CANCELLATIONS IN SHORT SUMS RELATED TO HECKE-CUSP FORMS

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April 2024

A thesis submitted to the Indian Statistical Institute in partial fulfilment of the requirements for the award of the degree of Doctor of Philosophy in Mathematics

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To my parents, my wife, my advisor and my pet (Mimi) ....

ACKNOWLEDGEMENT

First I would like to thank my parents, my wife and my pet for their considerate efforts to make sure, that I have tranquillity of mind to perform my research. Even when my presence in family matters became essential, they would simply smile and tell me not to be bothered about things other than my research. It is for them that I embarked upon the world of mathematics. They left no stone unturned to ensure that no perturbation reached my workspace at home. I want to thank them for everything.

It is my privilege to have Prof. Ritabrata Munshi as my advisor. The impact he had on me is inexpressible in words. I would like to thank him for introducing me to the fascinating world of analytic number theory and L-functions. His wit and his impeccable sense of humour helped me to overcome anxiety, fear and mundane nature during my PhD life. His door was always open for me and he was always more than happy to discuss any issues (be it mathematical or personal) that came up. His faith in me motivated me to work hard with more enthusiasm. His helpful comments, interesting and fruitful discussions, insightful ideas and remarks helped me in my research. So I would like to dedicate my thesis to him also.

I want to thank Prof. Satadal Ganguly and Prof. Kummari Mallesham for acting as my unofficial advisors. They acted as my 'silent guardians' at Indian Statistical Institute, Kolkata. I also want to thank them for helping me with useful comments, and discussions and I wish to express my gratitude to them.

I would like to thank Prof. Mrinal Das, Prof. Mahuya Dutta, Prof. Neena Gupta, Prof. S. M. Srivastava, Prof. Rudra Pada Sarkar, Prof. Kingshook Biswas and Prof. Krishanu Maulik for teaching us some wonderful courses at ISI. Also, I thank Jyotisman Da for his friendly nature to whom, without any hesitation, I could approach for any help.

I would also like to express my gratitude to the Indian Statistical Institute for giving us excellent facilities (Library, canteen, auditorium, classrooms etc.) and giving me the stage to complete my Ph.D. Also, I thank all the administrative staff of the SMU office, account section, dean's office and others for their kind support, help and generosity.

I thank specially to my collaborators, Mallesham bhaiya, Saurabh bhaiya. I benefited from their helpful comments and various fruitful discussions that also helped during my research.

I express my deep gratitude to Prof. Roman Holowinsky, Prof. Stephan Baier and Prof.

Kaneenika Sinha for helping me and guiding me in getting jobs. Also, I am grateful to Prof. Philippe Michel for his helpful comments and suggestions.

I would also like to express my deep gratitude to all my teachers. I would like to mention Sitaram Sir, Ashoke Sir, Samit Sir and Bapi da for mentoring and inspiring me toward a good life from my school days. I want to thank Apurba Sir from my undergrads and other faculties at RKMVU, Belur, from my Masters.

I would also like to mention my friends without whom my Ph.D. life becomes incomplete. I am thankful to my batchmates, Mainak, Debonil Da, Priyanka, and Aparajita Di for being among some of my first few friends at ISI and also to Sampa di, Rachita, Sayan da, Prahlad, Sumit, Gopal, Sampurna, Ritwik Da, Mayukh, Pratim, Sayan and Mallesham da for spending a memorable time in the tea shop and my office. Also, I want to thank my junior cum friend Ankit for motivating me in any situation. I thank to my friends Raja, Souvik, Sayantan, Aaranya, Soham, Sumit, Jyoti, Koustav, Somnath and others.

In the end, I would like to express my love and respect towards my family members. I am dedicating this thesis to my parents, my wife, and my pet (Mimi). I am grateful to God for getting such a wonderful family to live with. I am indebted to my Bidhan kaka, my extended family, my late parents-in-law, my sister-in-law, my nephew, Tarak da, and Bapi da for their constant support and encouragement. I am grateful to my Goddess Tara Maa for her blessings on me. Also, the life of the great Indian mathematician, Srinivasa Ramanujan inspired me to pick up number theory and pursue higher studies.

Aritra Ghosh

Contents

Notations & Abbreviations

$e_p(z)$, $e\left(\frac{z}{p}\right)$	$e^{\frac{2\pi i z}{p}}$
$a \sim b$, $a \asymp b$	$cb \leq a \leq Cb$ for some absolute constants c, C, not depending on a, b
$f(x) \sim g(x)$	$f(x)/g(x) \to 1$, as $x \to \infty$
$A \ll B$	$\exists$ some absolute constant $c > 0$ such that $\|A\| \leq cB$
$A \ll_f B$	$\exists$ some $c(f) > 0$ depending on f such that $\|A\| \leq c(f)B$
$\mathbb{N}$	set of all positive integers
$\mathbb{N}_0$	set of all non-negative integers
$\mathbb{Z}$	set of all integers
$\mathbb{C}$	set of all complex numbers
$\mathbb{H}$	the upper half plane
$\mathbb{R}_0^+$	set of all non-negative real numbers
ε	arbitrary small positive real number
$\mathbb{I}_S$	the characteristic function of the set S
$\mathbb{Z}_p$	ring of p-adic integers
$\mathbb{Q}_p$	field of p-adic numbers
ρ_p	$\frac{1}{p-1}$ for any prime p

Chapter 1

Introduction

1.1 General correlation problem

A problem which arises in a variety of contexts is the cancellations in sums of the form

$$S = \sum_{n=1}^{\infty} a(n)b(n), \tag{1.1}$$

where $a = \{a(n)\}$, and $b = \{b(n)\}$ are arithmetic sequences of special interest. This type of problem is also known as the **general correlation problem**.

For most of the problems we can assume the following three assumptions (these are standard also) :

A. Finite support : The sequences are supported in some dyadic range $[N, 2N]$, i.e., $a(n) = b(n) = 0$ for all $n \notin [N, 2N]$.

B. The Ramanujan bound : Though for the sequences the "**pointwise Ramanujan bound**", i.e., $a(n), b(n) \ll n^{\varepsilon}$ may not be available to us, so we assume the "**Ramanujan bound on average**" at least in the L^2-sense, i.e., we consider $\sum_{N \leq n \leq 2N} \left|a(n)\right|^2 \ll_{\varepsilon} N^{1+\varepsilon}$ and $\sum_{N \leq n \leq 2N} \left|b(n)\right|^2 \ll_{\varepsilon} N^{1+\varepsilon}$.

C. Non-trivial cancellations : At least one of $\sum_{n=1}^{\infty} a(n)$ and $\sum_{n=1}^{\infty} b(n)$ is small, i.e., of size $\mathrm{O}(N^{1-\delta})$ for some $\delta > 0$.

Then from the first and second assumptions, indexed by A and B respectively, one can note that by the Cauchy-Schwarz inequality, we have,

$$\Big|S(N)\Big| = \Big|\sum_{n=1}^{\infty} a(n)b(n)\Big| \leq \left(\sum_{N\leq n\leq 2N} \Big|a(n)\Big|^2\right)^{1/2} \left(\sum_{N\leq n\leq 2N} \Big|b(n)\Big|^2\right)^{1/2} \ll_{\varepsilon} N^{1+\varepsilon}$$

Then the correlation is given by:

$$\frac{1}{N}\sum_{n=1}^{\infty} a(n)b(n) - \left[\frac{1}{N}\sum_{n=1}^{\infty} a(n)\right]\left[\frac{1}{N}\sum_{n=1}^{\infty} b(n)\right] = \frac{S}{N} + \mathrm{O}(N^{-\delta}).$$

Definition 1.1.1. We say that the sequences $a = \{a(n)\}$, and $b = \{b(n)\}$ are **independent** if there exists some $\delta > 0$, such that,

$$\textstyle\sum_{n=1}^{\infty} a(n)\, b(n) \ll N^{1-\delta}. \tag{1.2}$$

1.2 Examples :

For several problems of arithmetic interest, these correlation problems occur naturally. Let us discuss some of them here :

- Subconvexity problems : To break the convexity bound (coming from the Phragmen-Lindelof principle of complex analysis) of certain L-functions, we need to consider certain correlation problems as in the following examples.

(i) At first we take $a(n) = \chi(n)W\left(\frac{n}{N}\right)$ where χ is a primitive Dirichlet character modulo q and $b(n) = n^{it}V\left(\frac{n}{N}\right)$ where $t > 2$, for $n \in [N, 2N]$. Here W, V are finitely supported nice functions, which will be discussed later. Suppose we take $N = \sqrt{qt}$. Note that the pointwise bounds $a(n)$, $b(n) \ll 1$ hold trivially. Also $\sum_{N\leq n\leq 2N} b(n) \ll N^{1-\delta}$ follows from cancellations of exponential sums (Weyl differencing). In this case $\sum_{N\leq n\leq 2N} a(n) \ll N^{1-\delta}$ can also be established, following Burgess [11]. The correlation (1.2) in this case is related to the subconvexity problem of $L(\frac{1}{2} + it, \chi)$.

(ii) Now we take $a(n) = \chi(n)$, $b(n) = \lambda_f(n)$ where χ is a primitive Dirichlet character modulo q and $\lambda_f(n)$'s are normalized Fourier coefficients for a holomorphic cusp or Hecke-Maass cusp form f for the full modular group $\mathrm{SL}(2,\mathbb{Z})$. Here we take $N = q$. Then as discussed in the previous example, we have cancellations for the sequence $\{a(n)\}$ and also for $\{b(n)\}$ (see [18]). The correlation (1.2) in this case will be discussed in our next two chapters, where we show that it is related to the subconvexity of $L(\frac{1}{2}+it, f\otimes\chi)$.

- Dirichlet's divisor problem and its cuspidal analogue : Let d_2 be the usual divisor function. In 1849 (see [47]), using the hyperbola method, Dirichlet established that

$$D(N) := \sum_{n\leq N} d_2(n) = N\log N + N(2\gamma-1) + \Delta(N),$$

where γ is the Euler-Mascheroni constant and $\Delta(N) = \mathrm{O}(\sqrt{N})$ is the error term. The **Dirichlet divisor problem** is about improving this error term by finding the smallest value θ for which $\Delta(N) = \mathrm{O}(N^{\theta+\varepsilon})$. Hardy (see [47]) showed that $\inf\theta \geq \frac{1}{4}$.

In general, for $d_k(n)$, which counts the number of ways that n can be written as a product of exactly k numbers, the error term (see [32]) is related to

$$\sum_{n\sim N} d_k(n)\, e\left(k(nx)^{\frac{1}{k}}\right) \quad \textit{for } N\sim x^{\frac{k-1}{k+1}}.$$

For the above problem, it is natural to define $\left((Nx)^{1/k}\right)^k \asymp x^{\frac{2k}{k+1}}$ as the "conductor". Then note that the length of the sum N is smaller than the square root of the conductor. If we take a Hecke-Maass or holomorphic cusp form then our interest lies in estimating the following sum with the sharp cut :

$$\sum_{n<x} \lambda(n).$$

We expect that it is bounded by $\mathrm{O}\left(x^{\frac{1}{4}+\varepsilon}\right)$, which is conjectured, though the known bound is $x^{\frac{1}{3}+\varepsilon}$ (see [18]). To improve the known bound $\mathrm{O}\left(x^{\frac{1}{3}+\varepsilon}\right)$, one needs to study

$$\sum_{n\sim N} \lambda(n)\, e(a\sqrt{nx}) \quad \textit{for } N\sim x^{1/3}.$$

For the above problem we define $\left(\sqrt{Nx}\right)^2 \asymp N^4$ as the "conductor" naturally. One notes that this is the cuspidal analogue of the above problem with $k = 2$. Note that the length of the sum N is less than the square root of the conductor.

- Critical Zeros of Symmetric Square L-functions : To prove the existence of critical zeros of the symmetric square L-functions (analogue of Hardy's theorem, see [55]), one has to show strong cancellation in the following sum, namely

$$\sum_{n\sim N} \lambda(n^2)\, e(\alpha n^{2/3}) \ll N^{1-\frac{1}{6}-\delta} \quad \textit{for some } \delta > 0.$$

For the above problem we define $\left(N^{2/3}\right)^3 = N^2$ as the "conductor". Here one can note that the length of the sum N is less than the square root of the conductor.

- Higher rank exponential sums : For the GL(2) forms we know that

$$\sum_{n\sim N} \lambda(n)\, e(n\alpha) \ll N^{\frac{1}{2}+\varepsilon}.$$

The problem for **linear twists of** GL(3) **forms** is to estimate the following sums :

$$\sum_{n\sim N} \lambda(n,1)\, e(n\alpha),$$

with smooth weight. One conjecture is that the above sum is bounded by $\mathrm{O}\left(N^{1/2+\varepsilon}\right)$. But the best known bound is $\mathrm{O}\left(N^{\frac{3}{4}+\varepsilon}\right)$, due to Miller (see [46]).

If we consider a GL(d) automorphic form π then we are interested in estimating non-trivially

$$\sum_{n\sim N} \lambda_\pi(n, 1, \ldots, 1)\, e\left(\alpha n^{\beta}\right).$$

This problem is completely open for $d \geq 4$, even for $\beta = 1$ (**Linear twist**). For $d = 2$ one can see [27].

- Shifted convolution sums : Usually, these types of sums arise in the study of subconvexity problems using the moment method and also are some natural examples of this type of problem. A $\mathrm{GL}(2) \times \mathrm{GL}(2)$ shifted convolution sum problem is to estimate non-trivially the following sum :

$$\sum_{n \sim N} \lambda_f(n)\, \lambda_g(n+h),$$

where f, g are GL(2) cuspidal automorphic forms with $\lambda_f(n)$, $\lambda_g(n)$ are their respective normalized Fourier coefficients and $h > 0$. Here we must mention the **additive divisor problem** (see [16], [48]), which asks for asymptotics of the following sum (with power saving error term) :

$$\sum_{n \leq N} d_2(n)\, d_2(n+h),$$

where d_2 is the usual divisor function, as $N \to \infty$.

1.3 Long smooth sums vs Short smooth sums

In our examples, we defined "conductor" for many cases, whereas the notion of "**conductor**" occurs naturally in the case of automorphic L-functions which can be found in [36]. If the length of a smooth sum is greater than the square root of the conductor then we will call that sum to be a "**long smooth sum**" for which non-trivial cancellation exists due to the respective functional equation. But if the length of the sum is less than the square root of the conductor then we will call the sum "**short smooth sum**" for which, getting non-trivial cancellation is not easy and becomes one of the centres of attraction in analytic number theory.

Now consider a smooth sum

$$S(N) = \sum_{n=1}^{\infty} a(n)\, W\left(\frac{n}{N}\right), \tag{1.3}$$

where W is a nice function supported on $[\frac{1}{2}, 3]$ and taking value 1 on $[1, 2]$. Also let the corresponding Dirichlet series of the sequence $a = \{a(n)\}$ be

$$L_a(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

which we assume to satisfy the standard properties of an L-function as given in Chapter 5 of [35] (i.e., these L-functions are in Selberg class) with conductor $Q(a)$. Then by the Mellin inversion formula, (1.3) can be written as

$$S(N) = \frac{1}{2\pi i} \int_{(\sigma')} N^s \, \tilde{W}(s) \left(\sum_{n=1}^{\infty} \frac{a(n)}{n^s} \right) \mathrm{d}s \; = \; \frac{1}{2\pi i} \int_{(\sigma')} N^s \, \tilde{W}(s) \, L_a(s) \, \mathrm{d}s, \quad \sigma' > 1.$$

Suppose that the above L-function has polynomial growth, i.e.,

$$\Big| L_a(s) \Big| \leq \Big| L_a(\sigma) \Big| \, (2 + |t|)^{A+\varepsilon},$$

for some $A > 0$. Then by shifting the contour to the line $\mathcal{R}(s) = \sigma \, (< 1$ and $= \frac{1}{2}$ for our case) and using the properties of $\tilde{W}(s)$, we have,

$$S(N) \ll N^{\sigma} \Big| L_a(\sigma) \Big| \ll N^{\sigma} \, Q(a)^{\theta}.$$

This gives a non-trivial cancellation, i.e., $S(N) < N$, if and only if,

$$N^{\sigma} Q(a)^{\theta+\varepsilon} < N \, , \, i.e., \, N > Q(a)^{\mu+\varepsilon},$$

where $\mu = \frac{\theta}{1-\sigma}$. We will call this "**the range of cancellations**" of $S(N)$. Then we will follow some terminologies :

Trivial range : If the range of cancellations of $S(N)$ is $N > Q(a)^{1/4+\varepsilon}$, i.e., $\mu = \frac{1}{4}$, then we call this range of cancellations of $S(N)$ to be the "**Trivial range**". This corresponds to the trivial or the convexity bound of the corresponding L-function.

Burgess range : If the range of cancellations of $S(N)$ is $N > (Q(a))^{\frac{1}{4} - \frac{1}{16} + \varepsilon}$, i.e., $\mu = \frac{1}{4} - \frac{1}{16}$, then we call this range of cancellations of $S(N)$ to be the "**Burgess range**", as it corresponds to the Burgess strength bound of the L-function.

Weyl range : If the range of cancellations of $S(N)$ is $N > (Q(a))^{\frac{1}{4} - \frac{1}{12} + \varepsilon}$, i.e., $\mu = \frac{1}{4} - \frac{1}{12}$, then we call this range of cancellations of $S(N)$ to be the "**Weyl range**", as it corresponds to the Weyl strength bound of the L-function.

Sub-Weyl range : If the range of cancellations of $S(N)$ is $N > (Q(a))^{\frac{1}{4}-\frac{1}{12}-\delta+\varepsilon}$ for some $\delta > 0$, i.e., $\mu = \frac{1}{4} - \frac{1}{16} - \delta$, then we call this range of cancellations of $S(N)$ to be the "**Sub-Weyl range**".

Lindelöf range : If the range of cancellations of $S(N)$ is $N > (Q(a))^{\varepsilon}$, i.e., $\mu = 0$, then we call this range of cancellations of $S(N)$ to be the "**Lindelöf range**". Still now, this is a conjecture.

Now we will illustrate three examples in the next three following sections.

1.4 Short smooth sums related to Dirichlet characters

Let χ be a primitive Dirichlet character modulo p^r for $r \geq 1$. Then for this case, one can note that the convexity bound recovers the conclusion of the Polya-Vinogradov inequality. Hence subconvexity corresponds to cancellation in shorter sums.

Burgess (see [11]) proved that $L(1/2, \chi) \ll_{\varepsilon} p^{3/16+\varepsilon}$ which yields a non-trivial bound if and only if $N > p^{3/8+\varepsilon}$. His method used the Riemann Hypothesis for curves over finite fields. However, 3/16, the Burgess exponent is larger than the exponent 1/6 established by Weyl. However, Burgess's method yields a non-trivial bound for $S_{\chi}(N)$ for any $N > p^{1/4+\varepsilon}$ if p is cube-free (especially for primes). This does not come through the passage to L-functions as we have sketched above. But this basic idea applies to the scenarios as well, invoking higher-rank harmonics.

The Burgess-type subconvex bound is known only for some limited special cases. A Weyl quality bound for quadratic characters of the odd conductor was achieved by Conrey and Iwaniec (see [13]) using the techniques from automorphic forms and Deligne's solution of the Weil conjectures for varieties over finite fields. [4] and [25] considered the cases when the conductor q of χ runs over prime powers or otherwise has some special factorizations. For any Dirichlet L-function having a cube-free conductor, recently Petrow and Young (see [54]) proved a Weyl type subconvex bound where they also got subconvex bound having the same strength for certain L-functions of self-dual GL(2) automorphic forms which arise as twists of forms of smaller conductor. One can also see the recent work of Nelson (see [53]). The exponent 3/16 curiously often occurs in various problems, see [2], [6], [8], [12], [19], [63], [64] as examples. Also, related work on the Burgess-type bounds can be found in the paper of Munshi (see [51]). These we will discuss in our next section.

Recently, Milicévić (see [45]) got a sub-Weyl subconvex bound when $q = p^n$ with n large by developing the p-adic exponent pair method.

1.5 Short smooth sums related to Hecke-cusp forms

1.5.1 A brief history

Let us take a holomorphic or Hecke-Maass cusp form, f for $\mathrm{SL}(2,\mathbb{Z})$ having normalized Fourier coefficients $\lambda_f(n)$ and a primitive Dirichlet character χ of conductor p^r, with $r \geq 1$. Now consider the smooth sum

$$S_{f,\chi}(N) = \sum_{n=1}^{\infty} \chi(n)\, \lambda_f(n)\, W\left(\frac{n}{N}\right), \tag{1.4}$$

where W is a smooth bump function supported on $[1,2]$ and satisfies $W^{(j)}(x) \ll_j 1$.

One can use information (bounds for $L(1/2, f\otimes\chi)$) about L-values $L(1/2, f\otimes\chi)$ to show cancellations in the sum $S_{f,\chi}(N)$. Indeed, by the Mellin inversion formula, we have that

$$S_{f,\chi}(N) = \frac{1}{2\pi i}\int_{(\sigma)} N^s\, \tilde{W}(s)\, L\left(s, f\otimes\chi\right)\, \mathrm{d}s, \quad \sigma > 1,$$

where $L(s, f\otimes\chi)$ is given by

$$L(s, f\times\chi) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi(n)}{n^s}, \qquad \textit{for}\ \ \mathcal{R}(s) > 1.$$

Now shifting the contour to $\mathcal{R}(s) = 1/2$-line and estimating trivially, as done in the Section 1.3, we get that

$$S_{f,\chi}(N) \ll N^{1/2} \int_{|t|\leq N^{\varepsilon}} |L(1/2+it, f\otimes\chi)|\, \mathrm{dt}\; + \mathrm{O}\left(\mathrm{N}^{-\mathrm{A}}\right),$$

for any $A > 0$.

Now we fix the form f. Then the analytic conductor of $L(1/2+it, f\otimes\chi)$ becomes $p^{2r}t^2$. Then the convexity bound $L(1/2+it, f\times\chi) \ll_t p^{r/2}t^{1/2}$ would imply that

$$S_{f,\chi}(N) \ll_f N^{1/2}\, p^{r/2} N^{\varepsilon},$$

which is non-trivial if $N > p^{r+\varepsilon}$.

By the Burgess exponent (see [7], [8], [12], [19], [51]) we have

$$L(\frac{1}{2}, f \otimes \chi) \ll_{f,\varepsilon} p^{3/8+\varepsilon}.$$

Hence we have the Burgess range

$$S_{f,\chi}(N) \ll_{f,\varepsilon} \sqrt{N} p^{\frac{3}{8}+\varepsilon} < N \iff N > p^{3/4+\varepsilon}.$$

For $r > 1$, sufficiently large, currently Weyl bound is known for $L(1/2, f \otimes \chi)$ which is $L(1/2, f \otimes \chi) \ll p^{r/3+\varepsilon}$, due to Milićević and Blomer [10]; Munshi and Singh [52]. This would then imply that

$$S_{f,\chi}(N) \ll_f N^{1/2} p^{r/3+\varepsilon} N^{\varepsilon},$$

which is non-trivial if $N > p^{2r/3+\varepsilon}$.

Currently we do not know how to obtain sub-Weyl bounds for $L(1/2, f \otimes \chi)$, i.e., $L(1/2 + it, f \otimes \chi) \ll (p^{2r})^{\frac{1}{3}-\eta+\varepsilon}$ for some $\eta > 0$ which would give non-trivial bounds for $S_f, \chi(N)$ whenever $N > (p^r)^{\frac{2}{3}-2\eta+\varepsilon}$. It is needless to mention that Lindelöf hypothesis would give non-trivial bounds for $L(1/2, f \otimes \chi)$ if $N > p^{r\varepsilon}$.

Let $K(n)$ be a trace function modulo prime p. In [19], Fouvry, Kowalski and Michel showed that

$$\sum_n \lambda_f(n)\, K(n)\, W\left(\frac{n}{N}\right) \ll_f N^{1/2} p^{3/8} N^{\varepsilon}.$$

This is the Burgess type bound which gives non-trivial bounds for the above sums if $N > p^{3/4+\varepsilon}$. For general trace functions $K(n)$ the bound of Fouvry, Kowalski, and Michel [19] is the best-known result.

1.5.2 Our result on the Weyl-type cancellations range for $r = 1$

Let us take two holomorphic cusp forms g and f with weights k_g, k_f respectively or two Hecke-Maass cusp forms corresponding to the Laplacian eigenvalues $\frac{1}{4} + \nu_g^2$, $\nu_g \geq 1$ and

$\frac{1}{4}+\nu_f^2$, $\nu_f \geq 1$, respectively, for $\mathrm{SL}(2,\mathbb{Z})$, and χ be a primitive Dirichlet character of modulus p^r, where p is an odd prime and $r \geq 1$.

Still, the Weyl type bound is unknown. Without going into the theory of the L-function, directly analysing the twisted GL(2) short character sums in [20] we achieved the Weyl type range :

Theorem 1.5.1. *Let f, χ be as above with $r = 1$. Then for any $\varepsilon > 0$ and $0 < \theta < \frac{1}{10}$ we have*

$$S_{f,\chi}(N) \ll_{f,\varepsilon} N^{3/4+\theta/2} p^{1/6} (pN)^{\varepsilon} + N^{1-\theta} (pN)^{\varepsilon},$$

which becomes non-trivial if $p^{\frac{2}{3}+\alpha+\varepsilon} \leq N \leq p$, where $\alpha = \frac{4\theta}{1-6\theta}$.

Remark 1.5.2. Note that the Theorem 1.5.1 improves the range of cancellation from $N > p^{3/4+\varepsilon}$ (Burgess range) to $N > p^{2/3+\varepsilon}$ (Weyl range).

1.5.3 Our result on the sub-Weyl type range for $r > 1$

Similarly to the previous one, we do not have the sub-Weyl type bound for the twisted GL(2) L-function. But directly analysing the twisted GL(2) short character sums in [22] with Mallesham, we got that

Theorem 1.5.3. *Let f, χ be as above with $r > 1$ and p be an odd prime such that $p \geq 5$. Then we have*

$$S_{f,\chi}(N) \ll_f N^{\frac{5}{9}} \, p^{\frac{13r}{45}} \, N^{\varepsilon},$$

where the implied constant depends on f only, provided $p^{13r/20+\varepsilon} \leq N \leq p^{4r/5}$.

Remark 1.5.4. Our result is a counterpart result to that of Holowinsky, Munshi, and Qi [27] where they showed a sub-Weyl type range for cancellations in an analytic twist of $\lambda_f(n)$.

1.6 Rankin-Selberg L-functions

1.6.1 A brief history

Let us take two holomorphic cusp forms f and g having weights k_f and k_g respectively or Hecke-Maass cusp forms corresponding to the Laplace eigenvalues $1/4+\nu_f^2$ and $1/4+\nu_f^2$ respectively for $\mathrm{SL}(2,\mathbb{Z})$, having normalized Fourier coefficients $\lambda_f(n)$ and $\lambda_g(n)$ respectively. Then the Rankin-Selberg L-function corresponding to f and g is

$$L(s, f\otimes g) = \zeta(2s)\sum_{n=1}^{\infty}\frac{\lambda_f(n)\lambda_g(n)}{n^s}, \quad \mathcal{R}(s)>1. \tag{1.5}$$

Note that the last series can be continued analytically to all over $\mathbb{C}$, except when $g=\overline{f}$ (in which case, the above L-series has a simple pole at $s=1$). It is also known that the above Rankin-Selberg L-function is an example of a degree four automorphic L-function (see [57]).

If the weight k_g of a holomorphic cusp form g gets fixed and the weight k_f of a holomorphic cusp form f varies then in [58], Sarnak got the following subconvexity bound in the weight aspect

$$L(1/2, f\otimes g) \ll_{g,\varepsilon} k_f^{1-7/165+\varepsilon}. \tag{1.6}$$

Further, the above exponent in (1.6) was improved to $2/3+\varepsilon$ in [42] using similar ideas.

In the t-aspect, using the representation theoretic approach Michel and Venkatesh [44] proved subconvexity bounds whereas using a similar method, recently Blomer, Jana and Nelson [9] got the Weyl type subconvexity bound

$$L(1/2+it, f\otimes g) \ll_{f,g,\varepsilon} (1+|t|)^{2/3+\varepsilon}.$$

In this context, one can see [1] also.

For the level aspect subconvex bound one can see [23], [40], [43], [44] when the level of g gets fixed and the level of f varies. If we vary the levels P_g and P_f of the forms g and f, respectively, simultaneously, then Holowinsky and Munshi [28] settled the subconvexity problem in a certain range using the amplified second-moment method and got that for some $\delta(\eta)>0$:

$$L(1/2, g\otimes f) \ll_{f,g,\varepsilon} (P_gP_f)^{1/2-\delta(\eta)+\varepsilon}.$$

when $(P_g, P_f) = 1$ and $P_f \sim P_g^\eta$ where $0 < \eta < 2/21$. Assuming that the form has a smaller level being holomorphic, Ye [65] extended this result for all η. For both holomorphic and Maass forms, Raju generalised Ye's result in his thesis [56] using the delta method approach and got a better exponent.

For us, we have considered the case of GL(1) twists of Rankin-Selberg L-functions. If χ is a primitive Dirichlet character of modulus p, then the L-function associated with $f \otimes g \otimes \chi$ is given by

$$L(s, g \otimes f \otimes \chi) = L(2s, \chi^2) \sum_{n=1}^{\infty} \frac{\chi(n)\lambda_g(n)\lambda_f(n)}{n^s},$$

for $\mathcal{R}(s) > 1$, which can be analytically extended to $\mathbb{C}$ and also satisfies a functional equation relating s and $1 - s$ (see [35]). In this context the convexity bound is

$$L(\frac{1}{2}, f \otimes g \otimes \chi) \ll_{f,g,\varepsilon} p^{1+\varepsilon},$$

for any $\varepsilon > 0$ which can be obtained by using the approximate functional equation and the Phragmen-Lindelöf principle.

Sun considered the same type of problem in [59] when the character has modulus p^κ where $\kappa > 12$ and then fixing the modulus p, she varied it over κ (which is known as depth aspect). Using [3, Proposition 3.1.], one can see that $g \otimes \chi \in M_k(p^2, \chi^2)$ and then one can relate this problem with the problems that have been considered in [40], [43] (where in both of them they assumed that at least one of the forms to be holomorphic) and also in [23] (where they got the bound to be $1 - \frac{1}{1324}$) though main purpose of our article is to address the problem as a GL(2) × GL(2) × GL(1) problem using delta method and getting a better result.

This problem is different from the problem considered in the thesis of Raju (see [56]) where he considered the problem when both f, g have trivial nebentypus but for us, using [3, Proposition 3.1.], we can say that $g \otimes \chi \in M_k(p^2, \chi^2)$. Eventually, we followed the path chosen in [56].

1.6.2 Our result on the GL(1) twists of Rankin-Selberg L-functions

In [21], we got the following

Theorem 1.6.1. *Let f, g be as given above and let χ be a primitive Dirichlet character of modulus p, an odd prime. Then we have*

$$L(\frac{1}{2}, f \otimes g \otimes \chi) \ll_{f,g,\varepsilon} p^{\frac{22}{23}+\varepsilon},$$

for any $\varepsilon > 0$.

Application 1.6.2. As an application of the Theorem 1.6.1, we improve the bounds obtained in [40] for the problem of distinguishing modular forms based on their first Fourier coefficients.

Corollary 1.6.3. *Let us consider a Hecke cusp form f for $\mathrm{SL}(2,\mathbb{Z})$, as given in the Theorem 1.6.1. Then there exists a constant $C = C(f, \varepsilon)$ such that for any primitive cuspidal newform g for $\mathrm{SL}(2,\mathbb{Z})$ and for any primitive Dirichlet character $\chi(n)$ having modulus $N(\geq 3)$, an odd natural number, there exists $n \leq C\, N^{\frac{22}{23}+\varepsilon}$ with $(n, N) = 1$, such that*

$$\lambda_f(n) \neq \lambda_g(n)\chi(n).$$

Proof. The proof is identical to the proof of [40, Corollary 1.3]. Use the Theorem 1.6.1 in the place of [40, Theorem 1.1], in the proof. □

1.7 Preliminary lemmas

Let us record some lemmas in this section that we have used in the proofs of our results.

1.7.1 Automorphic form for GL(2)

First let us see some basic results on automorphic forms for $\mathrm{SL}(2,\mathbb{Z})$ (for more details see [33], [35]).

Holomorphic cusp forms :

Let f be a holomorphic Hecke cusp form for $\mathrm{SL}(2,\mathbb{Z})$, the full modular group, having weight k_f and normalized Fourier coefficients $\lambda_f(n)$'s, i.e., $\lambda_f(1) = 1$ so that it has a Fourier expansion at ∞ which is given by

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e\left(nz\right), \quad z \in \mathbb{H}.$$

Then the corresponding Hecke L-function is given by

$$L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \lambda_f(p) p^{-s} + p^{-2s}\right)^{-1}, \quad \mathcal{R}(s) > 1,$$

which has an analytic continuation to the whole complex plane $\mathbb{C}$, proved by Hecke, satisfying the following functional equation

$$\Lambda(s,f) = \varepsilon(f) \Lambda(1-s, \overline{f}),$$

where $\overline{f}$ is the dual form of f and $\varepsilon(f)$ is the root number. So we have the following complete L-function :

$$\Lambda(s,f) = \frac{1}{\pi^s} \, \Gamma\left(\frac{s + (1+k_f)/2}{2}\right) \Gamma\left(\frac{s + (k_f - 1)/2}{2}\right) L(s,f).$$

Now for the holomorphic cusp form f let us recall the Voronoi summation formula.

Lemma 1.7.1. *For $a, q \in \mathbb{Z}$ with $(a,q) = 1$ and a compactly supported, smooth function on $(0,\infty)$, say u, we have*

$$\sum_{n=1}^{\infty} \lambda_f(n) e\left(\frac{na}{q}\right) u(n) = \frac{2\pi i^{k_f}}{q} \sum_{n=1}^{\infty} \lambda_f(n) e\left(-\frac{nd}{q}\right) v(n),$$

where $da \equiv 1 \bmod q$ and

$$v(y) = \int_0^{\infty} u(x) J_{k_f - 1}\left(\frac{4\pi\sqrt{yx}}{q}\right) \mathrm{d}x.$$

Proof. See [17, P. 792]. See [40, Theorem A.4] also for the general level. □

Remark 1.7.2. From the work of Deligne (see [14]) and Deligne–Serre (see [15]) (the latter is for $k = 1$), the Petersson-Ramanujan conjecture for the holomorphic cusp forms is now well-known:

$$|\lambda_f(n)| \leq d(n) \ll n^{\varepsilon},$$

where $d(n)$ denotes the divisor function.

Maass cusp forms :

Let us take a Hecke-Maass cusp form f for $\mathrm{SL}(2, \mathbb{Z})$, the full modular group, corresponding to the Laplacian eigenvalue $\frac{1}{4} + \nu_f^2$ and normalized Fourier coefficients $\lambda_f(n)$'s, i.e., $\lambda_f(1) = 1$ so that the Fourier series expansion of f at ∞ becomes

$$f(z) = \sqrt{y} \sum_{n \neq 0} \lambda_f(n) K_{i\nu_f}(2\pi|n|y) e(nx),$$

where $K_{i\nu_f}(y)$ is the Bessel function of third kind. Then we have the following associated L-function :

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s}, \quad \mathcal{R}(s) > 1,$$

which can be extended as an entire function, satisfying the following functional equation :

$$\Lambda(s, f) = \varepsilon(f) \Lambda(1 - s, \overline{f}),$$

where $\overline{f}$ is the dual form of f, $\varepsilon(f)$ is the root number of f with $|\varepsilon(f)| = 1$. Then we have the following complete L-function :

$$\Lambda(s, f) = \frac{L(s, f)}{\pi^s} \, \Gamma\left(\frac{s + \varepsilon + i\nu}{2}\right) \Gamma\left(\frac{s + \varepsilon - i\nu}{2}\right).$$

where $\varepsilon = 1$ if f is odd and $\varepsilon = 0$ if f is even. Let us now state the following lemma concerning the Rankin-Selberg bound for the Fourier coefficients.

Lemma 1.7.3. *We have*

$$\sum_{1 \leq n \leq x} \left| \lambda_f(n) \right|^2 \ll_\varepsilon C(f)^\varepsilon x^{1+\varepsilon},$$

where $C(f) = k_f^2$ if f is holomorphic and $C(f) = 1 + \nu_f^2$ if f is a Maass form.

Proof. See [34, Lemma 1]. □

Remark 1.7.4. Here we note that the Ramanujan-Petersson conjecture predicts that

$$|\lambda_f(n)| \leq d(n),$$

where $d(n)$ is the divisor function. Now if we assume $H_\theta : \lambda_f(n) \ll_{f,\varepsilon} d(n) n^\theta$ where d is the divisor function, as a bound towards the Petersson-Ramanujan conjecture for Maass forms then towards this, the current record is given by $\theta = \frac{7}{64}$, by the work of Kim and Sarnak [39], [41].

Now we recall the Voronoi summation formula for Maass cusp forms.

Lemma 1.7.5. *For $a, q \in \mathbb{Z}$ with $(a, q) = 1$ and a compactly supported, smooth function on $(0, \infty)$, say u, we have*

$$\sum_{n=1}^{\infty} \lambda_f(n) e\left(\frac{an}{q}\right) u(n) = q \sum_{\pm} \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n} e\left(\mp \frac{dn}{q}\right) H^{\pm}\left(\frac{n}{q^2}\right),$$

where we have $ad \equiv 1 \bmod q$ and

$$H^{\pm}(y) = \frac{\varepsilon_f^{(1 \mp 1)/2}}{4\pi^2 i} \int_\sigma (\pi^2 x)^{-s} \, \tilde{g}(-s) \, \left(h^+(-s) \pm h^-(-s)\right) \, \mathrm{d}s,$$

where

$$h^{+}(s)=\frac{\Gamma\left(\frac{i\nu_f+1+s}{2}\right)\Gamma\left(\frac{-i\nu_f+s+1}{2}\right)}{\Gamma\left(\frac{-s+i\nu_f}{2}\right)\Gamma\left(\frac{-s-i\nu_f}{2}\right)},\quad h^{-}(s)=\frac{\Gamma\left(\frac{i\nu_f+s+2}{2}\right)\Gamma\left(\frac{-i\nu_f+2+s}{2}\right)}{\Gamma\left(\frac{-s+i\nu_f+1}{2}\right)\Gamma\left(\frac{-s-i\nu_f+1}{2}\right)}.$$

Here $\varepsilon_f=\pm 1$ *depending on whether* f *is odd or even.*

Proof. For details, see [40, Theorem A.4]. □

1.7.2 Bessel function

Now let us recall some expressions of Bessel functions of the first kind. For an integer $k\geq 2$, let us consider a Bessel function J_{k-1} of first kind of order $k-1$ which is given by

$$J_{k-1}(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e\left(\frac{-x\sin\tau+(k-1)\tau}{2\pi}\right)\mathrm{d}\tau.$$

If the order k is fixed then we have the following expression :

Lemma 1.7.6. *We have*

$$J_{k-1}(2\pi x)=e(-x)\,\overline{W}_{k-1}(x)+e(x)\,W_{k-1}(x),$$

where the smooth function W_{k-1} *satisfies*

$$x^j\,W_{k-1}^{(j)}(x)\ll_{j,k}\frac{1}{\sqrt{x}},$$

whenever $j\geq 0$ *and* $x\gg 1$.

Proof. See [29, Section 4.5]. □

1.7.3 The Mellin transform

The Mellin transform (see [60]) is closely connected to the theory of the Dirichlet series and is often used in number theory. It is an integral transform of a function $F : \mathbb{R} \to \mathbb{R}$ (wherever it exists) given by

$$\mathcal{M}(F)(s) := \phi(s) := \int_0^\infty F(x)x^{s-1}\mathrm{d}x,$$

for $s \in \mathbb{C}$ (wherever it exists).

Also, we can define the inverse Mellin transform (if it exists) by

$$\mathcal{M}^{-1}(\phi)(x) := F(x) = \int_{x-I\infty}^{x+I\infty} \frac{\phi(s)}{x^s}\mathrm{d}s,$$

where $x \in \mathbb{R}$.

For example, one can see that, the gamma function $\Gamma(z)$ ($Re(z) > 0$) is the Mellin transform of $F(x) = e^{-x}$ for $x \in \mathbb{R}_0^+$.

1.7.4 Stationary phase method

We will also utilize a result related to estimating the exponential integrals of the form:

$$\mathfrak{I} = \int_a^b G(x)e(F(x))\mathrm{d}x, \tag{1.7}$$

where F and G are real valued smooth functions on the interval $[a, b]$. Let us recall the following lemma on exponential integrals.

Lemma 1.7.7. *Let F and G be real valued twice differentiable functions and let $F'' \geq r > 0$ or $F'' \leq -r < 0$, throughout the interval $[a, b]$. Let $G(x)/F'(x)$ be monotonic and also let $|G(x)| \leq A$. Then we have*

$$\mathfrak{I} \leq \frac{8A}{\sqrt{r}}.$$

Proof. See [61, Lemma 4.5, P. 72] □

1.7.5 Kloosterman Sum

Let a, b, m be natural numbers. Then we define a Kloosterman sum which is a particular kind of exponential sum, given by the following :

$$S(a,\, b;\, m) \;=\; \sum_{\substack{0\le x\le m-1\\(x,m)=1}} e_m(ax+b\overline{x}),$$

where $\overline{x}$ denotes the inverse of x modulo m. The most famous estimate is due to Weil (see [33]) and states that :

$$\Big|S(a,\, b;\, m)\Big| \le\; \tau(m)\,\sqrt{gcd(a,\, b,\, m)}\,\sqrt{m}.$$

For more information, one can see [33], [35].

1.7.6 Shifted convolution sum

Let $N, k, l_1, l_2, h \in \mathbb{N}$; M_1, M_2, P_1, P_2 be real numbers greater than 1. Let χ_1 be a character (not necessarily primitive) to modulus N, and let

$$f(z) \;=\; \sum_{m=1}^{\infty} a(m)\, m^{(k-1)/2}\, e(mz) \;\in S_k\left(N,\, \chi_1\right),$$

be a primitive cusp form (i.e., an eigenfunction for all the Hecke operators, arithmetically normalized by $a(1) = 1$) of weight k and character χ_1 for the congruence subgroup $\Gamma_0(N)$. Now consider the shifted convolution sum :

$$D_g\left(l_1,\, l_2,\, h\right) \;:=\; \sum_{l_1m_1-l_2m_2=h} a\left(m_1\right)\, \overline{a\left(m_2\right)}\, g\left(m_1,\, m_2\right),$$

where g be a smooth function, supported on $[M_1, 2M_1] \times [M_2, 2M_2]$ such that $\left|\left|g^{(ij)}\right|\right|_{\infty} \ll_{i,j} \left(P_1/M_1\right)^{i} \left(P_2/M_2\right)^{j}$ for all $i,\, j\, \ge 0$. Then we have (see [5, Theorem 1.3], [6, Theorem 1.3])

$$D_g\left(l_1,\, l_2,\, h\right) \,\ll_{\varepsilon,P_1,P_2,N,k}\, \left(l_1M_1 + l_2M_2\right)^{1/2+\theta+\varepsilon},$$

for θ as in the Remark 1.7.4, uniformly in l_1, l_2, h and also note that here the dependence on P_1, P_2, N, and k is polynomial.

1.7.7 p-Adic exponent pair method

In this section, we will discuss the p-adic exponent pair method which is the p-adic analogue of the analytic exponent pair method, developed by Milicévić (see [45]). Using these ready-to-use observations, we can estimate short exponential sums with p-adic analytic phase. Here we will give a short description, for more theory see [45]. To start with, let us discuss some notations.Let $b(t) = \sum_{k=0}^{\infty} b_k t^k$ be a power series with coefficients $b_k \in \mathbb{Z}_p$. Now for any given $\lambda \in \mathbb{R}_{\geq 0}$, let us consider :

$$\mathbf{I}_0(\mathbb{Z}_p) := \{b(t) : b_k \in \mathbb{Z}_p (k \geq 0), \lim |b_k|_p = 0\},$$

$$\mathbf{I}_0[\lambda](\mathbb{Z}_p) := \{b(t) \in \mathbf{I}_0(\mathbb{Z}_p) : ord_p(b_k) \geq \lceil k\lambda \rceil \ (k \in \mathbb{N}_0)\}.$$

For $x \in \mathbb{Q}^+$, let $\iota(x) = \max(ord_p(x^{-1}), 0)$ and $\iota'(x) = \max(ord_p(x), 0)$, so we have $ord_p(x) = \iota'(x) - \iota(x)$.

Definition 1.7.8. . Let $u, \kappa \in \mathbb{N}$ where $1 + \iota'(2) \leq \kappa$, $w \in \mathbb{Z}$, $\lambda \in \rho_p\mathbb{N}$, $y \in \mathbb{Q}^+$, and also let $\iota' = \iota'(y)$, $\iota = \iota(y)$, $\omega', \omega \in \mathbb{Z}_p^\times$. If $f \in \mathbb{Q}_p^\times \mathbf{I}_0(\mathbb{Z}_p)$ satisfies the following condition :

$$f'(t) = p^w \omega' (1 + p^{\iota+\kappa} \omega t)^{-y} + p^w \gamma_0 + p^{u+w} g(t),$$

where $g \in \mathbf{I}_0[\lambda](\mathbb{Z}_p)$ and for some $\gamma_0 \in \mathbb{Z}_p$, then we say that f belongs to the class $\mathbf{F}(w, y, \kappa, \lambda, u, \omega, \omega')$. We say that f belongs to class $F(w, y, \kappa, \lambda, u)$ if for some ω', $\omega \in \mathbb{Z}_p^\times$, we have $f \in \mathbf{F}(w, y, \kappa, \lambda, u, \omega, \omega')$.

Let us consider the set of prime numbers, P and for any sets X, Y, and any family of subsets $X_p \subset X$ ($p \in P$), let $\mathbf{J}(X_p; Y)$ be the set of all functions $g : \mathbb{Q}^+ \times \bigsqcup_{p \in P} (\{p\} \times X_p) \to Y$ such that, for every $y \in \mathbb{Q}^+$, there is a finite subset $P_0(y) \subset P$ and a function $g_0 : (P \backslash P_0(y)) \times X \to Y$ such that $g(y, p, x) = g_0(p, x)$ for every $p \in P \backslash P_0(y)$ and every $x \in X_p$.

In particular, we consider $\mathbb{J}(Y) := \mathbb{J}(\emptyset; Y)$ for the set of functions $g(y, p) : \mathbb{Q}^+ \times P \to Y$ with the above properties, and $\mathbb{J}_1(Y) := \mathbb{J}\left(\mathbb{N}_p' \times \rho_p\mathbb{N};\, Y\right)$ (with $X = \mathbb{R}^+$ and $\mathbb{N}_p' = \iota'(2) + \mathbb{N}$) for the set of such functions $g(y, p, \kappa, \lambda) : \mathbb{Q}^+ \times \bigsqcup_{p \in P} \left(\{p\} \times \mathbb{N}_p' \times \rho_p\mathbb{N}\right) \to Y$.

Definition 1.7.9. Let us consider the set of all quintuples, $\mathbf{Q}$ given by

$$q = (k, \ell, r, \delta, (n_0, u_0, \kappa_0, \lambda_0)), \tag{1.8}$$

where $\ell, k \in \mathbb{R}$, $0 \le k \le \frac{1}{2} \le \ell \le 1$, $\delta \in \mathbb{R}_0^+$, $r \in \mathbf{J}_1(\mathbb{R})$, $u_0, n_0 \in \mathbf{J}_1(\mathbb{N})$, $\kappa_0 \in \mathbf{J}(\mathbb{N})$, $\lambda_0 \in \mathbf{J}(\mathbb{R}^+)$, and $\kappa + \iota'(y) < n_0(y, p, \kappa, \lambda)$. Then we say that a quintuple $q \in \mathbb{Q}$ as given in (1.8) to be a p-adic exponent datum if, for each $y \in \mathbb{Q}^+$, $p \in P$, $w \in \mathbb{Z}$, $\kappa \in \mathbb{N}$ with $\kappa \ge 1 + \iota_0(2)$, $\lambda \in \rho_p\mathbb{N}$, $u, n \in \mathbb{N}$ satisfying

$$\kappa_0(y, p) \le \kappa,\ \lambda_0(y, p) \le \lambda,\ w + n_0(y, p, \kappa, \lambda) \le n,\ u_0(y, p, \kappa, \lambda) \le u,$$

and for every $f \in \mathbf{F}(w, y, \kappa, \lambda, u)$, $M \in \mathbb{Z}$, and $0 < B \le p^{n-w-\kappa-\iota'}$ we have the following estimate

$$\sum_{M<m\le M+B} e\left(\frac{f(m)}{p^n}\right) \ll p^r \left(\frac{p^{n-w-\kappa-\iota'}}{B}\right)^k B^\ell \left(\log p^{n-w-\kappa-\iota'}\right)^\delta,$$

where $r = r(y, p, \kappa, \lambda)$, and the implied constant depends only on the datum q.

Definition 1.7.10. We call a pair $\pi = (k, \ell)$, of non-negative numbers a p-adic exponent pair if $q = (k, \ell, r, \delta, (n_0, u_0, \kappa_0, \lambda_0))$ is a p-adic exponent datum for some $r \in \mathbf{J}_1(\mathbb{R})$, $\delta \in \mathbb{R}_0^+$, $n_0, u_0 \in \mathbf{J}_1(\mathbb{N})$, $\kappa_0 \in \mathbf{J}(\mathbb{N})$, $\lambda_0 \in \mathbf{J}(\mathbb{R}_0^+)$.

Then we observe that $(0, 1)$ is an exponent pair.

Theorem 1.7.11. *If (k, ℓ) is a p-adic exponent pair, then so are*

$$A(k, \ell) = \left(\frac{k}{2(k+1)}, \frac{k+\ell+1}{2(k+1)}\right) \text{ and } B(k, \ell) = \left(\ell - \frac{1}{2}, k + \frac{1}{2}\right).$$

Proof. For the proof see [45]. □

So using these A-process and B-process, we can generate an exponent pair $A^3B(0, 1) = \left(\frac{1}{30}, \frac{13}{15}\right)$ whenever $p \ge 5$.

1.8 An approach towards the correlation problems

Here we will discuss mainly the "Delta symbol approach" towards these correlation problems. This works as a separation of variables.

Here we follow the following steps :

- At first we consider the space $\mathcal{B}$ of all functions : $(N, 2N] \mapsto \mathbb{C}$ where $N \in \mathbb{Z}$ (we can also take $N \in \mathbb{R}$ and we can proceed similarly). Then this is a vector space of dimension N with natural Hermitian inner product $<,>$.

- Then note that $a = \{a(n)\}$, $b = \{b(n)\} \in \mathcal{B}$ and $S =< a, \bar{b} >$.

- Suppose $\{\{\psi_v(n)\} : v \in \mathrm{B}\}$ be an orthonormal basis of the space $\mathcal{B}$. Then $((\psi_v(n)))$ is an $N \times N$ orthonormal matrix so that we have

$$\sum_{v \in \mathrm{B}} \psi_v(n)\overline{\psi}_v(m) =: \delta(n, m) := \begin{cases} 1 & \text{if } n = m \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

  Here δ is the Kronecker delta symbol.

- The last equation helps us to write

$$S =< a, \bar{b} >= \sum_{v \in \mathrm{B}} < a, \psi_v >< \psi_v, \bar{b} > . \tag{1.9}$$

- The **circle method / delta method** gives a way to come up with such basis functions.

- Individual pieces in (1.9) can be "evaluated" using summation formulae (Poisson, Voronoi etc.) so we will get a sum having dual coefficients. For example

$$< a, \psi_v >= \sum_{n \sim N} a(n)\overline{\psi}_v(n) \mapsto \sum_{n \sim N^*} a^*(n)\psi_v^*(n),$$

  and

$$< b, \psi_v >= \sum_{m \sim M} b(m)\overline{\psi}_v(m) \mapsto \sum_{m \sim M^\dagger} b^\dagger(m)\psi_v^\dagger(m).$$

- So we arrived at

$$S = \sum_{v \in \mathrm{B}} \sum_{n \sim N^*} a^*(n) \psi_v^*(n) \sum_{m \sim M^\dagger} b^\dagger(m) \psi_v^\dagger(m).$$

- Now applying the Cauchy-Schwarz inequality to get rid of one of the sequences using the bound on average in L^2-sense we have

$$S \ll \sqrt{N^*} \left\{ \sum_{n \sim N^*} \Big| \sum_{v \in \mathrm{B}} \psi_v^*(n) \sum_{m \sim M^\dagger} b^\dagger(m) \psi_v^\dagger(m) \Big|^2 \right\}^{1/2}.$$

- Finally opening the absolute value square, we mainly need to estimate the sum

$$\sum_{n \sim N^*} \sum\sum_{v, v' \in \mathrm{B}} \psi_v^*(n) \overline{\psi_{v'}^*(n)} \sum\sum_{m, m' \sim M^\dagger} b^\dagger(m) \psi_v^\dagger(m) \overline{b^\dagger(m')} \overline{\psi_{v'}^\dagger(m')}, \tag{1.10}$$

  using some properties or using some summation formula for the sum

$$\sum_{n \sim N^*} \psi_v^*(n) \overline{\psi_{v'}^*(n)}.$$

- After estimating the above sum, we get some pattern and use that to estimate the sum (1.10), so that we can apply summation formulae or the cancellation properties of "short sums" (for example, see [5], [6]). Then we look at what else can be done.

1.9 A short discussion on the Circle method

In the classical circle method, we try to show that these basis vectors, discussed above, $\psi_v(n)$'s can be taken as the trigonometric functions. The classical form is based on trigonometric functions or harmonics of the circle group. For the higher rank group, we use harmonics of that higher rank group and try to get different expressions for the delta symbol $\delta(m, n)$. Though sometimes it may be good to look also at the classical continuous circle method (see [62]) where the notion of major and minor arc is used. In the delta method, we usually focus on major arcs and try to approximate the results.

The skeleton of the GL(1) delta symbol formula is :

$$\frac{1}{N}\sum_{q}\sideset{}{^*}\sum_{a \bmod q} e\left(\frac{an}{q}\right)\overline{e\left(\frac{am}{q}\right)} \approx \delta(n,m),$$

where the $*$ on the sum denotes that the sum is taken over a satisfying the condition $(a,q)=1$ and q is to be chosen appropriately.

Here we basically take $v=(a,q)$ (index) and the basis vectors $\psi_v(n)$'s as

$$\psi_{(v)}(n) =: \psi_{(a,q)}(n) = \frac{1}{\sqrt{N}}e\left(\frac{an}{q}\right),$$

so that we have

$$\sum_{v}\psi_v(n)\overline{\psi_v(m)} \approx \delta(n,m).$$

The circle method was first introduced by Hardy and Ramanujan in their celebrated paper (see [24]) on the partition function. Over the years there have been many forms of circle methods given by many eminent mathematicians (see [25], [33, Chapter 20], [49] etc.) though for our need we will discuss only two of them - Jutila's circle method (see [37], [38]) and DFI delta method (see [26], [33, Chapter 20]).

1.10 Jutila's Circle method

Let us consider the delta function $\delta:\mathbb{Z}\to\{0,1\}$ which is defined as

$$\delta(n) := \begin{cases} 0 & \text{if } n \neq 0 \ , \\ 1 & \text{if } n = 1 \ . \end{cases}$$

As discussed in the previous sections, we need a nice Fourier expansion of $\delta(n)$ or at least a nice approximation of $\delta(n)$ in some L^p norm (for us $p=2$). For any $S\subset\mathbb{R}$, let us consider

$\mathbb{I}_S$ to be the respective indicator function or the characteristic function given by $\mathbb{I}_S(x) = 1$ for $x \in S$ and 0 otherwise. Note that

$$\delta(n) = \int_{\mathbb{R}} \mathbb{I}_{[0,1]}(x)\, e(nx)\, \mathrm{d}x. \tag{1.11}$$

Then $\Phi \subset [Q, 2Q]$, for any collection of positive integers defined as the set of moduli, where $Q \geq 1$ and δ, a positive real number, satisfying $Q^{-2} \ll \delta \ll Q^{-1}$, let us define the following function

$$\tilde{\mathbb{I}}_{\Phi,\delta}(x) := \frac{1}{2\delta L} \sum_{q \in \Phi} \sideset{}{^*}\sum_{d \bmod q} \mathbb{I}_{[\frac{d}{q}-\delta, \frac{d}{q}+\delta]}(x),$$

where $\mathbb{I}_{[\frac{d}{q}-\delta, \frac{d}{q}+\delta]}$ is the indicator function of the interval $[\frac{d}{q} - \delta, \frac{d}{q} + \delta]$. Here $L := \sum_{q \in \Phi} \phi(q)$ (then we have $L \ll Q^2$) and we will choose Φ in such a way that $L \asymp Q^{2-\varepsilon}$.

Then this becomes an approximation of $\mathbb{I}_{[0,1]}$ in the following sense:

Lemma 1.10.1. *We have*

$$\int_{\mathbb{R}} \left| \mathbb{I}_{[0,1]}(x) - \tilde{\mathbb{I}}_{\Phi,\delta}(x) \right|^2 \mathrm{d}x \ll \frac{Q^{2+\varepsilon}}{\delta L^2},$$

where I is the indicator function of $[0, 1]$.

Proof. This becomes a consequence of the Parseval's identity from Fourier analysis. For the proof one can see [37], [49]. □

1.11 DFI delta method

First let us mention an expression for $\delta(n)$ which is due to Duke, Friedlander and Iwaniec (see [26], [33, Chapter 20]). Let $L \geq 1$ be a large real number. Then for $n \in \mathbb{Z} \cap [-2L, 2L]$, we have

$$\delta(n) = \frac{1}{Q} \sum_{1 \leq q \leq Q} \sideset{}{^*}\sum_{a \bmod q} e\left(\frac{an}{q}\right) \int_{\mathbb{R}} g(q, x) e\left(\frac{nx}{qQ}\right) \mathrm{d}x, \tag{1.12}$$

where $Q = 2L^{1/2}$ and $e(z) = e^{2\pi iz}$ and $*$ on the previous sum denotes that the sum is taken over a with the restricted condition that $(a, q) = 1$. In the above formula, the function g is not given explicitly (though one can if one wishes to). The properties of g (for proof, see [1, Lemma 2.1] and [30, Lemma 15]) that we need in our analysis are given below. For any $A > 1$, we have

1. $g(q, x) = 1 + h(q, x)$ where $h(q, x) = \mathrm{O}\left(\frac{1}{qQ}\left(\frac{q}{Q} + |x|\right)^{A}\right)$.

2. $|x|^j \frac{\partial^j}{\partial x^j} g(q, x) \ll_j \log Q \min\left\{\frac{Q}{q}, \frac{1}{|x|}\right\}$, $j \geq 1$.

3. $g(q, x) \ll |x|^{-A}$.

4. $$\int_{\mathbb{R}} (|g(q, x)| + |g(q, x)|^2)\mathrm{d}x \ll_\varepsilon Q^\varepsilon. \tag{1.13}$$

Then from the third property given above, the effective range of the x-integral in (1.12) becomes $[-Q^\varepsilon, Q^\varepsilon]$. From the above observations, we have :

Lemma 1.11.1. *Let us take a large parameter $L \geq 1$ and δ as above. Then, for any $n \in [-2L, 2L]$ with $Q = 2L^{1/2}$, we have the following expression*

$$\delta(n) = \frac{1}{Q} \sum_{1 \leq q \leq Q} \ \sideset{}{^*}\sum_{a \bmod q} e\left(\frac{an}{q}\right) \int_{\mathbb{R}} W\left(\frac{x}{Q^\varepsilon}\right) g(q, x) e\left(\frac{nx}{qQ}\right) \mathrm{d}x + \mathrm{O}\left(L^{-2023}\right),$$

where $W(x)$ is a smooth non-negative bump function supported in $[-2, 2]$, satisfying $W(x) = 1$ when $x \in [-1, 1]$ and $W^{(j)}(x) \ll_j 1$, for $j \geq 0$ and the function g satisfies (1.11).

Proof. For the details see [1, Lemma 2.1], [30, Lemma 15] and [35, Chapter 20]. □

Chapter 2

Weyl-type bounds for twisted GL(2) short character sums

In this chapter we will analyse the sum $S_{f,\chi}(N)$, given by (1.4), using a version of δ-method, without going into L-functions. Our method improves the range of cancellation from $N > p^{3/4+\varepsilon}$ (Burgess range) to $N > p^{2/3+\varepsilon}$ (Weyl range). Here let us state our result (see [20]) again:

Theorem 2.0.1. *Let f be any Hecke-Maass cusp form for* $\mathrm{SL}(2,\mathbb{Z})$ *and χ be a primitive Dirichlet character modulo p, an odd prime. Then for any $\varepsilon > 0$ and $0 < \theta < \frac{1}{10}$ we have*

$$S_{f,\chi}(N) \ll_{f,\varepsilon} N^{3/4+\theta/2} p^{1/6} (pN)^{\varepsilon} + N^{1-\theta}(pN)^{\varepsilon},$$

which becomes non-trivial if $p^{\frac{2}{3}+\alpha+\varepsilon} \leq N \leq p$, where $\alpha = \frac{4\theta}{1-6\theta}$.

Though this result is implicit in the work of Munshi (see [50]), we are doing it here explicitly. Actually in that work (see [50]) he aims to get a subconvexity bound for $L(1/2+it, f \otimes \chi)$ but here we aim to get a range for N to have a non-trivial bound or more precisely getting cancellation in our twisted GL(2) short character sum. Here we are using the same strategy and ideas developed in the work of Munshi (see [50]). We will only present the case of holomorphic cusp forms for $\mathrm{SL}(2,\mathbb{Z})$ as for the Maass forms one can see Munshi's work (see [50]) which carried out the Maass form case in detail. The case for Maass forms is similar as we only need the Ramanujan bound in the L^2-sense.

Remark 2.0.2. Here we are considering p to be a prime number for simplicity but also one can do for p when p is not a prime (one has to deal with the coprimality issues carefully) using the same method.

2.1 Sketch of the proof

We will describe our method briefly by taking a holomorphic primitive cusp form f, for $\mathrm{SL}(2,\mathbb{Z})$. Here we are using the method of Munshi (see [50]). At first, we consider the sum

$$\mathbf{S} := \sum_{n\sim N} \lambda_f(n)\chi(n),$$

for $N > p^{\frac{2}{3}+\varepsilon}$, where $\lambda_f(n)$'s are the normalized Fourier coefficients of f and p is the conductor of χ. Here in the sketch, we will suppress the weight function for notational simplicity. Then we can write the above sum as

$$\mathbf{S} = \sum\sum_{n,m\sim N} \lambda_f(n)\chi(m)\delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker δ-symbol. To get an inbuilt bilinear structure that comes from the circle method - which was found by Jutila (see [37], [38]), we use a flexible version of the circle method though it comes with a satisfactory error term, as long as we allow the moduli of the circle method to be slightly larger than $\sqrt{N}$ (see the Section 2.2). Up to an admissible error, we see that $\mathbf{S}$ is given by

$$\mathbf{S} = \sum\sum_{n,m\sim N} \chi(m)\lambda_f(n) \int_{\mathbb{R}} \tilde{\mathbb{I}}(\alpha) e((n-m)\alpha)\mathrm{d}\alpha,$$

where $\widetilde{\mathbb{I}}(\alpha) := \frac{1}{2\delta L}\sum_{q\in\Phi}\sum^{\star}_{d(\mod q)} \mathbb{I}_{d/q}(\alpha)$, $\mathbb{I}_{d/q}$ is the indicator function of the interval $[\frac{d}{q}-\delta, \frac{d}{q}+\delta]$, $Q := N^{1/2+\epsilon}$, Φ is fixed later in a suitable way and $L \asymp Q^{2-\epsilon}$ (see the Section 1.10).

Trivial bound at this stage yields $N^{2+\varepsilon}$ and we need to establish the bound $N^{1-\theta}$ for some $\theta > 0$, i.e., roughly speaking we need to save N. Observe that by our choice of Q, there is no analytic oscillation in the weight function $e((n-m)\alpha)$. Hence their weights can be dropped in our sketch. At first using the GL(2) Voronoi summation formula on the n sum we get that

$$\sum_{n\sim N} \lambda_f(n) e\left(\frac{na}{q}\right) \approx \frac{N}{q} \sum_{n\sim \frac{Q^2}{N}} \lambda_f(n) e\left(\frac{-n\bar{a}}{q}\right),$$

where q is of size $Q = N^{1/2+\theta}$. The left-hand side is trivially bounded by N, whereas the right-hand side is trivially bounded by Q. Hence we have "saved"

$$\sqrt{\frac{\textit{Initial length}}{\textit{Final length}}} = \frac{\textit{Initial length}}{\sqrt{\textit{Conductor}}} = \frac{N}{Q} = \sqrt{\frac{N}{N^{2\theta}}}.$$

Now applying the Poisson summation formula to the m sum we arrive at

$$\sum_{m\sim N} \chi(m) e\left(-\frac{ma}{q}\right) \approx \frac{N\tau_\chi}{p} \sum_{|m|\ll \frac{pQ}{N}} \bar{\chi}(m)\chi(q)\mathbb{I}_{a\equiv m\bar{p} \bmod q},$$

where $\mathbb{I}_{a\equiv m\bar{p} \bmod q}$ is the indicator function for $a \equiv m\bar{p} \bmod q$ on $\mathbb{Z}$. Comparing the trivial bounds for the two sides we observe that we have "saved" $\frac{N}{\sqrt{pQ}}$.

From the a mod q-sum, which will be a Kloosterman sum, we will "save" $\sqrt{Q}$.

With this, the above sum is roughly reduced to

$$\mathbf{S} \approx \frac{N^2}{Q^3 p^{1/2}} \sum_{q\in\Phi} \sum_{n\sim N^{2\theta}} \sum_{m\sim \frac{pN^\theta}{\sqrt{N}}} \lambda_f(n)\bar{\chi}(m)\chi(q) e\left(-\frac{\bar{m}np}{q}\right).$$

So far we have "saved" $N^{1/2-\theta} \times \frac{N}{\sqrt{pQ}} \times \sqrt{Q} = \frac{N^{3/2-\theta}}{\sqrt{p}}$. Hence our job is to "save" $\frac{N}{\frac{N^{3/2-\theta}}{\sqrt{p}}} = \frac{\sqrt{p}}{N^{1/2-\theta}}$ in the above sum.

Next we choose $Q = Q_1Q_2$ and take the set of moduli Φ to be a product of two sets of primes so that (as discussed in the Subsection 1.10.1 and the Section 2.4) $q = q_1q_2$ in a certain unique way with $q_1 \leq Q_1$ and $q_2 \leq Q_2$ (see the Section 2.4). Then applying the Cauchy-Schwarz inequality we arrive at

$$\sum_{q_1\sim Q_1} \sum_{m\sim \frac{pN^\theta}{\sqrt{N}}} \Big| \sum_{n\sim N^{2\theta}} \sum_{q_2\sim Q_2} \lambda_f(n)\chi(q_2) e\left(-\frac{\bar{m}np}{q_1q_2}\right) \Big|^2.$$

Now we open the absolute value square and apply the Poisson summation formula to the m-sum (after appropriate smoothing). Here the diagonal is of length $Q_2N^{2\theta}$ and so the contribution of the zero frequency is given by $\ll pN^{4\theta}$. Hence the diagonal contribution is satisfactory if

$$Q_2N^{2\theta} > \frac{p}{N^{1-2\theta}} \text{ , i.e., } Q_2 > \frac{p}{N}.$$

Also, the contribution of the off-diagonal is given by $\ll pN^{6\theta}$. Note that this is satisfactory if

$$\frac{pN^{\theta/2}}{N^{3/4}\sqrt{Q_2}} > \frac{p}{N^{1-2\theta}} \iff Q_2 < N^{1/2-3\theta}.$$

So we have a choice for Q_2 if

$$\frac{p}{N} < N^{1/2-3\theta} \Rightarrow p < N^{3/2-3\theta}.$$

Hence as long as $N > p^{2/3+\varepsilon}$ for some $\varepsilon > 0$ then the above method yields a non-trivial bound for $\mathbf{S}$.

Notation

In this section, by '$\ll$' we mean the implied constant will depend on ε, f only, whenever it occurs and the notation '$X \asymp Y$' will mean that $Yp^{-\varepsilon} \leq X \leq Yp^{\varepsilon}$.

2.2 Setting-up the circle method :

Now we apply the circle method to the following smooth sum directly

$$S_{f,\chi}(N) = \sum_{n\in\mathbb{Z}} \lambda_f(n)\chi(n)h_1\left(\frac{n}{N}\right),$$

where the function h_1 is smooth, supported in $[1,2]$ with $h_1^{(j)}(x) \ll_j 1$. Now we will approximate the above sum $S_{f,\chi}(N)$ using Jutila's circle method (see [37], [38]) by the following sum :

$$\tilde{S}(N) = \frac{1}{L}\sum_{q\in\Phi}\ \sideset{}{^*}\sum_{a \bmod q}\ \sum\sum_{m,n\in\mathbb{Z}} \lambda_f(n)\chi(m)e\left(\frac{a(n-m)}{q}\right)F(n,m),$$

where we denote $e_q(x) = e^{2\pi i x/q}$, and

$$F(x,y) = h_1\left(\frac{x}{N}\right)h_2\left(\frac{y}{N}\right)\frac{1}{2\delta}\int_{-\delta}^{\delta} e(\alpha(n-m))\,\mathrm{d}\alpha.$$

Here we consider another smooth function h_2 having compact support in $(0, \infty)$, with $h_2(x) = 1$ for x whenever comes from the support of h_1. Also we choose $\delta = N^{-1}$ so that we have

$$\frac{\partial^{i+j}}{\partial^i x \partial^j y} F(x, y) \ll_{i,j} \frac{1}{N^{i+j}}.$$

Then we have the following lemma :

Lemma 2.2.1. *Let $\Phi \subset [Q, 2Q]$, with*

$$L = \sum_{q \in \Phi} \phi(q) \gg Q^{2-\varepsilon},$$

and $\delta = \frac{1}{N} \gg \frac{N^{2\theta}}{Q^2}$. Then we must have

$$S_{f,\chi}(N) = \tilde{S}(N) + \mathrm{O}_{f,\varepsilon}\left(\sqrt{N} \frac{N(QN)^{\varepsilon}}{Q}\right).$$

Proof. For the proof of this lemma, one can see the proof of [50, Lemma 3]. □

We will choose the size of the moduli in the Section 2.4. We shall pick the set of the moduli to be $Q = N^{1/2+\theta}$. Hence the error term getting from the previous lemma is $O(N^{1-\theta+\varepsilon})$ for some $\theta > 0$. Now we will estimate the sum $\tilde{S}(N)$.

2.3 Estimation of $\tilde{S}(N)$

2.3.1 Application of the summation formulae

At first, we assume that each element of the set Φ is coprime to p, the modulus of the character χ. Let us define

$$\tilde{S}_x(N) = \frac{1}{L} \sum_{q \in \Phi} \sideset{}{^*}\sum_{a \bmod q} S\left(a, q, x, f\right) T\left(a, q, x, \chi\right), \tag{2.1}$$

where

$$S\left(a,q,x,f\right) := \sum_{n\in\mathbb{Z}} \lambda_f(n) h_1\left(\frac{n}{N}\right) e\left(\frac{an}{q}\right) e(nx),$$

and

$$T\left(a,q,x,\chi\right) := \sum_{m\in\mathbb{Z}} \chi(m) e\left(-\frac{am}{q}\right) h_2\left(\frac{m}{N}\right) e(-xm),$$

with $|x| < \delta$. Then from (1.11) of the Section 1.10 we have

$$\tilde{S}(N) = (2\delta)^{-1} \int_{-\delta}^{\delta} \tilde{S}_x(N)\, \mathrm{d}x.$$

Let us first study the n-sum using the Voronoi summation formula 1.7.1.

$$S\left(a,q,x,f\right) = \sum_{n=1}^{\infty} \lambda_f(n) h_1\left(\frac{n}{N}\right) e\left(\frac{an}{q}\right) e(nx). \tag{2.2}$$

Then we have the following lemma:

Lemma 2.3.1. *We have*

$$S\left(a,q,x,f\right) = \frac{N^{3/4}}{q^{1/2}} \sum_{|n| \ll \frac{Q^2}{N}} \frac{\lambda_f(n)}{n^{1/4}} e\left(-\frac{\bar{a}n}{q}\right) \mathcal{I}_1(n,x,q) + \mathrm{O}(N^{-2021}), \tag{2.3}$$

where $q \in [Q, 2Q]$, coprime with p and $\mathcal{I}_1(n,x,q)$ is given by

$$\mathcal{I}_1(n,x,q) := \int_{\mathbb{R}} h_1(y) e\left(Nxy \pm \frac{4\pi}{q}\sqrt{Nny}\right) W\left(\frac{4\pi\sqrt{Nny}}{q}\right) \mathrm{d}y,$$

where W is a smooth nice function.

Proof. Applying the Voronoi summation formula 1.7.1 to the n-sum of the equation (2.2), then we have

$$\sum_{n\in\mathbb{Z}} \lambda_f(n) e\left(\frac{an}{q}\right) e(nx) h_1\left(\frac{n}{N}\right) = \frac{1}{q}\sum_{n\in\mathbb{Z}} \lambda_f(n) e\left(-\frac{\bar{a}n}{q}\right)$$
$$\times \int_{\mathbb{R}} h_1\left(\frac{y}{N}\right) e(xy) J_{k-1}\left(\frac{4\pi\sqrt{ny}}{q}\right) \mathrm{d}y,$$

where J_{k-1} is the Bessel function. By changing $y \mapsto Ny$ and using the decomposition 1.7.6,

$$J_{k-1}(x) = \frac{W(x)}{\sqrt{x}} e(x) + \frac{\bar{W}(x)}{\sqrt{x}} e(-x),$$

where $W(x)$ is a nice function, the right-hand side integration becomes

$$N^{3/4} q^{1/2} \int_{\mathbb{R}} h_1(y) e\left(Nxy \pm \frac{4\pi}{q}\sqrt{Nny}\right) W\left(\frac{4\pi\sqrt{Nny}}{q}\right) \mathrm{d}y.$$

By repeated integral by parts we see that this integral is negligibly small if $|n| \gg \frac{Q^2 N^{\varepsilon}}{N}$. Hence the Lemma 2.3.1 follows. □

Remark 2.3.2. Note that $x \asymp \frac{\sqrt{n}}{\sqrt{N}q}$, otherwise $\mathcal{I}_1(n, x, q)$ is negligibly small.

Now let us consider the m-sum of (2.1) given by

$$T(a, q, x, \chi) = \sum_{m\in\mathbb{Z}} \chi(m) e\left(-\frac{am}{q}\right) h_2\left(\frac{m}{N}\right) e(-xm), \tag{2.4}$$

for which we have the following lemma:

Lemma 2.3.3. *We have*

$$T(a, q, x, \chi) = \frac{N\tau_\chi}{p} \sum_{\substack{|m| \ll \frac{pQ}{N} \\ m\bar{p} \equiv a \bmod q}} \overline{\chi}(m)\chi(q)\mathcal{I}_2(m, x, q) + \mathrm{O}(N^{-2021}), \tag{2.5}$$

where

$$\mathcal{I}_2(m,x,q) := \int_{\mathbb{R}} h_2(y)e(-Nxy)e\left(-\frac{mNy}{pq}\right)\,\mathrm{d}y.$$

Proof. To the m-sum in the equation (2.4), we apply the Poisson summation formula to get that

$$T(a,q,x,\chi) = \frac{N}{pq}\sum_{m\in\mathbb{Z}}\mathcal{I}_2(m,x,q)\sum_{\beta \bmod q}\chi(\beta)e\left(-\frac{a\beta}{q}\right)e\left(\frac{m\beta}{pq}\right),$$

where

$$\mathcal{I}_2(m,x,q) := \int_{\mathbb{R}} h_2(y)e(-Nxy)e\left(-\frac{mNy}{pq}\right)\,\mathrm{d}y.$$

Here note that this integral is negligibly small if $|m| \gg \frac{pQ}{N}N^{\varepsilon}$.

So we have

$$T(a,q,x,\chi) = \frac{N}{pq}\sum_{|m|\ll\frac{pQ}{N}N^{\epsilon}}\mathcal{I}_2(m,x,q)\sum_{\beta \bmod pq}\chi(\beta)e\left(-\frac{a\beta}{q}\right)e\left(\frac{m\beta}{pq}\right) + \mathrm{O}(N^{-2021}).$$

As we know $(p,q)=1$, so that we can write β as $\beta = \beta_1 q\bar{q} + \beta_2 p\bar{p}$, where β_1, β_2 runs through a complete set of residue classes congruent to p, q respectively. Then substituting these in the place of β we have

$$T(a,q,x,\chi) = \frac{N\tau_\chi}{p}\sum_{\substack{|m|\ll\frac{pQ}{N} \\ m\bar{p}\equiv a \bmod q}}\overline{\chi}(m)\chi(q)\mathcal{I}_2(m,x,q) + \mathrm{O}(N^{-2021}).$$

This completes the proof. □

From (2.3) and (2.5) we get:

Proposition 2.3.4. *We have*

$$\tilde{S}_x(N) = \frac{N^{7/4}}{\sqrt{p}L}\sum_{q\in\Phi}\frac{\chi(q)}{q^{1/2}}\sum_{|n|\ll\frac{Q^2}{N}}\sum_{\substack{|m|\ll\frac{pQ}{N}\\(m,q)=1}}\frac{\lambda_f(n)}{n^{1/4}}\overline{\chi}(m)e\left(-\frac{p\bar{m}n}{q}\right)\mathcal{I}_1(n,x,q)\mathcal{I}_2(m,x,q) + \mathrm{O}(N^{-2021}), \tag{2.6}$$

where $\mathcal{I}_1(n,x,q)$, $\mathcal{I}_2(m,x,q)$ *are given by* (2.3), (2.5) *respectively.*

2.4 Further estimation

2.4.1 Application of the Cauchy-Schwarz and Poisson summation formulae

Le $\Phi = \Phi_1\Phi_2$ be the set of moduli, where Φ_i consists of primes in the dyadic segment $[Q_i, 2Q_i]$, coprime to p, for $i \in \{1,2\}$ and $Q_1Q_2 = Q = N^{1/2+\theta}$. Also, we consider Q_1 and Q_2 whose optimal sizes will be chosen later so that the collections Φ_1 and Φ_2 become disjoint. Now consider $M_0 := \frac{pQ}{N}, N_0 := \frac{Q^2}{N}$. Here we note that, as $0 < \theta < 1/10$ so that we have $Q_2 > N_0$ and also we have $Q_1 > N_0$.

Now applying the Cauchy-Schwarz inequality to the equation (2.6), we arrive at

$$\tilde{S}_x(N) \ll \frac{N^{7/4}\sqrt{M_0}}{\sqrt{p}L\sqrt{Q_1}}\sum_{q_1\in\Phi_1} \times\left(\sum_{|m|\ll M_0}\left|\sum_{q_2\in\Phi_2}\frac{\chi(q_2)}{q_2^{1/2}}\sum_{|n|\ll N_0}\frac{\lambda_f(n)}{n^{1/4}}\mathcal{I}_1(n,x,q_1q_2)\mathcal{I}_2(m,x,q_1q_2)e\left(-\frac{p\bar{m}n}{q_1q_2}\right)\right|^2\right)^{1/2} \tag{2.7}$$

$$\ll \frac{N^{7/4}\sqrt{M_0}}{\sqrt{p}L\sqrt{Q_1}}\sum_{q_1\in\Phi_1}\Omega(N_0,q_1,Q_2,x)^{1/2},$$

where $\Omega(N_0, q_1, Q_2, x)$ is defined as

$$\sum_{\substack{|m| \ll M_0 \\ (m,q)=1}} \left| \sum_{q_2 \in \Phi_2} \frac{\chi(q_2)}{q_2^{1/2}} \sum_{|n| \ll N_0} \frac{\lambda_f(n)}{n^{1/4}} \mathcal{I}_1(n, x, q_1 q_2) \mathcal{I}_2(m, x, q_1 q_2) e\left(-\frac{p \bar{m} n}{q_1 q_2} \right) \right|^2 . \tag{2.8}$$

Now we apply the Poisson summation formula to the m-sum with the modulus $q_1 q_2 q_2'$ in the equation (2.8). To this end, we first split the sum over m into dyadic blocks $m \sim M_1$, $M_1 \ll M_0$ and then open the absolute value square in the equation (2.8), to get the following,

$$\Omega(N_0, q_1, Q_2, x) = \sum_{q_2, q_2' \in \Phi_2} \frac{\chi(q_2 \bar{q_2'})}{(q_2 q_2')^{1/2}} \sum_{|n|, |n|' \ll N_0} \frac{\lambda_f(n) \lambda_f(n')}{(nn')^{1/4}} \mathcal{I}_1(n, x, q_1 q_2) \overline{\mathcal{I}_1(n', x, q_1 q_2')} \Delta,$$

where

$$\Delta = \sum_{M_1 \ll M_0} \sum_{m \in \mathbb{Z}} W'\left(\frac{m}{M_1} \right) e\left(\frac{\bar{m} p (q_2' n - n' q_2)}{q_1 q_2 q_2'} \right) \mathcal{I}_2(m, x, q_1 q_2) \overline{\mathcal{I}_2(m, x, q_1 q_2')},$$

and $W'(x)$ is a non-negative smooth function supported on $[2/3, 3]$ with $W'(x) = 1$ for $x \in [1, 2]$ and $W'^{(j)}(x) \ll_j 1$.

Now applying the Poisson summation formula to the m-sum it transforms into

$$\frac{M_1}{q_1 q_2 q_2'} \sum_{m \in \mathbb{Z}} S(p(q_2' n - n' q_2), m; q_1 q_2 q_2') \mathcal{I}(m, x, q_1, q_2, q_2'),$$

where

$$\mathcal{I}(m, x, q_1, q_2, q_2') := \int_{\mathbb{R}} W'(y) \, \mathcal{I}_2(M_1 y, x, q_1 q_2) \, \overline{\mathcal{I}_2(M_1 y, x, q_1 q_2')} \, e\left(-\frac{m M_1 y}{q_1 q_2 q_2'} \right) \mathrm{d}y.$$

Here note that the integral $\mathcal{I}$ is negligibly small if $|m| \gg \frac{Q_1 Q_2^2}{M_1} N^{\varepsilon} = \frac{Q_2 Q}{M_1} N^{\varepsilon}$.
Let $R_1 = \frac{Q_2 Q}{M_1}$. So we get

$$\tilde{S}_x(N) = \frac{N^{7/4} \sqrt{M_0}}{\sqrt{p} L \sqrt{Q_1}} \sum_{q_1 \in \Phi_1} \Omega(N_0, q_1, Q_2, x)^{1/2} + \mathrm{O}(N^{-2021}), \tag{2.9}$$

where

$$\Omega(N_0, q_1, Q_2, x) = \sum_{M_1 \ll M_0} \frac{M_1}{q_1} \sum_{q_2, q_2' \in \Phi_2} \frac{\chi(q_2 \bar{q_2'})}{(q_2 q_2')^{3/2}} \sum_{|n|, |n|' \ll N_0} \frac{\lambda_f(n) \lambda_f(n')}{(nn')^{1/4}}$$

$$\times \mathcal{I}_1(n, x, q_1 q_2) \overline{\mathcal{I}_1(n', x, q_1 q_2')} \sum_{|m| \ll R_1} S(p(q_2' n - n' q_2), m; q_1 q_2 q_2') \mathcal{I}(m, x, q_1, q_2, q_2').$$

Lemma 2.4.1. *We have*

$$\Omega(N_0, q_1, Q_2, x) \ll (M_0 N_0^{1/2} + N_0^{3/2} Q_2^2 \sqrt{Q_1})(Np)^{\epsilon}. \tag{2.10}$$

The proof of this lemma is given below. The first term of the right-hand side of (2.10) is coming from $m = 0$ and the second term is coming from other m's, i.e., for the terms with $m \neq 0$. For the proof, at first, we consider the zero frequency case, i.e., when $m = 0$.

The zero frequency

The zero frequency $m = 0$ has to be treated differently. Let Σ_0 denote the contribution of the zero frequency to $\tilde{S}_x(N)$, i.e.,

$$\Sigma_0 = \sum_{M_1 \ll M_0} \frac{M_1}{q_1} \sum_{q_2, q_2' \in \Phi_2} \frac{\chi(q_2 \bar{q_2'})}{(q_2 q_2')^{3/2}} \sum_{|n|, |n|' \ll N_o} \frac{\lambda_f(n) \lambda_f(n')}{(nn')^{1/4}} S(p(q_2' n - n' q_2), 0; q_1 q_2 q_2')$$

$$\times \mathcal{I}_1(n, x, q_1 q_2) \overline{\mathcal{I}_1(n', x, q_1 q_2')} \mathcal{I}(0, x, q_1, q_2, q_2')$$

$$= \sum_{M_1 \ll M_0} \frac{M_1}{q_1} \sum_{q_2, q_2' \in \Phi_2} \frac{\chi(q_2 \bar{q_2'})}{(q_2 q_2')^{3/2}} \sum_{|n|, |n|' \ll N_0} \frac{\lambda_f(n) \lambda_f(n')}{(nn')^{1/4}}$$

$$\times \sum_{d | (q_1 q_2 q_2', p(q_2' n - n' q_2))} d\mu \left(\frac{p(q_2' n - n' q_2)}{d} \right) \mathcal{I}_1(n, x, q_1 q_2) \overline{\mathcal{I}_1(n', x, q_1 q_2')} \mathcal{I}(0, x, q_1, q_2, q_2')$$

$$= \sum_{M_1 \ll M_0} \frac{M_1}{q_1} \sum_{q_2, q_2' \in \Phi_2} \frac{\chi(q_2 \bar{q_2'})}{(q_2 q_2')^{3/2}} \sum_{d | q_1 q_2 q_2'} d \sum_{\substack{|n|, |n'| \ll N_0 \\ q_2' n - n' q_2 \equiv 0 \bmod d}} \frac{\lambda_f(n) \lambda_f(n')}{(nn')^{1/4}}$$

$$\times \mathcal{I}_1(n, x, q_1 q_2) \overline{\mathcal{I}_1(n', x, q_1 q_2')} \mathcal{I}(0, x, q_1, q_2, q_2'). \tag{2.11}$$

Lemma 2.4.2. *We have*

$$\Sigma_0 \ll M_0 N_0^{1/2} (Np)^{\epsilon}. \tag{2.12}$$

Proof. For $m = 0$ we have six cases according to the divisors of $q_1 q_2 q_2'$ and note that $Q = Q_1 Q_2$.

Case 1

Let $d = q_1 q_2 q_2'$. Then note that size of d for this case is $Q_1 Q_2^2$. But size of $q_2' n - n' q_2$ is $Q_2 N_0$. So in this case

$$q_2' n - n' q_2 \equiv 0 \bmod d \iff q_2 = q_2' \text{ and } n = n',$$

as size of n is smaller than size of Q_2. Hence we have, using the well-known pointwise Ramanujan bound, given in the Remark 1.7.3,

$$\Sigma_0 \ll \sup_{M_1 \ll M_0} M_1 N_0^{1/2},$$

as there are atmost $\log M_0 \,(\ll p^\varepsilon)$ many M_1's. Hence we have

$$\Sigma_0 \ll M_0 N_0^{1/2}.$$

Case 2

Let $d = 1$. Then we get that, as done in the previous case,

$$\Sigma_0 \ll \sup_{M_1 \ll M_0} \frac{M_1 N_0^{3/2}}{Q} \ll \frac{M_0 N_0^{3/2}}{Q}.$$

But as $N_0 \ll Q$ for this case, we must have,

$$\Sigma_0 \ll M_0 N_0^{1/2}.$$

Case 3

Let $d = q_1$. For this case, we have,

$$\Sigma_0 \ll \sup_{M_1 \ll M_0} \frac{M_1 N_0^{3/2}}{Q_2} \ll \frac{M_0 N_0^{3/2}}{Q_2}.$$

But as $N_0 < Q_2$ for this case again we have,

$$\Sigma_0 \ll M_0 N_0^{1/2}.$$

Case 4

Now consider $d = q_2$. For this case we have,

$$\Sigma_0 \ll \sup_{M_1 \ll M_0} \frac{M_1 N_0^{3/2}}{Q_1} \ll \frac{M_0 N_0^{3/2}}{Q_1}.$$

But as $N_0 < Q_1$ for this case, we must have,

$$\Sigma_0 \ll M_0 N_0^{1/2}.$$

For the case $d = q_2'$ we have to proceed similarly and we will get the same bound.

Case 5

Now take $d = q_1 q_2$. But as size of $q_2' n - n' q_2$ is $Q_2 N_0$ which is less than the size of d, i.e., $Q_1 Q_2$ so for this case we have

$$q_2' n - n' q_2 \equiv 0 \bmod d \iff q_2 = q_2' \text{ and } n = n'.$$

Hence we have

$$\Sigma_0 \ll \sup_{M_1 \ll M_0} \frac{M_1 N_0^{1/2}}{Q_2} \ll \frac{M_0 N_0^{1/2}}{Q_2}.$$

But then again we have, for this case,

$$\Sigma_0 \ll M_0 N_0^{1/2}.$$

For $d = q_1 q_2'$ if we proceed similarly then we will get the same bound.

Case 6

For the last case we have $d = q_2^2$. This case will be similar to the previous case. By considering the size of d for this case again we can say that

$$q_2' n - n' q_2 \equiv 0 \bmod d \iff q_2 = q_2' \text{ and } n = n'.$$

So for this case, we have

$$\Sigma_0 \ll \sup_{M_1 \ll M_0} \frac{M_1 N_0^{1/2}}{Q_1} \ll \frac{M_0 N_0^{1/2}}{Q_1}.$$

Hence we have, for this case,

$$\Sigma_0 \ll M_0 N_0^{1/2}.$$

This completes the proof of the Lemma 2.4.2. □

Non-zero frequency

Now we will consider the non-zero frequency case, i.e., when $m \neq 0$. In this case, we will need the following basic lemma:

Lemma 2.4.3. *For any $x, y \in \mathbb{R}$ with $x, y \geq 1$ and $c \in \mathbb{N}$, we have,*

$$\sum_{1 \leq a \leq x} \sum_{1 \leq b \leq y} (a, b, c) = \mathrm{O}(xy).$$

Proof. We have

$$\sum_{1 \leq a \leq x} \sum_{1 \leq b \leq y} (a, b, c) \leq \sum_{1 \leq a \leq x} \sum_{1 \leq b \leq y} (a, b)$$

$$\leq \sum_{1 \leq d \leq \min\{x, y\}} \left(\sum_{\substack{d \mid a \\ 1 \leq a \leq x}} 1 \right) \left(\sum_{\substack{d \mid a \\ 1 \leq b \leq y}} 1 \right) \leq \sum_{d=1}^{\infty} \frac{x}{d} \cdot \frac{y}{d} = \mathrm{O}(xy).$$

This completes the proof of this lemma. □

Now the contribution of the non-zero frequency to $\tilde{S}_x(N)$ is given by the following:

$$\Sigma_{\neq 0} = \sum_{M_1 \ll M_0} \frac{M_1}{q_1} \sum_{q_2, q_2' \in \Phi_2} \frac{\chi(q_2 \bar{q_2'})}{(q_2 q_2')^{3/2}} \sum_{|n|,|n|' \ll N_0} \frac{\lambda_f(n)\lambda_f(n')}{(nn')^{1/4}} \mathcal{I}_1(n, x, q_1 q_2) \overline{\mathcal{I}_1(n', x, q_1 q_2')}$$
$$\times \sum_{1 \leq |m| \ll R_1} S(p(q_2' n - n' q_2), m; q_1 q_2 q_2') \mathcal{I}(m, x, q_1, q_2, q_2'). \tag{2.13}$$

By the Weil's bound for Kloosterman sums and estimating the integral trivially, we arrive at

$$\sum_{1 \leq |m| \ll R_1} S(p(q_2' n - n' q_2), m; q_1 q_2 q_2') \mathcal{I}(m, x, q_1, q_2, q_2')$$
$$\ll Q_2 \sqrt{Q_1} \sum_{1 \leq |m| \ll R_1} (p(q_2' n - n' q_2), m; q_1 q_2 q_2')^{1/2},$$

Then by the previous lemma 2.4.3, we have,

$$\sum_{1 \leq |m| \ll R_1} (p(q_2' n - n' q_2), m; q_1 q_2 q_2')^{1/2} \ll R_1^{1+\epsilon}.$$

Hence we get that

$$\sum_{|m| \ll R_1} S(p(q_2' n - n' q_2), m; q_1 q_2 q_2')\, \mathcal{I}(m, x, q_1, q_2, q_2') \ll R_1^{1+\epsilon} Q_2 \sqrt{Q_1}. \tag{2.14}$$

Now putting values of R_1 we get that, using the well-known pointwise Ramanujan bound, given in the Remark 1.7.3,

$$\Sigma_{\neq 0} \ll (Np)^{\epsilon} \sup_{M_1 \ll M_0} \frac{M_1}{Q_1} \times \frac{1}{Q_2} \times N_0^{3/2} \times \frac{Q_2 Q}{M_1} \times Q_2 \sqrt{Q_1},$$

as there are atmost $\log M_0 \, (\ll p^{\varepsilon})$ many M_1's. So we have,

$$\Sigma_{\neq 0} \ll N_0^{3/2} Q_2^2 \sqrt{Q_1} (Np)^{\epsilon}.$$

This completes the proof of the Lemma 2.4.1.

2.5 Final estimation

From the equations (2.9) and (2.10) we get the following

$$\tilde{S}_x(N) \ll \frac{N^{7/4}\sqrt{M_0}}{\sqrt{p}L\sqrt{Q_1}} \sum_{q_1 \sim Q_1} \left(M_0 N_0^{1/2} + N_0^{3/2} Q_2^2 \sqrt{Q_1} \right)^{1/2} (Np)^{\epsilon}$$
$$\ll \frac{N^{7/4}\sqrt{Q_1 M_0}}{\sqrt{p}L} \left(M_0^{1/2} N_0^{1/4} + N_0^{3/4} Q_2 Q_1^{1/4} \right) (Np)^{\epsilon}. \tag{2.15}$$

Now the optimal choice of Q_1 is obtained by equating the two terms of the equation (2.10) and using the relations $Q_1 Q_2 = Q = N^{1/2+\theta}$, $N_0 = \frac{Q^2}{N}$ and $M_0 = \frac{pQ}{N}$, so that we have

$$Q_1 = \frac{N^{1+2\theta}}{p^{2/3}}. \tag{2.16}$$

This satisfies our requirement that $\frac{p}{N} < Q_2 < N^{1/2-3\theta}$. Now putting this value of Q_1 in the equation (2.15), we have

$$\tilde{S}_x(N) \ll N^{3/4+\theta/2} p^{1/6} (Np)^{\epsilon}.$$

Therefore we arrive at

$$S_{f,\chi}(N) \ll (Np)^{\epsilon} (N^{3/4+\theta/2} p^{1/6} + N^{1-\theta}) \tag{2.17}$$

with $0 < \theta < 1/10$. We choose $\theta = \frac{1}{6} - \frac{\log p}{9 \log N}$ so that

$$N^{3/4+\theta/2} p^{1/6} = N^{1-\theta} \tag{2.18}$$

Our θ will satisfy the condition $0 < \theta < \frac{1}{10}$ if $p^{2/3+\alpha+\varepsilon} < N < p$ where $\alpha = \frac{4\theta}{1-6\theta}$ which is fine.

This completes the proof of the Theorem 2.0.1.

Chapter 3

Sub-Weyl type range for twisted GL(2) short sums

First let us state our second result 1.5.3 here again :

Theorem 3.0.1. *(see [22]) Let p be an odd prime such that $p \geq 5$. Then we have*

$$S_{f,\chi}(N) \ll_f N^{\frac{5}{9}}\, p^{\frac{13r}{45}}\, N^{\varepsilon},$$

where the implied constant depends on f only, and provided $p^{13r/20+\varepsilon} \ll N \leq p^{4r/5}$.

Remark 3.0.2. We need condition $p \geq 5$ on the prime p to apply the p-adic exponent pair $\Big(1/30, 13/15\Big)$, as one can see in [45, P. 871] that this p-adic exponent pair is only valid when $p \geq 5$. By applying other exponent pairs one can get a good range but here our main concern is to get the sub-Weyl range for N.

Remark 3.0.3. Here for simplicity, we are considering f to be a holomorphic Hecke eigenform for $\mathrm{SL}(2, \mathbb{Z})$. We can get the same kind of bounds for $S_f, \chi(N)$ even if we consider Fourier coefficients $\lambda_f(n)$ of Hecke-Maass cusp form f as we only require Ramanujan bound for the coefficients in L^2-sense.

3.1 Sketch of the proof

We take the path of the delta method to bound the sum $S_{f,\chi}(n)$. Our approach is inspired by the approach of Munshi and Singh [52]. At first, we separate the oscillations $\lambda_f(n)$ and $\chi(n)$ using the delta symbol. While separating these oscillations we introduce extra additive harmonics in the sum which serve as conductor lowering in the delta method. Our next step is to dualize these sums. To do that we apply the Voronoi summation (see the Lemma 1.7.1) and the Poisson summation formulae accordingly. Now we remove the Fourier coefficients appearing in the dualized sum by applying the Cauchy-Schwarz inequality. In the resulting expression, we open the absolute square and again we employ the Poisson summation formula.

To proceed further, we treat the cases of zero frequency and non-zero frequencies separately. In the zero frequency, we can not do much better than evaluating trivially. But in the non-zero frequencies, we observe that we have a sum of the form

$$\sum_{R \leq m \leq 2R} e\left(\frac{f(m)}{p^r}\right)$$

with $R \leq p^r/N$ and a "nice" phase function f. The main step in our proof is to get some cancellations in the above sum which we achieve by appealing to the p-adic exponent pair $(1/15, 13/15)$. The novelty of this paper is the application of p-adic exponent pair to get better bounds. Note that this p-adic exponent pair has been developed by Milićević in [45]. For a brief introduction to this, one can see the Subsection 1.7.7.

Notations

We write $p^s \parallel m$ to denote that $p^s \mid m$ and $p^{s+1} \nmid m$. Also by $A \ll B$, we mean that $|A| \leq C|B|$ for some absolute constant $C > 0$, depending on f, ε only.

3.2 An application of the circle method

Separating the oscillations $\lambda_f(n)$ and $\chi(n)$ in the sum $S_{f,\chi}(N)$ by using the delta symbol "δ", as given in the Section 1.11, we have

$$S_{f,\chi}(N) = \sum_{\substack{m,n=1 \\ p^\ell \mid (n-m)}}^{\infty}\sum \lambda_f(n)\chi(m)\, \delta\left(\frac{n-m}{p^\ell}\right) W\left(\frac{n}{N}\right) V\left(\frac{m}{N}\right),$$

with the condition that $p^\ell \leq N$ and $\ell \leq r$. Now by writing the expression for δ, with the choice $Q=\sqrt{N/p^\ell}$, in the above sum we arrive at

$$S_{f,\chi}(N)=\frac{1}{Qp^\ell}\int_{\mathbb{R}}\sum_{1\le q\le Q}\frac{g(q,x)}{q}\sum_{a \bmod q}^{\star}\sum_{b \bmod p^\ell}\mathcal{S}_f(N;a,b,q,x)\,\mathcal{S}_\chi(N;a,b,q,x)\,\mathrm{d}x, \tag{3.1}$$

where

$$\mathcal{S}_f(N;a,b,q,x)=\sum_{n=1}^{\infty}\lambda_f(n)e\left(\frac{(a+bq)n}{p^\ell q}\right)e\left(\frac{nx}{p^\ell qQ}\right)W\left(\frac{n}{N}\right), \tag{3.2}$$

and

$$\mathcal{S}_\chi(N;a,b,q,x)=\sum_{m=1}^{\infty}\chi(m)e\left(-\frac{(a+bq)m}{p^\ell q}\right)e\left(\frac{-mx}{p^\ell qQ}\right)V\left(\frac{m}{N}\right). \tag{3.3}$$

Now in the following section, we dualize these sums by applying summation formulae.

3.3 Application of summation formulae

3.3.1 Applying the Poisson summation formula

We shall apply the Poisson summation formula to the sum over m in the equation (3.3) so that we have :

Lemma 3.3.1.

$$\mathcal{S}_\chi(N;a,b,q,x)=\frac{N}{p^rq}\sum_{\substack{m\in\mathbb{Z}\\|m|\le M_0}}\mathcal{C}(a,b,q,m)\,\mathcal{I}(x,q,m)+\mathrm{O}\left(N^{-2022}\right),$$

with $M_0:=\frac{p^rQ}{N}N^\varepsilon$, *where*

$$\mathcal{C}(a,b,q,m)=\sum_{\beta \bmod p^rq}\chi(\beta)e\left(-\frac{(a+bq)\beta}{p^\ell q}+\frac{m\beta}{p^rq}\right),$$

and

$$\mathcal{I}(x,q,m) = \int_{\mathbb{R}} V(z) e\left(\frac{-Nxz}{p^\ell qQ}\right) e\left(\frac{-Nmz}{p^r q}\right) \mathrm{d}z.$$

Proof. We split the m-sum in (3.3) into congruence classes modulo $p^r q$. Indeed, we write $m = \beta + cp^r q$ with β mod $p^r q$, and $c \in \mathbb{Z}$ to get

$$\mathcal{S}_\chi(N;a,b,q,x) = \sum_{\beta \bmod p^r q} \chi(\beta) e\left(-\frac{(a+bq)\beta}{p^\ell q}\right) \sum_{c\in\mathbb{Z}} V\left(\frac{\beta + cp^r q}{N}\right) e\left(\frac{-(\beta+cp^r q)x}{p^\ell qQ}\right)$$

$$= \sum_{\beta \bmod p^r q} \chi(\beta) e\left(-\frac{(a+bq)\beta}{p^\ell q}\right) \sum_{m\in\mathbb{Z}} \int_{\mathbb{R}} V\left(\frac{\beta + yp^r q}{N}\right) e\left(\frac{-(\beta+yp^r q)x}{p^\ell qQ}\right) e(-my)\mathrm{d}y,$$

where the second equality follows by applying the Poisson summation formula. We now make the change of variable $(\beta + yp^r q)/N = z$ to obtain the value of $\mathcal{S}_\chi(N;a,b,q,x)$ to be

$$\frac{N}{p^r q} \sum_{m\in\mathbb{Z}} \left\{ \sum_{\beta \bmod p^r q} \chi(\beta) e\left(-\frac{(a+bq)\beta}{p^\ell q} + \frac{m\beta}{p^r q}\right) \right\} \int_{\mathbb{R}} V(z) e\left(\frac{-Nxz}{p^\ell qQ}\right) e\left(\frac{-Nmz}{p^r q}\right) \mathrm{d}z$$

$$= \frac{N}{p^r q} \sum_{m\in\mathbb{Z}} \mathcal{C}(a,b,q,m)\,\mathcal{I}(x,q,m),$$

where $\mathcal{C}(a,b,q,m)$, $\mathcal{I}(x,q,m)$ are given by

$$\mathcal{C}(a,b,q,m) := \sum_{\beta \bmod p^r q} \chi(\beta) e\left(-\frac{(a+bq)\beta}{p^\ell q} + \frac{m\beta}{p^r q}\right),$$

and

$$\mathcal{I}(x,q,m) := \int_{\mathbb{R}} V(z) e\left(\frac{-Nxz}{p^\ell qQ}\right) e\left(\frac{-Nmz}{p^r q}\right) \mathrm{d}z.$$

We see, by repeated integration by parts, that

$$\mathcal{I}(x,q,m) \ll_j \left(1+\frac{N|x|}{p^\ell qQ}\right)^j \left(\frac{p^r q}{Nm}\right)^j,$$

for any $j \geq 0$. Thus, $\mathcal{I}(x,q,m)$ is negligibly small unless

$$|m| \leq M_0 := \frac{p^r Q}{N} N^\varepsilon.$$

□

Since the character sum $\mathcal{C}(a,b,q,m)$ involves prime power moduli we can easily evaluate this sum. The following subsection is devoted to these calculations.

3.3.2 Evaluation of the character sum

We have the following lemma.

Lemma 3.3.2. *Let $q = p^{r_1}q'$ with $(p,q')=1$ (i.e., $p^{r_1} \parallel q$). Then we have*

$$\mathcal{C}(a,b,q,m) = \begin{cases} q\ \chi(q')\ \overline{\chi}\left(\frac{m-(a+bq)p^{r-\ell}}{p^{r_1}}\right)\tau_\chi & \text{if } a \equiv m\,\overline{p^{r-\ell}} \bmod q', \text{ and } p^{r_1} \parallel m, \\ 0 & \text{otherwise}, \end{cases}$$

where τ_χ denotes the Gauss sum.

Proof. Since $q = p^{r_1}q'$ with $(p,q')=1$, the character sum $\mathcal{C}(a,b,q,m)$ is given by

$$\sum_{\beta \bmod p^{r+r_1}q'} \chi(\beta) e\left(\frac{-(a+bq)\beta}{p^{\ell+r_1}q'} + \frac{m\beta}{p^{r+r_1}q'}\right).$$

By writing $\beta = \alpha_1 q' \overline{q'} + \alpha_2\, p^{r+r_1}\overline{p^{r+r_1}}$ with α_1 mod p^{r+r_1} and α_2 mod q' in the above sum we see that the above character sum changes to

$$\sum_{\alpha_1 \bmod p^{r+r_1}} \chi(\alpha_1) e\left(\frac{-(a+bq)\alpha_1\overline{q'}}{p^{\ell+r_1}} + \frac{m\alpha_1\overline{q'}}{p^{r+r_1}}\right)$$
$$\times \sum_{\alpha_2 \bmod q'} e\left(\frac{-(a+bq)\alpha_2\overline{p^{r+r_1}}p^{r-\ell}}{q'} + \frac{m\alpha_2\overline{p^{r+r_1}}}{q'}\right).$$

Again, considering $\alpha_1 = \beta_1 p^r + \beta_2$, where β_2 is modulo p^r and β_1 modulo p^{r_1}, the above sum becomes

$$\sum_{\beta_2 \bmod p^r} \chi(\beta_2) e\left(\frac{-(a+bq)p^{r-\ell}\beta_2\overline{q'}}{p^{r+r_1}} + \frac{m\beta_2\overline{q'}}{p^{r+r_1}}\right) \sum_{\beta_1 \bmod p^{r_1}} e\left(\frac{-(a+bq)\beta_1\overline{q'}p^{r-\ell}}{p^{r_1}} + \frac{m\beta_1\overline{q'}}{p^{r_1}}\right)$$
$$\times \sum_{\alpha_2 \bmod q'} e\left(\frac{-(a+bq)\alpha_2\overline{p^{r+r_1}}p^{r-\ell}}{q'} + \frac{m\alpha_2\overline{p^{r+r_1}}}{q'}\right).$$

We execute sums over β_1 and α_2 to transform the above sum to

$$q\, \mathbb{I}_{(m-ap^{r-\ell}\equiv 0 \bmod p^{r_1})}\, \mathbb{I}_{(m-ap^{r-\ell}\equiv 0 \bmod q')} \sum_{\beta_2 \bmod p^r} \chi(\beta_2) e\left(\frac{\left(m-(a+bq)p^{r-\ell}\right)\beta_2\overline{q'}}{p^{r+r_1}}\right).$$

Since $N \le p^r$, we have the inequality $p^{r_1} \le q \le Q = \sqrt{N/p^\ell} \le p^{(r-\ell)/2} < p^{r-\ell}$. Thus, $\min\{r_1, r-\ell\} = r_1$. Hence the congruence $m - ap^{r-\ell} \equiv 0 \bmod p^{r_1}$ is same as $p^{r_1} \mid m$.

Note that, since χ is a primitive character modulo p^r, the sum over β_2 is Gauss sum which vanishes unless

$$\left(\frac{m-(a+bq)p^{r-\ell}}{p^{r_1}}, p\right) = 1 \iff (m/p^{r_1}, p) = 1,$$

as $r_1 < r - \ell$. In this case, we have

$$\sum_{\beta_2 \bmod p^r} \chi(\beta_2) e\left(\frac{\left(m-(a+bq)p^{r-\ell}\right)\beta_2\overline{q'}}{p^{r+r_1}}\right) = \chi(q')\overline{\chi}\left(\frac{m-(a+bq)p^{r-\ell}}{p^{r_1}}\right)$$

$$\times \sum_{\beta_2 \bmod p^r} \chi(\beta_2) e\left(\frac{\beta_2}{p^r}\right).$$

Note that the last sum over β_2 is the Gauss sum which completes the proof of the lemma. □

After applying the Poisson summation formula, the sum $S_{f,\chi}(N)$ becomes

$$S_{f,\chi}(N) = \frac{1}{Qp^\ell}\int_{\mathbb{R}} \sum_{r_1=0}^{\lfloor \frac{\log Q}{\log p} \rfloor} \sum_{\substack{1\leq q'\leq Q/p^{r_1} \\ (q',p)=1}} \frac{g(p^{r_1}q',x)}{p^{r_1}q'} \sideset{}{^\star}\sum_{a_1 \bmod p^{r_1}} \sum_{b \bmod p^\ell}$$

$$\times \left\{ \frac{\tau_\chi \chi(q') N}{p^r} \sum_{\substack{m' \ll M_0/p^{r_1} \\ (m',p)=1}} \overline{\chi}\left(m'-(a+bq)p^{r-\ell-r_1}\right) \mathcal{I}(x, p^{r_1}q', p^{r_1}m') \right\}$$

$$\times \left\{ \sum_{n=1}^{\infty} \lambda_f(n) e\left(\frac{(a+bq)n}{p^\ell q}\right) e\left(\frac{xn}{p^\ell qQ}\right) W\left(\frac{n}{N}\right) \right\} \mathrm{d}x + \mathrm{O}_A\left(N^{-A}\right),$$

for any real $A > 0$, where $a \bmod q$ is determined in terms of $a_1 \bmod p^{r_1}$ and m'. Indeed we have $a \equiv m' p^{2r_1}\overline{p^{r-\ell+r_1}} + a_1 q'\bar{q'} \bmod p^{r_1}q'$ and also $q = p^{r_1}q'$ where $\overline{p^{r-\ell+r_1}}$ is the multiplicative inverse of $p^{r-\ell+r_1}$ modulo q'; $\bar{q'}$ is the multiplicative inverse of q' modulo p^{r_1}.

We now split the above expression for $S_{f,\chi}(N)$ as follows

$$S_{f,\chi}(N) = S_{f,\chi}(N; r_1 = 0) + S_{f,\chi}(N; r_1 \geq 1).$$

Here note that $r_1 \geq 1$ implies $(a+bq, p^\ell q) = 1$ which allows us to apply the Voronoi summation formula directly. But, if $r_1 = 0$, then $a+bq$ may not be coprime to p^ℓ. Therefore we can not directly apply the Voronoi summation formula. So we need to work with these two situations separately. Note that except for very small modifications, these two cases can be dealt with similarly.

The rest of this chapter only focuses on the estimation of $S_{f,\chi}(N; r_1 = 0)$. Similar approach towards the sum $S_{f,\chi}(N; r_1 \geq 1)$ gives even better estimates.

$$S_{f,\chi}(N; r_1 = 0) = \frac{1}{Qp^\ell} \int_{\mathbb{R}} \sum_{\substack{1 \leq q \leq Q \\ (q,p)=1}} \frac{g(q,x)}{q} \sum_{b \bmod p^\ell}$$

$$\times \left\{ \frac{\tau_\chi \chi(q) N}{p^r} \sum_{\substack{m \ll M_0 \\ (m,p)=1}} \overline{\chi}\left(m - (a+bq)p^{r-\ell}\right) \mathcal{I}(x,q,m) \right\} \tag{3.4}$$

$$\times \left\{ \sum_{n=1}^{\infty} \lambda_f(n) e\left(\frac{(a+bq)n}{p^\ell q}\right) e\left(\frac{xn}{p^\ell q Q}\right) W\left(\frac{n}{N}\right) \right\} \mathrm{d}x,$$

where $a \equiv m\,\overline{p^{r-\ell}} \bmod q$. Note that from this congruence relation, it follows that $(m,q) = 1$.

3.3.3 Application of the Voronoi summation formula

Now we appeal to an application of the Voronoi summation formula to the n-sum in (3.4). Recall from (3.2) that $\mathcal{S}_f(N; a, b, q, x)$ is same as this n-sum. An application of the Voronoi summation formula leads to the following lemma.

Lemma 3.3.3. *Let $(a+bq, p^\ell) = p^{\ell_1}$ for some $0 \leq \ell_1 \leq \ell$. Then we have*

$$\mathcal{S}_f(N; a, b, q, x) = \frac{2\pi i^k N^{3/4}}{p^{(\ell-\ell_1)/2} q^{1/2}} \sum_{\varepsilon' \in \{\pm\}} \sum_{1 \leq n \ll N_0} \frac{\lambda_f(n)}{n^{1/4}} e\left(-\frac{\overline{((a+bq)/p^{\ell_1})} n}{p^{\ell-\ell_1} q}\right)$$

$$\times \mathcal{J}(\varepsilon', q, x, n),$$

where

$$\mathcal{J}(\varepsilon', q, x, n) = \int W_{1,\varepsilon'}(y)\, e\left(\frac{xNy}{p^\ell q Q}\right) e\left(\frac{\varepsilon' 2\sqrt{nNy}}{p^{\ell-\ell_1} q}\right) \mathrm{d}y,$$

and $N_0 = p^{\ell-2\ell_1} N^{\varepsilon}$.

Proof. An application of Voronoi summation formula (see the Lemma 1.7.1) transforms the sum $\mathcal{S}_f(N; a, b, q, x)$ into

$$\frac{2\pi i^k}{p^{\ell-\ell_1} q} \sum_{n=1}^{\infty} \lambda_f(n)\, e\left(-\frac{\overline{((a+bq)/p^{\ell_1})} n}{p^{\ell-\ell_1} q}\right) \int_{\mathbb{R}} W\left(\frac{y}{N}\right) e\left(\frac{xy}{p^{\ell} q Q}\right) J_{k-1}\left(\frac{4\pi\sqrt{yn}}{p^{\ell-\ell_1} q}\right) \mathrm{d}y,$$

where k is the weight of the holomorphic Hecke eigenform f and $J_{k-1}(x)$ is the Bessel function. We make a change of variables $y/N \mapsto z$ in the above integration and use the expression for the Bessel function (see the Subsection 1.7.6)

$$J_{k-1}(x) = \frac{1}{\sqrt{x}} \sum_{\varepsilon' \in \{\pm\}} W_{k,\varepsilon'}(x)\, e^{i\varepsilon' x},$$

where $x^j W_{k,\varepsilon'}^{(j)}(x) \ll_{k,j} 1$ for $x \gg 1$, for Bessel function to get

$$\frac{N^{3/4} p^{(\ell-\ell_1)/2} q^{1/2}}{n^{1/4}} \sum_{\varepsilon' \in \{\pm\}} \int W_{1,\varepsilon'}(y)\, e\left(\frac{xNy}{p^{\ell} q Q}\right) e\left(\frac{\varepsilon' 2\sqrt{nNy}}{p^{\ell-\ell_1} q}\right) \mathrm{d}y,$$

where $W_{1,\varepsilon'}(y) = W(y) W_{k,\varepsilon'}\left(\frac{4\pi\sqrt{nNy}}{p^{\ell-\ell_1} q}\right)$ satisfying $y^j W_{1,\varepsilon'}^{(j)}(y) \ll_{k,j} 1$ and $\varepsilon' \in \{\pm 1\}$. Using integration by parts repeatedly, we can see that the above integral becomes negligibly small unless

$$1 \leq n \ll N_0 = p^{\ell-2\ell_1} N^{\varepsilon}.$$

Thus the lemma follows.

□

Note that we have the following identities

$$\frac{\left(\overline{(a+bq)/p^{\ell_1}}\right)}{p^{\ell-\ell_1} q} = \frac{\bar{a} p^{\ell_1} p^{\ell-\ell_1} \overline{p^{\ell-\ell_1}} + \overline{((a+bq)/p^{\ell_1})} q \bar{q}}{p^{\ell-\ell_1} q}$$

$$= \frac{\overline{((a+bq)/p^{\ell_1})}\, \bar{q}}{p^{\ell-\ell_1}} + \frac{\bar{m} p^r \overline{p^{2(\ell-\ell_1)}}}{q}.$$

In the second equality, we have used the fact that $a \equiv m\overline{p^{r-\ell}} \mod q$. The following proposition is obtained by using this identity and by rearranging all the terms.

Proposition 3.3.4. *We have*

$$S_{f,\chi}(N; r_1 = 0) = \frac{\tau_\chi N^{7/4} 2\pi i^k}{Qp^{r+\frac{3\ell}{2}}} \sum_{\varepsilon' \in \{\pm\}} \sum_{\ell_1=0}^{\ell} p^{\ell_1/2}\, T(\varepsilon', \ell_1, N),$$

where

$$T(\varepsilon', \ell_1, N) = \sum_{1 \le n \ll N_0} \frac{\lambda_f(n)}{n^{1/4}} \sum_{\substack{1 \le q \le Q \\ (q,p)=1}} \frac{\chi(q)}{q^{3/2}} \sum_{\substack{m \ll M_0 \\ (m,p)=1}} \sideset{}{^\dagger}\sum_{\beta \bmod p^{\ell-\ell_1}}$$

$$\times\, \overline{\chi}\left(m - \beta p^{r-\ell+\ell_1}\right) e\left(-\frac{\overline{\beta}\,\bar{q} n}{p^{\ell-\ell_1}} - \frac{\bar{m} p^r \overline{p^{2(\ell-\ell_1)}} n}{q}\right) \mathfrak{I}(\varepsilon', q, n, m),$$

$a + bq = \beta p^{\ell_1}$ *and* $\varepsilon' \in \{\pm 1\}$. *The symbol* $\dagger$ *on the* β*-sum means that* $((a+bq)/p^{\ell_1}, p) = 1$ *and*

$$\mathfrak{I}(\varepsilon', q, n, m) = \int_{\mathbb{R}} g(q, x)\, \mathcal{J}(\varepsilon', q, x, n)\, \mathcal{I}(q, x, m)\, \mathrm{d}x.$$

3.3.4 Bounds for the integrals $\mathfrak{I}(\varepsilon', q, n, m)$

In this subsection, we give bounds for the integrals $\mathfrak{I}(\varepsilon', q, n, m)$, where $\varepsilon' \in \{\pm 1\}$, which are useful when we will deal with small q (note that the phase functions, appeared in $\mathfrak{I}(\varepsilon', q, n, m)$, oscillate when q is small).

Lemma 3.3.5. *We have*

$$\mathfrak{I}(\varepsilon', q, n, m) \ll \frac{p^\ell q Q}{N} N^\varepsilon.$$

Proof. Recall that $\mathfrak{I}(\varepsilon', q, n, m)$ is given by

$$\mathfrak{I}(\varepsilon', q, n, m) = \int_{\mathbb{R}} g(q, x)\, \mathcal{J}(\varepsilon', q, x, n)\, \mathcal{I}(q, x, m)\, \mathrm{d}x.$$

The function $g(q, x)$ is negligible unless $|x| \le N^{\varepsilon}$. Therefore we have

$$\begin{aligned}\mathfrak{I}(\varepsilon', q, n, m) = \int_{|x| \le N^{\varepsilon}} g(q, x) \int_{\mathbb{R}} W_{1,\varepsilon'}(y)\, e\left(\frac{xNy}{p^{\ell} qQ}\right) e\left(\frac{\varepsilon' 2\sqrt{nNy}}{p^{\ell-\ell_1} q}\right) \\ \times \int_{\mathbb{R}} V(z) e\left(\frac{-Nxz}{p^{\ell} qQ}\right) e\left(\frac{-Nmz}{p^{r} q}\right) \mathrm{d}z\, \mathrm{d}y\, \mathrm{d}x + \mathrm{O}\left(N^{-2022}\right).\end{aligned} \tag{3.5}$$

Now we consider the z integral. By repeated application of integration by parts, we see that the z integral is negligible unless

$$\left|\frac{Nx}{p^{\ell} qQ} + \frac{Nm}{p^{r} q}\right| \ll N^{\varepsilon} \iff \left|x + \frac{mQ}{p^{r-\ell}}\right| \ll \frac{p^{\ell} qQ}{N} N^{\varepsilon}.$$

We know, by properties of the function $g(q, x)$, see the Section 1.11, that

$$g(q, x) = 1 + \mathrm{O}\left(N^{-2022}\right),$$

if $q \le Q^{1-\varepsilon}$ or $|x| \le N^{-\varepsilon}$. We can assume that $q \le Q^{1-\varepsilon}$ otherwise the statement of the Lemma 3.3.5 follows trivially from the bound $g(q, x) \ll N^{\varepsilon}$ if $Q^{1-\varepsilon} \le q \le Q$. Therefore we divide x-integral into two parts and write

$$\mathfrak{I}(\varepsilon', q, n, m) = \left(\int_{\substack{|x| \le N^{-\varepsilon} \\ \left|x + \frac{mQ}{p^{r-\ell}}\right| \ll \frac{p^{\ell} qQ}{N} N^{\varepsilon}}} + \int_{\substack{N^{-\varepsilon} \le |x| \le N^{\varepsilon} \\ \left|x + \frac{mQ}{p^{r-\ell}}\right| \ll \frac{p^{\ell} qQ}{N} N^{\varepsilon}}} \right) g(q, x) \int_{\mathbb{R}} \int_{\mathbb{R}} \{y, z\}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

$$+\mathrm{O}\left(N^{-2022}\right),$$

where $\{y, z\}$ is the remaining integrand in the variables y, z of (3.5).

In the first integral, we can replace $g(q, x)$ by 1 up to a negligible error term. We treat everything else trivially in the first x-integral to get this integral to be

$$\ll \frac{p^{\ell} q Q}{N} N^{\varepsilon}.$$

In the second x-integral, we have the condition that $N^{-\varepsilon} \leq |x| \leq N^{\varepsilon}$. In this case, we consider y-integral in (3.5). In this integral we make a change of variable $y \to y^2$, and then the resulting expression of this integral is given by

$$\int_{\mathbb{R}} 2\, y\, W_{1,\varepsilon'}(y^2)\; e\left(f(y)\right) \mathrm{d}y,$$

where the phase function is as follows

$$f(y) := \frac{xNy^2}{p^{\ell} q Q} - \frac{\varepsilon' 2\sqrt{nN} y}{p^{\ell-\ell_1} q}.$$

The stationary point y_0 of $f(y)$ is given by

$$y_0 = \varepsilon' \sqrt{n} Q p^{\ell_1} / x\sqrt{N}.$$

Hence we have

$$f^{('')}(y_0) = \frac{2xN}{p^{\ell} q Q}.$$

Thus we get the following second derivative bound

$$\frac{1}{\sqrt{|f^{('')}(y_0)|}} \ll \sqrt{\frac{p^{\ell} q Q}{N}} N^{\varepsilon}.$$

Therefore, using the Lemma 1.7.7, this y-integral is at most

$$\ll \sqrt{\frac{p^{\ell} q Q}{N}} N^{\varepsilon}.$$

Thus, the second x-integral is at most

$$\ll \sqrt{\frac{p^\ell qQ}{N}} N^\varepsilon \int_{\substack{N^{-\varepsilon}\le|x|\le N^\varepsilon \\ \left|x+\frac{mQ}{p^{r-\ell}}\right|\ll \frac{p^\ell qQ}{N}N^\varepsilon}} |g(q,x)|\,\mathrm{d}x$$

$$\ll \frac{p^\ell qQ}{N} N^\varepsilon \int_{\mathbb{R}} |g(q,x)|^2\,\mathrm{d}x$$

$$\ll \frac{p^\ell qQ}{N} N^\varepsilon.$$

Here we have used the L^2-bound for the function $g(q,x)$ from the Section 1.11. □

3.4 Cauchy-Schwarz and Poisson summation formulae

An application of the Cauchy-Schwarz inequality on the n-sum in $T(\varepsilon',\ell_1,N)$ along with the Ramanujan bound for the Fourier coefficients $\lambda_f(n)$ gives that

$$T(\varepsilon',\ell_1,N) \ll N_0^{1/4}\,\Theta^{1/2}, \tag{3.6}$$

where

$$\Theta = \sum_n W_2\left(\frac{n}{N_0}\right)\Big|\sum_{\substack{1\le q\le Q\\(q,p)=1}} \frac{\chi(q)}{q^{3/2}} \sum_{\substack{m\ll M_0\\(m,p)=1}} \sum_{\beta \bmod p^{\ell-\ell_1}}^{\dagger}$$

$$\times\, \overline{\chi}\left(m-\beta p^{r-\ell+\ell_1}\right) e\left(-\frac{\overline{\beta}\,\bar{q}n}{p^{\ell-\ell_1}} - \frac{\bar{m}p^r\overline{p^{2(\ell-\ell_1)}}n}{q}\right) \mathfrak{I}(\varepsilon',q,n,m)\Big|^2,$$

where W_2 is smooth bump function supported on $[1,2]$. After opening the absolute square and interchanging sums we arrive at

$$\Theta = \sum_{\substack{1\le q_1,q_2\le Q\\(q_1q_2,p)=1}}\sum \frac{\chi\left(q_1q_2\right)}{q_1^{3/2}q_2^{3/2}} \sum_{\substack{m_1,m_2\ll M_0\\(m_1m_2,p)=1}}\sum \sideset{}{^\dagger}\sum_{\beta_1 \bmod p^{\ell-\ell_1}} \sideset{}{^\dagger}\sum_{\beta_2 \bmod p^{\ell-\ell_1}}$$

$$\times\, \overline{\chi}\left(m_1-\beta_1 p^{r-\ell+\ell_1}\right)\chi\left(m_2-\beta_2 p^{r-\ell+\ell_1}\right)\, T(m_1,m_2,q_1,q_2,b_1,b_2),$$

where

$$T(m_1,m_2,q_1,q_2,b_1,b_2) = \sum_{n\in\mathbb{Z}} W_2\left(\frac{n}{N_0}\right)\, e\left(\frac{\overline{\beta_1}\,\bar{q}_1 n - \overline{\beta_2}\,\bar{q}_2 n}{p^{\ell-\ell_1}}\right)$$

$$\times\, e\left(\frac{\bar{m}_1 p^r \overline{p^{2(\ell-\ell_1)}} n}{q_1} - \frac{\bar{m}_2 p^r \overline{p^{2(\ell-\ell_1)}} n}{q_2}\right)\mathfrak{I}(\varepsilon', q_1, n, m_1)\overline{\mathfrak{I}(\varepsilon', q_2, n, m_2)}.$$

Now we split the sum over n into congruence classes modulo $p^{\ell-\ell_1}q_1q_2$. Indeed for any α modulo $p^{\ell-\ell_1}q_1q_2$ we write $n = \alpha + kp^{\ell-\ell_1}q_1q_2$ with $k\in\mathbb{Z}$. Then by applying the Poisson summation on the k variable, we arrive at the expression

$$\frac{N_0}{p^{\ell-\ell_1}q_1q_2}\sum_{n\in\mathbb{Z}}\sum_{\alpha \bmod p^{\ell-\ell_1}q_1q_2} e\left(\frac{\left(\overline{\beta_1}\bar{q}_1 q_1q_2 - \overline{\beta_2}\bar{q}_2 q_1 q_2\right)\alpha}{p^{\ell-\ell_1}q_1q_2}\right)$$

$$\times\, e\left(\frac{\left(p^r\overline{p^{2(\ell-\ell_1)}}p^{\ell-\ell_1}\bar{m}_1q_2 - p^r\overline{p^{2(\ell-\ell_1)}}p^{\ell-\ell_1}\bar{m}_2q_1 + n\right)\,\alpha}{p^{\ell-\ell_1}q_1q_2}\right)\mathfrak{I}_1(n,q_i,m_i,\varepsilon'),$$

where

$$\mathfrak{I}_1(n,q_i,m_i,\varepsilon') = \int W_2(y)\,\mathfrak{I}(\varepsilon',q_1,N_0y,m_1)\overline{\mathfrak{I}(\varepsilon',q_2,N_0y,m_2)}\, e\left(-\frac{nN_0y}{p^{\ell-\ell_1}q_1q_2}\right)\mathrm{d}y,$$

for $T(m_1,m_2,q_1,q_2,b_1,b_2)$. Again by repeated integration by parts, we get, $\mathfrak{I}_1(n,q_i,m_i,\varepsilon')$ is negligibly small unless

$$|n| \le \frac{q_1q_2p^{\ell-\ell_1}}{N_0}N^{\varepsilon}.$$

Therefore after executing the sum over α, the value of $T(m_1, m_2, q_1, q_2, b_1, b_2)$ is given by

$$N_0 \sum_{\substack{|n| \leq \frac{q_1 q_2 p^{\ell-\ell_1}}{N_0} N^\epsilon \\ \mathcal{L}(condition)}} \mathfrak{I}_1(n, q_i, m_i, \varepsilon'),$$

up to a negligible error term and $\mathcal{L}(condition)$ denotes the following condition :

$$\overline{\beta_1}\bar{q}_1 q_1 q_2 - \overline{\beta_2}\bar{q}_2 q_1 q_2 + p^r \overline{p^{2(\ell-\ell_1)}} p^{\ell-\ell_1} \bar{m}_1 q_2 - p^r \overline{p^{2(\ell-\ell_1)}} p^{\ell-\ell_1} \bar{m}_2 q_1 + n \equiv 0 \mod p^{\ell-\ell_1} q_1 q_2 \,.$$

The above congruence relation shows that

$$\overline{\beta_1}\, q_2 - \overline{\beta_2}\, q_1 + n \equiv 0 \bmod p^{\ell-\ell_1},$$

and

$$p^{r-\ell+\ell_1} \bar{m}_1 q_2 - p^{r-\ell+\ell_1} \bar{m}_2 q_1 + n \equiv 0 \bmod q_1 q_2.$$

After changing the variables $(a_1 + b_1 q_1)/p^{\ell_1} \mapsto \alpha_1$ and $(a_2 + b_2 q_2)/p^{\ell_1} \mapsto \alpha_2$, we see that Θ is given by

$$\Theta = N_0 \sum_{\substack{1 \leq q_1, q_2 \leq Q \\ (q_1 q_2, p)=1}} \sum \frac{\chi(q_1 q_2)}{q_1^{3/2} q_2^{3/2}} \sum_{\substack{m_1, m_2 \ll M_0 \\ (m_1 m_2, p)=1}} \sum \sideset{}{^\star}\sum_{\alpha_1 \bmod p^{\ell-\ell_1}} \sideset{}{^\star}\sum_{\alpha_2 \bmod p^{\ell-\ell_1}} \sum_{\substack{|n| \leq \frac{q_1 q_2 p^{\ell-\ell_1}}{N_0} N^\epsilon \\ \bar{\alpha}_1 q_2 - \bar{\alpha}_2 q_1 + n \equiv 0 \bmod p^{\ell-\ell_1} \\ p^{r-\ell+\ell_1} \bar{m}_1 q_2 - p^{r-\ell+\ell_1} \bar{m}_2 q_1 + n \equiv 0 \bmod q_1 q_2}} \tag{3.7}$$

$$\times\, \overline{\chi}\left(m_1 - \alpha_1 p^{r-\ell+\ell_1}\right) \chi\left(m_2 - \alpha_2 p^{r-\ell+\ell_1}\right) \mathfrak{I}_1(n, q_i, m_i, \varepsilon').$$

To estimate Θ we deal with two separate cases depending on the value of n. We treat the zero frequency ($n = 0$) and the non-zero frequencies ($n \neq 0$) in the following subsections.

3.4.1 Zero frequency $n = 0$:

We write Θ_{zero} for the contribution of the zero frequency to Θ. In the following lemma, we give estimates for Θ_{zero}.

Lemma 3.4.1. *We have*

$$\Theta_{zero} \ll \frac{p^{r+\frac{5\ell}{2}-3\ell_1}}{N^{3/2}} N^{\varepsilon},$$

provided that $N \geq p^{r-(\ell-\ell_1)}$.

Proof. For $n = 0$, the congruence conditions in (3.7) reduces to

$$\bar{\alpha}_1 q_2 - \bar{\alpha}_2 q_1 \equiv 0 \bmod p^{\ell-\ell_1}, \text{and}$$

$$\bar{m}_1 q_2 - \bar{m}_2 q_1 \equiv 0 \bmod q_1 q_2.$$

From the second congruence we infer that $q_1 \mid q_2$ and $q_2 \mid q_1$ which implies that $q_1 = q_2 = q$, and we also have that $q \mid m_1 - m_2$. Then from the first congruence we immediately conclude that $\alpha_1 \equiv \alpha_2 \bmod p^{\ell-\ell_1}$. Therefore Θ_{zero} is given by

$$N_0 \sum_{\substack{1 \leq q \leq Q \\ (q,p)=1}} \frac{\chi(q^2)}{q^3} \sum\sum_{\substack{m_1, m_2 \ll M_0 \\ q \mid m_1 - m_2 \\ (m_1 m_2, p)=1}} \chi(\bar{m}_1 m_2) \sideset{}{^\star}\sum_{\alpha_1 \bmod p^{\ell-\ell_1}}$$

$$\times \overline{\chi}\left(1 - \alpha_1 \bar{m}_1 p^{r-(\ell-\ell_1)}\right) \chi\left(1 - \alpha_1 \bar{m}_2 p^{r-(\ell-\ell_1)}\right) \mathfrak{I}_1(n, q, m_i, \varepsilon').$$

For $m_1 \neq m_2$, we evaluate the character sum over α_1. To this end note that we have

$$\chi(1 + z p^{r-(\ell-\ell_1)}) = e\left(\frac{-A_1 p^{2r-2(\ell-\ell_1)} z^2 - A_2 p^{r-(\ell-\ell_1)} z}{p^r}\right),$$

for some integers A_1 and A_2 which are coprime to p, as our choice of ℓ satisfies the condition $r-\ell \geq r/3$ which is same as $\ell \leq 2r/3$ (see [45, Lemma 13]). Thus, the α_1 sum is given by

$$\sum_{\alpha \bmod \mathrm{p}^{\ell-\ell_1}}^{\star} e\left(\frac{Y_1\alpha^2+Y_2\alpha}{p^r}\right), \tag{3.8}$$

where

$$Y_1 = A_1 p^{2r-2(\ell-\ell_1)}\left(\bar{m}_2^2-\bar{m}_1^2\right), \text{and}$$

$$Y_2 = A_2 p^{r-(\ell-\ell_1)}\left(\bar{m}_1-\bar{m}_2\right).$$

Note that this character sum is the same as

$$\begin{aligned}&\sum_{\alpha \bmod p^{\ell-\ell_1}} e\left(\frac{A_1p^{r-(\ell-\ell_1)}(\bar{m}_1^2-\bar{m}_2^2)\alpha^2+A_2(\bar{m}_1-\bar{m}_2)\alpha}{p^{\ell-\ell_1}}\right)\\ &-\sum_{\alpha \bmod p^{\ell-\ell_1-1}} e\left(\frac{A_1p^{r-(\ell-\ell_1)+1}(\bar{m}_1^2-\bar{m}_2^2)\alpha^2+A_2(\bar{m}_1-\bar{m}_2)\alpha}{p^{\ell-\ell_1-1}}\right).\end{aligned} \tag{3.9}$$

We will proceed with the first term only as treatment for the second term will be similar and it will be dominated by the first sum. To estimate the first sum, let us write $\alpha=\alpha_1+p^{r-(\ell-\ell_1)}\beta_1$ where α_1 is modulo $p^{r-(\ell-\ell_1)}$ and β_1 is modulo $p^{2(\ell-\ell_1)-r}$. As we can take $\ell<\frac{2r}{3}$, the first term in (3.9) becomes

$$\begin{aligned}&\left(\sum_{\alpha_1 \bmod p^{r-(\ell-\ell_1)}} e\left(\frac{A_1p^{r-(\ell-\ell_1)}(\bar{m}_1^2-\bar{m}_2^2)\alpha_1^2+A_2(\bar{m}_1-\bar{m}_2)\alpha_1}{p^{(\ell-\ell_1)}}\right)\right)\\ &\times\left(\sum_{\beta_1 \bmod p^{2(\ell-\ell_1)-r}} e\left(\frac{A_2(\bar{m}_1-\bar{m}_2)\beta_1}{p^{2(\ell-\ell_1)-r}}\right)\right).\end{aligned} \tag{3.10}$$

From the above, as $\text{g.c.d}(A_2,p)=1$ we can see that the second sum will be

$$p^{2(\ell-\ell_1)-r}\mathbb{I}_{m_1\equiv m_2 \bmod p^{2(\ell-\ell_1)-r}}.$$

Now let $\bar{m}_1 - \bar{m}_2 = c_{m_1,m_2}\, p^{2(\ell-\ell_1)-r}$. Putting this in the first sum of (3.10), it reduces to

$$\sum_{\alpha_1 \bmod p^{r-(\ell-\ell_1)}} e\left(\frac{A_2\, c_{m_1,m_2}\, \alpha_1}{p^{r-(\ell-\ell_1)}}\right) = p^{r-(\ell-\ell_1)}\mathbb{I}_{m_1\equiv m_2 \bmod p^{r-(\ell-\ell_1)}}.$$

Hence (3.10) reduces to

$$p^{\ell-\ell_1}\ \mathbb{I}_{m_1\equiv m_2 \bmod \mathrm{l.c.m}\,(p^{2(\ell-\ell_1)-r},\, p^{r-(\ell-\ell_1)})}.$$

By substituting the bound for the character sum over α from the above and the bound for the integral

$$\mathfrak{I}_1(n,q,m_i,\varepsilon') \ll \frac{p^{2\ell}q^2Q^2}{N^2}N^{\varepsilon},$$

in Θ_{zero}, we see that

$$\Theta_{\mathrm{zero}} \ll N_0\left\{p^{\ell-\ell_1}\sum_{\substack{1\leq q\leq Q\\ (q,p)=1}}\frac{1}{q^3}\mathop{\sum\sum}_{\substack{m_1,m_2\ll M_0\\ q|m_1-m_2\\ (m_1m_2,p)=1}}\frac{p^{2\ell}q^2Q^2}{N^2}\mathbb{I}_{m_1\equiv m_2 \bmod \mathrm{l.c.m}\,(p^{2(\ell-\ell_1)-r},\, p^{r-(\ell-\ell_1)})}\right.$$

$$\left. + \sum_{\substack{1\leq q\leq Q\\ (q,p)=1}}\frac{1}{q^3}\mathop{\sum\sum}_{\substack{m_1,m_2\ll M_0\\ q|m_1-m_2\\ (m_1m_2,p)=1}}\frac{p^{2\ell}q^2Q^2}{N^2}\right\}N^{\varepsilon}$$

$$\ll \frac{p^{r+\frac{5\ell}{2}-3\ell_1}}{N^{3/2}}N^{\varepsilon} + \frac{p^{2r+\frac{3\ell}{2}-2\ell_1}}{N^{5/2}}N^{\varepsilon}$$

$$\ll \frac{p^{r+\frac{5\ell}{2}-3\ell_1}}{N^{3/2}}N^{\varepsilon},$$

where in the second inequality the first term corresponds to the contribution of $m_1 = m_2$, and the second one corresponds to the contribution of $m_1 \neq m_2$, and in the last inequality we have used the assumption that $N \geq p^{r-(\ell-\ell_1)}$. This concludes the proof of the lemma. □

Let $T_0(\varepsilon', \ell_1, N)$ and $S_{f,\chi}(N; r_1 = 0, \Theta_{\text{zero}})$ denote the contribution of Θ_{zero} to $T(\varepsilon', \ell_1, N)$ and $S_{f,\chi}(N; r_1 = 0)$ respectively. Then we have that

$$T_0(\varepsilon', \ell_1, N) \ll \frac{p^{\frac{r+3\ell-4\ell_1}{2}}}{N^{3/4}} N^{\varepsilon},$$

and consequently, we have that

$$S_{f,\chi}(N; r_1 = 0, \Theta_{\text{zero}}) \ll \sqrt{N} p^{\ell/2} N^{\varepsilon},$$

provided $N \geq p^{r-\ell}$. We record this as the following proposition.

Proposition 3.4.2. *We have*

$$S_{f,\chi}(N; r_1 = 0, \Theta_{zero}) \ll \sqrt{N} p^{\ell/2} N^{\varepsilon},$$

provided $N \geq p^{r-\ell}$.

3.4.2 Non-zero frequency $n \neq 0$:

Assume that $n \neq 0$ and also $(n, pq_1q_2) = 1$ otherwise we can take the gcd out and then proceed with a similar approach and we can see that the resulting term will be dominated by the following case. In this case we have determined that α_2 mod p^ℓ and write m_1, m_2 in terms of q_1, q_2 and n modulo q_1, q_2, respectively using the congruences. Indeed, we have

$$\alpha_2 \equiv q_1 \overline{(\bar{\alpha}_1 q_2 + n)} \bmod p^{\ell-\ell_1},$$

and

$$m_1 \equiv -\bar{n} p^{r-(\ell-\ell_1)} q_2 \bmod q_1, \quad m_2 \equiv \bar{n} p^{r-(\ell-\ell_1)} q_1 \bmod q_2.$$

By writing $m_1 = -\bar{n}p^{r-(\ell-\ell_1)}q_2 + r_1q_1$ and $m_2 = \bar{n}p^{r-(\ell-\ell_1)}q_1 + r_2q_2$, we see that

$$\Theta_{\text{non-zero}} = N_0 \sum_{\substack{1\le q_1,q_2\le Q \\ (q_1q_2,p)=1}}\sum \frac{\chi\left(q_1q_2\right)}{q_1^{3/2}q_2^{3/2}}$$

$$\times \sum_{0<|n|\le \frac{q_1q_2p^{\ell-\ell_1}}{N_0}N^{\varepsilon}} \sum_{\substack{|r_1|\le \frac{p^r}{N}N^{\varepsilon} \\ |r_2|\le \frac{p^r}{N}N^{\varepsilon}}}\sum \mathcal{C}(r_1,r_2,q_1,q_2,n)\, \mathfrak{I}_1(n,q_i,m_i,\varepsilon'),$$

where $\mathcal{C}(r_1,r_2,q_1,q_2,n)$ is given by

$$\sum_{\alpha \bmod p^{(\ell-\ell_1)}}^{\star} \overline{\chi}\left(-\bar{n}p^{r-(\ell-\ell_1)}q_2 + r_1q_1 - \alpha p^{r-(\ell-\ell_1)}\right)$$

$$\times\, \chi\left(\bar{n}p^{r-(\ell-\ell_1)}q_1 + r_2q_2 - q_1\overline{(\bar{\alpha}q_2+n)}p^{r-(\ell-\ell_1)}\right).$$

3.4.3 Evaluation of the sum over α

The α sum is given by

$$\mathcal{C}(r_1,r_2,q_1,q_2,n) = \sum_{\alpha \bmod p^{(\ell-\ell_1)}}^{\star} \overline{\chi}\left(r_1q_1 + \left(-\alpha-\bar{n}q_2\right)p^{r-(\ell-\ell_1)}\right)$$

$$\times\, \chi\left(r_2q_2 + \left(-q_1\overline{(\bar{\alpha}q_2+n)} + \bar{n}q_1\right)p^{r-(\ell-\ell_1)}\right).$$

Note that

$$\chi(1+zp^{r-(\ell-\ell_1)}) = e\left(\frac{-A_1p^{2r-2(\ell-\ell_1)}\,z^2 - A_2p^{r-(\ell-\ell_1)}\,z}{p^r}\right),$$

for some integers A_i's which are coprime to p, as our choice of ℓ satisfies the condition $r-\ell \geq r/3$ which is same as $\ell \leq 2r/3$. Thus, the character sum $\mathcal{C}(r_1, r_2, q_1, q_2, n)$ is same as

$$\overline{\chi}(r_1q_1)\,\chi(r_2q_2)\; e\left(\frac{A_1\overline{n}\,(\overline{r_1q_1}\,q_2+\overline{r_2q_2}\,q_1)+(-A_2\bar{n}^2\bar{q}_1^2\bar{r}_1^2q_2^2+A_2\bar{n}^2\bar{r}_2^2\bar{q}_2^2q_1^2)\,p^{r-(\ell-\ell_1)}}{p^{\ell-\ell_1}}\right)$$

$$\times \sideset{}{^\star}\sum_{\substack{\alpha \bmod p^{\ell-\ell_1}\\ \bar{\alpha}q_2+n\not\equiv 0 \bmod p^{\ell-\ell_1}}} e\left(\frac{\left(A_1\overline{r_1q_1}-2A_2\bar{n}\bar{r}_1^2\bar{q}_1^2q_2p^{r-(\ell-\ell_1)}\right)\alpha}{p^{\ell-\ell_1}}\right)$$

$$\times\, e\left(\frac{-\left(A_1\overline{r_2q_2}q_1+2A_2\bar{n}\bar{r}_2^2\bar{q}_2^2q_1^2p^{r-(\ell-\ell_1)}\right)\overline{(\bar{\alpha}q_2+n)}}{p^{\ell-\ell_1}}\right)$$

$$\times\, e\left(\frac{-A_2\bar{r}_1^2\bar{q}_1^2p^{r-(\ell-\ell_1)}\alpha^2+A_2\bar{r}_2^2\bar{q}_2^2q_1^2p^{r-(\ell-\ell_1)}\overline{(\bar{\alpha}q_2+n)}^2}{p^{\ell-\ell_1}}\right).$$

The above α sum reduces to

$$\sideset{}{^\star}\sum_{\substack{\alpha \bmod p^{\ell-\ell_1}\\ \alpha+1\not\equiv 0 \bmod p^{\ell-\ell_1}}} e\left(\frac{\left(A_1\bar{n}\overline{r_1q_1}-2A_2\bar{n}^2\bar{r}_1^2\bar{q}_1^2q_2p^{r-(\ell-\ell_1)}\right)\bar{\alpha}}{p^{\ell-\ell_1}}\right)$$

$$\times\, e\left(\frac{-\left(A_1\bar{n}\overline{r_2q_2}q_1+2A_2\bar{n}^2\bar{r}_2^2\bar{q}_2^2q_1^2p^{r-(\ell-\ell_1)}\right)\overline{(\alpha+1)}}{p^{\ell-\ell_1}}\right)$$

$$\times\, e\left(\frac{-A_2\bar{n}^2\bar{r}_1^2\bar{q}_1^2p^{r-(\ell-\ell_1)}\bar{\alpha}^2+A_2\bar{n}^2\bar{r}_2^2\bar{q}_2^2q_1^2p^{r-(\ell-\ell_1)}\overline{(\alpha+1)}^2}{p^{\ell-\ell_1}}\right).$$

We assume that $(\ell-\ell_1)$ is an even positive integer to make the exposition simpler and to keep the ideas clear. The odd case also can be treated similarly. We now evaluate the above sum by splitting the α variable. We write

$$\alpha=\alpha_1+\,\alpha_2\,p^{(\ell-\ell_1)/2},\ \text{with}\quad \alpha_1\,(\neq 0,\neq -1) \bmod p^{(\ell-\ell_1)/2},\quad \alpha_2 \bmod p^{(\ell-\ell_1)/2}.$$

Thus the α sum can be written as

$$\sideset{}{^\star}\sum_{\substack{\alpha_1 \bmod p^{(\ell-\ell_1)/2} \\ \alpha_1+1\not\equiv 0 \bmod p^{(\ell-\ell_1)/2}}} e\left(\frac{X_1\,\bar{\alpha_1} + X_2\,\overline{(\alpha_1+1)} + X_3\,\bar{\alpha_1}^2 + X_4\,\overline{(\alpha_1+1)}^2}{p^{\ell-\ell_1}}\right)$$

$$\times \sum_{\alpha_2 \bmod p^{(\ell-\ell_1)/2}} e\left(-\frac{\left(X_1\,\bar{\alpha}_1^2 + X_2\,\overline{(\alpha_1+1)}^2\right)\alpha_2}{p^{(\ell-\ell_1)/2}}\right)$$

$$= p^{(\ell-\ell_1)/2} \sideset{}{^\star}\sum_{\substack{\alpha_1 \bmod p^{(\ell-\ell_1)/2} \\ \alpha_1+1\not\equiv 0 \bmod p^{(\ell-\ell_1)/2} \\ X_1\,\bar{\alpha}_1^2+X_2\,\overline{(\alpha_1+1)}^2\equiv 0 \bmod p^{(\ell-\ell_1)/2}}} e\left(\frac{X_1\,\bar{\alpha_1} + X_2\,\overline{(\alpha_1+1)} + X_3\,\bar{\alpha_1}^2 + X_4\,\overline{(\alpha_1+1)}^2}{p^{\ell-\ell_1}}\right),$$

where

$$X_1 = A_1\bar{n}\overline{r_1q_1} - 2A_2\bar{n}^2\bar{r}_1^2\bar{q}_1^2q_2^2p^{r-(\ell-\ell_1)},$$

$$X_2 = -\left(A_1\bar{n}\overline{r_2q_2}q_1 + 2A_2\bar{n}^2\bar{r}_2^2\bar{q}_2^2q_1^2p^{r-(\ell-\ell_1)}\right),$$

$$X_3 = -A_2\bar{n}^2\bar{r}_1^2\bar{q}_1^2p^{r-(\ell-\ell_1)},$$

$$X_4 = A_2\bar{n}^2\bar{r}_2^2\bar{q}_2^2q_1^2p^{r-(\ell-\ell_1)}.$$

Note that $X_1 \equiv A_1\bar{n}\overline{r_1q_1} \bmod p^{(\ell-\ell_1)/2}$, and $X_2 \equiv -A_1\bar{n}\overline{r_2q_2}q_1 \bmod p^{(\ell-\ell_1)/2}$ as $r-(\ell-\ell_1) \geq$

$(\ell-\ell_1)/2$. Thus this α term inside the above sum is given by

$$p^{(\ell-\ell_1)/2}\, e\left(\frac{\left(A_1\bar{n}\overline{r_1 q_1} - 2A_2\bar{n}^2\bar{r}_1^2\bar{q}_1^2 q_2^2 p^{r-(\ell-\ell_1)}\right)\left(\left(r_2\bar{r}_1\right)^{1/2}(q_1\bar{q}_2) - 1\right)}{p^{\ell-\ell_1}}\right)$$

$$\times\, e\left(-\frac{\left(A_1\bar{n}\overline{r_2 q_2} q_1 + 2A_2\bar{n}^2\bar{r}_2^2\bar{q}_2^2 q_1^2 p^{r-(\ell-\ell_1)}\right)\left(1+\overline{\left(r_2\bar{r}_1\right)^{1/2}(q_1\bar{q}_2) - 1}\right)}{p^{\ell-\ell_1}}\right)$$

$$\times\, e\left(\frac{-A_2\bar{n}^2\bar{r}_1^2\bar{q}_1^2 q_2^2 p^{r-(\ell-\ell_1)}\left(\left(r_2\bar{r}_1\right)^{1/2}(q_1\bar{q}_2) - 1\right)^2}{p^{\ell-\ell_1}}\right)$$

$$\times\, e\left(\frac{A_2\bar{n}^2\bar{r}_2^2 q_1^2\bar{q}_2^2 p^{r-(\ell-\ell_1)}\left(1+\overline{\left(r_2\bar{r}_1\right)^{1/2}(q_1\bar{q}_2) - 1}\right)^2}{p^{\ell-\ell_1}}\right),$$

if $r_2\bar{r}_1 \equiv \square \bmod p^{(\ell-\ell_1)/2}$, otherwise the α sum becomes zero. Note that $r_2\bar{r}_1$ is square modulo $p^{(\ell-\ell_1)/2}$ if and only if $r_2\bar{r}_1$ is square modulo p. Any m modulo p be such that $r_2\bar{r}_1 \equiv m^2 \bmod p$ can be uniquely extended to modulo $p^{(\ell-\ell_1)/2}$ with the property that

$r_2\bar{r}_1 \equiv m^2 \bmod p^{(\ell-\ell_1)/2}$, by Hensel's lemma.

Therefore, we have

$$\begin{aligned}
\mathcal{C}(r_1, r_2, q_1, q_2, n) &= p^{(\ell-\ell_1)/2}\ \mathbb{I}_{r_2\bar{r}_1 \equiv \square \bmod p} \quad \overline{\chi}(r_1 q_1)\,\chi(r_2 q_2) \\
&\times e\left(\frac{A_1\bar{n}\left(\overline{r_1 q_1}\, q_2 + \overline{r_2 q_2}\, q_1\right)}{p^{\ell-\ell_1}}\right) e\left(\frac{\left(-A_2\bar{n}^2\bar{q}_1^2\bar{r}_1^2 q_2^2 + A_2\bar{n}^2\bar{r}_2^2\bar{q}_2^2 q_1^2\right) p^{r-(\ell-\ell_1)}}{p^{\ell-\ell_1}}\right) \\
&\times e\left(\frac{\left(A_1\bar{n}\overline{r_1 q_1} - 2A_2\bar{n}^2\bar{r}_1^2\bar{q}_1^2 q_2^2 p^{r-(\ell-\ell_1)}\right)\left(\left(r_2\bar{r}_1\right)^{1/2}\left(q_1\bar{q}_2\right) - 1\right)}{p^{\ell-\ell_1}}\right) \\
&\times e\left(-\frac{\left(A_1\bar{n}\overline{r_2 q_2} q_1 + 2A_2\bar{n}^2\bar{r}_2^2\bar{q}_2^2 q_1^2 p^{r-(\ell-\ell_1)}\right)\left(1 + \overline{\left(r_2\bar{r}_1\right)^{1/2}\left(q_1\bar{q}_2\right) - 1}\right)}{p^{\ell-\ell_1}}\right) \\
&\times e\left(\frac{-A_2\bar{r}_1^2\bar{n}^2\bar{q}_1^2 q_2^2 p^{r-(\ell-\ell_1)}\left(\left(r_2\bar{r}_1\right)^{1/2}\left(q_1\bar{q}_2\right) - 1\right)^2}{p^{\ell-\ell_1}}\right) \\
&\times e\left(\frac{A_2\bar{r}_2^2\bar{n}^2 q_1^2\bar{q}_2^2 p^{r-(\ell-\ell_1)}\left(1 + \overline{\left(r_2\bar{r}_1\right)^{1/2}\left(q_1\bar{q}_2\right) - 1}\right)^2}{p^{\ell-\ell_1}}\right).
\end{aligned}$$

3.4.4 The sum over r_2

We now consider the r_2 sum which is given by

$$\Delta(n, q_i, r_1, N, \varepsilon') = \sum_{\substack{|r_2| \leq \frac{p^r}{N} \\ r_2\bar{r}_1 \equiv \square \bmod p}} \chi(r_2)\, e\left(\frac{g(r_2)}{p^{\ell-\ell_1}}\right) \mathfrak{I}_1(n, q_i, r_1, r_2, \varepsilon'),$$

where

$$
\begin{aligned}
g(r_2) =& A_1\bar{n}\bar{q}_2q_1\bar{r}_2 + A_2\bar{n}^2\bar{q}_2^2q_1^2p^{r-(\ell-\ell_1)}\bar{r}_2^2 \\
&+ \left(A_1\bar{n}\overline{r_1q_1} - 2A_2\bar{n}^2\bar{r}_1^2\bar{q}_1^2q_2p^{r-(\ell-\ell_1)}\right)(r_2\bar{r}_1)^{1/2} \\
&- \left(A_1\bar{n}\overline{r_2q_2}q_1 + 2A_2\bar{n}^2\bar{r}_2^2\bar{q}_2^2q_1^2p^{r-(\ell-\ell_1)}\right)\left(1 - \overline{(r_2\bar{r}_1)^{1/2}}\right) \\
&- A_2\bar{r}_1^2\bar{q}_1^2p^{r-(\ell-\ell_1)}\left(r_2\bar{r}_1 - 2(r_2\bar{r}_1)^{1/2}\right) - A_2\bar{r}_1^2\bar{q}_1^2p^{r-(\ell-\ell_1)}\left(r_1\bar{r}_2 - 2(r_2\bar{r}_1)^{1/2}\right).
\end{aligned}
$$

By taking dyadic sub-division we see that this sum is at most

$$
\Delta(n, q_i, r_1, N, \varepsilon') \ll N^{\varepsilon} \sup_{R \leq \frac{p^r}{N}} |T(R)|,
$$

where

$$
T(R) = \sum_{\substack{R \leq r_2 \leq 2R \\ r_2\bar{r}_1 \equiv \square \bmod p}} \chi(r_2)\, e\left(\frac{g(r_2)}{p^{\ell-\ell_1}}\right) \mathfrak{I}_1(n, q_i, r_1, r_2, \varepsilon').
$$

Remark 3.4.3. Note that we have

$$
\frac{\partial}{\partial r_2}\mathfrak{I}_1(n, q_i, r_1, r_2, \varepsilon') \ll \frac{N}{p^r}\,\frac{p^{2\ell}q^2Q^2}{N^2}N^{\varepsilon},
$$

so we can ignore the integral $\mathfrak{I}_1(n, q_i, r_1, r_2, \varepsilon')$ using partial summation, while estimating $T(R)$.

In the following lemma, we give an estimate for $T(R)$.

Lemma 3.4.4. *We have*

$$
T(R) \ll p^{14/15}\, p^{r/30}\, R^{1/5} N^{\varepsilon}.
$$

Proof. Let κ be a large positive integer but fixed. Then we have that

$$\begin{aligned} T(R) &= \frac{1}{2} \sum_{1 \le m \le p^\kappa} \sum_{\substack{R \le r_2 \le 2R \\ r_2 \equiv r_1\, m^2 \bmod p^\kappa}} \chi\left(r_2\right) \, e\left(\frac{g(r_2)}{p^{\ell-\ell_1}}\right) \\ &= \frac{1}{2} \sum_{1 \le m \le p^\kappa} \chi(r_1 m^2) \sum_{\frac{R - r_1 m^2}{p^\kappa} \le t \le \frac{2R - r_1 m^2}{p^\kappa}} \chi\left(1 + \bar{r}_1 \bar{m}^2 p^\kappa t\right) \, e\left(\frac{g(r_1 m^2 + t p^\kappa)}{p^{\ell-\ell_1}}\right) \\ &= \frac{1}{2} \sum_{1 \le m \le p^\kappa} \chi(r_1 m^2) \sum_{\frac{R - r_1 m^2}{p^\kappa} \le t \le \frac{2R - r_1 m^2}{p^\kappa}} e\left(\frac{f(t)}{p^r}\right), \end{aligned}$$

where $f(t) = a_0 \log_p \left(1 + p^\kappa r_1 \bar{m}^2 t\right) + p^{r-(\ell-\ell_1)} g(r_1 m^2 + t p^\kappa)$. Note that

$$f'(t) = p^\kappa \, a_0 r_1 \bar{m}^2 \left(1 + p^\kappa r_1 \bar{m}^2 t\right)^{-1} + p^{r-(\ell-\ell_1)} \, h(t),$$

where $h(t) = p^\kappa \, g'(r_1 m^2 + p^\kappa t)$. Our phase function f is in the class $\mathbf{F}(\kappa, 1, \kappa, \lambda, u)$ for arbitrarily large positive λ and positive integer u but fixed, from the Section 1.7.7, so that we can apply p-adic exponent pair $(1/30, 13/15)$, when $p \neq 2, 3$, to the above inner sum to get

$$\begin{aligned} T(R) &\ll_p \left(\frac{p^{r-2\kappa}}{R}\right)^{1/30} R^{13/15} N^\varepsilon \\ &\ll_p \, p^{\frac{r}{30}} \, R^{1/5} N^\varepsilon, \end{aligned}$$

where the absolute constant depends on prime p. This concludes the lemma. ∎

As a consequence of the above lemma we have

$$\Delta(n, q_i, r_1, N, \varepsilon') \ll p^{\frac{13r}{15}} \, N^{-5/6} N^\varepsilon. \tag{3.11}$$

The following lemma gives an estimate for $\Theta_{\text{non-zero}}$.

Lemma 3.4.5. *We have*

$$\Theta_{\textit{non-zero}} \ll \frac{p^{\frac{28r}{15}+\ell-\frac{3\ell_1}{2}}}{N^{4/3}} N^{\varepsilon}.$$

Proof. Observe that

$$\Theta_{\text{non-zero}} \ll \frac{p^{r+\ell-\frac{3\ell_1}{2}}}{\sqrt{N}} \sup_{\substack{q_i \leq Q \\ |r_1| \leq \frac{p^r}{N} \\ 0<|n| \leq \frac{q_1 q_2 p^{\ell-\ell_1}}{N_0}}} |\Delta(n, q_i, r_1, N, \varepsilon')| \, N^{\varepsilon}.$$

By substituting the bound for $\Delta(n, q_i, r_1, N, \varepsilon')$ from the equation (3.11) in the above inequality we get

$$\Theta_{\text{non-zero}} \ll \frac{p^{\frac{28r}{15}+\ell-\frac{3\ell_1}{2}}}{N^{4/3}} N^{\varepsilon}.$$

□

Let $T_{\neq 0}(\varepsilon', \ell_1, N)$ and $S_{f,\chi}(N; r_1 = 0, \Theta_{\text{non-zero}})$ denote the contribution of $\Theta_{\text{non-zero}}$ to $T(\varepsilon', \ell_1, N)$ and $S_{f,\chi}(N; r_1 = 0)$ respectively. Then we have

$$T_{\neq 0}(\varepsilon', \ell_1, N) \ll \frac{p^{14r/15} p^{\frac{3\ell - 5\ell_1}{4}}}{N^{2/3}} N^{\varepsilon},$$

and consequently, we have

$$S_{f,\chi}(N;\, r_1 = 0, \Theta_{\text{non-zero}}) \ll \frac{p^{13r/30} N^{7/12}}{p^{\ell/4}} \, N^{\varepsilon}.$$

Therefore we have the following proposition.

Proposition 3.4.6. *We have*

$$S_{f,\chi}(N;\, r_1 = 0,\, \Theta_{\textit{non-zero}}) \ll \frac{p^{13r/30} N^{7/12}}{p^{\ell/4}}\, N^{\varepsilon}.$$

3.5 Conclusion

In this section, we complete the proof of our main Theorem 3.0.1. Since the zero frequency and the non-zero frequency cases together give $S_{f,\chi}(N; r_1 = 0)$, we have

$$S_{f,\chi}(N; r_1 = 0) = S_{f,\chi}(N; r_1 = 0, \Theta_{\text{zero}}) + S_{f,\chi}(N; r_1 = 0, \Theta_{\text{non-zero}}).$$

From the Propositions 3.4.2 and 3.4.6, we infer that

$$S_{f,\chi}(N; r_1 = 0) \ll \left(\sqrt{N} p^{\ell/2} + \frac{p^{13r/30} N^{7/12}}{p^{\ell/4}} \right) N^{\varepsilon},$$

provided $\max\{p^{\ell}, p^{r-\ell}\} \leq N$ and $\ell \leq 2r/3$. By equating two terms in parenthesis we get the value of ℓ which is given by

$$\ell = \left[\frac{26r}{45} + \frac{1}{9} \log_p N \right].$$

Note that this choice of ℓ satisfies above conditions and $\ell \leq 2r/3$ provided that $N \geq p^{13r/20}$ and $N \leq p^{4r/5}$ respectively. Therefore we conclude that

$$S_{f,\chi}(N) \ll N^{\frac{5}{9}}\, p^{\frac{13r}{45}}\, N^{\varepsilon},$$

provided $p^{13r/20+\epsilon} \ll_{\varepsilon} N \leq p^{4r/5}$ and absolute value may depend on the prime p. This concludes the proof of the Theorem 3.0.1.

Chapter 4

Subconvexity for $\mathrm{GL}(1)$ twists of Rankin-Selberg L-functions

Let f and g be two Hecke-Maass or holomorphic primitive cusp forms for $\mathrm{SL}(2,\mathbb{Z})$ and χ be a primitive Dirichlet character of modulus p, an odd prime. In this chapter, we will prove a subconvex bound for $L(s, f \otimes g \otimes \chi)$. Let us state our result (see [21]) here again :

Theorem 4.0.1. *Let f and g be two holomorphic or Hecke-Maass primitive cusp forms for* $\mathrm{SL}(2,\mathbb{Z})$ *and let χ be a primitive Dirichlet character of modulus p, an odd prime. Then we have*

$$L(\frac{1}{2}, f \otimes g \otimes \chi) \ll_{f,g,\varepsilon} p^{\frac{22}{23}+\varepsilon},$$

for any $\varepsilon > 0$.

Remark 4.0.2. The main ingredient of this chapter is to use Jutila's circle method to reduce the original problem to a $\mathrm{GL}(2) \times \mathrm{GL}(2)$ shifted convolution sum problem and then appeal to some available bounds for those shifted sums. An anonymous referee has pointed out that a similar method was used in Raju's PhD thesis (see [56]) in the case when $n \mapsto n^{it}$ and the method can be generalised for Dirichlet characters.

Remark 4.0.3. If we assume the Remark 1.7.4 then we would have

$$L(\frac{1}{2}, f \otimes g \otimes \chi) \ll_{f,g,\varepsilon} p^{\frac{19}{20}+\frac{202}{100}\theta+\varepsilon}.$$

However, for our case, we only need that $\theta < \frac{1}{5}$, which is consistent with the current record for θ as discussed in the Remark 1.7.4.

Notation

In this chapter, by $A \ll B$ we mean that $|A| \leq C|B|$ for some absolute constant $C > 0$, depending on f, ε only and the notation '$X \asymp Y$' will mean that $Yp^{-\varepsilon} \leq X \leq Yp^{\varepsilon}$.

4.1 Sketch of the proof

At first, we use the approximate functional equation to truncate the sum and then we use dyadic subdivision to arrive at (see [23, equation 3.8])

$$L(\frac{1}{2}, f \otimes g \otimes \chi) \ll_{\varepsilon,A} \sup_{N \ll p^{2+\varepsilon}} \frac{|S(N)|}{\sqrt{N}} + \mathrm{O}\left(p^{-A}\right), \tag{4.1}$$

for arbitrary large $A > 0$, where

$$S(N) = \sum_{n \sim N} \lambda_f(n) \lambda_g(n) \chi(n).$$

Here in the sketch, we will consider the case when $N = p^2$, at the boundary and also suppress the weight function for notational simplicity and we are using the method of Munshi (see [50]) to get the following sum

$$\mathbf{S} = \sum\sum_{n,m \sim p^2} \lambda_f(n) \lambda_g(m) \chi(m) \delta_{n,m},$$

where $\delta_{n,m}$ denotes the Kronecker δ-symbol. Here to get an inbuilt bilinear structure in the circle method itself, we need to use a more flexible version of the circle method - the one investigated by Jutila (see [37], [38]). This version comes with a satisfactory error term, as we will find out, as long as we allow the moduli to be slightly larger than $\sqrt{N}$. Upto an admissible error, we see that $\mathbf{S}$ is given by

$$\mathbf{S} = \sum\sum_{n,m \sim N} \lambda_f(n) \lambda_g(m) \chi(m) \int_{\mathbb{R}} \tilde{\mathbb{I}}(\alpha) e((n-m)\alpha) \mathrm{d}\alpha,$$

where $\widetilde{\mathbb{I}}(\alpha) := \frac{1}{2\delta L} \sum_{q \in \Phi} \sum^{\star}_{d (\bmod q)} \mathbb{I}_{d/q}(\alpha)$ and $\mathbb{I}_{d/q}$ is the indicator function of the interval $[\frac{d}{q} - \delta, \frac{d}{q} + \delta]$, $Q := N^{1/2+\varepsilon}$ and $L \asymp Q^{2-\varepsilon}$.

Trivial bound at this stage yields $N^{2+\varepsilon}$ and we need to establish the bound $N^{1-\theta}$ for some $\theta > 0$, i.e., roughly speaking we need to save $N+$ something. Observe that by our choice of Q, there is no analytic oscillation in the weight function $e((n-m)\alpha)$. Hence these weights can be dropped in our sketch. At first using the GL(2) Voronoi summation formula on the n sum we get that

$$\sum_{n\sim N}\lambda_f(n)e\left(\frac{na}{q}\right)\approx\frac{N}{q}\sum_{n\sim\frac{Q^2}{N}}\lambda_f(n)e\left(\frac{-n\bar{a}}{q}\right),$$

where q is of size $Q\approx\sqrt{N}p^{\eta/2}$. The left-hand side is trivially bounded by N, whereas the right-hand side is trivially bounded by Q. Hence we have "saved" $\frac{N}{Q}$.

Again applying the GL(2) Voronoi summation formula 1.7.1 on the m sum we will save $\frac{N}{pQ}$. Also as the a mod q sum will be a Ramanujan sum we will "save" Q in the a-sum. Up to this step, our total savings becomes

$$\frac{N}{Q}\times\frac{N}{pQ}\times Q=\frac{N^2}{pQ}=\frac{N}{p^{\eta/2}},$$

so we have already reached the boundary and any savings will work. After this, we used the Cauchy-Schwarz inequality and then we opened the absolute value squares after that, we used the shifted convolution sum result done in [6] and also analysed a short twisted GL(2) character sum where the length of the sum is greater than the size of the conductor. Hence we got our savings, giving our subconvexity result 4.0.1.

4.2 Setting-up the circle method :

Let us apply the circle method directly to the smooth sum

$$S_{f,g,\chi}(N)=\sum_{n\in\mathbb{Z}}\lambda_f(n)\lambda_g(n)\chi(n)h\left(\frac{n}{N}\right),$$

where the function h is smooth, supported in $[1,2]$ with $h^{(j)}(x)\ll_j 1$. Now we will approximate the above sum $S_{f,g,\chi}(N)$ using Jutila's circle method (see [37], [38]) by the following sum :

$$\tilde{S}(N)=\frac{1}{L}\sum_{q\in\Phi}\ \sideset{}{^*}\sum_{a \bmod q}\ \sum\sum_{n,m\in\mathbb{Z}}\lambda_f(n)\lambda_g(m)\chi(m)e\left(\frac{a(n-m)}{q}\right)F(n,m),$$

where

$$F(x,y) = h\left(\frac{x}{N}\right) h^*\left(\frac{y}{N}\right) \frac{1}{2\delta} \int_{-\delta}^{\delta} e(\alpha(n-m))\, \mathrm{d}\alpha.$$

Here we consider another smooth function h^* having compact support in $(0,\infty)$, with $h_2(x) = 1$ whenever x comes from the support of h. Also choosing $\delta = N^{-1}$ we have

$$\frac{\partial^{i+j}}{\partial^i x \partial^j y} F(x,y) \ll_{i,j} \frac{1}{N^{i+j}}.$$

Then we have the following lemma :

Lemma 4.2.1. *Let $\Phi \subset [1,Q]$, with*

$$L = \sum_{q\in\Phi} \phi(q) \gg Q^{2-\varepsilon},$$

and $\delta \gg \frac{1}{N}$. So we arrive at

$$S_{f,g,\chi}(N) = \tilde{S}(N) + \mathrm{O}_{f,\varepsilon}\left(N\sqrt{\frac{Q^2}{\delta L^2}}\right). \tag{4.2}$$

Proof. Consider

$$G(x) = \sum\sum_{n,m\in\mathbb{Z}} \lambda_f(n)\lambda_g(m)\chi(m) h\Big(\frac{n}{N}\Big) h^*\Big(\frac{m}{N}\Big) e\left(x(n-m)\right)).$$

One can see that $S_{f,g,\chi}(N) = \int_0^1 G(x)\mathrm{d}x$ and $\tilde{S}(N) = \int_0^1 \tilde{I}_{\Phi,\delta}(x) G(x)\mathrm{d}x$. Hence

$$\Big|S_{f,g,\chi}(N) - \tilde{S}(N)\Big| \leq \int_0^1 \Big|1 - \tilde{\mathbb{I}}_{\Phi,\delta}(x)\Big| \Big|\sum_{n\in\mathbf{Z}} \lambda_f(n) e(xn) h\Big(\frac{n}{N}\Big)\Big| \Big|\sum_{m\in\mathbf{Z}} \lambda_g(m)\chi(m) e(xm) h^*\Big(\frac{m}{N}\Big)\Big| \mathrm{d}x.$$

Here we have used the point-wise bound given by $\sum_{n\in\mathbf{Z}}\lambda_f(n)e(xn)h\left(\frac{n}{N}\right) \ll_{f,\varepsilon} N^{\frac{1}{2}+\varepsilon}$ for the middle sum. Using the Cauchy-Schwarz inequality we get

$$\left|S_{f,g,\chi}(N)-\tilde{S}(N)\right| \ll_{f,\epsilon} N^{\frac{1}{2}+\epsilon}\left[\int_0^1 \left|1-\tilde{\mathbb{I}}_{\Phi,\delta}(x)\right|^2 \mathrm{d}x\right]^{1/2}\left[\int_0^1 \left|\sum_{m\in\mathbf{Z}}\lambda_g(m)\chi(m)e(xm)h^*\left(\frac{m}{N}\right)\right|^2 \mathrm{d}x\right]^{1/2}.$$

For the last sum, we open the absolute value square and execute the integral. So we are left with the diagonal only, which has size N. For the other sum, we use the Lemma 1.10.1. It follows that

$$\left|S_{f,g,\chi}(N)-\tilde{S}(N)\right| \ll_{f,\varepsilon} N\sqrt{\frac{Q^2}{\delta L^2}}.$$

□

As for subconvexity, we need to save N with something more, i.e., we must have $\delta L^2 > Q^2 p^\eta$, i.e., $\frac{1}{N} \gg \delta \gg \frac{p^\eta}{Q^2}$, i.e., $Q \gg \sqrt{N}p^{\eta/2}$. Here we choose $Q = \sqrt{N}p^{\eta/2+\varepsilon}$ and prime to p. Hence the error term in the Lemma 4.2.1 is bounded by $\mathrm{O}\left(\frac{N}{p^{\eta/2}}\right)$.

4.3 Estimation of $\tilde{S}(N)$

Here we will estimate $\tilde{S}(N)$.

4.3.1 Application of the Voronoi and Poisson summation formulae

At first, we assume that each member of Φ is coprime to p, the modulus of the character χ. Now let us consider

$$\tilde{S}_x(N) = \frac{1}{L}\sum_{q\in\Phi}\ \sideset{}{^*}\sum_{a \bmod q} S(a,q,x,f)T(a,q,x,g), \tag{4.3}$$

where

$$S(a,q,x,f) := \sum_{n=1}^{\infty}\lambda_f(n)e_q(an)e(xn)h\left(\frac{n}{N}\right),$$

and

$$T(a,q,x,g) := \sum_{m=1}^{\infty} \lambda_g(m)\chi(m)e_q(-am)e(-xm)h^*\left(\frac{m}{N}\right).$$

Hence from the Section 1.10 we have

$$\tilde{S}(N) = \frac{1}{2\delta}\int_{-\delta}^{\delta} \tilde{S}_x(N)\mathrm{d}x. \tag{4.4}$$

Lemma 4.3.1. *We have*

$$S\left(a,q,x,f\right) = \frac{N^{3/4}}{q^{1/2}} \sum_{1\leq n\ll N_0} \frac{\lambda_f(n)}{n^{1/4}}\, e\left(-\frac{\bar{a}n}{q}\right)\mathcal{I}_1(n,x,q) + \mathrm{O}(N^{-2023}), \tag{4.5}$$

where $N_0 := \frac{Q^2}{N}$ *and* $\mathcal{I}_1(n,x,q)$ *is given by*

$$\mathcal{I}_1(n,x,q) := \int_{\mathbb{R}} h(y)\, e\left(Nxy \pm \frac{4\pi}{q}\sqrt{Nny}\right) W_f\left(\frac{4\pi\sqrt{Nny}}{q}\right)\mathrm{d}y,$$

where W_f *is a smooth nice function.*

Proof. Applying the Voronoi summation formula 1.7.1 to the n-sum of the equation (4.3), we have

$$\sum_{n\in\mathbb{Z}} \lambda_f(n)\, e\left(\frac{an}{q}\right) e(nx)\, h\left(\frac{n}{N}\right) = \frac{1}{q}\sum_{n\in\mathbb{Z}} \lambda_f(n)\, e\left(-\frac{\bar{a}n}{q}\right)$$

$$\times \int_{\mathbb{R}} h\left(\frac{y}{N}\right) e(xy)\, J_{k_f-1}\left(\frac{4\pi\sqrt{ny}}{q}\right)\mathrm{d}y,$$

where J_{k_f-1} is the Bessel function. By changing $y \mapsto Ny$ and using the decomposition 1.7.6,

$$J_{k_f-1}(x) = \frac{W_f(x)}{\sqrt{x}}\, e(x) + \frac{\bar{W_f}(x)}{\sqrt{x}}\, e(-x),$$

where $W_f(x)$ is a nice smooth function, the right-hand side integral becomes

$$N^{3/4}q^{1/2}\int_{\mathbb{R}} h(y)\, e\left(Nxy \pm \frac{4\pi}{q}\sqrt{Nny}\right) W_f\left(\frac{4\pi\sqrt{Nny}}{q}\right) \mathrm{d}y.$$

By repeated integral by parts we see that this integral is negligibly small if $|n| \gg \frac{Q^2 N^{\varepsilon}}{N} =: N_0$. Hence the lemma follows. □

Lemma 4.3.2. *We have*

$$\begin{aligned} T(a,q,x) =& \frac{N^{3/4}}{\tau_{\overline{\chi}}\sqrt{pq}} \sideset{}{^*}\sum_{\beta \bmod p} \overline{\chi}(-\beta) \sum_{1\le m \ll M_0}^{\infty} \frac{\lambda_g(m)}{m^{1/4}}\, e\left(\frac{\bar{c}m}{pq}\right) I_2(q,m,x) \\ &+ \mathrm{O}(N^{-2023}), \end{aligned} \tag{4.6}$$

where $c = ap + \beta q, (c, pq) = 1, M_0 := \frac{(pq)^2}{N}$, *and*

$$I_2(q,m,x) = \int_0^{\infty} h^*(y)\, e\left(Nxy \pm \frac{4\pi}{pq}\sqrt{Nmy}\right) W_g\left(\frac{4\pi\sqrt{Nmy}}{pq}\right) \mathrm{d}y. \tag{4.7}$$

Proof. At first, let us expand $\chi(m)$ in terms of additive characters so that we have

$$\chi(m) = \frac{1}{\tau_{\overline{\chi}}} \sum_{\beta \bmod p} \overline{\chi}(\beta)\, e\left(\frac{\beta m}{p}\right),$$

where $\tau_{\overline{\chi}}$ is the Gauss sum associated to $\overline{\chi}$. Hence the m-sum in (4.3) transforms into

$$\begin{aligned} T :&= T(a,q,x,g) \\ &= \sum_{m=1}^{\infty} \lambda_g(m)\chi(m)\, e\left(\frac{-am}{q}\right) e(-mx) h^*\left(\frac{m}{N}\right) \\ &= \frac{1}{\tau_{\overline{\chi}}} \sum_{\beta \bmod p} \overline{\chi}(\beta) \sum_{m=1}^{\infty} \lambda_g(m)\, e\left(\frac{(\beta q - ap)m}{pq}\right) e(-mx) h^*\left(\frac{m}{N}\right). \end{aligned} \tag{4.8}$$

Now let us consider $c = ap - \beta q$ so that $(c, pq) = 1$. Now applying the Voronoi summation formula 1.7.1 to the sum S_2 with the modulus pq and $g(m) = e(-mx)h^*\left(\frac{m}{N}\right)$, we arrive at

$$T = \frac{1}{\tau_{\overline{\chi}}\, pq} \sum_{\beta \bmod p}^{*} \overline{\chi}(\beta) \sum_{m=1}^{\infty} \lambda_g(m)\, e\left(\frac{\bar{c}m}{pq}\right) u(m), \tag{4.9}$$

where

$$u(m) = \int_0^\infty h^*\left(\frac{y}{N}\right) e\left(-yx\right) J_{k_g-1}\left(\frac{4\pi\sqrt{my}}{pq}\right) \mathrm{d}y. \tag{4.10}$$

Now by changing the variables $y \mapsto Ny$ and using the decomposition 1.7.6,

$$J_{k_g-1}(x) = \frac{W_g(x)}{\sqrt{x}} e(x) + \frac{\bar{W}_g(x)}{\sqrt{x}} e(-x),$$

where $W_f(x)$ is a nice smooth function, we get that,

$$\begin{aligned} u(m) &= \frac{N^{\frac{3}{4}}\sqrt{p}\sqrt{q}}{m^{1/4}} \int_0^\infty h^*(y)\, e\left(Nxy \pm \frac{4\pi}{pq}\sqrt{Nmy}\right) W_g\left(\frac{4\pi\sqrt{Nmy}}{pq}\right) \mathrm{d}y \\ &:= \frac{N^{\frac{3}{4}}\sqrt{pq}}{m^{1/4}}\, I_2(q, m, x). \end{aligned} \tag{4.11}$$

Here note that by abuse of notation we are using the same notation for the weight functions (the weight function h^* appearing above is different from the one we started with, which satisfies $h^{*(j)}(x) \ll_{j,k_f} \frac{1}{x^j}$ and also $\mathrm{supp}(h^*) \subset [1/2, 5/2]$). Now using integration by parts repeatedly, we have

$$I_2(q, m, x) \ll_j 1.$$

Hence the integral $I_2(q, m, x)$ is notably small if

$$m \gg \frac{(pq)^2}{N} := M_0.$$

Now plugging in the expression (4.11) of $u(m)$ into (4.9), we get our lemma.

□

Hence we have

Lemma 4.3.3.

$$\tilde{S}_x(N) = \frac{N^{3/2}}{\tau_{\overline{\chi}}\, L\sqrt{p}} \sum_{q\in\Phi} \frac{1}{q} \sum_{1\leq n\ll N_0} \frac{\lambda_f(n)}{n^{1/4}} \mathcal{I}_1(n,x,q) \sum_{1\leq m\ll M_0} \frac{\lambda_g(m)}{m^{1/4}} I_2(q,m,x)\, \mathcal{C}, \tag{4.12}$$

where the character sum $\mathcal{C}$ *is given by*

$$\sideset{}{^*}\sum_{a \bmod q} \sideset{}{^*}\sum_{\beta \bmod p} \overline{\chi}(\beta)\, e\left(\frac{\overline{(c)}m}{pq}\right) e\left(-\frac{\bar{a}n}{q}\right). \tag{4.13}$$

4.3.2 Evaluation of the character sum

Now let us further simplify the character sum so that we get

Lemma 4.3.4.

$$\mathcal{C} = \tau_{\overline{\chi}}\, \overline{\chi}(m)\, \overline{\chi}(q^2) \sum_{\substack{d|q \\ m\equiv p^2 n \bmod d}} d\mu\left(\frac{q}{d}\right) \quad \textit{if } (p,q)=1. \tag{4.14}$$

Proof. Here $(q,p)=1$. So we have

$$\mathcal{C} = \sideset{}{^*}\sum_{a \bmod q} \sideset{}{^*}\sum_{\beta \bmod p} \overline{\chi}(\beta) e\left(\frac{\overline{(-\beta q + ap)}m}{pq}\right) e\left(-\frac{\bar{a}n}{q}\right)$$

$$= \sideset{}{^*}\sum_{\beta \bmod p} \overline{\chi}(\beta) \sideset{}{^*}\sum_{a \bmod q} e\left(\frac{\overline{ap^2}}{q}m\right) e\left(-\frac{\overline{\beta q^2}}{p}m\right) e\left(-\frac{\bar{a}n}{q}\right)$$

$$= \sideset{}{^*}\sum_{a \bmod q} e\left(-\frac{\overline{ap^2}}{q}m\right) e\left(-\frac{\bar{a}n}{q}\right) \sideset{}{^*}\sum_{\beta \bmod p} \overline{\chi}(\beta) e\left(-\frac{\overline{\beta q^2}}{p}m\right)$$

$$= \tau_{\overline{\chi}}\, \overline{\chi}(m)\, \overline{\chi}(q^2) \sideset{}{^*}\sum_{a \bmod q} e\left(\frac{\overline{ap^2}m - \bar{a}n}{q}\right)$$

$$= \tau_{\overline{\chi}}\, \overline{\chi}(m)\, \overline{\chi}(q^2) \sum_{\substack{d|q \\ m \equiv p^2 n \bmod d}} d\mu\left(\frac{q}{d}\right).$$

□

4.4 Further estimation

In this section, first, we apply the Cauchy-Schwarz inequality to the n-sum in (4.15) to get rid of one GL(2) Fourier coefficients $\lambda_f(n)$, given below:

$$\tilde{S}_x(N) = \frac{N^{3/2}}{L\sqrt{p}} \sum_{q \in \Phi} \frac{\overline{\chi}(q^2)}{q} \sum_{d|q} d$$

$$\times \sum_{\substack{1 \leq n \ll N_0 \\ 1 \leq m \ll M_0 \\ m \equiv p^2 n \bmod d}} \frac{\lambda_f(n)\lambda_g(m)}{(nm)^{1/4}} \chi(m) \mathcal{I}_1(n,x,q) \mathcal{I}_2(q,m,x). \tag{4.15}$$

4.4.1 The Cauchy-Schwarz inequality

At first we choose the set of moduli $\Phi = \Phi_1\Phi_3\Phi_4$, where Φ_i consists of primes q_i's in the dyadic segment $[Q_i, 2Q_i]$ (and coprime to p) for $i = 1, 3, 4$ with $q = q_1q_3q_4$, and $Q_1Q_3Q_4 = Q = \sqrt{N}p^{\eta/2}$ with $Q_1^{1+\varepsilon} \ll Q_3 \ll Q_4^{1-\varepsilon}$. Also, we choose Q_1, Q_3, Q_4 (whose optimal sizes will be determined later) in such a way that the collections Φ_1, Φ_3, Φ_4 are disjoint. Also let $m = p^2n + dr$ so size of r becomes $\frac{M_0 - p^2n}{d} \ll \frac{M_0}{d}$ and call $q_2 = q_3q_4$ with $Q_2 = Q_3Q_4$ and $q_2 \in \Phi_2 = \Phi_3\Phi_4$ so (4.15) reduces to

$$\begin{aligned}\tilde{S}_x(N) &= \frac{N^{3/2}}{L\sqrt{p}} \sum_{q\in\Phi} \frac{\overline{\chi}(q^2)}{q} \sum_{d\mid q_1q_2} d \sum_{1\leq r\ll \frac{M_0}{d}} \chi(dr) \sum_{1\leq n\ll N_0} \frac{\lambda_f(n)\lambda_g(p^2n+dr)}{(n(p^2n+dr))^{1/4}} \\ &\quad\times \mathcal{I}_1(n,x,q)\mathcal{I}_2(q,p^2n+dr,x) \\ &= \frac{N^{3/2}}{L\sqrt{p}} \sum_{q\in\Phi} \frac{\overline{\chi}(q^2)}{q} \sum_{d\mid q_1q_2} d \sum_{1\leq r\ll \frac{M_0}{d}} \chi(dr) \sum_{1\ll n\ll N_0} \left(\lambda_f(n)\lambda_g(p^2n+dr)\right) \\ &\quad\times \left(\frac{\mathcal{I}_1(n,x,q)\mathcal{I}_2(q,p^2n+dr,x)}{(n(p^2n+dr))^{1/4}}\right)\end{aligned} \tag{4.16}$$

Now we have several cases.

Case (4.4.1)**.** Let $d = q$. For this case we have

$$\begin{aligned}\Sigma_q &:= \frac{N^{3/2}}{L\sqrt{p}} \sum_{\substack{q\in\Phi \\ q=q_1q_2}} \frac{q\overline{\chi}(q^2)}{q} \sum_{1\leq r\ll \frac{M_0}{Q}} \chi(qr) \sum_{1\leq n\ll N_0} \left(\lambda_f(n)\lambda_g(p^2n+qr)\right) \\ &\quad\times \left(\frac{\mathcal{I}_1(n,x,q)\mathcal{I}_2(q,p^2n+qr,x)}{(n(p^2n+qr))^{1/4}}\right) \\ &= \frac{N^{3/2}}{L\sqrt{p}} \sum_{q_2\in\Phi_2} \overline{\chi}(q_2) \sum_{1\leq r\ll \frac{M_0}{Q}} \chi(r) \left(\sum_{1\leq n\ll N_0} \frac{\lambda_f(n)}{n^{1/4}}\right) \\ &\quad\times \left(\sum_{q_1\in\Phi_1} \overline{\chi}(q_1)\lambda_g(p^2n+q_1q_2r) \frac{\mathcal{I}_1(n,x,q_1q_2)\mathcal{I}_2(q,p^2n+q_1q_2r,x)}{(p^2n+q_1q_2r)^{1/4}}\right).\end{aligned} \tag{4.17}$$

Now applying the Cauchy-Schwarz inequality to (4.17), as the integrals $\mathcal{I}_1$, $\mathcal{I}_2$ do not oscillate, we get that

$$\begin{aligned}
\left|\Sigma_q\right| &\ll \frac{N^{3/2}}{L\sqrt{p}}\left(\sum_{q_2\in\Phi_2}\sum_{1\leq r\ll\frac{M_0}{Q}}\sum_{1\leq n\ll N_0}\frac{|\lambda_f(n)|^2}{n^{1/2}}\right)^{1/2} \\
&\times\left(\sum_{q_2\in\Phi_2}\sum_{1\leq r\ll\frac{M_0}{Q}}\Big|\sum_{q_1\in\Phi_1}\overline{\chi}(q_1)\lambda_g(p^2n+q_1q_2r)\frac{\mathcal{I}_2(q,p^2n+q_1q_2r,x)}{(p^2n+q_1q_2r)^{1/4}}\Big|^2\right)^{1/2} \\
&\ll \frac{N^{3/2}}{L\sqrt{p}}\left(Q_2\times\frac{M_0}{Q}\times N_0^{1/2}\right)^{1/2} \qquad (4.18)\\
&\times\left(\sum_{q_2\in\Phi_2}\sum_{1\leq r\ll\frac{M_0}{Q}}\sum_{1\leq n\ll N_0}\Big|\sum_{q_1\in\Phi_1}\overline{\chi}(q_1)\lambda_g(p^2n+q_1q_2r)\frac{\mathcal{I}_2(q,p^2n+q_1q_2r,x)}{(p^2n+q_1q_2r)^{1/4}}\Big|^2\right)^{1/2} \\
&\ll \frac{N^{3/2}}{L\sqrt{p}}\left(\frac{M_0}{Q_1}\times N_0^{1/2}\right)^{1/2}\times S_q^{1/2},
\end{aligned}$$

where for the bound in the second step, we have used the partial summation formula and the Ramanujan bound on average and

$$S_q := \sum_{q_2\in\Phi_2}\sum_{1\leq r\ll\frac{M_0}{Q}}\sum_{1\leq n\ll N_0}\Big|\sum_{q_1\in\Phi_1}\overline{\chi}(q_1)\lambda_g(p^2n+q_1q_2r)\frac{\mathcal{I}_2(q,p^2n+q_1q_2r,x)}{(p^2n+q_1q_2r)^{1/4}}\Big|^2.$$

At first let us club the variables $q_2r\mapsto r$, we have

$$S_q = \sum_{Q_2\leq r\ll\frac{M_0}{Q_1}}\sum_{1\leq n\ll N_0}\Big|\sum_{q_1\in\Phi_1}\overline{\chi}(q_1)\lambda_g(p^2n+q_1r)\frac{\mathcal{I}_2(q,p^2n+q_1r,x)}{(p^2n+q_1r)^{1/4}}\Big|^2.$$

Now opening the absolute value square we have

$$S_q = \sum_{Q_2 \leq r \ll \frac{M_0}{Q_1}} \sum_{1 \leq n \ll N_0} \sum_{q_1 \in \Phi_1} \overline{\chi}(q_1) \lambda_g(p^2 n + q_1 r) \frac{\mathcal{I}_2(q, p^2 n + q_1 r, x)}{(p^2 n + q_1 r)^{1/4}}$$

$$\times \sum_{q_1' \in \Phi_1'} \chi(q_1') \lambda_g(p^2 n + q_1' r) \overline{\frac{\mathcal{I}_2(q, p^2 n + q_1' r, x)}{(p^2 n + q_1' r)^{1/4}}} \tag{4.19}$$

$$= S_{q,\,diag} + S_{q,\,off},$$

where $S_{q,\,diag}$ is the diagonal part when $q_1 = q_1'$ and $S_{q,\,off}$ is the off-diagonal when $q_1 \neq q_1'$.

For the diagonal case, i.e., for $q_1 = q_1'$, we have

$$S_{q,\,diag} \ll \sum_{Q_2 \leq r \ll \frac{M_0}{Q_1}} \sum_{1 \leq n \ll N_0} \sum_{q_1 \in \Phi_1} \frac{\left|\lambda_g(p^2 n + q_1 r)\right|^2 \left|\mathcal{I}_2(q, p^2 n + q_1 r, x)\right|^2}{(p^2 n + q_1 r)^{1/2}}. \tag{4.20}$$

Now changing the variables $q_1 r \mapsto r$, this reduces to,

$$S_{q,\,diag} \ll \sum_{1 \leq n \ll N_0} \sum_{Q \leq r \ll M_0} \frac{\left|\lambda_g(p^2 n + r)\right|^2}{(p^2 n + r)^{1/2}} \ll N_0 \times \frac{M_0}{M_0^{1/2}} = N_0 M_0^{1/2}, \tag{4.21}$$

by the Ramanujan bound on average and the fact that the maximum size of $p^2 n =$ maximum size of $q_1 r = M_0$.

For the off-diagonal case, i.e., for $q_1 \neq q_1'$, (4.19) becomes

$$\begin{aligned} S_{q,off} &\ll \sum_{1\le n\ll N_0} \sum_{q_1\in\Phi_1} \sum_{q_1'\in\Phi_1'} \chi(q_1')\overline{\chi}(q_1) \\ &\times \sum_{Q_2\le r\ll \frac{M_0}{Q_1}} \lambda_g(p^2n+q_1r)\lambda_g(p^2n+q_1'r)\frac{\mathcal{I}_2(q,p^2n+q_1r,x)\overline{\mathcal{I}_2(q,p^2n+q_1'r,x)}}{(p^2n+q_1r)^{1/4}(p^2n+q_1'r)^{1/4}} \\ &\ll \sum_{1\le n\ll N_0} \sum_{q_1\in\Phi_1} \sum_{q_1'\in\Phi_1'} \\ &\times \Big| \sum_{Q_2\le r\ll \frac{M_0}{Q_1}} \lambda_g(p^2n+q_1r)\lambda_g(p^2n+q_1'r)\frac{\mathcal{I}_2(q,p^2n+q_1r,x)\overline{\mathcal{I}_2(q,p^2n+q_1'r,x)}}{(p^2n+q_1r)^{1/4}(p^2n+q_1'r)^{1/4}}\Big|. \end{aligned} \tag{4.22}$$

Now using the partial summation formula to eliminate the non-oscillating weight function, as

$$\frac{\partial}{\partial y}\mathcal{I}_2(q,p^2n+q_1y,x) \ll_j 1 \ \ \textit{and} \ \ \frac{\partial}{\partial y}\mathcal{I}_2(q,p^2n+q_1'y,x) \ll_j 1,$$

Our problem boils down to an estimate

$$\sum_{1\le n\ll N_0} \sum_{q_1\in\Phi_1} \sum_{q_1'\in\Phi_1'} \frac{1}{M_0^{1/2}} \Big| \sum_{Q_2\le r\ll \frac{M_0}{Q_1}} \lambda_g(p^2n+q_1r)\lambda_g(p^2n+q_1'r)\Big|. \tag{4.23}$$

Indeed define $u := p^2n+q_1r,\ v := p^2n+q_1'r$, we see that $q_1'u - q_1v = (q_1' - q_1)p^2n$. Conversely, given u, v satisfying the equation $u := p^2n+q_1r,\ v := p^2n+q_1'r$, we can obtain r in the following manner:

$$q_1 \mid q_1'(u-p^2n) \implies q_1 \mid u - p^2n \implies \textit{there exists } r \textit{ such that } u - p^2n = q_1r.$$

Also, this gives

$$q_1'rq_1 = q_1v - q_1p^2n \implies q_1'r = v - p^2n.$$

Using the above idea, we can relate the inner sum of the equation (4.23) to a shifted convolution sum given by

$$\sum\sum_{\substack{M\le u,v\ll 2M\\ q_1'u-q_1v=(q_1'-q_1)p^2n}} \lambda_g(u)\lambda_g(v)W\left(\frac{u}{M}\right)V\left(\frac{v}{M}\right). \tag{4.24}$$

where W, V are nice smooth functions supported on $[1/2, 3]$, taking value 1 on $[1, 2]$ and $W^i(x) \ll_i \frac{1}{x^i}$, $V^j(y) \ll_j \frac{1}{x^j}$ and $Q + p^2n \ll M \ll M_0 + p^2n$.

Now using [11, Theorem 1.3], estimating the shifted convolution sum (4.24), we have

$$\sum\sum_{\substack{M\le u,v\ll 2M\\ q_1'u-q_1v=(q_1'-q_1)p^2n}} \lambda_g(u)\lambda_g(v)W\left(\frac{u}{M}\right)V\left(\frac{v}{M}\right) \ll (q_1'M+q_1M)^{1/2+\theta} \ll Q_1^{1/2+\theta}M^{1/2+\theta}. \tag{4.25}$$

where θ is given in [39], Kim-Sarnak exponent (for holomorphic cusp forms, $\theta = 0$).

So (4.22) reduces to

$$\begin{aligned} S_{q,off} &\ll N_0 \times Q_1^2 \times Q_1^{1/2+\theta}M_0^{\theta} \\ &\ll N_0Q_1^{5/2}(Q_1M_0)^{\theta}. \end{aligned} \tag{4.26}$$

Then from (4.18), (4.19), (4.21) and (4.26), we have

$$\begin{aligned} \left|\Sigma_q\right| &\ll \frac{N^{3/2}}{L\sqrt{p}}\left(\frac{M_0}{Q_1}\times N_0^{1/2}\right)^{1/2} \times \left(N_0M_0^{1/2} + N_0Q_1^{5/2}(Q_1M_0)^{\theta}\right)^{1/2} \\ &\ll \frac{N^{3/2}}{L\sqrt{p}}\left(\frac{M_0}{Q_1}\times N_0^{1/2}\right)^{1/2} \times \left(N_0M_0^{1/2}\right)^{1/2} \\ &\ll \sqrt{N}\times\frac{p^{1+\frac{\eta}{2}}}{Q_1^{1/2}}, \end{aligned} \tag{4.27}$$

which happens if, putting $M_0 = p^{2+\eta}$,

$$\begin{aligned} N_0 Q_1^{5/2}(Q_1 M_0)^{\theta} &\ll N_0 M_0^{1/2} \\ \iff Q_1 &\ll p^{\frac{2}{5}+\frac{\eta}{5}-\frac{12}{25}\theta(\eta+2)}. \end{aligned} \tag{4.28}$$

Case (4.4.2)**.** Let $d = q_1$ (or q_3 or q_4). For this case we have

$$\begin{aligned} \Sigma_{q_1} := & \frac{N^{3/2}}{L\sqrt{p}} \sum_{\substack{q\in\Phi \\ q=q_1q_2}} \frac{q_1\overline{\chi}(q^2)}{q} \sum_{1\le r\ll \frac{M_0}{Q_1}} \chi(q_1 r) \\ & \times \sum_{1\le n\ll N_0} \left(\frac{\mathcal{I}_1(n,x,q)\mathcal{I}_2(q,p^2n+q_1r,x)}{(n(p^2n+q_1r))^{1/4}} \right) \left(\lambda_f(n)\lambda_g(p^2n+q_1r)\right) \\ & \ll \frac{N^{3/2}}{L\sqrt{p}} \left(\sum_{q_2\in\Phi_2} \frac{1}{q_2^2} \sum_{1\le r\ll \frac{M_0}{Q_1}} \sum_{1\le n\ll N_0} \frac{|\lambda_f(n)|^2}{n^{1/2}} \right)^{1/2} \\ & \times \left(\sum_{q_2\in\Phi_2} \sum_{1\le r\ll \frac{M_0}{Q_1}} \Big| \sum_{q_1\in\Phi_1} \overline{\chi}(q_1r)\lambda_g(p^2n+q_1r) \frac{\mathcal{I}_1(n,x,q_1q_2)\mathcal{I}_2(q,p^2n+q_1r,x)}{(p^2n+q_1r)^{1/4}} \Big|^2 \right)^{1/2}. \end{aligned} \tag{4.29}$$

Now applying the Cauchy-Schwarz inequality to (4.29), as the integrals $\mathcal{I}_1, \mathcal{I}_2$ do not oscillate, we get that

$$\begin{aligned}
\left|\Sigma_{q_1}\right| &\ll \frac{N^{3/2}}{L\sqrt{p}} \left(\frac{Q_2}{Q_2^2} \sum_{1 \le r \ll \frac{M_0}{Q_1}} \sum_{1 \le n \ll N_0} \frac{|\lambda_f(n)|^2}{n^{1/2}} \right)^{1/2} \\
&\times \left(Q_2 \sum_{1 \le r \ll \frac{M_0}{Q_1}} \sum_{1 \le n \ll N_0} \Big| \sum_{q_1 \in \Phi_1} \overline{\chi}(q_1) \lambda_g(p^2 n + q_1 r) \frac{\mathcal{I}_2(q, p^2 n + q_1 r, x)}{(p^2 n + q_1 r)^{1/4}} \Big|^2 \right)^{1/2} \\
&\ll \frac{N^{3/2}}{L\sqrt{p}} \left(\frac{M_0}{Q_1} \times N_0^{1/2} \right)^{1/2} \qquad (4.30) \\
&\times \left(\sum_{1 \le r \ll \frac{M_0}{Q_1}} \sum_{1 \le n \ll N_0} \Big| \sum_{q_1 \in \Phi_1} \overline{\chi}(q_1) \lambda_g(p^2 n + q_1 r) \frac{\mathcal{I}_2(q, p^2 n + q_1 r, x)}{(p^2 n + q_1 r)^{1/4}} \Big|^2 \right)^{1/2} \\
&\ll \frac{N^{3/2}}{L\sqrt{p}} \left(\frac{M_0}{Q_1} \times N_0^{1/2} \right)^{1/2} \times S_{q_1}^{1/2},
\end{aligned}$$

where for the bound in the second step, we have used the partial summation formula and the Ramanujan bound on average and

$$S_{q_1} := \sum_{1 \le r \ll \frac{M_0}{Q_1}} \sum_{1 \le n \ll N_0} \Big| \sum_{q_1 \in \Phi_1} \overline{\chi}(q_1) \lambda_g(p^2 n + q_1 r) \frac{\mathcal{I}_2(q, p^2 n + q_1 r, x)}{(p^2 n + q_1 r)^{1/4}} \Big|^2.$$

Now opening the absolute value square we have

$$\begin{aligned}
S_{q_1} = &\sum_{1 \le r \ll \frac{M_0}{Q_1}} \sum_{1 \le n \ll N_0} \sum_{q_1 \in \Phi_1} \overline{\chi}(q_1) \lambda_g(p^2 n + q_1 r) \frac{\mathcal{I}_2(q, p^2 n + q_1 r, x)}{(p^2 n + q_1 r)^{1/4}} \\
&\times \sum_{q_1' \in \Phi_1'} \chi(q_1') \lambda_g(p^2 n + q_1' r) \frac{\overline{\mathcal{I}_2(q, p^2 n + q_1' r, x)}}{(p^2 n + q_1' r)^{1/4}}.
\end{aligned} \qquad (4.31)$$

For the diagonal case, i.e., for $q_1 = q_1'$, this reduces to

$$S_{q_1,diag} = \sum_{1 \le r \ll \frac{M_0}{Q_1}} \sum_{1 \le n \ll N_0} \sum_{q_1 \in \Phi_1} \frac{\left|\lambda_g(p^2n + q_1r)\right|^2 \left|\mathcal{I}_2(q, p^2n + q_1r, x)\right|^2}{(p^2n + q_1r)^{1/2}}. \tag{4.32}$$

Now clubbing the variables q_1, r and then changing the variables $q_1r + p^2n \mapsto r$, we have,

$$S_{q_1,diag} \ll \sum_{1 \le n \ll N_0} \sum_{Q_1 \le r \ll M_0} \frac{\left|\lambda_g(r)\right|^2}{r^{1/2}} \ll N_0 \times \frac{M_0}{M_0^{1/2}} = N_0 M_0^{1/2}, \tag{4.33}$$

by the Ramanujan bound on average.

For the off-diagonal case, i.e., for $q_1 \neq q_1'$, (4.31) becomes

$$\begin{aligned} S_{q_1,off} \ll & \sum_{1 \le n \ll N_0} \sum_{q_1 \in \Phi_1} \sum_{q_1' \in \Phi_1'} \chi(q_1')\overline{\chi}(q_1) \\ & \times \sum_{1 \le r \ll \frac{M_0}{Q_1}} \lambda_g(p^2n + q_1r)\lambda_g(p^2n + q_1'r) \frac{\mathcal{I}_2(q, p^2n + q_1r, x)\overline{\mathcal{I}_2(q, p^2n + q_1'r, x)}}{(p^2n + q_1r)^{1/4}(p^2n + q_1'r)^{1/4}} \\ & \ll \sum_{1 \le n \ll N_0} \sum_{q_1 \in \Phi_1} \sum_{q_1' \in \Phi_1'} \\ & \times \left| \sum_{1 \le r \ll \frac{M_0}{Q_1}} \lambda_g(p^2n + q_1r)\lambda_g(p^2n + q_1'r) \frac{\mathcal{I}_2(q, p^2n + q_1r, x)\overline{\mathcal{I}_2(q, p^2n + q_1'r, x)}}{(p^2n + q_1r)^{1/4}(p^2n + q_1'r)^{1/4}} \right|. \end{aligned} \tag{4.34}$$

Then in a similar fashion as done in Case (4.4.1), we have

$$S_{q_1,off} \ll N_0 Q_1^{5/2} (Q_1 M_0)^{\theta}. \tag{4.35}$$

So from (4.31), (4.33), (4.35) we have

$$S_{q_1} \ll \left(N_0 M_0^{1/2} + N_0 Q_1^{5/2} (Q_1 M_0)^{\theta} \right) \ll N_0 M_0^{1/2}, \tag{4.36}$$

by Case (4.4.1).

By similar arguments as done in Case (4.4.1), we have from (4.30) and (4.36),

$$\begin{aligned}\left|\Sigma_{q_1}\right| &\ll \frac{N^{3/2}}{L\sqrt{p}} \times \left(\frac{M_0}{Q_1} \times N_0^{1/2}\right)^{1/2} \times \left(N_0 M_0^{1/2}\right)^{1/2} \\ &\ll \sqrt{N} \times \frac{p^{1+\frac{\eta}{2}}}{Q_1^{1/2}}.\end{aligned} \tag{4.37}$$

Case (4.4.3)**.** Let $d = q_2$ with $q_2 = q_3 q_4$ (or $q_3 q_1$ or $q_1 q_4$). Replacing q_1 by q_2 in the previous case, we have

$$\begin{aligned}\Sigma_{q_2} := \frac{N^{3/2}}{L\sqrt{p}} \sum_{\substack{q \in \Phi \\ q = q_1 q_2}} \frac{q_2 \overline{\chi}(q^2)}{q} \sum_{1 \le r \ll \frac{M_0}{Q_2}} \chi(q_2 r) \\ \times \sum_{1 \le n \ll N_0} \left(\frac{\mathcal{I}_1(n, x, q)\mathcal{I}_2(q, p^2 n + q_2 r, x)}{(n(p^2 n + q_2 r))^{1/4}}\right) \left(\lambda_f(n)\lambda_g(p^2 n + q_2 r)\right)\end{aligned} \tag{4.38}$$

$$\ll \frac{N^{3/2}}{L\sqrt{p}} \left(\frac{M_0}{Q_2} \times N_0^{1/2}\right)^{1/2} \times S_{q_2}^{1/2},$$

where

$$S_{q_2} := \sum_{1 \le r \ll \frac{M_0}{Q_2}} \sum_{1 \le n \ll N_0} \left| \sum_{q_2 \in \Phi_2} \overline{\chi}(q_2)\lambda_g(p^2 n + q_2 r) \frac{\mathcal{I}_2(q, p^2 n + q_2 r, x)}{(p^2 n + q_2 r)^{1/4}} \right|^2.$$

Opening the absolute value square we have,

$$\begin{aligned}S_{q_2} = \sum_{1 \le r \ll \frac{M_0}{Q_2}} \sum_{1 \le n \ll N_0} \sum_{q_2 \in \Phi_2} \overline{\chi}(q_2)\lambda_g(p^2 n + q_2 r) \frac{\mathcal{I}_2(q, p^2 n + q_2 r, x)}{(p^2 n + q_2 r)^{1/4}} \\ \times \sum_{q_2' \in \Phi_2'} \chi(q_2')\lambda_g(p^2 n + q_2' r) \frac{\overline{\mathcal{I}_2(q, p^2 n + q_2' r, x)}}{(p^2 n + q_2' r)^{1/4}}.\end{aligned} \tag{4.39}$$

As done in the previous case, for the diagonal case, i.e., for $q_2 = q_2'$, we have,

$$S_{q_2,\,diag} \ll \sum_{1\leq n\ll N_0}\sum_{Q_2\leq r\ll M_0}\frac{\left|\lambda_g(r)\right|^2}{r^{1/2}} \ll N_0\times\frac{M_0}{M_0^{1/2}} = N_0M_0^{1/2}, \tag{4.40}$$

Now for the off-diagonal case, i.e., for $q_2 \neq q_2'$, we have,

$$\begin{aligned}
S_{q_2,\,off} \ll & \sum_{1\leq n\ll N_0}\sum_{1\leq r\ll\frac{M_0}{Q_2}}\sum_{\substack{q_2\in\Phi_2\\ q_2=q_3q_4}}\chi(q_2)\lambda_g(p^2n+q_2r)\frac{\mathcal{I}_2(q,p^2n+q_2r,x)}{(p^2n+q_2r)^{1/4}}\\
& \times\sum_{q_2'\in\Phi_2'}\overline{\chi}(q_2')\lambda_g(p^2n+q_2'r)\overline{\frac{\mathcal{I}_2(q,p^2n+q_2'r,x)}{(p^2n+q_2'r)^{1/4}}}\\
& \ll\left(\sum_{1\leq n\ll N_0}\sum_{q_4\in\Phi_4}\sum_{q_2'\in\Phi_2'}\sum_{1\leq r\ll\frac{M_0}{Q_2}}\frac{\left|\lambda_g(p^2n+q_2'r)\right|^2}{(p^2n+q_2'r)^{1/2}}\right)^{1/2}\\
& \times\left(\sum_{1\leq n\ll N_0}\sum_{q_2'\in\Phi_2'}\sum_{q_4\in\Phi_4}\sum_{1\leq r\ll\frac{M_0}{Q_2}}\left|\sum_{q_3\in\Phi_3}\overline{\chi}(q_3)\lambda_g(p^2n+q_3q_4r)\frac{\mathcal{I}_2(q,p^2n+q_3q_4r,x)}{(p^2n+q_3q_4r)^{1/4}}\right|^2\right)^{1/2}
\end{aligned}\tag{4.41}$$

Now changing the variables $u := p^2n + q_2'r$ for the first term and clubbing the variables

$v := q_4 r$ for the second term we have, we arrive at

$$S_{q_2, off} \ll \left(\sum_{1 \leq n \ll N_0} \sum_{q_4 \in \Phi_4} \sum_{Q_2 \leq u \ll M_0} \frac{\left| \lambda_g(u) \right|^2}{u^{1/2}} \right)^{1/2}$$

$$\times \left(\sum_{1 \leq n \ll N_0} \sum_{q_2' \in \Phi_2'} \sum_{Q_4 \leq v \ll \frac{M_0}{Q_3}} \Big| \sum_{q_3 \in \Phi_3} \overline{\chi}(q_3) \lambda_g(p^2 n + q_3 v) \frac{\mathcal{I}_2(q, p^2 n + q_3 v, x)}{(p^2 n + q_3 v)^{1/4}} \Big|^2 \right)^{1/2} \tag{4.42}$$

$$\ll \left(N_0 Q_4 M_0^{1/2} \right)^{1/2} \times (S_{q_2, of2})^{1/2}$$

where for the last inequality we have used the Ramanujan bound on average and the partial summation formula and

$$S_{q_2, of2} := \sum_{1 \leq n \ll N_0} \sum_{q_2' \in \Phi_2'} \sum_{Q_4 \leq v \ll \frac{M_0}{Q_3}} \Big| \sum_{q_3 \in \Phi_3} \overline{\chi}(q_3) \lambda_g(p^2 n + q_3 v) \frac{\mathcal{I}_2(q, p^2 n + q_3 v, x)}{(p^2 n + q_3 v)^{1/4}} \Big|^2.$$

Opening the absolute value square we have

$$S_{q_2, of2} = Q_2 \sum_{1 \leq n \ll N_0} \sum_{Q_4 \leq v \ll \frac{M_0}{Q_3}} \sum_{q_3 \in \Phi_3} \sum_{q_3' \in \Phi_3'} \overline{\chi}(q_3') \chi(q_3) \lambda_g(p^2 n + q_3 v) \lambda_g(p^2 n + q_3' v)$$
$$\times \frac{\mathcal{I}_2(q, p^2 n + q_3 v, x)}{(p^2 n + q_3 v)^{1/4}} \frac{\overline{\mathcal{I}_2(q, p^2 n + q_3' v, x)}}{(p^2 n + q_3' v)^{1/4}}. \tag{4.43}$$

Now for the diagonal case $q_3' = q_3$, using the partial summation formula and the Ramanujan bound on average and also clubbing the variables $u := p^2 n + q_3 v$, we have,

$$S_{q_2, of2d} = Q_2 \sum_{1 \leq n \ll N_0} \sum_{Q_2 \leq u \ll \frac{M_0}{Q_3}} \left| \lambda_g(u) \right|^2 \frac{1}{u^{1/2}}$$
$$\ll N_0 Q_2 M_0^{1/2}. \tag{4.44}$$

For the off-diagonal case $q_3' \neq q_3$, we have,

$$\begin{aligned} S_{q_2,of2od} &= Q_2 \sum_{1\le n \ll N_0} \sum_{Q_4 \le v \ll \frac{M_0}{Q_3}} \sum_{q_3\in\Phi_3} \sum_{q_3'\in\Phi_3'} \overline{\chi}(q_3')\chi(q_3)\lambda_g(p^2n+q_3v)\lambda_g(p^2n+q_3'v) \\ &\quad \times \frac{\mathcal{I}_2(q,p^2n+q_3v,x)}{(p^2n+q_3v)^{1/4}} \frac{\overline{\mathcal{I}_2(q,p^2n+q_3'v,x)}}{(p^2n+q_3'v)^{1/4}} \\ &\ll Q_2 \sum_{1\le n\ll N_0} \sum_{q_3\in\Phi_3} \sum_{q_3'\in\Phi_3'\in\Phi_2^*} \Bigg| \sum_{Q_4\le v\ll \frac{M_0}{Q_3}} \lambda_g(p^2n+q_3v)\lambda_g(p^2n+q_3'v) \\ &\quad \times \frac{\mathcal{I}_2(q,p^2n+q_3v,x)}{(p^2n+q_3v)^{1/4}} \frac{\overline{\mathcal{I}_2(q,p^2n+q_3'v,x)}}{(p^2n+q_3'v)^{1/4}} \Bigg|. \end{aligned} \tag{4.45}$$

Estimating in the similar manner as done in Case (4.4.1), (4.45) reduces to

$$S_{q_2,of2od} \ll N_0Q_2Q_3^{5/2}(Q_3M_0)^{\theta}. \tag{4.46}$$

So from (4.43), (4.44) and (4.46) we have

$$S_{q_2,of2} \ll \left(N_0Q_2M_0^{1/2} + N_0Q_2Q_3^{5/2}(Q_3M_0)^{\theta}\right) \ll N_0Q_2M_0^{1/2},. \tag{4.47}$$

where we will take $Q_3 \ll p^{\frac{2}{5}+\frac{\eta}{5}-\frac{12}{25}\theta(\eta+2)}$.

Then from (4.38), (4.40), (4.42) and (4.47), we have

$$\begin{aligned}
\left|\Sigma_{q_2}\right| &\ll \frac{N^{3/2}}{L\sqrt{p}} \times \left(\frac{M_0}{Q_2} \times N_0^{1/2}\right)^{1/2} \times \left(N_0 M_0^{1/2} + \left(\left(N_0 Q_4 M_0^{1/2}\right)^{1/2} \times \left(N_0 Q_2 M_0^{1/2}\right)^{1/2}\right)\right)^{1/2} \\
&\lll \frac{N^{3/2}}{L\sqrt{p}} \times \left(\frac{M_0}{Q_2} \times N_0^{1/2}\right)^{1/2} \times \left(N_0 M_0^{1/2} Q_4^{1/2} Q_2^{1/2}\right)^{1/2} \\
&\ll \frac{N^{3/2}}{L\sqrt{p}} \times \frac{M_0^{3/4} N_0}{Q_3^{1/4}} \\
&\ll \sqrt{N} \times \frac{p^{1+\frac{3\eta}{4}}}{Q_3^{1/4}}.
\end{aligned} \tag{4.48}$$

Case (4.4.4)**.** Let $d = 1$. For this case we have

$$\begin{aligned}
\Sigma_1 &:= \frac{N^{3/2}}{L\sqrt{p}} \sum_{q\in\Phi} \frac{\overline{\chi}(q^2)}{q} \sum_{1\leq r\ll M_q-p^2n} \chi(r) \\
&\times \sum_{1\leq n\ll N_q} \left(\frac{\mathcal{I}_1(n,x,q)\mathcal{I}_2(q,p^2n+r,x)}{(n(p^2n+r))^{1/4}}\right)\left(\lambda_f(n)\lambda_g(p^2n+r)\right) \\
&= \frac{N^{3/2}}{L\sqrt{p}} \sum_{q\in\Phi} \frac{\overline{\chi}(q^2)}{q} \sum_{1\leq n\ll N_q} \lambda_f(n)\left(\frac{\mathcal{I}_1(n,x,q)}{n^{1/4}}\right) \\
&\times \sum_{1\leq r\ll M_q-p^2n} \lambda_g(r+p^2n)\chi(r+p^2n)\left(\frac{\mathcal{I}_2(q,p^2n+r,x)}{(p^2n+r)^{1/4}}\right).
\end{aligned} \tag{4.49}$$

Now by the Cauchy-Schwarz inequality and partial summation formula and changing variables $r + p^2 n \mapsto r$, (4.49) reduces to

$$\Sigma_1 \ll \frac{N^{3/2}}{L\sqrt{p}} \left(\sum_{q \in \Phi} \frac{1}{q^2} \sum_{1 \le n \ll N_0} \frac{1}{N_0^{1/2}} \left| \lambda_f(n) \right|^2 \right)^{1/2}$$
$$\times \left(\sum_{q \in \Phi} \sum_{1 \le n \ll N_0} \left| \frac{1}{M_0^{1/4}} \sum_{r \in \mathbf{Z}} \lambda_g(r)\chi(r) W\left(\frac{r}{M_0}\right) \right|^2 \right)^{1/2}$$
$$\ll \frac{N^{3/2}}{L\sqrt{p}} \times \left(\frac{1}{Q} \times N_0^{1/2} \right)^{1/2} \times \left(\frac{QN_0}{M_0^{1/2}} \right)^{1/2} \times \left| \sum_{r \in \mathbf{Z}} \lambda_g(r)\chi(r) W\left(\frac{r}{M_0}\right) \right| \tag{4.50}$$
$$\ll \frac{N^{3/2}}{L\sqrt{p}} \times \frac{N_0^{3/4}}{M_0^{1/4}} \times \left| \sum_{r \in \mathbf{Z}} \lambda_g(r)\chi(r) W\left(\frac{r}{M_0}\right) \right|,$$

where W is a nice function supported on $[\frac{1}{2}, 3]$ and equals to 1 on $[1, 2]$. Here we have used the Ramanujan bound on average.

Now consider the twisted GL(2) short character sum of the right-hand side of (4.50). For this case note that conductor$= p^2 < M_0 = p^{2+\eta}$. Also applying the Mellin inversion formula,

$$\sum_{r \in \mathbf{Z}} \lambda_g(r)\chi(r) W\left(\frac{r}{M_0}\right) = \frac{1}{2\pi i} \int_{(\sigma)} M_0^s \tilde{W}(s) L(s, f \otimes \chi) ds + O(M_0^{-A}),$$

for any $A > 0$.

As the Mellin transform $\tilde{W}(s)$ decays rapidly on the vertical line, so shifting the contour to the negative side (i.e., $\sigma < 0$), and noting that $L(s, f \otimes \chi) \ll (p^2)^{\frac{1}{2} - \sigma}$, we have,

$$\sum_{r \in \mathbf{Z}} \lambda_g(r)\chi(r) W\left(\frac{r}{M_0}\right) = O\left(\left(\frac{M_0}{p^2} \right) \right) = O(p^{\eta\sigma}).$$

As $0 < \eta < 1$, so shifting the contour $\sigma \mapsto -\infty$, we have

$$\sum_{r \in \mathbf{Z}} \lambda_g(r)\chi(r) W\left(\frac{r}{M_0}\right) \ll p^{\varepsilon}. \tag{4.51}$$

Using (4.51), (4.50) reduces to

$$\Sigma_1 \ll \frac{N^{3/2}}{L\sqrt{p}} \times \frac{N_0^{3/4}}{M_0^{1/4}} \times p^{\epsilon} \ll \sqrt{N}\,\frac{1}{p^{1+\frac{\eta}{2}}}. \tag{4.52}$$

4.5 Final estimation

Now from (4.16), (4.27), (4.37), (4.48) and (4.52) we have

$$\begin{aligned}\tilde{S}_x(N) \ll \sqrt{N} \times \Big(&\frac{p^{1+\frac{\eta}{2}}}{Q_1^{1/2}} + \frac{p^{1+\frac{\eta}{2}}}{Q_1^{1/2}} + \frac{p^{1+\frac{\eta}{2}}}{Q_3^{1/2}} + \frac{p^{1+\frac{\eta}{2}}}{Q_4^{1/2}} + \frac{p^{1+\frac{3\eta}{4}}}{Q_3^{1/4}} + \frac{p^{1+\frac{3\eta}{4}}}{Q_1^{1/4}} \\ &+ \frac{p^{1+\frac{3\eta}{4}}}{Q_1^{1/4}} + \frac{1}{p^{1+\frac{\eta}{2}}} + \frac{p}{p^{\eta/2}}\Big).\end{aligned} \tag{4.53}$$

For the best possible estimate let us take

$$\frac{p^{1+\frac{\eta}{2}}}{Q_1^{1/2}} = \frac{p^{1+\frac{3\eta}{4}}}{Q_3^{1/4}}$$

$$\iff Q_3 = Q_1^2 p^{\eta},$$

with $Q_3 = Q_4$. So this with $Q_1 \leq Q_3 \leq Q_4$ and $Q = Q_1Q_3Q_4$ gives

$$Q_1^5 p^{2\eta} \ll p^{1+\frac{\eta}{2}}$$

$$\iff Q_1 \ll p^{\frac{1}{5}-\frac{3\eta}{10}}.$$

We take $Q_1 = p^{\frac{1}{5}-\frac{3\eta}{10}}$ which is compatiable with the condition (4.28) for all $\eta > 0$ for $0 < \theta < \frac{1}{5}$ (for now we know that $\theta = \frac{7}{64}$, see [39]). Also, we have from $Q_1Q_3Q_4 = Q = p^{1+\frac{\eta}{2}}$

we have $Q_3 = Q_4 = p^{\frac{2}{5}+\frac{2\eta}{5}}$.

So that (4.53) reduces to

$$\tilde{S}_x(N) \ll \sqrt{N}\left(p^{\frac{9}{10}+\frac{13\eta}{20}} + \frac{p}{p^{\eta/2}}\right). \tag{4.54}$$

For optimal choice of η, equating these we get $\eta = \frac{2}{23}$. Putting this in the previous equation we have

$$\tilde{S}_x(N) \ll \sqrt{N} p^{\frac{22}{23}+\epsilon}. \tag{4.55}$$

Hence (4.55) along with (4.1) gives the Theorem 4.0.1.

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