APPLICATIONS OF EXPONENTIAL MAPS TO EPIMORPHISM AND CANCELLATION PROBLEMS

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DECLARATION

The work presented in this thesis has been carried out by me under the supervision of Professor Neena Gupta of Indian Statistical Institute, Kolkata.

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Date: 13/05/24

To

Baba, Maa, Baro Mama and Bapi dada

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Notation

$\mathbb{N}$	:	Set of Natural Numbers.
$\mathbb{Z}$	:	Ring of Integers.
$\mathbb{Q}$	:	Field of Rational numbers.
$\mathbb{R}$	:	Field of Real numbers.
$\mathbb{C}$	:	Field of Complex numbers.
DVR	:	Discrete Valuation Ring.
PID	:	Principal Ideal Domain.
UFD	:	Unique Factorization Domain.

For a commutative ring R, a prime ideal p of R and an R-algebra A, the following notation will be used:

R^*	:	Group of units of R.
$R^{[n]}$	:	Polynomial ring in n variables over R.
$\mathrm{Spec}(R)$	:	The set of all prime ideals of R.
$\mathrm{ht}(p)$	:	Height of p.
$k(p)$	:	Residue field R_p/pR_p.
A_p	:	$S^{-1}A$ where $S = R \backslash p$; also identified with $A \otimes_R R_p$.

For integral domains $R \subseteq A$,

$\mathrm{tr.deg}_R(A)$	:	Transcendence degree of the field of fractions of A over that of R.

Contents

Chapter 1

Introduction

Aim

Throughout this thesis k will always denote a field. The main aims of this thesis are the following:

(a) To study "Generalised Asanuma varieties" and deduce Epimorphism results for a certain family of linear hyperplanes over fields of arbitrary characteristic.

(b) To determine isomorphism classes and automorphisms of Generalised Asanuma varieties and use the classification to demonstrate an infinite family of pairwise non-isomorphic varieties which are counter examples to the Zariski Cancellation Problem (ZCP) in higher dimensions (≥ 3) and in positive characteristic.

(c) To study Generalised Danielewski varieties in higher dimensions and in arbitrary characteristic and use the results to determine some invariants (*Derksen* and *Makar-Limanov* invariants) of some subfamilies of Generalised Asanuma varieties and to demonstrate a new infinite family of counterexamples to the General Cancellation Problem in arbitrary characteristic.

In Chapter 3 we discuss (a) under the heading "Triviality of a family of linear hyperplanes" and in Chapter 4 we discuss (b) under the title "An infinite family of higher dimensional counterexamples to ZCP". In Chapter 5, entitled "Generalised Danielewski varieties and invariants of generalised Asanuma varieties", we will study Danielewski varieties in a more general set up and thereby provide a new family of counterexamples to the General Cancellation Problem.

An overview of Chapters 3,4,5 and the main results are given below.

Overview

I. Triviality of a family of linear hyperplanes (Chapter 3)

We recall the Epimorphism Problem, one of the fundamental problems in the area of Affine Algebraic Geometry (cf. [15], [25]):

Question 1. If $\frac{k[X_1,\ldots,X_n]}{(H)} = k^{[n-1]}$, then is $k[X_1,\ldots,X_n] = k[H]^{[n-1]}$?

The famous Epimorphism Theorem of S.S. Abhyankar and T. Moh ([2]), also proved independently by M. Suzuki ([38]) for $k = \mathbb{C}$, provides an affirmative answer to Question 1 when k is a field of characteristic zero and $n = 2$. The Abhyankar-Sathaye Conjecture asserts an affirmative answer to Question 1 when k is of characteristic zero and $n > 2$; and this remains a formidable open problem in Affine Algebraic Geometry.

When k is a field of positive characteristic, explicit counterexamples to Question 1 had already been demonstrated by B. Segre ([36]) in 1957 and M. Nagata ([31]) in 1971. However, when the hyperplane H is of some specified form, it is possible to obtain affirmative answers to Question 1 even when k is of arbitrary characteristic. Thus, the Abhyankar-Sathaye Conjecture could be extended to fields of arbitrary characteristic for certain special cases of H.

The first (partial) affirmative solution to Question 1 was obtained for the case $n = 3$ and H a linear plane, i.e., linear in one of the three variables, by A. Sathaye ([35]) in characteristic zero and P. Russell ([32]) in arbitrary characteristic. They also proved that if $A = k^{[2]}$ and the hyperplane $H \in A[Y](= k^{[3]})$ is of the form $aY + b$, where $a, b \in A$, then the coordinates X, Z of A can be chosen such that $A = k[X,Z]$ with $a \in k[X]$; that is, linear planes were shown to be of the form $a(X)Y + b(X,Z)$.

For the case $n = 4$ and $k = \mathbb{C}$, S. Kaliman, S. Vènèreau and M. Zaidenberg proved the Abhyankar-Sathaye Conjecture for certain linear hyperplanes in $\mathbb{C}[X_1,X_2,Y,Z]$ of the type $a(X_1)Y + b(X_1,X_2,Z)$ and $a(X_1,X_2)Y + b(X_1,X_2,Z)$ under certain hypotheses ([26]). A general survey on other partial affirmative answers to the Abhyankar-Sathaye Conjecture has been made in [15, Section 2].

We consider certain types of linear hyperplanes in higher dimensions which arose out of investigations on the ZCP in [23]. We present the genesis below.

Consider a ring of the form

$$A := \frac{k\,[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T))}, \quad r_i > 1 \text{ for all } i, 1 \leqslant i \leqslant m, \tag{1.0.1}$$

where $F(0, \ldots, 0, Z, T) \neq 0$. Set $f(Z,T) := F(0, \ldots, 0, Z, T)$. Let $x_1, \ldots, x_m, y, z, t$ denote the images in A of $X_1, \ldots, X_m, Y, Z, T$ respectively. We shall call a variety defined by a ring of type (1.0.1) as a "Generalised Asanuma variety". In [3], T. Asanuma had constructed three dimensional rings of the above type as an illustration of non-trivial $\mathbb{A}^2$-fibrations (cf. Definition 2.1.2) over a PID not containing $\mathbb{Q}$. The original ring of Asanuma is obtained from (1.0.1) by taking $m = 1$, k a field of positive characteristic p and $F = Z^{p^e} + T + T^{sp}$, where e, s are positive integers such that $p^e \nmid sp$ and $sp \nmid p^e$. Now suppose $F = f(Z,T)$, where f is a *line* in $k[Z,T]$, i.e., $\frac{k[Z,T]}{(f)} = k^{[1]}$. For each integer $m \geqslant 1$, N. Gupta established the following two properties of the integral domain A under the above hypothesis ([23, Theorem 3.7]):

(a) $A^{[1]} = k^{[m+3]}$.

(b) If $A = k^{[m+2]}$ then $k[Z,T] = k[f]^{[1]}$.

Earlier, she had investigated the ring A for $m = 1$ and had shown that the condition (b) holds (i.e., $A = k^{[3]}$ implies $k[Z,T] = k[f]^{[1]}$) even without the hypothesis that $f(Z,T)$ is a line in $k[Z,T]$. In fact, she had proved the following general result [22, Theorem 3.11]:

Theorem A. *Let k be a field and $A' = \frac{k[X,Y,Z,T]}{(X^rY - F(X,Z,T))}$, where $r > 1$. Let x be the image of X in A'. If $G := X^rY - F(X,Z,T)$ and $f(Z,T) := F(0,Z,T) \neq 0$, then the following statements are equivalent:*

(i) $k[X,Y,Z,T] = k[X,G]^{[2]}$.

(ii) $k[X,Y,Z,T] = k[G]^{[3]}$.

(iii) $A' = k[x]^{[2]}$.

(iv) $A' = k^{[3]}$.

(v) $k[Z,T] = k[f]^{[1]}$.

Note that the equivalence of (ii) and (iv) above establishes a special case of the Abhyankar-Sathaye Conjecture for linear hyperplanes in $k^{[4]}$.

In view of the importance of Theorem A, A. K. Dutta had asked whether similar results as Theorem A hold over Generalised Asanuma varieties when $m > 1$, in particular:

Question 2. For $m > 1$, is the condition $k[Z,T] = k[f]^{[1]}$ both necessary and sufficient for the ring A (as in (1.0.1)) to be $k^{[m+2]}$?

Example 3.2.2 in Chapter 3 shows that when $m \geqslant 2$, the condition that f is a coordinate in $k[Z,T]$ is not sufficient for A to be $k^{[m+2]}$ in general. However, we will show that Question 2 indeed has an affirmative answer when F is of the form

$$F(X_1,\ldots,X_m,Z,T) = f(Z,T) + (X_1 \cdots X_m)g(X_1,\ldots,X_m,Z,T).$$

In fact, we prove the following generalisation of Theorem A (Chapter 3, Theorem 3.2.1):

Theorem A1. *Let k be a field and*

$$A = \frac{k\,[X_1,\ldots,X_m,Y,Z,T]}{(X_1^{r_1}\cdots X_m^{r_m}Y - F(X_1,\ldots,X_m,Z,T))}, \quad r_i > 1 \text{ for all } i, 1 \leqslant i \leqslant m.$$

Let $F(X_1,\ldots,X_m,Z,T) = f(Z,T) + (X_1 \cdots X_m)g(X_1,\ldots,X_m,Z,T)$ be such that $f(Z,T) \neq 0$. Let $G = X_1^{r_1}\cdots X_m^{r_m}Y - F(X_1,\ldots,X_m,Z,T)$ and $x_1,\ldots,x_m$ denote the images in A of $X_1,\ldots,X_m$ respectively. Then the following statements are equivalent:

(i) $k[X_1,\ldots,X_m,Y,Z,T] = k[X_1,\ldots,X_m,G]^{[2]}$.

(ii) $k[X_1,\ldots,X_m,Y,Z,T] = k[G]^{[m+2]}$.

(iii) $A = k[x_1,\ldots,x_m]^{[2]}$.

(iv) $A = k^{[m+2]}$.

(v) $k[Z,T] = k[f(Z,T)]^{[1]}$.

We actually prove the equivalence of each of the above statements with nine other technical statements (Theorems 3.2.1, 3.2.6), involving stable isomorphisms, affine fibrations and two invariants — the *Derksen invariant* and the *Makar-Limanov invariant* (cf. Definition 2.1.6) of A. Theorem 3.2.1 provides a connection between the Epimorphism Problem, Zariski Cancellation Problem and Affine Fibration Problem in higher dimensions. We also establish a generalisation of Theorem A1 over any commutative Noetherian domain R such that either R is seminormal or $\mathbb{Q} \subset R$ (Theorem 3.2.10).

The equivalence of (v) with any of the remaining statements in Theorem A1 shows that a certain property of the polynomial G in $m+3$ variables (the property of being a coordinate) is determined entirely by a property of the polynomial f in two variables, i.e., a question on a polynomial in $m+3$ variables reduces to a question on a polynomial in 2 variables. The equivalence of (iv) and (v) answers Question 2 for the particular structure of F; the equivalence of (ii) and (iv) gives an affirmative answer to Question 1 (Abhyankar-Sathaye Conjecture) for $n \geqslant 4$ and for a hypersurface of the form G. This result may be considered as a partial generalisation of the theorem on Linear Planes due to A. Sathaye ([35]) and P. Russell ([32, 2.3]), mentioned earlier.

II. An infinite family of higher dimensional counterexamples to ZCP (Chapter 4)

Our results in Chapter 4 are inspired by the Zariski Cancellation Problem (ZCP) for affine spaces which investigates the following.

Question 2. Let B be an n-dimensional affine k-domain such that $B^{[1]} = k^{[n+1]}$. Does this imply $B = k^{[n]}$?

The answer to Question 2 is affirmative for $n \leq 2$ over any field (cf. [1], [30], [17], [34], [7]). For $n \geq 3$, the problem is open over fields of characteristic zero. However, over fields of positive characteristic, Gupta has shown a certain subfamily of Generalised Asanuma varieties to be counterexample to this problem (cf. [21], [23]). Furthermore, in [22], Gupta has given an infinite family of pairwise non-isomorphic three dimensional varieties which are counterexamples to ZCP in positive characteristic. This motivates us to investigate the isomorphism classes of generalised Asanuma varieties of higher dimensions ($\geqslant 3$).

In this chapter, we first see the following result (Theorem 4.1.1).

Theorem A2. Let $(r_1, \ldots, r_m), (s_1, \ldots, s_m) \in \mathbb{Z}^m_{>1}$, and $F, G \in k[X_1, \ldots, X_m, Z, T]$, where $f(Z,T) := F(0, \ldots, 0, Z, T) \notin k$ and $g(Z,T) := G(0, \ldots, 0, Z, T) \notin k$. Suppose $\phi : A \to A'$ is an isomorphism, where

$$A = A(r_1, \ldots, r_m, F) := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T))}$$

and

$$A' = A(s_1, \ldots, s_m, G) := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{s_1} \cdots X_m^{s_m} Y - G(X_1, \ldots, X_m, Z, T))}.$$

Let $x_1, \ldots, x_m, y, z, t$ and $x'_1, \ldots, x'_m, y', z', t'$ denote the images of $X_1, \ldots, X_m, Y, Z, T$ in A and A' respectively. Let $E = k[x_1, \ldots, x_m]$ and $E' = k[x'_1, \ldots, x'_m]$. Suppose $B := \mathrm{DK}(A) = k[x_1, \ldots, x_m, z, t]$ and $B' := \mathrm{DK}(A') = k[x'_1, \ldots, x'_m, z', t']$. Then

(i) ϕ restricts to isomorphisms from B to B' and from E to E'.

(ii) For each $i, 1 \leqslant i \leqslant m$, there exists j, $1 \leqslant j \leqslant m$, such that $\phi(x_i) = \lambda_j x'_j$ for some $\lambda_j \in k^*$ and $r_i = s_j$. In particular, $(r_1, \ldots, r_m) = (s_1, \ldots, s_m)$ upto a permutation of $\{1, \ldots, m\}$.

(iii) $\phi\left(x_1^{r_1} \cdots x_m^{r_m}, F(x_1, \ldots, x_m, z, t)\right) = ((x'_1)^{s_1} \cdots (x'_m)^{s_m}, G(x'_1, \ldots, x'_m, z', t'))$

(iv) There exists $\alpha \in \mathrm{Aut}_k(k[Z, T])$ such that $\alpha(g) = \lambda f$ for some $\lambda \in k^*$.

Using Theorem A2, we characterise the automorphisms of a certain subfamily of Generalised Asanuma varieties (Theorem 4.1.2). Further as a consequence to Theorem A2, we establish the following (Theorem 4.2.1).

Theorem A3. Let $(r_1, \ldots, r_m), (s_1, \ldots, s_m) \in \mathbb{Z}^m_{>1}$, and $f, g \in k[Z, T]$ be non-trivial lines. Then $A(r_1, \ldots, r_m, f) \cong A(s_1, \ldots, s_m, g)$ if and only if $(r_1, \ldots, r_m) = (s_1, \ldots, s_m)$ upto a permutation of $\{1, \ldots, m\}$ and there exists $\alpha \in \mathrm{Aut}_k(k[Z, T])$ such that $\alpha(g) = \mu f$, for some $\mu \in k^*$.

The above theorem immediately yields the following result (Corollary 4.2.3).

Corollary A4. Let k be a field of positive characteristic. For each $n \geqslant 3$, there exists an infinite family of pairwise non-isomorphic rings C of dimension n, which are counter examples to the Zariski Cancellation Problem in positive characteristic, i.e., which satisfy that $C^{[1]} = k^{[n+1]}$ but $C \neq k^{[n]}$.

III. Generalised Danielewski varieties and invariants of generalised Asanuma varieties (Chapter 5)

The results in Chapter 5 have their genesis in the General Cancellation Problem in Affine Algebraic Geometry which asks the following (cf. [16]):

Question 3. Let D and E be two affine domains over a field k such that $D^{[1]} =_k E^{[1]}$. Does this imply $D \cong_k E$?

The answer to Question 3 is affirmative for one dimensional affine domains (cf. [1]). However, there are counterexamples in dimensions greater than or equal to two. In [11], Danielewski constructed a family of two dimensional pairwise non-isomorphic smooth complex varieties which are counterexamples to the Cancellation Problem. In [10], A. J. Crachiola extended Danielewski's examples over arbitrary characteristic. In [13], A. Dubouloz constructed higher dimensional ($\geqslant 3$) analogues of the Danielewski varieties, which are counterexamples to this problem. More precisely, for $\underline{r} := (r_1, \ldots, r_m) \in \mathbb{Z}^m_{\geqslant 1}$ and $F \in k^{[m+1]}$, he studied the affine varieties, defined by the following integral domains when $k = \mathbb{C}$:

$$B_{\underline{r}} := B(r_1, \ldots, r_m, F) = \frac{k[T_1, \ldots, T_m, U, V]}{(T_1^{r_1} \cdots T_m^{r_m} U - F(T_1, \ldots, T_m, V))},$$

where $F(T_1, \ldots, T_m, V)$ is monic in V and $d := \deg_V F > 1$. We note that setting $P(V) := F(0, \ldots, 0, V)$, we have $d = \deg_V P = \deg_V F = d (> 1)$. We call these varieties as "Generalised Danielewski varieties". In the above setting, Dubouloz proved ([13], Corollary 1.1, Corollary 1.2):

Theorem B. Suppose $F = \prod_{i=1}^{d} (V - \sigma_i(T_1, \ldots, T_m))$, where σ_i's are of the form

$$\sigma_i(T_1, \ldots, T_m) = a_i + T_1 \cdots T_m f_i(T_1, \ldots, T_m), \quad f_i \in \mathbb{C}^{[m]} \text{ for all } i, \quad 1 \leq i \leq d,$$

and $a_i \in \mathbb{C}$ are such that $a_i \neq a_j$ for $i \neq j$. Then the following statements hold:

(i) Suppose that either of the following conditions hold:
(a) $\underline{r} \in \mathbb{Z}^m_{>1}$ and $\underline{s} \in \mathbb{Z}^m_{\geqslant 1} \setminus \mathbb{Z}^m_{>1}$
(b) $\underline{r}, \underline{s} \in \mathbb{Z}^m_{>1}$ are such that the sets $\{r_1, \ldots, r_m\}$ and $\{s_1, \ldots, s_m\}$ are distinct.
Then $B_{\underline{r}} \ncong_k B_{\underline{s}}$.

(ii) $B_{\underline{r}}^{[1]} \cong_k B_{\underline{s}}^{[1]}$ for any $\underline{r}, \underline{s} \in \mathbb{Z}^m_{\geqslant 1}$.

We exhibit a larger class of varieties in higher dimensions ($\geqslant 2$) over fields of arbitrary characteristic which are counter examples to the General Cancellation Problem. These varieties accommodate the counterexamples due to Dubouloz over $\mathbb{C}$. More precisely, we establish the following (Chapter 5, Theorems 5.2.1 and 5.2.3):

Theorem B1. Let $F, F' \in k[T_1, \ldots, T_m, V]$ be such that

$$F = a_0 + a_1V + \cdots + a_{d-1}V^{d-1} + V^d, \quad \text{and}$$

$$F' = a_1 + 2a_2V + \cdots + (d-1)a_{d-1}V^{d-2} + dV^{d-1}$$

for some $a_i \in k[T_1, \ldots, T_m]$, $0 \leqslant i \leqslant d-1$. Then the following statements hold:

(i) Suppose that either of the following conditions hold:
(a) $\underline{r} \in \mathbb{Z}^m_{>1}$ and $\underline{s} \in \mathbb{Z}^m_{\geqslant 1} \setminus \mathbb{Z}^m_{>1}$
(b) $\underline{r}, \underline{s} \in \mathbb{Z}^m_{>1}$ are such that $\underline{r} = (r_1, \ldots, r_m)$ and $\underline{s} = (s_1, \ldots, s_m)$ are not permutation of each other.
Then $B_{\underline{r}} \not\cong_k B_{\underline{s}}$.

(ii) $B_{\underline{r}}^{[1]} \cong_k B_{\underline{s}}^{[1]}$ for any $\underline{r}, \underline{s} \in \mathbb{Z}^m_{\geqslant 1}$, whenever $(F, F') = k[T_1, \ldots, T_m, V]$.

As a consequence we prove the following result (Corollary 5.2.4).

Corollary B2. Let k be any field. For each $n \geqslant 2$, there exists an infinite family of pairwise non-isomorphic rings of dimension n, which are counterexamples to the Cancellation Problem.

In the rest of Chapter 5 (see 5.3) we will focus on some invariants of Generalised Asanuma varieties. As we have seen in I, special cases of these varieties have played crucial roles in solutions to central problems on affine spaces and the Makar-Limanov and Derksen invariants of these varieties were the key tools in some of the results. The case $m = 1$ accommodates the famous Russell-Koras threefold $x^2y + x + z^2 + t^3 = 0$ over $k = \mathbb{C}$ which arose in the context of the Linearisation Problem for $\mathbb{C}^{[3]}$ and which is now a potential candidate for a counterexample to ZCP for the affine three space in characteristic zero (see [25] for details).

Recall that for the affine domain A as in (1.0.1), when k is a field of positive characteristic, and $F = f(Z, T)$ is a nontrivial line in $k[Z, T]$, Gupta had shown ([21, Corollary 3.9], [23, Corollary 3.8]) that for each $m \geqslant 1$, the corresponding ring A is a counterexample to the ZCP in dimension $(m + 2)$, i.e., $A^{[1]} = k^{[m+3]}$ but $A \neq k^{[m+2]}$. A crucial step in the proof was a result on the Derksen invariant quoted as Proposition 3.1.5 in Chapter 3. Here we address the following converse of Proposition 3.1.5:

Question 4. Let A be as in (1.0.1). Suppose that there exists a system of coordinates $\{Z_1, T_1\}$ of $k[Z,T]$ such that $f(Z,T) = a_0(Z_1) + a_1(Z_1)T_1$. Is the Derksen invariant of A equal to A?

We shall apply Theorem B1 to give an affirmative answer to the above question when F is of a certain form (Proposition 5.3.2). We also give a complete description of the Derksen and Makar-Limanov invariants for

$$A = \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - f(Z,T))},$$

when k is an infinite field and A is a regular domain (Corollary 5.3.3).

In the next chapter, we recall some definitions, well-known results, and basic properties of exponential maps and some K-theoretic aspects of Noetherian rings which will be used in the subsequent chapters.

Chapter 2

Preliminaries

Throughout this thesis all rings and algebras will be assumed to be commutative with unity. The letter k will always denote a field. For any ring R, $R^{[n]}$ denotes a polynomial ring in n variables over R. For some $p \in \operatorname{Spec}(R)$ and an R-algebra B, B_p denotes the ring $S^{-1}B$ where $S = R \setminus p$ and $\kappa(p)$ denotes the field $\frac{R_p}{pR_p}$. Capital letters like $X, Y, Z, T, U, V, X_1, \ldots, X_m$ etc. will denote indeterminates over the respective ground rings or fields.

2.1 Definitions and well known results

We first recall a few definitions.

Definition 2.1.1. *A polynomial $h \in k[X,Y]$ is said to be a line in $k[X,Y]$ if $\frac{k[X,Y]}{(h)} = k^{[1]}$. Furthermore, if $k[X,Y] \neq k[h]^{[1]}$, then h is said to be a non-trivial line in $k[X,Y]$.*

Definition 2.1.2. *A finitely generated flat R-algebra B is said to be an $\mathbb{A}^n$-fibration over R if $B \otimes_R k(p) = k(p)^{[n]}$ for every prime ideal p of R.*

Definition 2.1.3. *A polynomial $h \in R[X,Y]$ is said to be a residual coordinate, if for every $p \in \operatorname{Spec}(R)$,*

$$R[X,Y] \otimes_R \kappa(p) = (R[h] \otimes_R \kappa(p))^{[1]}.$$

Definition 2.1.4. *An integral domain R with field of fractions K is called a seminormal domain if for any $a \in K$, the conditions $a^2, a^3 \in R$ imply that $a \in R$.*

Definition 2.1.5. *A k-algebra B is said to be geometrically factorial over k if for every algebraic field extension L of k, $B \otimes_k L$ is a UFD.*

We now define an exponential map on a k-algebra B and two invariants related to it, namely, the *Makar-Limanov invariant* and the *Derksen invariant*.

Definition 2.1.6. *Let B be a k-algebra and $\phi : B \to B^{[1]}$ be a k-algebra homomorphism. For an indeterminate U over B, let ϕ_U denote the map $\phi : B \to B[U]$. Then ϕ is said to be an exponential map on B, if the following conditions are satisfied:*

(i) $\epsilon_0 \phi_U = id_B$, *where $\epsilon_0 : B[U] \to B$ is the evaluation map at $U = 0$.*

(ii) $\phi_V \phi_U = \phi_{U+V}$, *where $\phi_V : B \to B[V]$ is extended to a k-algebra homomorphism $\phi_V : B[U] \to B[U, V]$, by setting $\phi_V(U) = U$.*

The ring of invariants of ϕ is a subring of B defined as follows:

$$B^{\phi} := \{b \in B \,|\phi(b) = b\}.$$

The map ϕ is said to be non-trivial if $B^{\phi} \neq B$.

Let $\mathrm{EXP}(B)$ *denote the set of all exponential maps on B. The Makar-Limanov invariant of B is defined to be*

$$\mathrm{ML}(B) := \bigcap_{\phi \in EXP(B)} B^{\phi}$$

and the Derksen invariant is a subring of B defined as

$$\mathrm{DK}(B) := k\left[b \in B^{\phi} \,|\, \phi \in \mathrm{EXP}(B) \text{ and } B^{\phi} \subsetneqq B\right].$$

Next we record some useful results on exponential maps. The following two lemmas can be found in [29, Chapter I], [9] and [21].

Lemma 2.1.1. *Let B be an affine domain over k and ϕ be a non-trivial exponential map on B. Then the following statements hold:*

(i) *B^{ϕ} is a factorially closed subring of B, i.e., for any non-zero $a, b \in B$, if $ab \in B^{\phi}$, then $a, b \in B^{\phi}$. In particular, B^{ϕ} is algebraically closed in B.*

(ii) $\mathrm{tr.deg}_k\, B^{\phi} = \mathrm{tr.deg}_k\, B - 1$.

(iii) *For a multiplicatively closed subset S of $B^{\phi}\setminus\{0\}$, ϕ induces a non-trivial exponential map $S^{-1}\phi$ on $S^{-1}B$ such that $(S^{-1}B)^{S^{-1}\phi} = S^{-1}(B^{\phi})$.*

Lemma 2.1.2. *Let k be a field and $B = k^{[n]}$. Then $\mathrm{DK}(B) = B$ for $n \geq 2$ and $\mathrm{ML}(B) = k$.*

Next we recall the definition of a *rigid k-domain.*

Definition 2.1.7. *A k-domain D is said to be rigid if there does not exist any non-trivial exponential map on D.*

For convenience, we record below an easy lemma.

Lemma 2.1.3. *Let D be a k-domain which is not rigid. Then $\mathrm{DK}(D[W]) = D[W]$.*

Proof. Consider the exponential map $\phi : D[W] \to D[W,U]$ defined by

$$\phi(a) = a \text{ for all } a \in D, \text{ and } \phi(W) = W + U.$$

It is easy to see that $(D[W])^{\phi} = D$. Again as D is not rigid, we have a non-trivial exponential map ψ on D. We extend ψ to an exponential map $\widetilde{\psi}$ on $D[W]$ such that $\widetilde{\psi}|_D = \psi$ and $\widetilde{\psi}(W) = W$. Then $W \in (D[W])^{\widetilde{\psi}}$. Therefore, it follows that $\mathrm{DK}(D[W]) = D[W]$. $\square$

Next we define *proper* and *admissible $\mathbb{Z}$-filtration* on an affine domain.

Definition 2.1.8. *Let k be a field and B an affine k-domain. A collection $\{B_n \mid n \in \mathbb{Z}\}$ of k-linear subspaces of B is said to be a proper $\mathbb{Z}$-filtration if*

(i) *$B_n \subseteq B_{n+1}$, for every $n \in \mathbb{Z}$.*

(ii) *$B = \bigcup_{n\in\mathbb{Z}} B_n$.*

(iii) *$\bigcap_{n\in\mathbb{Z}} B_n = \{0\}$.*

(iv) *$(B_n \setminus B_{n-1}).(B_m \setminus B_{m-1}) \subseteq B_{m+n} \setminus B_{m+n-1}$ for all $m, n \in \mathbb{Z}$.*

A proper $\mathbb{Z}$-filtration $\{B_n\}_{n\in\mathbb{Z}}$ of B is said to be admissible if there is a finite generating set Γ of B such that for each $n \in \mathbb{Z}$, every element in B_n can be written as a finite sum of monomials from $B_n \cap k[\Gamma]$.

A proper $\mathbb{Z}$-filtration $\{B_n\}_{n\in\mathbb{Z}}$ of B defines an associated graded domain defined by

$$gr(B) := \bigoplus_{n\in\mathbb{Z}} \frac{B_n}{B_{n-1}}.$$

It also defines the natural map $\rho : B \to gr(B)$ such that $\rho(b) = b + B_{n-1}$, if $b \in B_n \setminus B_{n-1}$.

We now record a result on homogenization of exponential maps due to H. Derksen, O. Hadas and L. Makar-Limanov [12]. The following version can be found in [9, Theorem 2.6].

Theorem 2.1.4. *Let B be an affine domain over a field k with an admissible proper $\mathbb{Z}$-filtration and $gr(B)$ be the induced $\mathbb{Z}$-graded domain. Let ϕ be a non-trivial exponential map on B. Then ϕ induces a non-trivial homogeneous exponential map $\overline{\phi}$ on $gr(B)$ such that $\rho(B^{\phi}) \subseteq gr(B)^{\overline{\phi}}$.*

We quote below a criterion for flatness from [27, (20.G)].

Lemma 2.1.5. *Let $R \to C \to D$ be local homomorphisms of Noetherian local rings, κ the residue field of R and M a finite D module. Suppose C is R-flat. Then the following statements are equivalent:*

(i) *M is C-flat.*

(ii) *M is R-flat and $M \otimes_R \kappa$ is $C \otimes_R \kappa$-flat.*

The following is a well known result ([1]).

Theorem 2.1.6. *Let k be a field and R be a normal domain such that $k \subset R \subset k^{[n]}$. If $\operatorname{tr.deg}_k R = 1$, then $R = k^{[1]}$.*

Next we state a version of the Russell-Sathaye criterion [33, Theorem 2.3.1], as presented in [6, Theorem 2.6].

Theorem 2.1.7. *Let $R \subset C$ be integral domains such that C is a finitely generated R-algebra. Let S be a multiplicatively closed subset of $R \setminus \{0\}$ generated by some prime elements of R which remain prime in C. Suppose $S^{-1}C = (S^{-1}R)^{[1]}$ and, for every prime element $p \in S$, we have $pR = pC \cap R$ and $\frac{R}{pR}$ is algebraically closed in $\frac{C}{pC}$. Then $C = R^{[1]}$.*

We now recall a result on separable $\mathbb{A}^1$-forms over a PID ([14, Theorem 7]).

Lemma 2.1.8. *Let $f \in k[Z,T]$ such that $L[Z,T] = L[f]^{[1]}$, for some separable field extension L of k. Then $k[Z,T] = k[f]^{[1]}$.*

We quote below a fundamental result on residual coordinates ([5, Theorem 3.2]).

Theorem 2.1.9. *Let R be a commutative Noetherian domain such that either $\mathbb{Q} \subset R$ or R is seminormal. Then the following statements are equivalent:*

(i) *$h \in R[X,Y]$ is a residual coordinate.*

(ii) *$R[X,Y] = R[h]^{[1]}$.*

2.2 Basic facts of K-theory

In this section we will consider a Noetherian ring R and note some K-theoretic aspects of R (cf. [4], [8]). Let $\mathscr{M}(R)$ denote the category of finitely generated R-modules and $\mathscr{P}(R)$ the category of finitely generated projective R-modules. Let $G_0(R)$ and $G_1(R)$ respectively denote the *Grothendieck group* and the *Whitehead group* of the category $\mathscr{M}(R)$. Let $K_0(R)$ and $K_1(R)$ respectively denote the *Grothendieck group* and the *Whitehead group* of the category $\mathscr{P}(R)$. For $i \geqslant 2$, the definitions of $G_i(R)$ and $K_i(R)$ can be found in ([37], Chapters 4 and 5). The following results can be found in [37, Propositions 5.15 and 5.16, Theorem 5.2].

Lemma 2.2.1. *Let t be a regular element of R. Then the inclusion map $j : R \hookrightarrow R[t^{-1}]$ and the natural surjection map $\pi : R \to \frac{R}{tR}$ induce the following long exact sequence of groups:*

$$\longrightarrow G_i(\tfrac{R}{tR}) \xrightarrow{\pi_*} G_i(R) \xrightarrow{j^*} G_i(R[t^{-1}]) \longrightarrow \cdots \longrightarrow G_0(R[t^{-1}]) \longrightarrow 0$$

Lemma 2.2.2. *Let t be a regular element of R and $\phi : R \to C$ be a flat ring homomorphism such that $u = \phi(t)$. Then we get the following commutative diagram:*

$$\begin{tikzcd}
\cdots \arrow[r] & G_i(\frac{R}{tR}) \arrow[r] \arrow[d, "\overline{\phi}^*"] & G_i(R) \arrow[r] \arrow[d, "\phi^*"] & G_i(R[t^{-1}]) \arrow[r] \arrow[d, "(t^{-1}\phi)^*"] & G_{i-1}(\frac{R}{tR}) \arrow[r] \arrow[d, "\overline{\phi}^*"] & \cdots \\
\cdots \arrow[r] & G_i(\frac{C}{uC}) \arrow[r] & G_i(C) \arrow[r] & G_i(C[u^{-1}]) \arrow[r] & G_{i-1}(\frac{C}{uC}) \arrow[r] & \cdots ,
\end{tikzcd}$$

where ϕ induces the vertical maps.

Lemma 2.2.3. *For an indeterminate T over R, the following hold:*

(a) *For every $i \geqslant 0$, the maps $G_i(R) \to G_i(R[T])$, which are induced by $R \hookrightarrow R[T]$, are isomorphisms.*

(b) *Let $j : R \to R[T, T^{-1}]$ be the inclusion map. Then for every i, $i \geqslant 1$, the following sequence is split exact.*

$$0 \longrightarrow G_i(R[T]) \xrightarrow{j^*} G_i(R[T, T^{-1}]) \longrightarrow G_{i-1}(R) \longrightarrow 0 \ .$$

In particular, for $i = 0$, j^* is an isomorphism and for $i \geqslant 1$, $G_i(R[T, T^{-1}]) \cong G_i(R[T]) \oplus G_{i-1}(R) \cong G_i(R) \oplus G_{i-1}(R)$.

We end this section with the following remark.

Remark 2.2.4. For a regular ring R, $G_i(R) = K_i(R)$, for every $i \geqslant 0$. In particular, $G_1(k[X]) = K_1(k[X]) = k^*$ and $G_0(k) = K_0(k) = \mathbb{Z}$. For $C_m := k[X_1, \ldots, X_m, X_1^{-1}, \ldots, X_m^{-1}]$, $G_0(C_m) = K_0(C_m) = \mathbb{Z}$ (since every finitely generated projective module over C_m is free) and by repeated application of Lemma 2.2.3(b), we get that $G_1(C_m) \cong k^* \oplus \mathbb{Z}^m$, for every $m \geqslant 1$.

Chapter 3

Triviality of a family of linear hyperplanes

The main objective of this chapter is to investigate the Epimorphism Problem and prove an extended version of Theorem A1 which includes fourteen equivalent statements (cf. Theorems 3.2.1 , 3.2.6). We will prove the results in two parts.
We first prove some preparatory lemmas, propositions and prove eleven equivalent statements in Theorem 3.2.1. The remaining 3 statements will be proved in Theorem 3.2.6. The results discussed in this chapter can be found in [19] and [20].

3.1 Properties of Generalised Asanuma varieties

We begin with the following lemma.

Lemma 3.1.1. *Let R be an integral domain. Let $E := R[X_1, \ldots, X_m, T]$, $C = E[Z, Y]$, $g \in E[Z]$ and $G = X_1^{r_1} \cdots X_m^{r_m} Y - X_1 \cdots X_m g + Z \in C$. Then $C = E[G]^{[1]}$.*

Proof. Let $D = E[G] \subseteq C$. Let S be the multiplicatively closed subset of $D \setminus \{0\}$, which is generated by $X_1, \ldots, X_m$. From the expression of G, it is clear that $S^{-1}C = (S^{-1}D)[Z]$. Again $\frac{C}{X_iC} = (\frac{D}{X_iD})^{[1]}$ for every $i, 1 \leqslant i \leqslant m$. Therefore, by Theorem 2.1.7, we get that $C = D^{[1]}$. $\square$

We now fix some notation. Throughout this chapter, A will denote the coordinate ring of Generalised Asanuma varieties, i.e.,

$$A = \frac{k\,[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T))}, \quad r_i > 1 \text{ for all } i, 1 \leqslant i \leqslant m, \tag{3.1.1}$$

where $f(Z,T) := F(0, \ldots, 0, Z, T) \neq 0$. Let $G := X_1^{r_1} \cdots X_m^{r_m} Y - F$. Note that A is an integral domain. Recall that $x_1, \ldots, x_m, y, z, t$ denote the images of $X_1, \ldots, X_m, Y, Z, T$ in A. The notation E and B will denote the following subrings of A:

$$\mathbf{E} = \mathbf{k}[\mathbf{x_1}, \ldots, \mathbf{x_m}] \text{ and } \mathbf{B} = \mathbf{k}[\mathbf{x_1}, \ldots, \mathbf{x_m}, \mathbf{z}, \mathbf{t}].$$

Note that

$$B = k[x_1, \ldots, x_m, z, t] \hookrightarrow A \hookrightarrow B[x_1^{-1}, \ldots, x_m^{-1}].$$

Fix $(e_1, \ldots, e_m) \in \mathbb{Z}^m$. The ring $B[x_1^{-1}, \ldots, x_m^{-1}]$ can be given the following $\mathbb{Z}$-graded structure:

$$B[x_1^{-1}, \ldots, x_m^{-1}] = \bigoplus_{n \in \mathbb{Z}} B_n,$$

where

$$B_n = \bigoplus_{\substack{(i_1, \ldots, i_m) \in \mathbb{Z}^m, \\ e_1 i_1 + \cdots + e_m i_m = n}} k[z,t] x_1^{i_1} \cdots x_m^{i_m}.$$

Now every $b \neq 0 \in B[x_1^{-1}, \ldots, x_m^{-1}]$ can be written uniquely as

$$b = \sum_{i=d_l}^{d_h} b_i,$$

where $b_i \in B_i$, d_l, d_h are some integers and $b_{d_l}, b_{d_h} \neq 0$. If $b \in B$, then each $b_i \in B$. We call d_h the degree of b and hence b_{d_h} is the highest degree homogeneous summand of b. For every $n \in \mathbb{Z}$, if $A_n = \bigoplus_{i \leqslant n} B_n \cap A$, then $\{A_n\}_{n \in \mathbb{Z}}$ defines a $\mathbb{Z}$-filtration on A. For every $j, 1 \leqslant j \leqslant m$, $x_j \in A_{e_j} \setminus A_{e_j - 1}$. If d denotes the degree of $F(x_1, \ldots, x_m, z, t)$, then $y \in A_l \setminus A_{l-1}$, where $l = d - (r_1 e_1 + \cdots + r_m e_m)$.

With respect to the notation as above, we first quote the following two results [23, Lemmas 3.1, 3.2].

Lemma 3.1.2. *The k-linear subspaces $\{A_n\}_{n \in \mathbb{Z}}$ define a proper admissible $\mathbb{Z}$-filtration on A with the generating set $\Gamma = \{x_1, \ldots, x_m, y, z, t\}$, and the associated graded ring $gr(A) = \bigoplus_{n \in \mathbb{Z}} \frac{A_n}{A_{n-1}}$ is generated by the image of Γ in*

$gr(A)$.

Lemma 3.1.3. *Let d denote the degree of $F(x_1, \dots, x_m, z, t)$ and F_d denote the highest degree homogeneous summand of F. Suppose that, for each j, $1 \leqslant j \leqslant m$, $x_j \nmid F_d$. Then the associated graded ring $gr(A)$ is isomorphic to*

$$\frac{k[X_1, \dots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - F_d(X_1, \dots, X_m, Z, T))}$$

as k-algebras.

The following result is proved in [23, Lemma 3.3].

Lemma 3.1.4. $k[x_1, \dots, x_m, z, t] \subseteq \mathrm{DK}(A)$.

The following proposition is stated in [23, Proposition 3.4(i)] under the hypothesis $\mathrm{DK}(A) = A$. However the proof uses only the consequence that $k[x_1, \dots, x_m, z, t] \subsetneqq \mathrm{DK}(A)$. Below we quote the result under this modified hyopthesis.

Proposition 3.1.5. *Suppose that k is infinite and $k[x_1, \dots, x_m, z, t] \subsetneqq \mathrm{DK}(A)$. Then there exist $Z_1, T_1 \in k[Z, T]$ and $a_0, a_1 \in k^{[1]}$ such that $k[Z, T] = k[Z_1, T_1]$ and $f(Z, T) = a_0(Z_1) + a_1(Z_1)T_1$.*

Remark 3.1.6. For $m = 1$, the hypothesis that k is an infinite field in the above proposition can be dropped (cf. [22, Proposition 3.7]).

The next result shows that if f is a non-trivial line, then $\mathrm{DK}(A) = k[x_1, \dots, x_m, z, t]$.

Proposition 3.1.7. *Suppose that $k[x_1, \dots, x_m, z, t] \subsetneqq \mathrm{DK}(A)$ and*

$$k[Z, T]/(f) = k^{[1]}.$$

Then $k[Z, T] = k[f]^{[1]}$.

Proof. If k is infinite, then the assertion follows from [23, Proposition 3.4(ii)]. Now suppose k is a finite field. Let $\overline{k}$ be an algebraic closure of k and $\overline{A} = A \otimes_k \overline{k}$. Since $k[x_1, \dots, x_m, z, t] \subsetneqq \mathrm{DK}(A)$, we have $\overline{k}[x_1, \dots, x_m, z, t] \subsetneqq \mathrm{DK}(\overline{A})$. If $k[Z, T]/(f) = k^{[1]}$, then $\overline{k}[Z, T]/(f) = \overline{k}^{[1]}$. As $\overline{k}$ is infinite, by Proposition 3.1.5, there exist $Z_1, T_1 \in \overline{k}[Z, T]$ and $a_0, a_1 \in \overline{k}^{[1]}$ such that $\overline{k}[Z, T] = \overline{k}[Z_1, T_1]$ and $f = a_0(Z_1) + a_1(Z_1)T_1$. Now since f is a line in $\overline{k}[Z, T]$, we have $\left(\overline{k}[Z, T]/(f)\right)^* = \overline{k}^*$. We now have the following two cases

Case 1: If $a_1(Z_1) = 0$, $f = a_0(Z_1)$ must be linear in Z_1 as f is irreducible in $\overline{k}[Z,T]$.

Case 2: If $a_1(Z_1) \neq 0$, then $\gcd(a_0(Z_1), a_1(Z_1)) = 1$ as f is irreducible in $\overline{k}[Z,T]$. Hence $a_1(Z_1) \in \left(\overline{k}[Z,T]/(f)\right)^* = \overline{k}^*$.

Therefore, by above two cases $\overline{k}[Z,T] = \overline{k}[f]^{[1]}$. Hence by Lemma 2.1.8, we get that $k[Z,T] = k[f]^{[1]}$. $\square$

Next we prove a result which describes $\mathrm{ML}(A)$ when $\mathrm{DK}(A)$ is exactly equal to $B(= k[x_1, \ldots, x_m, z, t])$.

Proposition 3.1.8. *Let A be the affine domain as in* (3.1.1). *Then the following hold:*

(a) *Suppose, for every $i \in \{1, \ldots, m\}$, $x_i \notin A^*$, and $F \notin k[X_1, \ldots, X_m]$. Then $\mathrm{ML}(A) \subseteq E(= k[x_1, \ldots, x_m])$.*

(b) *If $\mathrm{DK}(A) = B$, then $\mathrm{ML}(A) = E$.*

Proof. (a) Since $F \notin k[X_1, \ldots, X_m]$, without loss of generality, suppose $\deg_T F > 0$. Consider the map $\phi_1 : A \to A[U]$ defined as follows:

$$\phi_1(x_i) = x_i \,(1 \leqslant i \leqslant m), \;\; \phi_1(z) = z, \;\; \phi_1(t) = t + x_1^{r_1} \cdots x_m^{r_m} U,$$

and

$$\phi_1(y) = \frac{F(x_1, \ldots, x_m, z, t + x_1^{r_1} \cdots x_m^{r_m} U)}{x_1^{r_1} \cdots x_m^{r_m}} = y + Uv(x_1, \ldots, x_m, z, t, U),$$

for some $v \in k[x_1, \ldots, x_m, z, t, U]$. It is easy to see that $\phi_1 \in \mathrm{EXP}(A)$. Now

$$k[x_1, \ldots, x_m, z] \subseteq A^{\phi_1} \subseteq A \subseteq k[x_1, \ldots, x_m, (x_1 \cdots x_m)^{-1}, z, t].$$

Since $\deg_T F > 0$, and for every i, $1 \leqslant i \leqslant m$, $x_i \notin A^*$, it follows that

$$A \cap k[x_1, \ldots, x_m, (x_1 \cdots x_m)^{-1}, z] = k[x_1, \ldots, x_m, z].$$

Therefore, $k[x_1, \ldots, x_m, z]$ is algebraically closed in A. Also $\operatorname{tr.deg}_k(k[x_1, \ldots, x_m, z]) = \operatorname{tr.deg}_k(A^{\phi_1}) = m + 1$ (cf. Lemma 2.1.1(ii)). Hence $A^{\phi_1} = k[x_1, \ldots, x_m, z]$.

Again consider the map $\phi_2 : A \to A[U]$ defined as follows:

$$\phi_2(x_i) = x_i \,(1 \leqslant i \leqslant m), \;\; \phi_2(t) = t, \;\; \phi_2(z) = z + x_1^{r_1} \cdots x_m^{r_m} U,$$

and

$$\phi_2(y) = \frac{F(x_1,\dots,x_m,z+x_1^{r_1}\cdots x_m^{r_m}U,t)}{x_1^{r_1}\cdots x_m^{r_m}} = y + Uw(x_1,\dots,x_m,z,t,U),$$

for some $w \in k[x_1,\dots,x_m,z,t,U]$. It follows that $\phi_2 \in \mathrm{EXP}(A)$. Clearly

$$k[x_1,\dots,x_m,t] \subseteq A^{\phi_2} \subseteq k[x_1,\dots,x_m,(x_1\cdots x_m)^{-1},t].$$

Therefore, $\mathrm{ML}(A) \subseteq A^{\phi_1} \cap A^{\phi_2} \subseteq k[x_1,\dots,x_m] = E$.

(b) Suppose $\mathrm{DK}(A) = B$. Note that for every i, $1 \leqslant i \leqslant m$, $x_i \notin A^*$, otherwise $x_i^{-1} \in \mathrm{DK}(A)$. Further, if either $\deg_T F = 0$ or $\deg_Z F = 0$, then either $y \in A^{\phi_1}$ or $y \in A^{\phi_2}$ respectively, where $\phi_1,\phi_2 \in \mathrm{EXP}(A)$ are as defined in part (a). This contradicts our assumption that $\mathrm{DK}(A) = B$. Therefore, $F \notin k[X_1,\dots,X_m]$, and hence by part (a), it is clear that $\mathrm{ML}(A) \subseteq E$.

Let $\phi \in \mathrm{EXP}(A)$. We show that $E \subseteq A^\phi$. Now $\operatorname{tr.deg}_k A^\phi = m+1$ (cf. Lemma 2.1.1(ii)) and $A^\phi \subseteq k[x_1,\dots,x_m,z,t]$. Suppose $\{f_1,\dots,f_{m+1}\}$ is an algebraically independent set of elements in A^ϕ. We fix some $j \in \{1,\dots,m\}$ and let

$$f_i = g_i(x_1,\dots,x_{j-1},x_{j+1},\dots,x_m,z,t) + x_j h_i(x_1,\dots,x_m,z,t) \qquad (3.1.2)$$

for every $i \in \{1,\dots,m+1\}$. We show that the set $\{g_1,\dots,g_{m+1}\}$ is algebraically dependent.

Suppose not. We consider the $\mathbb{Z}$-filtration on A induced by the element $(0,\dots,0,-1,0,\dots,0) \in \mathbb{Z}^m$, where the j-th entry is -1. If f_{id} denotes the highest degree homogeneous summand of f_i, then from (3.1.2), we get that $f_{id} = g_i$. By Theorem 2.1.4, ϕ will induce a non-trivial exponential map ϕ_j on the associated graded ring A_j and $\{g_i \mid 1 \leqslant i \leqslant m+1\} \subseteq A_j^{\phi_j}$. By Lemma 3.1.3,

$$A_j \cong \frac{k[X_1,\dots,X_m,Y,Z,T]}{(X_1^{r_1}\cdots X_m^{r_m}Y - F(X_1,\dots X_{j-1},0,X_{j+1},\dots,X_m,Z,T))}. \qquad (3.1.3)$$

Since $\{g_i \mid 1 \leqslant i \leqslant m+1\}$ is algebraically independent, $k[x_1,\dots,x_{j-1},x_{j+1},\dots,x_m,z,t]$ is algebraic over $k[g_i \mid 1 \leqslant i \leqslant m+1]$. As $k[g_i \mid 1 \leqslant i \leqslant m+1] \subseteq A_j^{\phi_j}$ and $A_j^{\phi_j}$ is algebraically closed, we have $k[x_1,\dots,x_{j-1},x_{j+1},\dots,x_m,z,t] \subseteq A_j^{\phi_j}$. Therefore, from (3.1.3) we get that $x_j, y \in A_j^{\phi_j}$, which contradicts that ϕ_j is non-trivial. Thus, $\{g_i \mid 1 \leqslant i \leqslant m+1\}$ is algebraically dependent. Hence there exists a polynomial $P \in k^{[m+1]}$ such that

$$P(g_1, \ldots, g_{m+1}) = 0.$$

Therefore, from (3.1.2), we get that there exists $H \in k[x_1, \ldots, x_m, z, t]$ such that $x_j H = P(f_1, \ldots, f_{m+1}) \in A^{\phi}$. As A^{ϕ} is factorially closed (cf. Lemma 2.1.1(i)), we have $x_j \in A^{\phi}$. Since j is arbitrarily chosen from $\{1, \ldots, m\}$, we have $E = k[x_1, \ldots, x_m] \subseteq A^{\phi}$. As $\phi \in \mathrm{EXP}(A)$ is arbitrary, we have $E \subseteq \mathrm{ML}(A)$. Therefore, $\mathrm{ML}(A) = E$. □

Remark 3.1.9. From Proposition 3.1.5, it follows that if k is an infinite field and there is no system of coordinates $\{Z_1, T_1\}$ of $k[Z, T]$ such that $f(Z, T) = a_0(Z_1) + a_1(Z_1)T_1$, then $\mathrm{DK}(A) = B$ and hence $\mathrm{ML}(A) = E$.

Remark 3.1.10. In Proposition 3.1.8(a) both the conditions that $x_i \notin A^*$ and $F \notin k[X_1, \ldots, X_m]$ are necessary. If some $x_i \in A^*$, then $x_i^{-1} \in \mathrm{ML}(A)$ which shows that $\mathrm{ML}(A) \nsubseteq k[x_1, \ldots, x_m]$.

Now consider the ring $A = \dfrac{k[X, Y, Z, T]}{(X^2Y - 1)}$, where $F = 1 \in k[X]$. Note that $A = k[x, x^{-1}, z, t]$ and hence $\mathrm{ML}(A) = k[x, x^{-1}] \nsubseteq k[x]$.

We now prove a criterion for a simple birational extension of a UFD to be a UFD.

Proposition 3.1.11. *Let R be a UFD, $a, b \in R \setminus \{0\}$ and $C := \frac{R[Y]}{(aY-b)}$ be an integral domain. Let $a := \prod_{i=1}^{m} a_i^{r_i}$ be a prime factorization of a in R. Suppose that for every $i, 1 \leqslant i \leqslant m$, whenever $(a_i, b)R$ is a proper ideal, then $\prod_{j \neq i} {a_j}^{s_j} \notin (a_i, b)R$, for any integer $s_j \geqslant 0$. Then the following are equivalent.*

(i) *C is a UFD.*

(ii) *For each i, $1 \leqslant i \leqslant m$, either a_i is prime in C or $a_i \in C^*$.*

(iii) *For each i, $1 \leqslant i \leqslant m$, either $(a_i, b)R$ is a prime ideal of R or $(a_i, b)R = R$.*

Proof. (ii) ⇔ (iii) : For every $j, 1 \leqslant j \leqslant m$, we have

$$\frac{C}{a_j C} \cong \left(\frac{R}{(a_j, b)} \right)^{[1]}. \tag{3.1.4}$$

Note that a_j is either a prime element or a unit in C according as $\frac{C}{a_j C}$ is either an integral domain or a zero ring. Hence the equivalence follows from (3.1.4).

(i) $\Rightarrow$ (ii) : Note that $R \hookrightarrow C \hookrightarrow R[a_1^{-1}, \ldots, a_m^{-1}]$. Suppose $a_j \notin C^*$ for some $j, 1 \leqslant j \leqslant m$. Since C is a UFD, it is enough to show that a_j is irreducible. Suppose $a_j = c_1c_2$ for some $c_1, c_2 \in C$. If $c_1, c_2 \in R$, then either $c_1 \in R^*$ or $c_2 \in R^*$, as a_j is irreducible in R. Therefore, we can assume that at least one of them is not in R. Suppose $c_1 \notin R$. Let $c_1 = \frac{h_1}{a_1^{i_1} \cdots a_m^{i_m}}$ and $c_2 = \frac{h_2}{a_1^{l_1} \cdots a_m^{l_m}}$, for some $h_1, h_2 \in R$ and $i_s, l_s \geqslant 0$, $1 \leqslant s \leqslant m$. Therefore, we have

$$h_1h_2 = a_j(a_1^{i_1+l_1} \cdots a_m^{i_m+l_m}). \tag{3.1.5}$$

As $c_1 \notin R$, using (3.1.5), without loss of generality, we can assume that

$$c_1 = \lambda \frac{\prod_{i \leqslant s} a_i^{p_i}}{\prod_{i=s+1}^{m} a_i^{p_i}}, \text{for some } \lambda \in k^* \text{ and } s < m, \tag{3.1.6}$$

where $p_i \geqslant 0$ for $i \leqslant s$, and $p_i > 0$ for $i \geqslant s + 1$.

Now when $m = 1$ or $p_i = 0$ for every $i \leqslant s$, then $c_1 \in C^*$, and we are done. If not, then $m > 1$ and without loss of generality, we assume that $p_1 > 0$.

Therefore, from (3.1.6), we have $a_1^{p_1} \ldots a_s^{p_s} \in a_iC \cap R = (a_i, b)\, R$ for every $i \geqslant s + 1$. Hence by the given hypothesis, for every $i \geqslant s + 1$, we get that $(a_i, b)R = R$, i.e., $a_i \in C^*$. Thus we get

$$c_1 = \mu \prod_{i \leqslant s} a_i^{p_i} \tag{3.1.7}$$

for some $\mu \in C^*$ and hence

$$\mu c_2 = \frac{a_j}{\prod_{i \leqslant s} a_i^{p_i}}.$$

If $\prod_{i \leqslant s} a_i^{p_i} \in a_jR$, then $c_2 \in C^*$ and we are done. If not, then $a_j \in a_iC \cap R = (a_i, b)R$ for every $i \leqslant s$ with $p_i > 0$. If for such an a_i with $i \leqslant s$ and $p_i > 0$, $(a_i, b)R$ is a proper ideal then we get a contradiction by the given hypothesis. Therefore, for all such a_i, $(a_i, b)R = R$ and hence by (3.1.7), $c_1 \in C^*$ and we are done.

Therefore, we obtain that a_j must be an irreducible element in C, hence prime in C.

(ii) $\Rightarrow$ (i) : Without loss of generality we assume that $a_1, \ldots, a_{i-1} \in C^*$ and $a_i, \ldots, a_m$ are primes in C for some i, $1 \leqslant i \leqslant m$. Since $C[a_1^{-1}, \ldots, a_m^{-1}] = C[a_i^{-1}, \ldots, a_m^{-1}] = R[a_1^{-1}, \ldots, a_m^{-1}]$ is a UFD, by Nagata's criterion for UFD

([28, Theorem 20.2]), we obtain that C is a UFD. $\square$

As a consequence we obtain the following equivalent conditions for the ring A to be a UFD.

Proposition 3.1.12. *The following statements are equivalent:*

(i) *A is a UFD.*

(ii) *For each $j, 1 \leqslant j \leqslant m$, either x_j is prime in A or $x_j \in A^*$.*

(iii) *For each $j, 1 \leqslant j \leqslant m$, $F_j := F(X_1, \ldots, X_{j-1}, 0, X_{j+1}, \ldots, X_m, Z, T)$ is either an irreducible element in $k[X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_m, Z, T]$ or $F_j \in k^*$.*

In particular, if $F(X_1, \ldots, X_m, Z, T) = f(Z,T) + (X_1 \cdots X_m)g$, for some $g \in k[X_1, \ldots, X_m, Z, T]$, then the following statements are equivalent:

(i′) *A is a UFD.*

(ii′) *For each $j, 1 \leqslant j \leqslant m$, either x_j is prime in A or $x_j \in A^*$.*

(iii′) *$f(Z,T)$ is either an irreducible element in $k[Z,T]$ or $f(Z,T) \in k^*$.*

Proof. Putting $R = k[X_1, \ldots, X_m, Z, T]$, $a_i = X_i$ for $1 \leq i \leq m$ and $b = F(X_1, \ldots, X_m, Z, T)$ in Proposition 3.1.11, we have $A = \frac{R[Y]}{(aY-b)}$ where $a = X_1^{r_1} \cdots X_m^{r_m}$. Since $f(Z,T) = F(0, \ldots, 0, Z, T) \neq 0$, it follows that for every i, $\prod_{j \neq i} X_j^{s_j} \notin (X_i, F(X_1, \ldots, X_m, Z, T))k[X_1, \ldots, X_m, Z, T]$, for $j \neq i$, whenever it is a proper ideal. Therefore, the result follows from Proposition 3.1.11. $\square$

The next result gives a condition for A to be flat over the subring E.

Lemma 3.1.13. *Let $F(X_1, \ldots, X_m, Z, T) = f(Z,T) + (X_1 \cdots X_m)g$, for some $g \in k[X_1, \ldots, X_m, Z, T]$. Then A is a flat E-algebra.*

Proof. Let $\mathfrak{q} \in \operatorname{Spec}(A)$ and $\mathfrak{p} = \mathfrak{q} \cap E \in \operatorname{Spec}(E)$. Note that $A[x_1^{-1}, \ldots, x_m^{-1}] = E[x_1^{-1}, \ldots, x_m^{-1}, z, t]$. Hence $A_\mathfrak{q}$ is a flat $E_\mathfrak{p}$ algebra if $x_i \notin \mathfrak{p}$ for every $i, 1 \leqslant i \leqslant m$. Now suppose $x_i \in \mathfrak{p}$ for some i. Consider the following maps:

$$k[x_i]_{(x_i)} \hookrightarrow E_\mathfrak{p} \hookrightarrow A_\mathfrak{q}.$$

We observe that both $E_\mathfrak{p}$ and $A_\mathfrak{q}$ are flat over $k[x_i]_{(x_i)}$ and

$$\frac{A}{x_i A} \cong \frac{(E/x_i E)[Y, Z, T]}{(f(Z,T))}.$$

Since $\frac{k[Y,Z,T]}{(f(Z,T))}$ is a flat k-algebra, it follows that $\frac{A}{x_iA}\left(=\frac{k[Y,Z,T]}{(f(Z,T))}\otimes_k \frac{E}{x_iE}\right)$ is a flat $\frac{E}{x_iE}$-algebra. Hence it follows that $\frac{A_q}{x_iA_q}$ is flat over $\frac{E_p}{x_iE_p}$ and therefore, by Lemma 2.1.5, we get that A_q is flat over E_p. Thus, A is locally flat over E, and hence A is flat over E. $\square$

The next result gives some necessary and sufficient conditions for A to be an affine fibration over E.

Proposition 3.1.14. *Let $F(X_1,\ldots,X_m,Z,T)=f(Z,T)+(X_1\cdots X_m)g$, for some $g\in k[X_1,\ldots,X_m,Z,T]$. Then the following statements are equivalent:*

(i) *A is an $\mathbb{A}^2$-fibration over E.*

(ii) $\dfrac{A}{(x_1,\ldots,x_m)A}=k^{[2]}$.

(iii) *$f(Z,T)$ is a line in $k[Z,T]$, i.e., $\frac{k[Z,T]}{(f(Z,T))}=k^{[1]}$.*

Proof. (i) $\Rightarrow$ (ii) : Since A is an $\mathbb{A}^2$-fibration over E, for every $p\in \operatorname{Spec}(E)$, we have

$$A\otimes_E \frac{E_p}{pE_p}=\left(\frac{E_p}{pE_p}\right)^{[2]}.$$

Hence for $p=(x_1,\ldots,x_m)E$, we get $\frac{A}{(x_1,\ldots,x_m)A}=k^{[2]}$.

(ii) $\Rightarrow$ (iii) : Since

$$k\hookrightarrow \frac{k[Z,T]}{(f(Z,T))}\hookrightarrow \frac{A}{(x_1,\ldots,x_m)A}=\left(\frac{k[Z,T]}{(f(Z,T))}\right)^{[1]}=k^{[2]},$$

we obtain that $\frac{k[Z,T]}{(f(Z,T))}$ is a one dimensional normal domain. Hence, $f(Z,T)$ is a line in $k[Z,T]$ by Theorem 2.1.6.

(iii) $\Rightarrow$ (i) : By Lemma 3.1.13, A is a flat E-algebra. Let $p\in\operatorname{Spec}(E)$ and A_p denotes the localisation of the ring A with respect to the multiplicatively closed set $E\setminus p$. We now show A is an $\mathbb{A}^2$-fibration over E.

Case 1: If $x_i\notin p$ for every $i=1,\ldots,m$, then $A_p=E_p[z,t]$. Hence,

$$\frac{A_p}{pA_p}=A\otimes_E\frac{E_p}{pE_p}=\left(\frac{E_p}{pE_p}\right)^{[2]}. \tag{3.1.8}$$

Case 2: Suppose $x_i\in p$ for some $i, 1\leqslant i\leqslant m$. Since $f(Z,T)$ is a line in $k[Z,T]$, we have

$$\frac{A}{x_iA} = \frac{(E/x_iE)[Y, Z, T]}{(f(Z, T))} = \left(\frac{E}{x_iE}\right)^{[2]}.$$

Hence, $A \otimes_E \kappa(p) = \kappa(p)^{[2]}$, where $\kappa(p) = \frac{E_p}{pE_p}$.

Therefore, by the above two cases, we obtain that A is an $\mathbb{A}^2$-fibration over E. □

The following lemma may be known but in the absence of a ready reference, we give below a proof.

Lemma 3.1.15. *Let $E = k[x_1, \ldots, x_m]$, $u = x_1 \cdots x_m$ and $R = \frac{E}{uE}$. Then $G_0(R) \neq 0$.*

Proof. By Lemma 2.2.1 the inclusion $E \hookrightarrow E[u^{-1}]$ induces the exact sequence:

$$G_1(E) \longrightarrow G_1(E[u^{-1}]) \longrightarrow G_0(R) \longrightarrow G_0(E) \xrightarrow{h} G_0(E[u^{-1}]) \longrightarrow 0. \tag{3.1.9}$$

By Remark 2.2.4 and repeated application of Lemma 2.2.3, we see that the map

$$G_1(E) \cong k^* \hookrightarrow G_1(E[u^{-1}]) \cong k^* \oplus \mathbb{Z}^m$$

is a split inclusion and hence can not be surjective. Therefore, $G_0(R) \neq 0$ by (3.1.9). □

3.2 Main theorems

We now prove an extended version of Theorem A1 with 11 equivalent statements.

Theorem 3.2.1. *Let $F(X_1, \ldots, X_m, Z, T) = f(Z, T) + (X_1 \cdots X_m)g$, for some $g \in k[X_1, \ldots, X_m, Z, T]$. Then the following statements are equivalent:*

(i) $k[X_1, \ldots, X_m, Y, Z, T] = k[X_1, \ldots, X_m, G]^{[2]}$.

(ii) $k[X_1, \ldots, X_m, Y, Z, T] = k[G]^{[m+2]}$.

(iii) $A = k[x_1, \ldots, x_m]^{[2]}$.

(iv) $A = k^{[m+2]}$.

(v) $k[Z, T] = k[f(Z, T)]^{[1]}$.

(vi) $A^{[l]} = k^{[l+m+2]}$ *for some* $l \geqslant 0$ *and* $k[x_1, \ldots, x_m, z, t] \subsetneqq \mathrm{DK}(A)$.

(vii) A *is an* $\mathbb{A}^2$*-fibration over* $k[x_1, \ldots, x_m]$ *and* $k[x_1, \ldots, x_m, z, t] \subsetneqq \mathrm{DK}(A)$.

(viii) $f(Z,T)$ *is a line in* $k[Z,T]$ *and* $k[x_1, \ldots, x_m, z, t] \subsetneqq \mathrm{DK}(A)$.

(ix) $f(Z,T)$ *is a line in* $k[Z,T]$ *and* $\mathrm{ML}(A) = k$.

(x) $A^{[l]} = k^{[l+m+2]}$ *for* $l \geqslant 0$ *and* $\mathrm{ML}(A) = k$.

(xi) A *is an* $\mathbb{A}^2$*-fibration over* $k[x_1, \ldots, x_m]$ *and* $\mathrm{ML}(A) = k$.

Proof. Note that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (vi), (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv), (iii) $\Rightarrow$ (vii), (iii) $\Rightarrow$ (x) and (iii) $\Rightarrow$ (xi) follow trivially.

(vi) $\Rightarrow$ (v) : We first assume that k is algebraically closed. Since $k[x_1, \ldots, x_m, z, t] \subsetneqq \mathrm{DK}(A)$, by Proposition 3.1.5, we may assume that

$$f(Z,T) = a_0(Z) + a_1(Z)T.$$

Suppose $a_1(Z) = 0$, i.e., $f(Z,T) = a_0(Z)$. As A is a UFD, by Proposition 3.1.12, we obtain that $a_0(Z)$ is irreducible in $k[Z,T]$ or $a_0(Z) \in k^*$. If $a_0(Z) \in k^*$, then for every i, $1 \leqslant i \leqslant m$, $x_i \in A^*$. This contradicts the fact that $A^* = (A^{[l]})^* = (k^{[m+l+2]})^* = k^*$. Therefore, $a_0(Z)$ is irreducible in $k[Z,T]$ and hence a linear polynomial as k is algebraically closed. Thus f is a coordinate of $k[Z,T]$.

Now suppose $a_1(Z) \neq 0$. We show that $a_1(Z) \in k^*$. Note that $\gcd(a_0(Z), a_1(Z)) = 1$ in $k[Z]$, as $f(Z,T)$ is irreducible in $k[Z,T]$.

Consider the inclusions $k \xhookrightarrow{\alpha} E \xhookrightarrow{\beta} A \xhookrightarrow{\gamma} A^{[l]}$. Since $E = k^{[m]}$ and $A^{[l]} = k^{[m+l+2]}$, by Lemma 2.2.3(a), the inclusions α, γ and $\gamma\beta\alpha$ induce isomorphisms

$$G_i(k) \xrightarrow[G_i(\alpha)]{\cong} G_i(E), \quad G_i(A) \xrightarrow[G_i(\gamma)]{\cong} G_i(A^{[l]}) \quad \text{and} \quad G_i(k) \xrightarrow[G_i(\gamma\beta\alpha)]{\cong} G_i(A^{[l]}),$$

respectively. Hence we get that β induces an isomorphism $G_i(E) \xrightarrow[G_i(\beta)]{\cong} G_i(A)$, for every $i \geqslant 0$. Let $u = x_1 \cdots x_m \in E$. Note that $A[u^{-1}] = E[u^{-1}, z, t]$. Therefore, by Lemma 2.2.3(a), the inclusion $E[u^{-1}] \hookrightarrow A[u^{-1}]$ induces isomorphisms $G_i(E[u^{-1}]) \xrightarrow{\cong} G_i(A[u^{-1}])$ for $i \geqslant 0$. By Lemma 3.1.13, the inclusion map $\beta : E \longrightarrow A$ is a flat map. Therefore, by Lemma 2.2.2, $\beta : E \hookrightarrow A$

induces the following commutative diagram for $i \geqslant 1$:

$$\begin{tikzcd}
G_i(E) \arrow[r] \arrow[d, "G_i(\beta)"', "\cong"] & G_i(E[u^{-1}]) \arrow[r] \arrow[d, "\cong"] & G_{i-1}(\frac{E}{uE}) \arrow[r] \arrow[d] & G_{i-1}(E) \arrow[r] \arrow[d, "\cong"] & G_{i-1}(E[u^{-1}]) \arrow[d, "\cong"] \\
G_i(A) \arrow[r] & G_i(A[u^{-1}]) \arrow[r] & G_{i-1}(\frac{A}{uA}) \arrow[r] & G_{i-1}(A) \arrow[r] & G_{i-1}(A[u^{-1}]).
\end{tikzcd}$$

From the above diagram, applying the Five Lemma we obtain that the map $\frac{E}{uE} \xrightarrow{\overline{\beta}} \frac{A}{uA}$, induced by $E \xhookrightarrow{\beta} A$, induces isomorphism of groups

$$G_i(\tfrac{E}{uE}) \xrightarrow[G_i(\overline{\beta})]{\cong} G_i(\tfrac{A}{uA}), \tag{3.2.1}$$

for every $i \geqslant 0$. Let $R = \frac{E}{uE}$. Now using the structure of $f(Z,T)$ we get an isomorphism as follows

$$\tfrac{A}{uA} \xrightarrow[\eta]{\cong} R\left[Y, Z, \tfrac{1}{a_1(Z)}\right] .$$

Further, note that $\overline{\beta}$ factors through the following maps

$$\overline{\beta} : R \xrightarrow{\gamma_1} R[Z] \xrightarrow{\gamma_2} R\left[Z, \frac{1}{a_1(Z)}\right] \xrightarrow{\gamma_3} R\left[Y, Z, \frac{1}{a_1(Z)}\right] \xrightarrow{\eta^{-1}} A/uA.$$

Since $\overline{\beta}, \gamma_1, \gamma_3, \eta^{-1}$ induce isomorphisms of G_i-groups for $i \geqslant 0$, we obtain that γ_2 induces an isomorphisms of groups

$$G_i(R[Z]) \xrightarrow[G_i(\gamma_2)]{\cong} G_i\left(R\left[Z, \tfrac{1}{a_1(Z)}\right]\right), \tag{3.2.2}$$

for every $i \geqslant 0$. Since k is algebraically closed, if $a_1(Z) \notin k^*$, then we have $a_1(Z) = \lambda \prod_{i=1}^{n} (Z - \lambda_i)^{m_i}$ where $\lambda \in k^*, \lambda_i \in k$, $m_i \geqslant 1$ and $\lambda_i \neq \lambda_j$ for $i \neq j$. Now we have

$$R\left[Z, \frac{1}{a_1(Z)}\right] = R\left[Z, \frac{1}{Z - \lambda_1}, \dots, \frac{1}{Z - \lambda_n}\right].$$

Let $R_i = R\left[Z, \frac{1}{Z-\lambda_1}, \dots, \frac{1}{Z-\lambda_i}\right]$, for $i \geqslant 1$ and $R_0 = R[Z]$. For every $i \geqslant 1$, we have the map $R[Z] \hookrightarrow R_{i-1}$ is flat. Therefore, applying Lemma 2.2.2, we get the following commutative diagram for $j \geqslant 1$:

$$\begin{array}{ccccccccc}
G_j\left(\frac{R[Z]}{(Z-\lambda_i)}\right) & \longrightarrow & G_j(R[Z]) & \longrightarrow & G_j\left(R\left[Z, \frac{1}{Z-\lambda_i}\right]\right) & \longrightarrow & G_{j-1}\left(\frac{R[Z]}{(Z-\lambda_i)}\right) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \mu_j} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mu_{j-1}} & & \\
G_j\left(\frac{R_{i-1}}{Z-\lambda_i}\right) & \longrightarrow & G_j(R_{i-1}) & \longrightarrow & G_j\left(R_{i-1}\left[\frac{1}{Z-\lambda_i}\right]\right) & \longrightarrow & G_{j-1}\left(\frac{R_{i-1}}{Z-\lambda_i}\right) & \longrightarrow & \cdots
\end{array} \tag{3.2.3}$$

Now by Lemma 2.2.3(b) we get the following split exact sequence for each $j \geqslant 1$:

$$0 \longrightarrow G_j(R[Z]) \longrightarrow G_j\left(R\left[Z, \frac{1}{Z-\lambda_i}\right]\right) \longrightarrow G_{j-1}\left(\frac{R[Z]}{(Z-\lambda_i)}\right) \longrightarrow 0.$$

Since the inclusion $R[Z] \hookrightarrow R_{i-1}$ induces isomorphism of the rings $\frac{R[Z]}{(Z-\lambda_i)} \xrightarrow{\cong} \frac{R_{i-1}}{(Z-\lambda_i)}$, for every i, $1 \leqslant i \leqslant n$, the maps μ_{j-1}'s are isomorphisms for every $j \geqslant 1$. Hence from (3.2.3), the following exact sequence is also split exact:

$$0 \longrightarrow G_j(R_{i-1}) \longrightarrow G_j\left(R_{i-1}\left[\frac{1}{Z-\lambda_i}\right]\right) \longrightarrow G_{j-1}\left(\frac{R_{i-1}}{Z-\lambda_i}\right) \longrightarrow 0$$

Thus the inclusion $R_{i-1} \hookrightarrow R_i = R_{i-1}[\frac{1}{Z-\lambda_i}]$ will induce a group isomorphism

$$G_j(R_i) \cong G_j(R_{i-1}) \oplus G_{j-1}(R)$$

for every $j \geqslant 1$ and i, $1 \leqslant i \leqslant n$. In particular, for $j = 1$, inductively we get that the inclusion $R[Z] \xhookrightarrow{\gamma_2} R_n = R\left[Z, \frac{1}{a_1(Z)}\right]$ induces the isomorphism

$$G_1\left(R\left[Z, \frac{1}{a_1(Z)}\right]\right) \cong G_1(R[Z]) \oplus (G_0(R))^n. \tag{3.2.4}$$

As $G_0(R) \neq 0$ by Lemma 3.1.15, (3.2.4) contardicts (3.2.2) if $n > 0$ i.e., if $a_1(Z) \notin k^*$. Therefore, we have $a_1(Z) \in k^*$. Hence we obtain that $k[Z,T] = k[f]^{[1]}$.

When k is not algebraically closed, consider the ring $\overline{A} = A \otimes_k \overline{k}$, where $\overline{k}$ is an algebraic closure of k. Note that $\overline{A}^{[l]} = \overline{k}^{[m+l+2]}$ and $\overline{k}[x_1, \ldots, x_m, z, t] \subsetneqq \mathrm{DK}(\overline{A})$. Hence by the above argument, we have $\overline{k}[Z,T] = \overline{k}[f]^{[1]}$.
Case 1: If k is a finite field, then by Lemma 2.1.8, $k[Z,T] = k[f]^{[1]}$.

Case 2: Let k be an infinite field. Then by Proposition 3.1.5 we can assume that $f(Z,T) = a_0(Z) + a_1(Z)T$ in $k[Z,T]$. As f is a coordinate in $\overline{k}[Z,T]$, this is possible only if either $a_1(Z) = 0$ and $a_0(Z)$ is linear in Z or if $a_1(Z) \in \overline{k}^*$. Hence in either case

$$k[Z,T] = k[f]^{[1]}.$$

(v) $\Rightarrow$ (i) : Let $h \in k[Z,T]$ such that $k[Z,T] = k[f,h]$. Therefore, without loss of generality we can assume that $f = Z$ and $h = T$. Hence $G = X_1^{r_1} \cdots X_m^{r_m} Y - X_1 \cdots X_m g - Z$, for some $g \in k[X_1, \ldots, X_m, Z, T]$. Now by Lemma 3.1.1, we get that

$$k[X_1, \ldots, X_m, Y, Z, T] = k[X_1, \ldots, X_m, G, T]^{[1]} = k[X_1, \ldots, X_m, G]^{[2]}.$$

Therefore the equivalence of the first six statements are established.

By Proposition 3.1.7, (viii) $\Rightarrow$ (v). By Proposition 3.1.14, (vii) $\Leftrightarrow$ (viii) and (ix) $\Leftrightarrow$ (xi). Therefore, (viii) $\Rightarrow$ (v) $\Leftrightarrow$ (iii) $\Rightarrow$ (vii) $\Leftrightarrow$ (viii).

By Lemma 3.1.4 and Proposition 3.1.8(b), (x) $\Rightarrow$ (vi), (xi) $\Rightarrow$ (vii). We now see that the following hold:

(ix) $\Leftrightarrow$ (xi) $\Rightarrow$ (vii) $\Leftrightarrow$ (iii) $\Rightarrow$ (xi).

(x) $\Rightarrow$ (vi) $\Leftrightarrow$ (iii) $\Rightarrow$ (x).

Hence equivalence of all the statements are established. □

The following example shows that the answer to Question 2 (as in Introduction) is not affirmative in general, i.e., without the hypothesis that F is of the form as in Theorem 3.2.1, the condition $k[Z,T] = k[f]^{[1]}$ is not sufficient for A to be $k^{[m+2]}$, for $m \geqslant 2$. In particular, it is not sufficient to ensure $k[X_1, \ldots, X_m, Y, Z, T] = k[X_1, \ldots, X_m, G]^{[2]}$.

Example 3.2.2. Let

$$A = \frac{k[X_1, X_2, Y, Z, T]}{(X_1^2 X_2^2 Y - F)},$$

where $F(X_1, X_2, Z, T) = X_1 Z + X_2 + Z$. Note that here $f(Z,T) = F(0,0,Z,T) = Z$ is a coordinate of $k[Z,T]$. Also it is easy to see that A is regular. But as $F(X_1, 0, Z, T) = X_1 Z + Z$ is neither irreducible nor a unit, by Proposition 3.1.12, A is not even a UFD. Therefore, $A \neq k^{[4]}$.

We shall now prove (Theorem 3.2.6) three additional statements which are equivalent to each of the eleven statements in Theorem 3.2.1. We start with two technical lemmas which will be used in the proof of Theorem 3.2.6. The aim of these lemmas is to show that if $\mathrm{DK}(A) = A$, then we can construct another affine domain A_l of similar structure to A, such that $\mathrm{DK}(A_l) = A_l$ and $\dim(A_l) < \dim(A)$.

Lemma 3.2.3. *Suppose there exists an exponential map ϕ on A such that $A^\phi \nsubseteq k[x_1, \dots, x_m, z, t]$. Then there exists an integral domain*

$$\widehat{A} \cong \frac{k[X_1, \dots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - h(Z, T))}$$

and a non-trivial exponential map $\widehat{\phi}$ on $\widehat{A}$, induced by ϕ, such that $\widehat{y} \in \widehat{A}^{\widehat{\phi}}$, where $\widehat{y}$ denote the image of Y in $\widehat{A}$. Moreover, if $x_1, \dots, x_m \in A^\phi$, then $\widehat{x_1}, \dots, \widehat{x_m} \in \widehat{A}^{\widehat{\phi}}$, where $\widehat{x_1}, \dots, \widehat{x_m}$ denote the images of $X_1, \dots, X_m$ in $\widehat{A}$.

Proof. Since $A^\phi \nsubseteq k[x_1, \dots, x_m, z, t]$, there exists $g \in A^\phi$ and g has a monomial summand of the form $g_{\bar{\iota}} = x_1^{i_1} \cdots x_m^{i_m} y^j z^p t^q$ where $\bar{\iota} = (i_1, \dots, i_m) \in \mathbb{Z}^m_{\geqslant 0}$, $j \geqslant 1, p \geqslant 0, q \geqslant 0$ and there exists some i_s such that $i_s < r_s$. Without loss of generality, we assume that $i_s = i_1$.

We now consider the proper $\mathbb{Z}$-filtration on A with respect to $(-1, 0, \dots, 0) \in \mathbb{Z}^m$. Let $\widetilde{A}$ be the associated graded ring and $\widetilde{F}(x_1, \dots, x_m, z, t)$ denote the highest degree homogeneous summand of $F(x_1, \dots, x_m, z, t)$. Then $\widetilde{F}(x_1, \dots, x_m, z, t) = F(0, x_2, \dots, x_m, z, t)$. Since $f(z, t) \neq 0$, we have $x_i \nmid \widetilde{F}$, for every $1 \leqslant i \leqslant m$. Therefore, by Lemma 3.1.3, we have

$$\widetilde{A} \cong \frac{k[X_1, \dots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - \widetilde{F})}.$$

For every $h \in A$, let $\widetilde{h}$ denote its image in $\widetilde{A}$. By Theorem 2.1.4, ϕ will induce a non-trivial homogeneous exponential map $\widetilde{\phi}$ on $\widetilde{A}$ such that $\widetilde{g} \in \widetilde{A}^{\widetilde{\phi}}$. From the chosen filtration on A, it is clear that $\widetilde{y} \mid \widetilde{g}$ and hence $\widetilde{y} \in \widetilde{A}^{\widetilde{\phi}}$.

We now consider the $\mathbb{Z}$-filtration on $\widetilde{A}$ with respect to $(-1, \dots, -1) \in \mathbb{Z}^m$. By Theorem 2.1.4, we have a homogeneous non-trivial exponential map $\widehat{\phi}$ on the associated graded ring $\widehat{A}$ induced by $\widetilde{\phi}$. By Lemma 3.1.3,

$$\widehat{A} \cong \frac{k[X_1, \dots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - f(Z, T))},$$

since $f(Z, T) = \widetilde{F}(0, \dots, 0, Z, T)$ is the highest degree homogeneous summand of $\widetilde{F}$. For every $a \in \widetilde{A}$, let $\widehat{a}$ denote its image in $\widehat{A}$. Since $\widetilde{y} \in \widetilde{A}^{\widetilde{\phi}}$, we have $\widehat{y} \in \widehat{A}^{\widehat{\phi}}$. The weights of $\widehat{x_1}, \dots, \widehat{x_m}, \widehat{y}, \widehat{z}, \widehat{t}$ are clear from the chosen filtration on $\widetilde{A}$. $\square$

Lemma 3.2.4. *Suppose $m > 1$ and $k[x_1, \ldots, x_m, z, t] \subsetneqq \mathrm{DK}(A)$. Then there exists an integer $l \in \{1, \ldots, m\}$ and an integral domain A_l such that*

$$A_l \cong \frac{k(X_l)[X_1, \ldots, X_{l-1}, X_{l+1}, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - f(Z,T))},$$

where the image y' of Y in A_l belongs to $\mathrm{DK}(A_l)$. In particular, $\mathrm{DK}(A_l) = A_l$.

Proof. By Lemma 3.2.3, there exists an integral domain

$$\widehat{A} \cong \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - f(Z,T))},$$

and a non-trivial homogeneous exponential map $\widehat{\phi}$ on $\widehat{A}$ such that for every $i \in \{1, \ldots, m\}$, $wt(\widehat{x_i}) = -1$, $wt(\widehat{z}) = wt(\widehat{t}) = 0$, $wt(\widehat{y}) = r_1 + \cdots + r_m$ and $\widehat{y} \in \widehat{A}^{\widehat{\phi}}$, where $\widehat{x_1}, \ldots, \widehat{x_m}, \widehat{y}, \widehat{z}, \widehat{t}$ denote the images of $X_1, \ldots, X_m, Y, Z, T$ respectively in $\widehat{A}$.

Since $m > 1$, $\operatorname{tr.deg}_k(\widehat{A}^{\widehat{\phi}}) \geqslant 3$. Hence if $\widehat{A}^{\widehat{\phi}} \subseteq k[\widehat{y}, \widehat{z}, \widehat{t}]$, then $\widehat{A}^{\widehat{\phi}} = k[\widehat{y}, \widehat{z}, \widehat{t}]$. But then $\widehat{x}_1^{r_1} \cdots \widehat{x}_m^{r_m} \widehat{y} = f(\widehat{z}, \widehat{t}) \in \widehat{A}^{\widehat{\phi}}$, that means $\widehat{\phi}$ is trivial, as $\widehat{A}^{\widehat{\phi}}$ is factorially closed (cf. Lemma 2.1.1(i)). This is a contradiction. Therefore, there exists $h_1 \in \widehat{A}^{\widehat{\phi}} \setminus k[\widehat{y}, \widehat{z}, \widehat{t}]$, which is homogeneous with respect to the grading on $\widehat{A}$. Let

$$h_1 = h'(\widehat{x_1}, \ldots, \widehat{x_m}, \widehat{z}, \widehat{t}) + \sum_{\bar{\imath}=(i_1,\ldots,i_m)\in\mathbb{Z}^m_{\geqslant 0},\, j>0\, p,q\geqslant 0} \lambda_{\bar{\imath}\, jpq}\, \widehat{x_1}^{i_1} \cdots \widehat{x_m}^{i_m} \widehat{y}^j \widehat{z}^p \widehat{t}^q$$

such that $\lambda_{\bar{\imath}\, jpq} \in k$ and for every $j > 0$ and $\bar{\imath} = (i_1, \ldots, i_m) \in \mathbb{Z}^m_{\geqslant 0}$, there exists $s_j \in \{1, \ldots, m\}$ such that $i_{s_j} < r_{s_j}$. Now we have the following two cases:

Case 1: If $h' \notin k[\widehat{z}, \widehat{t}]$, then it has a monomial summand h_2 such that $\widehat{x_l} \mid h_2$, for some $l, 1 \leqslant l \leqslant m$.

Case 2: If $h' \in k[\widehat{z}, \widehat{t}]$, then each of the monomial summands of the form $\lambda_{\bar{\imath}\, jpq} \widehat{x_1}^{i_1} \cdots \widehat{x_m}^{i_m} \widehat{y}^j \widehat{z}^p \widehat{t}^q$ of h_1 has degree zero. That means, $j(r_1 + \cdots + r_m) = i_1 + \cdots + i_m$. Therefore, for every $j > 0$ and $\bar{\imath} \in \mathbb{Z}^m_{\geqslant 0}$, there exists $l_j \in \{1, \ldots, m\}, l_j \neq s_j$ such that $i_{l_j} > j r_{l_j}$, as $i_{s_j} < r_{s_j}$. We choose one of these l_j's and call it l.

We consider the $\mathbb{Z}$-filtration on $\widehat{A}$ with respect to $(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^m$, where the l-th entry is 1. Let $\overline{A}$ be the associated graded ring of $\widehat{A}$, and by Lemma 3.1.3,

$$\overline{A} \cong \frac{k[X_1, \dots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - f(Z,T))}.$$

For every $a \in \widehat{A}$, let $\overline{a}$ denote its image in $\overline{A}$. By Theorem 2.1.4, $\widehat{\phi}$ induces a non-trivial exponential map $\overline{\phi}$ on the associated graded ring $\overline{A}$ such that $\overline{h_1} \in \overline{A}^{\overline{\phi}}$. From cases 1 and 2, it is clear that $\overline{h_1}$ is divisible by $\overline{x_l}$. Hence we get that $\overline{x_l} \in \overline{A}^{\overline{\phi}}$. Therefore, by Lemma 2.1.1(iii), $\overline{\phi}$ will induce a non-trivial exponential map ϕ_l on $A_l = \overline{A} \otimes_{k[\overline{x_l}]} k(\overline{x_l})$, where

$$A_l \cong \frac{k(X_l)[X_1, \dots, X_{l-1}, X_{l+1}, \dots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - f(Z,T))}.$$

Since $A_l^{\phi_l} = \overline{A}^{\overline{\phi}} \otimes_{k[\overline{x_l}]} k(\overline{x_l})$ and $\overline{y} \in \overline{A}^{\overline{\phi}}$, the image y' of Y in A_l is in $A_l^{\phi_l}$. Therefore, by Lemma 3.1.4, we get that $DK(A_l) = A_l$. □

The following lemma is an important step to prove (vi) ⇒ (v) of Theorem 3.2.6.

Lemma 3.2.5. *Let $F(X_1, \dots, X_m, Z, T) = f(Z,T) + (X_1 \cdots X_m)g$, for some $g \in k[X_1, \dots, X_m, Z, T]$. If $A = k^{[m+2]}$, then there exists an exponential map ϕ such that $k[x_1, \dots, x_m] \subseteq A^{\phi} \not\subseteq k[x_1, \dots, x_m, z, t]$.*

Proof. By Theorem 3.2.1, $k[Z,T] = k[f, f_1] = k[f]^{[1]}$ for some $f_1 \in k[Z,T]$. Now for $G = X_1^{r_1} \cdots X_m^{r_m} Y - F$, by Lemma 3.1.1 we have

$$k[X_1, \dots, X_m, Y, Z, T] = k[X_1, \dots, X_m, Y, f, f_1] = k[X_1, \dots, X_m, G, f_1, f_2]$$

for some $f_2 \in k[X_1, \dots, X_m, Y, Z, T]$. Hence $A = k[x_1, \dots, x_m, f_1(z,t), \overline{f_2}] = k[x_1, \dots, x_m]^{[2]}$ where $\overline{f_2}$ denote the image of f_2 in A. Since $k[x_1, \dots, x_m, z, t] \subsetneq A$ it follows that $\overline{f_2} \in A \setminus k[x_1, \dots, x_m, z, t]$. We now consider the exponential map $\phi : A \to A[W]$ such that

$$\phi(x_i) = x_i, \text{ for every } i, 1 \leqslant i \leqslant m, \quad \phi(f_1) = f_1 + W, \quad \phi(\overline{f_2}) = \overline{f_2}.$$

Therefore, $A^{\phi} = k[x_1, \dots, x_m, \overline{f_2}]$ and hence the assertion follows. □

We now add three more equivalent statements to Theorem 3.2.1.

Theorem 3.2.6. *Let A be the affine domain as in* (3.1.1) *and $F(X_1, \dots, X_m, Z, T) = f(Z,T) + (X_1 \cdots X_m)g$, for some $g \in k[X_1, \dots, X_m, Z, T]$. Then the following statements are equivalent.*

(i) *A is a UFD, $k[x_1, \ldots, x_m, z, t] \subsetneqq \mathrm{DK}(A)$ and $\left(\frac{A}{x_iA}\right)^* = k^*$, for every $i \in \{1, \ldots, m\}$.*

(ii) $k[Z, T] = k[f]^{[1]}$.

(iii) $A = k[x_1, \ldots, x_m]^{[2]}$.

(iv) *A is a UFD, $\mathrm{ML}(A) = k$ and $\left(\frac{A}{x_iA}\right)^* = k^*$ for every $i \in \{1, \ldots, m\}$.*

(v) *A is geometrically factorial over k and there exists an exponential map ϕ on A satisfying $k[x_1, \ldots, x_m] \subseteq A^{\phi} \nsubseteq k[x_1, \ldots, x_m, z, t]$.*

(vi) $A = k^{[m+2]}$.

Proof. (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (vi) follows from Theorem 3.2.1. (iii) $\Rightarrow$ (iv) holds trivially and (iv) $\Rightarrow$ (i) follows by Proposition 3.1.8(b). Therefore, it is enough to show (i) $\Rightarrow$ (ii) and (v) $\Leftrightarrow$ (vi).

(i) $\Rightarrow$ (ii) : We prove this by induction on m. We consider the case for $m = 1$. Since $k[x_1, z, t] \subsetneqq \mathrm{DK}(A)$, by Remark 3.1.6, without loss of generality we can assume that $f(Z, T) = a_0(Z) + a_1(Z)T$ for some $a_0, a_1 \in k^{[1]}$. Since A is a UFD, either $f(Z, T)$ is irreducible or $f(Z, T) \in k^*$ (cf. Proposition 3.1.12). If $f(Z, T) \in k^*$, then $x_1 \in A^*$, which contradicts that $\left(\frac{A}{x_1A}\right)^* = k^*$. Therefore, $f(Z, T)$ is irreducible in $k[Z, T]$. Note that $\frac{A}{x_1A} \cong \frac{k[Y,Z,T]}{(f(Z,T))}$. If $a_1(Z) = 0$, then $f(Z, T) = a_0(Z)$. Since $\left(\frac{A}{x_1A}\right)^* = \left(\frac{k[Y,Z,T]}{(a_0(Z))}\right)^* = k^*$, $a_0(Z)$ must be linear in Z. Hence $k[Z, T] = k[f]^{[1]}$. If $a_1(Z) \neq 0$, then $\gcd(a_0, a_1) = 1$, as $f(Z, T)$ is irreducible. Therefore, since $k^* = \left(\frac{A}{x_1A}\right)^* = \left(\frac{k[Y,Z,T]}{(f(Z,T))}\right)^* = \left(k\left[Y, Z, \frac{1}{a_1(Z)}\right]\right)^*$, $a_1(Z) \in k^*$. Thus, $k[Z, T] = k[f]^{[1]}$.

We now assume $m > 1$ and the result holds upto $m - 1$. Since $k[x_1, \ldots, x_m, z, t] \subsetneqq \mathrm{DK}(A)$, by Lemma 3.2.4, there exists an integral domain

$$A_l \cong \frac{k(X_l)[X_1, \ldots, X_{l-1}, X_{l+1}, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - f(Z, T))},$$

such that $\mathrm{DK}(A_l) = A_l$. Note that for every $i \in \{1, \ldots, m\}$, $\frac{A}{x_iA} \cong \frac{k[X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_m,Y,Z,T]}{(f(Z,T))}$. Now for every $i \neq l$,

$$\frac{A}{x_iA} \otimes_{k[x_l]} k(x_l) \cong \frac{k(X_l)[X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{l-1}, X_{l+1}, \ldots, X_m, Y, Z, T]}{(f(Z, T))} \cong \frac{A_l}{\overline{x_i}A_l},$$

where $\overline{x_i}$ denotes the image of X_i in A_l. Since $\left(\frac{A}{x_iA}\right)^* = k^*$, it follows that $\left(\frac{A_l}{\overline{x_i}A_l}\right)^* \cong k(X_l)^*$. Since A is a UFD and x_i's are not unit in A, by Proposition 3.1.12, $f(Z,T)$ is irreducible in $k[Z,T]$. As $\mathrm{DK}(A_l) = A_l$, by Proposition 3.1.5, there exist $a_0, a_1 \in k(X_l)^{[1]}$ such that $f(Z,T) = a_0(Z_1) + a_1(Z_1)T_1$, for some $Z_1, T_1 \in k(X_l)[Z,T]$ such that $k(X_l)[Z,T] = k(X_l)[Z_1,T_1]$.

We fix some $i, i \neq l$. Suppose $a_1(Z_1) = 0$. Since $\left(\frac{A_l}{\overline{x_i}A_l}\right)^* \cong \left(\frac{k(X_l)[Z_1,T_1]}{(a_0(Z_1))}\right)^* \cong k(X_l)^*$, and $a_0(Z_1)$ is irreducible in $k(X_l)[Z_1]$, it follows that $a_0(Z_1)$ is linear in Z_1. Therefore, $f(Z,T) = a_0(Z_1)$ is a coordinate in $k(X_l)[Z,T]$.

Suppose $a_1(Z_1) \neq 0$. Since $f(Z,T)$ is irreducible in $k(X_l)[Z_1,T_1]$, it follows that $\gcd(a_0(Z_1), a_1(Z_1)) = 1$ and hence

$$\frac{A_l}{\overline{x_i}A_l} \cong \frac{k(X_l)[Z_1,T_1]}{(a_0(Z_1) + a_1(Z_1)T_1)} \cong k(X_l)\left[Z_1, \frac{1}{a_1(Z_1)}\right].$$

Therefore, $a_1(Z_1) \in \left(\frac{A_l}{\overline{x_i}A_l}\right)^* \cong k(X_l)^*$.

Hence we have $k(X_l)[Z,T] = k(X_l)[f]^{[1]}$. Therefore, by Lemma 2.1.8, we have $k[Z,T] = k[f]^{[1]}$.

(v) $\Rightarrow$ (vi) : By Lemma 3.2.3, ϕ induces a non-trivial exponential map $\widehat{\phi}$ on

$$\widehat{A} \cong \frac{k[X_1, \ldots, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - f(Z,T))},$$

such that $\widehat{x_1}, \ldots, \widehat{x_m}, \widehat{y} \in \widehat{A}^{\widehat{\phi}}$, where $\widehat{x_1}, \ldots, \widehat{x_m}, \widehat{y}$ denote the images of $X_1, \ldots, X_m, Y$ in $\widehat{A}$, respectively. We now show that $k[Z,T] = k[f]^{[1]}$.

Case 1: Let k be an infinite field. Since $\widehat{y} \in \widehat{A}^{\widehat{\phi}}$, by Lemma 3.1.4 it follows that $\mathrm{DK}(\widehat{A}) = \widehat{A}$. Hence by Proposition 3.1.5, we can assume that

$$f(Z,T) = a_0(Z) + a_1(Z)T$$

for some $a_0, a_1 \in k^{[1]}$. Let $\overline{k}$ be an algebraic closure of the field k. As A is geometrically factorial $f(Z,T)$ is irreducible in $\overline{k}[Z,T]$ (cf. Proposition 3.1.12).

If $a_1(Z) = 0$, then $a_0(Z) (= f(Z,T))$ is irreducible in $\overline{k}[Z,T]$, hence it is linear in Z. Thus $k[Z,T] = k[f]^{[1]}$.

If $a_1(Z) \neq 0$, then $\gcd(a_0(Z), a_1(Z)) = 1$. Now $\widehat{\phi}$ induces a non-trivial exponential map on

$$\widetilde{A} = \widehat{A} \otimes_{k[\widehat{x_1},\ldots,\widehat{x_m},\widehat{y}]} k(\widehat{x_1},\ldots,\widehat{x_m},\widehat{y}) \cong \frac{L[Z,T]}{(\mu - f(Z,T))} = \frac{L[Z,T]}{(\mu - a_0(Z) - a_1(Z)T)},$$

where $L = k(X_1,\ldots,X_m,Y)$, $\mu \in L \setminus k$, and hence $\gcd(\mu - a_0, a_1) = 1$ in $L^{[1]}$. Since $\widetilde{A}$ is not rigid, $a_1 \in k^*$, and hence $k[Z,T] = k[f]^{[1]}$.

Case 2: Let k be a finite field. Now $\widehat{\phi}$ induces a non-trivial exponential map $\overline{\phi}$ on

$$\overline{A} := \widehat{A} \otimes_k \overline{k} \cong \frac{\overline{k}[X_1,\ldots,\ldots,X_m,Y,Z,T]}{(X_1^{r_1}\cdots X_m^{r_m}Y - f(Z,T))},$$

such that $\overline{x_1},\ldots,\overline{x_m},\overline{y} \in \overline{A}^{\overline{\phi}}$, where $\overline{x_1},\ldots,\overline{x_m},\overline{y}$ denote the images of $X_1,\ldots,X_m,Y$ in $\overline{A}$, respectively. By Case 1, we have $\overline{k}[Z,T] = \overline{k}[f]^{[1]}$, and hence by Lemma 2.1.8, $k[Z,T] = k[f]^{[1]}$.

Now from Theorem 2.1.7, it follows that $A = k^{[m+2]}$.

(vi) $\Rightarrow$ (v) : Since $A = k^{[m+2]}$, it is geometrically factorial over k and the rest follows from Lemma 3.2.5. $\square$

Remark 3.2.7. The above result shows that the condition "A is geometrically factorial" (i.e., $A \otimes_k \overline{k}$ is a UFD where $\overline{k}$ is an algebraic closure of k) in [22, Theorem 3.11(viii)] can be relaxed to "A is a UFD".

Remark 3.2.8. Note that the proof shows that the condition "A is geometrically factorial over k" may be replaced by a more specific condition that "$A \otimes_k \overline{k}$ is a UFD" where $\overline{k}$ is an algebraic closure of k. However, the following example shows that in statement (v) of Theorem 3.2.6, the condition "A is geometrically factorial" can not be relaxed to "A is a UFD".

Example 3.2.9. Let

$$A = \frac{\mathbb{R}[X_1,X_2,Y,Z,T]}{(X_1^2X_2^2Y - 1 - Z^2)},$$

where x_1,x_2,y,z,t denote the images of X_1,X_2,Y,Z,T respectively in A. Note that $A = C[t] = C^{[1]}$, where $C = \frac{\mathbb{R}[X_1,X_2,Y,Z]}{(X_1^2X_2^2Y-1-Z^2)}$.

We consider the exponential map $\phi : A \to A[W]$, such that $\phi|_C = id_C$ and $\phi(t) = t + W$. Then it follows that $A^\phi = C$. Therefore, $\mathbb{R}[x_1,x_2] \subseteq A^\phi \nsubseteq \mathbb{R}[x_1,x_2,z,t]$, as $y \in A^\phi$.

Now note that here $F = f = 1 + Z^2$, which is an irreducible polynomial in $\mathbb{R}[Z,T]$ but not irreducible $\mathbb{C}[Z,T]$. Hence by Proposition 3.1.12, A is a UFD but $A \otimes_{\mathbb{R}} \mathbb{C}$ is not a UFD. Therefore, $A \neq \mathbb{R}^{[4]}$.

We now generalise Theorem A1 over a larger class of integral domains. The $m = 1$ case of the following theorem has been proved in [15, Theorem 4.9].

Theorem 3.2.10. *Let R be a Noetherian integral domain such that either $\mathbb{Q}$ is contained in R or R is seminormal. Let*

$$A_R := \frac{R\,[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T))}, \quad r_i > 1 \text{ for all } i, 1 \leqslant i \leqslant m,$$

where $F(X_1, \ldots, X_m, Z, T) = f(Z,T) + (X_1 \cdots X_m)g(X_1, \ldots, X_m, Z, T)$ and $f(Z,T) \neq 0$. Let $G = X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T)$ and $\widetilde{x_1}, \ldots, \widetilde{x_m}$ denote the images in A_R of $X_1, \ldots, X_m$ respectively. Then the following statements are equivalent:

(i) $R[X_1, \ldots, X_m, Y, Z, T] = R[X_1, \ldots, X_m, G]^{[2]}$.

(ii) $R[X_1, \ldots, X_m, Y, Z, T] = R[G]^{[m+2]}$.

(iii) $A_R = R[\widetilde{x_1}, \ldots, \widetilde{x_m}]^{[2]}$.

(iv) $A_R = R^{[m+2]}$.

(v) $R[Z,T] = R[f(Z,T)]^{[1]}$.

Proof. Note that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv) follow trivially. Therefore it is enough to show (iv) $\Rightarrow$ (v) and (v) $\Rightarrow$ (i).

(iv) $\Rightarrow$ (v) : Let $p \in \operatorname{Spec} R$ and $\kappa(p) = \frac{R_p}{pR_p}$. Now $A \otimes_R \kappa(p) = \kappa(p)^{[m+2]}$. Now from (iv) $\Rightarrow$ (v) of Theorem 3.2.1, we have f is a residual coordinate in $R[Z,T]$. Hence $R[Z,T] = R[f]^{[1]}$ by Theorem 2.1.9.

(v) $\Rightarrow$ (i) : Let $h \in R[Z,T]$ be such that $R[Z,T] = R[f,h]$. Therefore, without loss of generality we can assume that $f = Z$ and $h = T$. Hence $G = X_1^{r_1} \cdots X_m^{r_m} Y - X_1 \cdots X_m g - Z$, for some $g \in R[X_1, \ldots, X_m, Z, T]$. Now by Lemma 3.1.1, we get that

$$R[X_1, \ldots, X_m, Y, Z, T] = R[X_1, \ldots, X_m, G, T]^{[1]} = R[X_1, \ldots, X_m, G]^{[2]}.$$

$\square$

Chapter 4

An infinite family of higher dimensional counterexamples to ZCP

In this chapter we will first describe the isomorphism classes and automorphisms of Generalised Asanuma varieties and use the classification to exhibit an infinite family of counterexamples to ZCP in positive characteristic. These results can be found in [19].

4.1 Isomorphism classes and Automorphisms

We first recall the structure of the coordinate rings of these varieties.

$$A := \frac{k\,[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T))}, \quad r_i > 1 \text{ for all } i, 1 \leqslant i \leqslant m, \tag{4.1.1}$$

where $F(0, \ldots, 0, Z, T) \neq 0$. Set $f(Z, T) := F(0, \ldots, 0, Z, T)$. Let $x_1, \ldots, x_m, y, z, t$ denote the images in A of $X_1, \ldots, X_m, Y, Z, T$ respectively. The following result describes some necessary conditions for two such rings to be isomorphic when $\mathrm{DK}(A) = k[x_1, \ldots, x_m, z, t]$.

Theorem 4.1.1. *Let* $(r_1, \ldots, r_m), (s_1, \ldots, s_m) \in \mathbb{Z}^m_{>1}$*, and* $F, G \in k[X_1, \ldots, X_m, Z, T]$*, where* $f(Z, T) := F(0, \ldots, 0, Z, T) \notin k$ *and* $g(Z, T) := G(0, \ldots, 0, Z, T) \notin k$*. Suppose* $\phi : A \to A'$ *is an isomorphism, where*

$$A = A(r_1, \ldots, r_m, F) := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \ldots, X_m, Z, T))}$$

and

$$A' = A(s_1, \ldots, s_m, G) := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{s_1} \cdots X_m^{s_m} Y - G(X_1, \ldots, X_m, Z, T))}.$$

Let $x_1, \ldots, x_m, y, z, t$ *and* $x_1', \ldots, x_m', y', z', t'$ *denote the images of* $X_1, \ldots, X_m, Y, Z, T$ *in* A *and* A' *respectively. Let* $E = k[x_1, \ldots, x_m]$ *and* $E' = k[x_1', \ldots, x_m']$. *Suppose* $B := \mathrm{DK}(A) = k[x_1, \ldots, x_m, z, t]$ *and* $B' := \mathrm{DK}(A') = k[x_1', \ldots, x_m', z', t']$. *Then*

(i) ϕ *restricts to isomorphisms from* B *to* B' *and from* E *to* E'.

(ii) *For each* $i, 1 \leqslant i \leqslant m$, *there exists* j, $1 \leqslant j \leqslant m$, *such that* $\phi(x_i) = \lambda_j x_j'$ *for some* $\lambda_j \in k^*$ *and* $r_i = s_j$. *In particular,* $(r_1, \ldots, r_m) = (s_1, \ldots, s_m)$ *upto a permutation of* $\{1, \ldots, m\}$.

(iii) $\phi\left(x_1^{r_1} \cdots x_m^{r_m}, F(x_1, \ldots, x_m, z, t)\right) = ((x_1')^{s_1} \cdots (x_m')^{s_m}, G(x_1', \ldots, x_m', z', t'))$

(iv) *There exists* $\alpha \in \mathrm{Aut}_k(k[Z, T])$ *such that* $\alpha(g) = \lambda f$ *for some* $\lambda \in k^*$.

Proof. (i) Since $\phi : A \to A'$ is an isomorphism, ϕ restricts to an isomorphism of the Derksen invariant and the Makar-Limanov invariant. Therefore, $\phi(B) = B'$. By Proposition 3.1.8(b), $\mathrm{ML}(A) = E$ and $\mathrm{ML}(A') = E'$. Hence $\phi(E) = E'$.

We now identify $\phi(A)$ *with* A *and assume that* $A' = A$, ϕ *is identity on* A, $B' = B$ *and* $E' = E$.

(ii) We first show that for every $i, 1 \leqslant i \leqslant m$, $x_i = \lambda_j x_j'$, for some $j, 1 \leqslant j \leqslant m$ and $\lambda_j \in k^*$. We now have

$$y' = \frac{G(x_1', \ldots, x_m', z', t')}{(x_1')^{s_1} \cdots (x_m')^{s_m}} \in A \setminus B.$$

Since $A \hookrightarrow k[x_1^{\pm 1}, \ldots, x_m^{\pm 1}, z, t]$, there exists $n > 0$ such that

$$(x_1 \cdots x_m)^n y' = \frac{(x_1 \cdots x_m)^n G(x_1', \ldots, x_m', z', t')}{(x_1')^{s_1} \cdots (x_m')^{s_m}} \in B.$$

Since for every $j \in \{1, \ldots, m\}$, x_j' is irreducible in B, and $x_j' \nmid G$ in B, we have $x_j' \mid (x_1 \cdots x_m)^n$. Since $x_1, x_2, \ldots, x_m$ are also irreducibles in B, we have

$$x_i = \lambda_j x_j', \tag{4.1.2}$$

for some $i \in \{1, \ldots, m\}$ and $\lambda_j \in k^*$.

We now show that $r_i = s_j$. Suppose $r_i > s_j$. Consider the ideal

$$\mathfrak{a}_i := x_i^{r_i} A \cap B = (x_i^{r_i}, F(x_1, \ldots, x_m, z, t))\, B.$$

Again by (4.1.2),

$$\mathfrak{a}_i = (x_j')^{r_i} A \cap B = (x_j')^{r_i} A' \cap B' = \left((x_j')^{r_i}, (x_j')^{r_i - s_j} G(x_1', \ldots, x_m', z', t')\right) B' \subseteq x_j' B',$$

which implies that $F(x_1, \ldots, x_m, z, t) \in x_j' B' = x_j' B$. But this is a contradiction. Therefore, $r_i \leqslant s_j$. By similar arguments, we have $s_j \leqslant r_i$. Hence $r_i = s_j$ and as $i \in \{1, \ldots, m\}$ is arbitrary, the assertion follows.

(iii) We now show that

$$(x_1^{r_1} \cdots x_m^{r_m}, F(x_1, \ldots, x_m, z, t))\, B = \left((x_1')^{s_1} \cdots (x_m')^{s_m}, G(x_1', \ldots, x_m', z', t')\right) B.$$

From (ii), it is clear that $x_1^{r_1} \cdots x_m^{r_m} = \mu (x_1')^{s_1} \cdots (x_m')^{s_m}$, for some $\mu \in k^*$. Since

$$(x_1^{r_1} \cdots x_m^{r_m}) A \cap B = (x_1^{r_1} \cdots x_m^{r_m}, F(x_1, \ldots, x_m, z, t))\, B$$

and

$$\left((x_1')^{s_1} \cdots (x_m')^{s_m}\right) A \cap B = \left((x_1')^{s_1} \cdots (x_m')^{s_m}, G(x_1', \ldots, x_m', z', t')\right) B,$$

the result follows.

(iv) Since $(x_1^{r_1} \cdots x_m^{r_m}, F(x_1, \ldots, x_m, z, t))\, B = ((x_1')^{s_1} \cdots (x_m')^{s_m}, G(x_1', \ldots, x_m', z', t'))\, B$, from (4.1.2), it follows that

$$g(z', t') = \lambda f(z, t) + H(x_1, \ldots, x_m, z, t), \tag{4.1.3}$$

for some $\lambda \in k^*$ and $H \in (x_1, \ldots, x_m)\, B$. Let $z' = h_1(x_1, \ldots, x_m, z, t)$ and $t' = h_2(x_1, \ldots, x_m, z, t)$. Then we have $k[z, t] = k[h_1(0, \ldots, 0, z, t), h_2(0, \ldots, 0, z, t)]$. Hence $\alpha : k[Z, T] \to k[Z, T]$ defined by $\alpha(Z) = h_1(0, \ldots, 0, Z, T)$ and $\alpha(T) = h_2(0, \ldots, 0, Z, T)$ gives an automorphism of $k[Z, T]$, and from (4.1.3), it follows that $\alpha(g) = \lambda f$. $\square$

The next result characterises the automorphisms of A when $\mathrm{DK}(A) = B = k[x_1, \ldots, x_m, z, t]$.

Theorem 4.1.2. *Let A be the affine domain as in* (4.1.1)*, where*

$$F(X_1,\ldots,X_m,Z,T) = f(Z,T) + h(X_1,\ldots,X_m,Z,T),$$

for some $h \in (X_1,\ldots,X_m)\,k[X_1,\ldots,X_m,Z,T]$ and $f(Z,T) \notin k$. As before, $x_1,\ldots,x_m,y,z,t$ denote the images of $X_1,\ldots,X_m,Y,Z,T$ in A. Suppose $\mathrm{DK}(A) = B = k[x_1,\ldots,x_m,z,t]$*. If $\phi \in \mathrm{Aut}_k(A)$, then the following hold:*

(a) *ϕ restricts to an automorphism of $E = k[x_1,\ldots,x_m]$ and $B = k[x_1,\ldots,x_m,z,t]$.*

(b) *For each i, $1 \leqslant i \leqslant m$, there exists j, $1 \leqslant j \leqslant m$ such that $\phi(x_i) = \lambda_j x_j$ where $\lambda_j \in k^*$ $(1 \leqslant i,j \leqslant m)$ and $r_i = r_j$.*

(c) *$\phi(I) = I$, where $I = (x_1^{r_1}\cdots x_m^{r_m}, F(x_1,\ldots,x_m,z,t))\,k[x_1,\ldots,x_m,z,t]$.*

Conversely, if $\phi \in \mathrm{End}_k(A)$ satisfies the conditions (a) and (c), then (b) holds and $\phi \in \mathrm{Aut}_k(A)$.

Proof. By Proposition 3.1.8(b), $\mathrm{ML}(A) = E = k[x_1,\ldots,x_m]$. Now the statements (a), (b), (c) follow from Theorem 4.1.1(i), (ii), (iii) respectively.

We now show the converse part. From (a) and (c), it follows that $\phi(B) = B$, $\phi(E) = E$ and $\phi(I) = I$. Hence $\phi(I \cap E) = I \cap E = (x_1^{r_1}\cdots x_m^{r_m})\,E$, and therefore,

$$\phi(x_1^{r_1}\cdots x_m^{r_m}) = \lambda x_1^{r_1}\cdots x_m^{r_m},$$

for some $\lambda \in k^*$. Fix $i \in \{1,\ldots,m\}$. Since x_i and $\phi(x_i)$ are irreducibles in E, $\phi(x_i) = \lambda_j x_j$, for some $\lambda_j \in k^*$ and $j \in \{1,\ldots,m\}$ and hence $r_i = r_j$ and (b) follows.

Since ϕ is an automorphism of B and $A \subseteq B[(x_1\cdots x_m)^{-1}]$, ϕ is an injective endomorphism of A, by (b). Therefore, it is enough to show that ϕ is surjective. For this, it is enough to find a preimage of y in A. Since $\phi(I) = I$, we have $F = x_1^{r_1}\cdots x_m^{r_m} u(x_1,\ldots,x_m,z,t) + \phi(F)v(x_1,\ldots,x_m,z,t)$, for some $u,v \in B$. Since $y = \frac{F(x_1,\ldots,x_m,z,t)}{x_1^{r_1}\cdots x_m^{r_m}}$, using (b),

$$y = u(x_1,\ldots,x_m,z,t) + \frac{\phi(F)v(x_1,\ldots,x_m,z,t)}{\lambda^{-1}\phi(x_1^{r_1}\cdots x_m^{r_m})}. \tag{4.1.4}$$

Since $\phi(B) = B$, there exist $\widetilde{u},\widetilde{v} \in B$ such that $\phi(\widetilde{u}) = u$ and $\phi(\widetilde{v}) = v$. And hence from (4.1.4), we get that $y = \phi(\widetilde{u} + \lambda y\widetilde{v})$ where $\widetilde{u} + \lambda y\widetilde{v} \in A$. □

4.2 ZCP in positive characteristic

We consider the following subfamily of the Generalised Asanuma varieties. Let $(r_1, \ldots, r_m) \in \mathbb{Z}^m_{>1}$ and $f(Z,T)$ be a non-trivial line in $k[Z,T]$. Let

$$A(r_1, \ldots, r_m, f) := \frac{k[X_1, \ldots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - f)}.$$

The following result determines the isomorphism classes among the family of rings defined above.

Theorem 4.2.1. *Let* $(r_1, \ldots, r_m), (s_1, \ldots, s_m) \in \mathbb{Z}^m_{>1}$*, and* $f, g \in k[Z,T]$ *be non-trivial lines. Then* $A(r_1, \ldots, r_m, f) \cong A(s_1, \ldots, s_m, g)$ *if and only if* $(r_1, \ldots, r_m) = (s_1, \ldots, s_m)$ *upto a permutation of* $\{1, \ldots, m\}$ *and there exists* $\alpha \in \operatorname{Aut}_k(k[Z,T])$ *such that* $\alpha(g) = \mu f$*, for some* $\mu \in k^*$*.*

Proof. Suppose $A(r_1, \ldots, r_m, f) \cong A(s_1, \ldots, s_m, g)$ and let $x_1, \ldots, x_m, y, z, t$ and
$x'_1, \ldots, x'_m, y', z', t'$ denote the images of $X_1, \ldots, X_m, Y, Z, T$ in $A(r_1, \ldots, r_m, f)$ and
$A(s_1, \ldots, s_m, g)$ respectively. As $f(Z,T), g(Z,T)$ are non-trivial lines in $k[Z,T]$, by Lemma 3.1.4 and Proposition 3.1.7, we have $\operatorname{DK}(A(r_1, \ldots, r_m, f)) = k[x_1, \ldots, x_m, z, t]$ and $\operatorname{DK}(A(s_1, \ldots, s_m, g)) = k[x'_1, \ldots, x'_m, z', t']$. Hence the result follows from (ii) and (iv) of Theorem 4.1.1.

The converse is obvious. $\square$

We now recall a result from [23], which shows that varieties in a certain subfamily of generalised Asanuma varieties are stably isomorphic to a polynomial ring.

Theorem 4.2.2. *Let* k *be a field of any characteristic and* A *an integral domain as in* (4.1.1)*. Suppose that*

$$\frac{k[X_1, \ldots, X_m, Z, T]}{(X_1 \cdots X_m, F(X_1, \ldots, X_m, Z, T))} = \left(\frac{k[X_1, \ldots, X_m]}{(X_1 \cdots X_m)} \right)^{[1]}$$

as $k[X_1, \ldots, X_m]$*-algebras. Then* $A^{[1]} = k[X_1, \ldots, X_m]^{[3]} = k^{[m+3]}$*. Moreover, if* $\operatorname{ch}.k > 0$ *and* $f(Z,T)$ *is a non-trivial line in* $k[Z,T]$ *then* $A \neq k^{[m+2]}$*.*

We now show how the results in this chapter yield an infinite family of pairwise non-isomorphic n dimensional rings which are counterexamples to ZCP, for each $n \geqslant 3$.

Corollary 4.2.3. *Let k be a field of positive characteristic. For each $n \geqslant 3$, there exists an infinite family of pairwise non-isomorphic rings C of dimension n, which are counter examples to the Zariski Cancellation Problem in positive characteristic, i.e., which satisfy that $C^{[1]} = k^{[n+1]}$ but $C \neq k^{[n]}$.*

Proof. Consider the family of rings

$$\Omega := \{A(r_1, \ldots, r_m, f) \mid (r_1, \ldots, r_m) \in \mathbb{Z}^m_{>1},\ f(Z,T) \text{ is a non-trivial line in } k[Z,T]\}.$$

By Theorem 4.2.2, for every $C \in \Omega$, we have $C^{[1]} = k^{[m+3]}$ but $C \neq k^{[m+2]}$. By Theorem 4.2.1, there exist infinitely many rings $C \in \Omega$ which are pairwise non-isomorphic. Taking $n = m + 2$, we get the result.

□

Chapter 5

Generalised Danielewski varieties and invariants of generalised Asanuma varieties

In this chapter we will prove Theorem B1 (Theorems 5.2.1 and 5.2.3) and apply it to describe Makar-Limanov and Derksen invariant of a certain subfamily of Generalised Asanuma varieties. The results discussed in this chapter can be found in [20].

We begin by proving some preparatory results. Before that we fix some notation which will be used throughout this chapter. For positive integers $r_1, \ldots, r_m$ and a polynomial $F = F(T_1, \ldots, T_m, V) \in k^{[m+1]}$ which is monic in V with $\deg_V F > 1$, $B(\underline{r}, F)$ will denote the ring

$$B(\underline{r}, F) := B(r_1, \ldots, r_m, F) = \frac{k[T_1, \ldots, T_m, U, V]}{(T_1^{r_1} \cdots T_m^{r_m} U - F(T_1, \ldots, T_m, V))},$$

where $\underline{r} := (r_1, \ldots, r_m) \in \mathbb{Z}^m_{\geqslant 1}$. Set $P(V) := F(0, \ldots, 0, V)$ and $d := \deg_V P = \deg_V F (> 1)$. Further, when F is understood from the context, we will use the notation $B_{\underline{r}}$ to denote the above ring $B(\underline{r}, F)$, i.e.,

$$B_{\underline{r}} := B(r_1, \ldots, r_m, F) = \frac{k[T_1, \ldots, T_m, U, V]}{(T_1^{r_1} \cdots T_m^{r_m} U - F(T_1, \ldots, T_m, V))}. \tag{5.0.1}$$

We call the varieties defined by these rings as "Generalised Danielewski varieties". Let $t_1, \ldots, t_m, u, v$ denote respectively the images of $T_1, \ldots, T_m, U, V$ in $B_{\underline{r}}$ and R denote the subring $k[t_1, \ldots, t_m, v]$ of $B_{\underline{r}}$.

5.1 Properties of Generalised Danielewski varieties

The aim of this section is to describe the $\mathrm{ML}(B_{\underline{r}})$. We first note that

$$R := k[t_1, \ldots, t_m, v](= k^{[m+1]}) \hookrightarrow B_{\underline{r}} \hookrightarrow B_{\underline{r}}[t_1^{-1}, \ldots, t_m^{-1}] = k[t_1^{\pm 1}, \ldots, t_m^{\pm 1}, v]$$

and $F \in R$. Fix $(e_1, \ldots, e_m) \in \mathbb{Z}^m$. This m-tuple defines a proper $\mathbb{Z}$-filtration $\{B_n\}_{n\in\mathbb{Z}}$ on $B_{\underline{r}}$ as follows:
Set $C_n := \bigoplus_{e_1 i_1 + \cdots + e_m i_m = n} k[v] t_1^{i_1} \ldots t_m^{i_m}$. Then the ring $k[t_1^{\pm 1}, \ldots, t_m^{\pm 1}, v]$ has the following $\mathbb{Z}$-graded structure (with $wt(t_i) = e_i$, $1 \leqslant i \leqslant m$):

$$k[t_1^{\pm 1}, \ldots, t_m^{\pm 1}, v] = \bigoplus_{n\in\mathbb{Z}} C_n = \bigoplus_{n\in\mathbb{Z}, e_1 i_1 + \cdots + e_m i_m = n} k[v] t_1^{i_1} \cdots t_m^{i_m}.$$

For every $n \in \mathbb{Z}$, set $B_n := \bigoplus_{i \leqslant n} C_n \cap B_{\underline{r}}$.

Then $\{B_n\}_{n\in\mathbb{Z}}$ defines a proper $\mathbb{Z}$-filtration on $B_{\underline{r}}$ induced by $(e_1, \ldots, e_m)$ and for every j, $1 \leqslant j \leqslant m$, $t_j \in B_{e_j} \setminus B_{e_j - 1}$.

Let $e := \deg(F)$ with respect to the given filtration. Then $u \in B_\ell \setminus B_{\ell-1}$, where $\ell = e - (r_1 e_1 + \cdots + r_m e_m)$. Set

$$\Lambda := \left\{ (\underline{i}, j, q) := (i_1, \ldots, i_m, j, q) \in \mathbb{Z}_{\geqslant 0}^m \times \mathbb{Z}_{>0} \times \mathbb{Z}_{\geqslant 0} \mid i_s < r_s \text{ for some } s, 1 \leqslant s \leqslant m \right\}.$$

Using the relation $t_1^{r_1} \cdots t_m^{r_m} u = F(t_1, \ldots, t_m, v)$, it can be noted that every element $b \in B_{\underline{r}}$ can be uniquely expressed as

$$b = \widetilde{b}(t_1, \ldots, t_m, v) + \sum_{(\underline{i}, j, q) \in \Lambda} \alpha_{\underline{i}\, jq}\, t_1^{i_1} \cdots t_m^{i_m} u^j v^q, \tag{5.1.1}$$

where $\widetilde{b} \in R(= k[t_1, \ldots, t_m, v])$ and $\alpha_{\underline{i}\, jq} \in k^*$.

Now since the filtration $\{B_n\}_{n\in\mathbb{Z}}$ is induced from the graded structure of the ring $B_{\underline{r}}[t_1^{-1}, \ldots, t_m^{-1}]$, from the expression (5.1.1) it follows that the filtration $\{B_n\}_{n\in\mathbb{Z}}$ is admissible with respect to the generating set $\Gamma = \{t_1, \ldots, t_m, u, v\}$ of $B_{\underline{r}}$ and the associated graded ring $gr(B_{\underline{r}}) = \bigoplus_{n\in\mathbb{Z}} \frac{B_n}{B_{n-1}}$ is generated by the image of Γ in $gr(B_{\underline{r}})$.

The following lemma exhibits the structure of $gr(B_{\underline{r}})$.

Lemma 5.1.1. *Let F_e denote the highest degree homogeneous summand of F.*

If, for every i, $t_i \nmid F_e$ in $R = k[t_1, \ldots, t_m, v]$, then

$$gr(B_{\underline{r}}) \cong \frac{k[T_1, \ldots, T_m, U, V]}{(T_1^{r_1} \cdots T_m^{r_m} U - F_e(T_1, \ldots, T_m, V))}.$$

Proof. Since $\deg(F) = e$, $t_1^{r_1} \cdots t_m^{r_m} u = F \in B_e \setminus B_{e-1}$ and hence,

$$\overline{t_1}^{r_1} \cdots \overline{t_m}^{r_m} \overline{u} = F_e(\overline{t_1}, \ldots, \overline{t_m}, \overline{v}) \tag{5.1.2}$$

in $gr(B_{\underline{r}})$, where $\overline{t_1}, \ldots, \overline{t_m}, \overline{u}, \overline{v}$ denote the images of $t_1, \ldots, t_m, u, v$ in $gr(B_{\underline{r}})$. Since $gr(B_{\underline{r}})$ can be identified with a subring of $gr(k[t_1^{\pm 1}, \ldots, t_m^{\pm 1}, v]) \cong k[t_1^{\pm 1}, \ldots, t_m^{\pm 1}, v]$, we have $\overline{t_1}, \ldots, \overline{t_m}, \overline{v}$ are algebraically independent in $gr(B_{\underline{r}})$ and hence $\dim gr(B_{\underline{r}}) = m+1$. As $t_i \nmid F_e$ in R for every i, $\frac{k[T_1, \ldots, T_m, U, V]}{(T_1^{r_1} \cdots T_m^{r_m} U - F_e(T_1, \ldots, T_m, V))}$ is an integral domain, and its dimension is $m+1$. Hence by (5.1.2), we have the isomorphism:

$$gr(B_{\underline{r}}) \cong \frac{k[T_1, \ldots, T_m, U, V]}{(T_1^{r_1} \cdots T_m^{r_m} U - F_e(T_1, \ldots, T_m, V))}.$$

□

In [13], Dubouloz showed that for $k = \mathbb{C}$, $\mathrm{ML}(B_{\underline{r}}) = \mathbb{C}[t_1, \ldots, t_m]$, when $\underline{r} \in \mathbb{Z}_{>1}^m$. When k is an algebraically closed field of characteristic zero and $F \in k[V]$, a similar result appears in [18, Lemma 6.2]. We extend this result over any field (of any characteristic) and arbitrary F. First we recall the following lemma ([22], Lemma 3.5).

Lemma 5.1.2. *Let $r_1 > 1$ and $q \in k[V]$ be such that $\deg_V q > 1$. Then there is no non-trivial exponential map ϕ on $B(r_1, q) = k[T_1, U, V]/(T_1^{r_1} U - q(V))$ such that $u \in B(r_1, q)^{\phi}$, where u denotes the image of U in $B(r_1, q)$.*

We now establish a generalisation of the above lemma.

Lemma 5.1.3. *Let $\underline{r} \in \mathbb{Z}_{>1}^m$, and $P(V) \in k[V]$ with $\deg_V P > 1$. Then there is no non-trivial exponential map ϕ on $B(\underline{r}, P)$, such that $u \in B(\underline{r}, P)^{\phi}$.*

Proof. Let

$$B := B(\underline{r}, P) = \frac{k[T_1, \ldots, T_m, U, V]}{(T_1^{r_1} \cdots T_m^{r_m} U - P(V))}.$$

We will prove the result by induction on m. For $m = 1$, the result holds by Lemma 5.1.2. Therefore we assume that $m \geqslant 2$. Suppose the assertion holds upto $m - 1$.

If possible, suppose there exists a non-trivial exponential map ϕ on B such that $u \in B^{\phi}$. By Theorem 2.1.4, with respect to the filtration induced by $(-1, \ldots, -1) \in \mathbb{Z}^m$ on B, we get a non-trivial exponential map $\overline{\phi}$ on the associated graded ring $\overline{B}$, which is isomorphic to B itself (cf. Lemma 5.1.1), and $\overline{u} \in \overline{B}^{\overline{\phi}}$, where $\overline{u}$ denotes the image of u in $\overline{B}$. Therefore, we can assume that B is a graded ring and ϕ is a homogeneous exponential map on B, such that $u \in B^{\phi}$ and the weights of the generators of B are as follows:

$$wt(t_i) = -1, \text{ for every } i, 1 \leqslant i \leqslant m, \; wt(u) = r_1 + \cdots + r_m, \; wt(v) = 0.$$

We first show that $B^{\phi} \nsubseteq k[u, v]$. Suppose, if possible, $B^{\phi} \subseteq k[u, v] (\subseteq B)$. Then $m = 2$ and $\operatorname{tr.deg}_k B^{\phi} = 2$ and hence $B^{\phi} = k[u, v]$ (cf. Lemma 2.1.1(i)). But then, it follows that $t_1, t_2 \in B^{\phi}$, as $t_1^{r_1} t_2^{r_2} u = P(v) \in B^{\phi}$ (Lemma 2.1.1(i)). This contradicts the fact that ϕ is non-trivial. Hence $B^{\phi} \nsubseteq k[u, v]$.

Therefore, there exists $g \in B^{\phi} \setminus k[u, v]$, which is homogeneous with respect to the grading on B and

$$g = \widetilde{g}(t_1, \ldots, t_m, v) + \sum_{(\underline{i}, j, q) \in \Lambda} \alpha_{\underline{i}\, jq}\, t_1^{i_1} \cdots t_m^{i_m} u^j v^q,$$

where $\widetilde{g} \in k^{[m+1]}$ and $\alpha_{\underline{i}\, jq} \in k$. Now the following two cases can occur. We choose a suitable index $l, 1 \leqslant l \leqslant m$ as follows:

Case 1: If $\widetilde{g} \notin k[v]$, then it has a monomial summand g_2 such that $t_l \mid g_2$, for some $l \in \{1, \ldots, m\}$.

Case 2: If $\widetilde{g} \in k[v]$, then there exist at least one nonzero summand of g of the form $\alpha_{\underline{i}\, jq}\, t_1^{i_1} \cdots t_m^{i_m} u^j v^q$, and each such summand has weight zero. We fix such a summand. Since its weight is zero, we have $j(r_1 + \cdots + r_m) = i_1 + \cdots + i_m$. Also there exists some $s \in \{1, \ldots, m\}$ such that $i_s < r_s$. Hence there exists some $l \in \{1, \ldots, m\}, l \neq s$ such that $i_l > jr_l$.

Now if we consider the filtration on B, induced by $(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^m$, where the l-th entry is 1, then ϕ will induce a non-trivial exponential map $\widehat{\phi}$ on the associated graded ring (cf. Lemma 5.1.1)

$$\widehat{B} \cong B = \frac{k[T_1, \ldots, T_m, U, V]}{(T_1^{r_1} \cdots T_m^{r_m} U - P(V))}.$$

For every $b \in B$, let $\widehat{b}$ denote its image in $\widehat{B}$. Note that $\widehat{u}, \widehat{g} \in \widehat{B}^{\widehat{\phi}}$. Further, one can see from *Cases* 1 and 2 that $\widehat{t_l} \mid \widehat{g}$. Hence $\widehat{t_l} \in \widehat{B}^{\widehat{\phi}}$ (cf. Lemma 2.1.1(i), Theorem 2.1.4). Therefore, by Lemma 2.1.1(iii), $\widehat{\phi}$ will induce a non-trivial exponential map $\widetilde{\phi}$ on

$$\widetilde{B} = \widehat{B} \otimes_{k[\widehat{t_l}]} k(\widehat{t_l}) \cong \frac{k(T_l)[T_1, \ldots, T_{l-1}, T_{l+1}, \ldots, T_m, U, V]}{(T_1^{r_1} \cdots T_{l-1}^{r_{l-1}} T_{l+1}^{r_{l+1}} \cdots T_m^{r_m} U - P(V))}.$$

Since $\widehat{u} \in \widehat{B}^{\widehat{\phi}}$, we have $\widetilde{u} \in \widetilde{B}^{\widetilde{\phi}}$, where $\widetilde{u}$ is the image of $\widehat{u}$ in $\widetilde{B}$. But this contradicts the induction hypothesis. Hence the result follows. □

The next result describes the Makar-Limanov invariant of the ring $B_{\underline{r}}$.

Theorem 5.1.4. *Let $B_{\underline{r}}$ be the ring as in* (5.0.1). *Then the following hold:*

(a) *If $\underline{r} \in \mathbb{Z}^m_{>1}$, then* $\mathrm{ML}(B_{\underline{r}}) = k[t_1, \ldots, t_m]$.

(b) *If $\underline{r} \in \mathbb{Z}^m_{\geqslant 1} \setminus \mathbb{Z}^m_{>1}$, then* $\mathrm{ML}(B_{\underline{r}}) \subsetneq k[t_1, \ldots, t_m]$, *and for* $\underline{1} = (1, \ldots, 1)$, $\mathrm{ML}(B_{\underline{1}}) = k$.

Proof. (a) We first show that for any non-trivial exponential map ϕ on $B_{\underline{r}}$, $B_{\underline{r}}^{\phi} \subseteq k[t_1, \ldots, t_m]$. Suppose, if possible, there exists a non-trivial exponential map ψ on $B_{\underline{r}}$ such that $B_{\underline{r}}^{\psi} \nsubseteq k[t_1, \ldots, t_m]$. Therefore, there exists $g \in B_{\underline{r}}^{\psi} \setminus k[t_1, \ldots, t_m]$. Further, suppose that $g \notin k[t_1, \ldots, t_m, v]$. Then, g can be uniquely expressed as

$$g = g_1(t_1, \ldots, t_m, v) + \sum_{(\underline{i}, j, q) \in \Lambda} \alpha_{\underline{i}\, j q}\, t_1^{i_1} \cdots t_m^{i_m} u^j v^q,$$

where $g_1 \in k^{[m+1]}$ and $\alpha_{\underline{i}\, j q} \in k^*$.

Let us choose a summand $\alpha_{\underline{i}\, j q} t_1^{i_1} \cdots t_m^{i_m} u^j v^q$ of g, where $i_s < r_s$ for some $s, 1 \leqslant s \leqslant m$. We now consider the proper $\mathbb{Z}$-filtration on $B_{\underline{r}}$, induced by $(0, \ldots, 0, -1, 0, \ldots, 0) \in \mathbb{Z}^m$, where the s-th entry is -1. Then ψ induces a non-trivial exponential map $\overline{\psi}$ on the associated graded ring $\overline{B_{\underline{r}}}$. By Lemma 5.1.1,

$$\overline{B_{\underline{r}}} \cong \frac{k[T_1, \ldots, T_m, U, V]}{(T_1^{r_1} \cdots T_m^{r_m} U - F(T_1, \ldots, T_{s-1}, 0, T_{s+1}, \ldots, T_m, V))}.$$

For every $b \in B_{\underline{r}}$, let $\overline{b}$ denote the image of b in $\overline{B_{\underline{r}}}$. With respect to the chosen filtration it is clear that $\overline{u} \mid \overline{g}$ and since $\overline{g} \in \overline{B_{\underline{r}}}^{\overline{\psi}}$ (Theorem 2.1.4), $\overline{u} \in \overline{B_{\underline{r}}}^{\overline{\psi}}$.

Further with respect to the $\mathbb{Z}$-filtration induced by $(-1, \ldots, -1) \in \mathbb{Z}^m_{>1}$ on $\overline{B_{\underline{r}}}$, $\overline{\psi}$ induces a non-trivial exponential map ψ' on the associated graded ring

$B'_{\underline{r}}$. By Lemma 5.1.1,

$$B'_{\underline{r}} \cong \frac{k[T_1, \ldots, T_m, U, V]}{(T_1^{r_1} \cdots T_m^{r_m} U - P(V))}.$$

Let u' denote the image of U in $B'_{\underline{r}}$. Since $\overline{u} \in \overline{B}_{\underline{r}}^{\overline{\psi}}$, $u' \in (B'_{\underline{r}})^{\psi'}$. But this contradicts Lemma 5.1.3. Therefore, we have $g \in k[t_1, \ldots, t_m, v] = R$.

Now note that $B_{\underline{r}} \hookrightarrow C := k[t_1^{\pm 1}, \ldots, t_m^{\pm 1}, v]$. Set $D_n := k[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]v^n$ for all $n \geq 0$. The ring $k[t_1^{\pm 1}, \ldots, t_m^{\pm 1}, v]$ can be given the following $\mathbb{Z}$-graded structure:

$$k[t_1^{\pm 1}, \ldots, t_m^{\pm 1}, v] = \bigoplus_{n \geqslant 0} D_n = \bigoplus_{n \geqslant 0} k[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]v^n.$$

This induces a proper $\mathbb{Z}$-filtration $\{(B_{\underline{r}})_n\}_{n \in \mathbb{Z}}$ on $B_{\underline{r}}$ such that $(B_{\underline{r}})_n = \left(\bigoplus_{i \leqslant n} D_n\right) \cap B_{\underline{r}}$. Set

$$\Lambda_1 := \left\{(\underline{i}, j) := (i_1, \ldots, i_m, j) \in \mathbb{Z}_{\geqslant 0}^m \times \mathbb{Z}_{>0} \mid i_s < r_s \text{ for some } s, 1 \leqslant s \leqslant m\right\}. \tag{5.1.3}$$

Using the relation $t_1^{r_1} \cdots t_m^{r_m} u = F(t_1, \ldots, t_m, v)$, one can see that every element $b \in B_{\underline{r}}$ can be uniquely expressed as

$$b = \sum_{n \geqslant 0} b_n(t_1, \ldots, t_m)v^n + \sum_{(\underline{i}, j) \in \Lambda_1} b_{\underline{i}\, j}(v)\, t_1^{i_1} \cdots t_m^{i_m} u^j, \tag{5.1.4}$$

such that $b_{\underline{i}j}(v) \in k[v] \setminus \{0\}$.

Now since the filtration $\{(B_{\underline{r}})_n\}_{n \in \mathbb{Z}}$ on $B_{\underline{r}}$ is induced from the graded structure of the ring C, from the expression (5.1.4), it follows that the filtration $\{(B_{\underline{r}})_n\}_{n \in \mathbb{Z}}$ is admissible with respect to the generating set $\Gamma = \{t_1, \ldots, t_m, u, v\}$ of $B_{\underline{r}}$ and the associated graded ring

$$E := \bigoplus_{n \in \mathbb{Z}} \frac{B_n}{B_{n-1}}$$

is generated by the image of Γ in E. For any $b \in B_{\underline{r}}$, let $\widetilde{b}$ denote its image in E. Note that $\widetilde{t_1}^{r_1} \cdots \widetilde{t_m}^{r_m} \widetilde{u} = \widetilde{v}^d$ in E, where $P(V) = F(0, \ldots, 0, V)$ and $d = \deg_V F = \deg_V P(V) (> 1)$.

As E can be identified with a subring of the graded domain $gr(C) \cong$

$k[t_1^{\pm 1}, \dots, t_m^{\pm 1}, v]$, we have $\widetilde{t_1}, \dots, \widetilde{t_m}, \widetilde{v}$ are algebraically independent in E and hence $\dim E = m+1$. Since $\frac{k[T_1,\dots,T_m,U,V]}{(T_1^{r_1}\cdots T_m^{r_m}U - V^d)}$ is an integral domain of dimension $m+1$, we have the following isomorphism

$$E \cong \frac{k[T_1, \dots, T_m, U, V]}{(T_1^{r_1} \cdots T_m^{r_m} U - V^d)}. \tag{5.1.5}$$

Now by Theorem 2.1.4, we have ψ induces a non-trivial exponential map $\widetilde{\psi}$ on E such that $\widetilde{g} \in E^{\widetilde{\psi}}$. Now from the grading on E it is clear that $\widetilde{v} \mid \widetilde{g}$ and hence $\widetilde{v} \in E^{\widetilde{\psi}}$. But then from (5.1.5) it follows that $\widetilde{t_1}, \dots, \widetilde{t_m}, \widetilde{u} \in E^{\widetilde{\psi}}$ (cf. Lemma 2.1.1(i)) and hence $\widetilde{\psi}$ is a trivial exponential map, which is a contradiction.

Therefore we obtain that for every non-trivial exponential map ϕ on $B_{\underline{r}}$, $B_{\underline{r}}^{\phi} \subseteq k[t_1, \dots, t_m]$. Since $B_{\underline{r}}^{\phi}$ is algebraically closed in $B_{\underline{r}}$ and $\operatorname{tr.deg}_k B_{\underline{r}}^{\phi} = m$ (cf. Lemma 2.1.1(ii)), we have $B_{\underline{r}}^{\phi} = k[t_1, \dots, t_m]$. Therefore $\mathrm{ML}(B_{\underline{r}}) = k[t_1, \dots, t_m]$.

(b) Let $\underline{r} = (r_1, \dots, r_m) \in \mathbb{Z}^m_{\geqslant 1} \setminus \mathbb{Z}^m_{>1}$. Suppose $r_j = 1$ for some $j, 1 \leqslant j \leqslant m$. Note that

$$B_{\underline{r}} = \frac{k[T_1, \dots, T_m, U, V]}{(T_1^{r_1} \cdots T_{j-1}^{r_{j-1}} T_{j+1}^{r_{j+1}} \cdots T_m^{r_m} T_j U - F(T_1, \dots, T_m, V))}.$$

Let $E_j := B_{\underline{r}}[t_1^{-1}, \dots, t_{j-1}^{-1}, t_{j+1}^{-1}, \dots, t_m^{-1}]$ and $C_j := k[t_1^{\pm 1}, \dots, t_{j-1}^{\pm 1}, t_{j+1}^{\pm 1}, \dots, t_m^{\pm 1}]$. Suppose that

$$F = T_j F_j(T_1, \dots, T_m, V) + F(T_1, \dots, T_{j-1}, 0, T_{j+1}, \dots, T_m, V).$$

Now for

$$u_j := u - \frac{F_j(t_1, \dots, t_m, v)}{t_1^{r_1} \cdots t_{j-1}^{r_{j-1}} t_{j+1}^{r_{j+1}} \cdots t_m^{r_m}} \in E_j, \tag{5.1.6}$$

we have

$$t_j u_j - \frac{F(t_1, \dots, t_{j-1}, 0, t_{j+1}, \dots, t_m, v)}{t_1^{r_1} \cdots t_{j-1}^{r_{j-1}} t_{j+1}^{r_{j+1}} \cdots t_m^{r_m}} = 0$$

and $E_j = C_j[t_j, u_j, v]$. We consider the exponential map $\phi_j : E_j \to E_j[W]$ such that

$$\phi_j|_{C_j} = id_{C_j}, \quad \phi_j(u_j) = u_j, \quad \phi_j(v) = v + u_j W \text{ and}$$

$$\phi_j(t_j) = \frac{F(t_1, \ldots, t_{j-1}, 0, t_{j+1}, \ldots, t_m, v + u_j W)}{t_1^{r_1} \cdots t_{j-1}^{r_{j-1}} t_{j+1}^{r_{j+1}} \cdots t_m^{r_m} u_j} = t_j + \sum_{i=1}^{r} \alpha_i W^i,$$

where $\alpha_i \in E_j$, for each $i, 1 \leqslant i \leqslant r$. Now since $\phi_j(u_j) = u_j$, using (5.1.6), it follows that

$$\phi_j(u) = u + \frac{F_j(t_1, \ldots, t_j + \sum_{i=1}^{r} \alpha_i W^i, \ldots, t_m, v + u_j W) - F_j(t_1, \ldots, t_m, v)}{t_1^{r_1} \cdots t_{j-1}^{r_{j-1}} t_{j+1}^{r_{j+1}} \cdots t_m^{r_m}} = u + \sum_{l=1}^{s} \beta_l W^l,$$

where $\beta_l \in E_j$ for every $l, 1 \leqslant l \leqslant s$.

Let $p_j := t_1 \cdots t_{j-1} t_{j+1} \cdots t_m$ and n be the smallest positive integer such that $p_j^n u_j, p_j^n \alpha_i, p_j^n \beta_l \in B_{\underline{r}}$ for every $i, 1 \leqslant i \leqslant r$ and every $l, 1 \leqslant l \leqslant s$. Since $\phi_j(p_j) = p_j$, ϕ_j induces an exponential map $\widetilde{\phi_j} : B_{\underline{r}} \to B_{\underline{r}}[W]$ such that

$$\begin{aligned}
&\widetilde{\phi_j}(t_i) = t_i, \text{ for every } i, 1 \leqslant i \leqslant m \text{ and } i \neq j,\\
&\widetilde{\phi_j}(t_j) = t_j + \sum_{i=1}^{r} \alpha_i (p_j^n W)^i,\\
&\widetilde{\phi_j}(v) = v + u_j p_j^n W \text{ and}\\
&\widetilde{\phi_j}(u) = u + \sum_{l=1}^{s} \beta_l (p_j^n W)^l.
\end{aligned}$$

Since $t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_m, u_j \in E_j^{\phi_j}$, it is clear that $\widetilde{u_j} \in E_j^{\phi_j}$, where

$$\widetilde{u_j} := t_1^{r_1} \cdots t_{j-1}^{r_{j-1}} t_{j+1}^{r_{j+1}} \cdots t_m^{r_m} u_j = t_1^{r_1} \cdots t_{j-1}^{r_{j-1}} t_{j+1}^{r_{j+1}} \cdots t_m^{r_m} u - F_j(t_1, \ldots, t_m, v).$$

Therefore, from the definition of $\widetilde{\phi_j}$, it follows that $\widetilde{u_j} \in B_{\underline{r}}^{\widetilde{\phi_j}}$, and hence

$$D_j := k[t_1, \ldots, t_{j-1}, \widetilde{u_j}, t_{j+1}, \ldots, t_m] \subseteq B_{\underline{r}}^{\widetilde{\phi_j}}.$$

Further, since $\operatorname{tr.deg}_k D_j = \operatorname{tr.deg}_k B_{\underline{r}}^{\widetilde{\phi_j}}$ and D_j is algebraically closed in $B_{\underline{r}}^{\widetilde{\phi_j}}$, we have $B_{\underline{r}}^{\widetilde{\phi_j}} = D_j$.

Again consider the following map $\phi : B_{\underline{r}} \to B_{\underline{r}}[W]$ such that

$$\phi(t_i) = t_i \text{ for every } i, 1 \leqslant i \leqslant m, \quad \phi(v) = v + t_1^{r_1} \cdots t_m^{r_m} W$$

and

$$\phi(u) = \frac{F(t_1, \ldots, t_m, v + t_1^{r_1} \cdots t_m^{r_m} W)}{t_1^{r_1} \cdots t_m^{r_m}} = u + W\alpha(t_1, \ldots, t_m, v, W),$$

where $\alpha \in k^{[m+2]}$. It is easy to see that $\phi \in \mathrm{EXP}(B_{\underline{r}})$ and $k[t_1, \ldots, t_m] \subseteq B_{\underline{r}}^{\phi}$. As $k[t_1, \ldots, t_m]$ is algebraically closed in $B_{\underline{r}}$, we have $B_{\underline{r}}^{\phi} = k[t_1, \ldots, t_m]$.

Since $B_{\underline{r}}^{\widetilde{\phi_j}} = k[t_1, \ldots, t_{j-1}, \frac{F(t_1,\ldots,t_{j-1},0,t_{j+1},\ldots,t_m,v)}{t_j}, t_{j+1}, \ldots, t_m]$ for every j, $1 \leqslant j \leqslant m$, and F is monic in v, we have

$$\mathrm{ML}(B_{\underline{r}}) \subseteq B_{\underline{r}}^{\phi} \bigcap_{\{j \,|\, r_j = 1\}} B_{\underline{r}}^{\widetilde{\phi_j}} = k[t_j \mid r_j \neq 1] \subsetneq k[t_1, \ldots, t_m].$$

In particular, for $\underline{r} = \underline{1}$ we have $k \subseteq \mathrm{ML}(B_{\underline{1}}) \subseteq B_{\underline{1}}^{\phi} \bigcap_{1 \leqslant j \leqslant m} B_{\underline{1}}^{\widetilde{\phi_j}} = k$. Hence the result follows. $\square$

Remark 5.1.5. From Theorem 5.1.4, it is clear that when $\underline{r} \in \mathbb{Z}_{>1}^m$ and $\underline{s} \in \mathbb{Z}_{\geqslant 1}^m \setminus \mathbb{Z}_{>1}^m$, then $B_{\underline{r}} \ncong B_{\underline{s}}$, as $\mathrm{ML}(B_{\underline{r}}) \neq \mathrm{ML}(B_{\underline{s}})$.

5.2 A family of counterexamples to the Cancellation Problem in arbitrary characteristic

In this section we will prove a certain subfamily (Ω_1) of the generalised Danielewski varieties to provide counterexamples to the Cancellation Problem in arbitrary characteristic.

The following theorem classifies the generalised Danielewski varieties $B_{\underline{r}}$ upto isomorphism when $\underline{r} \in \mathbb{Z}_{>1}^m$.

Theorem 5.2.1. *Let $(r_1, \ldots, r_m), (s_1, \ldots, s_m) \in \mathbb{Z}_{>1}^m$ and $F, G \in k[T_1, \ldots, T_m, V]$ be monic polynomials in V each of degree more than 1, such that $P(V) = F(0, \ldots, 0, V)$ and $Q(V) = G(0, \ldots, 0, V)$. Suppose*

$$B := B(r_1, \ldots, r_m, F) = \frac{k[T_1, \ldots, T_m, U, V]}{(T_1^{r_1} \cdots T_m^{r_m} U - F(T_1, \ldots, T_m, V))}$$

and

$$B' := B(s_1, \ldots, s_m, G) = \frac{k[T_1, \ldots, T_m, U, V]}{(T_1^{s_1} \cdots T_m^{s_m} U - G(T_1, \ldots, T_m, V))}$$

If $B(r_1, \ldots, r_m, F) \cong B(s_1, \ldots, s_m, G)$ then

(i) *$(r_1, \ldots, r_m) = (s_1, \ldots, s_m)$ upto a permutation of $\{1, \ldots, m\}$.*

(ii) *There exists $\alpha \in Aut_k(k[V])$ such that $\alpha(Q) = \lambda P$, for some $\lambda \in k^*$. Thus $\deg_V P = \deg_V Q$.*

(iii) *There exists a permutation* σ *of* $\{1, \dots, m\}$, $\lambda_1, \dots, \lambda_m, \beta, \gamma \in k^*$, $f \in k[T_1, \dots, T_m]$ *and* $\alpha \in k[T_1, \dots, T_m, V]$ *such that*

$$G(\lambda_{\sigma(1)}T_{\sigma(1)}, \dots, \lambda_{\sigma(m)}T_{\sigma(m)}, \gamma V + f(T_1, \dots, T_m)) = \\ T_1^{r_1} \cdots T_m^{r_m} \alpha(T_1, \dots, T_m, V) + \beta F(T_1, \dots, T_m, V).$$

Furthermore, the conditions (i) *and* (iii) *are sufficient for* $B(r_1, \dots, r_m, F)$ *to be isomorphic to* $B(s_1, \dots, s_m, G)$.

Proof. (i) Let $t_1, \dots, t_m, u, v$ and $t'_1, \dots, t'_m, u', v'$ denote the images of $T_1, \dots, T_m, U, V$ in B and B' respectively. Let $\rho : B \to B'$ be an isomorphism. Identifying $\rho(B)$ by B, we assume that $B' = B$. By Theorem 5.1.4(a), we have

$$\mathrm{ML}(B) = k[t_1, \dots, t_m] = k[t'_1, \dots, t'_m].$$

Therefore,

$$B \otimes_{k[t_1,\dots,t_m]} k(t_1, \dots, t_m) = B \otimes_{k[t'_1,\dots,t'_m]} k(t'_1, \dots, t'_m),$$

and hence

$$k(t_1, \dots, t_m)[v] = k(t'_1, \dots, t'_m)[v'] = k(t_1, \dots, t_m)[v']. \tag{5.2.1}$$

Note that $R := k[t_1, \dots, t_m, v] \hookrightarrow B \hookrightarrow k[t_1^{\pm 1}, \dots, t_m^{\pm 1}, v]$. We now show that $v' \in R$. Suppose $v' \in B \setminus R$. Then

$$v' = g(t_1, \dots, t_m, v) + \sum_{(\underline{i},j) \in \Lambda_1} b_{\underline{i}\,j}(v)\, t_1^{i_1} \cdots t_m^{i_m} u^j, \tag{5.2.2}$$

where Λ_1 is as in (5.1.3), $g \in R$ and $b_{\underline{i}\,j}(v) \in k[v] \setminus \{0\}$. Since $u = \frac{F(t_1,\dots,t_m,v)}{t_1^{r_1} \cdots t_m^{r_m}}$ and $\deg_v P(v) > 1$, from (5.2.2), it is clear that $\deg_v v' > 1$, when considered as an element in $k(t_1, \dots, t_m)[v]$. But this contradicts (5.2.1). Therefore, $v' \in R$. Now using the symmetry in (5.2.1), we obtain that

$$R = k[t_1, \dots, t_m, v] = k[t'_1, \dots, t'_m, v']. \tag{5.2.3}$$

Also from (5.2.1) and (5.2.3), it is clear that

$$v' = \gamma v + f(t_1, \dots, t_m), \tag{5.2.4}$$

for some $\gamma \in k^*$ and $f \in k^{[m]}$. Now

$$u' = \frac{G(t'_1, \ldots, t'_m, v')}{(t'_1)^{s_1} \cdots (t'_m)^{s_m}} \in B \setminus R.$$

Since $B \hookrightarrow k[t_1^{\pm 1}, \ldots, t_m^{\pm 1}, v]$, there exists $n > 0$ such that

$$(t_1 \cdots t_m)^n u' = \frac{(t_1 \cdots t_m)^n G(t'_1, \ldots, t'_m, v')}{(t'_1)^{s_1} \cdots (t'_m)^{s_m}} \in R.$$

Since for every $i \in \{1, \ldots, m\}$, t'_i is irreducible in R and $t'_i \nmid G$, we have $t'_i \mid (t_1 \ldots t_m)$. As $t_1, \ldots, t_m$ are also irreducibles in R, we have

$$t'_i = \lambda_j t_j, \tag{5.2.5}$$

for some $j \in \{1, \ldots, m\}$ and $\lambda_j \in k^*$. We now show that $s_i = r_j$. Suppose $s_i > r_j$. Consider the ideal

$$\mathfrak{a}_i := (t'_i)^{s_i} B \cap R = \left((t'_i)^{s_i}, G(t'_1, \ldots, t'_m, v')\right).$$

Again

$$\mathfrak{a}_i = t_j^{s_i} B \cap R = \left(t_j^{s_i}, t_j^{s_i - r_j} F(t_1, \ldots, t_m, v)\right),$$

which implies that $G(t'_1, \ldots, t'_m, v') \in t_j R$. But this is a contradiction. Therefore, $s_i \leqslant r_j$.

Again by similar arguments as above, we get that $s_i \geqslant r_j$. Therefore, we have $s_i = r_j$. Hence it follows that $(r_1, \ldots, r_m) = (s_1, \ldots, s_m)$ upto a permutation.

(ii) Since by (5.2.5) $(t_1 \cdots t_m) B \cap R = (t'_1 \cdots t'_m) B \cap R$, we have

$$(t_1 \cdots t_m, F) R = \left(t'_1 \cdots t'_m, G\right) R.$$

Therefore, it follows that

$$Q(v') = \lambda P(v) + Q_1(t_1, \ldots, t_m, v), \tag{5.2.6}$$

for some $\lambda \in k^*$ and $Q_1 \in (t_1, \ldots, t_m)R$. Therefore using (5.2.4), one can see that $\alpha : k[V] \to k[V]$ defined by $\alpha(V) = \gamma V + f(0, \ldots, 0)$ is an automorphism of $k[V]$, and from (5.2.6) it follows that $\alpha(Q) = \lambda P$.

(iii) From (i) we get a permutation σ of $\{1, \ldots, m\}$ such $t'_i = \lambda_{\sigma(i)} t_{\sigma(i)}$

(cf. (5.2.5)) and $s_i = r_{\sigma(i)}$ for every i, $1 \leqslant i \leqslant m$. Hence $t_1^{r_1} \cdots t_m^{r_m} = \lambda (t_1')^{s_1} \cdots (t_m')^{s_m}$ for some $\lambda \in k^*$. Now

$$\begin{aligned} t_1^{r_1} \cdots t_m^{r_m} B \cap R &= (t_1')^{s_1} \cdots (t_m')^{s_m} B \cap R \\ \text{i.e., } \left(t_1^{r_1} \cdots t_m^{r_m}, F(t_1, \ldots, t_m, v)\right) R &= \left((t_1')^{s_1} \cdots (t_m')^{s_m}, G(t_1', \ldots, t_m', v')\right) R \end{aligned}$$

Therefore,

$$G(t_1', \ldots, t_m', v') = t_1^{r_1} \cdots t_m^{r_m} \alpha(t_1, \ldots, t_m, v) + F(t_1, \ldots, t_m, v)\beta(t_1, \ldots, t_m, v)$$

for some $\alpha, \beta \in k^{[m+1]}$ such that no monomial summand of β is divisible by $t_1^{r_1} \cdots t_m^{r_m}$. Since both F and G are monic in v, by (ii) $\deg_v F = \deg_v G$, and hence it follows that $\beta \in k^*$. Therefore, the desired assertion follows.

We now prove the converse. We define $\rho : k[T_1, \ldots, T_m, U, V] \to B$ as follows:

$$\begin{aligned} \rho(T_i) &= \lambda_{\sigma(i)} t_{\sigma(i)}, 1 \leqslant i \leqslant m \\ \rho(V) &= \gamma v + f(t_1, \ldots, t_m) \\ \rho(U) &= \lambda^{-1}\beta u + \lambda^{-1}\alpha(t_1, \ldots, t_m, v), \end{aligned}$$

where $\lambda = \prod_{i=1}^m \lambda_{\sigma(i)}^{s_i}$. Note that ρ is surjective, and

$$\begin{aligned} \rho(T_1^{s_1} \cdots T_m^{s_m} U - G(T_1, \ldots, T_m, V)) = {}& \lambda \left(\prod_{i=1}^m t_{\sigma(i)}^{s_i} \right) \left(\lambda^{-1}\beta u + \lambda^{-1}\alpha(t_1, \ldots, t_m, v)\right) \\ & - G(\lambda_{\sigma(1)} t_{\sigma(1)} \ldots \lambda_{\sigma(m)} t_{\sigma(m)}, \gamma v + f(t_1, \ldots, t_m)). \end{aligned}$$

Now using (5.2.5) and (iii) it follows that

$$\begin{aligned} \rho(T_1^{s_1} \cdots T_m^{s_m} U - G(T_1, \ldots, T_m, V)) &= t_1^{r_1} \cdots t_m^{r_m} \left(\beta u + \alpha(t_1, \ldots, t_m, v)\right) \\ &\quad - \left(t_1^{r_1} \cdots t_m^{r_m} \alpha(t_1, \ldots, t_m, v) + \beta F(t_1, \ldots, t_m, v)\right) \\ &= 0. \end{aligned}$$

Therefore, ρ induces a surjective map $\overline{\rho} : B' \to B$, where B, B' both are affine domains whose dimensions are equal. Hence $\overline{\rho}$ is an isomorphism. □

We now record an elementary lemma.

Lemma 5.2.2. *Let E, D be integral domains such that $E \subseteq D$. Suppose there* *exists* $a(\neq 0) \in E$ *such that* $E[a^{-1}] = D[a^{-1}]$ *and* $aD \cap E = aE$. *Then* $E = D$.

The next theorem exhibits a certain sub-family of generalised Danielewski varieties which are stably isomorphic.

Theorem 5.2.3. *Let* $(r_1, \ldots, r_m), (s_1, \ldots, s_m) \in \mathbb{Z}^m_{\geqslant 1}$. *If* $(F, F_V) = k[T_1, \ldots, T_m, V]$, *then*

$$B(r_1, \ldots, r_m, F)^{[1]} \cong B(s_1, \ldots, s_m, F)^{[1]}.$$

Proof. Let $B(r_1, \ldots, r_m, F)$ be such that $r_j > 1$ for some $j \in \{1, \ldots, m\}$. Without loss of generality we assume $r_1 > 1$. We now show that

$$B(r_1 - 1, r_2, \ldots, r_m, F)^{[1]} \cong B(r_1, \ldots, r_m, F)^{[1]},$$

and therefore for any pair $(r_1, \ldots, r_m), (s_1, \ldots, s_m) \in \mathbb{Z}^m_{\geqslant 1}$, we get the result inductively.

Let $E = B(r_1, \ldots, r_m, F)[w] = B(r_1, \ldots, r_m, F)^{[1]}$. Consider the exponential map $\phi : E \to E[T] = E^{[1]}$ as follows:

$$\begin{aligned}
\phi(t_i) &= t_i, \text{ for all } i, 1 \leqslant i \leqslant m,\\
\phi(v) &= v + t_1^{r_1} \cdots t_m^{r_m} T,\\
\phi(u) &= \frac{F(t_1, \ldots, t_m, v + t_1^{r_1} \cdots t_m^{r_m} T)}{t_1^{r_1} \cdots t_m^{r_m}} = u + T\alpha(t_1, \ldots, t_m, v, T),\\
\phi(w) &= w - t_1 T,
\end{aligned}$$

where $\alpha \in k^{[m+2]}$. Now for

$$v_1 = v + t_1^{r_1 - 1} t_2^{r_2} \cdots t_m^{r_m} w, \tag{5.2.7}$$

we have $v_1 \in E^{\phi}$. Again,

$$\begin{aligned}
F(t_1, \ldots, t_m, v_1) &= F(t_1, \ldots, t_m, v + t_1^{r_1 - 1} \cdots t_m^{r_m} w)\\
&= F(t_1, \ldots, t_m, v) + t_1^{r_1 - 1} \cdots t_m^{r_m}(wF_V(t_1, \ldots, t_m, v) + bt_1),
\end{aligned}$$

for some $b \in k[t_1, \ldots, t_m, v, w]$. Therefore, as $F(t_1, \ldots, t_m, v) = t_1^{r_1} \cdots t_m^{r_m} u$, we have

$$F(t_1, \ldots, t_m, v_1) = t_1^{r_1 - 1} \cdots t_m^{r_m} u_1 \tag{5.2.8}$$

where

$$u_1 = t_1 u + wF_V(t_1, \ldots, t_m, v) + bt_1. \tag{5.2.9}$$

Since $t_1, \ldots, t_m, v_1 \in E^{\phi}$, by (5.2.8), $u_1 \in E^{\phi}$ (cf. Lemma 2.1.1(i)). Now since $(F, F_V) = k[T_1, \ldots, T_m, V]$, there exist $g_1, g_2 \in k^{[m+1]}$ such that

$$F(t_1, \ldots, t_m, v)g_1(t_1, \ldots, t_m, v) + F_V(t_1, \ldots, t_m, v)g_2(t_1, \ldots, t_m, v) = 1. \tag{5.2.10}$$

Note that $g_2(t_1, \ldots, t_m, v_1) - g_2(t_1, \ldots, t_m, v) \in t_1 E$. Therefore,

$$\begin{aligned} w - u_1 g_2(t_1, \ldots, t_m, v_1) &= w - g_2(t_1, \ldots, t_m, v_1)(t_1 u + wF_V(t_1, \ldots, t_m, v) + bt_1) \\ &= w(1 - F_V(t_1, \ldots, t_m, v)g_2(t_1, \ldots, t_m, v_1)) + t_1\theta \\ &= w(1 - F_V(t_1, \ldots, t_m, v)g_2(t_1, \ldots, t_m, v)) + t_1\delta \\ &= wF(t_1, \ldots, t_m, v)g_1(t_1, \ldots, t_m, v) + t_1\delta \\ &= wt_1^{r_1} \cdots t_m^{r_m} u g_1(t_1, \ldots, t_m, v) + t_1\delta \\ &= t_1\widetilde{w}, \end{aligned}$$

where $\theta, \delta \in E$ and $\widetilde{w} = wut_1^{r_1 - 1} \cdots t_m^{r_m} g_1(t_1, \ldots, t_m, v) + \delta \in E$. Therefore, we have

$$\widetilde{w} = \frac{w - u_1 g_2(t_1, \ldots, t_m, v_1)}{t_1} \in E. \tag{5.2.11}$$

Note that $\phi(\widetilde{w}) = \widetilde{w} - T$ and hence, by Lemma 2.1.1(ii), $E = E^{\phi}[\widetilde{w}] = (E^{\phi})^{[1]}$. Let $C := k[t_1, \ldots, t_m, v_1, u_1]$. Clearly $C \subseteq E^{\phi}$. By (5.2.8), $\operatorname{tr.deg}_k C = m + 1$ and hence $\dim C = m + 1$. We show that

(a) $C \cong B(r_1 - 1, r_2 \ldots, r_m, F) \cong \frac{k[T_1, \ldots, T_m, U, V]}{(T_1^{r_1 - 1} T_2^{r_2} \cdots T_m^{r_m} U - F(T_1, \ldots, T_m, V))}$,

(b) $C = E^{\phi}$.

(a) Consider the surjection $\psi : k[T_1, \ldots, T_m, V, U] \to C$ such that

$$\psi(T_i) = t_i \text{ for all } i, 1 \leqslant i \leqslant m, \ \psi(V) = v_1, \ \psi(U) = u_1.$$

From (5.2.8) it is clear that $(T_1^{r_1 - 1} T_2^{r_2} \cdots T_m^{r_m} U - F(T_1, \ldots, T_m, V)) \subseteq \ker \psi$. Therefore, ψ induces a surjection

$$\overline{\psi} : B(r_1 - 1, r_2, \ldots, r_m, F) = \frac{k[T_1, \ldots, T_m, U, V]}{(T_1^{r_1 - 1} T_2^{r_2} \cdots T_m^{r_m} U - F(T_1, \ldots, T_m, V))} \longrightarrow C.$$

Since $B(r_1 - 1, r_2, \ldots, r_m, F)$ is an integral domain and $\dim C = m + 1$, we have $\overline{\psi}$ is an isomorphism.

(b) We note that

$$\begin{aligned} E[t_1^{-1}] &= k[t_1^{\pm 1}, \dots, t_m, u, v, w] \\ &= k[t_1^{\pm 1}, \dots, t_m, u_1, v_1, w] \quad \text{by} \quad (5.2.7) \text{ and } (5.2.9) \\ &= k[t_1^{\pm 1}, \dots, t_m, u_1, v_1, \widetilde{w}] \quad \text{by} \quad (5.2.11) \\ &= C[t_1^{-1}][\widetilde{w}] \\ &= E^{\phi}[t_1^{-1}][\widetilde{w}] \end{aligned}$$

Since $C \subseteq E^{\phi}$, it follows that $C[t_1^{-1}] = E^{\phi}[t_1^{-1}]$. Therefore, to show that $C = E^{\phi}$, by Lemma 5.2.2, it is enough to show that $t_1 E^{\phi} \cap C = t_1 C$. Since $t_1 E \cap E^{\phi} = t_1 E^{\phi}$, it is therefore enough to show that $t_1 E \cap C = t_1 C$, i.e., the kernel of the map $\pi : C \to E/t_1 E$ is $t_1 C$. For every $b \in E$, let $\overline{b}$ denote its image in $E/t_1 E$, and for every $c \in C$, let $\widehat{c}$ denote its image in $C/t_1 C$. We note that by (5.2.8) and (a),

$$C/t_1 C = k[\widehat{t_2}, \dots, \widehat{t_m}, \widehat{v_1}, \widehat{u_1}] \cong \left(\frac{k[T_2, \dots, T_m, V]}{(F(0, T_2, \dots, T_m, V))} \right)^{[1]}. \qquad (5.2.12)$$

Also,

$$E/t_1 E \cong \frac{k[T_2, \dots, T_m, U, V, W]}{(F(0, T_2, \dots, T_m, V))} = \left(\frac{k[T_2, \dots, T_m, V]}{(F(0, T_2, \dots, T_m, V))} \right) [U, W] = k[\overline{t_2}, \dots, \overline{t_m}, \overline{v}, \overline{u}, \overline{w}]. \qquad (5.2.13)$$

Now by (5.2.7), we have $\pi(v_1) = \overline{v}$ as $r_1 > 1$ and by (5.2.9), $\pi(u_1) = \overline{w F_v(t_1, \dots, t_m, v)}$. By (5.2.10), $\overline{F_v(t_1, \dots, t_m, v)}$ is a unit in $E/t_1 E$ and hence

$$\pi(C) = k[\overline{t_2}, \dots, \overline{t_m}, \overline{v}, \overline{w}] \cong \left(\frac{k[T_2, \dots, T_m, V]}{(F(0, T_2, \dots, T_m, V))} \right)^{[1]}.$$

Therefore, from (5.2.12) it follows that π induces an isomorphism between $C/t_1 C$ and $\pi(C)$. Hence kernel of π is equal to $t_1 C$.

Thus, from (a) and (b) we have

$$B(r_1, \dots, r_m, F)[w] = E = (E^{\phi})^{[1]} = B(r_1 - 1, r_2, \dots, r_m, F)^{[1]}.$$

$\square$

As a consequence we have an infinite family of examples of varieties in arbitrary characteristic which are stably isomorphic but not isomorphic (c.f

Question 3).

Corollary 5.2.4. *For each $n \geqslant 2$, there exists an infinite family of pairwise non-isomorphic rings of dimension n in the class of Generalised Danielewski varieties over any field k of arbitrary characteristic, which are counterexamples to the General Cancellation Problem.*

Proof. Consider the family of rings

$$\Omega_1 := \big\{B(\underline{r}, F) \mid \underline{r} := (r_1, \dots, r_m) \in \mathbb{Z}^m_{\geqslant 1}, F \in k[T_1, \dots, T_m, V] \text{ is monic in } V \\ \text{and } (F, F_V) = k[T_1, \dots, T_m, V]\big\}.$$

For any pair $\underline{r} = (r_1, \dots, r_m), \underline{s} = (s_1, \dots, s_m) \in \mathbb{Z}^m_{\geqslant 1}$, by Theorem 5.2.3, $B(\underline{r}, F)^{[1]} \cong B(\underline{s}, F)^{[1]}$. Further, by Theorem 5.1.4(b) and Theorem 5.2.1, we get an infinite sub-family of Ω which contains pairwise non-isomorphic rings. Hence taking $n = m + 1$ we get the result. □

5.3 Invariants of Generalised Asanuma varieties

In this section we will see some applications of Theorem 5.2.3. We recall the coordinate ring of Generalised Asanuma varieties as follows:

$$A = \frac{k\,[X_1, \dots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - F(X_1, \dots, X_m, Z, T))}, \quad r_i > 1 \text{ for all } i, 1 \leqslant i \leqslant m, \tag{5.3.1}$$

where $f(Z, T) := F(0, \dots, 0, Z, T) \neq 0$ and $G := X_1^{r_1} \cdots X_m^{r_m} Y - F$. Let $x_1, \dots, x_m, y, z, t$ denote the images of $X_1, \dots, X_m, Y, Z, T$ in A, respectively.

We now deduce the structure of $\mathrm{DK}(A)$ and $\mathrm{ML}(A)$ for the special form of F given below:

$$F(X_1, \dots, X_m, Z, T) = f(Z, T) + (X_1 \cdots X_m)g, \tag{5.3.2}$$

for some $g \in k[X_1, \dots, X_m, Z, T]$.

Proposition 5.3.1. *Let A be the affine domain as in* (5.3.1) *with F as in* (5.3.2)*. Then the following hold:*

(a) *If $f(Z, T) \in k^*$, then $\mathrm{DK}(A) = A$ and $\mathrm{ML}(A) = k[x_1, \dots, x_m, x_1^{-1}, \dots, x_m^{-1}]$.*

(b) *If $f(Z, T) \notin k^*$, $\mathrm{DK}(A) \neq k[x_1, \dots, x_m, z, t]$ and A is geometrically factorial , then $\mathrm{ML}(A) \subsetneq k[x_1, \dots, x_m]$. Moreover, if $m = 1$ then $\mathrm{ML}(A) = k$.*

Proof. (a) If $f(Z,T) \in k^*$, then $A = k[x_1, \ldots, x_m, x_1^{-1}, \ldots, x_m^{-1}, z, t]$. Therefore,

$$\mathrm{DK}(A) = A \quad \text{and} \quad \mathrm{ML}(A) = k[x_1, \ldots, x_m, x_1^{-1}, \ldots, x_m^{-1}].$$

(b) Since $f(Z,T) \notin k^*$, for every i, $1 \leqslant i \leqslant m$, $x_i \notin A^*$. Therefore, by Proposition 3.1.8(a), $\mathrm{ML}(A) \subseteq k[x_1, \ldots, x_m]$. Since $\mathrm{DK}(A) \neq k[x_1, \ldots, x_m, z, t]$, there exists a non-trivial exponential map ϕ on A such that $A^\phi \nsubseteq k[x_1, \ldots, x_m, z, t]$. Suppose, if possible, $\mathrm{ML}(A) = k[x_1, \ldots, x_m]$. Then, $k[x_1, \ldots, x_m] \subseteq A^\phi \nsubseteq k[x_1, \ldots, x_m, z, t]$. But then, by Theorem 3.2.6, $A = k^{[m+2]}$, which is contradiction as $\mathrm{ML}(k^{[m+2]}) = k$.

Therefore, $\mathrm{ML}(A) \subsetneq k[x_1, \ldots, x_m]$, and since $\mathrm{ML}(A)$ is algebraically closed in A (cf. Lemma 2.1.1(i)), for $m = 1$, $\mathrm{ML}(A) = k$. □

We now answer Question 4 (see Chapter 1) for some special form of F.

Proposition 5.3.2. *Let $C := k[X_1, \ldots, X_m]$ and A be the affine domain as in* (5.3.1) *where*

$$F(X_1, \ldots, X_m, Z, T) = a_0(Z) + a_1(Z)T + \widetilde{F}(X_1, \ldots, X_m, Z)$$

and $\widetilde{F} \in (X_1, \ldots, X_m)C[Z]$. Then $\mathrm{DK}(A) = A$ *and* $\mathrm{ML}(A) = k$ *when any one of following holds:*

(i) $a_1(Z) \neq 0$.

(ii) $a_1(Z) = 0$*, F is a monic polynomial in Z and $(F, F_Z) = C[Z]$.*

Proof. (i) It is clear that for every $i \in \{1, \ldots, m\}$, $x_i \notin A^*$. Let $Q \in k^{[m+2]}$ be such that

$$Q(x_1, \ldots, x_m, y, z) = x_1^{r_1} \cdots x_m^{r_m} y - \widetilde{F}(x_1, \ldots, x_m, z) - a_0(z).$$

Note that $Q(x_1, \ldots, x_m, y, z) - a_1(z)t = 0$. As $a_1(Z) \neq 0$, for every $j \in \{1, \ldots, m\}$, we now define the following maps $\phi_j : A \to A[U]$ by

$$\phi_j(x_i) = x_i \text{ for } i, 1 \leqslant i \leqslant m, i \neq j,\ \phi_j(x_j) = x_j + a_1(z)U,\ \phi_j(y) = y,\ \phi_j(z) = z,$$

and

$$\phi_j(t) = \frac{Q(x_1, \ldots, x_{j-1}, x_j + a_1(z)U, x_{j+1}, \ldots, x_m, y, z)}{a_1(z)} = t + Uv_j(x_1, \ldots, x_m, y, z, U),$$

for some $v_j \in k[x_1, \dots, x_m, y, z, U]$. It is easy to see that $\phi_j \in \mathrm{EXP}(A)$, for every j. Since $y \in A^{\phi_j}$, by Lemma 3.1.4, we have $\mathrm{DK}(A) = A$.

Let $C_j := k[x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_m, y, z]$. Then $C_j \subseteq A^{\phi_j} \subseteq A$. As C_j is algebraically closed in A and $\operatorname{tr.deg}_k C_j = \operatorname{tr.deg}_k A^{\phi_j} = m + 1$ (cf. Lemma 2.1.1(ii)), we have $A^{\phi_j} = C_j$. By Proposition 3.1.8(a), $\mathrm{ML}(A) \subseteq k[x_1, \dots, x_m]$. Since j is arbitrarily chosen from $\{1, \dots, m\}$, we get that $\mathrm{ML}(A) \subseteq k[x_1, \dots, x_m] \bigcap_{1 \leqslant j \leqslant m} A^{\phi_j} = k$. Thus $\mathrm{ML}(A) = k$.

(ii) As $a_1(Z) = 0$, $A = B(r_1, \dots, r_m, F)^{[1]}$. Now by Theorem 5.2.3 we have $A \cong B(1, \dots, 1, F)^{[1]}$. Since by Theorem 5.1.4, $\mathrm{ML}(B(1, \dots, 1, F)) = k$, $\mathrm{ML}(A) = k$. As $B(1, \dots, 1, F)$ is not rigid, by Lemma 2.1.3, $\mathrm{DK}(A) = A$. □

As a consequence, we give the complete description of $\mathrm{DK}(A)$ and $\mathrm{ML}(A)$ when A (as in (5.3.1)) is a regular domain over an infinite field and $F = f(Z, T)$.

Corollary 5.3.3. *Let A be a regular domain defined by*

$$A = \frac{k[X_1, \dots, X_m, Y, Z, T]}{(X_1^{r_1} \cdots X_m^{r_m} Y - f(Z, T))}.$$

Then the following hold:

(a) *If $f(Z, T)$ is coordinate in $k[Z, T]$, then* $\mathrm{DK}(A) = A$ *and* $\mathrm{ML}(A) = k$.

(b) *If there exists a system of coordinates $\{Z_1, T_1\}$ of $k[Z, T]$ such that $f(Z, T) = a_0(Z_1) + a_1(Z_1)T_1$, then* $\mathrm{DK}(A) = A$ *and* $\mathrm{ML}(A) = k$.

(c) *If k is an infinite field and if $f(Z, T)$ can not be expressed in the form described in part (b) above, then* $\mathrm{DK}(A) = k[x_1, \dots, x_m, z, t]$ *and* $\mathrm{ML}(A) = k[x_1, \dots, x_m]$.

Proof. (a) As $k[Z, T] = k[f]^{[1]}$, $A = k^{[m+2]}$. Therefore, $\mathrm{DK}(A) = A$ and $\mathrm{ML}(A) = k$.

(b) If $a_1(Z_1) \neq 0$, then the assertion follows from Proposition 5.3.2(i).

If $a_1(Z_1) = 0$, then $f(Z, T) = a_0(Z_1)$. As A is regular we have $\gcd(a_0, (a_0)_{Z_1}) = 1$. Hence the result follows from Proposition 5.3.2(ii).

(c) A special case of Remark 3.1.9. □

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(i) P. Ghosh and N. Gupta, *On the triviality of a family of linear hyperplanes*, Advances in Mathematics, **428** (2023), 109166, https://doi.org/10.1016/j.aim.2023.109166.

(ii) P. Ghosh and N. Gupta, *On generalised Danielewski and Asanuma varieties*, Journal of Algebra, **632** (2023), 226–250, https://doi.org/10.1016/j.jalgebra.2023.05.028.