DETERMINATION OF A MATRIX BY ITS SUBCLASSES OF GENERALIZED INVERSES

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SUMMARY. The main result of the paper is : $AB^+A = A$ and $BA^+B = B \Longrightarrow A = B$, where A^+ and B^+ are the unique Moore-Penrose inverses of A and B respectively. This is a stronger result than uniqueness of Moore-Penrose inverse. Some other results are established generalizing the earlier result of Rao and Mitra (1971) that a matrix is uniquely determined by the entire class of its g-inverses.

1. Introduction

Rao and Mitra (1971) have shown that if every g-inverse of A is a g-inverse of B and vice-versa, then $A = B$ (see Theorem 2.4.2, p. 27) or, in other words, a matrix is completely determined by its class of g-inverses. In this paper, we explore the possibilities of characterizing a matrix by its subclasses of g-inverses.

We denote a particular choice of g-inverse of A by A^- and the entire class by $\{A^-\}$. Similarly the reflexive, minimum norm and least squares inverses and their classes are denoted by A_r^-, $\{A_r^-\}$, A_m^-, $\{A_m^-\}$, A_l^-, $\{A_l^-\}$, respectively. The symbols A_{mr}^-, $\{A_{mr}^-\}$, etc., have obvious meaning (see also Table 3.6, p. 16 of Rao and Mitra, 1971).

It is obvious that if $A^+ = B^+$ then $A = B$, since $(A^+)^+ = A$ and $(B^+)^+ = B$, and the Moore-Penrose inverse is unique. However, the more general result is proved in Theorem 4 : $A^+ \epsilon \{B^-\}$ and $B^+ \epsilon \{A^-\} \Longrightarrow A = B$. The other results are :

(i) $\{A_r^-\} \subset \{B^-\}$ and $R(A) = R(B) \Longrightarrow A = B$,

(ii) $\{A_{lr}^-\} \subset \{B_{lr}^-\} \Longrightarrow A = B$, and

(iii) $\{A_{mr}^-\} \subset \{B_{mr}^-\} \Longrightarrow A = B$.

It is also shown by counter examples :

(iv) $\{A_{lr}^-\} \subset \{B^-\}$ and $R(A) = R(B) \not\Rightarrow A = B$,

(v) $\{A_\rho^-\} = \{B_\rho^-\} \not\Rightarrow A = B$, and

(vi) $\{A_\chi^-\} = \{B_\chi^-\} \not\Rightarrow A = B$

where A_ρ^- and A_χ^- are ρ and χ-inverses. (See also Table 3.6, p. 16 of Rao and Mitra, 1971).

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2. Some preliminary lemmas

In the sequel, we use the following lemmas. Lemma 1 is given in Rao and Mitra (1971), (see Lemma 2.2.4 on p. 21 and example 14 on p. 43).

Lemma 1 : *BA^-C is invariant and non-null for all choices of g-inverses of A if and only if B and C are non-null matrices, $\mathcal{M}(C) \subset \mathcal{M}(A)$ and $\mathcal{M}(B^*) \subset \mathcal{M}(A^*)$.*

Proof : We observe that a general representation of a g-inverse G of A is

$$G = A^- + (I - P_{A^*})U + V(I - P_A) \qquad \dots \quad (2.1)$$

where U and V are arbitrary and A^- is a particular choice. (Two other representations are given in formulae (2.4.2) and (2.4.3) of Rao and Mitra, (1971)). Pre- and post-multiplying both sides of (2.1) by B and C respectively and noting that $BGC = BA^-C$, we find

$$B(I - P_{A^*})UC + BV(I - P_A)C = 0 \qquad \dots \quad (2.2)$$

for all U and V implying that

$$B(I - P_{A^*}) = 0 \text{ and } (I - P_A)C = 0 \qquad \dots \quad (2.3)$$

which establishes the desired results.

Lemma 2 : *Let A and B be matrices of the same order. Then the following statements* (i) *and* (ii) *are equivalent :*

(i) $A = B$ $\qquad \dots \quad (2.4)$

(ii) $A^*AA^* = A^*BA^*,\ B^*BB^* = B^*AB^*.$ $\qquad \dots \quad (2.5)$

Proof : Trivially (2.4) $\Longrightarrow$ (2.5). To prove the converse, observe that (2.5) $\Longrightarrow R(A) = R(B) = r$ (say).

Consider the singular value decompositions of A and B

$$A = U_1\Delta_1V_1^*, \quad B = U_2\Delta_2V_2^* \qquad \dots \quad (2.6)$$

where for $i = 1, 2$, U_i, V_i are unitary matrices,

$$\Delta_i = \begin{pmatrix} D_i & 0 \\ 0 & 0 \end{pmatrix}$$

and D_i is a diagonal matrix of order $r \times r$ with strictly positive diagonal elements.

Let $W_1 = U_1^*U_2$ and $W_2 = V_1^*V_2$. Let W_i be partitioned as $\begin{pmatrix} W_{i1} & W_{i2} \\ W_{i3} & W_{i4} \end{pmatrix}$ such that the submatrix W_{i1} is of order $r \times r$.

Check that (2.5) $\Longrightarrow$

$$\Delta_1^*\Delta_1\Delta_1^* = \Delta_1W_1\Delta_2W_2^*\Delta_1, \quad \Delta_2^*\Delta_2\Delta_2^* = \Delta_2W_1^*\Delta_1W_2\Delta_2$$

$$\Longrightarrow D_1^3 = D_1W_{11}D_2W_{21}^*D_1, \quad D_2^3 = D_2W_{11}^*D_1W_{21}D_2$$

$$\Longrightarrow |D_1| = |W_{11}W_{11}^*|\,|D_1|\,|W_{21}W_{21}^*| \qquad \dots \quad (2.7)$$

However, since $W_{i1}W_{i1}^*$ and $W_{i2}W_{i2}^*$ are n.n.d. matrices

$$W_{i1}W_{i1}^*+W_{i2}W_{i2}^* = I \Longrightarrow |W_{i1}W_{i1}^*| \leqslant 1$$

and the sign of equality holds iff $W_{i2} = 0$.

Hence from (2.7), $W_{i2} = 0$, $i = 1, 2$

$$\Longrightarrow \mathcal{M}(A) = \mathcal{M}(B),\ \mathcal{M}(A^*) = \mathcal{M}(B^*)$$

$$\Longrightarrow A = BKB,\ B = ALA \text{ for some } K \text{ and } L.$$

Thus in view of (2.5) we have

$$ALA = B = BB^+B = ALAB^*(B^*BB^*)^-B^*ALA$$

$$= ALAB^*(B^*AB^*)^-B^*ALA = ALALA \text{ and}$$

$$A = BKB = ALAKB = ALALAKB = ALA = B.$$

3. The main results

Theorem 1 : $\{A_r^-\} \subset \{B^-\}$ *and* $R(A) = R(B) \Longrightarrow A = B$.

Proof : Let G be a particular g-inverse of A. Then A^-AG and GAA^- are reflexive g-inverses of A for any $A^- \epsilon \{A^-\}$. Then, $R(A) = R(B)$ and $BA^-AGB = B$ for all $A^- \Longrightarrow \mathcal{M}(B^*) = \mathcal{M}(A^*)$ by applying Lemma 1. Again, $R(A) = R(B)$ and $BGAA^-B = B$ for all $A^- \Longrightarrow \mathcal{M}(A) = \mathcal{M}(B)$. Hence $B = DA = AE$ for some D and E. Now, $BA_r^-B = B \Longrightarrow B = BE = DB \Longrightarrow B(I-E) = 0 \Longrightarrow A(I-E) = 0 \Longrightarrow A = B$.

Corollary : $\{A^-\} \subset \{B^-\}$ *and* $R(A) = R(B) \Longrightarrow A = B$.

Note : However, $\{A_{lr}^-\} \subset \{B^-\}$ and $R(A) = R(B) \not\Rightarrow A = B$. This is demonstrated by the example

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}. \qquad \dots (3.1)$$

The same example demonstrates that $\{A_r^-\} = \{B_r^-\} \not\Rightarrow A = B$.

However, the following Theorem 2 is true.

Theorem 2 : $\{A_{lr}^-\} \subset \{B_{lr}^-\} \Longrightarrow A = B$.

Proof : First observe that under the given hypothesis $R(A) = R(B)$. Further, $B^*B(A^*A)^-A^* = B^*$ for all $(A^*A)^-$ and $R(A) = R(B) \Longrightarrow \mathcal{M}(A) = \mathcal{M}(B)$ and $\mathcal{M}(A^*) = \mathcal{M}(B)$. The rest of the proof follows as in Theorem 1.

Theorem 3 : $\{A_{mr}^-\} \subset \{B_{mr}^-\} \Longrightarrow A = B$.

Proof : The result is established as in Theorem 2.

Theorem 4 : $A^+ \epsilon \{B^-\}$ *and* $B^+ \epsilon \{A^-\} \Longrightarrow A = B$.

Proof : Clearly $A^+\epsilon\{B^-\}$, $B^+\epsilon\{A^-\} \Longrightarrow R(A) = R(B)$. Further $A^+\epsilon\{B^-\} \Longrightarrow BA^+B = B \Longrightarrow A^*AA^* = A^*BA^*$ using the representations

$$A = U_{11}D_1V_{11}^*, \quad B = U_{21}D_2V_{21}^*,$$

$$A^+ = V_{11}D_1^{-1}U_{11}^*, \; B^+ = V_{21}D_2^{-1}U_{21}^*,$$

where U_{11}, V_{11}, U_{21} and V_{21} are partitions of U_1, V_1, U_2 and V_2 defined in (2.6). Similarly $B^+\epsilon\{A^-\} \Longrightarrow B^*BB^* = B^*AB^*$. Hence Theorem 4 follows from Lemma 2.

Reference

RAO, C. RADHAKRISHNA and MITRA, SUJIT KUMAR (1971) : *Generalized Inverse of Matrices and its Applications*, Wiley, New York.

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