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ON TWO-PARAMETER FAMILY OF BIB DESIGNS

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SUMMARY. A method of construction of some BIB designs with the parameters $v = 4s^2$, $b = 4st$, $r = (2s-1)t$, $k = s(2s-1)$, $\lambda = (s-1)t$ for $t = 2s-1$ is given and the construction of symmetric BIB designs for $t = s = 7, 10, 27$ is included.

1. Introduction

Shrikhande (1962) has defined a two-parameter family of Balanced Incomplete Block (BIB) designs and gave a method of construction for some of these designs. In another paper, Shrikhande and Raghavarao (1963) gave construction for some such designs. Here we give a method of construction for some more designs. Finally, we include construction of three designs out of nine which Shrikhande (1962) left unsolved. After writing this paper we came to know that two of these were constructed in a different manner by Bose and Shrikhande*.

2. Notation and preliminary results

A BIB design with parameters v, b, r, k, λ is an arrangement of v treatments in b blocks such that (i) every block contains $k (< v)$ different treatments, (ii) every pair of distinct treatments occurs in exactly λ blocks, (iii) every treatment is replicated r times in the blocks.

Then we have

$$vr = bk, \quad \lambda(v-1) = r(k-1), \quad b \geq v. \qquad \dots \quad (2.1)$$

Let $N = ((n_{ij}))$ be the incidence matrix of the design with the above parameters. $N^* = E_{vb} - N$, where E_{mn} is the matrix of order $m \times n$ with positive units everywhere, is the incidence matrix of the complementary BIB design and its parameters are $v^* = v$, $b^* = b$, $r^* = b-r$, $k^* = v-k$, $\lambda^* = b-2r+\lambda$. $\qquad \dots \quad (2.2)$

Editorial Note : This paper is the first of the two posthumous publications of Dr. M. B. Rao written shortly before he died. The other paper will appear in the next issue.

* Bose, R. C. and Shrikhande, S. S. : Some further construction of $O_2(d)$ graphs. To appear in *Studia Scientiarum Mathematicarum Hungarica.*

A BIB design with the parameters v, b, r, k, λ belongs to a family (A) if $b = 4(r-\lambda)$. Define a sub-family (A_1) of (A) by the positive integers s and t where the parameters of the design $A_1(s, t)$ are

$$v = s^2,\ b = 2st,\ r = (s-1)t,\ k = \frac{s(s-1)}{2},\ \lambda = \frac{(s-2)t}{2} \qquad \dots \quad (2.3)$$

and s is even and $2t \geqslant s$. Replacing $s = 2s$ and t by s in (2.3) we get symmetric BIB design with the parameters

$$v = 4s^2 = b,\ r = s(2s-1) = k,\ \lambda = s(s-1). \qquad \dots \quad (2.4)$$

Define the sub-series (A_2) of (A) given by the positive integers s and t where the parameters of $A_2(s, t)$ are given by

$$v = s^2,\ b = 4st,\ r = 2(s-1)t,\ k = \frac{s(s-1)}{2},\ \lambda = (s-2)t \qquad \dots \quad (2.5)$$

and s is odd, $4t > s$.

Following Williamson (1944) we have that

(i) An n-th order square matrix having elements ± 1 off the diagonal and zeros in the diagonal is called an S_n matrix if

$$S_n S_n' = S_n' S_n = nI_n - E_{nn} \qquad \dots \quad (2.6)$$

where I_n is n-th order identity matrix. We can easily show that, if S_n exists, it is symmetrical.

(ii)
$$T_{n+1} = \begin{bmatrix} 0 & E_{1n} \\ E_{n1} & S_n \end{bmatrix} \qquad \dots \quad (2.7)$$

is an orthogonal matrix of order $n+1$ and

$$T_{n+1}T_{n+1}' = T_{n+1}' T_{n+1} = nI_{n+1}. \qquad \dots \quad (2.8)$$

When $n = p^h \equiv 1 \mod 4$ where p is an odd prime and h a positive integer, then there always exists an S_n matrix.

(iii) A skew symmetric matrix of order n having zeros as diagonal elements and ± 1 as non-diagonal elements is called Σ_n if

$$\Sigma_n \Sigma_n' = \Sigma_n' \Sigma_n = nI_n - E_{nn}. \qquad \dots \quad (2.9)$$

If $n = p^h$ where p is odd prime and h a positive integer, such that $p^h \equiv 3 \mod 4$, there always exists Σ_n matrix.

Also, we have that if Σ_t exists, then Σ_{2t+1} also exists and is given by

$$\Sigma_{2t+1} = \begin{bmatrix} \Sigma_t & \Sigma_t + I_t & -E_{t1} \\ \Sigma_t - I_t & \Sigma_t' & E_{t1} \\ E_{1t} & -E_{1t} & 0 \end{bmatrix} \qquad \dots \quad (2.10)$$

(iv) $$T^*_{n+1} = \begin{bmatrix} 0 & E_{1n} \\ -E_{n1} & \Sigma_n \end{bmatrix} \qquad \ldots \quad (2.11)$$

which is an orthogonal matrix of order $n+1 \equiv 0 \bmod 4$ and gives that

$$T^*_{n+1}\, T^{*\prime}_{n+1} = T^{*\prime}_{n+1}\, T^*_{n+1} = n I^*_{n+1}. \qquad \ldots \quad (2.12)$$

A Hadamard matrix H_n is a $(1,-1)$ square matrix of order n such that $H_n' H_n = H_n H_n' = n\, I_n$. It is conjectured by Bose and Shrikhande (1950) that H_n exists for every $n \equiv 0 \bmod 4$. If this conjecture is true, then there always exists a BIB design with the parameters

$$v = 2\,t,\ b = 4t-2,\ r = 2t-1,\ k = t,\ \lambda = t-1. \qquad \ldots \quad (2.13)$$

The conjecture has been verified for all n upto 1600. See, for example, Hall (1967).

3. A METHOD OF CONSTRUCTION OF $A_1\,(s, s-1)$

Theorem 3.1 : *The existence of T_s (or T_s^*) and the BIB design D with the parameters $v = s$, $b = 2(s-1)$, $r = s-1$, $k = \frac{s}{2}$, $\lambda = \frac{s-2}{2}$ implies the existence of $A_1(s, s-1)$.*

Proof : Let N and N^* be the incidence matrices of D and its compliment respectively. It is easy to verify that the complimentary design has also the same parameters as D. Consider T_s (or T_s^*). Replace 1^s and -1^s and zeros in T_s (or T_s^*) by N, N^* and 0_{vb} where 0_{vb} is null matrix of order $v \times b$, respectively. The resulting matrix (say M) of order $s^2 \times 2s\,(s-1)$ is the incidence matrix of $A_1\,(s, s-1)$. We will now prove this. For each row of T_s (or T_s^*) there corresponds a group of s rows in M and every row of it contains $(s-1)^2$ units. Since λ is the same in D as well as its complimentary design, any two rows of this group of s rows contain $\frac{s-2}{2}(s-1)$ units common between them. Now consider any 2 rows of T_s (or T_s^*). They correspond to two different groups in M. Since any two rows of T_s (or T_s^*) are orthogonal and there are $s-2$ non-zero elements common between them, they can be written in $\frac{s-2}{2}$ sub-matrices of order 2×2 of the type $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$. Since -1^s and 1^s are replaced by N^* and N in them, and also $r^* = r = s-1$, and $\lambda^* = \lambda$, considering a particular sub-matrix one finds that any two rows of different groups has $s-1$ units common between them and hence considering all the $\frac{s-2}{2}$ sub-matrices together one finds that there are $(s-1)\frac{(s-2)}{2}$ units common between any two rows of different groups in M. Further, it is easy to see that each column sum of M is $\frac{s(s-1)}{2}$.

Since D exists for all even $s \leqslant 100$, Theorem 3.1 can be used to provide solutions for those values of even $s \leqslant 100$, for which T_s or T_s^* exists.

4. Alternative construction of symmetric BIB designs of family (A)

The following theorem is due to Ehlich (1965).

Theorem 4.1 : *When $m = n+2$ and Σ_m, S_n (or S_m, Σ_n) exist, then Hadamard matrix of order $(n+1)^2$ exists and it is given as*

$$H_{(n+1)^2} = \left[\begin{array}{c|ccc} 1 & E_{1n} & \cdots & E_{1n} \\ \hline E_{n1} & & & \\ \vdots & & \Sigma^*_{mn} & \\ E_{n1} & & & \end{array}\right] \qquad \ldots \quad (4.1)$$

where

$$\Sigma^*_{mn} = \Sigma_m \times S_n - E_{mm} \times I_n + I_m \times E_{nn} + I_{mn}$$

or $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ldots \quad (4.2)$

$$= S_m \times \Sigma_n - E_{mm} \times I_n + I_m \times E_{nn} + I_{mn}$$

and $\times$ denotes the Kronecker product symbol.

Construction of symmetric BIB design with the parameters

$$v = b = 4s^2,\ r = k = s(2s+1),\ \lambda = s(s+1) \qquad \ldots \quad (4.3)$$

where $s = \frac{n+1}{2}$.

Multiply 2nd, 3rd, ..., $\left(\frac{n(n+1)}{2}+1\right)$-th rows and 2nd, 3rd, ..., $\left(\frac{n(n+1)}{2}+1\right)$-th columns by -1 in (4.2). The property of Hadamard matrix does not change. We can easily observe that every row and column of resulting matrix consists $\frac{(n+2)(n+1)}{2}$ positive units. If we replace -1's by zeros in this matrix, the matrix thus obtained is nothing but the incidence matrix of the symmetric BIB design with the parameters in (4.3).

We know that Σ_m exists for $m = 15, 39, 55$ and S_n exists for $n = 13, 37, 53$. Hence symmetric BIB designs for $s = \frac{n+1}{2} = 7, 19, 27$ exist which are the three cases out of nine left unsolved by Shrikhande (1962). Bose and Shrikhande (see footnote on page 259) present a different method of construction for $s = 7, 27$.

4. On the series of $A_1(s, t)$ and $A_2(s, t)$ for $s \leqslant 10$

We will fix s. When s is even, if one can find solutions of $A_1(s, t)$ for all t such that $\frac{s}{2} \leqslant t \leqslant s-1$, then $A_1(s, t)$ can be obtained for other values of t by suitable repetitions of $A_1\left(s, \frac{s}{2}\right)$, $A_1\left(s, \frac{s+2}{2}\right)$, ..., $A_1(s, s-1)$. And when s is odd, if one can construct designs $A_2(s, t)$ for all t, in the interval $\left[\frac{s}{4}\right] < t < 2\left[\frac{s}{4}\right]+2$ where $[x]$ is the greatest integer contained in x, then we can have $A_2(s, t)$ for other values of t by suitable repetitions of

$$A_2\left(s, \left[\frac{s}{4}\right]+1\right), A_2\left(s, \left[\frac{s}{4}\right]+2\right), \ldots, A_2\left(s, 2\left[\frac{s}{4}\right]+1\right).$$

Existence of $A_1(4, t)$ for all $t \geqslant 2$ is known (Shrikhande, 1962). Also, we know that the designs $A_1(6,, 3)$ (Sillito, 1957), $A_1(6, 4)$ (Shrikhande, 1962) are possible. By the Theorem 3.1 we have the existence of $A_1(6, 5)$. Hence by suitable repetitions of $A_1(6, 3)$, $A_1(6, 4)$, $A_1(6, 5)$ we will get $A_1(6, t)$ for all $t > 5$.

The following designs i.e. $A_2(5, 3)$; $A_2(7, 2)$, $A_2(9, 5)$; $A_1(8, 4)$, $A_1(8, 6)$, $A_2(9, 4)$, $A_1(10, 5)$, $A_1(10, 8)$ are from Sillito (1957); Shrikhande and Raghavarao (1963); Shrikhande (1962). By the Theorem 3.1 we have $A_1(8, 7)$ and $A_1(10, 9)$. Hence we may have $A_1(s, t)$, $A_2(s, t)$ for $s \leqslant 10$ if one finds the solutions of the following designs.

v	b	r	k	λ	
25	40	16	10	6	$A_2(5, 2)$
49	84	36	21	15	$A_2(7, 3)$
64	80	35	28	15	$A_1(8, 5)$
81	108	48	36	21	$A_2(9, 3)$
100	120	54	45	24	$A_1(10, 6)$
100	140	63	45	28	$A_1(10, 7)$

References

Bose, R. C. and Shrikhande, S. S. (1959): A note on a result in the theory of code construction. *Information and Control*, **2**, 183–194.

Ehlich, H. (1965): Neue Hadamard-Matrizen. *Arch. Math.*, **16**, 34–36.

Hall, M. (1967) : *Combinatorial Theory*, Blaisdell, London.

Raghavarao, D. (1960) : Some aspects of weighing designs. *Ann. Math. Stat.*, 31, 878–884.

Shrikhande, S. S. (1962) : On a two-parameter family of BIB designs. *Sankhyā*, Series A, 24, 33-40.

Shrikhande, S. S. and Raghavarao, D. (1963) : A method of construction of incomplete block designs. *Sankhyā*, Series A, 25, 399–402.

Sillitto, G. P. (1957) : An extension property of a class of BIB designs. *Biometrika*, 44, 278-279.

Williamson, J. (1944) : Hadamard's determinant theorem and the sum of four squares. *Duke, Math. Jour.*, 11, 65–81

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