1

Indian Statistical Institute
Mid Semestral Examination: (2006)
M. Math.–I Year
Measure Theory
Total marks=~~100~~ 80

Date: 21.08.06 Time: 3 hours

1. Let μ be a finite measure on a field F. Let λ be an extension of μ to $\sigma(F)$. Show that the extension is unique. You are NOT allowed to use the extension theorem. [10]

2. Consider a finite measure μ on $\sigma(F)$ where F is a field. Show that for every $\epsilon > 0$ and any set $A \in \sigma(F)$, there is a set $B \in F$ such that $\mu(A \Delta B) < \epsilon$. [10]

3. Consider the Borel sigma field B on R^2 (defined as the smallest sigma field containing all open sets.) Show that it is the smallest sigma field generated by the class of sets consisting of product of intervals. [20]

4. Suppose f is a nonnegative measurable function (that is, in the class L^+) on a finite measure space (Ω, A, μ). Define $\nu(E) = \int_E f d\mu$. Show that ν is a measure and for any $g \in L^+$, and for any $B \in A$, $\int_B g d\nu = \int_B f g d\mu$. [20]

5. Suppose F is a monotonically nondecreasing right continuous function. Let $F(\infty) = \lim_{x \to \infty} F(x)$ and $F(-\infty) = \lim_{x \to -\infty} F(x)$. For any $a < b$, define $\mu(a, b] = F(b) - F(a)$. Show that μ can be extended to a measure on the Borel sigma field. You may use Caratheodory's extension theorem. [20]

INDIAN STATISTICAL INSTITUTE
Mid-semester Examination : 2006-2007
M. Math. - First Year
Complex Analysis

Date : 24. 08. 2006   Maximum Score : 100   Time : 3 Hours

This paper carries questions worth a total of 112 marks. Answer as much as you can. The maximum you can score is 100 marks. Throughout this question paper, Ω stands for a region and γ a piecewise C^1-curve in $\mathbb{C}$.

(1) (a) Compute

$$\int_{|z|=2} \frac{dz}{z^2+1}.$$

(b) Let $f \in H(\Omega)$ such that $|f(z) - 1| < 1$. Compute

$$\int_\gamma \frac{f'(z)}{f(z)} dz,$$

γ a closed curve in Ω.

(c) Let $z_0 \in \Omega$ be a zero of $f \in H(\Omega)$ of order m. Show that $f^{1/m}$ has a continuous branch in a neighborhood of z_0.

(d) Let f and g be two entire maps such that $|f(z)| \leq |g(z)|$ for all z. Show that $f = \lambda \cdot g$ for some complex λ.

[5 + 8 + 8 + 8]

(2) (a) Let f and g be two linear fractional transformations from the open unit disc D onto D such that for some $|a| < 1$, $f(a) = 0 = g(a)$. Show that $f = \lambda \cdot g$ for some $|\lambda| = 1$.

(b) Show that any one-to-one conformal map from an open disc onto an open half plane is a linear fractional tranformation.

[15 + 12]

(3) (a) Let φ be a continuous map on γ. For $z \in \mathbb{C} \setminus \gamma$, define

$$\Phi(z) = \int_\gamma \frac{\varphi(\xi)}{\xi - z}.$$

Show that $\Phi(z)$ is holomorphic on $\mathbb{C} \setminus \gamma$.

(b) Let γ be closed and $a \in \mathbb{C} \setminus \gamma$. Show that

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} \in \mathbb{Z}.$$

[15 + 12]

P.T.O

(4) Let $f \in H(\Omega)$, $\overline{B_r(z_o)} \subset \Omega$ and

$$M = \sup_{|z-z_0|=r} |f(z)|.$$

(a) Show that

$$|f^{(n)}(z_0)| \leq \frac{n!M}{r^n}.$$

(b) Let $\{f_j\}$ be a sequence of holomorphic functions converging to f uniformly on each compact subsets of Ω. Show that for each $n \geq 1$, $\{f_j^{(n)}\}$ is uniformly Cauchy on each compact subsets of Ω.

(c) Show that for each $n \geq 1$, $f_j^{(n)} \to f^{(n)}$ uniformly on each compact subsets of Ω.

[5 + 12 + 12]

Indian Statistical Institute

Mid-semestral examination : (2006-2007)

M. Math I year

Algebra I

Date : 28.8.06 Maximum marks : 80 Duration : 3 hours.

Answer **ALL** questions. Marks are indicated in brackets.

(1) S and T are solvable subgroups of a group G and S is normal in G. Prove that ST is a solvable subgroup of G. [10]

(2) If p, q are prime numbers with $p < q$ and n is a positive integer, then prove that any group of order pq^n is solvable. [15]

(3) Let G be a nilpotent group, $o(G) = n$, and m is a positive integer which divides n. Prove that G has a subgroup of order m. [10]

(4) Prove that if an initial object exists in a category then it is unique upto isomorphism. [10]

(5) For two group K and Q with Q finite, we define the *wreath product* of K by Q, denoted by $K \diamond Q$, as follows. Let K^Q be the set of all functions $f : Q \to K$, and equip K^Q the group structure given by pointwise multiplication, i.e. $(f_1 \cdot f_2)(q) := f_1(q)f_2(q)$, where the multiplication on the right hand side is that of the group K. Q has a natural action $\theta : Q \to Aut(K^Q)$ given by

$$(\theta(q)f)(q') = f(q'q), \quad q, q' \in Q, \ f \in K^Q.$$

Define the wreath product of $K \diamond Q$ to be semidirect product of K^Q by Q with respect to the action θ. Now answer the questions below.

(a) Let G be a group which is an extension of K by Q, that is, G is a group containing K as a group such that there is a homomorphism π from G onto Q with $K = Ker\ \pi$. For each $q \in Q$, choose and fix an element $l(q) \in G$ such that $\pi(l(q)) = q$ and define $\pi : G \to K \diamond Q$ by

$$\phi(g) = (f_g, \pi(g)),$$

where $f_g \in K^Q$ denotes the function $f_g(q) = l(q)\ g\ (l(q\pi(g))^{-1}$. Prove that π is a one-to-one homomorphism, and hence $K \diamond Q$ contains a subgroup isomorphic with G.

P.T.O

(b) Using (a), prove the following :

Let $\mathcal{A}$ be a collection if finite groups satisfying (i) if G belongs to $\mathcal{A}$, then every subgroup o
G also belongs to $\mathcal{A}$, and (ii) if G, H are two groups in $\mathcal{A}$, H acting on G, then the semidire
product of G by H will also belong to $\mathcal{A}$. Prove that if $K, Q \in \mathcal{A}$, and G is any extension
K by Q, then there exists a group isomorphic with G in $\mathcal{A}$.

[20+10=30]

(6) Prove that $F(A \bigcup B) \cong F(A) * F(B)$ if A, B are disjoint sets, where $*$ denotes the fr
product and $F(X)$ denotes the free group generated by a set X. [15]

INDIAN STATISTICAL INSTITUTE

M. Math First Year

General Topology

31. 8. 06 Mid-Semester Examination Time : 2 hrs.

This paper carries 50 marks.The maximum you can score is 40. You may use any theorem proved in the class.

1. Let X be the real line. Let $\mathbf{B} = \{[\mathrm{a}, \mathrm{b}) : \mathrm{a} < \mathrm{b}\}$ Show that
   (a) $\mathbf{B}$ is the base for a topology $\mathbf{T}$ on X. [5]
   (b) Is $(X, \mathbf{T})$ connected? Justify your answer. [5]
   (c) Is $[0, 1]$ with the subspace topology compact? Justify your answer. [5]
2. Let (X, d) be a complete metric space with no finite open subset. Show that there is no countable open subset. [10]
3. Let (X, d) be a compact metric space and $f : X \to X$ an isometry. Show that f is onto X. [10]
4. Let X be a metrizable space. Show that X is compact if and only if every continuous real valued function on X is bounded. [15]

INDIAN STATISTICAL INSTITUTE

M. MATH FIRST YEAR

General Topology

Date: 23. 11. 06 Semestral-I Examination Time : 3 hrs.

This paper carries 70 marks.The maximum you can score is 60. You may use any theorem proved in the class.

1. Let X, Y be Hausdorff spaces and Y compact. Define p on $X \times Y$ onto X by $p(x, y) = x$. Show that p is a closed , continuous map. [10]

2. Let X, Y be Hausdorff spaces and f a closed continuous map from X onto Y. If X is normal, show that Y is normal. [10]

3. Let $p : E \to B$ be a covering map with B connected. Show that if $p^{-1}(b)$ has m elements for some b in B then $p^{-1}(x)$ has m elements for all x in B. [10]

4. Suppose $G \subset \mathbf{R}^3$ is a topological group . Further suppose G has a relatively open subset homeomorphic to the open disc B^2 . Show that G is a surface. [10]

5. Calculate the Fundamental Group of a Mobius Strip. [10]

6. (a) Let X, Y be topological spaces. Let $x_0 \in X, y_0 \in Y$. Define $(X, x_0) \vee (Y, y_0)$ and show it is homeomorphic to $\{x_0\} \times Y \cup X \times \{y_0\}$ [8]

(b) Let $X_i = S^1, x_i = (1,0) \in S^1$ for all i. Show $\vee(X_i, x_i)$ is not homeomorphic to $\bigcup_i \{x_1\} \times \cdots \times \{x_{i-1}\} \times X_i \times \{x_{i+1}\} \times \cdots \subset \Pi X_i$ [12]

INDIAN STATISTICAL INSTITUTE
Semestral Examination : 2006-2007
M. Math. - First Year
Complex Analysis

Date : 27. 11. 2006 Maximum Score : 100 Time : 4 Hours

This paper carries questions worth a total of 130 marks. Answer as much as you can. The maximum you can score is 100 marks.
Throughout this question paper, Ω stands for a region, H the open upper half plane and D the open unit disc in $\mathbb{C}$.
You are free to use any theorem from Topology and Complex Analysis proved or presented without proof in the class. However, you must state such a result at least once in the answer script because the evaluation will take that into account.

(1) (a) Let A be a discrete, relatively closed subset of Ω and $f \in H(\Omega \setminus A)$ such that $\operatorname{res}(f, z) = 0$ for each $z \in A$. Show that there is a $g \in H(\Omega \setminus A)$ whose derivative is f.

(b) Let $\{f_n\}$ be a sequence of injective holomorphic functions on Ω converging to f uniformly on compacta. Show that f is injective.

(c) Let $f_n : \Omega \to D$ be a sequence of holomorphic functions. Show that there is a holomorphic function $f : \Omega \to D$ such that a subsequence of $\{f_n\}$ converges to f uniformly on compacta.

(d) Let f be a complex-valued continuous map on D such that $f|D \cap H$ is holomorphic and $f(z) = \overline{f(\overline{z})}$ for all $|z| < 1$. Show that there is a $g \in H(D)$ whose derivative is f.

(e) Assume that $z \in \Omega \Rightarrow \overline{z} \in \Omega$ and f is a continuous map on $\Omega \cap \overline{H}$ which is holomorphic on $\Omega \cap H$ and real on the real axis. Show that there is a $g \in H(\Omega)$ that extends f.

[8 + 8 + 10 + 10 + 10]

(2) (a) Show that the germs $[\mathbb{C}, z]_1$ and $[\mathbb{C} \setminus \{0\}, 1/z]_1$ at 1 do not belong to a connected component of the sheaf of germs of holomorphic functions on $\mathbb{C}$.

(b) Let γ_0 and γ_1 be two paths in Ω from a to b such that $\gamma_0 \cong \gamma_1$ (rel $\{0,1\}$) and $f \in H(\Omega)$. Show that

$$\int_{\gamma_0} f = \int_{\gamma_1} f.$$

(c) Show that every harmonic function on Ω is a C^∞-map.

P.T.O

[8 + 14 + 10]

(3) (a) Let X be a compact Riemann surface and $p_1, \cdots, p_n \in X$. Show that the range of every non-constant, complex-valued function on $X \setminus \{p_1, \cdots, p_n\}$ is dense in $\mathbb{C}$.

(b) Let X, Y be Riemann surfaces with X compact. Show that every unbranched holomorphic function $p : X \to Y$ is a covering map.

(c) Show by an example that (b) need not be true if p were not unbranched.

[10 + 10 + 8]

(4) (a) Assume that Ω is simply connected, $a \in \Omega$ and f_a is a germ of holomorphic function at a that admits an analytic continuation along every path in Ω beginning from a. Show that there is a holomorphic function g on Ω such that $f_a = [\Omega, g]_a$.

(b) Let f be an entire function such that $|\mathbb{C} \setminus f(\mathbb{C})| > 1$. Show that f is a constant map.

[10 + 14]

Indian Statistical Institute
Semestral Examination: (2006)
M. Math.-I Year
Measure Theory
Total marks=120

Date: 05.12.2006 Time: 3.5 hours

The question paper carries 130 marks. The maximum you may score is 120.

Notations are as usual. In particular R is the real line, $\lambda =$ Lebesgue measure on R, L_p= space of all measurable functions whose pth power is integrable, $||f||_p = L_p$ norm of f and (p, q) is a conjugate pair.

1. Show that for every Borel subset B of R,
$\lambda(B) = \inf\{\lambda(G) : G \text{ is open and } B \subset G\} = \sup\{\lambda(F) : F \text{ is closed and } F \subset B\}$. [20]

2. Suppose $\{f_n\}$ is a sequence of measurable functions. Show that $\{x : \{f_n(x)\}$ *is a Cauchy sequence*$\}$ is a measurable set. [10]

3. Suppose $(\Omega, \mathcal{A}, \mu)$ is a σ-finite measure space and $p > 1$. For a given $g \in L_q$, and every $f \in L_p$ define $T(f) = \int fg d\mu$. Show that $\sup_{||f||_p=1} |T(f)| = ||g||_q$. Show that this is also true for $p = 1$. [20]

4. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. Let $\{f_n\}$ be a sequence of integrable functions such that $f_n \to f$ almost everywhere. Further suppose that for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\sup_n |\int_A f_n d\mu| < \epsilon$ whenever $\mu(A) < \delta$. Show that f is integrable and $\lim \int f_n d\mu = \int f d\mu$. [20]

5. Suppose $f \in L_1(\lambda)$. Define $g(t) = \int |f(x+t) - f(x)| d\lambda(x)$. Show that g is continuous at 0. [20]

6. Suppose μ is a measure on the Borel sets of $(0, 1]$ such that $\mu(k/2^n, (k+1)/2^n] = 1/2^n$ for every $n \geq 1$ and every $k = 0, 1, \ldots 2^n - 1$. Show that μ is the Lebesgue measure on $(0, 1]$. [20]

7. Suppose $(\Omega, \mathcal{A}, \mathcal{P})$ is a probability space. Let $\mathcal{B}$ and $\mathcal{C}$ be any two sub semi fields of $\mathcal{A}$. Show that they are independent iff $\sigma(\mathcal{B})$ and $\sigma(\mathcal{C})$ are independent. [20]

Indian Statistical Institute
Semestral Examination: (2006)
M. Math.–I Year
Measure Theory
Total marks=120

Date: 05.12.2006 Time: 3.5 hours

The question paper carries 130 marks. The maximum you may score is 120.

Notations are as usual. In particular R is the real line, λ = Lebesgue measure on R, L_p= space of all measurable functions whose pth power is integrable, $||f||_p = L_p$ norm of f and (p, q) is a conjugate pair.

1. Show that for every Borel subset B of R,
$\lambda(B) = \inf\{\lambda(G) : G \text{ is open and } B \subset G\} = \sup\{\lambda(F) : F \text{ is closed and } F \subset B\}$. [20]

2. Suppose $\{f_n\}$ is a sequence of measurable functions. Show that
$\{x : \{f_n(x)\}$ *is a Cauchy sequence*$\}$ is a measurable set. [10]

3. Suppose $(\Omega, \mathcal{A}, \mu)$ is a σ-finite measure space and $p > 1$. For a given $g \in L_q$, and every $f \in L_p$ define $T(f) = \int fg d\mu$. Show that $\sup_{||f||_p=1} |T(f)| = ||g||_q$. Show that this is also true for $p = 1$. [20]

4. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space. Let $\{f_n\}$ be a sequence of integrable functions such that $f_n \to f$ almost everywhere. Further suppose that for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\sup_n |\int_A f_n d\mu| < \epsilon$ whenever $\mu(A) < \delta$. Show that f is integrable and $\lim \int f_n d\mu = \int f d\mu$. [20]

5. Suppose $f \in L_1(\lambda)$. Define $g(t) = \int |f(x+t) - f(x)| d\lambda(x)$. Show that g is continuous at 0. [20]

6. Suppose μ is a measure on the Borel sets of $(0, 1]$ such that $\mu(k/2^n, (k+1)/2^n] = 1/2^n$ for every $n \geq 1$ and every $k = 0, 1, \ldots 2^n - 1$. Show that μ is the Lebesgue measure on $(0, 1]$. [20]

7. Suppose $(\Omega, \mathcal{A}, \mathcal{P})$ is a probability space. Let $\mathcal{B}$ and $\mathcal{C}$ be any two sub semi fields of $\mathcal{A}$. Show that they are independent iff $\sigma(\mathcal{B})$ and $\sigma(\mathcal{C})$ are independent. [20]

Indian Statistical Institute

First semsetral examination : (2006-2007)

M. Math I year

Algebra I

Date : 1.12.06 Maximum marks : 60 Duration : $3\frac{1}{2}$ hours.

Answer all questions. Marks are indicated in brackets.

All algebras are assumed to be associative and unital, and all rings are unital, but not necessarily commutative unless specifically mentioned to be so. Homomorphisms of rings and algebras are assumed to be identity preserving.

(1) Let R be a PID and x, y be elements of R, d be g.c.d. of x, y. Prove that $R/(x) \otimes_R R/(y) \cong R/(d)$. [5]

(2) Compute $K_0(R)$ when R is a PID (state clearly any result proved in the class that you use). [5]

(3) Let E be a (right) R-module which is both noetherian and artinian. Prove that the ring $S := \text{End}(E)$ is connected (that is, it does not have any nontrivial idempotent elements) if and only if the set of non-invertible elements of S form an ideal. [8]

(4) Let V be a vector space over a field F and let Q be the quadratic form on V defined by $Q(v) = 0$ for all $v \in V$. Prove that the Clifford algebra $C(Q)$ is isomorphic as a unital F-algebra with the exterior algebra $\Lambda(V)$. [8]

(5) Prove or disprove :

If E is a projective R module over a commutative ring R then the dual module $E' := \text{Hom}(E, R)$ is injective. [8]

(6) Let E be a noetherian right R-module and $\text{ann}(E) := \{a \in R \,:\, xa = 0 \;\forall x \in E\} = \{0\}$. Then prove that R is a (right) noetherian ring. Give a counterexample to prove that the result fails if noetherian is replaced by artinian. [9+6=15]

(7) Let G be a group and and K be a commutative unital ring. Consider the category $\mathcal{C}$, whose objects are pairs $(\mathcal{A}, \beta)$ where $\mathcal{A}$ is a unital K-algebra and $\beta : G \to \mathcal{A}$ be a map satisfying $\beta(g_1 g_2) = \beta(g_1)\beta(g_2)$, $\beta(e) = 1$, where e is the identity of the group G. The set of morphisms from $(\mathcal{A}_1, \beta_1)$ to $(\mathcal{A}_2, \beta_2)$ consists of K-algebra homomorphisms $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ sat-

P.T.O

isfying $\phi o \beta_1 = \beta_2$. Prove that this category has an initial object. [8]

(8) Let G_1, G_2 be groups containing a common subgroup H. Prove that the following category $\mathcal{F}$ has an initial object :

The object class Obj($\mathcal{F}$) consists of triples (G, j_1, j_2) where G is a group, $j_i : G_i :\to G$ $(i = 1,2)$ are group homomorphisms satisfying $j_1(h) = j_2(h)$ for all $h \in H$. The set of morphisms from (G, j_1, j_2) to (G', j_1', j_2') consists of all group homomorphisms $\pi : G \to G'$ satisfying $\pi \circ j_i = j_i'$ for $i = 1, 2$.

(Note : this initial object is called the amalgamated free product of G_1 and G_2 with respect to H, denoted by $G_1 *_H G_2$. You may try to construct this as a suitable quotient of the free product $G_1 * G_2$.) [10]

INDIAN STATISTICAL INSTITUTE
First Semestral Examination: 2006-07
M. Math. I Year
Graph Theory & Combinatorics

Date: 08.12.06 Maximum Marks: 60 Duration: 3:30 Hours

This paper carries a total of 75 marks. Answer as much as your can.
In case you score more than 60, your final score will be 60.

1.(a) Draw all the nonisomorphic trees of order 7.

(b) Describe a tournament on 7 vertices with exactly 14 dicycles of length 3. Justify your answer.

(c) Prove that the Ramsey number $R(3,4) = 9$.

[4+6+5=15]

2.(a) Prove that the sequence $s = (5,5,5,5,2,2,2,2)$ is a potentially Hamiltonian graphic sequence but s is not a forcibly Hamiltonian graphic sequence.

(b) Prove that the edges of any 3-regular graph of order $2n$, can be decomposed into k pairwise edge-disjoint paths when $k = n$ but not when $k < n$.

(c) Suppose that S is a set of 2006 points inside a unit cube in R^3. Show that there is a subset A of S of cardinality 32 such that every (possibly degenerate) closed polygon with these 32 points of A as vertices has perimeter less than $8\sqrt{3}$ units.

[5+5+5=15]

3.(a) Find the vertex, edge and total chromatic numbers of the n-dimensional cube graph $Q(n)$. Justify your answer in each case.

(b) Prove that the graph $Q(n)$ as in 3(a) is hamiltonian for all $n \geq 2$; and is planar if and only if $n \leq 3$.

[9+6=15]

P.T.O.

4.(a) Find all the integral maximum flows in the network N of the Figure 1 below, with the unique source s and the unique sink t. Justify your answer.

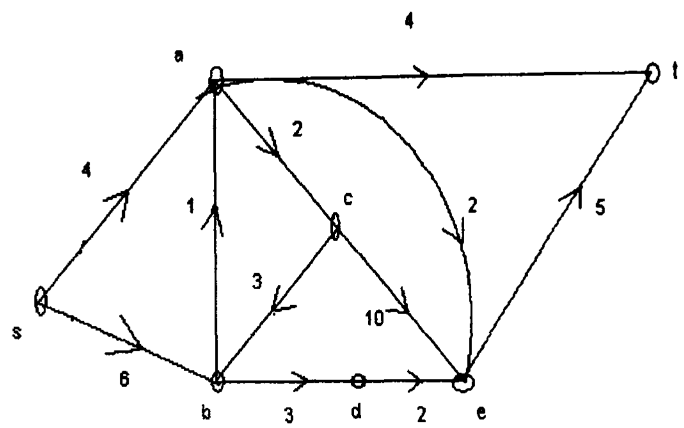

Figure 1: The network N

(b) Construct a connected 4-regular planar graph on even number of vertices without a perfect matching. Justify your answer.

[9+6=15]

5.(a) Prove that there is no graph whose energy number is equal to 3.

(b) State and prove the Prim's algorithm to find a minimum spanning tree (MST) in a connected cost graph. Show that this algorithm implemented appropriately gives all the MSTs.

[8+7=15]

***

Indian Statistical Institute
Backpaper Examination: (2006)
M. Math.–I Year
Measure Theory
Total marks=100

Date: 5.2.07 Time: 3.5 hours

Notations are as usual. In particular $\lambda =$ Lebesgue measure on the real line, $L_p =$ space of all measurable functions whose pth power is integrable, $||f||_p = L_p$ norm of f and (p,q) is a conjugate pair.

1. If $B \in \sigma(\mathcal{A})$, show that there is a countable subclass $\mathcal{A}_B$ of $\mathcal{A}$ such that $B \in \mathcal{A}_B$. [15]

2. Suppose f is a measurable function on R, $\int |f| d\lambda < \infty$ and $\int_K f d\lambda = 0$ for every compact subset K of R. Show that $f = 0$ almost everywhere. [15]

3. Suppose μ is a finite measure on $([0,\infty), \mathcal{B})$ where $\mathcal{B}$ is the Borel σ-field. Show that $x \mapsto \mu[x,\infty)$ is a Borel measurable function and $\int_{[0,\infty]} x d\mu(x) = \int_0^\infty \mu[x,\infty) d\lambda(x)$. [10]

4. Suppose ν and μ are two finite measures on the same space $(\Omega, \mathcal{A})$. Suppose $S = \{f \geq 0 : \int_E f d\mu \leq \nu(E)$ for all $E \in \mathcal{A}\}$. Show that there exists $g \geq 0$, such that $\int g d\mu = \sup\{\int f d\mu : f \in S\}$. [20]

5. For any Borel set B, define $\mu(B) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\lambda(B \cap [-n,n])}{2n}$. Show that for every $x \in R$, $\mu_x(B) = \mu(B+x)$ defines a measure on the Borel sets of R and they are all equivalent to each other. [20]

6. Suppose f and g are two positive measurable functions on a probability space such that $fg \geq 1$. Show that $(\int f d\mu)(\int g d\mu) \geq 1$. [20]

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination: 2006-07

M. Math. I Year
Algebra II

Date: 19.02.07 **Maximum Marks: 60** **Duration: 3 Hours**

The paper carries 72 marks. The maximum you can score is 60.

1. Let $\mathbb{R}$ and $\mathbb{C}$ denote the field of real numbers and the field of complex numbers respectively and let

$$A = \mathbb{R}[X] / (X^4 - 1).$$

Prove that A is isomorphic (as a ring) to the product ring $\mathbb{R} \times \mathbb{R} \times \mathbb{C}$ and identify the element (a, b, c) of the product which corresponds to the element $\overline{X}$ of A under an isomorphism. Quote the result(s) that you use.

[7]

2. Let R be a UFD with field of fractions K $(\neq R)$. Show that K can not be an algebraically closed field.

[7]

3. Let k be a field. Prove that k (X) is not finitely generated as a k-algebra.

[7]

4. Let $\mathbb{Q}$ denote the field of rational numbers. Let $\alpha = \cos\frac{\pi}{6} + i \sin\frac{\pi}{6}$.

Show that $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 4$ and determine the minimal polynomial f (X) of α over $\mathbb{Q}$. Determine a splitting field E of f (X) and resolve f (X) into factors over E.

[10]

5. Let p be a prime and $f(X) = X^{p-1} + X^{p-2} + \ldots + X + 1$.

(i) Show that f (X) is irreducible in $\mathbb{Q}[X]$ where $\mathbb{Q}$ denotes the field of rationals. Clearly state the result(s) that you use.

(ii) Prove that $\mathbb{Q}[X] / (f(X))$ is a normal field extension of $\mathbb{Q}$.

[6+8=14]

P.T.O

6. Let k be a field.

(i) Let f(X) and g(X) be coprime polynomials in k[X]. Show tha f(X) – Yg (X) is irreducible in the polynomial ring k[X, Y] and hence i irreducible in k (Y) [X]. Clearly state the result(s) that you use.

(ii) Let L = k (t) where t is transcendental over k and F=k(u) wher $u = f(t)/g(t)$, (f, g) = 1. Show that L is algebraic over F and comput [L : F].

(iii) Deduce that if φ is a k-automorphism of L, then φ is defined by φ(t) = $(at+b)/(ct+d)$ for some a, b, c, d, ε k with ad - bc ≠ 0.

[5+5+5=15

7. Let ℝ denote the field of real numbers and let $A = \mathbb{R}[X,Y,Z]/(X^2+Y^2+Z^2-1)$.

(i) Show that A is an integral domain.
(ii) Prove that $\overline{X}-1$ is a prime element in A.
(iii) Examine if A is a PID.

[4+4+4=12

*****

INDIAN STATISTICAL INSTITUTE
Mid-semester Examination (2006–2007)
M. Math. I
Functional Analysis

Date : 21.02.2007 Maximum Marks : 100 Time : 3 hrs.

This paper carries 105 marks. Maximum you can score is 100. Precisely justify all your steps. Carefully state all the results you are using.

1. Let X be a normed linear space. Show that X is a Banach space if and only if whenever $\{x_n\} \subseteq X$ and $\sum_n \|x_n\| < \infty$, the series $\sum_n x_n$ converges in norm. [15]

2. Show that if f is a linear functional (need not be continuous) on a normed linear space X, then $\ker f = \{x \in X : f(x) = 0\}$ is either closed or dense in X. [10]

3. (a) Let Y be a linear subspace of a normed linear space X, and let $x_0 \in X$ be such that
$$d = d(x_0, Y) = \inf_{y \in Y} \|y - x_0\| > 0.$$
Show that there exists $f \in X^*$ such that $f(x_0) = 1$, $\|f\| = \frac{1}{d}$, and $f|_Y \equiv 0$. [12]

(b) Let Y be a closed linear subspace of a normed linear space X. Show that
$$Y = \bigcap \{\ker f : f \in X^*, Y \subseteq \ker f\}.$$
[8]

4. A subset $A \subseteq X^*$ is said to separate points of X if whenever $x_1 \neq x_2 \in X$, then there exists $f \in A$ such that $f(x_1) \neq f(x_2)$.

(a) For any normed linear space X, show that X^* separates points of X. [10]

(b) Let X, Y be Banach spaces. Let $A \subseteq Y^*$ separate points of Y. Let $T : X \to Y$ be a linear map. Show that $T \in \mathcal{L}(X,Y)$ if and only if for all $y^* \in A$, $y^* \circ T \in X^*$. [15]

5. Let X, Y be Banach spaces. Let $T \in \mathcal{L}(X,Y)$. Show that there is a constant $c > 0$ such that $\|Tx\| \geq c\|x\|$ for all $x \in X$ if and only if T is 1-1 and $T(X)$ is closed in Y. [20]

6. Let X be an infinite dimensional normed linear space. Show that X^* cannot have a countable Hamel basis. [15]

Indian Statistical Institute
Mid Semestral Examination: (2007)
Probability and Stochastic Processes
Total marks=80

Date: 23.2.07 M. Math.- 1st yr. Time: 3 hours

1. Suppose X and Y are two random variables (on the same probability space). Show that X and Y are independent if and only if for all measurable functions g and f for which $E(|f(X)|) < \infty$ and $E(|g(Y)|) < \infty$, we have $E(f(X)g(Y)) = E(f(X))E(g(Y))$. [10]

2. Suppose $(X_1, \ldots X_n)$ is a multivariate normal random variable. Give a necessary and sufficient condition for $X_1, \ldots X_n$ to be exchangeable. [10]

3. Suppose X_n are mean zero independent random variables such that $\sup_n P(|X_n| > x) \leq CP(|X| > x)$ where C is a constant and $E(|X|) < \infty$. Show that the SLLN holds for $\{X_n\}$. [10]

4. Suppose X is a Poisson random variable with mean 1. If $X = k$, then I take a fair coin and toss it k times. Let N be the number of heads. What are the possible values of N? What are the corresponding probabilities?[10]

5. Suppose you toss a coin independently till you get two *successive* heads. Let T be the number of tosses needed. What is the probability distribution of T and what is $E(T)$? [10]

6. Suppose $X_1, \ldots X_n$ are i.i.d. random variables, with density function $f(x) = kx^{k-1}$, $0 \leq x \leq 1$, for some $k > 0$. Show that $n(1 - X_{(n)})$ converges in distribution and find the limit. Here $X_{(n)} = \max\{X_1, \ldots X_n\}$. [10]

7. Suppose X and Y are two independent random variables with distribution functions F and G. Show further that if any one of the X or Y is absolutely continuous, then $X + Y$ also is absolutely continuous and find a formula for the density. [10]

8. Suppose $\{X_n\}$ and $\{Y_n\}$ are independent and each of them converge in distribution. What can you say about the convergence in distribution of (X_n, Y_n)? [10]

INDIAN STATISTICAL INSTITUTE
Mid-semester Examination : 2006-2007
M. Math. - First Year
Algebraic Topology

Date : 27. 02. 2007 Maximum Score : 100 Time :3 Hours

1. This paper carries questions worth a total of 116 marks. Answer as much as you can. The maximum you can score is 100 marks.
2. Unless otherwise stated, X, Y will denote arbitrary topological spaces, maps between topological spaces will be continuous and all homology groups are over the coefficient group $\mathbb{Z}$.

(1) State and prove five lemma.

[15]

(2) Let $f, g : X \to Y$ be homotopic maps. Show that for all $p \geq 0$,

$$f_* = g_* : H_p(X) \longrightarrow H_p(Y).$$

[15]

(3) Consider the 1-cycle

$$\sigma((1-t)e_1 + te_2) = e^{2\pi i t}, \quad 0 \leq t \leq 1,$$

in S^1. Show that $\sigma_* \in H_1(S^1)$ is a free generator of $H_1(S^1)$.

[15]

(4) Let $A \subset X$ be a non-empty, closed set that has a neighbourhood in X that strong deformation retracts onto A. Show that for all $p \geq 0$,

$$H_p(X, A) = \widetilde{H}_p(X/A).$$

[12]

(5) (a) Let M be a manifold, ∂M its (manifold) boundary and M° the (manifold) interior of M. Show that

$$\partial M \cap M^\circ = \emptyset.$$

(b) Show that the inclusion map

$$i : (D^n, S^{n-1}) \hookrightarrow (D^n, D^n \setminus \{0\})$$

is not a homotopy equivalence in the category of pairs of topological spaces, where $D^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ is the closed unit disc in $\mathbb{R}^n$.

P.T.O

(c) Let CX be the cone of X. Suppose $f : X \to Y$ be a map such that there is a map $g : CX \longrightarrow Y$ such that $g \circ q = f$, where $q : X \longrightarrow CX$ is the quotient map. Compute

$$f_* : H_p(X) \to H_p(Y),$$

$p \geq 0$.

[6+12+6]

(6) (a) Let $f : S^{2n} \longrightarrow S^{2n}$ be a map. Show that there is a $x \in S^{2n}$ such that $f(x) = x$ or $f(x) = -x$.

(b) Show that every map $\mathbb{R}P^{2n} \longrightarrow \mathbb{R}P^{2n}$ has a fixed point.

(c) Let M be a n-manifold without boundary and $A \subset M$ a finite subset with k elements. Compute $H_n(M, M \setminus A)$ for all $p \geq 0$.

[8+12+15]

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination : 2006-2007
M. Math. - I Year
Differential Geometry

Date : 02. 03. 2007 | Maximum Score : 40 | Time :3 Hours

This paper carries questions worth a total of 50 marks. Answer as much as you can. The maximum you can score is 40 marks.

(1) Define a Smooth manifold. Describe an atlas on the real projective space $\mathbb{R}P^2$ to show that it is a smooth manifold of dimension 2.

[4+5=9]

(2) Let $f : M \longrightarrow \mathbb{R}$ be a smooth function on a manifold M. Explain what do you mean by differential of f at a point $p \in M$. Suppose M is given to be connected and the differential of f vanishes at all points of M. Prove that f must be a constant function.

[2+4=6]

(3) Define a tangent vector at a point p of a smooth manifold M. Let

$$\sigma : (-1, 1) \longrightarrow M$$

be a smooth curve. Prove that the velocity vector of σ at $p = \sigma(0)$ is a tangent vector at p.

[2+3=5]

(4) Let f be a smooth function defined on an open set U containing a point p of a smooth manifold M. Prove that there is an open set V, $p \in V \subset \bar{V} \subset U$ and a smooth function g on M which agrees with f on V.

[5]

(5) Define the notion of tangent bundle of a smooth manifold M. Show that the tangent bundle of the unit circle S^1 can be identified with $S^1 \times \mathbb{R}$.

[5+4=9]

(6) Prove that the set $\chi(M)$ of vector fields on a smooth manifold M is a module over the ring $C^\infty(M)$ of smooth functions on M.

[4]

(7) Define a 1-form on a smooth manifold M. Let ω be a 1-form on M. Describe a local expression of ω on a chart $(U; x_1, x_2, \cdots, x_n)$ in M. Deduce a change of basis formula for 1-forms on M.

[2+2+4=8]

(8) Prove that the space of 1-forms on M is isomorphic to the dual of the $C^\infty(M)$-module $\chi(M)$.

[4]

INDIAN STATISTICAL INSTITUTE
SECOND SEMESTER EXAMINATION 2006-07
M. Math. I Year
Algebra II

Date : 08.05.07 **Maximum Marks : 70** **Time : 4 hours**

Attempt all questions from Group A, any FOUR from Group B and any THREE from Group C. Maximum score :70.

Throughout the paper, k denotes a field.

GROUP – A

1. State whether the following statements are TRUE or FALSE with brief justification. Each statement carries 3 marks.

(i) If L/K and F/K are Galois radical extensions, then so is L/K.

(ii) The Galois group of the polynomial $X^3 + X + 1$ over $\mathbf{Q}$ is S_3.

(iii) Cos 20° is not solvable by radicals over $\mathbf{Q}$.

(iv) $\mathbf{Q}(\sqrt[3]{2})$ is not a subfield of any cyclotomic field over $\mathbf{Q}$.

(v) The simple extension $k(x)/k$ (where x is transcendental over k) has infinitely many intermediary fields.

$[3 \times 5 = 15]$

GROUP – B

Attempt ANY FOUR questions.
Each question carries 8 marks.

(i) Show that the polynomial
$f(X,Y) = X^5 + 3X^2Y^2 + 2X^2Y + Y^4 + Y^2 + 7Y$ is irreducible in ¢ $[X,Y]$.

(ii) Let p be a prime number and $f(X)$ a polynomial in $\mathbb{Z}[X]$ whose image in $\mathbf{F}_p[X]$ is an irreducible polynomial of $\mathbf{F}_p[X]$.
Show that $(p, f(X))$ is a maximal ideal of $\mathbb{Z}[X]$.

(i) Let k be a field of characteristic p>0 , and let α be algebraic over k. Prove that α is separable over k if and only it $k(\alpha^p) = k(\alpha)$.

i) Give an example of an inseparable field extension L/k which does not have purely inseparable elements outside k. (Detailed proof not required.)

P.T.O

3. Suppose that the Galois group of an irreducible and separable polynomial $f(X) \in k[X]$ is Abelian. Let E be a splitting field of $f(X)$ over k and let $\alpha_1,...,\alpha_n$ be the roots of $f(X)$ in E. Show that $E = k(\alpha_i)$ for each i, $1 \leq i \leq n$, and $[E:k] = \deg f$.

4. Let $f(X)$ be an irreducible polynomial of degree n over a field k. Let $g(X) \in k[X]$ and $h(X)$ an irreducible factor of $f(g(X))$ in $k[X]$. Prove that the degree of $h(X)$ is divisible by n. [Hint : Consider field extensions.]

5. Construct a polynomial of degree 7 in $\mathbf{Q}[X]$ which cannot be solved by radicals over $\mathbf{Q}$. (Give proof; clearly quoting the results you use in your arguments.)

6. Let n be a natural number and p a prime number such that p does not divide n. Let $\Phi_n(X)$ denote the n^{th} cyclotomic polynomial over **Q**. Show that, for $a \in \mathbb{Z}$, we have $\Phi_n(a) \equiv 0 \pmod{p}$ if and only if $\bar{a}$ is an element of order n in the multiplicative group $\mathbf{F}_p^*$, where $\bar{a}$ is the image of a in $\mathbf{F}_p$. Clearly state the results that you use.

$[8 \times 4 = 32]$

GROUP – C

Attempt ANY THREE questions
Each question carries 12 marks.

1. Let G be a finite group of automorphism of a field L and let k be the fixed field of G. Prove that L/k is a finite Galois extension whose Galois group is G.

2. Let ρ be a root of $X^4 + 1$ over **Q**.

   (i) Resolve $X^4 + 1$ into factors in $\mathbf{Q}(\rho)$.
   (ii) Determine the Galois group of $X^4 + 1$ over **Q**.
   (iii) Describe all subfields of $\mathbf{Q}(\rho)$.

3. (i) Suppose that a field extension L/k of degree 2^n is of the from $L = k(\sqrt{a_1},...,\sqrt{a_n})$ with $a_i \in k$. Prove that, if ch k $\neq 2$, then $L = k(\alpha)$ where $\alpha = \sqrt{a_1} + ... + \sqrt{a_n}$. [Hint : Show that L/k is Galois and consider $G\left(L/k(\alpha)\right)$]

   (ii) If $a_1...,a_n$ are pairwise relatively prime square–free integers such that $|a_i| > 1 \; \forall$ i, then show that $\left[\mathbf{Q}(\sqrt{a_1},...,\sqrt{a_n}) : \mathbf{Q}\right] = 2^n$.

P.T.O

4. (i) Show that the polynomial $X^4 - 10X^2 + 1$ is irreducible in $\mathbb{Z}[X]$ but is reducible in $\mathbf{F}_p[X]$ for every prime p.

   (ii) If α is a root of $X^3 + X + 1 \in \mathbf{F}_2[X]$, then show that α is a primitive 7^{th} root of unity over $\mathbf{F}_2$. If ω is a primitive 7^{th} root of unity over $\mathbf{F}_2$, does it follow that ω is a root of $X^3 + X + 1$? Explain.

5. Let $f \in k[X]$ be irreducible and let L/k be a finite normal field extension. Show that if g, h are monic irreducible factors of f in L[X], then there exists a k-automorphism σ of L such that $\sigma(g) = h$. Deduce that in the prime factorisation of f in L [X], all prime factors have the same degree and exponent. Give an example to show that the assumption of normality is required.

$[12 \times 3 = 36]$

******

INDIAN STATISTICAL INSTITUTE
Second Semester Examination: 2006-07
M. Math. I Year
Differential Geometry

Date: 11.05.07 Maximum Marks: 60 Duration: 3 Hours

1.(a) State inverse function Theorem in $\mathbb{R}^n$. Use it to show the following: Let $\psi : M \to N$ be a smooth map between smooth manifolds. Assume that for $p \in M, d\psi_p : T_pM \to T_{\psi(p)}N$ is an isomorphism. Then there exists an open set U containing p in M such that ψ/U is a diffeomorphism onto its image.

(b) Let M be a smooth manifold and U an open subset of M containing $p \in U$. Let $\{y_1, y_2, \ldots\ldots\ldots, y_n\}$ be a collection of smooth functions on U such that $\left\{ dy_i|_p \right\}_{1 \le i \le n}$ is an independent set in T_p^*M. Prove that $\{y_1, y_2, \ldots\ldots, y_n\}$ forms a set of coordinates on an open set containing p.

[2+5+5=12]

2. Let θ be smooth action of $\mathbb{R}$ on a smooth manifold M. Show that there exists a vector field on M which is invariant under the action θ.

[10]

3.(a) Define the notion of orientation on a smooth manifold.

(b) Let M_1 and M_2 be orientable manifolds. Prove that $M_1 \times M_2$ with the product manifold structure is also orientable.

[4+8=12]

4.(a) Define a Riemannian manifold.

(b) Let $f : M \to N$ be an immersion. Assume that N has a Riemannian metric. Prove that f induce a Riemannian metric on M.

[2+6=8]

5.(a) Define a Riemannian connection on a Riemannian manifold.

(b) Let M be a Riemannian manifold. Let D_t be the covariant derivative with respect to the Riemannian connection. Let U be a coordinate chart in M and V be a vector field along a smooth curve $r : I \to U$. Compute the components of D_tV at $t = t_0$.

[4+8=12]

P.T.O.

6.(a) Explain what do you mean by a parallel vector field along a smooth curve on a Riemannian manifold.

(b) Let $r:[a,b]\to M$ be a smooth curve on a Riemannian manifold M and V,W be vector fields which are parallel along r. Prove that the function $f:[a,b]\to\mathbb{R}$ given by $f(t)=\langle V(t),W(t)\rangle_{r(t)}$ is Constant.

[2+6=8]

7. Let M be a closed embedded submanifold of $\mathbb{R}^N$. Show how the Riemannian connection on (M,g) is obtained from the standard Riemannian connection on $\mathbb{R}^N$, where g is the induced metric M.

[12]

8.(a) State Gauss-Bonnet Theorem.

(b) Prove that a compact orientable surface of genus $g\geq 1$ does not admit any metric of strictly positive Gaussian curvature.

[2+4=6]

***

Indian Statistical Institute
Second Semestral Examination: (2007)
M. Math- 1st Year
Advanced Probability
Total marks=100

Date: 15.05.07 Time: 3 hours

Note: The question paper carries 105 marks. The maximum you can score is 100.

1. Suppose $\{X_n\}$ is a sequence of i.i.d. random variables, $P\{X_n=1\}=\frac{1}{2}=1-P\{X_n=0\}$. Determine the behaviour of the series $\sum_{n=1}^{\infty}\frac{X_n}{n}$. [10]

2. Suppose $\{X_n\}$ are independent $E(X_n)=0$ for all n and $\sup E(X_n^4)<\infty$. Show that $n^{-1}(X_1+\ldots X_n)\to 0$ almost surely. [10]

3. Suppose X is in L_1. Show that

$$E(X)=\int_0^{\infty}P(X>t)dt-\int_{-\infty}^{0}P(X<t)dt. \quad [15]$$

4. Suppose $\{X_n\}$ is a sequence of i.i.d. random variables such that $P\{X_n=1\}=p=1-P\{X_n=0\}$, $0<p<1$. Let A be the event that in the sequence $\{X_n\}$, the triplet $(1,0,1)$ appears infinitely many times. Show that $P(A)=1$. [15]

5. Suppose X_1 is a $U(0,1)$ random variable. Define $\{X_n\}$ recursively as follows: given $X_1=x_1,X_2=x_2,\ldots,X_{n-1}=x_{n-1}$, X_n is uniformly distributed on $(0,x_{n-1})$. Show that $\{X_n\}$ is a supermartingale. Show also that $X_n\to 0$ almost surely. [15]

6. Suppose F is a continuous distribution function. Show that F is uniformly continuous. If F_n converges to F weakly, and F is everywhere continuous, show that $\sup_x|F_n(x)-F(x)|\to 0$. [20]

7. Suppose X_n is a sequence of i.i.d. random variables such that $EX_1^2<\infty$. Show that for any $\epsilon>0$, $nP\{|X_n|>\epsilon\sqrt{n}\}\to 0$. Using this, show that $\max_{1\leq i\leq n}X_i/\sqrt{n}\to 0$ almost surely. [20]

INDIAN STATISTICAL INSTITUTE
Second Semestral Examination : (2006-2007)
M. Math. - First Year
Algebraic Topology

Date : 18. 05. 2007 Maximum Score : 100 Time :3 1/2 Hours

1. This paper carries questions worth a total of 120 marks. Answer as much as you can. The maximum you can score is 100 marks.
2. Unless otherwise stated, R will denote a commutative ring with identity.

(1) (a) State the universal coefficient theorem for homology for a topological pair (X, A).
(b) Let X be a n-manifold without boundary. Show that for every $x \in X$,

$$H_n(X, X \setminus \{x\}) \otimes_{\mathbb{Z}} R \approx H_n(X, X \setminus \{x\}: R).$$

(c) If X is an orientable n-manifold without boundary, show that X is R-orientable.

[2 + 4 + 4]

(2) (a) Show that every simply connected manifold without boundary is orientable.
(b) Let X be an oriented manifold, G a group that acts on X such that for every $g \in G$, $x \to g \cdot x$ is orientation preserving. Show that the quotient space X/G is orientable.

[6 + 6]

(3) (a) Show that $S^2 \times S^2$ is not homotopically equivalent to S^4.
(b) Let X and Y be n-manifolds without boundary and $X\#Y$ their connected sum. Compute the Euler characteristic of $X\#Y$ in terms of the Euler characteritics of X and Y.

[7 + 7]

(4) Let X be a topological space, $X \supset A \supset B$, A, B good closed subsets of X and $\mathbb{F}$ a field. Assume that

$$\tilde{H}_p(X/A: \mathbb{F}) \approx \tilde{H}_p(X/B: \mathbb{F}).$$

Show that

$$H^p(A: \mathbb{F}) \approx H^p(B: \mathbb{F}).$$

[10]

(5) (a) Let I_k be a homeomorph of $[0,1]^k$ in S^n. Show that $S^n \setminus I_k$ is connected.

P.T.O

(b) Show that $S^k \setminus s_{k-1}$ has exactly two components, where s_{k-1} is a homeomorph of S^{k-1}.

[10 + 10]

(6) (a) For $p \geq 0$, compute $H_c^p(\mathbb{R}^n)$.
(b) Prove the Poincaré duality theorem for $\mathbb{R}^n$.
(c) Show that the Euler characteristic of an odd dimensional closed manifold is zero.

[5 + 7 + 10]

(7) (a) Let X be a compact n-manifold with boundary and X_1 and X_2 two copies of it. Suppose Y is the topological sum of X_1, $\partial X \times [0,1]$ and X_2. Identify any $x' \in \partial X_1$ with $(x', 0)$ and any $x' \in \partial X_2$ with $(x', 1)$. Let $2X$ be the corresponding quotient space.
Show that $2X$ is a union of two open sets A_1 and A_2 such that X_i is a strong deformation retract of A_i, $i = 1, 2$, and ∂X is a strong deformation retract of $A_1 \cap A_2$.
(b) Show that

$$\chi(2X) = 2\chi(X) - \chi(\partial X).$$

(c) Show that $\mathbb{R}P^{2n}$ is not the boundary of any compact orientable manifold.

[12 + 10 + 10]

INDIAN STATISTICAL INSTITUTE
2nd Semestral Examination (2006–2007)
M. MATH. I
Functional Analysis

Date : 21.05.2007 Maximum Marks : 100 Time : $3\frac{1}{2}$ hrs.

The question carries 130 marks. Maximum you can score is 100. Precisely justify all your steps. Carefully state all the results you are using.

1. Given a sequence $\{\alpha_n\}$ of scalars, show that there exists a complex measure μ on $[0,1]$ such that $\int_0^1 x^n d\mu = \alpha_n$ for all $n \geq 0$ and $\|\mu\| \leq 1$ if and only if for any polynomial $P(x) = \sum_{k=0}^{n} a_k x^k$,

$$\left|\sum_{k=0}^{n} a_k \alpha_k\right| \leq \sup\{|P(x)| : x \in [0,1]\}.$$ [10]

2. Let c_0 be the space of all sequences of real numbers converging to zero equipped with the usual sup norm. Let A be the subset of c_0 consisting of all sequences of *rational* numbers converging to zero. Show that A cannot be a G_δ set in c_0. [10]

3. (a) The space ℓ_1 of all summable sequences of scalars is clearly a subspace of the space c_0 of all scalar sequences converging to zero. Is this a closed subspace, if c_0 is equipped with the usual sup norm? Justify! [5]
(b) Clearly explain what is meant by the expression "$\ell_1^* = \ell_\infty$" and prove it. [15]
(c) Show that ℓ_1 is separable, while ℓ_∞ is not. [8]
(d) Show that ℓ_1 with its usual norm is not a Hilbert space. [7]
(e) Is there any equivalent norm on ℓ_1 that will make it a Hilbert space? Justify! [10]

4. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces and $T : X \to Y$ be a linear map. Define

$$\|x\|_1 = \|T(x)\|_Y, \quad x \in X.$$

(a) Is $\|\cdot\|_1$ a norm on X? If yes, prove it. If not, can you find a necessary and sufficient condition on T that will make it a norm on X? [5]
(b) Given that $\|\cdot\|_1$ is a norm on X (i.e., the condition of (a) holds), find a necessary and sufficient condition on the range of T that will make $\|\cdot\|_1$ a complete norm. [6]

P.T.O

(c) Given that $(X, \|\cdot\|_1)$ is a Banach space (i.e., the conditions of both (a) and (b) hold), prove that $\|\cdot\|_1$ is equivalent to $\|\cdot\|_X$ if and only if $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is continuous. [10]

5. Let Y be a w*-closed subspace of a dual Banach space X^*. Let $x^* \in X^*$. Show that there exists $y^* \in Y$ that is nearest to x^*, i.e., $\|x^* - y^*\| = \inf\{\|x^* - z^*\| : z^* \in Y\}$.

[Hint : The norm on a dual space is w*-lsc.] [10]

6. Let (Ω, Σ, μ) be a σ-finite measure space. Let $f \in L^\infty(\mu)$. Define $M_f : L^2(\mu) \longrightarrow L^2(\mu)$ by

$$M_f(g) = fg, \quad g \in L^2(\mu).$$

Prove that M_f is a bounded, normal operator on $L^2(\mu)$ with $\|M_f\| = \|f\|_\infty$. [10]

(a) M_f is unitary if and only if $|f(t)| = 1$ a.e. [8]

(b) M_f is self-adjoint if and only if $f(t) \in \mathbb{R}$ a.e. [8]

(c) M_f is an orthogonal projection if and only if $f(t) \in \{0, 1\}$ a.e. [8]