4. *Let $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ be a quotient module. Let*

$$\theta_{\mathcal{Q}} = P_{\mathcal{Q}} 1.$$

If $\theta_{\mathcal{Q}} \in H^\infty(\mathbb{D}^n)$, then $S_{\theta_{\mathcal{Q}}} = I_{\mathcal{Q}}$.

Proof. Since $\theta_{\mathcal{Q}} = P_{\mathcal{Q}} 1 \in \mathcal{Q} \cap H^\infty(\mathbb{D}^n)$, in view of the decomposition $H^2(\mathbb{T}^n) = \mathcal{Q} \oplus \mathcal{Q}^\perp$, there exists $\varphi \in H^\infty(\mathbb{D}^n) \cap \mathcal{Q}^\perp$ such that

$$1 = \theta_{\mathcal{Q}} \oplus \varphi \in \mathcal{Q} \oplus \mathcal{Q}^\perp.$$

Fix $f \in \mathcal{Q}$. In particular, since $f \in H^2(\mathbb{T}^n)$, there exists a sequence

$$\{p_j\}_{j=1}^\infty \subseteq \mathbb{C}[z_1, \ldots, z_n],$$

such that

$$p_j \longrightarrow f \quad \text{in } H^2(\mathbb{T}^n).$$

Since $\varphi \in H^\infty(\mathbb{D}^n) \cap \mathcal{Q}^\perp$ is a multiplier, the above implies

$$\varphi p_j \longrightarrow \varphi f \quad \text{in } H^2(\mathbb{T}^n).$$

Moreover, $\varphi \in \mathcal{Q}^\perp$ implies that

$$\{p_j \varphi\}_{j=1}^\infty \subseteq \mathcal{Q}^\perp,$$

as $\mathcal{Q}^\perp$ is a submodule, and hence $\varphi f \in \mathcal{Q}^\perp$. Equivalently, we have

$$P_{\mathcal{Q}}(\varphi f) = 0.$$

Finally, since $\theta_{\mathcal{Q}}, \varphi \in H^\infty(\mathbb{D}^n)$, it follows that

$$\begin{aligned} f &= \theta_{\mathcal{Q}} f + \varphi f \\ &= P_{\mathcal{Q}}(\theta_{\mathcal{Q}} f + \varphi f) \qquad (\text{as } f \in \mathcal{Q}) \\ &= P_{\mathcal{Q}}(\theta_{\mathcal{Q}} f) + 0 \\ &= (P_{\mathcal{Q}}(P_{\mathcal{Q}} 1)|_{\mathcal{Q}}) f, \end{aligned}$$

which yields $S_{\theta_{\mathcal{Q}}} f = f$, and completes the proof of the proposition. $\square$

We are now ready for the weak lifting. It asserts, in essence, that a module map on a finite-dimensional zero-based quotient module always admits a lift to $H^\infty(\mathbb{D}^n)$.

Corollary 2.5.5. *Let $\mathcal{Z} = \{z_1, \ldots, z_m\} \subset \mathbb{D}^n$ be m distinct points, and let $X \in \mathcal{B}(\mathcal{Q}_{\mathcal{Z}})$. Then*

$$X S_{z_i} = S_{z_i} X,$$

for all $i = 1, \ldots, n$, if and only if there exists $\varphi \in H^\infty(\mathbb{D}^n)$ such that

$$X = S_\varphi.$$

Moreover, the function φ is given by

$$\varphi = X(P_{\mathcal{Q}_\mathcal{Z}} 1),$$

and, in particular

$$\varphi \in H^\infty(\mathbb{D}^n) \cap \mathcal{Q}_\mathcal{Z}.$$

Proof. The sufficient part is trivial. We prove the necessary part. For simplicity of notation, we set $\mathcal{Q} = \mathcal{Q}_\mathcal{Z}$. Suppose $X \in \mathcal{B}(\mathcal{Q})$ and suppose that $XS_{z_i} = S_{z_i}X$ for all $i = 1, \ldots, n$. As in Proposition 2.5.4, set

$$\theta_\mathcal{Q} = P_\mathcal{Q} 1.$$

As observed earlier, $\mathbb{S}(\cdot, w) \in H^\infty(\mathbb{D}^n)$ for all $w \in \mathbb{D}^n$ implies that $\mathcal{Q} \subseteq H^\infty(\mathbb{D}^n)$. In particular, $\theta_\mathcal{Q} \in H^\infty(\mathbb{D}^n)$. By Proposition 2.5.4, we have

$$S_{\theta_\mathcal{Q}} = I_\mathcal{Q}.$$

Since $X \in \mathcal{B}(\mathcal{Q})$, it follows that

$$\varphi := X\theta_\mathcal{Q} \in H^\infty(\mathbb{D}^n).$$

Therefore

$$\begin{aligned} S_\varphi \theta_\mathcal{Q} &= P_\mathcal{Q}(\varphi\theta_\mathcal{Q}) \\ &= S_{\theta_\mathcal{Q}} \varphi \\ &= \varphi \\ &= X\theta_\mathcal{Q}. \end{aligned}$$

The remainder of the proof is based on the standard property of the module map X. Indeed, we first observe that

$$\mathcal{Q} = \overline{\text{span}}\{P_\mathcal{Q} z^k P_\mathcal{Q} 1 : k \in \mathbb{Z}_+^n\}.$$

On the other hand, for $k \in \mathbb{Z}_+^n$, since $XS_z^k = S_z^k X$, we have

$$\begin{aligned} X(P_{\mathcal{Q}}(z^k \theta_{\mathcal{Q}})) &= X(S_z^k \theta_{\mathcal{Q}}) \\ &= S_z^k X \theta_{\mathcal{Q}} \\ &= P_{\mathcal{Q}} z^k \varphi \\ &= P_{\mathcal{Q}} z^k S_\varphi \theta_{\mathcal{Q}} \\ &= S_\varphi (P_{\mathcal{Q}}(z^k \theta_{\mathcal{Q}})). \end{aligned}$$

This completes the proof of the fact that $X = S_\varphi$. The final assertion follows from the definition of $\theta_{\mathcal{Q}}$. □

As already pointed out, the weak lifting does not touch the delicate structure of the Schur functions on $\mathbb{D}^n$, $n > 1$.

We will now look at the interpolation problem. Recall once again that

$$\mathbb{S}(z, w) = \prod_{i=1}^m \frac{1}{1 - z_i \bar{w}_i} \qquad (z, w \in \mathbb{D}^n),$$

is the Szegö kernel of $\mathbb{D}^n$, and

$$\mathbb{S}(z, w) = \Big\langle \mathbb{S}(\cdot, w), \mathbb{S}(\cdot, z) \Big\rangle_{H^2(\mathbb{T}^n)} \qquad (z, w \in \mathbb{D}^n).$$

Theorem 2.5.6. *Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{D}^n$ be m distinct points, and let $\{w_i\}_{i=1}^m \subset \mathbb{D}$ be m scalars. Set*

$$\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}} = \mathcal{Q}_{\mathcal{Z}}^{conj} \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)).$$

Then there exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that

$$\varphi(z_i) = w_i,$$

for all $i = 1, \ldots, m$, if and only if

$$M_{\mathcal{Z},\mathcal{W}} f = \int_{\mathbb{T}^n} \psi_{\mathcal{Z},\mathcal{W}} f \, d\mu \qquad (f \in \mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}),$$

defines a contraction $M_{\mathcal{Z},\mathcal{W}} : (\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}, \| \cdot \|_1) \to \mathbb{C}$, where

$$\psi_{\mathcal{Z},\mathcal{W}} = \sum_{i=1}^m c_i \mathbb{S}(\cdot, z_i),$$

and the scalar coefficients $\{c_i\}_{i=1}^m$ are given by the identity

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} \mathbb{S}(z_1, z_1) & \mathbb{S}(z_1, z_2) & \cdots & \mathbb{S}(z_1, z_m) \\ \mathbb{S}(z_2, z_1) & \mathbb{S}(z_2, z_2) & \cdots & \mathbb{S}(z_2, z_m) \\ \vdots & \ddots & \ddots & \vdots \\ \mathbb{S}(z_m, z_1) & \mathbb{S}(z_m, z_2) & \cdots & \mathbb{S}(z_m, z_m) \end{bmatrix}^{-1} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}.$$

Proof. Consider the module map $X_{\mathcal{Z},\mathcal{W}}$ on the quotient module $\mathcal{Q}_{\mathcal{Z}}$ as (see Lemma 2.5.3)

$$X^*_{\mathcal{Z},\mathcal{W}} \mathbb{S}(\cdot, z_j) = \bar{w}_j \mathbb{S}(\cdot, z_j),$$

for all $j = 1, \ldots, m$. Define

$$\psi_{\mathcal{Z},\mathcal{W}} = X_{\mathcal{Z},\mathcal{W}}(P_{\mathcal{Q}_{\mathcal{Z}}} 1). \tag{2.5.1}$$

We note the crucial fact that (as $\mathcal{Q}_{\mathcal{Z}} \subset H^\infty(\mathbb{D}^n)$, or see Corollary 2.5.5)

$$\psi_{\mathcal{Z},\mathcal{W}} \in H^\infty(\mathbb{D}^n).$$

Claim: A function $\varphi \in \mathcal{S}(\mathbb{D}^n)$ interpolates $\{z_i\}_{i=1}^m \subset \mathbb{D}^n$ and $\{w_i\}_{i=1}^m \subset \mathbb{D}$, that is

$$\varphi(z_i) = w_i,$$

for all $i = 1, \ldots, m$, if and only if

$$S_\varphi = X_{\mathcal{Z},\mathcal{W}}.$$

Indeed, since $\mathbb{S}(\cdot, z_i) \in \mathcal{Q}_{\mathcal{Z}}$, it follows that $P_{\mathcal{Q}_{\mathcal{Z}}} \mathbb{S}(\cdot, z_i) = \mathbb{S}(\cdot, z_i)$ and hence

$$\begin{aligned} S^*_\varphi \mathbb{S}(\cdot, z_i) &= P_{\mathcal{Q}_{\mathcal{Z}}} T^*_\varphi \mathbb{S}(\cdot, z_i) \\ &= \overline{\varphi(z_i)} \mathbb{S}(\cdot, z_i), \end{aligned}$$

for all $i = 1, \ldots, m$. The definition of $X_{\mathcal{Z},\mathcal{W}}$ now supports the claim. Of course, φ is a lift of $X_{\mathcal{Z},\mathcal{W}}$. Then, by Theorem 2.4.1, it follows that $\varphi \in \mathcal{S}(\mathbb{D}^n)$ interpolates $\{z_i\}_{i=1}^m$ and $\{w_i\}_{i=1}^m$ if and only if

$$M_{\mathcal{Z},\mathcal{W}} f = \int_{\mathbb{T}^n} \psi_{\mathcal{Z},\mathcal{W}} f \, d\mu \qquad (f \in \mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}),$$

defines a contraction $M_{\mathcal{Z},\mathcal{W}} : (\mathcal{M}_{\mathcal{Q}_{\mathcal{Z}}}, \|\cdot\|_1) \to \mathbb{C}$. This proves the first half of the theorem. Now all that is left to do is calculate the representation of $\psi_{\mathcal{Z},\mathcal{W}}$. Corollary 2.5.5 says that

$$X_{\mathcal{Z},\mathcal{W}} = S_\varphi = S_{\psi_{\mathcal{Z},\mathcal{W}}}.$$

Since $\psi_{\mathcal{Z},\mathcal{W}} \in \mathcal{Q}_{\mathcal{Z}}$, there exists scalars $\{c_i\}_{i=1}^m$ such that

$$\psi_{\mathcal{Z},\mathcal{W}} = \sum_{i=1}^m c_i \mathbb{S}(\cdot, z_i).$$

To compute the coefficients $\{c_i\}_{i=1}^m$ of the preceding expansion, we employ both reproducing kernel Hilbert space methods and conventional linear algebra. Fix $j \in \{1, \ldots, m\}$. Then

$$\begin{aligned} X^*_{\mathcal{Z},\mathcal{W}} \mathbb{S}(\cdot, z_j) &= S^*_{\psi_{\mathcal{Z},\mathcal{W}}} \mathbb{S}(\cdot, z_j) \\ &= \overline{\psi_{\mathcal{Z},\mathcal{W}}(z_j)} \mathbb{S}(\cdot, z_j), \end{aligned}$$

where, on the other hand, $X^*_{\mathcal{Z},\mathcal{W}} \mathbb{S}(\cdot, z_j) = \bar{w}_j \mathbb{S}(\cdot, z_j)$. Therefore

$$w_j = \psi_{\mathcal{Z},\mathcal{W}}(z_j),$$

and hence, by the reproducing property of kernel functions (2.2.4), it follows that

$$\begin{aligned} w_j &= \psi_{\mathcal{Z},\mathcal{W}}(z_j) \\ &= \Big\langle \psi_{\mathcal{Z},\mathcal{W}}, \mathbb{S}(\cdot, z_j) \Big\rangle_{H^2(\mathbb{T}^n)} \\ &= \Big\langle \sum_{i=1}^m c_i \mathbb{S}(\cdot, z_i), \mathbb{S}(\cdot, z_j) \Big\rangle_{H^2(\mathbb{T}^n)} \\ &= \sum_{i=1}^m c_i \mathbb{S}(z_j, z_i), \end{aligned}$$

for all $j = 1, \ldots, m$. In other words, we have

$$\begin{bmatrix} \mathbb{S}(z_1, z_1) & \mathbb{S}(z_1, z_2) & \cdots & \mathbb{S}(z_1, z_m) \\ \mathbb{S}(z_2, z_1) & \mathbb{S}(z_2, z_2) & \cdots & \mathbb{S}(z_2, z_m) \\ \vdots & \ddots & \ddots & \vdots \\ \mathbb{S}(z_m, z_1) & \mathbb{S}(z_m, z_2) & \cdots & \mathbb{S}(z_m, z_m) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix},$$

equivalently

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} \mathbb{S}(z_1, z_1) & \mathbb{S}(z_1, z_2) & \cdots & \mathbb{S}(z_1, z_m) \\ \mathbb{S}(z_2, z_1) & \mathbb{S}(z_2, z_2) & \cdots & \mathbb{S}(z_2, z_m) \\ \vdots & \ddots & \ddots & \vdots \\ \mathbb{S}(z_m, z_1) & \mathbb{S}(z_m, z_2) & \cdots & \mathbb{S}(z_m, z_m) \end{bmatrix}^{-1} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}.$$

The above $m \times m$ matrix is nothing but the Gram matrix of the linearly independent kernel functions $\{\mathbb{S}(\cdot, z_i) : i = 1, \ldots, m\}$. The invertibility of the matrix is now immediate. $\square$

For solutions to the interpolation problem in the setting of bounded harmonic functions and H^p functions, we refer the reader to [47]

2.6 Quantitative interpolation and examples

This section is a continuation of our investigation into the interpolation problem. To begin, we will provide a quantitative solution to the interpolation problem on $\mathbb{D}^n$. The quantitative solution will then be employed to generate examples of interpolation with interpolating functions in $\mathcal{S}(\mathbb{D}^n)$, $n \geq 2$.

Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{D}^n$ be m distinct points, and let $\{w_i\}_{i=1}^m \subset \mathbb{D}$ be m scalars. As usual, define the m-dimensional zero-based quotient module $\mathcal{Q}_{\mathcal{Z}}$ of $H^2(\mathbb{T}^n)$ by

$$\mathcal{Q}_{\mathcal{Z}} = \text{span}\{\mathbb{S}(\cdot, z_i) : i = 1, \ldots, m\}, \tag{2.6.1}$$

and $X_{\mathcal{Z},\mathcal{W}} \in \mathcal{B}(\mathcal{Q}_{\mathcal{Z}})$ by

$$X_{\mathcal{Z},\mathcal{W}}^* \mathbb{S}(\cdot, z_j) = \bar{w}_j \mathbb{S}(\cdot, z_j) \qquad (j = 1, \ldots, m).$$

As observed in Lemma 2.5.3, $X_{\mathcal{Z},\mathcal{W}}$ is a module map, and hence Corollary 2.5.5 implies

$$X_{\mathcal{Z},\mathcal{W}} = S_\psi, \tag{2.6.2}$$

where

$$\psi := X_{\mathcal{Z},\mathcal{W}}(P_{\mathcal{Q}_{\mathcal{Z}}} 1).$$

Recall that

$$\psi \in \mathcal{Q}_{\mathcal{Z}} \subset H^\infty(\mathbb{D}^n).$$

On the other hand, as observed in the proof of Theorem 2.5.6 (more specifically, the claim part in the proof of Theorem 2.5.6), there exists a function $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that

$$\varphi(z_i) = w_i,$$

for all $i = 1, \ldots, m$, if and only if

$$S_\varphi = X_{\mathcal{Z},\mathcal{W}}.$$

Equivalently, $X_{\mathcal{Z},\mathcal{W}}$ on $\mathcal{Q}_{\mathcal{Z}}$ is a contraction and admits a lift (namely, $\varphi \in \mathcal{S}(\mathbb{D}^n)$). Based on Theorem 2.4.2 and the fact that $\psi = X_{\mathcal{Z},\mathcal{W}}(P_{\mathcal{Q}_{\mathcal{Z}}} 1)$, this is the same as saying that

$$\text{dist}_{L^1(\mathbb{T}^n)}\Big(\frac{\bar{\psi}}{\|\psi\|_2^2}, \tilde{\mathcal{M}}_{\mathcal{Q}_{\mathcal{Z}}}\Big) \geq 1,$$

where $\tilde{\mathcal{M}}_{\mathcal{Q}_{\mathcal{Z}}} = (\mathcal{Q}_{\mathcal{Z}}^{conj} \ominus \{\bar{\psi}\}) \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n))$. This results in the quantitative solution to the interpolation problem:

Theorem 2.6.1. *Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{D}^n$ be m distinct points, and let $\{w_i\}_{i=1}^m \subset \mathbb{D}$ be m scalars. Suppose $\psi := X_{\mathcal{Z},\mathcal{W}}(P_{\mathcal{Q}_\mathcal{Z}} 1)$ and*

$$\tilde{\mathcal{M}}_{\mathcal{Q}_\mathcal{Z}} = (\mathcal{Q}_\mathcal{Z}^{conj} \ominus \{\bar{\psi}\}) \dot{+} (\mathcal{M}_n \dot{+} H_0^2(\mathbb{T}^n)).$$

Then there exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that

$$\varphi(z_i) = w_i,$$

for all $i = 1, \dots, m$, if and only if

$$dist_{L^1(\mathbb{T}^n)}\Big(\frac{\bar{\psi}}{\|\psi\|_2^2}, \tilde{\mathcal{M}}_{\mathcal{Q}_\mathcal{Z}}\Big) \geq 1.$$

Moreover, in this case, we have

$$\psi(z_i) = w_i,$$

for all $i = 1, \dots, m$.

Here is how the proof of the final assertion works: For each $i = 1, \dots, m$, in view of the definition of $X_{\mathcal{Z},\mathcal{W}}$ and (2.6.2), we compute

$$\begin{aligned}
\bar{w}_i \mathbb{S}(\cdot, z_i) &= X_{\mathcal{Z},\mathcal{W}}^* \mathbb{S}(\cdot, z_i) \\
&= S_\psi^* \mathbb{S}(\cdot, z_i) \\
&= P_{\mathcal{Q}_\mathcal{Z}} T_\psi^* \mathbb{S}(\cdot, z_i) \\
&= \overline{\psi(z_i)} \mathbb{S}(\cdot, z_i),
\end{aligned}$$

as $\mathbb{S}(\cdot, z_i) \in \mathcal{Q}_\mathcal{Z}$. Therefore, $\psi(z_i) = w_i$ for all $i = 1, \dots, m$, which completes the proof. The final assertion will play an important role in the discussion that follows.

The rest of this section will be devoted to exploring examples of interpolation. We need to prove two lemmas. Before doing so, let us standardize some notations. We will set aside $m \geq 2$ as the number of nodes of the given interpolation data. We use bold letters such as $\boldsymbol{a}$, $\boldsymbol{v}$, $\boldsymbol{w}$, etc. to denote vectors in $\mathbb{C}^m$. For instance

$$\boldsymbol{a} = (a_1, \dots, a_m) \in \mathbb{C}^m.$$

Also, denote by $\langle \cdot, \cdot \rangle_{\mathbb{C}^m}$ the standard inner product on $\mathbb{C}^m$. In particular

$$\|\boldsymbol{a}\|_{\mathbb{C}^m} = (\sum_{i=1}^m |a_i|^2)^{\frac{1}{2}}.$$

We write

$$\boldsymbol{a}^\perp = \{\boldsymbol{v} \in \mathbb{C}^m : \langle \boldsymbol{a}, \boldsymbol{v} \rangle_{\mathbb{C}^m} = 0\}.$$

In view of the above notation, for each $\boldsymbol{a} \in \mathbb{C}^m$, we have the orthogonal decomposition

$$\mathbb{C}^m = \mathbb{C}\boldsymbol{a} \oplus \boldsymbol{a}^{\perp}.$$

We will work in the following general setting: Fix m distinct points $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{D}^n$ and m scalars $\{w_i\}_{i=1}^m \subset \mathbb{D}$. The quotient module of interest will be $\mathcal{Q}_{\mathcal{Z}} \subset H^\infty(\mathbb{D}^n)$ as defined in (2.6.1).

Lemma 2.6.2. *Let $\psi \in \mathcal{Q}_{\mathcal{Z}}$, and suppose $\psi(z_i) = w_i$ for all $i = 1, \ldots, m$. Then there exits $\boldsymbol{v} \in \boldsymbol{w}^{\perp}$ such that*

$$\psi = \frac{\|\psi\|_2^2}{\|\boldsymbol{w}\|_{\mathbb{C}^m}^2} \sum_{i=1}^m w_i \mathbb{S}(\cdot, z_i) + \sum_{i=1}^m v_i \mathbb{S}(\cdot, z_i).$$

Proof. Since $\psi \in \mathcal{Q}_{\mathcal{Z}}$, there exists $\boldsymbol{c} \in \mathbb{C}^m$ such that

$$\psi = \sum_{i=1}^m c_i \mathbb{S}(\cdot, z_i).$$

Moreover, there exist a scalar $\alpha \in \mathbb{C}$ and a vector $\boldsymbol{v} \in \boldsymbol{c}^{\perp}$ such that $\boldsymbol{c} = \alpha \boldsymbol{w} \oplus \boldsymbol{v}$. Then

$$\psi = \alpha \sum_{i=1}^m w_i \mathbb{S}(\cdot, z_i) + \sum_{i=1}^m v_i \mathbb{S}(\cdot, z_i).$$

By assumption, $\psi \in H^\infty(\mathbb{D}^n)$ and $\psi(z_i) = w_i$ for all $i = 1, \ldots, m$. The above equality then results in

$$\begin{aligned}
\|\psi\|_2^2 &= \Big\langle \alpha \sum_{i=1}^m w_i \mathbb{S}(\cdot, z_i) + \sum_{i=1}^m v_i \mathbb{S}(\cdot, z_i), \psi \Big\rangle_{H^2(\mathbb{T}^n)} \\
&= \alpha \Big\langle T_\psi^* \Big(\sum_{i=1}^m w_i \mathbb{S}(\cdot, z_i) + \sum_{i=1}^m v_i \mathbb{S}(\cdot, z_i) \Big), 1 \Big\rangle_{H^2(\mathbb{T}^n)} \\
&= \alpha \Big\langle \Big(\sum_{i=1}^m w_i \overline{\psi(z_i)} \mathbb{S}(\cdot, z_i) + \sum_{i=1}^m v_i \overline{\psi(z_i)} \mathbb{S}(\cdot, z_i) \Big), 1 \Big\rangle_{H^2(\mathbb{T}^n)} \\
&= \alpha \Big\langle \Big(\sum_{i=1}^m |w_i|^2 \mathbb{S}(\cdot, z_i) + \sum_{i=1}^m v_i \bar{w}_i \mathbb{S}(\cdot, z_i) \Big), 1 \Big\rangle_{H^2(\mathbb{T}^n)} \\
&= \alpha \|\boldsymbol{w}\|_{\mathbb{C}^m}^2 + \sum_{i=1}^m v_i \bar{w}_i \\
&= \alpha \|\boldsymbol{w}\|_{\mathbb{C}^m}^2,
\end{aligned}$$

as $\boldsymbol{v} \perp \boldsymbol{w}$. We have also used the general property that $\langle \mathbb{S}(\cdot, w), 1 \rangle_{H^2(\mathbb{T}^n)} = 1$ for all $w \in \mathbb{D}^n$. The above identity yields

$$\alpha = \frac{\|\psi\|_2^2}{\|\boldsymbol{w}\|_{\mathbb{C}^m}^2},$$

which completes the proof of the lemma. □

The proof of the following lemma is similar to the proof of the previous one.

Lemma 2.6.3. *Let $\psi \in \mathcal{Q}_{\mathcal{Z}}$, and suppose $\psi(z_i) = w_i$ for all $i = 1, \ldots, m$. Then*

$$\mathcal{Q}_{\mathcal{Z}} \ominus \{\psi\} = \Big\{ \sum_{i=1}^{m} v_i \mathbb{S}(\cdot, z_i) : \boldsymbol{v} \in \boldsymbol{w}^{\perp} \Big\}.$$

Proof. Given $\boldsymbol{v} \in \mathbb{C}^m$, observe that

$$\sum_{i=1}^{m} v_i \mathbb{S}(\cdot, z_i) \perp \psi,$$

if and only if

$$\begin{aligned} 0 &= \Big\langle \sum_{i=1}^{m} v_i \mathbb{S}(\cdot, z_i), \psi \Big\rangle_{H^2(\mathbb{T}^n)} \\ &= \Big\langle T_{\psi}^{*} \Big(\sum_{i=1}^{m} v_i \mathbb{S}(\cdot, z_i) \Big), 1 \Big\rangle_{H^2(\mathbb{T}^n)} \\ &= \Big\langle \sum_{i=1}^{m} v_i \overline{\psi(z_i)} \mathbb{S}(\cdot, z_i), 1 \Big\rangle_{H^2(\mathbb{T}^n)} \\ &= \sum_{i=1}^{m} v_i \bar{w}_i \\ &= \langle \boldsymbol{v}, \boldsymbol{w} \rangle_{\mathbb{C}^m}. \end{aligned}$$

This completes the proof of the lemma. □

Now we are ready for examples of interpolation on $\mathbb{D}^n$, $n \geq 2$. First, we elaborate on the construction of the 3-point interpolation problem. Suppose:

1. $\{\boldsymbol{b}_0, \boldsymbol{b}_1, \boldsymbol{b}_2\}$ is an orthogonal basis for $\mathbb{C}^3$, where

$$\begin{cases} \boldsymbol{b}_0 = (1, 1, 1) \\ \boldsymbol{b}_1 = (\zeta_{11}, \zeta_{12}, \zeta_{13}) \\ \boldsymbol{b}_2 = (\zeta_{21}, \zeta_{22}, \zeta_{23}). \end{cases}$$

2. $\|\boldsymbol{b}_1\|_{\mathbb{C}^3}, \|\boldsymbol{b}_2\|_{\mathbb{C}^3} \geq 1$.

3. $\{z_1, z_2, z_3\} \subset \mathbb{D}^n$ are three distinct points such that

$$\begin{cases} z_1 = (\zeta_{11}, \zeta_{21}, \tilde{z}_1) \\ z_2 = (\zeta_{12}, \zeta_{22}, \tilde{z}_2) \\ z_3 = (\zeta_{13}, \zeta_{23}, \tilde{z}_3), \end{cases}$$

for some (arbitrary) $\tilde{z}_1, \tilde{z}_1, \tilde{z}_3 \in \mathbb{D}^{n-2}$.

4. $\boldsymbol{w} = (w_1, w_2, w_3) \in \mathbb{D}^3$ such that $\|\boldsymbol{w}\|_{\mathbb{C}^3} \leq \frac{1}{\sqrt{3}}$.

Claim: There exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that $\varphi(z_i) = w_i$ for all $i = 1, 2, 3$.

Here is how the proof of the claim goes: In view of Theorem 2.6.1, it is enough to prove that

$$\mathrm{dist}_{L^1(\mathbb{T}^n)}\Big(\frac{\bar{\psi}}{\|\psi\|_2^2}, \tilde{\mathcal{M}}_{\mathcal{Q}_{\mathcal{Z}}}\Big) \geq 1,$$

where $\psi = X_{\mathcal{Z},\mathcal{W}}(P_{\mathcal{Q}_{\mathcal{Z}}}1)$ and

$$\tilde{\mathcal{M}}_{\mathcal{Q}_{\mathcal{Z}}} = (\mathcal{Q}_{\mathcal{Z}}^{conj} \ominus \{\bar{\psi}\}) \dotplus (\mathcal{M}_n \dotplus H_0^2(\mathbb{T}^n)).$$

Recall that $X_{\mathcal{Z},\mathcal{W}} \in \mathcal{B}(\mathcal{Q}_{\mathcal{Z}})$ is defined by

$$X_{\mathcal{Z},\mathcal{W}}^* \mathbb{S}(\cdot, z_i) = \bar{w}_i \mathbb{S}(\cdot, z_i),$$

for all $i = 1, \ldots, m$. Also recall the crucial fact that (see Theorem 2.6.1)

$$\psi(z_i) = w_i \qquad (i = 1, \ldots, m).$$

Using the conjugation invariance property of L^1-norm (that is, $\|f\|_{L^1(\mathbb{T}^n)} = \|\bar{f}\|_{L^1(\mathbb{T}^n)}$ for all $f \in L^1(\mathbb{T}^n)$), we infer that

$$\mathrm{dist}_{L^1(\mathbb{T}^n)}\Big(\frac{\bar{\psi}}{\|\psi\|_2^2}, \tilde{\mathcal{M}}_{\mathcal{Q}_{\mathcal{Z}}}\Big) = \mathrm{dist}_{L^1(\mathbb{T}^n)}\Big(\frac{\psi}{\|\psi\|_2^2}, \tilde{\mathcal{M}}_{\mathcal{Q}_{\mathcal{Z}}}^{conj}\Big),$$

where (recall that $\mathcal{M}_n^{conj} = \mathcal{M}_n$)

$$\tilde{\mathcal{M}}_{\mathcal{Q}_{\mathcal{Z}}}^{conj} = (\mathcal{Q}_{\mathcal{Z}} \ominus \{\psi\}) \dotplus (\mathcal{M}_n \dotplus H_0^2(\mathbb{T}^n)^{conj}).$$

It will be convenient (as well as enough) to prove that

$$\mathrm{dist}_{L^1(\mathbb{T}^n)}\Big(\frac{\psi}{\|\psi\|_2^2}, \tilde{\mathcal{M}}_{\mathcal{Q}_{\mathcal{Z}}}^{conj}\Big) \geq 1.$$

Also, to avoid notational confusion, we use $\{Z_1, \ldots, Z_n\}$ for the variables of $\mathbb{C}^n$. By the definition of Szegö kernel, we have

$$\begin{cases} \mathbb{S}(\cdot, z_1) = 1 + \bar{\zeta}_{11} Z_1 + \bar{\zeta}_{21} Z_2 + \cdots \\ \mathbb{S}(\cdot, z_2) = 1 + \bar{\zeta}_{12} Z_1 + \bar{\zeta}_{22} Z_2 + \cdots \\ \mathbb{S}(\cdot, z_3) = 1 + \bar{\zeta}_{13} Z_1 + \bar{\zeta}_{23} Z_2 + \cdots . \end{cases} \tag{2.6.3}$$

We will need to prove the following inequality

$$\Big\| \frac{\psi}{\|\psi\|_2^2} + f \Big\|_{L^1(\mathbb{T}^n)} \geq 1 \qquad (f \in \tilde{\mathcal{M}}_{\mathcal{Q}_{\mathcal{Z}}}^{conj}).$$

Since $\psi(z_i) = w_i$ for all $i = 1, \ldots, m$, in view of Lemma 2.6.3, an element $f \in \tilde{\mathcal{M}}_{\mathcal{Q}_{\mathcal{Z}}}^{conj}$ admits the following representation

$$f = \sum_{i=1}^{3} v_i \mathbb{S}(\cdot, z_i) + \tilde{f},$$

for some $\boldsymbol{v} \in \boldsymbol{w}^{\perp}$ and $\tilde{f} \in \mathcal{M}_n \dotplus H_0^2(\mathbb{T}^n)^{conj}$. Therefore, for each $f \in \tilde{\mathcal{M}}_{\mathcal{Q}_{\mathcal{Z}}}^{conj}$, by Lemma 2.6.2, we conclude that

$$\frac{\psi}{\|\psi\|_2^2} + f = \sum_{i=1}^{3} \frac{w_i}{\|\boldsymbol{w}\|_{\mathbb{C}^3}^2} \mathbb{S}(\cdot, z_i) + \sum_{i=1}^{3} v_i \mathbb{S}(\cdot, z_i) + \tilde{f},$$

for some $\boldsymbol{v} \in \boldsymbol{w}^{\perp}$ and $\tilde{f} \in \mathcal{M}_n \dotplus H_0^2(\mathbb{T}^n)^{conj}$. For each $\boldsymbol{v} \in \boldsymbol{w}^{\perp}$, we set

$$F_{\boldsymbol{v}} = \frac{1}{\|\boldsymbol{w}\|_{\mathbb{C}^3}^2} \boldsymbol{w} \oplus \boldsymbol{v}.$$

It is important to keep in mind that $\boldsymbol{v}$ and $\tilde{f}$ depend on f. By assumption, $\|\boldsymbol{w}\|_{\mathbb{C}^3} \leq \frac{1}{\sqrt{3}}$, and hence

$$\|F_{\boldsymbol{v}}\|_{\mathbb{C}^3} \geq \sqrt{3}.$$

Using the kernel functions' power series expansion as in (2.6.3), we find

$$\begin{aligned}\frac{\psi}{\|\psi\|_2^2} + f &= \sum_{i=1}^{3} \frac{w_i}{\|\boldsymbol{w}\|_{\mathbb{C}^3}^2} \mathbb{S}(\cdot, z_i) + \sum_{i=1}^{3} v_i \mathbb{S}(\cdot, z_i) + \tilde{f} \\ &= \Big(\langle F_{\boldsymbol{v}}, \boldsymbol{b}_0 \rangle_{\mathbb{C}^3} 1 + \langle F_{\boldsymbol{v}}, \boldsymbol{b}_1 \rangle_{\mathbb{C}^3} Z_1 + \langle F_{\boldsymbol{v}}, \boldsymbol{b}_2 \rangle_{\mathbb{C}^3} Z_2 + \cdots \Big) + \tilde{f}.\end{aligned}$$

There exists $i \in \{0, 1, 2\}$ such that

$$|\langle F_{\boldsymbol{v}}, \boldsymbol{b}_i \rangle_{\mathbb{C}^3}| \geq 1.$$

If not, suppose $|\langle F_{\boldsymbol{v}}, \boldsymbol{b}_i \rangle_{\mathbb{C}^m}| < 1$ for all $i = 0, 1, 2$. Then

$$\Big| \Big\langle F_{\boldsymbol{v}}, \|\boldsymbol{b}_i\|_{\mathbb{C}^m} \Big(\frac{1}{\|\boldsymbol{b}_i\|_{\mathbb{C}^3}} \boldsymbol{b}_i \Big) \Big\rangle \Big| < 1,$$

implies

$$\Big| \Big\langle F_{\boldsymbol{v}}, \Big(\frac{1}{\|\boldsymbol{b}_i\|_{\mathbb{C}^3}} \boldsymbol{b}_i \Big) \Big\rangle \Big| < \frac{1}{\|\boldsymbol{b}_i\|_{\mathbb{C}^3}} \leq 1,$$

for all $i = 0, 1, 2$. Since

$$\Big\{ \frac{1}{\|\boldsymbol{b}_i\|_{\mathbb{C}^3}} \boldsymbol{b}_i \Big\}_{i=0}^{2},$$

is an orthonormal basis for $\mathbb{C}^3$, the above inequality contradicts the fact that $\|F_{\boldsymbol{v}}\|_{\mathbb{C}^3} \geq \sqrt{3}$. On the other hand, since $\tilde{f}$ does not have an analytic part and

$$\langle \tilde{f}, 1 \rangle_{L^2(\mathbb{T}^n)} = 0,$$

it follows that

$$\left\langle \frac{\psi}{\|\psi\|_2^2} + f, g \right\rangle_{L^2(\mathbb{T}^n)} = \begin{cases} \langle F_{\boldsymbol{v}}, \boldsymbol{b}_0 \rangle_{\mathbb{C}^3} & \text{if } g = 1 \\ \langle F_{\boldsymbol{v}}, \boldsymbol{b}_1 \rangle_{\mathbb{C}^3} & \text{if } g = Z_1 \\ \langle F_{\boldsymbol{v}}, \boldsymbol{b}_2 \rangle_{\mathbb{C}^3} & \text{if } g = Z_2. \end{cases}$$

Therefore

$$\left| \left\langle \frac{\psi}{\|\psi\|_2^2} + f, g \right\rangle_{L^2(\mathbb{T}^n)} \right| \geq 1,$$

for some $g \in \{1, Z_1, Z_2\}$. On the other hand (see the duality (2.4.1))

$$\left\langle \frac{\psi}{\|\psi\|_2^2} + f, g \right\rangle_{L^2(\mathbb{T}^n)} = \begin{cases} \chi_1\left(\frac{\psi}{\|\psi\|_2^2} + f\right) & \text{if } g = 1 \\ \chi_{\bar{Z}_1}\left(\frac{\psi}{\|\psi\|_2^2} + f\right) & \text{if } g = Z_1 \\ \chi_{\bar{Z}_2}\left(\frac{\psi}{\|\psi\|_2^2} + f\right) & \text{if } g = Z_2. \end{cases}$$

However

$$\|\chi_{\bar{g}}\| = 1,$$

for all $g \in \{1, Z_1, Z_2\}$, and hence

$$\left\| \frac{\psi}{\|\psi\|_2^2} + f \right\|_{L^1(\mathbb{T}^n)} \geq 1,$$

for all $f \in \tilde{\mathcal{M}}^{conj}_{\mathcal{Q}_{\mathcal{Z}}}$. This completes the proof of the claim. Furthermore, in this case, we can specify an explicit interpolating function. Note that $\{\boldsymbol{e}_i\}_{i=0}^2$ is an orthonormal basis for $\mathbb{C}^3$, where

$$\boldsymbol{e}_i = \frac{1}{\|\boldsymbol{b}_i\|_{\mathbb{C}^3}} \boldsymbol{b}_i,$$

for all $i = 0, 1, 2$. We write

$$\boldsymbol{w} = \sum_{i=0}^{2} \alpha_i \boldsymbol{e}_i,$$

and set

$$\varphi(Z) = \frac{\alpha_0}{\|\boldsymbol{b}_0\|_{\mathbb{C}^3}} + \frac{\alpha_1}{\|\boldsymbol{b}_1\|_{\mathbb{C}^3}} Z_1 + \frac{\alpha_2}{\|\boldsymbol{b}_2\|_{\mathbb{C}^3}} Z_2,$$

for all $Z = (Z_1, \ldots, Z_n) \in \mathbb{D}^n$. Since $\|\boldsymbol{w}\|_{\mathbb{C}^3}^2 \leq \frac{1}{3}$, it follows that

$$\sum_{i=0}^{2} |\alpha_i|^2 \leq \frac{1}{3},$$

and hence, by the Cauchy-Schwarz inequality, we conclude that

$$\sum_{i=0}^{2} |\alpha_i| \leq 1.$$

Moreover, since $\|\boldsymbol{b}_i\|_{\mathbb{C}^3} \geq 1$ for all $i = 0, 1, 2$, for each $Z \in \mathbb{D}^n$, we infer that

$$\begin{aligned}
|\varphi(Z)| &= \Big| \frac{\alpha_0}{\|\boldsymbol{b}_0\|_{\mathbb{C}^3}} + \frac{\alpha_1}{\|\boldsymbol{b}_1\|_{\mathbb{C}^3}} Z_1 + \frac{\alpha_2}{\|\boldsymbol{b}_2\|_{\mathbb{C}^3}} Z_2 \Big| \\
&\leq \frac{|\alpha_0|}{\|\boldsymbol{b}_0\|_{\mathbb{C}^3}} + \frac{|\alpha_1|}{\|\boldsymbol{b}_1\|_{\mathbb{C}^3}} |Z_1| + \frac{|\alpha_2|}{\|\boldsymbol{b}_2\|_{\mathbb{C}^3}} |Z_2| \\
&\leq \sum_{i=0}^{2} |\alpha_i| \\
&\leq 1,
\end{aligned}$$

and consequently, $\varphi \in \mathcal{S}(\mathbb{D}^n)$. Finally, we compute

$$\begin{aligned}
\varphi(z_i) &= \frac{\alpha_0}{\|\boldsymbol{b}_0\|_{\mathbb{C}^3}} + \frac{\alpha_1}{\|\boldsymbol{b}_1\|_{\mathbb{C}^3}} \zeta_{1i} + \frac{\alpha_2}{\|\boldsymbol{b}_2\|_{\mathbb{C}^3}} \zeta_{2i} \\
&= \alpha_0 \pi_i(\boldsymbol{e}_0) + \alpha_1 \pi_i(\boldsymbol{e}_1) + \alpha_2 \pi_i(\boldsymbol{e}_2) \\
&= \pi_i(\sum_{j=0}^{2} \alpha_j \boldsymbol{e}_j) \\
&= \pi_i(\boldsymbol{w}) \\
&= w_i,
\end{aligned}$$

for all $i = 1, 2, 3$. Therefore, φ is a solution to the interpolation problem with data $\{z_i\}_{i=1}^3 \subset \mathbb{D}^n$ and $\{w_i\}_{i=1}^3 \subset \mathbb{D}$.

For general m-point interpolation, $m \geq 2$, the same proof concept applies, but the computation would be more laborious. We only report the general result and leave the other details to the interested readers.

Theorem 2.6.4. *Let $n \geq 2$, $m \geq 3$, and suppose $n \geq m - 1$. Let $\{z_i\}_{i=1}^m \subset \mathbb{D}^n$ be m distinct points, and let $\{w_i\}_{i=1}^m \subset \mathbb{D}$ be m scalars. Suppose $\{\boldsymbol{b}_i\}_{i=0}^{m-1} \subset \mathbb{C}^m$, where $\boldsymbol{b}_0 = (1, \dots, 1)$, and*

$$\boldsymbol{b}_j = (\zeta_{j1}, \zeta_{j2}, \dots, \zeta_{jm}),$$

for all $j = 1, \dots, m - 1$. Assume that:

1. *$\{\boldsymbol{b}_i\}_{i=0}^{m-1}$ is an orthogonal basis for $\mathbb{C}^m$.*
2. *$\|\boldsymbol{b}_i\|_{\mathbb{C}^m} \geq 1$ for all $i = 1, \dots, m - 1$.*
3. *$z_j = (\zeta_{1j}, \zeta_{2j}, \dots, \zeta_{m-1,j}, \tilde{z}_j)$, where $\tilde{z}_j \in \mathbb{D}^{n-m+1}$ arbitrary, and $j = 1, \dots, m$.*
4. *$\|\boldsymbol{w}\|_{\mathbb{C}^m} \leq \frac{1}{\sqrt{n}}$, where $\boldsymbol{w} = (w_1, \dots, w_n)$.*

Then there exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that

$$\varphi(z_i) = w_i,$$

for all $i = 1, \dots, m$. Furthermore, φ can be chosen as a polynomial.

Evidently, there is no dearth of examples of data that meet the aforementioned conditions. The following remark elaborates on this:

Remark 2.6.1. *If the number of variables $n(\geq 2)$ and the number of nodes $m(\geq 3)$ satisfies the condition $n \geq m-1$, and if one restricts the first $m-1$ slots of the coordinates of the interpolation nodes $\{z_i\}_{i=1}^m$ (so that the corresponding columns along with the constant vector 1 forms a basis of $\mathbb{C}^m$) along with the norm bound on $\boldsymbol{w}$ as*

$$\|\boldsymbol{w}\|_{\mathbb{C}^m} \leq \frac{1}{n},$$

then one can ensure that interpolation will occur for any choice of $\{\tilde{z}_i\}_{i=1}^m \subset \mathbb{D}^{n-m+1}$. The relationship between the orthogonal set of vectors $\{\boldsymbol{b}_i\}_{i=1}^{m-1} \subset \mathbb{C}^m$ and interpolation nodes $\{z_i\}_{i=1}^m \subset \mathbb{D}^n$ can be represented by the formal matrix:

$$\begin{array}{c} \\ z_1 \\ z_2 \\ \vdots \\ z_m \end{array}
\begin{array}{c}
\begin{array}{cccccc} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}_3 & \cdots & \boldsymbol{b}_{m-1} & \end{array} \\
\begin{pmatrix}
\zeta_{11} & \zeta_{21} & \zeta_{31} & \cdots & \zeta_{m-1,1} & \cdots \\
\zeta_{12} & \zeta_{22} & \zeta_{32} & \cdots & \zeta_{m-1,2} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\zeta_{1m} & \zeta_{2m} & \zeta_{3m} & \cdots & \zeta_{m-1,m} & \cdots
\end{pmatrix}
\end{array}.$$

What this means is that there is an abundance of examples of interpolation in hand in several variables.

We refer the reader to [75] for interpolation from operator algebraic perspective.

2.7 Commutant lifting and examples

This section contains illustrations of commutant lifting on quotient modules of $H^2(\mathbb{T}^n)$, $n > 1$. Our first aim is to validate the examples in Section 2.3 using our commutant lifting theorem. We begin with a lemma.

Lemma 2.7.1. *Let $h \in H^2(\mathbb{T}^n)$. Then $\|h\|_1 = \|h\|_2 = 1$ if and only if h is inner.*

Proof. Suppose $\|h\|_1 = \|h\|_2 = 1$. In particular, $h \in H^1(\mathbb{T}^n) \subseteq L^1(\mathbb{T}^n)$. By the Hahn–Banach theorem, there exists $\varphi \in L^\infty(\mathbb{T}^n)$ such that $\|\varphi\|_\infty = 1$ (as $\|h\|_1 = 1$) and

$$\int_{\mathbb{T}^n} h\varphi d\mu = \|h\|_1 = 1.$$

In the above, we used the duality $(L^1(\mathbb{T}^n))^* \cong L^\infty(\mathbb{T}^n)$ once more. We claim that φ is unimodular. Indeed, if

$$|\varphi| < 1 \text{ on } A,$$

for some measurable set $A \subseteq \mathbb{T}^n$ such that $\mu(A) > 0$, then

$$\begin{aligned}
1 &= \Big| \int_{\mathbb{T}^n} h\varphi d\mu \Big| \\
&\leq \Big| \int_{A^c} h\varphi d\mu \Big| + \Big| \int_{A} h\varphi d\mu \Big| \\
&\leq \int_{A^c} |h||\varphi| d\mu + \int_{A} |h||\varphi| d\mu \\
&< \int_{A^c} |h| d\mu + \int_{A} |h| d\mu \\
&= \|h\|_1,
\end{aligned}$$

that is, $1 < \|h\|_1$, a contradiction. Since $h \in H^2(\mathbb{D}^n) \subseteq L^2(\mathbb{T}^n)$, we find a scalar c and a function $g \in L^2(\mathbb{T}^n)$ such that

$$\varphi = c\bar{h} \oplus g.$$

Observe that $\langle \bar{h}, g \rangle = \langle h, \bar{g} \rangle = 0$. Therefore

$$\begin{aligned}
1 &= \int_{\mathbb{T}^n} h\varphi d\mu \\
&= \int_{\mathbb{T}^n} h(c\bar{h} \oplus g) d\mu \\
&= \langle h, \bar{c}h \oplus \bar{g} \rangle_{L^2(\mathbb{T}^n)} \\
&= c,
\end{aligned}$$

and hence, $\varphi = \bar{h} \oplus g$. Then

$$\begin{aligned}
1 + \|g\|_2^2 &= \|h\|_2^2 + \|g\|_2^2 \\
&= \|\varphi\|_2^2 \\
&\leq \|\varphi\|_\infty^2 \\
&= 1,
\end{aligned}$$

implies that $g = 0$, and hence $\varphi = \bar{h}$. Since $\varphi \in L^\infty(\mathbb{T}^n)$, it follows that $h \in H^\infty(\mathbb{D}^n)$ is an inner function. The converse simply follows from the integral representation of norms on $H^2(\mathbb{T}^n)$ and $H^1(\mathbb{T}^n)$ and the fact that $|h| = 1$ a.e. on $\mathbb{T}^n$. $\square$

Now we follow the setting of Corollary 2.3.3: For a fixed $m \in \mathbb{N}$, we consider the homogeneous quotient module

$$\mathcal{Q}_m = \bigoplus_{t=0}^{m} H_t,$$

a homogeneous polynomial $p \in \mathcal{Q}_m$ as

$$p = \sum_{|k|=m} a_k z^k,$$

with $\|p\|_2 = 1$, and that $a_k, a_l \neq 0$ for some $k \neq l$ in $\mathbb{Z}_+^n$. We know, by Theorem 2.4.1, that S_p is liftable if and only if

$$X_{\mathcal{Q}_m}(f) = \int_{\mathbb{T}^n} \psi f d\mu \qquad (f \in \mathcal{M}_{\mathcal{Q}_m}),$$

defines a contraction on $(\mathcal{M}_{\mathcal{Q}_m}, \|\cdot\|_1)$, where $\psi = S_p(P_{\mathcal{Q}_m}1)$. Since 1 and p are in $\mathcal{Q}_m$, it follows that $\psi = p$, and hence

$$\begin{aligned} X_{\mathcal{Q}_m}(\bar{p}) &= \int_{\mathbb{T}^n} \bar{p}\psi d\mu \\ &= \int_{\mathbb{T}^n} |p|^2 d\mu \\ &= 1. \end{aligned}$$

However

$$\|\bar{p}\|_1 = \|p\|_1 < 1.$$

Indeed, since $a_k, a_l \neq 0$, Lemma 2.3.1 ensures that p is not inner. This, together with the fact that $\|p\|_2 = 1$ and Lemma 2.7.1 completes the proof of the claim. Therefore, $X_{\mathcal{Q}_m}$ on $(\mathcal{M}_{\mathcal{Q}_m}, \|\cdot\|_1)$ is not a contraction, and hence S_p is not liftable. As a result, we recover Corollary 2.3.3 using Theorem 2.4.1.

The idea used in the preceding example can be extended to provide further nontrivial examples of module maps that do not admit any lift. The following is an example, and this time we will use Theorem 2.4.1 directly to prove that such a module map does not lift. Suppose $n > 1$. Consider the submodule

$$\mathcal{S} = z_1 \cdots z_n H^2(\mathbb{T}^n).$$

We will be working on the corresponding quotient module $\mathcal{Q} = \mathcal{S}^\perp$. Clearly

$$\mathcal{Q} = \ker\Big(\prod_{i=1}^n T_{z_i}^*\Big).$$

We observe that

$$\mathcal{Q} = H_{z_1}^2(\mathbb{T}^n) + \cdots + H_{z_n}^2(\mathbb{T}^n),$$

where $H_{z_i}^2(\mathbb{T}^n)$, $i = 1, \ldots, n$, is the closed subspace of $H^2(\mathbb{T}^n)$ of functions that are independent of the z_i variable, or equivalently

$$H_{z_i}^2(\mathbb{T}^n) = \ker T_{z_i}^*.$$

Indeed, it is clear that $H_{z_1}^2(\mathbb{T}^n) + \cdots + H_{z_n}^2(\mathbb{T}^n) \subseteq \mathcal{Q}$. Let $f \in \ker(\prod_{i=1}^n T_{z_i}^*)$, and suppose

$$f = \sum_{k \in \mathbb{Z}_+^n} a_k z^k.$$

Then

$$\begin{aligned} 0 &= T^*_{z_1\cdots z_n} f \\ &= P_{H^2(\mathbb{T}^n)}\Big(\sum_{k\in\mathbb{Z}^n_+} a_k \bar{z}_1\cdots\bar{z}_n z^k\Big), \end{aligned}$$

implies

$$\sum_{k\in\mathbb{Z}^n_+} a_k \bar{z}_1\cdots\bar{z}_n z^k \in (H^2(\mathbb{T}^n))^{\perp}.$$

In other words, if $a_k \neq 0$ for some $k = (k_1,\ldots,k_n) \in \mathbb{Z}^n_+$, then we must have $k_i = 0$ for some $i = 1,\ldots,n$. Therefore, there exists $f_i \in H^2_{z_i}(\mathbb{T}^n)$, $i = 1,\ldots,n$, such that $f = f_1 + \cdots + f_n$. This proves the claim. Now for each $i = 1,\ldots,n$, set

$$\zeta_i := \prod_{j\neq i} z_j,$$

and pick inner function $\varphi_i \in H^2_{z_i}(\mathbb{T}^n)$. Let $z_0 \in \mathbb{T}^n$ and suppose $\varphi_i(z_0)$ is well defined and

$$|\varphi_i(z_0)| = 1,$$

for all $i = 1,\ldots,n$. Choose $\{\alpha_1,\ldots,\alpha_n\} \subset \mathbb{R}_{\geq 0}$ such that $\alpha_p, \alpha_q \neq 0$ for some $p \neq q$, and

$$\sum_{i=1}^n \alpha_i^2 = 1.$$

The preceding set of assumptions ensures that

$$\sum_{i=1}^n \alpha_i > 1.$$

Finally, define $\varphi \in \mathcal{S}(\mathbb{D}^n)$ by

$$\varphi = \sum_{i=1}^n \alpha_i \bar{\beta}_i \zeta_i \varphi_i,$$

where $\beta_i = (\zeta_i\varphi_i)(z_0)$ for all $i = 1,\ldots,n$. We claim that φ is not inner. Indeed, since

$$\varphi(z_0) = \sum_{i=1}^n \alpha_i \bar{\beta}_i \beta_i,$$

and $|\beta_i| = 1$, it follows that

$$\varphi(z_0) = \sum_{i=1}^n \alpha_i > 1.$$

Therefore, there exists $r \in (0,1)$ such that (note that φ is well defined at z_0)

$$|\varphi(rz_0)| > 1,$$

and hence, by the maximum modulus theorem, we conclude that $\|\varphi\|_\infty > 1$, which completes the proof of the claim. Next, we claim that S_φ is a contraction. Fix $f \in \mathcal{Q}$. For each $i \in \{1, \dots, n\}$, we have

$$\mathcal{Q} \ominus H^2_{z_i}(\mathbb{T}^n) = (\ker T^*_{z_i})^\perp \cap \mathcal{Q} = \{z_i g \in \mathcal{Q} : g \in H^2(\mathbb{T}^n)\},$$

and hence there exist $f_i \in H^2_{z_i}(\mathbb{T}^n)$ and $g_i \in H^2(\mathbb{T}^n)$ such that

$$f = f_i \oplus z_i g_i \in H^2_{z_i}(\mathbb{T}^n) \oplus (\mathcal{Q} \ominus H^2_{z_i}(\mathbb{T}^n)).$$

Then

$$\begin{aligned} S_{\zeta_i\varphi_i} f &= S_{\zeta_i\varphi_i}(f_i + z_i g_i) \\ &= S_{\zeta_i\varphi_i} f_i + P_{\mathcal{Q}}(\zeta_i \varphi_i z_i g_i) \\ &= S_{\zeta_i\varphi_i} f_i + P_{\mathcal{Q}}(z_1 \cdots z_n \varphi_i g_i) \\ &= S_{\zeta_i\varphi_i} f_i, \end{aligned}$$

as $z_1 \cdots z_n \varphi_i g_i \in \mathcal{S}$, and hence

$$S_{\zeta_i\varphi_i} f = S_{\zeta_i\varphi_i} P_{H^2_{z_i}(\mathbb{T}^n)} f.$$

Observe moreover that $\zeta_i \varphi_i H^2_{z_i}(\mathbb{T}^n) \subseteq H^2_{z_i}(\mathbb{T}^n)$. In view of $H^2_{z_i}(\mathbb{T}^n) \subseteq \mathcal{Q}$, we conclude that $S_{\zeta_i\varphi_i} P_{H^2_{z_i}(\mathbb{T}^n)} f = \zeta_i \varphi_i P_{H^2_{z_i}(\mathbb{T}^n)} f$, which yields

$$S_{\zeta_i\varphi_i} f = \zeta_i \varphi_i P_{H^2_{z_i}(\mathbb{T}^n)} f.$$

Therefore, for $i \neq j$, we have

$$\begin{aligned} \langle S_{\zeta_i\varphi_i} f, S_{\zeta_j\varphi_j} f \rangle &= \langle \zeta_i \varphi_i P_{H^2_{z_i}(\mathbb{T}^n)} f, \zeta_j \varphi_j P_{H^2_{z_j}(\mathbb{T}^n)} f \rangle \\ &= \langle T^*_{\zeta_j} \zeta_i \varphi_i P_{H^2_{z_i}(\mathbb{T}^n)} f, \varphi_j P_{H^2_{z_j}(\mathbb{T}^n)} f \rangle \\ &= \langle T^*_{z_i} z_j \varphi_i P_{H^2_{z_i}(\mathbb{T}^n)} f, \varphi_j P_{H^2_{z_j}(\mathbb{T}^n)} f \rangle \\ &= 0, \end{aligned}$$

as $z_j \varphi_i P_{H^2_{z_i}(\mathbb{T}^n)} f \in \ker T^*_{z_i}$. So we find

$$S_{\zeta_i\varphi_i} f \perp S_{\zeta_j\varphi_j} f \qquad (i \neq j). \tag{2.7.1}$$

This allows us to compute the norm of $S_\varphi f$ as follows (note that $|\beta_i| = 1$ and $S_{\zeta_i\varphi_i}$ is a contraction for all $i = 1, \dots, n$):

$$\begin{aligned}\|S_\varphi f\|^2 &= \|\sum_{i=1}^n \alpha_i \bar{\beta}_i S_{\zeta_i\varphi_i} f\|^2 \\ &= \sum_{i=1}^n \alpha_i^2 \|S_{\zeta_i\varphi_i} f\|^2 \\ &\le \sum_{i=1}^n \alpha_i^2 \|f\|^2 \\ &= \|f\|^2.\end{aligned}$$

This means that S_φ is a contraction. Our final claim is that S_φ is incapable of admitting any lift, which, in view of Theorem 2.4.1, is equivalent to the assertion that $X_\mathcal{Q} : (\mathcal{M}_\mathcal{Q}, \|\cdot\|_1) \to \mathbb{C}$ is not a contraction, where

$$X_\mathcal{Q} f = \int_{\mathbb{T}^n} \psi f d\mu \qquad (f \in \mathcal{M}_\mathcal{Q}),$$

and

$$\psi = S_\varphi(P_\mathcal{Q} 1).$$

Indeed, since $1, \varphi \in \mathcal{Q}$, it follows that

$$\psi = \varphi.$$

On the other hand, since $\bar{\varphi} \in \mathcal{M}_\mathcal{Q}$ (recall that $\mathcal{M}_\mathcal{Q} = \mathcal{Q}^{conj} \dotplus (\mathcal{M}_n \dotplus H_0^2(\mathbb{T}^n)))$, we observe that

$$\begin{aligned}X_\mathcal{Q}\bar{\varphi} &= \int_{\mathbb{T}^n} \varphi\bar{\varphi} d\mu \\ &= \|\varphi\|^2_{H^2(\mathbb{T}^n)} \\ &= 1.\end{aligned}$$

Finally, applying (2.7.1) to $f = 1 \in \mathcal{Q}$, we obtain that

$$\|\varphi\|^2_{H^2(\mathbb{T}^n)} = 1.$$

This also follows from the equalities following (2.7.1) corresponding to the choice $f = 1$ along with the fact that $\zeta_i\varphi_i$ is inner for all $i = 1, \dots, n$. Since φ is not an inner function, by Lemma 2.7.1, we conclude that

$$\|\bar{\varphi}\|_1 = \|\varphi\|_1 < 1,$$

and hence $X_\mathcal{Q} : (\mathcal{M}_\mathcal{Q}, \|\cdot\|_1) \to \mathbb{C}$ is not a contraction. This proves the following result:

Proposition 2.7.2. *Let $\{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{R}_{\geq 0}$, suppose $\alpha_p, \alpha_q \neq 0$ for some $p \neq q$, and*

$$\sum_{i=1}^{n} \alpha_i^2 = 1.$$

Let $z_0 \in \mathbb{T}^n$, and let φ_i be an inner function independent of the variable z_i, and suppose $\varphi_i(z_0)$ is well defined and

$$|\varphi_i(z_0)| = 1,$$

for all $i = 1, \ldots, n$. Define $\varphi \in \mathcal{S}(\mathbb{D}^n)$ by

$$\varphi = \sum_{i=1}^{n} \alpha_i \bar{\beta}_i \zeta_i \varphi_i,$$

where $\beta_i = (\zeta_i \varphi_i)(z_0)$ and $\zeta_i := \prod_{j \neq i} z_j$ for all $i = 1, \ldots, n$. Then S_φ on $\mathcal{Q} = \ker(\prod_{i=1}^{n} T_{z_i}^*)$ does not admit any lift.

2.8 Recovering Sarason's lifting theorem

In this section, we explain how to recover Sarason's commutant lifting theorem from Theorem 2.4.1. We will employ several tools (just like Sarason) that are commonly used and are valid only in one variable function theory. We start with the Beurling theorem [23]. Let $\mathcal{Q} \subsetneq H^2(\mathbb{T})$ be a closed subspace. Then $\mathcal{Q}$ is a quotient module if and only if there exists an inner function $\theta \in H^\infty(\mathbb{D})$ such that $\mathcal{Q} = \mathcal{Q}_\theta$, where

$$\mathcal{Q}_\theta := H^2(\mathbb{T}) \ominus \theta H^2(\mathbb{T}).$$

Observe that $\theta H^2(\mathbb{T})$ is a closed subspace (as T_θ is an isometry on $H^2(\mathbb{T})$) and

$$\mathcal{Q}_\theta \cong H^2(\mathbb{T})/\theta H^2(\mathbb{T}).$$

Therefore, quotient modules of $H^2(\mathbb{T})$ are inner function based - a typical one variable phenomenon (see Rudin [87] for counterexamples in several variables). In the following, we prove a key result.

Lemma 2.8.1. *Let $\theta \in H^\infty(\mathbb{D})$ be an inner function. Then*

$$\mathcal{Q}_\theta^{conj} \oplus zH^2(\mathbb{T}) = \overline{\theta}(zH^2(\mathbb{T})).$$

Proof. Let $g \in \mathcal{Q}_\theta$. Then $\bar{g} \in \mathcal{Q}_\theta^{conj}$, and hence, for each $m \geq 0$, we have

$$\begin{aligned} \langle \theta \bar{g}, \bar{z}^m \rangle_{L^2(\mathbb{T})} &= \overline{\langle \bar{\theta} g, z^m \rangle}_{L^2(\mathbb{T})} \\ &= \overline{\langle g, \theta z^m \rangle}_{H^2(\mathbb{T})} \\ &= 0, \end{aligned}$$

as $\theta z^m \in \mathcal{Q}_\theta^\perp$. This implies $\theta \mathcal{Q}_\theta^{conj} \subseteq zH^2(\mathbb{T})$ and hence $\mathcal{Q}_\theta^{conj} \subseteq \bar{\theta}(zH^2(\mathbb{T}))$. Also, for all $h \in H^2(\mathbb{T})$, since

$$zh = \overline{\theta}(\theta z h) = \overline{\theta}(z\theta h),$$

it follows that $zH^2(\mathbb{T}) \subseteq \overline{\theta}(zH^2(\mathbb{T}))$. Therefore

$$\mathcal{Q}_\theta^{conj} \oplus zH^2(\mathbb{T}) \subseteq \overline{\theta}(zH^2(\mathbb{T})).$$

For the reverse inclusion, first we observe that for $f \in \mathcal{Q}_\theta$ and $m \geq 1$, since

$$\begin{aligned}\langle \overline{\theta} z f, z^m \rangle_{L^2(\mathbb{T})} &= \langle zf, z\theta z^{m-1} \rangle_{H^2(\mathbb{T})} \\ &= \langle f, \theta z^{m-1} \rangle_{H^2(\mathbb{T})} \\ &= 0,\end{aligned}$$

it follows that $\bar{\theta} z \mathcal{Q}_\theta \perp zH^2(\mathbb{T})$, and hence $\bar{\theta} z \mathcal{Q}_\theta \subseteq H^2(\mathbb{T})^{conj}$. On the other hand, we know

$$H^2(\mathbb{T})^{conj} = \mathcal{Q}_\theta^{conj} \oplus (\theta H^2(\mathbb{T}))^{conj}.$$

In view of this, for each $f \in \mathcal{Q}_\theta$ and $g \in H^2(\mathbb{T})$, we further compute

$$\begin{aligned}\langle \overline{\theta} z f, \overline{\theta} \overline{g} \rangle_{L^2(\mathbb{T})} &= \langle zf, \overline{g} \rangle_{L^2(\mathbb{T})} \\ &= \langle f, \bar{z}\overline{g} \rangle_{L^2(\mathbb{T})} \\ &= 0,\end{aligned}$$

which implies that $\overline{\theta} z \mathcal{Q}_\theta \perp (\theta H^2(\mathbb{T}))^{conj}$. As a result, $\overline{\theta} z \mathcal{Q}_\theta \subseteq \mathcal{Q}_\theta^{conj}$. Finally

$$zH^2(\mathbb{T}) = z\mathcal{Q}_\theta \oplus z\theta H^2(\mathbb{T}),$$

yields

$$\begin{aligned}\overline{\theta} z H^2(\mathbb{T}) &= \overline{\theta} z \mathcal{Q}_\theta + zH^2(\mathbb{T}) \\ &\subseteqq \mathcal{Q}_\theta^{conj} + zH^2(\mathbb{T}),\end{aligned}$$

and completes the proof of the lemma. □

We are now almost ready to prove Sarason's commutant lifting theorem. Just one more result is required with regard to representations of polynomials as the sum of $H^\infty(\mathbb{D})$-functions. Since this result holds true in several variables and is of independent interest, we prove it in the later part of this chapter (see Proposition 2.9.7).

Theorem 2.8.2. *Contractive module maps on quotient modules of $H^2(\mathbb{T})$ are liftable.*

Proof. Since we are dealing with one variable quotient module, we fix a quotient module $\mathcal{Q}_\theta$ of $H^2(\mathbb{T})$ corresponding to an inner function $\theta \in H^\infty(\mathbb{D})$. Since $\mathcal{M}_1 = \{0\}$ and

$H_0^2(\mathbb{T}) = zH^2(\mathbb{T})$, it follows that

$$\mathcal{M}_{\mathcal{Q}_\theta} = \mathcal{Q}_\theta^{conj} \oplus zH^2(\mathbb{T}),$$

and hence Lemma 2.8.1 yields a compact form of $\mathcal{M}_{\mathcal{Q}_\theta}$ as

$$\mathcal{M}_{\mathcal{Q}_\theta} = \overline{\theta}(zH^2(\mathbb{T})).$$

Let $X \in \mathcal{B}_1(\mathcal{Q})$ and suppose $\psi = X(P_{\mathcal{Q}_\theta}1)$. In view of the above and Theorem 2.4.1, it is enough to prove that $X_{\mathcal{Q}_\theta} : (\mathcal{M}_{\mathcal{Q}_\theta}, \|\cdot\|_1) \to \mathbb{C}$ is a contraction, where

$$X_{\mathcal{Q}_\theta}(\bar{\theta} f) = \int_{\mathbb{T}} \psi \bar{\theta} f \, d\mu,$$

for all $f \in zH^2(\mathbb{T})$. To this end, fix $f \in zH^2(\mathbb{T})$. Then $f \in H^2(\mathbb{T})$ and $f(0) = 0$. There exists a sequence of polynomials $\{p_m\}_{m\geq 0} \subseteq \mathbb{C}[z]$ such that

$$p_m(0) = 0,$$

for all $m \geq 0$, and

$$p_m \longrightarrow f \text{ in } H^2(\mathbb{T}).$$

Using the contractive containment $H^2(\mathbb{T}) \hookrightarrow H^1(\mathbb{T})$, we see that

$$p_m \to f \text{ in } H^1(\mathbb{T}).$$

It also follows that

$$\bar{\theta} p_m \to \bar{\theta} f, \tag{2.8.1}$$

in both $L^2(\mathbb{T})$ and $L^1(\mathbb{T})$. Then

$$\int_{\mathbb{T}} \psi \bar{\theta} p_m d\mu \to \int_{\mathbb{T}} \psi \bar{\theta} f d\mu,$$

and

$$\|\bar{\theta} p_m\|_1 \to \|\bar{\theta} f\|_1,$$

and hence it is enough to prove that

$$\left| \int_{\mathbb{T}} \psi \bar{\theta} p \, d\mu \right| \leq \|\bar{\theta} p\|_1,$$

for all $p \in \mathbb{C}[z]$ such that $p(0) = 0$. Fix such a polynomial p. Consider the inner-outer factorization of p as

$$p = \eta h,$$

where η is an inner function, h is outer, and $\eta(0) = 0$. Since $p \in H^\infty(\mathbb{D})$, it follows that $h \in H^\infty(\mathbb{D})$. Using the fact that $\sqrt{h} \in H^\infty(\mathbb{D}) \subseteq H^2(\mathbb{T})$, we rewrite p as

$$p = (\eta\sqrt{h})\sqrt{h}.$$

It is easy to see that

$$\|p\|_1 = \|\sqrt{h}\|_2^2.$$

Moreover, we have a sequence of polynomials $\{q_t\}_{t\geq 0} \subseteq \mathbb{C}[z]$ such that

$$q_t \longrightarrow \sqrt{h} \text{ in } H^2(\mathbb{T}).$$

As $\eta\sqrt{h} \in H^\infty(\mathbb{D})$, we have

$$\langle \psi, \theta\overline{q_t\eta\sqrt{h}}\rangle \longrightarrow \langle \psi, \theta\overline{\sqrt{h}\eta\sqrt{h}}\rangle,$$

and then, rewriting $\sqrt{h}\eta\sqrt{h} = p$, we conclude that

$$\langle \psi, \theta\overline{q_t\eta\sqrt{h}}\rangle \longrightarrow \langle \psi, \theta\bar{p}\rangle = \int_{\mathbb{T}} \psi\bar{\theta}p d\mu,$$

as $t \to \infty$. Since $(\eta\sqrt{h})(0) = 0$, Lemma 2.8.1 implies

$$\theta\overline{\eta\sqrt{h}} \in \mathcal{Q}_\theta \oplus \overline{zH^2(\mathbb{T})},$$

and consequently

$$\tilde{h} := P_{H^2(\mathbb{T})}(\theta\overline{\eta\sqrt{h}}) \in \mathcal{Q}_\theta.$$

Then, recalling $\psi = X(P_{\mathcal{Q}_\theta}1)$, we compute

$$\begin{aligned}
\langle \psi, \theta\overline{q_t\eta\sqrt{h}}\rangle &= \langle \psi q_t, \theta\overline{\eta\sqrt{h}}\rangle \\
&= \langle P_{H^2(\mathbb{T})}\psi q_t, P_{H^2(\mathbb{T})}\theta\overline{\eta\sqrt{h}}\rangle \\
&= \langle P_{H^2(\mathbb{T})}\psi q_t, \tilde{h}\rangle \\
&= \langle P_{\mathcal{Q}_\theta}\psi q_t, \tilde{h}\rangle.
\end{aligned}$$

We also observe, for a general polynomial $r \in \mathbb{C}[z]$, that

$$\begin{aligned}
XP_{\mathcal{Q}_\theta}r &= Xr(S_z)P_{\mathcal{Q}_\theta}1 \\
&= r(S_z)XP_{\mathcal{Q}_\theta}1 \\
&= r(S_z)\psi,
\end{aligned}$$

that is, $XP_{\mathcal{Q}_\theta}r = P_{\mathcal{Q}_\theta}r\psi$. It is important to note that (by virtue of Proposition 2.9.7)

$$P_{\mathcal{Q}_\theta}r \in H^\infty(\mathbb{D}).$$

Since $\{q_t\}_{t\geq 0} \subseteq \mathbb{C}[z]$, we conclude

$$\langle \psi, \theta\overline{q_t\eta\sqrt{h}}\rangle = \langle XP_{\mathcal{Q}_\theta}q_t, \tilde{h}\rangle,$$

and hence

$$\left|\langle XP_{\mathcal{Q}_\theta}q_t, \tilde{h}\rangle\right| \longrightarrow \left|\int_{\mathbb{T}} \psi\bar{\theta}p d\mu\right|,$$

as $t \to \infty$. But, $\|X\| \leq 1$, and $\|\tilde{h}\| \leq \|\sqrt{h}\|$, and hence

$$\left|\langle XP_{\mathcal{Q}_\theta}q_t, \tilde{h}\rangle\right| \leq \|q_t\|_2\|\sqrt{h}\|_2.$$

As $t \to \infty$, we have (note that θ is an inner function)

$$\|q_t\|_2\|\sqrt{h}\|_2 \to \|\sqrt{h}\|_2^2 = \|p\|_1 = \|\bar{\theta}p\|_1,$$

and hence

$$\left|\int_{\mathbb{T}} \psi\bar{\theta}p d\mu\right| \leq \|\bar{\theta}p\|_1,$$

which completes the proof of the theorem. □

Sarason's proof of the above theorem used similar one-variable tools.

2.9 Other results

In this section, we present a variety of results with varying flavors. First, we present a solution to the Carathéodory-Fejér interpolation problem on $\mathbb{D}^n$. Then we discuss the interpolation problem from the standpoint of Pick matrix positivity. The lifting theorem for the Bergman space over $\mathbb{D}^n$ is then compared, followed by decompositions of polynomials in light of Beurling-type quotient modules of $H^2(\mathbb{T}^n)$.

2.9.1 Carathéodory-Fejér interpolation

We use the notations that were introduced in Section 2.3. Recall that for $t \in \mathbb{Z}_+$, $H_t \subseteq \mathbb{C}[z_1, \ldots, z_n]$ is the complex vector space of homogeneous polynomials of degree t. Moreover, for each $m \in \mathbb{N}$, define the finite-dimensional homogeneous quotient module $\mathcal{Q}_m$ of $H^2(\mathbb{T}^n)$ by

$$\mathcal{Q}_m := \bigoplus_{t=0}^{m} H_t.$$

Fix a natural number m. Given $p \in \mathbb{C}[z_1, \ldots, z_n]$, it follows that $p \in \mathcal{Q}_m$ if and only if $\mathrm{deg} p \leq m$. In the context of $\mathcal{S}(\mathbb{D}^n)$, the Carathéodory-Fejér interpolation problem asks the following: Given a polynomial $p \in \mathcal{Q}_m$, when does there exist a function $f \in \mathcal{Q}_m^\perp$ such that

$$p \oplus f \in \mathcal{S}(\mathbb{D}^n)?$$

Here and in what follows, $p \oplus f$ is in the sense of the direct sum $\mathcal{Q}_m \oplus \mathcal{Q}_m^\perp$. This formulation of the Carathéodory-Fejér interpolation problem is more appropriate for the case of $n > 1$, see [21, page 670].

The following is an interpretation of the Carathéodory-Fejér problem in terms of commutant lifting.

Proposition 2.9.1. *Let $p \in \mathcal{Q}_m$. There exists $f \in \mathcal{Q}_m^\perp$ such that $p \oplus f \in \mathcal{S}(\mathbb{D}^n)$ if and only if S_p is a contraction and admits a lift.*

Proof. Suppose there exists a function $f \in \mathcal{Q}_m^\perp$ such that

$$\varphi := p \oplus f \in \mathcal{S}(\mathbb{D}^n).$$

For each $q \in \mathcal{Q}_m$, we have

$$\begin{aligned} S_\varphi q &= P_{\mathcal{Q}_m} T_\varphi q \\ &= P_{\mathcal{Q}_m} (p \oplus f) q \\ &= P_{\mathcal{Q}_m}(pq) + P_{\mathcal{Q}_m}(fq). \end{aligned}$$

But, $\mathcal{Q}_m^\perp$ is a submodule and q is a polynomial. This implies $fq \in \mathcal{Q}_m^\perp$, and consequently

$$S_\varphi q = P_{\mathcal{Q}_m}(pq).$$

On the other hand, $q \in \mathcal{Q}_m$ and

$$p \in \mathcal{Q}_m \subseteq \mathbb{C}[z_1, \dots, z_n]$$

yield

$$\begin{aligned} P_{\mathcal{Q}_m}(pq) &= P_{\mathcal{Q}_m} T_p|_{\mathcal{Q}_m} q \\ &= S_p q, \end{aligned}$$

which proves that $S_\varphi = S_p$. The contractivity of S_p also follows from the same of S_φ (recall that $\|\varphi\|_\infty \leq 1$).

For the reverse direction, suppose $S_p \in \mathcal{B}_1(\mathcal{Q}_m)$ admits a lift. Then there exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that $S_p = S_\varphi$. Using $1 \in \mathcal{Q}_m$, it follows that

$$\begin{aligned} p &= S_p 1 \\ &= S_\varphi 1 \\ &= P_{\mathcal{Q}_m} \varphi, \end{aligned}$$

and hence there exists $f \in \mathcal{Q}_m^\perp$ such that $\varphi = p \oplus f$. This completes the proof of the proposition. $\square$

We are now ready for the solution to the Carathéodory-Fejér interpolation problem. We will apply our commutant lifting theorem to the above. In view of Theorem 2.4.1, we set

$$\mathcal{M}_{\mathcal{Q}_m} = \mathcal{Q}_m^{conj} \dotplus (\mathcal{M}_n \dotplus H_0^2(\mathbb{T}^n)).$$

Recall that

$$\mathcal{M}_n = L^2(\mathbb{T}^n) \ominus (H^2(\mathbb{T}^n)^{conj} \dotplus H^2(\mathbb{T}^n)).$$

Corollary 2.9.2. *Given $p \in \mathcal{Q}_m$, there exists $f \in \mathcal{Q}_m^{\perp}$ such that $p \oplus f \in \mathcal{S}(\mathbb{D}^n)$ if and only if $\mathfrak{C}_{\mathcal{Z},\mathcal{W}} : (\mathcal{M}_{\mathcal{Q}_m}, \|\cdot\|_1) \to \mathbb{C}$ is a contraction, where*

$$\mathfrak{C}_{\mathcal{Z},\mathcal{W}}(g) = \int_{\mathbb{T}^n} pg \, d\mu \qquad (g \in \mathcal{M}_{\mathcal{Q}_m}).$$

Proof. By Theorem 2.4.1 and the preceding proposition, the assertion is equivalent to the contractivity of the functional $\chi_{\mathcal{Q}_m}$ on $(\mathcal{M}_{\mathcal{Q}_m}, \|\cdot\|_1)$, where

$$\chi_{\mathcal{Q}_m} g = \int_{\mathbb{T}^n} \psi g \, d\mu \qquad (g \in \mathcal{M}_{\mathcal{Q}_m}),$$

and $\psi = S_p(P_{\mathcal{Q}_m} 1)$. However, $1 \in \mathcal{Q}_m$ implies $P_{\mathcal{Q}_m}(1) = 1$, and $p \in \mathcal{Q}_m$ implies $S_p(1) = p$. Then

$$\chi_{\mathcal{Q}_m} = \mathfrak{C}_{\mathcal{Z},\mathcal{W}} \text{ on } \mathcal{M}_{\mathcal{Q}_m},$$

completes the proof of the corollary. $\square$

We refer the reader to Eschmeier, Patton and Putinar [49], and Woerdeman [102] for the Carathéodory interpolation problem in the context of Agler-Herglotz class functions and Agler-Herglotz-Nevanlinna formula on the polydisc. Also see the paper by Kalyuzhnyi-Verbovetzkii [64].

2.9.2 Weak interpolation

Given $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{D}^n$ and $\mathcal{W} = \{w_i\}_{i=1}^m \subset \mathbb{D}$, we define the $m \times m$ *Pick matrix* $\mathfrak{P}_{\mathcal{Z},\mathcal{W}}$ as

$$\mathfrak{P}_{\mathcal{Z},\mathcal{W}} = \Big((1 - w_i \bar{w}_j)\mathbb{S}(z_i, z_j)\Big)_{i,j=1}^m.$$

Recall that a matrix $(a_{ij})_{m\times m}$ is positive semi-definite (in short $(a_{ij})_{m\times m} \geq 0$) if

$$\sum_{i,j=1}^m \bar{\alpha}_i \alpha_j a_{ij} \geq 0,$$

for all scalars $\{\alpha_i\}_{i=1}^m \subseteq \mathbb{C}$.

Definition 2.9.3. *A set of distinct points $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{D}^n$ is said to be a Pick set if, for $\mathcal{W} = \{w_i\}_{i=1}^m \subset \mathbb{D}$ satisfying*

$$\mathfrak{P}_{\mathcal{Z},\mathcal{W}} \geq 0,$$

there exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that $\varphi(z_i) = w_i$ for all $i = 1, \ldots, m$.

This definition is in view of the classical Pick positivity and the Nevanlinna-Pick interpolation on $\mathbb{D}$. We need another definition along the lines of Sarason's commutant lifting theorem:

Definition 2.9.4. *A quotient module $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ satisfies the commutant lifting property if every contraction on $\mathcal{Q}$ admits lifting.*

In other words, for a module map $X \in \mathcal{B}_1(\mathcal{Q})$, there exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that $X = S_\varphi$. Now we use Sarason's trick to prove the Nevanlinna-Pick interpolation but in the setting of $\mathcal{S}(\mathbb{D}^n)$ for any $n \geq 1$. The proof is standard and follows in Sarason's footsteps.

Proposition 2.9.5. *Let $\mathcal{Z} = \{z_j\}_{j=1}^m \subset \mathbb{D}^n$ be a set of m distinct points. Then $\mathcal{Z}$ is a Pick set if and only if $\mathcal{Q}_\mathcal{Z}$ satisfies the commutant lifting property, where*

$$\mathcal{Q}_\mathcal{Z} = span\{\mathbb{S}(\cdot, z_i) : i = 1, \ldots, m\}.$$

Proof. We begin with a simple observation. Given $\mathcal{W} = \{w_i\}_{i=1}^m \subset \mathbb{D}$, we define $X \in \mathcal{B}(\mathcal{Q}_\mathcal{Z})$ by (note that $\mathcal{Q}_\mathcal{Z}$ is a finite-dimensional Hilbert space)

$$X\mathbb{S}(\cdot, z_i) = \bar{w}_i \mathbb{S}(\cdot, z_i) \qquad (i = 1, \ldots, m).$$

By Lemma 2.5.3, it follows that X^* is a module map. Moreover, we have

$$\Big\langle (I_{\mathcal{Q}_\mathcal{Z}} - X^*X)\Big(\sum_{j=1}^m \alpha_j \mathbb{S}(\cdot, z_j)\Big), \Big(\sum_{i=1}^m \alpha_i \mathbb{S}(\cdot, z_i)\Big)\Big\rangle = \sum_{i,j=1}^m \alpha_j \bar{\alpha}_i (1 - w_i \bar{w}_j) \mathbb{S}(z_i, z_j),$$

for all scalars $\{\alpha_i\}_{i=1}^m \subset \mathbb{C}$. It follows that X is a contraction if and only if

$$\mathfrak{P}_{\mathcal{Z},\mathcal{W}} \geq 0.$$

Now suppose that $\mathcal{Z}$ is a Pick set, and suppose $Y \in \mathcal{B}_1(\mathcal{Q}_\mathcal{Z})$ is a module map. We claim that Y has a lift. If we define $X := Y^*$, then we are precisely in the setting of the above discussion. The contractivity of X (as $\|Y^*\| \leq 1$) then implies that the Pick matrix is positive, that is, $\mathfrak{P}_{\mathcal{Z},\mathcal{W}} \geq 0$. There exists $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that $\varphi(z_i) = w_i$ for all $i = 1, \ldots, m$. Then

$$Y^*\mathbb{S}(\cdot, z_j) = T_\varphi^* \mathbb{S}(\cdot, z_j) \qquad (j = 1, \ldots, m),$$

and we conclude that $Y^* = T_\varphi^*|_{\mathcal{Q}_\mathcal{Z}}$, or equivalently, $Y = S_\varphi$.

To show the converse, assume that $\mathcal{Q}_\mathcal{Z}$ satisfies the commutant lifting property. Suppose $\mathcal{W} = \{w_i\}_{i=1}^m \subset \mathbb{D}$, and let $\mathfrak{P}_{\mathcal{Z},\mathcal{W}} \geq 0$. Then X, as defined at the beginning of the proof, is a contraction, and hence $X = S_\varphi$ for some $\varphi \in \mathcal{S}(\mathbb{D}^n)$. It is now routine to check that $\varphi(z_i) = w_i$ for all $i = 1, \ldots, m$. $\square$

In the case of $n = 1$, the classical Nevanlinna Pick interpolation theorem now follows directly from Sarason's lifting theorem. The above formulation also works verbatim the same way as for multiplier spaces for general reproducing kernel Hilbert spaces over domains in $\mathbb{C}^n$ (including the open unit ball in $\mathbb{C}^n$).

In view of the above proposition, we conclude that the solution to the interpolation problem in terms of Pick positivity is simply equivalent to the commutant lifting problem for quotient modules of the form $\mathcal{Q}_{\mathcal{Z}}$ for finite subsets $\mathcal{Z} \subseteq \mathbb{D}^n$. Again, this is true for general multiplier spaces.

2.9.3 Bergman space and lifting

Although all of the observations in this subsection hold true for weighted Bergman spaces (even for a large class of reproducing kernel Hilbert spaces) over $\mathbb{D}^n$ along with verbatim proofs, we will stick to the Bergman space only. Denote by $A^2(\mathbb{D}^n)$ the Bergman space over $\mathbb{D}^n$. Recall that an analytic function f on $\mathbb{D}^n$ is in $A^2(\mathbb{D}^n)$ if and only if

$$\|f\|_{A^2(\mathbb{D}^n)} := \Big(\int_{\mathbb{D}^n} |f(z)|^2 \, d\sigma(z) \Big)^{\frac{1}{2}} < \infty,$$

where $d\sigma(z)$ denotes the normalized volume measure on $\mathbb{D}^n$. We know that $A^2(\mathbb{D}^n)$ is a reproducing kernel Hilbert space corresponding to the Bergman kernel

$$K(z, w) = \prod_{i=1}^{n} \frac{1}{(1 - z_i \bar{w}_i)^2} \qquad (z, w \in \mathbb{D}^n).$$

Recall that the multiplier space of $A^2(\mathbb{D}^n)$ is again $H^\infty(\mathbb{D}^n)$, which for simplicity of notation (or, to avoid confusion), we denote by $\mathcal{M}(A^2(\mathbb{D}^n))$. In other words

$$\mathcal{M}(A^2(\mathbb{D}^n)) = H^\infty(\mathbb{D}^n).$$

For each $\varphi \in \mathcal{M}(A^2(\mathbb{D}^n))$, the map $f \in A^2(\mathbb{D}^n) \mapsto \varphi f \in A^2(\mathbb{D}^n)$ defines a multiplication operator on $A^2(\mathbb{D}^n)$, which we denote by M_φ.

Let $\mathcal{Q} \subseteq A^2(\mathbb{D}^n)$ be a quotient module (that is, $\mathcal{Q}$ is closed and $M_{z_i}^* \mathcal{Q} \subseteq \mathcal{Q}$ for all $i = 1, \dots, n$). For each $\varphi \in H^\infty(\mathbb{D}^n)$, set

$$B_\varphi = P_{\mathcal{Q}} M_\varphi|_{\mathcal{Q}}.$$

Suppose $X \in \mathcal{B}_1(\mathcal{Q})$ is a module map, that is, $X B_{z_i} = B_{z_i} X$ for all $i = 1, \dots, n$. We say that X is *liftable* or X has a *lift* if there exists $\varphi \in H^\infty(\mathbb{D}^n) = \mathcal{M}(A^2(\mathbb{D}^n))$ such that

$$X = B_\varphi,$$

and

$$\|B_\varphi\|_{\mathcal{B}(A^2(\mathbb{D}^n))} \leq 1. \tag{2.9.1}$$

We are interested in the commutant lifting for finite-dimensional zero-based quotient modules of $A^2(\mathbb{D}^n)$. For a set of distinct points $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{D}^n$, we define (following Section 2.5) the m-dimensional zero-based quotient module $\mathcal{B}_{\mathcal{Z}} \subseteq A^2(\mathbb{D}^n)$ as

$$\mathcal{B}_{\mathcal{Z}} = \text{span}\{K(\cdot, z_i) : i = 1, \dots, m\} \subseteq A^2(\mathbb{D}^n).$$

At the same time, keep in mind that $\mathcal{Q}_{\mathcal{Z}}$ is also a zero-based quotient module of $H^2(\mathbb{T}^n)$ (again, see the preceding subsection or Section 2.5), where

$$\mathcal{Q}_{\mathcal{Z}} = \text{span}\{\mathbb{S}(\cdot, z_i) : i = 1, \dots, m\} \subseteq H^2(\mathbb{T}^n).$$

Note that module maps on $\mathcal{Q}_{\mathcal{Z}}$ are parameterized by m scalars. To be more precise, let $X \in \mathcal{B}(\mathcal{Q}_{\mathcal{Z}})$. Then X is a module map if and only if there exists $\{w_i\}_{i=1}^m \subset \mathbb{C}$ such that

$$X^* \mathbb{S}(\cdot, z_i) = w_i \mathbb{S}(\cdot, z_i),$$

for all $i = 1, \dots, m$. This was observed in Lemma 2.5.3. The same conclusion and proof apply to $\mathcal{B}_{\mathcal{Z}}$. Therefore, a module map $X \in \mathcal{B}(\mathcal{Q}_{\mathcal{Z}})$ is associated with $\{w_i\}_{i=1}^m \subseteq \mathbb{C}$, which further defines a module map $\tilde{X} \in \mathcal{B}(\mathcal{B}_{\mathcal{Z}})$ as

$$\tilde{X}^* K(\cdot, z_i) = w_i K(\cdot, z_i),$$

for all $i = 1, \dots, m$. Consequently, we have the bijective correspondence

$$X \in \mathcal{B}(\mathcal{Q}_{\mathcal{Z}}) \longleftrightarrow \tilde{X} \in \mathcal{B}(\mathcal{B}_{\mathcal{Z}}).$$

In the case of $n = 1$, the problem of commutant lifting for quotient module $\mathcal{B}_{\mathcal{Z}}$ of $A^2(\mathbb{D})$ was studied in the thesis of Sultanic [98]. While she was focused on finite-dimensional quotient modules of $A^2(\mathbb{D})$, but the zero-based quotient modules played the most crucial role. Here we aim at proving the following proposition:

Proposition 2.9.6. *Let $\mathcal{Z} = \{z_i\}_{i=1}^m \subset \mathbb{D}^n$ be a set of distinct points, and let $X \in \mathcal{B}_1(\mathcal{Q}_{\mathcal{Z}})$ be a module map. Then X on $\mathcal{Q}_{\mathcal{Z}}$ is liftable if and only if $\tilde{X}$ on $\mathcal{B}_{\mathcal{Z}}$ is liftable.*

Proof. We start by stating a general (and well known) fact: Let $\varphi \in \mathcal{M}(A^2(\mathbb{D}^n))$. Then the operator norm (or multiplier norm) of M_φ on $A^2(\mathbb{D}^n)$ is given by

$$\|M_\varphi\|_{\mathcal{B}(A^2(\mathbb{D}^n))} = \|\varphi\|_\infty.$$

Indeed, for $f \in A^2(\mathbb{D}^n)$, we have

$$\begin{aligned}\|\varphi f\|_{A^2(\mathbb{D}^n)} &= \Big(\int_{\mathbb{D}^n} |\varphi f|^2 d\sigma\Big)^{\frac{1}{2}}\\ &\le \Big(\int_{\mathbb{D}^n} \|\varphi\|_\infty^2 |f|^2 d\sigma\Big)^{\frac{1}{2}}\\ &\le \|\varphi\|_\infty \Big(\int_{\mathbb{D}^n} |f|^2 d\sigma\Big)^{\frac{1}{2}},\end{aligned}$$

that is, $\|M_\varphi\|_{\mathcal{B}(A^2(\mathbb{D}^n))} \le \|\varphi\|_\infty$. On the other hand, for each $w \in \mathbb{D}^n$,

$$\begin{aligned}\varphi(w) &= \frac{1}{\|K(\cdot,w)\|^2}\langle K(\cdot,w), \overline{\varphi(w)}K(\cdot,w)\rangle\\ &= \frac{1}{\|K(\cdot,w)\|^2}\langle K(\cdot,w), T_\varphi^* K(\cdot,w)\rangle\\ &= \Big\langle T_\varphi\Big(\frac{K(\cdot,w)}{\|K(\cdot,w)\|}\Big), \frac{K(\cdot,w)}{\|K(\cdot,w)\|}\Big\rangle,\end{aligned}$$

implies that $|\varphi(w)| \le \|M_\varphi\|_{\mathcal{B}(A^2(\mathbb{D}^n))}$, and completes the proof of the claim. Now, suppose that $\tilde{X}$ on $\mathcal{B}_{\mathcal{Z}}$ is liftable, that is $\tilde{X} = B_\varphi$ for some $\varphi \in \mathcal{M}(A^2(\mathbb{D}^n)) = H^\infty(\mathbb{D}^n)$ with $\|M_\varphi\|_{\mathcal{B}(A^2(\mathbb{D}^n))} \le 1$. In view of the above observation, we have $\varphi \in \mathcal{S}(\mathbb{D}^n)$. Suppose $\{w_i\}_{i=1}^m \subset \mathbb{C}$ be the scalars corresponding to $\tilde{X}$, that is

$$\tilde{X}^* K(\cdot, z_i) = w_i K(\cdot, z_i),$$

for all $i = 1, \ldots, m$. This and the equality $\tilde{X} = B_\varphi$ imply that

$$\varphi(z_i) = \bar{w}_i \qquad (i = 1, \ldots, m),$$

and hence $X^* = S_\varphi^*$. Therefore, $X = S_\varphi$, and hence φ is a lift of X. Proof of the reverse direction is similar. $\square$

In other words, the lifting problem on zero-based quotient modules of $A^2(\mathbb{D}^n)$ is equivalent to the lifting problem on zero-based quotient modules of $H^2(\mathbb{T}^n)$. In the case $n = 1$, for a module map $\tilde{X} \in \mathcal{B}_1(\mathcal{B}_{\mathcal{Z}})$, if $\|X\|_{\mathcal{B}(\mathcal{Q}_{\mathcal{Z}})} \le 1$, then $\tilde{X}$ can be lifted (thanks to Sarason). On the other hand, if $X \in \mathcal{B}_1(\mathcal{Q}_{\mathcal{Z}})$ is a module map, then automatically $\tilde{X} \in \mathcal{B}_1(\mathcal{B}_{\mathcal{Z}})$, and hence X has a lift.

2.9.4 Decompositions of polynomials

In this subsection, we decompose polynomials with respect to Beurling-type quotient modules of $H^2(\mathbb{T}^n)$. This result has already been used ($n = 1$ case) to recover Sarason's commutant lifting theorem (see Theorem 2.8.2).

A quotient module $\mathcal{Q} \subseteq H^2(\mathbb{T}^n)$ is said to be of *Beurling type* if there exists an inner function $\varphi \in \mathcal{S}(\mathbb{D}^n)$ such that

$$\mathcal{Q} = (\varphi H^2(\mathbb{T}^n))^\perp.$$

Recall that all one variable quotient modules are of Beurling type [23].

Proposition 2.9.7. *Let $\varphi \in \mathcal{S}(\mathbb{D}^n)$ be an inner function, and let $p \in \mathbb{C}[z_1, \ldots, z_n]$. Write*

$$p = f \oplus g \in \varphi H^2(\mathbb{T}^n) \oplus (\varphi H^2(\mathbb{T}^n))^\perp.$$

Then $f, g \in H^\infty(\mathbb{D}^n)$.

Proof. It is enough to prove that $f \in H^\infty(\mathbb{D}^n)$. It is also enough to consider p as a monomial. Fix $k \in \mathbb{Z}_+^n$, and suppose

$$z^k = f \oplus g \in \varphi H^2(\mathbb{T}^n) \oplus (\varphi H^2(\mathbb{T}^n))^\perp.$$

Let $\boldsymbol{\lambda} \in \mathbb{Z}_+^n$, and suppose $l_i > k_i$ for some $i = 1, \ldots, n$. Since $T_z^{*\boldsymbol{\lambda}}(z^k) = 0$, it follows that

$$T_z^{*\boldsymbol{\lambda}} f = -T_z^{*\boldsymbol{\lambda}} g.$$

Since g is in the quotient module $(\varphi H^2(\mathbb{T}^n))^\perp$, we conclude that

$$T_z^{*\boldsymbol{\lambda}} f \in (\varphi H^2(\mathbb{T}^n))^\perp.$$

Now there exists $f_1 \in H^2(\mathbb{T}^n)$ such that $f = \varphi f_1$. Consequently

$$T_z^{*\boldsymbol{\lambda}} f = T_z^{*\boldsymbol{\lambda}} \varphi f_1 \in (\varphi H^2(\mathbb{T}^n))^\perp,$$

and hence

$$\langle T_z^{*\boldsymbol{\lambda}} \varphi f_1, \varphi h \rangle = 0,$$

for all $h \in H^2(\mathbb{T}^n)$. Then, $T_\varphi^* T_z^{*\boldsymbol{\lambda}} = T_z^{*\boldsymbol{\lambda}} T_\varphi^*$ and $T_\varphi^* T_\varphi = I$ yield

$$\begin{aligned}\langle T_z^{*\boldsymbol{\lambda}} f_1, h \rangle &= \langle T_z^{*\boldsymbol{\lambda}} \varphi f_1, \varphi h \rangle \\ &= 0,\end{aligned}$$

for all $h \in H^2(\mathbb{T}^n)$ and $l \in \mathbb{Z}_+^n$ such that $l_i > k_i$ for some $i = 1, \ldots, n$. Therefore

$$f_1 \in \bigcap_{|l| = |k|+1} \ker T_z^{*l}.$$

and hence

$$f_1 \in \operatorname{span}\{z^t : t \in \mathbb{Z}_+^n, |t| \leq |k| + 1\}.$$

We conclude that

$$f \in \operatorname{span}\{z^t \varphi : t \in \mathbb{Z}_+^n, |t| \leq |k| + 1\} \subseteq H^\infty(\mathbb{D}^n).$$

This completes the proof of the proposition. □

A similar question could be posed for other classes of functions. What about the decomposition of a rational function with respect to a Beurling decomposition, for example?

2.10 Concluding remarks

We start off by commenting on the commutant lifting theorem. Let us recall Ball, Li, Timotin, and Trent's commutant lifting theorem [20, Theorem 5.1], which is only relevant for $n = 2$ case in our context.

Theorem 2.10.1. *Let $\mathcal{Q} \subseteq H^2(\mathbb{T}^2)$ be a quotient module, and let $X \in \mathcal{B}_1(\mathcal{Q})$ be a module map. Then X admits a lift if and only if there exist positive operators $G_1, G_2 \in \mathcal{B}(\mathcal{Q})$ such that $G_1 - S_{z_2} G_1 S_{z_2}^* \geq 0$ and $G_2 - S_{z_1} G_2 S_{z_1}^* \geq 0$, and*

$$I - XX^* = G_1 + G_2.$$

The proof is based on Agler's transfer function realization formula for functions in $\mathcal{S}(\mathbb{D}^2)$ (which we will comment on more about below). In contrast to the preceding theorem, however, our commutant lifting theorem appears to be more explicit. For instance, Theorem 2.4.1 has been validated for the examples constructed in Corollary 2.3.3 (see Section 2.7).

Now we turn to the interpolation problem. We already mentioned in Section 2.1 that the traditional approach to solving the interpolation problem in terms of the positivity of the Pick matrix (or family of Pick matrices) in higher variables produces only limited results. There is, however, likely to be one notable exception: interpolation on $\mathbb{D}^2$, which Agler [5, 6] pioneered in his seminal papers in the late '80s (also see [8, Theorem 1.3]):

Theorem 2.10.2. *Let $\{(\alpha_i, \beta_i)\}_{i=1}^m$ be a set of distinct points in $\mathbb{D}^2$ and let $\{w_i\}_{i=1}^m \subset \mathbb{D}$. There exists $\varphi \in \mathcal{S}(\mathbb{D}^2)$ such that*

$$\varphi(\alpha_i, \beta_i) = w_i,$$

for all $i = 1, \ldots, m$, if and only if there exist positive semi-definite $m \times m$ matrices $\Gamma = (\Gamma_{ij})$ and $\Delta = (\Delta_{ij})$ such that

$$(1 - \bar{w}_i w_j) = (1 - \bar{\alpha}_i \alpha_j)\Gamma_{ij} + (1 - \bar{\beta}_i \beta_j)\Delta_{ij},$$

for all $i, j = 1, \ldots, m$.

This is clearly an analogue of the solution to the classical Nevanlinna–Pick interpolation problem (also see Cole and Wermer [32, 33, 34]). In a slightly different context, see Kosiński [65] for three-point interpolation problem (also, see Cotlar and Sadosky [35, 36, 37]). Whereas the above result appears to be abstract (particularly the existence of positive semi-definite matrices), the approach is useful in a variety of other problems. Indeed, based on the Ando dilation and the von Neumann inequality for pairs of commuting contractions [11], Agler derived a realization formula for Schur functions in terms of colligation matrices, which leads to the above solution to the interpolation problem. His realization formula has proven very useful in operator theory and function theory on $\mathbb{D}^n$, $n \geq 2$. Whereas we believe Theorem 2.5.6 is more concrete and provides a new perspective on the interpolation problem in general, we are unsure how to relate it to Theorem 2.10.2. We are also unclear about using Theorem 2.10.2 to validate the examples of interpolation in Theorem 2.6.4 for the specific case of $n = 2$.

Finally, we remark that, unlike the present case of scalar functions, the earlier lifting theorem and the solutions to the interpolation problem work equally well for the operator or vector-valued functions [8, 20, 18]. The powerful n-variables von Neumann inequality (which is automatic in the case of $n = 2$ but not so when $n > 2$), like the Sz.-Nagy and Foiaş [70] effective dilation theoretic approach appears to be a key factor. However, as previously stated, we followed a function theoretic route pioneered by Sarason in his work [91]. The results reported here, we think, will be also helpful in building related theories like isometric dilations for commuting contractions, several variables von Neumann inequality, Nehari problem on $\mathbb{D}^n$, etc., similar to Sarason's classic result.

Chapter 3

Perturbations Of Analytic Functions On The Polydisc

3.1 Introduction

Our aim in this chapter is to present a classification of $H^2(\mathbb{T}^n)$-functions that can be perturbed by $H^2(\mathbb{T}^n)$-functions so that the resultant functions are in $\mathcal{S}(\mathbb{D}^n)$. Our perturbation result is of independent interest and not directly related to the commutant lifting theorem. However, the technique involved here is motivated by the one used in the proof of the lifting theorem in the preceding chapter.

Our interest is in the following question: Given a nonzero function $f \in H^2(\mathbb{T}^n)$, does there exist $g \in H^2(\mathbb{T}^n)$ such that

$$f + g \in \mathcal{S}(\mathbb{D}^n)?$$

Of course, to avoid triviality (that $g = -f$, for instance), we assume that $g \in \{f\}^{\perp}$. Set

$$\mathcal{L}_n = \mathcal{M}_n \oplus H_0^2(\mathbb{T}^n),$$

and treat it as a subspace of $L^1(\mathbb{T}^n)$.

To answer this, we first formalize some notations. Throughout the sequel, we denote

$$\mathcal{L}_n = \mathcal{M}_n \oplus H_0^2(\mathbb{T}^n).$$

Recall that $H_0^2(\mathbb{T}^n) = H^2(\mathbb{T}^n) \ominus \{1\}$ is the closed subspace of $H^2(\mathbb{T}^n)$ of functions vanishing at the origin. Recall also that

$$\mathcal{M}_n = L^2(\mathbb{T}^n) \ominus (H^2(\mathbb{T}^n)^{conj} + H^2(\mathbb{T}^n)),$$

the closed subspace of $L^2(\mathbb{T}^n)$ generated by all the trigonometric monomials that are neither analytic nor co-analytic. In particular, we have the crucial property that

$$\langle f, 1\rangle_{L^2(\mathbb{T}^n)} = 0 \qquad (f \in \mathcal{L}_n).$$

Finally, we recall a basic fact from Banach space theory: Let x be a vector in a Banach space B. Then

$$\|x\|_B = \sup\{|x^*(x)| : x^* \in B^*, \|x^*\| \leq 1\}.$$

Now we are ready for the perturbation theorem.

Theorem 3.1.1. *Let $f \in H^2(\mathbb{T}^n)$ be a nonzero function. There exists $g \in \{f\}^\perp$ such that $f + g \in \mathcal{S}(\mathbb{D}^n)$ if and only if*

$$dist_{L^1(\mathbb{T}^n)}\Big(\frac{\bar{f}}{\|f\|_2^2}, \mathcal{L}_n\Big) \geq 1.$$

Proof. We start by recalling the definition of distance function (in the present case):

$$\mathrm{dist}_{L^1(\mathbb{T}^n)}\Big(\frac{\bar{f}}{\|f\|_2^2}, \mathcal{L}_n\Big) = \inf\Big\{\Big\|\frac{\bar{f}}{\|f\|_2^2} + h\Big\|_1 : h \in \mathcal{L}_n\Big\}.$$

Suppose $g \in \{f\}^\perp$ be such that $\psi := f + g \in \mathcal{S}(\mathbb{D}^n)$. It is enough to prove that

$$\Big\|\frac{\bar{f}}{\|f\|_2^2} + h\Big\|_1 \geq 1 \qquad (h \in \mathcal{L}_n).$$

Fix $h \in \mathcal{L}_n$. Since $\psi \in \mathcal{S}(\mathbb{D}^n)$ and $\mathcal{S}(\mathbb{D}^n)$ is a subset of the closed unit ball of $L^\infty(\mathbb{T}^n)$, we have $\psi \in L^\infty(\mathbb{T}^n)$ and $\|\psi\|_\infty \leq 1$. By the duality (see (2.4.1))

$$(L^1(\mathbb{T}^n))^* \cong L^\infty(\mathbb{T}^n),$$

it follows that $\chi_\psi \in (L^1(\mathbb{T}^n))^*$ and

$$\|\psi\|_\infty = \|\chi_\psi\| \leq 1,$$

where

$$\chi_\psi g = \int_{\mathbb{T}^n} \psi g \, d\mu,$$

for all $g \in L^1(\mathbb{T}^n)$. In particular, for

$$g = \frac{\bar{f}}{\|f\|_2^2} + h \in L^1(\mathbb{T}^n),$$

we compute

$$\begin{aligned}\int_{\mathbb{T}^n} \psi\Big(\frac{\bar{f}}{\|f\|_2^2} + h\Big)\, d\mu &= \Big\langle f + g, \frac{f}{\|f\|_2^2} + \bar{h}\Big\rangle_{L^2(\mathbb{T}^n)} \\ &= 1 + \Big\langle g, \frac{f}{\|f\|_2^2} + \bar{h}\Big\rangle_{L^2(\mathbb{T}^n)} \\ &= 1.\end{aligned}$$

The last but one equality follows from the fact that (note that $\bar{h}$ has no analytic part)

$$\langle f, \bar{h}\rangle_{L^2(\mathbb{T}^n)} = 0,$$

and the last equality is due to the fact that $g \in \{f\}^{\perp}$ and

$$\langle g, \bar{h}\rangle_{L^2(\mathbb{T}^n)} = 0,$$

similar reason as in the preceding equality. We also have used the fact that f is analytic and $\langle h, 1\rangle_{L^2(\mathbb{T}^n)} = 0$. Therefore, $\chi_\psi \in (L^1(\mathbb{T}^n))^*$ with $\|\chi_\psi\| \le 1$ and

$$\Big|\chi_\psi\Big(\frac{\bar{f}}{\|f\|_2^2} + h\Big)\Big| = 1.$$

The norm identity for Banach spaces stated preceding the statement of this theorem immediately implies that

$$\Big\|\frac{\bar{f}}{\|f\|_2^2} + h\Big\|_1 \ge 1.$$

For the reverse direction, suppose the above inequality holds for all $h \in \mathcal{L}_n$. Equivalently

$$\|\lambda \bar{f} + h\|_1 \ge |\lambda| \|f\|_2^2,$$

for all $h \in \mathcal{L}_n$ and $\lambda \in \mathbb{C}$. Define $\mathcal{S}$ a subspace of $L^1(\mathbb{T}^n)$ as

$$\mathcal{S} := \operatorname{span}\{\bar{f}, \mathcal{L}_n\},$$

and then define a linear functional $\zeta_f : \mathcal{S} \to \mathbb{C}$ by

$$\zeta_f(\lambda \bar{f} + h) = \int_{\mathbb{T}^n} (\lambda \bar{f} + h) f \, d\mu,$$

for all $h \in \mathcal{L}_n$ and $\lambda \in \mathbb{C}$. As in the proof of the forward direction, we have

$$\begin{aligned}\int_{\mathbb{T}^n} f h \, d\mu &= \langle h, \bar{f}\rangle_{L^2(\mathbb{T}^n)} \\ &= 0,\end{aligned}$$

for all $h \in \mathcal{L}_n$. Moreover, since

$$\int_{\mathbb{T}^n} f\bar{f}\, d\mu = \|f\|_2^2,$$

it follows that

$$\begin{aligned} |\zeta_f(\lambda\bar{f} + h)| &= |\lambda|\|f\|_2^2 \\ &\leq \|\lambda\bar{f} + h\|_1, \end{aligned}$$

for all $h \in \mathcal{L}_n$ and $\lambda \in \mathbb{C}$. This ensures that ζ_f is a contractive functional on $\mathcal{S}$; hence, by the Hahn-Banach theorem, there exists $\zeta \in (L^\infty(\mathbb{T}^n))^*$ such that $\|\zeta\| \leq 1$ and

$$\zeta|_{\mathcal{S}} = \zeta_f.$$

Again, by the duality (2.4.1), there exists $\varphi \in L^\infty(\mathbb{T}^n)$ such that $\|\varphi\|_\infty \leq 1$ and

$$\chi_\varphi|_{\mathcal{S}} = \zeta|_{\mathcal{S}} = \zeta_f.$$

Therefore

$$\int_{\mathbb{T}^n} (\lambda\bar{f} + h) f\, d\mu = \int_{\mathbb{T}^n} (\lambda\bar{f} + h)\varphi\, d\mu, \tag{3.1.1}$$

for all $h \in \mathcal{L}_n$ and $\lambda \in \mathbb{C}$. We now claim that φ is analytic (which would clearly imply that $\varphi \in H^\infty(\mathbb{D}^n)$). As in the proof of Theorem 2.4.1, we consider a typical monomial F from $\mathcal{L}_n = \mathcal{M}_n \dotplus H_0^2(\mathbb{T}^n)$. Therefore

$$F = z^k,$$

for some $k \in \mathbb{N}^n$, or

$$F = z_A^{k_A} \bar{z}_B^{k_B},$$

for some $k_A \in \mathbb{Z}_+^{|A|}$ and $k_B \in \mathbb{Z}_+^{|B|}$, where $A, B \subseteq \{1, \ldots, n\}$, $A \cap B = \emptyset$, and $A, B \neq \emptyset$ (see the representation of $\mathcal{M}_n$ in (2.4.2)). We compute

$$\begin{aligned} 0 &= \langle f, \bar{F} \rangle_{L^2(\mathbb{T}^n)} \\ &= \int_{\mathbb{T}^n} fF\, d\mu \\ &= \int_{\mathbb{T}^n} \varphi F\, d\mu \\ &= \langle \varphi, \bar{F} \rangle_{L^2(\mathbb{T}^n)}, \end{aligned}$$

which proves the claim. Since $\|\varphi\|_\infty \leq 1$, we conclude that $\varphi \in \mathcal{S}(\mathbb{D}^n)$. Using the containment $H^\infty(\mathbb{D}^n) \subseteq H^2(\mathbb{D}^n)$, first we conclude $\varphi \in H^2(\mathbb{D}^n)$, and then write

$$\varphi = cf \oplus g,$$

for some scalar c and function $g \in H^2(\mathbb{D}^n)$ such that $g \in \{f\}^\perp$. It remains to show that $c = 1$. Observe, if $h = 0$, and

$$\lambda = \frac{1}{\|f\|_2^2},$$

then (3.1.1) along with the fact that $\langle g, f\rangle = 0$ yields

$$\begin{aligned} 1 &= \int_{\mathbb{T}^n} f \frac{\bar{f}}{\|f\|_2^2}\, d\mu \\ &= \int_{\mathbb{T}^n} \varphi \frac{\bar{f}}{\|f\|_2^2}\, d\mu \\ &= \Big\langle \varphi, \frac{f}{\|f\|_2^2} \Big\rangle_{H^2(\mathbb{T}^n)} \\ &= \Big\langle cf \oplus g, \frac{f}{\|f\|_2^2} \Big\rangle_{H^2(\mathbb{T}^n)} \\ &= c. \end{aligned}$$

This completes the proof of the theorem. $\square$

We know, in particular, that $\mathcal{M}_1 = \{0\}$ (see (2.4.4)). Moreover, as observed earlier, that $H_0^2(\mathbb{T}) = zH^2(\mathbb{T})$. Therefore

$$\mathcal{L}_1 = zH^2(\mathbb{T}),$$

and as a result, the preceding theorem is simplified as follows:

Corollary 3.1.2. *Given $f \neq 0$ in $H^2(\mathbb{T})$, there exists $g \in \{f\}^\perp$ such that*

$$f + g \in \mathcal{S}(\mathbb{D}),$$

if and only if

$$dist_{L^1(\mathbb{T})}\Big(\frac{\bar{f}}{\|f\|_2^2}, zH^2(\mathbb{T})\Big) \geq 1.$$

The following example illustrates the above theorem.

Example 3.1.3. *Fix a real number $0 < c < 1$, and pick $b \in (c^2, c)$. Also fix a multiindex $k_0 \in \mathbb{Z}_+^n$, $k_0 \neq (0, \ldots, 0)$, and set*

$$\Lambda := \mathbb{Z}_+^n \setminus \{k_0\}.$$

Finally, choose a sequence $\{a_k\}_{k\in\Lambda} \subseteq \mathbb{R}_+$ such that

1. $\sum\limits_{k\in\Lambda} a_k$ *diverges, and*

2. $\sqrt{\sum\limits_{k\in\Lambda} a_k^2 + b^2} = c.$

Set

$$f = \sum_{k \in \Lambda} a_k z^k + b z^{k_0}.$$

We want to show that f can be perturbed to become a Schur function. To this end, we first observe that $f(0, \ldots, 0) = a_0$ and

$$f(1, \ldots, 1) = b + \sum_{k \in \Lambda} a_k,$$

and hence (by continuity)

$$f(L) = (a_0, \infty),$$

where L is the line joining $(0, \ldots, 0)$ and $(1, \ldots, 1)$. We conclude, in particular, that

$$f \notin H^\infty(\mathbb{D}^n).$$

Moreover

$$\|f\|_2 = c,$$

by construction of f. We now consider the functional $\chi_{z^{k_0}} \in (L^1(\mathbb{T}^n))^*$ (see the duality (2.4.1)). Clearly

$$\|\chi_{z^{k_0}}\| = 1.$$

Given arbitrary functions $g \in \mathcal{M}_n$ and $h \in H_0^2(\mathbb{T}^n)$, we compute

$$\begin{aligned}
\chi_{z^{k_0}}\Big(\frac{\bar{f}}{c^2} + g + h\Big) &= \int_{\mathbb{T}^n} z^{k_0}\Big(\frac{\bar{f}}{c^2} + g + h\Big) d\mu \\
&= \Big\langle \frac{\bar{f}}{c^2} + g + h, \bar{z}^{k_0} \Big\rangle_{L^2(\mathbb{T}^n)} \\
&= \Big\langle \frac{\bar{f}}{c^2}, \bar{z}^{k_0} \Big\rangle_{L^2(\mathbb{T}^n)} \\
&= \frac{b}{c^2}.
\end{aligned}$$

Since $b > c^2$, it follows that

$$\chi_{z^{k_0}}\Big(\frac{\bar{f}}{c^2} + g + h\Big) \geq 1,$$

and consequently, the norm identity that was mentioned preceding the statement of Theorem 2.5.6 infers that

$$\Big\|\frac{\bar{f}}{c^2} + g + h\Big\|_1 \geq 1.$$

Given that $\mathcal{L}_n = \mathcal{M}_n \dotplus H_0^2(\mathbb{T}^n)$, the above is equivalent to saying that

$$dist_{L^1(\mathbb{T}^n)}\Big(\frac{\bar{f}}{\|f\|_2^2}, \mathcal{L}_n\Big) \geq 1,$$

and hence, by Theorem 3.1.1, we conclude that $f \oplus g \in \mathcal{S}(\mathbb{D}^n)$ for some $g \in \{f\}^\perp$.

It may appear to be a coincidence that the distance recipe in Theorem 3.1.1 as well as in Theorem 2.4.2 (and the quantitative interpolation theorem in Section 2.6) is similar to the well known Nehari theorem [79] for Hankel operators. Recall that the Hankel operator with symbol $\varphi \in L^\infty(\mathbb{T})$ is defined by

$$H_\varphi = P_{H^2_-(\mathbb{T})} L_\varphi|_{H^2(\mathbb{T})},$$

where $H^2_-(\mathbb{T}) = L^2(\mathbb{T}) \ominus H^2(\mathbb{T})$. The Nehari theorem states:

$$\|H_\varphi\| = \mathrm{dist}(\varphi, H^\infty(\mathbb{D})) = \|\varphi\|_\infty.$$

Furthermore, it is well known that the Nehari problem is related to the Nevanlinna-Pick interpolation problem for rational functions. See also the well known Adamyan, Arov, and Krein theorem, also known as the AAK step-by-step extension [81, Chapter 2]. Another important formula is due to Adamyan, Arov and Krein [4]:

$$\|H_\varphi\|_{ess} = \mathrm{dist}(\varphi, C(\mathbb{T}) + H^\infty(\mathbb{D})),$$

for all $\varphi \in L^\infty(\mathbb{T})$, where $C(\mathbb{T})$ denotes the space of all continuous functions on $\mathbb{T}$. Hankel operators in several variables [86] are also complex objects. We refer the reader to Coifman, Rochberg, and Weiss [30] for some progress to the theory of Hankel operators (also see [50]).

Chapter 4

Commutant Lifting And Nevanlinna-Pick Interpolation On The Unit Ball

4.1 Introduction

In this chapter we make a contribution to a commutant lifting theorem and a version of Nevanlinna-Pick interpolation in several variables. To be more precise, let $m \geq 1$ and let $\mathcal{H}_m$ denotes the reproducing kernel Hilbert space corresponding to the kernel k_m on $\mathbb{B}^n$, where

$$k_m(z, w) = (1 - \sum_{i=1}^{n} z_i \bar{w}_i)^{-m} \qquad (z, w \in \mathbb{B}^n),$$

and $\mathbb{B}^n = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{i=1}^{n} |z_i|^2 < 1\}$.

Our main result, restricted to $\mathcal{H}_m$, $m > 1$, can now be formulated as follows:

Commutant lifting theorem (Theorem 4.3.4): Suppose $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are joint $(M_{z_1}, \ldots, M_{z_n})$ co-invariant subspaces of $H^2_n(= \mathcal{H}_1)$ and $\mathcal{H}_m$, respectively. Let $X \in \mathcal{B}(\mathcal{Q}_1, \mathcal{Q}_2)$ and $\|X\| \leq 1$. If

$$X(P_{\mathcal{Q}_1} M_{z_i}|_{\mathcal{Q}_1}) = (P_{\mathcal{Q}_2} M_{z_i}|_{\mathcal{Q}_2})X,$$

for all $i = 1, \ldots, n$, then there exists a holomorphic function $\varphi : \mathbb{B}^n \to \mathbb{C}$ such that the multiplication operator $M_\varphi \in \mathcal{B}(H^2_n, \mathcal{H}_m)$, $\|M_\varphi\| \leq 1$ (that is, φ is a *contractive multiplier*), and

$$X = P_{\mathcal{Q}_2} M_\varphi|_{\mathcal{Q}_1}.$$

The chapter is organized as follows. Section 4.2 discusses some useful and known facts about reproducing kernel Hilbert spaces. Section 4.3 presents the commutant lifting theorem. Section 4.4 is devoted to factorizations of multipliers. The factorization results obtained here may be of independent interest. Section 4.5 provides the interpolation theorem.

4.2 Preliminaries

The Drury-Arveson space over the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ will be denoted by H^2_n. Recall that H^2_n is a reproducing kernel Hilbert space corresponding to the kernel function

$$k_1(\boldsymbol{z},\boldsymbol{w}) = (1 - \sum_{i=1}^{n} z_i \bar{w}_i)^{-1} \qquad (z, w \in \mathbb{B}^n).$$

Let $k : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$ be a kernel such that k is analytic in the first variables $\{z_1, \ldots, z_n\}$. We say that k is *regular* if there exists a kernel $\tilde{k} : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$, analytic in $\{z_1, \ldots, z_n\}$, such that

$$k(z, w) = k_1(z, w)\tilde{k}(z, w) \qquad (z, w \in \mathbb{B}^n).$$

If k is a regular kernel, then $\mathcal{H}_k$, the reproducing kernel Hilbert space corresponding to the kernel k, will be referred as a *regular reproducing kernel Hilbert space.*

In the case of a regular reproducing kernel Hilbert space $\mathcal{H}_k$, it follows [66] that M_{z_i}, the multiplication operator by the coordinate function z_i, is bounded. Note that

$$(M_{z_i} f)(w) = w_i f(w),$$

for all $f \in \mathcal{H}_k$, $\boldsymbol{w} \in \mathbb{B}^n$ and $i = 1, \ldots, n$. Moreover, it also follows that the commuting tuple $(M_{z_1}, \ldots, M_{z_n})$ on $\mathcal{H}_k$ is a *row contraction*, that is

$$\sum_{i=1}^{n} M_{z_i} M_{z_i}^* \leq I_{\mathcal{H}_k}.$$

If $\mathcal{E}$ is a Hilbert space, then we also say that $\mathcal{H}_k \otimes \mathcal{E}$ is a regular reproducing kernel Hilbert space. Note that the kernel function of $\mathcal{H}_k \otimes \mathcal{E}$ is given by

$$\mathbb{B}^n \times \mathbb{B}^n \ni (\boldsymbol{z},\boldsymbol{w}) \mapsto k(\boldsymbol{z},\boldsymbol{w}) I_{\mathcal{E}}.$$

The $\mathcal{E}$-valued Drury-Arveson space, denoted by $H^2_n(\mathcal{E})$, is the reproducing kernel Hilbert space corresponding to the $\mathcal{B}(\mathcal{E})$-valued kernel function

$$\mathbb{B}^n \times \mathbb{B}^n \ni (\boldsymbol{z},\boldsymbol{w}) \mapsto k_1(\boldsymbol{z},\boldsymbol{w}) I_{\mathcal{E}}.$$

To simplify the notation, we often identify $H^2_n(\mathcal{E})$ with $H^2_n \otimes \mathcal{E}$ via the unitary map defined by $z^{\boldsymbol{k}}\eta \mapsto z^{\boldsymbol{k}} \otimes \eta$ for all $\boldsymbol{k} \in \mathbb{Z}^n_+$ and $\eta \in \mathcal{E}$. This also enable us to identify $(M_{z_1}, \ldots, M_{z_n})$ on $H^2_n(\mathcal{E})$ with $(M_{z_1} \otimes I_{\mathcal{E}}, \ldots, M_{z_n} \otimes I_{\mathcal{E}})$ on $H^2_n \otimes \mathcal{E}$.

Typical examples of regular reproducing kernel Hilbert spaces arise from weighted Bergman spaces over $\mathbb{B}^n$. More specifically, let $\lambda > 1$, and let

$$k_\lambda(\boldsymbol{z}, \boldsymbol{w}) = (1 - \sum_{i=1}^{n} z_i \bar{w}_i)^{-\lambda} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n). \tag{4.2.1}$$

Then $\mathcal{H}_{k_\lambda}$ is a regular reproducing kernel Hilbert space. Note that $\mathcal{H}_{k_\lambda}$ is the Hardy space, Bergman space and weighted Bergman space for $\lambda = n$, $n+1$ and $n+1+\alpha$ for any $\alpha > 0$, respectively.

Suppose $\mathcal{H}$ and $\mathcal{E}_*$ are Hilbert spaces and $(T_1, \ldots, T_n)$ is a commuting tuple of bounded linear operators on $\mathcal{H}$. We say that $(T_1, \ldots, T_n)$ on $\mathcal{H}$ *dilates* to $(M_{z_1} \otimes I_{\mathcal{E}_*}, \ldots, M_{z_n} \otimes I_{\mathcal{E}_*})$ on $H^2_n \otimes \mathcal{E}_*$ if there exists an isometry $\Pi : \mathcal{H} \to H^2_n \otimes \mathcal{E}_*$ such that

$$\Pi T_i^* = (M_{z_i} \otimes I_{\mathcal{E}_*})^* \Pi,$$

for all $i = 1, \ldots, n$ (cf. [92]). We often say that $\Pi : \mathcal{H} \to H^2_n \otimes \mathcal{E}_*$ is a *dilation* of $(T_1, \ldots, T_n)$.

If $\mathcal{H} = \mathcal{H}_k$ is a regular reproducing kernel Hilbert space, then by [Theorem 6.1, [66]], it follows that $(M_{z_1} \otimes I_{\mathcal{E}}, \ldots, M_{z_n} \otimes I_{\mathcal{E}})$ on $\mathcal{H}_k \otimes \mathcal{E}$ dilates to $(M_{z_1} \otimes I_{\mathcal{E}_*}, \ldots, M_{z_n} \otimes I_{\mathcal{E}_*})$ on $H^2_n \otimes \mathcal{E}_*$ for some Hilbert space $\mathcal{E}_*$. More specifically:

Theorem 4.2.1. *Let $\mathcal{E}$ be a Hilbert space. If $\mathcal{H}_k$ is a regular reproducing kernel Hilbert space, then there exist a Hilbert space $\mathcal{E}_*$ and an isometry*

$$\Pi_k : \mathcal{H}_k \otimes \mathcal{E} \to H^2_n \otimes \mathcal{E}_*,$$

such that

$$\Pi_k (M_{z_i} \otimes I_{\mathcal{E}})^* = (M_{z_i} \otimes I_{\mathcal{E}_*})^* \Pi_k,$$

for all $i = 1, \ldots, n$.

Since $(M_{z_1} \otimes I_{\mathcal{E}}, \ldots, M_{z_n} \otimes I_{\mathcal{E}})$ on $\mathcal{H}_k \otimes \mathcal{E}$ is a pure row contraction [66], the above result also directly follows from Muller-Vasilescu [76] and Arveson [13].

In what follows, given a Hilbert space $\mathcal{H}$ and a closed subspace $\mathcal{Q}$ of $\mathcal{H}$, we will denote by $i_{\mathcal{Q}}$ the inclusion map

$$i_{\mathcal{Q}} : \mathcal{Q} \hookrightarrow \mathcal{H}.$$

Note that $i_{\mathcal{Q}}$ is an isometry and

$$i_{\mathcal{Q}} i_{\mathcal{Q}}^* = P_{\mathcal{Q}}.$$

We now recall the commutant lifting theorem in the setting of the Drury-Arveson space (see [10] or Theorem 5.1, page 118, [19]). A closed subspace $\mathcal{Q}$ of a regular reproducing kernel Hilbert space $\mathcal{H}_k \otimes \mathcal{E}$ is said to be *shift co-invariant* if

$$(M_{z_i} \otimes I_{\mathcal{E}})^* \mathcal{Q} \subseteq \mathcal{Q} \qquad (i = 1, \ldots, n).$$

Theorem 4.2.2. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert spaces. Suppose $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are shift co-invariant subspaces of $H^2_n(\mathcal{E}_1)$ and $H^2_n(\mathcal{E}_2)$, respectively, $X \in \mathcal{B}(\mathcal{Q}_1, \mathcal{Q}_2)$ and let $\|X\| \leq 1$. If*

$$X(P_{\mathcal{Q}_1} M_{z_i}|_{\mathcal{Q}_1}) = (P_{\mathcal{Q}_2} M_{z_i}|_{\mathcal{Q}_2})X,$$

for all $i = 1, \ldots, n$, then there exists a multiplier $\Phi \in \mathcal{M}(H^2_n(\mathcal{E}_1), H^2_n(\mathcal{E}_2))$ such that $\|M_\Phi\| \leq 1$ and $P_{\mathcal{Q}_2} M_\Phi|_{\mathcal{Q}_1} = X$.

Recall also that, given regular reproducing kernel Hilbert spaces $\mathcal{H}_{k_1} \otimes \mathcal{E}_1$ and $\mathcal{H}_{k_2} \otimes \mathcal{E}_2$, a function $\Phi : \mathbb{B}^n \to \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ is called a *multiplier* from $\mathcal{H}_{k_1} \otimes \mathcal{E}_1$ to $\mathcal{H}_{k_2} \otimes \mathcal{E}_2$ if

$$\Phi(\mathcal{H}_{k_1} \otimes \mathcal{E}_1) \subseteq \mathcal{H}_{k_2} \otimes \mathcal{E}_2.$$

The *multiplier space* $\mathcal{M}(\mathcal{H}_{k_1} \otimes \mathcal{E}_1, \mathcal{H}_{k_2} \otimes \mathcal{E}_2)$ is the set of all multipliers from $\mathcal{H}_{k_1} \otimes \mathcal{E}_1$ to $\mathcal{H}_{k_2} \otimes \mathcal{E}_2$. In what follows, $\mathcal{M}_1(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$ will denote the closed ball of radius one:

$$\mathcal{M}_1(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2) = \{\Phi \in \mathcal{M}(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2) : \|M_\Phi\| \leq 1\}.$$

We have the following useful characterization of multipliers (cf. Proposition 4.2, [92]): Let $\mathcal{H}_k$ be a regular reproducing kernel Hilbert space, and let $X \in \mathcal{B}(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$. Then

$$X(M_{z_i} \otimes I_{\mathcal{E}_1}) = (M_{z_i} \otimes I_{\mathcal{E}_2})X,$$

if and only if $X = M_\Phi$ for some $\Phi \in \mathcal{M}(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$.

4.3 Commutant lifting theorem

We begin with a general result concerning intertwiner of bounded linear operators.

Lemma 4.3.1. *Suppose $\Pi : \mathcal{H} \to \mathcal{K}$ and $\hat{\Pi} : \hat{\mathcal{H}} \to \hat{\mathcal{K}}$ are isometries, $V \in \mathcal{B}(\mathcal{K})$, $\hat{V} \in \mathcal{B}(\hat{\mathcal{K}})$, $T = \Pi^* V \Pi$ and $\hat{T} = \hat{\Pi}^* \hat{V} \hat{\Pi}$. Moreover, let $X \in \mathcal{B}(\mathcal{H}, \hat{\mathcal{H}})$ satisfies*

$$XT = \hat{T}X.$$

If we define

$$\mathcal{Q} = \Pi\mathcal{H} \quad \textit{and} \quad \hat{\mathcal{Q}} = \hat{\Pi}\hat{\mathcal{H}},$$

and

$$\tilde{X} = \hat{\Pi} X \Pi^*|_{\mathcal{Q}},$$

then $\tilde{X} \in \mathcal{B}(\mathcal{Q}, \hat{\mathcal{Q}})$ and

$$\tilde{X}(P_{\mathcal{Q}} V|_{\mathcal{Q}}) = (P_{\hat{\mathcal{Q}}} \hat{V}|_{\hat{\mathcal{Q}}})\tilde{X}.$$

Proof. Notice that $P_{\mathcal{Q}} = \Pi\Pi^*$ and $P_{\hat{\mathcal{Q}}} = \hat{\Pi}\hat{\Pi}^*$. Hence

$$\tilde{X} = (\hat{\Pi}\hat{\Pi}^*)\hat{\Pi} X \Pi^*|_{\mathcal{Q}} = P_{\hat{\mathcal{Q}}}(\hat{\Pi} X \Pi^*)|_{\mathcal{Q}},$$

and in particular

$$(\hat{\Pi} X \Pi^*)\mathcal{Q} \subseteq \hat{\mathcal{Q}},$$

which shows that $\tilde{X} \in \mathcal{B}(\mathcal{Q}, \hat{\mathcal{Q}})$. Moreover

$$\begin{aligned}\tilde{X}(P_{\mathcal{Q}} V|_{\mathcal{Q}}) &= \hat{\Pi} X \Pi^* P_{\mathcal{Q}} V|_{\mathcal{Q}} = \hat{\Pi} X \Pi^* V|_{\mathcal{Q}} = \hat{\Pi} X T \Pi^*|_{\mathcal{Q}} \\ &= \hat{\Pi} \hat{T} X \Pi^*|_{\mathcal{Q}} = \hat{\Pi}\hat{\Pi}^* \hat{V} \hat{\Pi}(\hat{\Pi}^* \hat{\Pi}) X \Pi^*|_{\mathcal{Q}} \\ &= P_{\hat{\mathcal{Q}}} \hat{V}|_{\hat{\mathcal{Q}}} \hat{\Pi} X \Pi^*|_{\mathcal{Q}} = (P_{\hat{\mathcal{Q}}} \hat{V}|_{\hat{\mathcal{Q}}})\tilde{X}.\end{aligned}$$ □

Now we are ready to prove a variation, in terms of dilations, of Theorem 4.2.2.

Theorem 4.3.2. *Let $\mathcal{H}$ and $\hat{\mathcal{H}}$ be Hilbert spaces. Suppose $T = (T_1, \ldots, T_n)$ and $\hat{T} = (\hat{T}_1, \ldots, \hat{T}_n)$ are commuting tuples on $\mathcal{H}$ and $\hat{\mathcal{H}}$, respectively, $X \in \mathcal{B}(\mathcal{H}, \hat{\mathcal{H}})$, $\|X\| \leq 1$, and*

$$X T_i = \hat{T}_i X,$$

for all $i = 1, \ldots, n$. If $\Pi : \mathcal{H} \to H^2_n \otimes \mathcal{E}$ and $\hat{\Pi} : \hat{\mathcal{H}} \to H^2_n \otimes \hat{\mathcal{E}}$ are dilations of T and $\hat{T}$, respectively, then there exists a multiplier $\Phi \in \mathcal{M}_1(H^2_n \otimes \mathcal{E}, H^2_n \otimes \hat{\mathcal{E}})$ such that

$$X = \hat{\Pi}^* M_\Phi \Pi.$$

Proof. Let

$$\mathcal{Q} = \Pi \mathcal{H} \quad \text{and} \quad \hat{\mathcal{Q}} = \hat{\Pi}\hat{\mathcal{H}}.$$

If

$$\tilde{X} = \hat{\Pi} X \Pi^*|_{\mathcal{Q}},$$

then by Lemma 4.3.1, it follows that $\tilde{X} \in \mathcal{B}(\mathcal{Q}, \hat{\mathcal{Q}})$ and

$$\tilde{X}(P_{\mathcal{Q}}(M_{z_i} \otimes I_{\mathcal{E}})|_{\mathcal{Q}}) = (P_{\hat{\mathcal{Q}}}(M_{z_i} \otimes I_{\hat{\mathcal{E}}})|_{\hat{\mathcal{Q}}})\tilde{X},$$

for all $i = 1, \ldots, n$. It then follows from the commutant lifting theorem, Theorem 4.2.2, that

$$\tilde{X} = P_{\hat{\mathcal{Q}}} M_\Phi|_{\mathcal{Q}},$$

for some $\Phi \in \mathcal{M}(H^2_n \otimes \mathcal{E}, H^2_n \otimes \hat{\mathcal{E}})$ and $\|M_\Phi\| \leq 1$. Then

$$\hat{\Pi} X \Pi^*|_{\mathcal{Q}} = P_{\hat{\mathcal{Q}}} M_\Phi|_{\mathcal{Q}}.$$

It then follows from

$$\mathcal{Q} = \text{ran } \Pi = \text{ran } \Pi\Pi^*,$$

that

$$(\hat{\Pi} X \Pi^*)(\Pi\Pi^*) = P_{\hat{\mathcal{Q}}} M_\Phi (\Pi\Pi^*).$$

Thus

$$\hat{\Pi} X = P_{\hat{\mathcal{Q}}} M_\Phi \Pi = (\hat{\Pi}\hat{\Pi}^*) M_\Phi \Pi,$$

and hence $X = \hat{\Pi}^* M_\Phi \Pi$. $\square$

Now let $\mathcal{Q}$ be a shift co-invariant subspace of $\mathcal{H}_k \otimes \mathcal{E}$. An isometry $\Pi : \mathcal{Q} \to H^2_n \otimes \mathcal{E}_*$ is said to be a *dilation of* $\mathcal{Q}$ if

$$\Pi(P_{\mathcal{Q}}(M_{z_i} \otimes I_{\mathcal{E}})|_{\mathcal{Q}})^* = (M_{z_i} \otimes I_{\mathcal{E}_*})^* \Pi,$$

for all $i = 1, \dots, n$, that is $(P_{\mathcal{Q}} M_{z_1}|_{\mathcal{Q}}, \dots, P_{\mathcal{Q}} M_{z_n}|_{\mathcal{Q}})$ on $\mathcal{Q}$ dilates to $(M_{z_1} \otimes I_{\mathcal{E}_*}, \dots, M_{z_n} \otimes I_{\mathcal{E}_*})$ on $H^2_n \otimes \mathcal{E}_*$ via the isometry Π.

Lemma 4.3.3. *Let $\mathcal{H}_k$ be a regular reproducing kernel Hilbert space, and let $\mathcal{E}$ and $\mathcal{E}_*$ be a Hilbert spaces. Suppose $\mathcal{Q}$ is a shift co-invariant subspace of $\mathcal{H}_k \otimes \mathcal{E}$. If $\Pi : \mathcal{H}_k \otimes \mathcal{E} \to H^2_n \otimes \mathcal{E}_*$ is a dilation of $\mathcal{H}_k \otimes \mathcal{E}$, then $\Pi_{\mathcal{Q}} : \mathcal{Q} \to H^2_n \otimes \mathcal{E}_*$, defined by*

$$\Pi_{\mathcal{Q}} = \Pi \circ i_{\mathcal{Q}},$$

is a dilation $\mathcal{Q}$.

Proof. We first observe that

$$\Pi_{\mathcal{Q}}^* \Pi_{\mathcal{Q}} = i_{\mathcal{Q}}^* \Pi^* \Pi i_{\mathcal{Q}} = I_{\mathcal{Q}}.$$

Now we compute

$$\begin{aligned} \Pi_{\mathcal{Q}}(P_{\mathcal{Q}}(M_{z_i} \otimes I_{\mathcal{E}})|_{\mathcal{Q}})^* &= \Pi i_{\mathcal{Q}} P_{\mathcal{Q}} (M_{z_i} \otimes I_{\mathcal{E}})^*|_{\mathcal{Q}} = \Pi (M_{z_i} \otimes I_{\mathcal{E}})^*|_{\mathcal{Q}} \\ &= (M_{z_i} \otimes I_{\mathcal{E}_*})^* \Pi|_{\mathcal{Q}} = (M_{z_i} \otimes I_{\mathcal{E}_*})^* (\Pi i_{\mathcal{Q}}) i_{\mathcal{Q}}^*|_{\mathcal{Q}} \\ &= (M_{z_i} \otimes I_{\mathcal{E}_*})^* \Pi_{\mathcal{Q}} i_{\mathcal{Q}}^*|_{\mathcal{Q}}. \end{aligned}$$

Now

$$i_{\mathcal{Q}}^*|_{\mathcal{Q}} = I_{\mathcal{Q}},$$

and so

$$\Pi_{\mathcal{Q}}(P_{\mathcal{Q}}(M_{z_i} \otimes I_{\mathcal{E}})|_{\mathcal{Q}})^* = (M_{z_i} \otimes I_{\mathcal{E}_*})^* \Pi_{\mathcal{Q}},$$

for all $i = 1, \dots, n$. $\square$

We are now ready to present and prove the commutant lifting theorem.

Theorem 4.3.4. *Let $\mathcal{H}_k$ be a regular reproducing kernel Hilbert space, $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert spaces, and let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be shift co-invariant subspaces of $H^2_n \otimes \mathcal{E}_1$ and $\mathcal{H}_k \otimes \mathcal{E}_2$, respectively. Let $X \in \mathcal{B}(\mathcal{Q}_1, \mathcal{Q}_2)$, and suppose that $\|X\| \leq 1$ and*

$$X(P_{\mathcal{Q}_1}(M_{z_i} \otimes I_{\mathcal{E}_1})|_{\mathcal{Q}_1}) = (P_{\mathcal{Q}_2}(M_{z_i} \otimes I_{\mathcal{E}_2})|_{\mathcal{Q}_2}) X,$$

for all $i = 1, \dots, n$. Then there exists a multiplier $\Phi \in \mathcal{M}_1(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$ such that

$$X = P_{\mathcal{Q}_2} M_\Phi|_{\mathcal{Q}_1}.$$

Proof. Observe that the inclusion map $i_{\mathcal{Q}_1} : \mathcal{Q}_1 \hookrightarrow H^2_n \otimes \mathcal{E}_1$ is a dilation of $\mathcal{Q}_1$. Let $\Pi_k : \mathcal{H}_k \otimes \mathcal{E}_2 \to H^2_n \otimes \hat{\mathcal{E}}$ be a dilation of $\mathcal{H}_k$ (see Theorem 4.2.1), that is, Π_k is an isometry and

$$\Pi_k (M_{z_i} \otimes I_{\mathcal{E}_2})^* = (M_{z_i} \otimes I_{\hat{\mathcal{E}}})^* \Pi_k, \tag{4.3.1}$$

for all $i = 1, \dots, n$ and some Hilbert space $\hat{\mathcal{E}}$. Set

$$\Pi_{\mathcal{Q}_2} = \Pi_k i_{\mathcal{Q}_2}.$$

By Lemma 4.3.3, it follows that $\Pi_{\mathcal{Q}_2} : \mathcal{Q}_2 \to H^2_n \otimes \hat{\mathcal{E}}$ is a dilation of $\mathcal{Q}_2$. Then Theorem 4.3.2 yields

$$X = \Pi^*_{\mathcal{Q}_2} M_{\Phi_1} i_{\mathcal{Q}_1},$$

for some multiplier $\Phi_1 \in \mathcal{M}(H^2_n \otimes \mathcal{E}_1, H^2_n \otimes \hat{\mathcal{E}})$. Hence

$$X = i^*_{\mathcal{Q}_2} (\Pi^*_k M_{\Phi_1}) i_{\mathcal{Q}_1}.$$

Since

$$M_{\Phi_1} (M_{z_i} \otimes I_{\mathcal{E}_1}) = (M_{z_i} \otimes I_{\hat{\mathcal{E}}}) M_{\Phi_1},$$

we have, using also the adjoint of (4.3.1),

$$\Pi^*_k M_{\Phi_1} (M_{z_i} \otimes I_{\mathcal{E}_1}) = \Pi^*_k (M_{z_i} \otimes I_{\hat{\mathcal{E}}}) M_{\Phi_1} = (M_{z_i} \otimes I_{\mathcal{E}_2}) \Pi^*_k M_{\Phi_1},$$

for all $i = 1, \dots, n$, that is, $\Pi^*_k M_{\Phi_1} : H^2_n \otimes \mathcal{E}_1 \to \mathcal{H}_k \otimes \mathcal{E}_2$ intertwines the shifts. Consequently

$$\Pi^*_k M_{\Phi_1} = M_\Phi,$$

for some multiplier $\Phi \in \mathcal{M}(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$. Hence

$$X = i^*_{\mathcal{Q}_2} M_\Phi i_{\mathcal{Q}_1},$$

and thus

$$i_{\mathcal{Q}_2} X = P_{\mathcal{Q}_2} M_\Phi i_{\mathcal{Q}_1}.$$

Hence, we have

$$X = P_{\mathcal{Q}_2} M_\Phi |_{\mathcal{Q}_1}.$$

Finally

$$\|M_\Phi\| \le \|M_{\Phi_1}\| \le 1. \qquad \square$$

A simpler way of presenting the above theorem, from Hilbert module point of view, is to say that the following diagram commutes:

$$
\begin{tikzcd}
& & H^2_n \otimes \hat{\mathcal{E}} \\
H^2_n \otimes \mathcal{E}_1 \arrow[r, dotted, "M_\Phi"'] \arrow[urr, dotted, bend left, "M_{\Phi_1}"] \arrow[d, "P_{\mathcal{Q}_1}"'] & \mathcal{H}_k \otimes \mathcal{E}_2 \arrow[ur, "\Pi_k"] \arrow[d, "P_{\mathcal{Q}_2}"'] & \\
\mathcal{Q}_1 \arrow[r, "X"'] & \mathcal{Q}_2 \arrow[uur, bend right, "\Pi_{\mathcal{Q}_2}"'] &
\end{tikzcd}
$$

4.4 Factorizations

Let k be a regular reproducing kernel on $\mathbb{B}^n$. Then there exists a positive definite kernel $\tilde{k} : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$ such that

$$k(\boldsymbol{z}, \boldsymbol{w}) = k_1(\boldsymbol{z}, \boldsymbol{w})\tilde{k}(\boldsymbol{z}, \boldsymbol{w}) \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

Let $\mathcal{H}_{\tilde{k}}$ be the reproducing kernel Hilbert space corresponding to the kernel $\tilde{k}$. Suppose $\boldsymbol{w} \in \mathbb{B}^n$ and $ev(\boldsymbol{w}) : \mathcal{H}_{\tilde{k}} \to \mathbb{C}$ is the evaluation map, that is

$$ev(\boldsymbol{w})(f) = f(\boldsymbol{w}) \qquad (f \in \mathcal{H}_{\tilde{k}}).$$

Then

$$\tilde{k}(\boldsymbol{z}, \boldsymbol{w}) = ev(\boldsymbol{z})ev(\boldsymbol{w})^* \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n),$$

and so

$$k(\boldsymbol{z}, \boldsymbol{w}) = k_1(\boldsymbol{z}, \boldsymbol{w})\Big(ev(\boldsymbol{z})ev(\boldsymbol{w})^*\Big) \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n). \tag{4.4.1}$$

From Corollary 4.2 in [66] it follows that the map

$$(\pi F)(\boldsymbol{z}) := F(\boldsymbol{z}, \boldsymbol{z}),$$

for all $F \in H^2_n \otimes \mathcal{H}_{\tilde{k}}$ and $\boldsymbol{z} \in \mathbb{B}^n$, defines a coisometry from $H^2_n \otimes \mathcal{H}_{\tilde{k}} = \mathcal{H}_{k_1} \otimes \mathcal{H}_{\tilde{k}}$ to $\mathcal{H}_k = \mathcal{H}_{k_1\tilde{k}}$. If we view $H^2_n \otimes \mathcal{H}_{\tilde{k}}$ as a reproducing kernel Hilbert space of functions with values in $\mathcal{H}_{\tilde{k}}$, then the map π is actually the multiplier M_{ev}; indeed, if we compute the action on reproducing kernels, we have

$$M_{ev}(f \otimes g)(\boldsymbol{w}) = f(\boldsymbol{w}) \otimes ev(\boldsymbol{w})(g) = f(\boldsymbol{w}) \otimes g(\boldsymbol{w}) = \pi(f \otimes g)(\boldsymbol{w}).$$

This formula may be extended by tensorizing with $I_{\mathcal{E}}$, where $\mathcal{E}$ is a Hilbert space. If we define $\Psi_k : \mathcal{H}_{\tilde{k}} \otimes \mathcal{E} \to \mathcal{E}$ by $\Psi_k := ev \otimes I_{\mathcal{E}}$, then Ψ_k is obviously also a coisometric multiplier. Taking into account (4.4.1), we obtain the following theorem (see also [29, Theorem 4.1] and [66, Theorem 6.2]):

Theorem 4.4.1. *Let $k : \mathbb{B}^n \times \mathbb{B}^n \to \mathbb{C}$ be a regular kernel, and let*

$$k(\boldsymbol{z}, \boldsymbol{w}) = k_1(\boldsymbol{z}, \boldsymbol{w})\tilde{k}(\boldsymbol{z}, \boldsymbol{w}) \qquad (z, \boldsymbol{w} \in \mathbb{B}^n),$$

for some kernel $\tilde{k}$ on $\mathbb{B}^n$. Suppose $\mathcal{H}_{\tilde{k}}$ is the reproducing kernel Hilbert space corresponding to the kernel $\tilde{k}$. If $\mathcal{E}$ is a Hilbert space, then there exists a co-isometric multiplier $\Psi_k \in \mathcal{M}(H^2_n \otimes (\mathcal{H}_{\tilde{k}} \otimes \mathcal{E}), \mathcal{H}_k \otimes \mathcal{E})$ such that

$$k(\boldsymbol{z}, \boldsymbol{w}) I_{\mathcal{E}} = \frac{\Psi_k(\boldsymbol{z})\Psi_k(\boldsymbol{w})^*}{1 - \langle \boldsymbol{z}, \boldsymbol{w} \rangle} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

It is worth noting that except the explicit identification of the state space $\mathcal{H}_{\tilde{k}}$ and the fact that $\Psi_k \in \mathcal{M}(H^2_n \otimes (\mathcal{H}_{\tilde{k}} \otimes \mathcal{E}), \mathcal{H}_k \otimes \mathcal{E})$, Theorem 4.4.1 essentially follows from the Kolmogorov decomposition of a positive definite kernel.

It is instructive to consider, in particular, the familiar case: weighted Bergman spaces over $\mathbb{B}^n$. Let $m > 1$ be an integer and let

$$k_m(\boldsymbol{z}, \boldsymbol{w}) = (1 - \sum_{i=1}^{n} z_i \bar{w}_i)^{-m} \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n).$$

Then

$$\tilde{k}_m(\boldsymbol{z}, \boldsymbol{w}) = k_{m-1}(\boldsymbol{z}, \boldsymbol{w}),$$

and hence $\Psi_{k_m}(\boldsymbol{w})^* : \mathcal{E} \to \mathcal{H}_{k_{m-1}} \otimes \mathcal{E}$ is given by

$$\Psi_{k_m}(\boldsymbol{w})^* \eta = k_{m-1}(\cdot, \boldsymbol{w}) \otimes \eta,$$

for all $\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n$ and $\eta \in \mathcal{E}$. Note also that

$$\langle \Psi_{k_m}(\boldsymbol{w})(f \otimes \eta), \zeta \rangle = f(\boldsymbol{w}) \langle \eta, \zeta \rangle,$$

for all $f \in \mathcal{H}_{k_{m-1}}$, $\eta, \zeta \in \mathcal{E}$ and $\boldsymbol{w} \in \mathbb{B}^n$.

For this particular case, the representation of Ψ_{k_m} has been computed explicitly in [24, Section 4] and [16].

Now suppose $\mathcal{E}_1$ and $\mathcal{E}_2$ are Hilbert spaces, and k is a regular kernel on $\mathbb{B}^n$. Let $\Theta : \mathbb{B}^n \to \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ be an analytic function. From [84, Theorem 6.28] it follows that $\Theta \in \mathcal{M}_1(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$ if and only if

$$k(\boldsymbol{z}, \boldsymbol{w}) - k_1(\boldsymbol{z}, \boldsymbol{w})\Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^*$$

is a positive definite kernel. By virtue of Theorem 4.4.1, this is equivalent to positive definiteness of the kernel

$$(\boldsymbol{z}, \boldsymbol{w}) \mapsto k_1(\boldsymbol{z}, \boldsymbol{w})(\Psi_k(\boldsymbol{z})\Psi_k(\boldsymbol{w})^* - \Theta(\boldsymbol{z})\Theta(\boldsymbol{w})^*).$$

We may then apply [7, Theorem 8.57($(i) \Rightarrow (ii)$)] to obtain the following theorem.

Theorem 4.4.2. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert spaces, and let $\Theta : \mathbb{B}^n \to \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$ be an analytic function. In the setting of Theorem 4.4.1, the following conditions are equivalent:*

(i) $\Theta \in \mathcal{M}_1(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$,

(ii) there exists $\tilde{\Theta} \in \mathcal{M}_1(H^2_n \otimes \mathcal{E}_1, H^2_n \otimes (\mathcal{H}_{\tilde{k}} \otimes \mathcal{E}_2))$ *such that*

$$M_\Theta = M_{\Psi_k} M_{\tilde{\Theta}}.$$

More specifically, the multiplier Ψ_k makes the following diagram commutative:

$$\begin{tikzcd}
 & H^2_n \otimes (\mathcal{H}_{\tilde{k}} \otimes \mathcal{E}_2) \arrow[d, "M_{\Psi_k}"] \\
H^2_n \otimes \mathcal{E}_1 \arrow[ur, dashed, "M_{\tilde{\Theta}}"] \arrow[r, "M_\Theta"] & \mathcal{H}_k \otimes \mathcal{E}_2
\end{tikzcd}$$

The above factorization theorem, in the scalar-valued multiplier case, is due to Aleman, Hartz, McCarthy and Richter (see Proposition 4.10 in [9]). The proof relies solely on Leech's theorem. One should also compare Theorems 4.4.1 and 4.4.2 with Lemma 4.1 and Theorem 4.2 in [24] and Theorem 2.1 in [16].

4.5 Nevanlinna-Pick interpolation

We now turn to the interpolation problem. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert spaces. We denote by $\mathcal{B}_1(\mathcal{E}_1, \mathcal{E}_2)$ the open unit ball of $\mathcal{B}(\mathcal{E}_1, \mathcal{E}_2)$, that is

$$\mathcal{B}_1(\mathcal{E}_1, \mathcal{E}_2) = \{A \in \mathcal{B}(\mathcal{E}_1, \mathcal{E}_2) : \|A\| < 1\}.$$

We aim to solve the following version of Pick-type interpolation problem: Suppose $\{\boldsymbol{z}_i\}_{i=1}^m \subseteq \mathbb{B}^n$, $\{W_i\}_{i=1}^m \subseteq \mathcal{B}_1(\mathcal{E}_1, \mathcal{E}_2)$ and $m \geq 1$. Find necessary and sufficient conditions (on $\{\boldsymbol{z}_i\}_{i=1}^m$ and $\{W_i\}_{i=1}^m$) for the existence of a multiplier $\Phi \in \mathcal{M}_1(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$ such that

$$\Phi(\boldsymbol{z}_i) = W_i, \tag{4.5.1}$$

for all $i = 1, \ldots, m$.

Given such data $\{\boldsymbol{z}_i\}_{i=1}^m \subseteq \mathbb{B}^n$, $\{W_i\}_{i=1}^m \subseteq \mathcal{B}_1(\mathcal{E}_1, \mathcal{E}_2)$, set

$$\mathcal{Q}_1 = \{\sum_{i=1}^m k_1(\cdot, \boldsymbol{z}_i)\zeta_i : \zeta_i \in \mathcal{E}_1\},$$

and

$$\mathcal{Q}_2 = \{\sum_{i=1}^m k(\cdot, \boldsymbol{z}_i)\eta_i : \eta_i \in \mathcal{E}_2\}.$$

Obviously $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are shift co-invariant subspaces of $H^2_n \otimes \mathcal{E}_1$ and $\mathcal{H}_k \otimes \mathcal{E}_2$, respectively. Define $X : \mathcal{Q}_2 \to \mathcal{Q}_1$ by

$$Xk(\cdot, \boldsymbol{z}_i)\eta = k_1(\cdot, \boldsymbol{z}_i)(W_i^* \eta),$$

for all $i = 1, \ldots, m$ and $\eta \in \mathcal{E}_2$. Then

$$X(M_{z_i} \otimes I_{\mathcal{E}_2})^*|_{\mathcal{Q}_2} = (M_{z_i} \otimes I_{\mathcal{E}_1})^*|_{\mathcal{Q}_1} X,$$

for all $i = 1, \ldots, n$. Then, by Theorem 4.3.4, X is a contraction if and only if there exists $\Phi \in \mathcal{M}_1(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$ such that

$$P_{\mathcal{Q}_2} M_\Phi|_{\mathcal{Q}_1} = X^*.$$

In particular

$$\begin{aligned} k_1(\cdot, \boldsymbol{z}_i)(W_i^* \eta) &= X(k(\cdot, \boldsymbol{z}_i)\eta) \\ &= M_\Phi^*(k(\cdot, \boldsymbol{z}_i)\eta) \\ &= k_1(\cdot, \boldsymbol{z}_i)(\Phi(\boldsymbol{z}_i)^* \eta), \end{aligned}$$

for all $\eta \in \mathcal{E}_2$ and $i = 1, \ldots, m$, and so Φ satisfies (4.5.1). Conversely, if Φ satisfies (4.5.1), then it is easy to see that X defines a contraction from $\mathcal{Q}_2$ to $\mathcal{Q}_1$.

Now X is a contraction if and only if

$$\begin{aligned} & 0 \leq \langle (I - X^*X) \sum_{i=1}^m k(\cdot, \boldsymbol{z}_i)\eta_i, \sum_{i=1}^m k(\cdot, \boldsymbol{z}_i)\eta_i \rangle \\ & \Rightarrow \sum_{1 \leq i,j \leq m} \langle k(\boldsymbol{z}_i, \boldsymbol{z}_j)\eta_j, \eta_i \rangle - \sum_{1 \leq i,j \leq m} \langle W_i k_1(\boldsymbol{z}_i, \boldsymbol{z}_j) W_j^* \eta_j, \eta_i \rangle \geq 0 \\ & \Rightarrow \sum_{1 \leq i,j \leq m} \langle \Big(k(\boldsymbol{z}_i, \boldsymbol{z}_j) I_{\mathcal{E}_2} - \frac{W_i W_j^*}{1 - \langle \boldsymbol{z}_i, \boldsymbol{z}_j \rangle} \Big) \eta_j, \eta_i \rangle \geq 0, \end{aligned}$$

for all $\eta_1, \ldots, \eta_m \in \mathcal{E}_2$, where the last equality follows from Theorem 4.4.1.

On the other hand, Theorem 4.4.2 says that $\Phi \in \mathcal{M}_1(H^2_n \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$ if and only if there exists $\tilde{\Phi} \in \mathcal{M}_1(H^2_n \otimes \mathcal{E}_1, H^2_n \otimes (\mathcal{H}_{\tilde{k}} \otimes \mathcal{E}_2))$ such that

$$\Phi(\boldsymbol{z}) = \Psi_k(\boldsymbol{z}) \tilde{\Phi}(\boldsymbol{z}),$$

for all $\boldsymbol{z} \in \mathbb{B}^n$. Summarizing, we have established the following interpolation theorem:

Theorem 4.5.1. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be Hilbert spaces, k be a regular kernel on $\mathbb{B}^n$, and let*

$$k(\boldsymbol{z}, \boldsymbol{w}) = k_1(\boldsymbol{z}, \boldsymbol{w}) \tilde{k}(\boldsymbol{z}, \boldsymbol{w}) \qquad (\boldsymbol{z}, \boldsymbol{w} \in \mathbb{B}^n),$$

for some kernel $\tilde{k}$ on $\mathbb{B}^n$. Suppose $\{\boldsymbol{z}_i\}_{i=1}^m \subseteq \mathbb{B}^n$ and $\{W_i\}_{i=1}^m \subseteq \mathcal{B}_1(\mathcal{E}_1, \mathcal{E}_2)$. Then the following conditions are equivalent:

(i) There exists a multiplier $\Phi \in \mathcal{M}_1(H_n^2 \otimes \mathcal{E}_1, \mathcal{H}_k \otimes \mathcal{E}_2)$ *such that* $\Phi(\boldsymbol{z}_i) = W_i$ *for all* $i = 1, \ldots, m$.

(ii) $\sum\limits_{1\leq i,j\leq m} \langle \Big(k(\boldsymbol{z}_i, \boldsymbol{z}_j) I_{\mathcal{E}_2} - \dfrac{W_i W_j^*}{1 - \langle \boldsymbol{z}_i, \boldsymbol{z}_j \rangle} \Big) \eta_j, \eta_i \rangle$ *for all* $\eta_1, \ldots, \eta_m \in \mathcal{E}_2$.

(iii) There exists a multiplier $\tilde{\Phi} \in \mathcal{M}_1(H_n^2 \otimes \mathcal{E}_1, H_n^2 \otimes (\mathcal{H}_{\tilde{k}} \otimes \mathcal{E}_2))$ *such that*

$$\Psi_k(\boldsymbol{z}_i)\tilde{\Phi}(\boldsymbol{z}_i) = W_i \qquad (i = 1, \ldots, n).$$

As we pointed out before, in the case of scalar-valued multipliers (that is, $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{C}$), the equivalence of (i) and (ii) in Theorem 4.5.1 is due to Aleman, Hartz, McCarthy and Richter (see Proposition 4.4 in [9]). Moreover, if $n = 1$ and $\tilde{k}(z, w) = (1 - z\bar{w})^{-m}$, $m \in \mathbb{N}$ (that is, weighted Bergman space over $\mathbb{D}$ with an integer weight), then the equivalence of (i) and (ii) in Theorem 4.5.1 was proved by Ball and Bolotnikov [16].

Note that, the positivity condition in part (ii) of Theorem 4.5.1 does not hold in general:

Example: Consider the regular kernel k as the Bergman kernel on $\mathbb{D}$, that is

$$k(z, w) = \frac{1}{(1 - z\bar{w})^2} \qquad (z, w \in \mathbb{D}).$$

Here

$$k(z, w) = \tilde{k}(z, w) = \Psi_k(z)\Psi_k^*(w) = \frac{1}{(1 - z\bar{w})} \qquad (z, w \in \mathbb{D}).$$

Then, for a given pair of points $\{w_1, w_2\} \subseteq \mathbb{D}$, condition (ii) in Theorem 4.5.1 holds for some pair $\{z_1, z_2\} \subseteq \mathbb{D}$ if and only if

$$\begin{bmatrix} \frac{1}{1-|z_1|^2} - |w_1|^2 & \frac{1}{1-z_1\bar{z}_2} - w_1\bar{w}_2 \\ \frac{1}{1-z_2\bar{z}_1} - w_2\bar{w}_1 & \frac{1}{1-|z_2|^2} - |w_2|^2 \end{bmatrix} \diamond \begin{bmatrix} \frac{1}{1-|z_1|^2} & \frac{1}{1-z_1\bar{z}_2} \\ \frac{1}{1-z_2\bar{z}_1} & \frac{1}{1-|z_2|^2} \end{bmatrix} \geq 0,$$

where '$\diamond$' denotes the Schur product of matrices. However, if $z_1 = w_2 = 0$ and $z_2 \neq 0$, then it is easy to see that the positivity condition fails to hold for any $w_1 \in \mathbb{D}$ such that

$$\frac{1 - |w_1|^2}{1 - |z_2|^2} < 1.$$

Bibliography

[1] M. Abrahamse, *The Pick interpolation theorem for finitely connected domains*, Michigan Math. J. 26 (1979), 195–203.

[2] V. Adamyan, D. Arov and M. Krein, *Analytic properties of the Schmidt pairs of a Hankel operator and the generalized Schur-Takagi problem*, Mat. Sb. 86 (1971), 34–75.

[3] V. Adamyan, D. Arov and M. Krein, *Infinite Hankel block matrices and related problems of extension*, Izv. Akad. Nauk Armjan. SSR Ser. Mat. 6 (1971), 87–112.

[4] V. Adamyan, D. Arov and M. Krein, *Infinite Hankel matrices and generalized problems of Carathéodory-Fejér and F. Riesz*, Funkcional. Anal. i Priložen. 2 (1968), 1–19.

[5] J. Agler, *On the representation of certain holomorphic functions defined on a polydisk*, in Topics in Operator Theory: Ernst D. Hellinger Memorial Volume (L. de Branges, I. Gohberg, and J. Rovnyak, Eds.), Operator Theory and Applications, Vol. 48, pp. 47–66, Birkhäuser-Verlag, Basel, 1990.

[6] J. Agler, *Interpolation*, unpublished manuscript (1988).

[7] J. Agler, J. McCarthy, *Pick Interpolation and Hilbert Function Spaces*, AMS, 2002.

[8] J. Agler and J. McCarthy, *Nevanlinna–Pick interpolation on the bidisk*, J. Reine Angew. Math. 506 (1999), 191–204.

[9] A. Aleman, M. Hartz, J. McCarthy and S. Richter, *Interpolating sequences in spaces with the complete Pick property*, Int. Math. Res. Not. IMRN. 2019, 3832—3854.

[10] C. Ambrozie and D. Timotin, *On an intertwining lifting theorem for certain reproducing kernel Hilbert spaces*, Integral Equations Operator Theory. 42 (2002), 373–384.

[11] T. Ando, *On a pair of commutative contractions*, Acta Sci. Math. (Szeged). 24 (1963), 88–90.

[12] R. Arocena, *Unitary extensions of isometries and contractive intertwining dilations*, The Gohberg anniversary collection, Vol. II (Calgary, AB, 1988), 13–23, Oper. Theory Adv. Appl., 41, Birkhäuser, Basel, 1989.

[13] W. Arveson, *Subalgebras of C^*-algebras. III. Multivariable operator theory*, Acta Math. 181 (1998), 159–228.

[14] S. Axler, *Toeplitz operators*, A glimpse at Hilbert space operators, 125–133, Oper. Theory Adv. Appl., 207, Birkhäuser Verlag, Basel, 2010.

[15] N. Bala, K. Dhara, J. Sarkar and A. Sensarma, *A Bishop-Phelps-Bollobás theorem for bounded analytic functions*, J. Funct. Anal. 284 (2023), no. 6, Paper No. 109834, 20 pp.

[16] J. Ball and V. Bolotnikov, *Contractive multipliers from Hardy space to weighted Hardy space*, Proc. Amer. Math. Soc. 145 (2017), 2411–2425.

[17] J. Ball and V. Bolotnikov, *Interpolation in the noncommutative Schur-Agler class*, J. Operator Theory. 58 (2007), 83–126.

[18] J. Ball and T. Trent, *Unitary colligations, reproducing kernel Hilbert spaces, and Nevanlinna-Pick interpolation in several variables*, J. Funct. Anal. 157 (1998), 1–61.

[19] J. Ball, T. Trent and V. Vinnikov, *Interpolation and commutant lifting for multipliers on reproducing kernel Hilbert spaces*, Operator theory and analysis, Operator Theory: Advances and Applications 122 (eds H. Bart, I. Gohberg and A. Ran; Birkhäuser, Basel, 2001) 89–138.

[20] J. Ball, W.S. Li, D. Timotin and T. Trent, *A commutant lifting theorem on the polydisc*, Indiana Univ. Math. J. 48 (1999), 653–675.

[21] S. Barik, B. Das, K. Haria and J. Sarkar, *Isometric dilations and von Neumann inequality for a class of tuples in the polydisc*, Trans. Amer. Math. Soc. 372 (2019), 1429–1450.

[22] H. Bercovici, C. Foiaş and A. Tannenbaum, *A spectral commutant lifting theorem*, Trans. Amer. Math. Soc. 325 (1991), 741–763.

[23] A. Beurling, *On two problems concerning linear transformations in Hilbert space*, Acta Math., 81 (1949), 239-255.

[24] M. Bhattacharjee, B. K. Das, J. Sarkar, *Hypercontractions and factorizations of multipliers in one and several variables*, Journal of Operator Theory, 88 (2022), 479-510.

[25] L. Bracci and L. Picasso, *Representations of semigroups of partial isometries*, Bull. Lond. Math. Soc. 39 (2007), 792–802.

[26] A. Brown and P. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. 213 (1963–64), 89–102.

[27] A. Brown and R. Douglas, *Partially isometric Toeplitz operators*, Proc. Amer. Math. Soc. (1965), 681–682.

[28] J. Bourgain, *New Banach space properties of the disc algebra and H^∞*, Acta Math. 152 (1984), 1–48.

[29] R. Clouâtre and M. Hartz, *Multiplier algebras of complete Nevanlinna-Pick spaces: dilations, boundary representations and hyperrigidity*, J. Funct. Anal. 274 (2018), 1690—1738.

[30] R. Coifman, R, Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. (2) 103 (1976), 611–635.

[31] B. Cole, K. Lewis and J. Wermer, *Pick conditions on a uniform algebra and von Neumann inequalities*, J. Funct. Anal. 107 (1992), 235–254.

[32] B. Cole and J. Wermer, *Ando's theorem and sums of squares*, Indiana Univ. Math. J. 48 (1999), 767–791.

[33] B. Cole and J. Wermer, *Interpolation in the bidisk*, J. Funct. Anal. 140 (1996), 194–217.

[34] B. Cole and J. Wermer, *Pick interpolation, von Neumann inequalities, and hyperconvex sets*, Complex potential theory (Montreal, PQ, 1993), 89–129, NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci., 439, Kluwer Acad. Publ. Dordrecht, 1994.

[35] M. Cotlar and C. Sadosky, *Nehari and Nevanlinna-Pick problems and holomorphic extensions in the polydisk in terms of restricted BMO*, J. Funct. Anal. 124 (1994), 205–210.

[36] M. Cotlar and C. Sadosky, *Transference of metrics induced by unitary couplings, a Sarason theorem for the bidimensional torus, and a Sz.-Nagy-Foias theorem for two pairs of dilations*, J. Funct. Anal. 111 (1993), 473–488.

[37] M. Cotlar and C. Sadosky, *Lifting properties, Nehari theorem and Paley lacunary inequality*, Rev. Mat. Iberoamericana 2 (1986), 55–71.

[38] K. Davidson, V. Paulsen, M. Raghupathi and D. Singh, *A constrained Nevanlinna-Pick interpolation problem*, Indiana Univ. Math. J. 58 (2009), 709–732.

[39] B. Das and J. Sarkar, *Ando dilations, von Neumann inequality, and distinguished varieties*, J. Funct. Anal. 272 (2017), 2114–2131.

[40] K. Davidson and T. Le, *Commutant lifting for commuting row contractions*, Bull. Lond. Math. Soc. 42 (2010), 506–516.

[41] Deepak K. D, D. Pradhan and J. Sarkar, *Partially isometric Toeplitz operators on the polydisc*, Bull. Lond. Math. Soc. 54 (2022), 1350–1362.

[42] X. Ding, *Products of Toeplitz operators on the polydisk*, Integr. Equ. Oper. Theory, 45 (2003), 389–403.

[43] X. Ding, S. Sun and D. Zheng, *Commuting Toeplitz operators on the bidisk*, J. Funct. Anal. 263 (2012), 3333–3357.

[44] M. Dritschel, S. Marcantognini and S. McCullough, *Interpolation in semigroupoid algebras*, J. Reine Angew. Math. 606 (2007), 1–40.

[45] R. Douglas, *Banach algebra techniques in operator theory*, Second edition. Graduate Texts in Mathematics, 179. Springer-Verlag, New York, 1998.

[46] R. Douglas, P. Muhly and C. Pearcy, *Lifting commuting operators*, Michigan Math. J. 15 (1968), 385–395.

[47] P. Duren and D. Williams, *Interpolation problems in function spaces*, J. Functional Analysis 9 (1972), 75–86.

[48] J. Eschmeier and M. Putinar, *Spherical contractions and interpolation problems on the unit ball*, J. Reine Angew. Math. 542 (2002), 219–236.

[49] J. Eschmeier, L. Patton and M. Putinar, *Carathéodory-Fejér interpolation on polydisks*, Math. Res. Lett. 7 (2000), 25–34.

[50] S. Ferguson and C. Sadosky, *Characterizations of bounded mean oscillation on the polydisk in terms of Hankel operators and Carleson measures*, J. Anal. Math. 81 (2000), 239–267.

[51] C. Foias, A. Frazho, I. Gohberg and M. Kaashoek, *Metric constrained interpolation, commutant lifting and systems*, Operator Theory: Advances and Applications, 100. Birkhäuser Verlag, Basel, 1998.

[52] N. Fowler and I. Raeburn, *The Toeplitz algebra of a Hilbert bimodule*, Indiana Univ. Math. J. 48 (1999), 155–181.

[53] C. Foias and A. Frazho, *The commutant lifting approach to interpolation problems*, Operator Theory: Advances and Applications, 44. Birkhäuser Verlag, Basel, 1990.

[54] J. Garnett, *Bounded analytic functions*, Revised first edition. Graduate Texts in Mathematics, 236. Springer, New York, 2007.

[55] A. Grinshpan, D. Kaliuzhnyi-Verbovetskyi, V. Vinnikov and H. Woerdeman, *Classes of tuples of commuting contractions satisfying the multivariable von Neumann inequality*, J. Funct. Anal. 256 (2009), 3035–3054.

[56] C. Gu, *Some algebraic properties of Toeplitz and Hankel operators on polydisk*, Arch. Math. (Basel), 80 (2003), 393–405.

[57] P. Halmos, *Normal dilations and extensions of operators*, Summa Brasil. Math. 2 (1950), 125–134.

[58] P. Halmos and J. McLaughlin, *Partial isometries*, Pacific J. Math. 13 (1963), 585–596.

[59] P. Halmos and L. Wallen, *Powers of partial isometries*, Indiana Univ. Math. J. 19 (1970), 657–663.

[60] R. Hamilton, *Pick interpolation in several variables*, Proc. Am. Math. Soc. 141 (2013), 2097–2103.

[61] J. W. Helton, *Operator theory, analytic functions, matrices and electrical engineering*, CBMS Regional Conference Series No. 68, AMS, Providence 1987.

[62] N. Higson and J. Roe, *Analytic K-homology*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[63] A. an Huef, I. Raeburn and I. Tolich, *Structure theorems for star-commuting power partial isometries*, Linear Algebra Appl. 481 (2015), 107–114.

[64] D. Kalyuzhnyi-Verbovetzkii, *Carathéodory interpolation on the non-commutative polydisk*, J. Funct. Anal. 229 (2005) 241–276.

[65] L. Kosiński, *Three-point Nevanlinna–Pick problem in the polydisc*, Proc. Lond. Math. Soc. 111 (2015), 887–910.

[66] R. Kumari, J. Sarkar, S. Sarkar and D. Timotin, *Factorization of kernels and reproducing kernel Hilbert spaces*, Integral Equations Operator Theory. 87 (2017), 225–244.

[67] Y. Lu and Y. Yang, *Invariant subspaces of of Hardy space on the polydisk*, Advances Math (China), 41(2012), 313-319.

[68] R. Martínez-Avendaño and P. Rosenthal, *An introduction to operators on the Hardy-Hilbert space*, Graduate Texts in Mathematics, 237. Springer, New York, 2007.

[69] B. Sz.-Nagy, *Sur les contractions de l'espace de Hilbert*, Acta Sci. Math. (Szeged) 15 (1953), 87–92.

[70] B.Sz.-Nagy and C. Foiaş, *Dilatation des commutants d'operateurs*, C.R. Acad. Sci. Paris, Serie A 266 (1968), 493–495.

[71] A. Maji, A. Mundayadan, J. Sarkar and T. R. Sankar, *Characterization of invariant subspaces in the polydisc*, J. Operator Theory, 82 (2019), 445–468.

[72] A. Maji, J. Sarkar and S. Sarkar, *Toeplitz and asymptotic Toeplitz operators on* $H^2(\mathbb{D}^n)$, Bull. Sci. Math. 146 (2018), 33–49.

[73] D. Marshall, *An elementary proof of the Pick-Nevanlinna interpolation theorem*, Michigan Math. J. 21 (1975), 219–223.

[74] S. McCullough, *Nevanlinna-Pick type interpolation in a dual algebra*, J. Funct. Anal. 135 (1996), 93–131.

[75] S. McCullough and V. Paulsen, *C^*-envelopes and interpolation theory*, Indiana Univ. Math. J. 51 (2002), 479–505.

[76] V. Muller and F.-H. Vasilescu, *Standard models for some commuting multioperators*, Proc. Amer. Math. Soc. 117 (1993), 979–989.

[77] R. Nevanlinna, *Über beschränkte Funktionen, die in gegebenen Punkten vorgeschriebene Werte annehmen*, Ann. Acad. Sci. Fenn. Sel A. 13 (1919), 1–72.

[78] N. Nikolski, *Operators, functions, and systems: an easy reading. Vol. 1. Hardy, Hankel, and Toeplitz*, Translated from the French by Andreas Hartmann. Mathematical Surveys and Monographs, 92. American Mathematical Society, Providence, RI, 2002.

[79] Z. Nehari, *On bounded bilinear forms*, Ann. of Math. (2) 65 (1957), 153–162.

[80] A. Nicolau, *The Nevanlinna-Pick interpolation problem*, Proceedings of the Summer School in Complex and Harmonic Analysis, and Related Topics, 1–28, Publ. Univ. East. Finl. Rep. Stud. For. Nat. Sci., 22, Univ. East. Finl., Fac. Sci. For., Joensuu, 2016.

[81] N. Nikolski, *Operators, functions, and systems: an easy reading. Vol. 1. Hardy, Hankel, and Toeplitz. Translated from the French by Andreas Hartmann*, Mathematical Surveys and Monographs, 92. American Mathematical Society, Providence, RI, 2002.

[82] N. Nikolski, *Treatise on the shift operator*, Grundlehren der mathematischen Wissenschafte, vol. 273, Springer-Verlag, Berlin, 1986.

[83] N. Nikolski and A. Volberg, *Tangential and approximate free interpolation*, Analysis and partial differential equations, 277–299, Lecture Notes in Pure and Appl. Math., 122, Dekker, New York, 1990.

[84] V.I. Paulsen, M. Raghupathi, *An introduction to the theory of reproducing kernel Hilbert spaces*. Cambridge Studies in Advanced Mathematics, 152. Cambridge University Press, Cambridge, 2016.

[85] G. Pick, *Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden*, Math. Ann. 77 (1916), 7–23.

[86] R. Rochberg, *Higher-order Hankel forms and commutators*, Holomorphic spaces (Berkeley, CA, 1995), 155–178, Math. Sci. Res. Inst. Publ. 33, Cambridge Univ. Press, Cambridge, 1998.

[87] W. Rudin, *Function Theory in Polydiscs*, W. A. Benjamin, Inc., New York–Amsterdam, 1969.

[88] D. Sarason, *Algebraic properties of truncated Toeplitz operators*, Oper. Matrices 1 (2007), 491–526.

[89] D. Sarason, *Nevanlinna-Pick interpolation with boundary data*, Dedicated to the memory of Mark Grigorievich Krein (1907–1989). Integral Equations Operator Theory 30 (1998), 231–250.

[90] D. Sarason, *New Hilbert spaces from old*, Paul Halmos Celebrating 50 Years of Mathematics, edited by J. H. Ewing and F. W. Gehring, Springer, New York, 1991, pp. 195–204.

[91] D. Sarason, *Generalized interpolation in H^∞*, Trans. Amer. Math. Soc. 127 (1967), 179–203.

[92] J. Sarkar, *An invariant subspace theorem and invariant subspaces of analytic reproducing kernel Hilbert spaces - II*, Complex Analysis and Operator Theory. 10 (2016), 769–782.

[93] J. Sarkar, *Submodules of the Hardy module over polydisc*, Isr. J. Math. 205 (2015), 317-336.

[94] I. Schur, *Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind I*, J. Reine Angew. Math. 147 (1917) 205–232.

[95] I. Schur, *Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind II*, J. Reine Angew. Math. 148 (1918) 122–145.

[96] K. Seip, *Interpolation and sampling in spaces of analytic functions*, University Lecture Series, 33. American Mathematical Society, Providence, RI, 2004.

[97] B. Shabat, *Introduction to complex analysis. Part II. Functions of several variables*, Translated from the third (1985) Russian edition by J. S. Joel. Translations of Mathematical Monographs, 110. American Mathematical Society, Providence, RI, 1992.

[98] S. Sultanic, *Commutant lifting theorem for the Bergman space*, Thesis (Ph.D.)–Washington University in St. Louis. 2005. 102 pp. ISBN: 978-0542-34513-5.

[99] S. Sun and D. Zheng, *Toeplitz operators on the polydisk*, Proc. Amer. Math. Soc. 124 (1996), 3351–3356.

[100] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North Holland, Amsterdam, 1970

[101] M. Tsuji, *On the boundary value of a bounded analytic function of several complex variables*, Proc. Japan Acad. 21 (1945), 308–312.

[102] H. Woerdeman, *Positive Carathéodory interpolation on the polydisc*, Integral Equations Operator Theory 42 (2002), no. 2, 229–242.

List of Publications

1. *Commutant lifting and Nevanlinna-Pick interpolation in several variables.* Deepak KD, Deepak pradhan, D Timotin, Jaydeb Sarkar (Integral Equations and Operator Theory, 92, Article number: 27 (2020)).

2. *Partially isometric Toeplitz operators on the polydisc* .Deepak KD, Deepak pradhan, Jaydeb Sarkar (Bulletin of the London Mathematical Society, Volume 54, (2022), 1350-1362).

3. *Commutant lifting, interpolation, and perturbations on the polydisc.* Deepak KD,Jaydeb Sarkar (arXiv).