SOME PROPERTIES OF ADDITIVE ARITHMETICAL FUNCTIONS

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SUMMARY. Using measure-theoretic methods, it is proved that if an additive arithmetical function has a distribution, 0 must be in the spectrum of that distribution; so also every value assumed by the function.

Again, it is shown that if an additive function has a distribution locally on one interval, it has a distribution on $(-\infty, \infty)$.

Finally, the joint distribution of infinitely many additive arithmetical functions is considered.

Preliminaries

Theorem A: *Let $p_1 = 2, p_2, p_3, \ldots$ be the prime numbers in ascending order of magnitude. Let X_1 consist of the points $0, 1, 2, \ldots$ and $X_2, X_3, \ldots$ be copies of X_1. The point r of X_i carries probability $\left(1-\frac{1}{p_i}\right)\frac{1}{p_i^r}$; $r = 0, 1, 2, \ldots$; $i = 1, 2, 3, \ldots$. Let X be the cartesian product $X_1 \times X_2 \times \ldots$ and P the product measure. The additive arithmetical function f has a distribution if and only if $\sum_1^\infty f\left(p_n^{x_n}\right)$, $x_n \epsilon X_n$, converges almost everywhere on X.* (See Paul, 1963, p. 278).

Theorem B: *Let S be any set of positive integers. In the space X above, let $M_L(S)$ be the set of points $(x_1, x_2, x_3, \ldots)$ such that $2^{x_1}. 3^{x_2} \ldots p_n^{x_n} \epsilon S$ for all sufficiently large n (how large n should be may vary from point to point). Then $P\{M_L(S)\} \leqslant$ the lower logarithmic density of S.*

This theorem is proved in Paul (1962). $M_L(S)$ is there called the lower magnification of S.

Theorem C: *As in Theorem A, consider the space X and the measure P in it. Let f and g be two additive arithmetical functions, each having a distribution. Then f and g have a joint distribution in the sense of logarithmic density and this is the same as the joint distribution of the random variables $\sum_n f\left(p_n^{x_n}\right)$ and $\sum_n g\left(p_n^{x_n}\right)$* (Paul, 1963; Theorem 3).

This may be extended to any finite number of additive arithmetical functions such that each has a distribution.

1. On the spectrum of the distribution

We recall that the spectrum of a probability distribution consists of points x such that every open interval containing x carries positive probability. In this section, we shall prove that if an additive arithmetical function f has a distribution, every value assumed by f is in the spectrum of the distribution. Roughly, if f assumes a value once, f assumes that value or neighbouring values fairly frequently.

In particular, since $f(1) = 0$, the number 0 invariarbly occurs in the spectrum of f.

Theorem 1 : *Let f be an additive arithmetical function having a distribution. Then $f(1) = 0, f(2), f(3)$, all belong to the spectrum of the distribution.*

Proof : Let $\sum_n f\left(p_n^{x_n}\right) = g(x)$; x stands for the point $(x_1, x_2, \ldots)$. Take any $\varepsilon > 0$ and δ such that $0 < \delta < 1$. Let H be a subset of X such that $P(H) < \delta$ and such that on H', the series $\sum_n f\left(p_n^{x_n}\right)$ converges uniformly.

Let N be so large that

$$\left| \sum_{N+1}^{\infty} f\left(p_n^{x_n}\right) \right| < \varepsilon \text{ on } H'.$$

Denote

$$\sum_1^N f\left(p_n^{x_n}\right) \text{ by } \gamma(x), \sum_{N+1}^{\infty} f\left(p_n^{x_n}\right) \text{ by } \phi(x).$$

Then

$$\gamma(x)+\phi(x) = g(x) p.p \text{ on } X.$$

$$P\{|\phi(x)| < \varepsilon\} \geqslant P(H') > 1-\delta.$$

$$P\{\gamma(x) = 0\} \geqslant \left(1-\frac{1}{2}\right) \ldots \left(1-\frac{1}{p_n}\right) > 0.$$

Now consider the joint distribution of the random variables $\phi(x)$ and $\gamma(x)$; in this two-dimensional distribution, the interval $(-\varepsilon, \varepsilon)$ on the ϕ-axis carries probability $> \left(1-\frac{1}{2}\right) \ldots \left(1-\frac{1}{P_N}\right)$. $(1-\delta)$. So $P\{|g(x)| < \varepsilon\} > 0$. But distribution of $g(x)$ is the same as that of f. So the point 0 belongs to the spectrum of the distribution. The rest of the theorem is proved similarly.

2. Local distributions

In this section, we prove that if an additive arithmetical function f assumes any given value on a set of positive integers having positive density, then f must have a distribution. In fact we prove more general theorems.

Theorem 2 : *Let f be a nonnegative valued additive function. If there is a finite interval I(open or closed, non-degenerate or not) such that $f^{-1}(I)$ has positive upper natural density, then f has a distribution.*

Proof : If f has no distribution, then $\sum_n f\left(p_n^{x_n}\right)$ diverges towards $+\infty$ with probability 1, by the zero-one law. So the sequence of distributions of the partial sums escapes towards $+\infty$.

Let $g_M(n)$ be the truncated additive function defined over all the positive integers by

$$g_M\left(2^{a_1} 3^{a_2} \ldots p_r^{a_r}\right) = f\left(2^{a_1}\right)+\ldots+f\left(P_M^{a_M}\right);$$

here the α's are arbitrary nonnegative integers and $r > M$. g_M has a distribution of the discrete type. Let $\beta > 0$ be a common continuity point of the distributions of $g_1, g_2, g_3, \ldots$ and let $[0, \beta]$ cover the interval I. As $M \to \infty$, density of $g_M^{-1}[0, \beta]$ tends to 0. Hence $f^{-1}[0, \beta]$ has density 0 contradiction.

If we remove the hypothesis that f be nonnegative, the theorem eases to be true. However, we salvage the result by imposing a stronger condition.

Let f be an arithmetical function. Let I be an open interval on $(-\infty, \infty)$ and let μ be a nonnegative measure defined on the Borel sets on I such that $\mu(I) > 0$. We shall say that $f(n)$ has distribution μ on I in case $f^{-1}(\alpha, \beta)$ has density equal to $\mu(\alpha, \beta)$ for all α, β such that

(i) $\alpha \epsilon I, \beta \epsilon I, \alpha < \beta$,

and

(ii) $\mu(\alpha) = \mu(\beta) = 0$.

Theorem 3 : *Let $f(n)$ be an additive arithmetical function and let $f(n)$ have a distribution μ on an open interval I, in the sense of logarithmic or natural density. Then if this distribution is not uniform on I (that is, the μ-measure should not be proportional to Lebesgue measure), f has a distribution.*

Remark : I believe the theorem is true even when the distribution on I is uniform; but I am unable to prove this.

Proof : The proof leans heavily on the results and methods of Paul (1963). As in the proof of Theorem 2 of Paul (1963), we see that there are constants $d_1, d_2 \ldots$ such that $\sum_n \left\{f\left(p_n^{r_n}\right)+d_n\right\}$ is convergent with probability 1. Now if $g_1(w), g_2(w), \ldots$ is a sequence of random variables (on any probability space) and if $\sum_n g_n(w)$ converges with probability 1, then, as $n \to \infty$, the distribution of $g_n(w)$ converges weakly to the distribution in which the whole probability is concentrated at 0; this is easily proved by using Egoroff's theorem. It follows that $d_n \to 0$.

As in the proof of Theorem 1 of Paul (1963), the partial sums of the infinite series $(d_1+d_2+\ldots)$ are bounded. Moreover, since $d_n \to 0$, every number between the lower and upper limits of the sequence of partial sums is a limit point of this sequence of partial sums (we assume that the lower limit is less than the upper limit since otherwise we shall be assuming the truth of the theorem we are now proving). Let θ be any number between these lower and upper limits. Let us take a sequence $J = \{j_m\}$ of positive integers such that $\sum_{r=1}^{j_m} d_r \to \theta$ as $m \to \infty$. Then $\sum_{n=1}^{j_m} f\left(p_n^{r_n}\right)$ converges with probability 1 to a random variable $g(x)$, as $m \to \infty$. As in the proof of Theorem 1 of Paul (1963), the random variable $g(x)+\theta$ has distribution μ on I. We may vary θ continuously and deduce that μ is uniform. Thus using the contrary hypothesis, we conclude that $\sum_n d_n$ is convergent. This proves the theorem.

Theorem 4 : *Let f be an additive arithmetical function and let f assume the value α on a set of positive integers having positive density. Then f has a distribution.*

Remark : α may be any real number. It is sufficient if the condition stated above holds for *one* value α.

The proof is practically the same as the foregoing one. However, it may be noted that this theorem is not exactly a special case of the previous theorem.

3. The joint distribution of infinitely many additive arithmetical functions

Let $f_1, f_2, \ldots$ be a sequence of additive arithmetical functions, each having a distribution. Then any finite collection of these functions has a joint distribution in the sense of logarithmic density (see Paul, 1963, p. 280).

We now consider an infinite dimensional product space $Y = Y_1 \times Y_2 \times \ldots$, each Y_i being the space of real numbers. In the space of $Y_1 \times \ldots \times Y_k$, we introduce the joint distribution of $f_1, \ldots, f_k$; we do this for each k. The sequence of finite-dimensional distributions thus introduced is easily seem to be consistent in the sense that the distribution in the $Y_1 \ldots Y_k$ space is the orthogonal projection (marginal distribution) of the distribution in the space $Y_1 \times \ldots \times Y_{(k+1)}$. Hence the fundamental theorem discovered by Daniell, and by Kolmogorov, tells us that there is a unique probability distribution in the infinite-dimensional product space which has the given finite-dimensional distributions as marginals. Since density is generally only finitely additive, it is of interest to examine whether this infinite-dimensional distribution Q has any arithmetical significance. Here we prove a preliminary theorem in this direction.

Theorem 5 : *Let $f_1, f_2, \ldots$ be all nonnegative additive arithmetical functions, each having a distribution. Let $\alpha_k > 0$ be a continuity point in the distribution of f_k, for $k = 1, 2, 3, \ldots$. Then the set S of positive integers n such that simultaneoulsy $f_1(n) \leqslant \alpha_1, f_2(n) \leqslant \alpha_2, f_3(n) \leqslant \alpha_3, \ldots$ has a logarithmic density and this density is equal to the Q-measure of the 'box'*

$$B = [0, \alpha_1] \times [0, \alpha_2] \times [0, \alpha_3] \times \ldots .$$

Proof : Denoting by $\bar{\lambda}, \lambda$ upper and lower logarithmic density, we obtain from the countable additivity (rather, from the consequent continuity) of Q the fact $\bar{\lambda}(s) \leqslant Q(B)$. Let us now consider the space X and the measure P in it (see Preliminaries). Consider $M_L(S)$, the lower magnification of S. Also let $f_k(2^{x_1}) + f_k(3^{x_2}) + \ldots$, which converges with probability 1, be $g_k(x)$; $k = 1, 2, 3, \ldots$.

$$\text{So } \lambda(S) \geqslant P\{M_L(S)\}, \text{ by Theorem B},$$
$$\leqslant P \underset{x}{E}\{g_1(x) < \alpha_1, g_2(x) \geqslant \alpha_2, \ldots\} = Q(B),$$

by Theorem C, and countable additivity of Q. This proves the theorem.

References

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