Sankhyā : The Indian Journal of Statistics
1988, Volume 50, Series A, Pt. 1, pp. 98-110

DISCRETE MULTIVARIATE DISTRIBUTIONS AND GENERALIZED LOG-CONCAVITY

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SUMMARY. For discrete multivariate distributions we introduce a notion called generalized log-concavity and then show that several standard distributions satisfy that property. The multiparameter multinomial density is shown to be generalized log-convave and the proof depends on results from the theory of permanents. The multiparameter negative multinomial density is expressed in terms of permanents and it is shown that the multiparameter negative binomial density is log-concave. The Alexandroff inequality for permanents is used to show that certain sequences associated with the multiparameter multinomial and with order statistics for nonidentically distributed variables are log-concave.

1. Introduction

Let $N = \{0, 1, 2, ...\}$ and let

$$N^r = \{k = (k_1, ..., k_r) : k_i \geqslant 0, \text{ integers}\}.$$

A function $f : N \to (0, \infty)$ is said to be log-concave if

$$f(x)^2 \geqslant f(x-1)\, f(x+1),\ x = 1, 2, ...$$

Such functions arise in a number of situations, for example, in statistics and combinatorics, and their importance partly stems from the simple fact that a log-concave function must be unimodal.

Several standard discrete distributions on N have log-concave densities. For example, if X has either the binomial or the poisson distribution, then the density function of X can be seen to be log-concave. If f is a continuous density defined on R^n, $n \geqslant 2$, then f can be said to be log-concave if $\log f$ is a concave function. Thus, if X has the multivariate normal distribution, it is easy to see that the density of X is log-concave. However, if X has a discrete multivariate distribution, it is not obvious as to what should be the appropriate notion of log-concavity and several possibilities exist.

In this paper, we first propose a certain generalization of the concept of log-concavity, called generalized log-concavity (g.l.c.) for positive functions defined on a subset of N^r, $r \geqslant 2$. The concept turns out to be stronger than a generalization considered by Karlin and Rinott (1981). Then we show

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AMS (1980) *subject classification :* 60E15.

Key words and phrases : discrete multivariate distributions, log-concavity, permanents.

that a certain family of discrete multivariate distributions satisfy the g.l.c. property. The family includes the hypergeometric, the negative hypergeometric, the multinomial and the negative multinomial densities. The multiparameter multinomial density, which will be defined later, also has the g.l.c. property. This fact is based on certain results from the theory of permanents.

The multiparameter negative multinomial density, considered in Section 4, is conjectured to have the g.l.c. property. This conjecture is stronger than a conjecture due to Karlin and Rinott (1981). It is shown that the density function of the multiparameter negative multinomial distribution can be expressed in terms of permanents and this suggests that the theory of permanents might have a role in attacking the conjectures mentioned above. It is also shown that the multiparameter negative binomial density is log-concave.

In Section 5, certain immediate consequences of the Alexandroff inequality for permanents are given. These results show that certain functons associated with the multiparameter multinomial distribution are log-concave It has been observed by Vaughan and Venables (1972) that densities of order statistics for independent variables coming from different populations, can be expressed in terms of a permanent. This fact, combined with the Alexandroff inequality shows that certain sequences associated with such order statistics are log-concave. This will also be shown in Section 5.

2. Generalized log-concavity

Let e_i denote the i-th row of the identity matrix, the order of which will be clear from the context.

Definition: Let $S \subset N^r$, $r \geqslant 2$, and suppose $f: S \to (0, \infty)$. We will say that f is generalized log-concave (g.l.c.) on S if for any $k \in N^r$ such that

$$k+e_i+e_j \in S,\ i, j = 1, 2, \ldots, r;$$

it is true that the symmetric, positive $r \times r$ matrix

$$((f(k+e_i+e_j)))$$

has exactly one (simple) positive eigenvalue.

We propose to show how to construct g.l.c. functions in a simple way. But we need some preliminary results.

Lemma 1: *Let $A = ((a_{ij}))$ be a positive, symmetric $r \times r$ matrix with exactly one positive eigenvalue and let $B = ((b_{ij}))$ be the $r \times r$ matrix with $b_{ij} = a_{ij}$, $i \neq j$; $b_{ii} = \theta_i a_{ii}$, where $0 < \theta_i \leqslant 1$, $i = 1, 2, \ldots, r$. Then B has exactly one positive eigenvalue.*

Proof: Since B is a positive matrix, it has at least one positive eigenvalue. Suppose B has two positive eigenvalues λ, μ with x, y as the corresponding eigenvectors. Here, either $\lambda \neq \mu$ or $\lambda = \mu$, in which case it is an eigenvalue of multiplicity greater than one. Thus we may assume $x'y = 0$.

Let α be the only positive eigenvalue of A with corresponding eigenvector u. Then by the Spectral Theorem, we can write

$$A = \alpha\, uu' + C,$$

where $z'Cz \leqslant 0$ for any z satisfying $z'u = 0$.

First suppose that $x'u = 0$. Then,

$$x'Bx = x'Ax + \sum_{i=1}^{r} (\theta_i - 1)x_i^2 \leqslant 0.$$

But, $x'Bx = \lambda x'x > 0$, which is a contradiction.

Hence we may assume that $x'u$ as well as $y'u$ are both nonzero, and then, by a suitable normalization, we assume that $(x-y)'u = 0$.

Now

$$(x-y)'B(x-y) = (x-y)'A(x-y) + \sum_{i=1}^{r} (\theta_i - 1)\,(x_i - y_i)^2 \leqslant 0,$$

whereas,

$$\begin{aligned}(x-y)'B(x-y) &= x'Bx + y'By \\ &= \lambda x'x + \mu y'y > 0,\end{aligned}$$

since $x'y = 0$. This is a contradiction and the result is proved.

Corollary 2: *Let $B = ((b_{ij}))$ be an $r \times r$ matrix with $b_{ij} = 1$, $i \neq j$ and $0 < b_{ii} < 1$, $i = 1, 2, \ldots, r$. Then B has exactly one positive eigenvalue.*

Proof: Let A be the $r \times r$ matrix with all entries equal to 1 and use Lemma 1.

The next result justifies, to some extent, the term "generalized log-concave".

Theorem 3: *Let $S \subset N^r$, $r \geqslant 2$, and let $f : S \to (0, \infty)$. Suppose there exist log-concave functions $f^i : N \to (0, \infty)$, $i = 1, 2, \ldots, r$; such that*

$$f(k) = \prod_{i=1}^{r} f^i(k_i),\ k = (k_1, \ldots, k_r) \in S,$$

then f is g.l.c.

Proof: Fix $k \in N^r$ such that $k + e_i + e_j \in S$, $i, j \in \{1, 2, \ldots, r\}$. Then

$$f(k + e_i + e_j) = f(k)\, \frac{f^i(k_i+1)}{f^i(k_i)}\, \frac{f^j(k_j+1)}{f^j(k_j)}\ ,\ 1 \leqslant i \neq j \leqslant r,$$

and

$$f(k+2e_i) = f(k)\,\frac{f^i(k_i+2)}{f^i(k_i)}\,,\ 1 \leqslant i \leqslant r.$$

Let D be the $r \times r$ diagonal matrix with its i-th diagonal entry equal to

$$\frac{f^i(k_i+1)}{f^i(k_i)}\,,\ i = 1, 2, \ldots, r.$$

Then $((f(k+e_i+e_j))) = DBD$, where $B = ((b_{ij}))$ is the $r \times r$ matrix given by

$$b_{ii} = \frac{f^i(k_i+2)f^i(k_i)}{\{f^i(k_i+1)\}^2}\,,\ 1 \leqslant i \leqslant r$$

and

$$b_{ij} = 1,\ 1 \leqslant i \neq j \leqslant r$$

Since f^i is log-concave, we have $0 < b_{ii} \leqslant 1$, $i = 1, 2, \ldots, r$. It follows by Lemma 1 that B, and hence the matrix $((f(k+e_i+e_j)))$ has exactly one positive eigenvalue.

We now give several examples to which Theorem 3 is applicable.

Examples (a) : Suppose X_i is a random variable taking values in N, $i = 1, 2, \ldots, r$; and suppose $X_1, \ldots, X_r$ are idenpendent. If each X_i has a log-concave density function, then clearly, by Theorem 3, $X = (X_1, \ldots, X_r)$ has a density which is g.l.c.

(b) *The hypergeometric distribution* : Let $N_i > n$ be positive integers, $i = 1, 2, \ldots, r$; and let $X = (X_1, \ldots, X_r)$ have the density

$$f(k_1, \ldots, k_r) = \frac{\binom{N_1}{k_1} \cdots \binom{N_r}{k_r}}{\binom{N_1+\ldots+N_r}{n}}\,,\ k \in N^r,\ \Sigma\, k_i = n.$$

Since $\binom{N_i}{k_i}$ is a log-concave function of k_i, it follows that f is g.l.c.

The argument is similar in the next three examples.

(c) *The negative hypergeometric distribution* : Let $\alpha_1, \ldots, \alpha_r$ be positive integers and let $X = (X_1, \ldots, X_r)$ have the density

$$f(k_1, \ldots, k_r) = \frac{\prod_{i=1}^{r} \binom{k_i+\alpha_i+1}{k_i}}{\binom{n+\Sigma\alpha_i+r}{n}}\,,\ k \in N^r,\ \Sigma\, k_i = n.$$

Then f is g.l.c.

(d) *The multinomial distribution*: Let $\theta_1, \ldots, \theta_r$ be positive numbers adding to 1 and let $X = (X_1, \ldots, X_r)$ have the density

$$f(k_1, \ldots, k_r) = n! \prod_{i=1}^{r} \frac{\theta_i^{k_i}}{k_i!}, \; k \in N^r, \; \Sigma k_i = n.$$

Then f is g.l.c.

(e) *The negative multinomial distribution*: Let n be a positive integer, let $p_0, \ldots, p_r$ be positive numbers adding to 1 and let $X = (X_1, \ldots, X_r)$ have the density

$$f(k_1, \ldots, k_r) = \frac{(n-1+\Sigma k_j)!}{(n-1)!} p_0^n \prod_{j=1}^{r} \frac{p_j^{k_j}}{k_j!}, \; k \in N^r$$

Then f is g.l.c.

The class of symmetric, positive matrices with exactly one positive eigenvalue has several interesting properties. It has been studied in the context of quasi-concave quadratic functions by Martos (1969) and by Cottle and Ferland (1972). It is also related to the class of distance matrices (see, for example, Micchelli (1986)).

Definition: If A is a real, syymetric $n \times n$ matrix, it is said to be conditionally negative definite (c.n.d.) if for any $x \in R^n$ with $\Sigma x_i = 0$, we have $x'Ax \leqslant 0$.

It has been shown in Bapat (1986) that if A is a symmetric, positive matrix with exactly one positive eigenvalue, then the matrix $((\log a_{ij}))$ is c.n.d. Thus, if f is defined as in Examples (b)–(e), then it follows that for any $k \in N^r$, $\Sigma k_i = n-2$, the matrix

$$((\log f(k+e_i+e_j))$$

is c.n.d. This latter fact was proved by Karlin and Rinott (1981), Theorem 2.2, and was implicity interpreted there as a log-concavity statement for the multivariate situation.

3. The multiparameter multinomial distribution

Consider an experiment which can result in any one of r possible outcomes and suppose n trials of the experiment are performed. Let p_{ij} be the probability that the experiment results in the j-th outcome at the i-th trial, $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, r$. Let P denote the $n \times r$ matrix $((p_{ij}))$, which, of course, is row-stochastic. We will assume throughout that the matrix P is positive and that $n \geqslant 2$, $r \geqslant 2$.

Let X_j denote the number of times the j-th outcome is obtained in the n trials, $j = 1, 2, \ldots, r$; and let $X = (X_1, \ldots, X_r)$. In this setup we say that X has the multiparameter multinomial distribution with the parameter matrix P. Clearly, if the rows of P are all identical, then X has the multinomial distribution of Example (d), Section 2. For an example of a multiparameter binomial, see Feller (1966), p. 257.

The probability generating function of X can be shown to be

$$\prod_{i=1}^{n} \left(\sum_{j=1}^{r} p_{ij} s_j \right). \qquad \ldots \ (1)$$

We now introduce a notation. If $k \in N^r$, $\Sigma k_i = n$, let $P(k)$ denote the $n \times n$ matrix obtained by taking k_j copies of the j-th column of P, $j = 1, 2, \ldots, r$.

If A is an $n \times n$ matrix, the permanent of A, denoted by per A, is defined as

$$\text{per } A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where S_n is the set of all permutation of $1, 2, \ldots, n$. Two comprehensive references on permanents are Minc (1978, 1983).

It is known that the probability generating function (1) can be expressed as a polynomial in $s_1, \ldots, s_r$ with coefficients given in terms of permanents of suitable matrices. We state this result as the next theorem and refer to Bebiano (1982) for a proof. The result has also been used by Gyires (1973). If $k \in N^r$, set $k! = k_1! \ldots k_r!$.

Theorem 4 : *Let P be an $n \times r$ matrix. Then*

$$\prod_{i=1}^{n} \left(\sum_{i=1}^{r} p_{ij} s_j \right) = \sum_{k \in N^r, \Sigma k_i = n} \frac{s_1^{k_1} \ldots s_r^{k_r}}{k!} \text{ per } P(k).$$

In other words, if $X = (X_1, \ldots, X_r)$ has the multiparameter multinomial distribution with the $n \times r$ parameter matrix P, then

$$Pr(X = k) = \frac{1}{k!} \ \text{ per } P(k), \ k \in N^r, \Sigma k_i = n. \qquad \ldots \ (2)$$

Using the representation (2) and some recent results from the theory of permanents, the following result was established in Bapat (1986).

Theorem 5 : *Let X have the multiparameter multinomial distribution with the $n \times r$ parameter matrix P. Let $k \in N^r$, $\Sigma k_i = n-2$. Then the matrix $((k_{ij}!\, Pr(X = k_{ij})))$ has exactly one positive eigenvalue, where $k_{ij} = k + e_i + e_j$ for all i, j.*

Now we have the following.

Theorem 6 : *Let X have the multiparameter multinomial distribution with the $n\times r$ parameter matrix P. Then the density function of X is g.l.c.*

Proof : Let $k\in N^r$, $\Sigma k_i = n-2$. Let D be the $r\times r$ diagonal matrix with its i-th diagonal entry equal to $(k_i+1)^{-1}$, $i = 1, 2, \ldots, r$.

Let

$$A = ((k_{ij}\,!\,Pr(X = k_{ij}))),\ B = ((Pr(X = k_{ij}))),$$

and let

$$C = \frac{1}{k\,!}\,D\,A\,D.$$

By Theorem 5, A, and hence C, have exactly one positive eigenvalue.

Observe that

$$b_{ij} = c_{ij}, \qquad i\neq j$$

and

$$b_{ii} = \frac{k_i+1}{k_i+2}\,c_{ii}, \qquad i = j.$$

It follows by Lemma 1 that B has exactly one positive eigenvalue and the proof is complete.

4. The multiparameter negative multinomial distribution

Suppose we have m dice, each with $r+1$ faces. We assume that each die has one face marked with an asterisk $\bullet$, whereas the remaining faces carry 1, 2, ..., r spots. Suppose $p_{ij} > 0$ is the probability of getting j spots when the i-th die is rolled, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, r$. Let P denote the $m\times r$ matrix $((p_{ij}))$. Let p_{i0} be the probability of getting a $\bullet$ when the i-th die is rolled, $i = 1, 2, \ldots, m$. Then, clearly,

$$p_{i0} = 1-\sum_{j=1}^{r} p_{ij},\ i = 1, 2, \ldots, m.$$

The following experiment is conducted. The first die is rolled until it shows a $\bullet$. Then we switch over to the second die and roll it until a $\bullet$ is obtained, whence we take up the third die. The process is repeated and the experiment stops when we obtain a $\bullet$ m times. Let X_i denote the number of times we get i spots in the experiment, $i = 1, 2, \ldots, r$; and let $X = (X_1, \ldots, X_r)$. In this setup we say that X has the multiparameter negative multinomial distribution with the $m \times r$ parameter martrix P. Clearly, X takes values in N^r and the probability generating funrction of X is known to be (see, for example, Karlin and Rinott, 1981, p. 339).

$$\prod_{i=1}^{r} \frac{p_{i0}}{\left(1-\sum_{j=1}^{r} p_{ij} s_j\right)},$$

Karlin and Rinott (1980), Section 5, obtained a representation for the density function of X in terms of a multivariate integral. Our next result shows that the density function of X can be expressed in terms of permanents.

Theorem 7 : *For any $m \times r$ matrix p,*

$$\prod_{i=1}^{m} \frac{1}{\left(1-\sum_{j=1}^{r} p_{ij} s_j\right)} = \sum_{n=1}^{\infty} \sum_{k \in N^r,\, \Sigma k_i = n} \frac{s_1^{k_1} \cdots s_r^{k_r}}{k\,!} \sum_{l \in N^m,\, \Sigma l_i = n} \text{per}[P(k)]'(l).$$

Proof: We can write formally,

$$\prod_{i=1}^{m} \frac{1}{\left(1-\sum_{j=1}^{r} p_{ij} s_j\right)} = \prod_{i=1}^{m} \left\{ \sum_{u=0}^{\infty} \left(\sum_{j=1}^{r} p_{ij} s_j \right)^{u} \right\}$$

$$= \sum_{l \in N^m} \prod_{i=1}^{m} \left(\sum_{j=1}^{r} p_{ij} s_j \right)^{l_i}$$

$$= \sum_{n=1}^{\infty} \sum_{l \in N^m,\, \Sigma l_i = n} \prod_{i=1}^{m} \left(\sum_{j=1}^{r} p_{ij} s_j \right)^{l_i} \qquad \ldots \quad (3)$$

For any $l \in N^m$ with $\Sigma l_i = n$, let

$$Q^l = ((q_{ij}^l))$$

denote the $n \times r$ matrix obtained by taking l_i copies of the i-th row of P, $i = 1, 2, \ldots, m$.

An application of Theorem 4 gives

$$\prod_{i=1}^{m} \left(\sum_{j=1}^{r} p_{ij} s_j \right)^{l_i} = \prod_{i=1}^{n} \left(\sum_{j=1}^{r} q_{ij}^l s_j \right)$$

$$= \sum_{k \in N^r,\, \Sigma k_i = n} \frac{s_1^{k_1} \cdots s_r^{k_r}}{k\,!} \text{ per } Q^l(k). \qquad \ldots \quad (4)$$

The proof is completed by substituting (4) in (3), then making a simple rearrangement of terms and by observing that

$$\text{per } Q^l(k) = \text{per}[P(k)]'(l).$$

It is clear from Theorem 7 that if X has the multiparameter negative multinomial distribution with the $m \times r$ parameter matrix P and if $k \,\epsilon\, N^r$, then the density function of X is given by

$$f(k) = \frac{p_{10} \cdots p_{m0}}{k\,!} \sum_{l \,\epsilon\, S^{m}, \Sigma l_i = n} \text{per}[P(k)]'(l). \quad \ldots \quad (5)$$

If $r \geqslant 2$, then it has been conjectured by Karlin and Rinott (1981), conjecture 3.2, that for any $k \,\epsilon\, N^r$, the matrix $((\log f(k_{ij})))$ is c.n.d. where f is as in (5) and, as before, $k_{ij} = k + e_i + e_j$ for all i, j. It seems reasonable to make the following conjecture, which is stronger in view of the discussion at the end of Section 2.

Conjecture: Let $r \geqslant 2$ and let f be as in (5). Then for any $k \,\epsilon\, N^r$, the matrix $((f(k_{ij})))$ has exactly one positive eigenvalue.

As discussed earlier, the conjecture simply says that when $r \geqslant 2$, the density function of X is g.l.c. When $r = 1$, X has the multiparameter negative binomial distribution and we now show that the density function of X is log-concave.

Suppose, then, that we have m coins, the i-th coin showing heads with probability p_i and tails with probability $q_i = 1 - p_i$, $i = 1, 2, \ldots, m$. We begin by tossing the first coin until it shows heads. Then we switch over over to the second coin and toss it until heads are obtained. The process continues until m heads are shown. Let X denote the number of tails obtained in the experiment. Then X has the multiparameter negative binomial distribution and the density function of X is easily seen to be

$$f(k) = \Pr(X = k) = q_1 \ldots q_m \sum_{u_1 + \cdots + u_m = k} p_1^{u_1} \ldots p_m^{u_m}, \; k \epsilon N. \quad \ldots \quad (6)$$

Note that the formula for f can either be derived directly or can be seen as a special case of (5) when $r = 1$. Also, when the coins are all identical, the density f reduces to the usual negative binomial density.

It will follow from our next result that the function f defined in (6) is log-concave.

Theorem 8: *Let p_i, $i = 1, 2, \ldots$ be positive numbers and for positive integers n, k let*

$$g^n(k) = \sum_{u_1 + \cdots + u_n = k} p_1^{u_1} \ldots p_n^{u_n}.$$

Then for any integers $n_1 \geqslant 1,\ n_2 \geqslant 1,\ k_1 \geqslant 2,\ k_2 \geqslant 2,$

$$2g^{n_1}(k_1)g^{n_2}(k_2) \geqslant g^{n_1}(k_1+1)g^{n_2}(k_2-1)+g^{n_1}(k_1-1)g^{n_2}(k_2+1). \quad \dots \quad (7)$$

Proof: If $n_1 = n_2 = 1$ and $k_1 = k_2 = 2$, it is easy to see that both sides of (7) are equal to $2p_1^4$. We then proceed by induction.

Note that the following recurrence relation holds of any $n \geqslant 1$, $k \geqslant 2$.

$$g^n(k) = \sum_{i=1}^{n} p_i\, g^i(k-1) \quad \dots \quad (8)$$

Now for $n_1 \geqslant 1$, $n_2 \geqslant 1$, $k_1 \geqslant 2$, $k_2 \geqslant 2$, we have

$$g^{n_1}(k_1)g^{n_2}(k_2) = \left\{\sum_{i=1}^{n_1} p_i g^i(k_1-1)\right\}\left\{\sum_{j=1}^{n_2} p_j g^j(k_2-1)\right\}$$

$$= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} p_i p_j g^i(k_1-1)g^j(k_2-1)$$

$$\geqslant \frac{1}{2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} p_i p_j\{g^i(k_1)g^j(k_2-2)+g^i(k_1-2)g^j(k_2)\} \quad \dots \quad (9)$$

$$= \frac{1}{2}\{g^{n_1}(k_1+1)g^{n_2}(k_2-1)+g^{n_1}(k_1-1)g^{n_2}(k_2+1)\}. \quad \dots \quad (10)$$

Step (9) follows by the induction assumption and (10) holds in view of (8). That completes the proof.

The log-cocavity of f in (6) is obtained by setting $n_1 = n_2$ and $k_1 = k_2$ in Theorem 8.

5. Alexandroff's inequality and log-concave sequences

It was conjectured by van der Waerden in 1926 that the permanent achieves a minimum over the set of $n \times n$ doubly stochastic matrices only at the $n \times n$ matrix with each entry equal to $\frac{1}{n}$. The conjecture was proved by Egorychev and independently by Falikman around 1979. We refer to Minc (1983) for the proof. An important tool used in the proof is the following inequality proved by Alexandroff in 1938 in a more general seting.

Theorem 9: *Let* $A = (a_1, \dots, a_n)$ *be a nonnegative* $n \times n$ *matrix. Then*

$$(\text{per } A)^2 \geqslant \text{per}(a_1, a_1, a_3, \dots, a_n)\ \text{per}\,(a_2, a_2, a_3, \dots, a_n).$$

Therorem 9 can be applied to show that certain sequences are log-concave. For example, let x, y be nonnegative vectors in R^n and let

$$\alpha_k = \text{per}\,(\overbrace{x, \dots, x}^{k}, \overbrace{y, \dots, y}^{n-k}),\quad k = 0, 1, \dots, n.$$

Then it is immediate from Theorem 9 that the sequence $\alpha_0, \alpha_1, \ldots, \alpha_n$ is log-concave (i.e., $\alpha_k^2 \geqslant \alpha_{k-1}\alpha_{k+1}$, $k = 1, 2, \ldots, n-1$).

Using the same idea we now show that certain sequences arising from the multiparameter multinomial distribution are log-concave.

As noted in Section 1, it is easy to see that if X has the binomial distribution, then the density function of X is log-concave. Our next result gives a generalization of this fact to the multiparameter multinomial situation.

Theorem 10 : *Let $X = (X_1, \ldots, X_r)$ have the multiparameter multinomial distribtion with the $n \times r$ parameter matrix P. Let $x_3, \ldots, x_r$ be nonnegative integers such that $y = x_3 + \ldots + x_r \leqslant n$ and let*

$$f(x) = Pr(X_1 = x, X_2 = n-y-x \,|\, X_3 = x_3, \ldots, X_r = x_r),\ x = 0, 1, \ldots, n-y.$$

Then f is log-concave.

Proof: Let

$$g(x) = Pr(X_1 = x, X_2 = n-y-x, X_3 = x_3, \ldots, X_r = x_r).$$

Then by (2),

$$g(x) = \frac{1}{x!\,(n-y-x)!\,x_3!\ldots x_r!}\ \text{per}\ P(x, n-y-x, x_3, \ldots, x_r).$$

By theorem 9, $x!\,(n-y-x)!\,g(x)$, $x = 0, 1, \ldots, n-y$ is log-concave. A simple calculation shows that $g(x)$ and hence $f(x)$, $x = 0, 1, \ldots, n-y$ must be log-concave.

The next result uses the same technique.

Theorem 11 : *Suppose there are n coins, of which, m are identical with the same probability of heads equal to p whereas the remaining $n-m$ haave the same probability of heads equal to p'. Let x be fixed, $0 \leqslant x \leqslant n$, and let $f(m)$ denote the probability of getting x heads when the n coins are tossed. Then f is log-concave.*

Proof: By (2),

$$f(m) = \frac{1}{x!\,(n-x)!}\ \text{per}\begin{bmatrix} \overbrace{p \ldots p}^{x} & \overbrace{p' \ldots p'}^{n-x} \\ \vdots \quad \vdots & \vdots \\ p \quad p & p' \ldots p' \end{bmatrix}$$

Hence, the result follows by Theorem 9.

Suppose $X_1, \ldots, X_n$ are independent, identically distributed, continuous random variables with density f and distribution function F. Let $X_{(1)} \leqslant X_{(2)} \leqslant \ldots \leqslant X_{(n)}$ denote the corresponding order statistics. The density function of $X_{(k)}$ is well-known to be

$$g_k(x) = \frac{n\,!}{(k-1)\,!\,(n-k)\,!}\, f(x)\,[F(x)]^{k-1}\,[1-F(x)]^{n-k},\ 1 \leqslant k \leqslant n. \quad \ldots \quad (11)$$

It may easily be verified from (11) that for any x, the sequence $g_k(x)$, $k = 1, 2, \ldots, n$ is log-concave. Note that the fact that $g_k(x)$, $k = 1, 2, \ldots, n$ should be unimodal is intuitively clear. We now show that the log-concavity of $g_k(x)$, $k = 1, 2, \ldots, n$ holds even when $X_1, \ldots, X_n$ are independent but not necessarily identically distributed. To prove this fact, we make use of a representation for g_k in terms of permanents given by Vaughan and Venables (1972).

Thus, let $X_1, \ldots, X_n$ be independent, continuous random variables where X_i has density f_i, $i = 1, 2, \ldots, n$; and let $X_{(1)} \leqslant \ldots \leqslant X_{(n)}$ be the corresponding order statistics. Let F_i denote the distribution function of X_i for each i. For any real x, consider the matrix

$$Z = \begin{bmatrix} f_1(x) & F_1(x) & 1-F_1(x) \\ \vdots & \vdots & \vdots \\ f_n(x) & F_n(x) & 1-F_n(x) \end{bmatrix}$$

If g_k denotes the density of $X_{(k)}$ then Vaughan and Venables (1972) have shown that

$$g_k\,(x) = \frac{n\,!}{(k-1)\,!\,(n-k)\,!}\ \text{per}\ Z(1, k-1, n-k) \qquad \ldots \quad (12)$$

Theorem 12 : *Let $X_1, \ldots, X_n$ be independent random variables where X_i has density f_i and distribution function F_i. Let g_k be the density function of $X_{(k)}$, $k = 1, 2, \ldots, n$. Then for any x, the sequence $g_k(x)$, $k = 1, 2, \ldots, n$ is log-concave.*

Proof : Using (12) and Theorem 9 we see that $(k-1)\,!\,(n-k)\,!\,g_k\,(x)$, $k = 1, 2, \ldots, n$ is log-concave. It follows that $g_k(x)$, $k = 1, 2, \ldots, n$ is log-concave and the proof is complete.

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Paper received: *May, 1987.*