2

Indian Statistical Institute
Semester 1 (2008-2009)
M. Math. 1st Year
Mid-semester Examination
Measure Theory

Date and Time: 1.9.08, 10:30 - 1:30 Total Points: $5 \times (3+3) = 30$

Answers must be justified with clear and precise arguments. If you use any theorem/proposition roved in class state it explicitly. All the functions are assumed measurable and the integrals are ebesgue integrals.

1. (a) If E is Lebesgue measurable then from the definition of measurability of a set show that $E + y$ is Lebesgue measurable for any fixed real number y.

   (b) If $E \subset [a, b]$ has measure zero and f is an increasing absolutely continuous function on $[a, b]$ with $f(a) = c, f(b) = d$, then show that $f(E)$ has measure zero.

2. (a) Suppose f is a nonnegative integrable function on $\mathbb{R}$ and t is a real number. Show that $\int_{\mathbb{R}} f(x)dx = \int_{\mathbb{R}} f(x-t)dx$. (You may assume that for any measurable set E, $m(E) = m(E+y)$ for any real number y.)

   (b) Prove the following form of Fatou's lemma: f_n is a sequence of nonnegative integrable functions. Then $\int \liminf f_n \leq \liminf \int f_n$.

3. Let f_n be a sequence of integrable functions such that $f_n \to f$ a.e. with f integrable. Then show that the following are equivalent:

   (a) $\int |f_n - f| \to 0$,

   (b) $\int |f_n| \to \int |f|$.

4. Suppose F is a continuous real valued increasing function on $[a, b]$. We know that F' exists a.e.

   (a) Considering the functions $g_n(x) = n(F(x + \frac{1}{n}) - F(x))$ show that for any $a \leq u < v \leq b$ we have $\int_u^v F' \leq F(v) - F(u)$. Explain how you use the assumptions on F (note that v may be less than b).

   (b) Prove that the function $G(x) = F(x) - F(a) - \int_a^x F'$ is singular, i.e. $G' = 0$ a.e. Is G nondecreasing on $[a, b]$?

5. (a) Let $f_n \to f$ in $L^p, 1 \leq p < \infty$, and let g_n be a sequence of measurable functions such that $|g_n| \leq M$ for all n, and $g_n \to g$ a.e. Then show that $g_n f_n \to gf$ in L^p.

   (b) Let $C[0, 1]$ be the space of all continuous functions on $[0, 1]$ and define the norm $||f|| = \max_{x \in [0,1]} |f(x)|$. Show that $C[0, 1]$ is complete.

INDIAN STATISTICAL INSTITUTE

Mid-semester Examination: 2008-2009

M. Math. I Year

Algebra I

Date: 3.9. 2008 **Maximum Marks: 40** **Duration: 2 hours**

Note: There are SIX questions on this test. Answer ANY FIVE. JUSTIFY your answers. Each question carries 8 marks.

1. Find a composition series for the symmetric group S_4. [8]
2. Show that every finite abelian group is isomorphic to a direct product of its Sylow subgroups. [8]
3. Suppose G is a group of order 105. Show that G is solvable. [8]
4. Construct a nonabelian group of order 375. [8]
5. Show that every finite group has a finite presentation. [8]
6. Suppose H is a proper subgroup of a finite group G. Show that $G \neq \bigcup_{g \in G} gHg^{-1}$. [8]

INDIAN STATISTICAL INSTITUTE

Mid- Semester Examination: 2008-2009
M. Math. I Year
Combinatorics & Graph Theory

Date: 5.9.08 **Maximum Marks: 40** **Duration: 2 hours**

Answer any five questions. Each question carries equal marks.
Class notes are allowed.

1. Consider weighted majority games and the two power indices: Shapley-Shubik & Coleman- Banzhaf. Compute these for the game : [65, 32, 32, 29, 22, 3,3]. Which index is computationally easier? Justify.

2. Find the number of sequence of length 3 from an alphabet $\{a,b,c,d,e,f\}$ if a,c,d & f occur an even number of times.

3. Consider a sequence $\{a_n : n \geq 0\}$ where

$$a_0 = 1, \quad a_1 = 1$$
$$a_n = a_{n-1} + a_{n-2}, \qquad n \geq 2$$

Prove that $F_{n+1} F_{n-1} - F_n^2 = (-1)^n$

4. Does there exist 4 mutually orthogonal latin squares of order 63 ? Justify.

5. Consider an SBIBD with parameters (ν, κ, λ). Delete one block & all the elements of that block from the design. What can you say about the resulting design?

6.(a) Show that the number 5 does not have the $R(3,3)$ Ramsey property.

(b) Find the Ramsey number $R(2,7)$.

7. Find the number of onto functions from a set with 6 elements to a set with 4 elements.

***

INDIAN STATISTICAL INSTITUTE
M. MATH. FIRST YEAR
M. STAT. SECOND YEAR
General Topology

Date: 8.9.08 Mid-Semestral Examination Time : 2 hrs.

This paper carries 50 marks.The maximum you can score is 40.

1. (a) Let X be a Hausdorff space. Show that $\{(x, x) : x \in X\}$ is closed in $X \times X$. [5]

   (b)X, Y are Hausdorff spaces and f, g are continuous functions from X into Y. Show that $\{x : f(x) = g(x)\}$ is closed in X. [7]

2. Let X be completely regular. Suppose A, B are disjoint closed subsets of X and A is compact. Show that there is a continuous function $f : X \to [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$. [15]

3. Let (X, d) be a metric space.

   (a) Let $A \subset X$. Show that $closure(A) = \{x : d(x, A) = 0\}$ [4]

   (b) Show that any closed subset of a metric space is the intersection of countably many open sets. [7]

4. Let X be a compact Hausdorff space. Show that the intersection of countably many dense open subsets of X is dense in X. [12]

Indian Statistical Institute

Mid-semsetral examination : (2008-2009)

M Math. I year

Complex Analysis

Date : 12.9.08 Maximum marks : 80 Duration : 3 hours.

Answer all the questions. Marks are indicated in brackets.

For a domain (open connected subset of the complex plane) D, we have denoted by $\mathcal{H}(D)$ the set of holomorphic functions on D.

(1) Let f be a function with an isolated singularity at a point c, and assume that the function $\mathrm{Re}(f)$ is bounded above in a neighbourhood of c. Prove that c must be a removable singularity. [20]

(2) Let D be the open unit disc around 0 and $\overline{D}$ be its closure. Suppose that f, g are two continuous complex-valued functions on $\overline{D}$ such that they are holomorphic in D and have no zeros in D. Moreover, assume that $|f(z)| = |g(z)|$ for all $|z| = 1$. Prove that there exists a complex number λ of unit modulus such that $f(z) = \lambda g(z)$ for all $z \in D$. [20]

(3) Let D be a domain, $f, g \in \mathcal{H}(D)$ be such that $f(g(z)) = 0$ for all $z \in D$. Prove that either f or g is identically constant on D. [20]

(4) How many solutions (counting multiplicities) does the following equation have in the annulus $\{z : 1 < |z| < 2\}$ (give arguments in support of your answer)? [10]

$$z^5 + iz^3 - 4z + i = 0$$

(5) For the following functions, compute the residues at all the points of singularity (with proof):

$$(i)\ \ f(z) = \frac{1}{(z^2+1)(z-i)^3},$$

$$(ii)\ \ f(z) = \frac{1}{\exp(z)+1}.$$

[5+5=10]

Indian Statistical Institute

Semestral Examinations, (2008-2009)

Master of Mathematics, First Year

Graph Theory and Combinatorics

Date: November 19, 2008 Maximum Marks: 80 Time: 3 Hours

Attempt all questions. *The paper carries a total of 93 marks. Maximum you can score is 80. Figures in the right margin indicate the marks on the different parts of a question. All notations used are as defined in the class.*

1. Suppose T and T' are two minimum weight spanning trees of a graph G. Prove that T' can be transformed into T by a sequence of steps that exchange one edge of T' for one edge of T, such that the edge set is always a spanning tree and the total weight never increases. [12]

2. Prove that for any graph G,
$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$
where the symbols have their usual meanings. [10]

3. A graph is said to be *arbitrarily traversable from a vertex* v_0 if the following procedure always results in an eulerian trail: Start at v_0, by traversing any incident edge; on arriving at a vertex u depart by using any incident edge not yet used, and continue until no new edges remain.
Prove that, an eulerian graph is arbitrarily traversable from v_0 if and only if every cycle contains v_0. [12]

4. Prove that for every nontrivial connected graph G,
$$\alpha_0 + \beta_0 = p = \alpha_1 + \beta_1$$
where the symbols have their usual meanings. [10]

5. Prove that every planar graph G is 5 colorable. [12]

6. Prove that any simple graph has at least one pair of vertices having equal degree. For $p = 3$ and 4, find the degree sequences of all graphs having exactly one pair of vertices with equal degree. Generalize the result to graphs with n vertices. [1+5+9=15]

7. Let G be an undirected graph with $2n$ vertices, such that the degree of every vertex is at least n. Show that G has a perfect matching. [10]

8. Prove that very strong tournament with p vertices has a cycle of length n, n =3,4, …, p. [12]

INDIAN STATISTICAL INSTITUTE

First Semestral Examination: 2008-2009

M. Math. I Year

Algebra I

Date: November 24, 2008 | Maximum Marks: 60 | Duration: 3.5 hours

Note: There are 9 questions on this test. Answer ANY EIGHT. JUSTIFY your answers. Each question carries 8 marks.

1. Show that a group of order 30 cannot be simple. [8]

2. Suppose G is a group of order p^n where p is a prime number and $n \geq 1$. Show that its center $Z(G)$ is nontrivial. Use this to prove that a group of order 625 is solvable. [4+4]

3. Construct short exact sequences of groups that are [2+3+3]

   (a) left-split.

   (b) right-split but not left-split.

   (c) not right-split.

4. Construct a nonabelian group of order 12 which is not isomorphic to either the alternating group A_4 or the dihedral group D_{12}. [8]

5. Let R be a commutative ring with 1. If $R^m \simeq R^n$, then show that $m = n$. [8]

6. Let R be a commutative ring with 1. Let M be an R-module with annihilator I. Suppose that M is a Noetherian R-module. [5+3]

   (a) Show that R/I is a Noetherian ring.

   (b) Does it follow that R is also a Noetherian ring?

7. Let R be a ring with 1. Define the tensor product $M \otimes_R N$ of two R-modules M and N. Show that it exists. Compute $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z}$. [2+4+2]

8. Let M be a finitely generated module over a P.I.D. and N a submodule of M. Show that $\text{rank}(M) = \text{rank}(N) + \text{rank}(M/N)$. [8]

9. Let V be a finite dimensional vector space and $T : V \to V$ be any linear transformation. Show that there exists a basis for V in which the matrix representing T has rational canonical form. [8]

Indian Statistical Institute

First semsetral examination : (2008-2009)

M. Math I year

Complex Analysis

Date : 28.11.08 Maximum marks : 60 Duration : 3 hours.

Answer **ALL** questions. Marks are indicated in brackets.

(1) Prove or disprove (with arguments): there exists a holomorphic function f defined on a nonempty open connected subset D of $\mathbb{C}$ satisfying $\mathrm{Re}(f)(x+iy) = x - 2y^2$ for all $x, y \in \mathbb{R}$ such that $x + iy \in D$. [10]

(2) Prove that the closure of the range of an entire function must be either a singleton set or the whole of $\mathbb{C}$. [10]

(3) Let $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ be two sequences of complex numbers such that $|a_n| \to \infty$ as $n \to \infty$. Prove that there exists an entire function f satisfying $f(a_n) = b_n$ for every n. [15]

(4) Let H denote the upper half plane in $\mathbb{C}$, i.e. $H = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, and D be the open unit disc. Suppose that $f : H \to D$ be a conformal (i.e. holomorphic bijective) map. Prove that f must be of the following form for some $\theta \in \mathbb{R}$ and $\lambda \in H$:

$$f(z) = e^{i\theta} \frac{z - \lambda}{z - \bar{\lambda}}.$$

[10]

(5) Let D be an open connected subset of $\mathbb{C}$ and $\mathcal{F}$ be a family of analytic functions on D such that for every $f \in \mathcal{F}$, we have either $\mathrm{Re}(f)(z) > 0 \ \forall z \in D$ or $\mathrm{Re}(f)(z) < 0 \ \forall z \in D$. Show that $\mathcal{F}$ is normal, i.e. pre-compact in the topology of $\mathcal{M}(D)$ (meromorphic functions on D) discussed in class. [15]

INDIAN STATISTICAL INSTITUTE
M. MATH. FIRST YEAR
M. STAT. SECOND YEAR
General Topology
Date: 1.12.08 Semestral Examination Time : 3 hrs.

This paper carries 70 marks.The maximum you can score is 60.

1. (a) If A, B are connected subsets of a topological space X and $A \cap B \neq \emptyset$, show that $A \cup B$ is connected. [4]

   (b) X, Y are connected topological spaces and A, B are proper subsets of X, Y respectively. Show that $(X \times Y) \setminus (A \times B)$ is connected. [8]

2. Let $\{A_i : i \in \mathbf{I}\}$ be a locally finite family of infinitely many compact subsets of a Hausdorff space X with $\bigcup A_i = X$. Show that X is locally compact but not compact. [10]

3. Show that a compact manifold is second countable. [15]

4. (a) Let G be a topological group with identity element e. Let U be an open neighbourhood of e. Show that there is an open neighbourhood V of e such that $\{xy^{-1} : x, y \in V\} \subset U$. [5]

   (b) Show any topological group is regular. [8]

5. Let E be a covering space for B. Show that E is a manifold if and only if B is a manifold. [10]

6. Let $X \subset \mathbf{R}^3$ be the union of two spheres with one common point. What is the fundamental group of X ? Justify your answer. [10]

Indian Statistical Institute
Semester 1 (2008-2009)
M. Math. 1st Year
Semestral Examination
Measure Theory

Date and Time: **9.1.09** , 10:30 - 1:30 Total Points: $5 \times (7 + 7) = 70$

Answers must be justified with clear and precise arguments. If you use any theorem/proposition proved in class state it explicitly. All functions are assumed measurable unless stated otherwise.

1. (a) Let f be a real valued measurable function defined on $(\mathbb{R}^2, \mathcal{M})$ where $\mathcal{M}$ denotes measurable sets wrt the Lebesgue measure on $\mathbb{R}^2$. If B is a Borel set of $\mathbb{R}$ then then show that $f^{-1}(B)$ is measurable.

   (b) For each Borel subset B of $\mathbb{R}$ consider the set $\{(x, y) : x - y \in B\}$. Show that this is a measurable subset of $\mathbb{R}^2$.

2. (a) Let f be integrable on $\mathbb{R}$. Then show that $\int_{-\infty}^{\infty} |f(x + 1/n) - f(x)|dx \to 0$, as $n \to \infty$. (Warning: f is only measurable and cannot be assumed continuous.)

   (b) Let $(X, \mathcal{B}, \mu)$ be a measure space and f_n be a sequence of nonnegative, measurable and integrable functions defined on X such that $f_n \to f$ a.e. wrt μ and $\int_X f_n d\mu \to \int_X f d\mu < \infty$. Then show that for each $E \in \mathcal{B}$ we have $\int_E f_n d\mu \to \int_E f d\mu$.

3. Let $(X, \mathcal{B}, \mu)$ be a measure space and f_n be a sequence of functions in $L^p(\mu)$, $1 \le p < \infty$, which converge a.e. wrt μ to a function f in $L^p(\mu)$. Recalling that the $L^p(\mu)$ norm is given by $||f||_p = \{\int_X |f|^p d\mu\}^{1/p}$, show that the following are equivalent.

   (a) f_n converges to f in $L^p(\mu)$,

   (b) $||f_n||_p \to ||f||_p$.

4. (a) Suppose μ and ν are measures on $(X, \mathcal{B})$ where μ is σ-finite and $\nu \ll \mu$. Assume the **Radon-Nikodym theorem for μ finite and** $\nu \ll \mu$, i.e. if μ is finite and $\nu \ll \mu$ then there is a measurable f such that $\nu(E) = \int_E f d\mu$ for $E \in \mathcal{B}$. Explain how this extends to the case of σ-finite μ.

   (b) Let μ, ν and λ be σ-finite. Show that the Radon-Nikodym derivative $d\nu/d\mu$ satisfies the following property: if $\nu \ll \mu \ll \lambda$, then $d\nu/d\lambda = (d\nu/d\mu)(d\mu/d\lambda)$.

5. (a) Use Fubini's theorem and the relation $1/x = \int_0^\infty e^{-xt}dt, x > 0$, to prove that $\lim_{A \to \infty} \int_0^A (\sin x/x)dx = \pi/2$. (Other methods of proving this will not be accepted.)

   (b) This problem is a continuation of part (b) of problem 1. Suppose E is a subset of measure zero of $\mathbb{R}$. Show that $\{(x, y) : x - y \in E\}$ is a measurable subset of $\mathbb{R}^2$. (Hint: Transform the axis by $u = (x - y)/\sqrt{2}, v = (x + y)/\sqrt{2}$ and show that in this new axis $u \in E/\sqrt{2}$ and v is arbitrary implies something about the measure of the required set.)

Indian Statistical Institute
Mid-Semestral Examination — February 2009
M.Math First Year Class Semester II (2008- 2009)
Subject: Differential Geometry

Marks 60 Date: 23.2.09 Time 3 hours

Answer all the questions, which carry equal marks

1. A curve is defined by

$$\overline{x}(t) = a \int \overline{f}(t) \times \overline{f}'(t)\, dt, \quad a = \text{constant} \neq 0,$$

where $\overline{f}(t)$ is a vector valued function with norm $|\overline{f}(t)| = 1$, scalar triple product $[\overline{f}(t), \overline{f}'(t), \overline{f}''(t)] \neq 0$ for all t. Show that the curvature and the torson functions are given respectively by

$$\kappa = \frac{|[\overline{f}, \overline{f}', \overline{f}'']|}{a|\overline{f}'|^3}, \quad \text{and } \tau = \frac{1}{a}.$$

2. Show that a curve is a general helix if and only if $\tau/\kappa =$ constant where $\kappa \neq 0$, and $\tau = 0$ whenever $\kappa = 0$.

3. Show that if a curve $\overline{x} = \overline{x}(s)$ lies on a sphere of radius a, then

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{\dot{\kappa}}{\kappa^2\tau}\right)^2 = a^2.$$

4. Let S be the cylinder $x_1^2 + x_2^2 = r^2$ of radius $r > 0$ in $\mathbb{R}^3$. Show that a curve σ is a geodesic of S if and only if σ is of the form

$$\sigma(t) = (r\cos(at+b), r\sin(at+b), ct+d),$$

for some $a, b, c, d \in \mathbb{R}$. Draw diagrams to depict the geodesics.

5. Let S be a 2-surface in $\mathbb{R}^3$, and $\sigma : I \longrightarrow S$ be a geodesic in S with $\dot{\sigma} \neq 0$. Show that a vector field $\overline{X}$ tangent to S long σ is parallel along S if and only if both $|\overline{X}|$ and the angle between $\overline{X}$ and $\dot{\sigma}$ are constant along σ.

6. Let S^2 denote the 2-sphere in $\mathbb{R}^3$, $p \in S^2$, and v, w be any two vectors in the tangent space to S^2 at p such that $|v| = |w|$. Show that there is a piecewise smooth parametrized curve σ in S^2 from p to p such that the parallel transport along σ takes v to w.

INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination

M. Math (First year), Second Semester

2008–2009

Algebraic Topology

Date: 27 February , 2009

Maximum Marks: 30

Duration: 2 hours

(1) Attempt any one of the following questions: 10

(a) Define a simplicial complex K whose polytope is homeomorphic with the real projective plane. Prove that the first homology group $H_1(K)$ is non-zero while $H_2(K)$ vanishes.

(b) Consider a simplicial complex L which consists of the following 2-simplexes and all their faces:

$$< a, b, f >, \ < b, c, d >, \ < f, d, e >$$

Compute the homology groups of L.

(2) Does there exist a simplicial complex K whose polytope $|K|$ is homeomorphic with the real line with its usual topology? Justify your answer. 6

(3) Let $\mathbb{Q}$ denote the set of rationals with the subspace topology. Does there exist a simplicial complex K such that $|K| = \mathbb{Q}$? Justify your answer. 4

(4) Let H_q stand for the q-th singular homology functor.

(a) Suppose P is a one point space. Find the singular homology groups of P.

(b) Suppse X is a topological space and $x \in X$. Prove that

$$H_q(X) \cong H_q(X, x) \text{ for } q > 0,$$

What happens when $q = 0$? 4+6

INDIAN STATISTICAL INSTITUTE
Semester Examination: (2008-09)

M. MATH. I YEAR
Algebra II

Date : 2.3.2009 | Maximum Marks : 60 | Duration : $3\frac{1}{2}$ Hours

Answer ANY FIVE questions.
k will denote a field.

1. Let $A = \mathbb{C}[X, Y, Z]/(XY - Z^2)$. Show that
(i) $(\bar{X} - 1)$ is a prime ideal of A which is not maximal.
(ii) $(\bar{X} - 1, \bar{Z} - 1)$ is a maximal ideal of A.
(iii) For $\lambda \in \mathbb{C}$, $\bar{Y} - \lambda \in (\bar{X} - 1, \bar{Z} - 1)$ if and only if $\lambda = 1$. [4+4+5=13]

2. (i) Show that $k[X]$ has infinitely many monic irreducible polynomials. Deduce that over any non-zero commutative ring R (with 1), $R[X]$ has infinitely many maximal ideals.
(ii) If t is an element in a field extension of k such that t is transcendental over k, then show that the polynomial $X^n - t$ is irreducible in $k(t)[X]$. [7+6=13]

3. (i) Let I be a proper ideal of the polynomial ring $k[X]$. Show that every element in $k[X]/I$ is either a unit or a zero-divisor.
(ii) If P is a prime ideal in a k-algebra A such that A/P is a finite-dimensional vector space over k, then show that P must be a maximal ideal of A. [6+7=13]

4. Let R be a UFD with field of fractions $K (\neq R)$. For $a \in R$, define $A = R[X]/(X^2 - a)$.
(i) Show that A is an integral domain if and only if a is not a square in R.
(ii) If A is an integral domain with field of fractions L, then show that L is a field extension of K and compute $[L : K]$.
(iii) Show that K cannot be an algebraically closed field. [5+4+4=13]

5. Let $L_1 = \mathbb{Q}(i)$, $L_2 = \mathbb{Q}(\omega)$ and $L = \mathbb{Q}(i, \omega)$, where ω is a non-real cube root of unity. Let $f(X) = X^{12} - 1 \in \mathbb{Q}[X]$ and $A = \mathbb{Q}[X]/(f(X))$.
(i) Show that $L = \mathbb{Q}(i\omega)$ and that L is a splitting field of $f(X)$ over $\mathbb{Q}$. Write down all the conjugates of $i\omega$ in L and show that one of them equals $cos(\frac{\pi}{6}) + i\ sin(\frac{\pi}{6})$.
(ii) Show that A is isomorphic, as a ring, to the direct product of fields $\mathbb{Q} \times \mathbb{Q} \times L_1 \times L_2 \times L_2 \times L$. Identify a 6-tuple $(a_1, a_2, a_3, a_4, a_5, a_6)$ in the product $\mathbb{Q} \times \mathbb{Q} \times L_1 \times L_2 \times L_2 \times L$ which corresponds to the element $\bar{X}$ of A under an isomorphism. [7+6=13]

6. (i) Determine a generator of the cyclic group $\mathbb{F}_{13}{}^{\star}$.
(ii) Show that if α is a root of the polynomial $f(X) = X^{13} - X - 1 \in \mathbb{F}_{13}[X]$, then $\mathbb{F}_{13}(\alpha)$ is the splitting field of $f(X)$ over $\mathbb{F}_{13}$.
(iii) Prove that the polynomial $Y - (X^{13} - Y^{12})^{13}$ is irreducible in $\mathbb{F}_{13}[X, Y]$. [3+5+5=13]

Indian Statistical Institute

Mid-semsetral examination : (2008-2009)

M. Math I year

Functional Analysis

Date : 4.3.09 Maximum marks : 60 Duration : 2 hours.

Answer any THREE questions. Each question carries 20 marks.

(1) Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M} \subseteq \mathcal{H}$ be a closed subspace. Prove that any bounded linear functional f on $\mathcal{M}$ has a unique norm-preserving extension on $\mathcal{H}$, i.e. there is a unique bounded linear functional $\tilde{f}$ on $\mathcal{H}$ satisfying $\tilde{f}|_{\mathcal{M}} = f$ and $\|\tilde{f}\| = \|f\|$. Give a counterexample to show that the uniqueness of norm-preserving extension is no longer true if the Hilbert space is replaced by a general Banach space.
[10+10=20]

(2) Prove that $L^2[0,1]$ is a subset of first category in $L^1[0,1]$, i.e. there is a countable family A_n, $n = 1, 2, \ldots$ of closed subsets of $L^1[0,1]$ with empty interior (w.r.t. the topology of L^1) such that $L^2[0,1] = \bigcup_{n=1}^{\infty} A_n$.
Hint: Consider $A_n = \{f : \int_0^1 |f(t)|^2 dt \leq n\}$ [20]

(3) Let X be a Banach space such that the dual X^* is reflexive, i.e. the canonical isometry from X^* into X^{***} is surjective. Prove that X is reflexive. [20]

(4) Let (S,d) be a metric space. We say that a real or complex valued function f on S is Lipschitz if there is a constant C such that $|f(x) - f(y)| \leq Cd(x,y)$ for all $x, y \in S$. Now, let X be a (real or complex) normed linear space and $F : S \to X$ be a function such that for every bounded linear functional ϕ on X, the function $\phi \circ F$ is Lipschitz. Prove that there exists a constant M such that $\|F(x) - F(y)\| \leq Md(x,y)$ for all $x, y \in S$. [20]

Indian Statistical Institute

Second semestral examination : (2008-2009)
M. Math I year
Functional Analysis

Date : 4.5.09 Maximum marks : 70 Duration : 3 hours.

Answer all the questions. The maximum you can score is 70. Marks are indicated in brackets.

(1) Let $\mathcal{H}$ be a Hilbert space and T be a bounded operator on $\mathcal{H}$ such that $\|T\| < 1$. Prove that T can be written as $T = \frac{1}{2}(U + V)$, where U, V are two unitary operators.
(Hint: use the polar decomposition and the spectral theorem) [15]

(2) Let $\mathcal{H} = l^2(\mathbb{N})$, where $\mathbb{N}$ denotes the set of natural numbers, and let $\{e_n\}$ be the canonical orthonormal basis of $l^2(\mathbb{N})$. Let T be the shift operator given by $Te_n = e_{n-1}$, with $e_{-1} := 0$.
(i) Let $\mathcal{P}$ denote the subalgebra of $\mathcal{B}(\mathcal{H})$ generated by T and T^*, and let $C \in \mathcal{P}$. Show that C can be written as

$$C = c_0 I + p(T) + q(T^*) + K,$$

where c_0 is a complex number, p, q polynomials in one variable with complex coefficients, and K is a compact operator.
(Hint: compute the commutator $[T, T^*] := TT^* - T^*T$.)
(iii) Prove that the norm-closure of the subalgebra $\mathcal{P}$ cannot contain the operator S given by $Se_n = (-1)^n e_{n-1}$. [10+10=20]

(3) Let X, Y be Banach spaces and $T : X \to Y$ be a compact operator. Prove that, if $x_n \xrightarrow{w} 0$ weakly (i.e. $x_n \to 0$ w.r.t. the weak topology of X), then $\|Tx_n\| \to 0$. Is this result valid without the assumption of compactness of T (justify your answer)? [12+3=15]

(4) Let E be a convex subset of a Banach space X, which is dense in X in the weak topology. Prove that E must be norm-dense in X. [15]

(5) Prove that the Banach space $C[0,1]$, with the usual norm given by $\|f\| = \sup_{x \in [0,1]} |f(x)|$, is not reflexive. [10]

INDIAN STATISTICAL INSTITUTE
Mid-Semester Examination – Semester II : 2008-2009
M. Math. I Year
ADVANCED PROBABILITY

Date : 06.03.09 | **Maximum Score : 30** | Time : 3 Hours

Note : This paper has **five** question with **8 marks each**, as shown against the questions. You may answer as many as you can. The **maximum** you can score is **30 marks**.

1. Let X be a real random variable with probability distribution function F given by

$$F(x) = \begin{cases} 0 & \text{if} \quad x \leq 1 \\ (3x^2 + 6x + 4)/12 & \text{if} \quad -1 < x < 0 \\ 5/12 & \text{if} \quad 0 \leq x \leq 1 \\ (x+9)/24 & \text{if} \quad 1 < x < 2 \\ (2x-3)/(2x-2) & \text{if} \quad x \geq 2 \end{cases}$$

Find the following probabilities:
(a) $P[X = 1]$, (b) $P[X \leq 0]$,
(c) $P[|X| < 1]$, (d) $P[X$ is not an integer]. (2×4)=[8]

2. (a) Let $X_1, \ldots, X_n$ be n real random variables on a probability space. Explain what is meant by saying that $(X_1, \ldots, X_n)$ has a joint density?
(b) Let X be a real random variable. Argue clearly that if there is a non-negative borel measurable function f such that the probability distribution function of X is given by $F(a) = \int_{-\infty}^{a} f(x)dx \; \forall \, a$, then X has density f.
(c) With X as in (b), show that the real random variable $Y = e^{-|X|}$ has a density and express this density in terms of f. (2+2+4)=[8]

3. Given a probability distribution function F on $\mathbf{R}$, define X on $(0,1)$ as $X(\omega) = \inf\{x \in \mathbf{R} : F(x) > \omega\}$, $\omega \in (0,1)$. Show that X defines a real random variable on the probability space $((0,1), \mathcal{B}, \lambda)$ and that F is the probability distribution function of X. [Here $\mathcal{B}$ is Borel σ-field on $(0,1)$ and λ the Lebesgue measure.] (5+3)=[8]

4. (a) State the necessary and sufficient conditions for a function $F : \mathbf{R}^2 \to \mathbf{R}$ to be a probability duistribution function.
(b) Show that if H and G are two probability distribution functions on $\mathbf{R}$, then the function $F : \mathbf{R}^2 \to \mathbf{R}$ defined as $F(x,y) = \min\{H(x), G(y)\}$ is a probability distribution function with marginals H and G. (2+6)=[8]

5. (a) Prove that if $\{A_n, n \geq 1\}$ is a sequence of events in a probability space $(\Omega, \mathcal{A}, P)$ with $\sum_n P(A_n) < \infty$, then $P[\limsup_n A_n] = 0$.
(b) Let $\{X_n, n \geq 1\}$ be a sequence of random variables on a probability space $(\Omega, \mathcal{A}, P)$ with X_n having density function $f_n(x) = \begin{cases} 1/(2n^2) & \text{if } x \in [-n^2, n^2] \\ 0 & \text{otherwise} \end{cases}$.
Prove that $P[|X_n| \to \infty] = 1$. (4+4)=[8]

Indian Statistical Institute
Semester 1 (2008-2009)
M. Math. 1st Year
Backpaper Examination
Measure Theory

Date and Time: 12.3.09 Time: 3 hrs. Total Points: $15 \times 7 = 105$

The maximum you can score is 100 Answers must be justified with clear and precise arguments. If you use any theorem/proposition proved in class state it explicitly. All functions are assumed measurable unless stated otherwise.

Questions 1 to 4 involve Lebesgue measure only.

1. Let f be a nonnegative integrable function. Show that the function F defined by $F(x) = \int_{-\infty}^{x} f(x)dx$ is continuous.

2. (a) Consider the sequence f_n defined by $f_n(x) = 1$ if $n \leq x < n+1$, with $f_n(x) = 0$ otherwise. Does this sequence show that we may have strict inequality in Fatou's lemma?

   (b) Show that the monotone convergence theorem need not hold for decreasing sequence of functions by considering the sequence $f_n(x) = 0$ if $x < n$, $f_n(x) = 1$ if $x \geq n$. 7+8 = 15 pts.

3. Let f_n be a sequence of integrable functions such that $f_n \to f$ a.e. with f integrable. Then $\int |f_n - f| \to 0$ iff $\int |f_n| \to \int |f|$. (Hint: you may use $||f_n| - |f|| \leq |f_n - f|$.)

4. Let f be a bounded measurable function on $[0,1]$. Then show that $\lim_{p\to\infty} ||f||_p = ||f||_\infty$.

5. (a) Let $(X, \mathcal{B}, \mu)$ be a measure space and g a nonnegative measurable function on X. Set $\nu(E) = \int_E g d\mu$. Show that ν is a measure on $\mathcal{B}$.

   (b) Under the same set up let f be a nonnegative measurable function on X. Then show that $\int f d\nu = \int f g d\mu$. 7+8 = 15 pts.

6. Let $X = [0,1]$, $\mathcal{B}$ be the class of Lebesgue measurable subsets of $[0,1]$, ν be Lebesgue measure and μ be the counting measure on $\mathcal{B}$.

   (a) Is ν finite and absolutely continuous wrt μ? Is there a function f such that $\nu(E) = \int_E f d\mu$ for all $E \in \mathcal{B}$?

   (b) Depending on your answer to (a), discuss if μ and ν give an example of the Radon-Nikodym theorem or point out which condition for the validity of Radon-Nikodym theorem is violated by μ and ν.

   10+5 = 15 pts.

7. (a) Let $X = Y = [0,1]$ with $\mathcal{A} = \mathcal{B}$ the class of Borel sets. Let μ be Lebesgue measure and ν the counting measure. Consider the measure space $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu)$. Then show that the diagonal $\Delta = \{(x,y) : x = y\}$ is measurable, but its indicator function fails to satisfy any of the equalities between iterated and product integrals.

   (b) Which conditions of the Tonelli theorem are violated in the above example? 10 + 5 = 15 pts.

SEMESTER EXAMINATIOM
M.Math. First Year Class
Semester II (2008-2009)
Subject. Differential Geometry

Full Marks: 70 11 May, 2009 Time: 3 hours

Answer any FIVE questions, each of which carries equal marks.

1. Let S be an n-surface in $\mathbb{R}^{n+1}$ oriented by the unit normal vector field $\overline{N}$. Suppose that $\overline{X}$ and $\overline{Y}$ are smooth tangent vector fields on S.

(a) Show that

$$(\nabla_{\overline{X}(p)}\overline{Y}) \cdot \overline{N}(p) = (\nabla_{\overline{Y}(p)}\overline{X}) \cdot \overline{N}(p)$$

for all $p \in S$.

(b) Conclude that the Lie bracket vector field $[\overline{X}, \overline{Y}]$ on S defined by

$$[\overline{X}, \overline{Y}](p) = \nabla_{\overline{X}(p)}\overline{Y} - \nabla_{\overline{Y}(p)}\overline{X}, \ p \in S,$$

is everywhere tangent to S.

2. Show that on each compact oriented n-surface S in $\mathbb{R}^{n+1}$ there exists a point $p \in S$ such that the second fundamental form at p is definite.

3. Let $\phi : U \longrightarrow \mathbb{R}^{n+1}$, $U \subset \mathbb{R}^n$ open, be a parametric n-surface in $\mathbb{R}^{n+1}$. Show that the Gauss-Kronecker curvature at $p \in U$ of the n-surface has absolute value

$$|K(p)| = \lim_{\epsilon \to 0} \frac{\text{Vol}(N|B_\epsilon)}{\text{Vol}(\phi|B_\epsilon)},$$

where B_ϵ is the open ball of radius ϵ about p contained in U, and $N : U \longrightarrow S^n$ is the Gauss map (S^n is the n-sphere in $\mathbb{R}^{n+1}$).

4. Find a parametric representation of a compact 2-surface of genus one in the form

$$\Phi(\phi, \theta) = \big((b + a\cos\phi)\cos\theta, (b + a\cos\phi)\sin\theta, a\sin\phi\big),$$

where (ϕ, θ) are the spherical coordinates.

Show that the principal curvatures of the surface S are given by

$$-\frac{\cos\phi}{b + a\cos\phi}, \quad -\frac{1}{a}.$$

Conclude without going into direct computations that

$$\int\int_S \frac{\cos\phi}{a(b + a\cos\phi)} dA = 0.$$

P.T.O

5. Show that if S is a compact surface, there is a point $p \in S$ at which the Gaussian curvature $K(p)$ is positive.

6. (a) Show that if a surface patch ϕ has Gaussian curvature ≤ 0 everywhere, then there is no simple closed geodesic in ϕ.

(b) Show that if a compact surface S has Gaussian curvature > 0 everywhere, then S is diffeomorphic to a sphere.

(c) Show that the Euler number χ of a compact surface of genus g is given by

$$\chi = 2 - 2g.$$

INDIAN STATISTICAL INSTITUTE

Semestral Examination – Semester II : 2008-2009

M. Math. I Year

ADVANCED PROBABILITY

Date : 15.05.09 **Maximum Score : 70** Time : $3\frac{1}{2}$ Hours

Note : This paper has six questions that carry a total of **82 marks**. The marks on each question are as shown. You may answer as many as you can. The **maximum** you can score is **70 marks**.

1. (a) Let F and G be two probability distribution functions on $\mathbb{R}$. Assuming usual notations, show that the function H defined on $\mathbb{R}$ by $H(z) = \int_{\mathbb{R}} F(z+x)dG(x)$, for $z \in \mathbb{R}$, is a probability distribution function.

   (b) Show that, in (a) above, if F is continuous everywhere, then so is H and that if F has a density function, then so does H. (6+6)=[12]

2. Let $(\Omega, \mathcal{A}, P)$ be a probability space.

   (a) If $\mathcal{C}_1, \ldots, \mathcal{C}_n$ are non-empty subclasses of $\mathcal{A}$, what is meant by saying that $\mathcal{C}_1, \ldots, \mathcal{C}_n$ are independent?

   (b) Show that if $\mathcal{C}_1, \ldots, \mathcal{C}_n$ are semi-fields contained in $\mathcal{A}$, then $\sigma(\mathcal{C}_1), \ldots, \sigma(\mathcal{C}_n)$ are independent if and only if $\mathcal{C}_1, \ldots, \mathcal{C}_n$ are independent.

   (c) What is meant by saying that a sequence $\{\mathcal{A}_n, n \geq 1\}$ of sub-σ-fields of $\mathcal{A}$ are independent? Show that if $\{\mathcal{A}_n, n \geq 1\}$ are independent, then the two σ-fields $\mathcal{B}_1 = \sigma\langle \bigcup_{n\geq 1} \mathcal{A}_{2n}\rangle$ and $\mathcal{B}_2 = \sigma\langle \bigcup_{n\geq 1} \mathcal{A}_{2n+1}\rangle$ are independent. (2+6+(2+5))=[15]

3. Let $\mathcal{B}^\infty = \otimes \mathcal{B}$ denote the product σ-field on $\mathbb{R}^\infty = \prod \mathbb{R}$.

   (a) Show that $\mathcal{B}^\infty$ is the smallest σ-field on $\mathbb{R}^\infty$ such that all the coordinate projections $\pi_n : \mathbb{R}^\infty \to \mathbb{R}$, $n \geq 1$, are measurable.

   (b) Let $(\Omega, \mathcal{A})$ be a measurable space and $\{f_n, n \geq 1\}$ a sequence of real-valued functions on Ω. Show that $f : (\Omega, \mathcal{A}) \to (\mathbb{R}^\infty, \mathcal{B}^\infty)$ defined by $f(\omega) = (f_1(\omega), f_2(\omega), \ldots)$ is measurable if and only if each $f_n : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B})$ is measurable. (6+6)=[12]

4. Let $\{X_n\}$ be a sequence of real random variables on a probability space.

   (a) Let F_n, for each n, be the probability distribution function of X_n. Show that if $\sum_{n\geq 1} (1 - F_n(a) < \infty$ for some a, then $P(\sup_n X_n < \infty) = 1$.

   (b) Show that the converse of part (a) holds in case the random variables $\{X_n\}$ are independent. (6+6)=[12]

5. (a) Define the "tail σ-field" for a sequence $\{X_n\}$ of random variables defined on a probability space. Determine, for each of the folllowing events, whether it is a tail event or not: (i) $\{\omega : \sum X_n(\omega)$ converges $\}$, (ii) $\{\omega : X_{n+1}(\omega) > X_n(\omega)$ for some $n\}$.

   (b) State Kolmogorov's Zero-One Law.

   (c) Show that if $\{X_n\}$ is a sequence of independent random variables, then the probability of the event $\{\{X_n\}$ converges $\}$ is either 0 or 1. Show that in case this probability is 1, then $\lim_{n\to\infty} X_n$ is a degenerate random variable. ((2+2+2)+2+8)=[16]

6. (a) If $(\Omega, \mathcal{A}, P)$ is a discrete probability space and X_n, $n \geq 1$ and X are random variables on $(\Omega, \mathcal{A}, P)$, then show that $X_n \xrightarrow{P} X$ implies $X_n \xrightarrow{\text{a.s.}} X$.

   (b) State the Strong Law of Large Numbers for a sequence of i.i.d. integrable random variables. Use an appropriate Strong Law of Large Numbers to find the limit $\lim_{n\to\infty} \int_0^1 \cdots \int_0^1 \left(\frac{x_1+\cdots+x_n}{n}\right)^4 dx_1 \ldots dx_n$. (7+(2+6))=[15]

INDIAN STATISTICAL INSTITUTE

Second Semestral Examination: 2008-2009

M. Math I Year

Algebra II

Date: May 18, 2009 **Maximum Marks: 70** **Duration: $3\frac{1}{2}$ hours**

1. Let L be an algebraic extension of a field k. Show that any k-endomorphism of L is an automorphism. Is the result true if L is not algebraic over k ? [10]

2. Let $f(X) = X^6 + 1 \in \mathbb{Q}[X]$. Compute the Galois group of $f(X)$. Describe all subfields of the splitting field of $f(X)$. [15]

3. Let k be a field and L be a finite Galois extension of k. Suppose that p is a prime divisor of $[L : k]$. Does there necessarily exist an intermediary field F ($k \subset F \subset L$) such that $[F : k] = p$? Justify your answer. [10]

4. Let k be a field and $f(X)$ be an irreducible and separable polynomial in $k[X]$. Assume that the Galois group of f over k is *abelian*. Let E be a splitting field of f over k and let $\alpha_1, \cdots, \alpha_n$ be the roots of f in E. Show that $E = k(\alpha_i)$ for any i. Clearly state the results you used. [10]

5. Let k be a field. Let $f(X)$ be a monic irreducible polynomial in $k[X]$ and let L be a finite normal extension of k. Suppose that $f(X) = g(X)h(X)$, where $g(X), h(X) \in L[X]$ and $g(X), h(X)$ are monic and irreducible in $L[X]$. Show that there exists a k-automorphism σ of L such that when σ is extended to $L[X]$ by defining $\sigma(X) = X$, we have $\sigma(g(X)) = h(X)$. [12]

6. Let k be a field and $f(X), g(X)$ be coprime in $k[X]$. Prove that

$$[k(t) : k(\frac{f(t)}{g(t)})] = \max\{\deg f, \deg g\}$$

[12]

7. (a) Let n be a positive integer and $\Phi_n(X)$ denote the nth cyclotomic polynomial over $\mathbb{Q}$. Prove that $\Phi_n(X) \in \mathbb{Z}[X]$.

   (b) Prove that the degree of the cyclotomic field extension $\mathbb{Q}(\zeta_n)$ is $\varphi(n)$, where φ is the Euler φ-function.

   (c) Prove that if the regular n-gon can be constructed by straightedge and compass, then $n = 2^k p_1 \cdots p_r$, where $k \geq 0$ and p_i are distinct Fermat primes.

[15]