ON GENERALIZED INVERSES OF PARTITIONED MATRICES

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SUMMARY. The object of this note is to prove the following theorem:

Let
$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$
and A^- be a g-inverse of A. Let
$$F = D - CA^-B \text{ and } G = \begin{pmatrix} A^- + A^-BF^-CA^- & -A^-BF^- \\ -F^-CA^- & F^- \end{pmatrix}$$
where F^- is some g-inverse F. Then G is a g-inverse of M if and only if

(i) $\mathcal{M}(C(I-A^-A)) \subset \mathcal{M}(F)$

(ii) $\mathcal{M}(((I-AA^-)B)') \subset \mathcal{M}(F')$ and

(iii) $(I-AA^-)BF^-C(I-A^-A) = 0$.

If A^- and F^- in F and G are replaced by A_r^- and F_r^- then M is a g-inverse of G.

Some interesting special cases are deduced which include the results of Rohde (1965).

0. Introduction

Rohde (1965) proved the following result on generalized inverses of partitioned matrices.

Let
$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (X_1 : X_2)^*(X_1 : X_2)$$

and
$$G = \begin{pmatrix} A^- + A^-BF^-CA^- & -A^-BF^- \\ -F^-CA^- & F^- \end{pmatrix}$$

where $F = D - CA^-B$. Then G is a g-inverse of M. If A^- and F^- in G are replaced by A_r^- and F_r^- then G is a reflexive g-inverse of M. Further if F is nonsingular and A^- and F^- in G are replaced by A^+ and F^{-1}, then G (considered above) is indeed M^+

The purpose of this note is to prove the following:

Theorem 1: *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *and* A^- *be a g-inverse of* A. *Let* $F = D - CA^-B$ *and*

$$G = \begin{pmatrix} A^- + A^-BF^-CA^- & -A^-BF^- \\ -F^-CA^- & F^- \end{pmatrix}$$

Then the following hold.

(a) *G is a g-inverse of M if and only if*

(i) $\mathcal{M}(C(I-A^-A)) \subset \mathcal{M}(F)$

(ii) $\mathcal{M}(((I-AA^-)B)') \subset \mathcal{M}(F')$

and (iii) $(I-AA^-)BF^-C(I-A^-A) = 0$.

(b) *If A^- and F^- in the expressions for F and G are replaced by A_r^- and F_r^-, then M is always (no further conditions being required) a g-inverse of G.*

We shall also give some interesting special cases which include the result of Rohde.

1. Notations

In this paper we consider matrices over the field of complex numbers. Matrices are denoted by bold face capital letters such as A, B, C, X, Y etc. Null matrix is denoted by 0. For a matrix A,

A^* denotes complex conjugate transpose,

A' denotes transpose,

$R(A)$ denotes the rank,

$\mathcal{M}(A)$ denotes the column space,

and A^-, A_r^-, A_l^-, A_m^- and A^+ denote a g-inverse, a reflexive g-inverse, a least squares g-inverse, a minimum norm g-inverse and Moore Penrose inverse respectively. (See Rao (1967)).

2. Proof of theorem 1

Matrix multiplication shows that

$$MGM = \begin{pmatrix} P & Q \\ R & D \end{pmatrix}$$

where
$$P = A+AA^-BF^-CA^-A-BF^-CA^-A-A^-ABF^-C+BF^-C,$$

$$Q = AA^-B-AA^-BF^-F+BF^-F$$

$$R = CA^-A-FF^-CA^-A+FF^-C$$

Thus $MGM = M$ if and only if

(i) $AA^-B-AA^-BF^-F+BF^-F = B \iff (I-AA^-)B = (I-AA^-)BF^-F$

$\iff \mathcal{M}(((I-AA^-)B)') \subset \mathcal{M}(F')$

(ii) $CA^-A-FF^-CA^-A+FF^-C = C \iff \mathcal{M}(C(I-A^-A)) \subset \mathcal{M}(F)$.

and

(iii) $AA^-BF^-CA^-A-BF^-CA^-A-AA^-BF^-C+BF^-C = 0$

$\iff (I-AA^-)BF^-C(I-A^-A) = 0$.

This completes the proof of (a). Proof of (b) follows by straightforward verification.

Note : In the presence of conditions (i) and (ii), if the condition (iii) holds for some choice of F^-, then it holds for every choice of F^-.

3. Special cases

In this section we give a few interesting special cases where the conditions of Theorem 1 indeed hold. We prove

Theorem 2 : *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

where $\mathcal{M}(C') \subset \mathcal{M}(A')$ *and* $\mathcal{M}(B) \subset \mathcal{M}(A)$ *and consider* F *and* G *as defined in Theorem* 1. *Then* (i) F *is invariant under the choices of g-inverse of* A *and* (ii) G *is a g-inverse of* M.

Proof : (i) follows trivially.

To prove (ii), observe that

$$\mathcal{M}(C') \subset \mathcal{M}(A') \Longleftrightarrow C = CA^-A, \text{ for any choice of g-inverse } A^- \text{ of } A.$$
$$\Longleftrightarrow (C - CA^-A) = 0$$

and $\mathcal{M}(B) \subset \mathcal{M}(A) \Longleftrightarrow (I - AA^-)B = 0$, for any choice of g-inverse A^- of A.

Thus if $\mathcal{M}(B) \subset \mathcal{M}(A)$ and $\mathcal{M}(C') \subset \mathcal{M}(A')$, the conditions of Theorem 1 are satisfied. Hence G is a g-inverse of M.

The following corollaries are easy to deduce.

Corollary 2.1 : *If*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (X_1 : X_2)^*(Y_1 : Y_2)$$

where $R(X_1^* Y_1) = R(X_1) = R(Y_1)$, *then* G *as defined in Theorem* 1 *is a g-inverse of* M.

Corollary 2.2 : (Rohde) : *If*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (X_1 : X_2)^* (X_1 : X_2)$$

then G *as defined in Theorem* 1 *is a g-inverse of* M.

Theorem 3 : *Let* M *be as defined in Theorem* 2 *and* G *be as defined in Theorem* 1.

(a) *If in* G, A^- *and* F^- *are replaced by* A_l^- *and* F_l^- *then* G *is* M_l^- *if and only if* $\mathcal{M}((CA_l^-)') \subset \mathcal{M}(F)$.

(b) *If in* G, A^- *and* F^- *are replaced by* A_m^- *then* $G = M_m^-$ *if and only if* $\mathcal{M}((A_m^-B)') \subset \mathcal{M}(F')$.

(c) *If in the expression for* G, A^- *and* F^- *are replaced by* A^+ *and* F^+, $G = M^+$ *if and ony if* $\mathcal{M}(C A^+) \subset \mathcal{M}(F)$ *and* $\mathcal{M}((A^+B)') \subset \mathcal{M}(F')$.

Proof of Theorem 3 is straightforward and we omit.

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