ON BOUNDED LENGTH CONFIDENCE INTERVAL FOR THE REGRESSION COEFFICIENT BASED ON A CLASS OF RANK STATISTICS[1]

By MALAY GHOSH[2]
Indian Statistical Institute
and
PRANAB KUMAR SEN
University of North Carolina, Chapel Hill

SUMMARY. Bounded length (sequential) confidence intervals for the regression coefficient (in a simple linear regression model) based on a class of robust rank statistics are considered here, and their various asymptotic properties are studied. In this context, several strong convergence results on some simple and weighted empirical processes as well as on a class of rank order processes are established. Comparison with an alternative procedure based on the least squares estimators is also made.

1. Introduction

Consider the simple linear regression model $X_i = \beta_0+\beta c_i+e_i$, $i = 1, 2, \ldots$, where c_i are known regression constants and the e_i are independent and identically distributed random variables (i.i.d.r.v.) with an absolutely continuous (unknown) cumulative distribution function (c.d.f.) $F(x)$, defined on the real line $(-\infty, \infty)$. We want to provide a robust confidence interval (of confidence coefficient $1-\alpha$, $0 < \alpha < 1$) for the regression coefficient β, such that the length of this confidence interval is bounded above by $2d$, for some specified $d > 0$. The proposed procedure rests on the use of a class of regression rank statistics (due to Hájek, 1962; 1968) for the derivation of robust confidence intervals for β (cf. Sen, 1969, Section 4), as extended here to the sequential case along the lines of Chow and Robbins (1965).

Several results, needed for this sequential extension, are derived here. First an elegant result of Jurečková (1969) on the weak convergence of a class of rank order processes to some appropriate linear processes is strengthened here to almost sure (a.s.) convergence. We need, however, more stringent regularity conditions on the c_i and the score-functions underlying the rank statistics. This result along with a martingale (or semi-martingale) property of the regression rank statistics, given in Section 3, guarantees the "asymptotic (as $d\to 0$) consistency" and "efficiency" (see Chow and Robbins, (1965)) of the proposed sequential procedure. Further, this enables one to study its asymptotic relative efficiency (ARE) with respect to the procedure by Gleser (1965) and Albert (1966), which are based on the least squares estimators. In this context, several well-known rank statistics are considered and the allied ARE results are beriefly presented. In particular, for the so-called normal scores statistic, it is

[1]Work supported by the National Institutes of Health, Grant GM-12868.

[2]Most of the work was done while the author was at the University of North Carolina at Chapel Hill.

shown that the ARE is bounded below by 1, uniformly in a borad class of $\{F\}$. It may be remarked at this point that a bounded length confidence interval for the location parameter in the one sample situation was considered in Sen and Ghosh (1971). Though in techniques and in definition of the score functions, there is some similarity, the basic results are quite different.

2. Notations, assumptions and the main theorem

In accordance with the model of Section 1, consider a sequence $\{X_1, X_2, \ldots\}$ of independent random variables for which

$$F_i(x) = P\{X_i \leqslant x\} = F(x-\beta_0-\beta c_i), \quad i = 1, 2, \ldots, \qquad \ldots \quad (2.1)$$

where β_0 is a nuisance parameter. We intend to determine a confidence interval $I_n = \{\beta : \hat{\beta}_{L,n} < \beta < \hat{\beta}_{U,n}\}$ (where $\hat{\beta}_{L,n}$ and $\hat{\beta}_{U,n}$ are statistics) such that

$$P\{\beta \in I_n\} = 1-\alpha, \text{ the presigned confidence coefficient,} \qquad \ldots \quad (2.2)$$

$$0 < \hat{\beta}_{U,n}-\hat{\beta}_{L,n} \leqslant 2d, \text{ for some predetermined } d(>0). \qquad \ldots \quad (2.3)$$

Since F is not known, no fixed-sample size procedure sounds valid for all F. It is therefore desired to determine sequentially a stopping variable N(a positive integer) and the corresponding $(\hat{\beta}_{L,N}, \hat{\beta}_{U,N})$, such that (2.2) and (2.3) hold. Our proposed procedure is based on the following class of regression rank statistics.

In a sample $X_n = (X_1, \ldots, X_n)$ of size $n(\geqslant 1)$, let $R_{ni} = \sum_{j=1}^{n} u(X_i - X_j)$ [where $u(t)$ is 1 or 0 according as $t \geqslant$ or < 0] be the rank of X_i, $1 \leqslant i \leqslant n$. Let

$$c^*_{ni} = (c_i - \bar{c}_n)/C_n, \; 1 \leqslant i \leqslant n, \text{ where } \bar{c}_n = n^{-1}\sum_{1}^{n} c_i \text{ and } C_n^2 = \sum_{1}^{n}(c_i-\bar{c}_n)^2. \qquad \ldots \quad (2.4)$$

Then, as in Hájek (1962, 1968), a regression rank statistic is defined as

$$T_n = T(X_n) = \sum_{i=1}^{n} c^*_{ni}\, J_n(R_{ni}/(n+1)), \qquad \ldots \quad (2.5)$$

where the "scores" $J_n(i/(n+1))$, $1 \leqslant i \leqslant n$, are generated by a "score-function" $\{J(u) : 0 < u < 1\}$ in the following manner. We let $J_n(u) = J_n\left(\frac{i}{n+1}\right)$, for $(i-1)/n < u \leqslant i/n$, $1 \leqslant i \leqslant n$, and define

$$J_n\left(\frac{i}{n+1}\right) \text{ as equal to } J\left(\frac{i}{n+1}\right) \text{ or } EJ(U_{ni}),\; 1 \leqslant i \leqslant n, \qquad \ldots \quad (2.6)$$

where $U_{n1} \leqslant \ldots \leqslant U_{nn}$ are the n ordered random variables in a sample of size n from the rectangular (0, 1) distribution. The score-function $J(u)$ is defined as $\Psi^{-1}(u)$: $0 < u < 1$, where $\Psi(x)$ is an absolutely continuous c.d.f. satisfying the condition that

$$\lim_{u \downarrow 0} \psi(\Psi^{-1}(u))/u > K, \quad \lim_{u \uparrow 1} \psi(\Psi^{-1}(u))/(1-u) > K, \qquad \ldots \quad (2.7)$$

where $\psi(x) = (d/dx)\Psi(x)$ and $0 < K < \infty$. As in (2.5) and (2.6) of Sen and Ghosh (1971), (2.7) implies that

$$|J(u)| \leqslant K_0[-\log u(1-u)],\ |J'(u)| \leqslant K_0[u(1-u)]^{-1},\ 0 < u < 1;\ 0 < K_0 < \infty,$$

and there exists a positive t_0 such that

$$M(t) = \int_{-\infty}^{\infty} \exp(tx)d\Psi(x) < \infty \text{ for all } 0 < t \leqslant t_0. \quad \ldots \quad (2.9)$$

Note that, by definition, $J(u)$ is $\uparrow$ in $u : 0 < u < 1$. These assumptions are more restrictive than those in Hájek (1962, 1968) and Jurečková (1969), where only the weak convergence results are studied. Since we are not merely interested in strong convergence of our stopping variable but also in 'convergence in mean' type results (see Theorem 2.1), the above conditions are needed. It may be noted that (2.7) holds for the entire class of normal, logistic, double-exponential, exponential, rectangular and many other c.d.f.'s. In fact, if we use $J(u) = u$[i.e., Ψ recengular c.d.f.] or $J(u)$ as the inverse of the standard normal c.d.f., we obtain respectively the *Wilcoxon* and the *normal scores*; the corresponding T_n are termed the *Wilcoxon* and the *normal scores linear (regression) rank statistics.*

The following assumptions are made regarding $\mathbf{c}_n = (c_1, \ldots, c_n)$. The first assumption is due to Hájek (1968).

(i) $\max_{1 \leqslant i \leqslant n} |c^*_{ni}| = O(n^{-\frac{1}{2}})$, ... (2.10)

(ii) $\liminf_n n^{-1} C_n^2 > K_0 > 0$, ... (2.11)

(iii) define $Q(x) = (n+1-x)C_n^2 + (x-n)C_{n+1}^2$, for $n \leqslant x \leqslant n+1$, $n = 0, 1, \ldots$, where we let $C_0^2 = 0$. Assume that $Q(x)$ is $\uparrow$ in x and

$$\lim_{n\to\infty} Q(na_n)/Q(n) = s(a) \text{ whenever } \lim_{n\to\infty} a_n = a, \quad \ldots \quad (2.12)$$

$s(a)$ being strictly monotone (increasing) with $s(1) = 1$. The condition (2.10) is again more stringent than the classical Noether-condition (cf. Hájeck, (1962; 1968)), but is satisfied in the majority of practical situations. (2.11) is less restrictive than the parallel condition : $\lim_{n\to\infty} n^{-1}C_n^2 = K_0 > 0$ assumed by Gleser (1965) and Albert (1966) in connection with the least squares theory. For example, if $c_i = a+ih$, $h > 0$, $i = 1, 2, \ldots$, (2.10) and (2.11) hold but Gleser's condition does not hold. Also, in this case, $Q(n) = nh^2(n^2-1)/12$, so that $s(a) = a^3$.

Finally, we assume that $F \epsilon \mathcal{F}(\Psi)$, where $\mathcal{F}(\Psi)$ is the class of all absolutely continuous F for which the density function $f(x)$ and its first derivative $f'(x)$ are bounded for almost all $x(a.e.x)$ and further

$$\lim_{x\to\pm\infty} f(x)J'[F(x)] \text{ are finite.} \quad \ldots \quad (2.13)$$

From (2.1), it follows that under $H_0 : \beta = 0$, implying that $X_1, \ldots, X_n$ are i.i.d.r.v., $T(X_n)$ has a completely specified distribution generated by the $n!$ equally likely realizations of $(R_{n1}, \ldots, R_{nn})$ over the permutations of $(1, \ldots, n)$. Hence, there exists a known constant $T_n^{(1)}$ (depending on $\mathbf{c}_n$), and an α_n (known) such that

$$P\{-T_n^{(1)} \leq T(X_n) \leq T_n^{(1)} \mid H_0\} = 1-\alpha_n(\rightarrow 1-\alpha \text{ as } n\rightarrow\infty). \qquad \ldots \quad (2.14)$$

[For small n, α_n may not be equal to α]. If we let

$$A_n^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} \left[J_n\left(\frac{i}{n+1}\right)-\bar{J}_n\right]^2; \; \bar{J}_n = \frac{1}{n} \sum_{i=1}^{n} J_n\left(\frac{i}{n+1}\right), \qquad \ldots \quad (2.15)$$

then from the classical Wald-Wolfowitz-Noether-Hájek permutational central limit theorem (viz., Hájek and Šidák, 1967, p. 160), we have

$$\lim_{n\rightarrow\infty} \{T_n^{(1)}/A_n\} = \tau_{\alpha/2}, \qquad \ldots \quad (2.16)$$

where $\Phi(\tau_\varepsilon) = 1-\varepsilon$ and $\Phi(x)$ is the standard normal c.d.f.

It follows from Sen (1969, Section 6) that $T(X_n-a.\mathbf{c}_n)$ is $\uparrow$ in $a : -\infty < a < \infty$. Hence, if we let

$$\hat{\beta}_{L,n} = \sup\{a : T(X_n-a.\mathbf{c}_n) > T_n^{(1)}\}, \hat{\beta}_{U,n} = \inf\{a : T(X_n-a.\mathbf{c}_n) < -T_n^{(1)}\} \quad \ldots \quad (2.17)$$

it follows as in Sen (1969, Section 4) that

$$P\{\hat{\beta}_{L,n} \leq \beta \leq \hat{\beta}_{U,n} \mid \beta\} = 1-\alpha_n(\rightarrow 1-\alpha \text{ as } n\rightarrow\infty). \qquad \ldots \quad (2.18)$$

We are now in a position to define our sequential procedure. For every $d > 0$, let $N(d)$, the stopping variable, be the smallest positive integer $\geqslant n_0$ [an initial sample size $(\geqslant 3)$] for which $\hat{\beta}_{U,N(d)}-\hat{\beta}_{L,N(d)} \leq 2d$. Then our proposed confidence interval for β is $I_{N(d)} = \{\beta : \hat{\beta}_{L,N(d)} \leq \beta < \hat{\beta}_{U,N(d)}\}$ and is based on the stopping variable $N(d)$. We justify the proposed procedure on the ground of its robustness (for outliers or gross errors etc.). Its asymptotic properties, considered in the following theorem, are sketched in the same fashion as in Chow and Robbins (1965).

Theorem 2.1 : *Under the assumptions made above $N(d)$ is a non-increasing function of $d(>0)$, it is finite a.s., $EN(d) < \infty$ for all $d > 0$, $\lim_{d\rightarrow 0} N(d) = \infty$ a.s., and $\lim_{d\rightarrow 0} EN(d) = \infty$. Further,*

$$\lim_{d\rightarrow 0} N(d)/Q^{-1}(\nu(d)) = 1 \text{ a.s.}, \qquad \ldots \quad (2.19)$$

$$\lim_{d\rightarrow 0} P\{\beta\epsilon I_{N(d)}\} = 1-\alpha \text{ for all } F\epsilon\mathcal{F}(\Psi), \qquad \ldots \quad (2.20)$$

$$\lim_{d\rightarrow 0} [EN(d)]/Q^{-1}(\nu(d)) = 1, \qquad \ldots \quad (2.21)$$

where

$$\nu(d) = [\{A\tau_{\alpha/2}\}/\{dB(F)\}]^2, \; B(F) = \int_{-\infty}^{\infty} (d/dx)J[F(x)]dF(x), \qquad \ldots \quad (2.22)$$

$$A^2 = \int_0^1 J^2(u)du-\mu^2 \text{ and } \mu = \int_0^1 J(u)du. \qquad \ldots \quad (2.23)$$

The proof of the theorem is postponed to Section 4; certain other results needed in this context and having importance of their own are derived in the next section.

3. Asymptotic behaviour of some empirical processes

It has been shown by Jurečková (1969) that under $H_0 : \beta = 0$, for all real and finite b, denoting $W(X_n, b) = T(X_n) - T(X_n - b\,c_n^*) - bB(F)$, for every $C > 0$

$$\sup_{b:\,|b|<C} |W(X_n, b)| \xrightarrow{P} 0 \text{ as } n \to \infty; \qquad \dots \quad (3.1)$$

our primary concern is not only to strengthen this statement to almost sure convergence but also to specify the order of convergence as well, extending the range of b to $|b| \leqslant C(\log n)^k$, $k \geqslant 1$, $0 < C < \infty$. We define now

$$H_n(x; b) = \frac{1}{n+1} \sum_1^n u(x + bc_{ni}^* - X_i), \quad S_n^*(x; b) = \sum_1^n c_{ni}^* u(x + bc_{ni}^* - X_i), \qquad \dots \quad (3.2)$$

for $-\infty < x < \infty$ and $-\infty < b < \infty$. Then, by (2.5) and (3.2), we have

$$T(X_n - b\,c_n^*) = \int_{-\infty}^{\infty} J_n[H_n(x; b)]dS_n^*(x; b). \qquad \dots \quad (3.3)$$

It will be convenient for us to study first the asymptotic behaviour of the two processes in (3.2). This will be useful in proving (3.1) with a specific order of convergence.

Let $\{Y_1, Y_2, \dots\}$ be a sequence of i.i.d.rv. having the rectangular (0, 1) distribution. For every $t : 0 < t < 1$, define

$$d_{ni}(t; b) = F(F^{-1}(t) + bc_{ni}^*) - t, \quad 1 \leqslant i \leqslant n, \quad -\infty < b < \infty. \qquad \dots \quad (3.4)$$

It follows that for every $i : 1 \leqslant i \leqslant n$ and $b : -\infty < b < \infty$,

(i) $0 < t + d_{ni}(t; b) \leqslant 1$; $t + d_{ni}(t; b)$ is $\uparrow$ in $t : 0 < t < 1$, ... (3.5)

(ii) $d_{ni}(t; b)$ and bc_{ni}^* have the same sign, ... (3.6)

(iii) $\bar{d}_n(t; b) = n^{-1} \sum_1^n d_{ni}(t, b) = b^2[O(n^{-1})] \;\forall\; F \epsilon \mathcal{F}(\Psi)$. ... (3.7)

Consider then the two stochastic processes

$$G_n^*(t; b) = \sum_1^n c_{ni}^* u(t + d_{ni}(t; b) - Y_i) - \sum_1^n c_{ni}^* d_{ni}(t; b), \quad -\infty < b < \infty, \; 0 < t < 1; \qquad \dots \quad (3.8)$$

$$L_n^*(t; b) = (n+1)^{-1} \sum_{i=1}^n u(t + d_{ni}(t; b) - Y_i), \quad -\infty < b < \infty, \; 0 \leqslant t \leqslant 1. \qquad \dots \quad (3.9)$$

Simple computations yield that

$$\zeta_n(t; b) = \sum_{i=1}^n c_{ni}^* d_{ni}(t; b) = bf(F^{-1}(t)) + b^2.O(n^{-1}); \qquad \dots \quad (3.10)$$

$$G_n^*(t; b) = S_n^*(F^{-1}(t); b) - \zeta_n(t; b) = S_n^*(F^{-1}(t); b) - bf(F^{-1}(t)) + b^2.O(n^{-1}); \qquad \dots \quad (3.11)$$

$$L_n^*(t; b) = (n+1)^1 H_n(F^{-1}(t); b), \quad 0 \leqslant t \leqslant 1 \quad -\infty < b < \infty. \qquad \dots \quad (3.12)$$

Theorem 3.1 : *For every $h(>0)$, there exist two positive constants K_1, K_2 and n^* (all of which may depend on h) such that for $n \geq n^*$, $k \geq 1$ and $0 < \delta < 1/4$,*

$$P\{\sup_{0<t<1} \sup_{b\in I_n^*} |G_n^*(t;b)-G_n^*(t;0)| \geq K_1 n^{-\delta} (\log n)^k\} \leq K_2 n^{-h}, \quad \ldots \quad (3.13)$$

$$P\{\sup_{0<t<1} \sup_{b\in I_n^*} |L_n^*(t;b)-L_n^*(t;0)| \geq K_1 n^{-\delta}(\log n)^k\} \leq K_2 n^{-h}, \quad \ldots \quad (3.14)$$

where $I_n^* = \{b : |b| \leq C(\log n)^k, 0 < C < \infty\}$.

Proof : We start with the proof of (3.13). Let $a_n = C(\log n)^k$, $r_n = [n^{-\delta_1}]$, and $\eta_{r,n} = ra_n/r_n$, for $r = 0, \pm 1, \ldots, \pm r_n$, where $0 < \delta < \delta_1 < \frac{1}{4}$ and $[u]$ denotes the integral part of $u(\geq 0)$. Then, $\eta_{r+1,n}-\eta_{r,n} = O(n^{-\delta_1} (\log n)^k)$, for all $r = -r_n, \ldots, r_n-1$. We complete the proof in several steps.

Step 1 : We first show that

$$\sup_{b\in I_n^*} |G_n^*(t;b)-G_n^*(t;0)| \leq \max_{-r_n\leq r\leq r_n} |G_n^*(t;\eta_{r,n})-G_n^*(t;0)|+O(n^{-\delta_1}(\log n)^k). \quad \ldots \quad (3.15)$$

To prove (3.15), we note that $\zeta_n(t;0) = 0$, so that $G_n^*(t;b)-G_n^*(t;0) = S_n^*(F^{-1}(t);b)-S_n^*(F^{-1}(t);0)-\zeta_n(t;b)$. But, for $b\epsilon[\eta_{r,n}, \eta_{r+1,n}]$,

$$c_{ni}^*[u(t+d_{ni}(t;\eta_{r,n})-Y_i)-u(t-Y_i)] \leq c_{ni}^*[u(t+d_{ni}(t;b)-Y_i)-u(t-Y_i)]$$
$$\leq c_{ni}^*[u(t+d_{ni}(t;\eta_{r+1,n})-Y_i)-u(t-Y_i)],\ 1 \leq i \leq n; \quad \ldots \quad (3.16)$$
$$\zeta_n(t;\eta_{r,n}) \leq \zeta_n(t;b) \leq \zeta_n(t;\eta_{r+1,n});$$

$$\zeta_n(t;\eta_{r+1,n})-\zeta_n(t;\eta_{r,n}) = O(n^{-\delta_1} (\log n)^k), \quad \ldots \quad (3.17)$$

for all $r = -r_n, \ldots, r_n-1$, $0 < t < 1$. Hence, for $b\epsilon[\eta_{r,n}, \eta_{r+1,n}]$,

$$|G_n^*(t;b)-G_n^*(t;0)| \leq \max_{j=r,r+1} |G_n^*(t;\eta_{j,n})-G_n^*(t;0)|+O(n^{-\delta_1}(\log n)^k), \quad \ldots \quad (3.18)$$

and (3.15) follows directly from (3.18).

Define now $s_n = [n^{1/2+\delta_1}]$, $\xi_{s,n} = s/s_n$, $s = 0, 1, \ldots, s_n$. Also, let

$$S_1^{(n)} = \{i : c_{ni}^* \geq 0,\ i = 1, \ldots, n\}\},\ S_2^{(n)} = \{i : c_{ni}^* < 0,\ i = 1, \ldots, n\};$$

$$U_{b,n}^{(s)} = \sum_{i\epsilon S_2^{(n)}} c_{ni}^*[u(\xi_{s,n}+d_{ni}(\xi_{s,n};b)-Y_i)-u(\xi_{s+1,n}+d_{ni}(\xi_{s+1,n};b)-Y_i)]$$

$$V_{b,n}^{(s)} = \sum_{i\epsilon S_1^{(n)}} c_{ni}^*[((\xi_{s+1,n}+d_{ni}(\xi_{s+1,n};b))-(\xi_{s,n}+d_{ni}(\xi_{s,n};b))].$$

Note that $U_{b,n}^{(s)} \geq 0$ and $V_{b,n}^{(s)} \geq 0$ for all $b\epsilon I_n^*$ and $s = 0, \ldots, s_n$.

Step 2 : We shall show that

$$\sup_{0<t<1} |G_n^*(t;b)-G_n^*(t,0)| \leq \max_{0\leq s\leq s_n} |G_n^*(\xi_{s,n};b)-G_n^*(\xi_{s,n};0)|+\max_{0\leq s\leq s_n} V_{b,n}^{(s)}+\max_{0\leq s\leq s_n} U_{b,n}^{(s)}.$$
$$\ldots \quad (3.19)$$

To prove (3.19), we note that by some simple algebraic manipulations, for

$$t\epsilon[\xi_{s,n}, \xi_{s+1,n}], \sum_{i=1}^{n} c_{ni}^{*}[u(\xi_{s,n}+d_{ni}(\xi_{s,n}; b)-Y_i)-[\xi_{s,n}+d_{ni}(\xi_{s,n}; b)]]$$

$$-V_{b,n}^{(s)}-U_{b,n}^{(s)} \leqslant \sum_{i=1}^{n} c_{ni}^{*}[u(t+d_{ni}(t; b)-Y_i)-[t+d_{ni}(t; b)]]$$

$$\leqslant \Sigma\, c_{ni}^{*}[u(\xi_{s+1,n}+d_{ni}(\xi_{s+1,n}; b)-Y_i)-[\xi_{s+1,n}+d_{ni}(\xi_{s+1,n}; b)]]+V_{b,n}^{(s)}+U_{b,n}^{(s)}.$$

Thus, for $b\epsilon I_n^{*}$ and $t\epsilon[\xi_{s,n}, \xi_{s+1,n}]$,

$$|G_n^{*}(t, b)-G_n^{*}(t; 0)| = \left|\sum_{i=1}^{n} c_{ni}^{*}[u(t+d_{ni}(t; b)-Y_i)-[t+d_{ni}(t; b)]]\right|$$

$$\leqslant \max_{j=s,\, s+1}\left|\sum_{i=1}^{n} c_{ni}^{*}[u(\xi_{j,n}+d_{ni}(\xi_{j,n}; b)-Y_i)-[\xi_{j,n}+d_{ni}(\xi_{j,n}; b)]]\right|$$

$$+V_{b,n}^{(s)}+U_{b,n}^{(s)} = \max_{j=s,\, s+1} |G_n^{*}(\xi_{j,n}; b)-G_n^{*}(\xi_{j,n}; 0)|+V_{b,n}^{(s)}+U_{b,n}^{(s)},$$

for $s = 0, 1, \ldots, s_n-1$, which imply (3.19)

Combining now steps 1 and 2, one gets

Step 3 :

$$\sup_{0<t<1}\ \sup_{|b|\leqslant a_n} |G_n^{*}(t; b)-G_n^{*}(t; 0)| \leqslant \max_{0\leqslant s\leqslant s_n}\ \max_{|r|\leqslant r_n} |G_n^{*}(\xi_{s,n}; \eta_{r,n})-G_n^{*}(\xi_{s,n}; 0)|$$

$$+\max_{0\leqslant s\leqslant s_n}\ \max_{-r_n\leqslant r\leqslant r_{n-1}} V_{\eta_{r,n},n}^{(s)}+\max_{0\leqslant s\leqslant s_n}\ \max_{-r_n\leqslant r\leqslant r_{n-1}} U_{\eta_{r,n},n}^{(s)}+O(n^{-\delta_1}(\log n)^k). \quad \ldots (3.20)$$

Returning now to the proof of (3.13), it follows that we are only to show that for every $h > 0$, there exist K_1, K_2 and n^{*} (depending on h), such that the first three terms on the right hand side of (3.20) are bounded above by $K_1 n^{-\delta}$ $(\log n)^k$ with probability $\geqslant 1-K_2 n^{-h}$ for all $n \geqslant n^{*}$.

Consider any fixed r (say $\geqslant 0$) and any fixed s. Then, $G_n^{*}(\xi_{s,n}; \eta_{r,n})-G_n^{*}(\xi_{s,n}; 0) = \sum_{i=1}^{n}(Z_{ni}-EZ_{ni})$, where, $Z_{ni} = c_n^{*}[u(\xi_{s,n}+d_{ni}(\xi_{s,n}; \eta_{r,n})-Y_i)-u(\xi_{s,n}-Y_i)]$, $i = 1, 2, \ldots, n$ are independent and Z_{ni} can assume the values 0 and $|c_{ni}^{*}|$ respectively with probability $1-|d_{ni}(\xi_{s,n}; \eta_{r,n})|$ and $|d_{ni}(\xi_{s,n}; \eta_{r,n})|$ respectively. Hence, for every $g > 0$, and $h > 0$,

$$P\Big\{\sum_{i=1}^{n}(Z_{ni}-EZ_{ni}) > g\Big\} \leqslant \exp\Big\{-h\Big[g+\sum_{i=1}^{n} EZ_{ni}\Big]\Big\}\prod_{i=1}^{n} E\{\exp(h\, Z_{ni})\}$$

$$= \exp\{-h[g+\zeta_n(\xi_{s,n}; \eta_{r,n})]\}\prod_{i=1}^{n}\{1+(\exp[h|c_{ni}^{*}|]-1)|d_{ni}(\xi_{s,n}; \eta_{r,n})|\}$$

$$= \chi_n(g, h), \text{ (say)} \quad \ldots (3.21)$$

Hence, $\log \chi_n(g, h) = -h[g+\zeta_n(\xi_{s,n};\eta_{r,n})]+\sum_{i=1}^{n}\log[1+(\exp[h|c^*_{ni}|]-1)\,|d_{ni}(\xi_{s,n};\eta_{r,n})|]$ $\leqslant -h[g+\zeta_n(\xi_{s,n};\eta_{r,n})+\sum_{i=1}^{n}(\exp[h|c^*_{ni}|]-1)|d_{ni}(\xi_{s,n},\eta_{r,n})| = -h[g+\zeta_n(\xi_{s,n};\eta_{r,n})]+h\sum_{i=1}^{n}$ $|c^*_{ni}|\,|d_{ni}(\xi_{s,n};\eta_{r,n})|+\frac{1}{2}h^2\sum_{i=1}^{n}(c^*_{ni})^2|d_{ni}(\xi_{s,n};\eta_{r,n})|e^{\theta h|c^*_{ni}|}$ (where $0<\theta<1$). This can be rewritten as $-hg+\frac{1}{2}h^2\sum_{i=1}^{n}(c^*_{ni})^2.|d_{ni}(\xi_{s,n};\eta_{r,n})|e^{\theta h|c^*_{ni}|}$, which upon choosing $g = g_n = K_1 n^{-\delta_1}(\log n)^k$, $h = h_n = n^{\delta_1}$, where $0<\delta<\delta_1<\frac{1}{4}$, $k>1$, $0<K_1<\infty$, simplifies to

$$\log \chi_n(g_n, h_n) = -K_1' n^{\delta_1-\delta}(\log n)^k[1+O(n^{-(\frac{1}{2}+\delta+\delta_1)})] \quad \ldots \quad (3.22)$$

uniformly in $s = 0, 1, \ldots, s_n$ and $|r| \leqslant r_n$. Thus, for n adequately large, we have, uniformly in r and s.

$$P\{G^*_n(\xi_{s,n};\eta_{r,n})-G^*_n(\xi_{s,n};0) > K_1 n^{-\delta}(\log n)^k\} \leqslant \exp\{-K_1 n^{\delta_1-\delta}(\log n)^k\}[1+o(1)], \quad \ldots \quad (3.23)$$

and a similar bound is easily obtained for the left hand tail. Thus, for large n, $0 \leqslant s \leqslant s_n$, $|r| \leqslant r_n$,

$$P\{|G^*_n(\xi_{s,n};\eta_{r,n})-G^*_n(\xi_{s,n};0)| > K_1 n^{-\delta}(\log n)^k\} \leqslant 2\exp\{-K_1 n^{\delta_1-\delta}(\log n)^k[1+o(1)]\}. \quad \ldots \quad (3.24)$$

Also, it follows obviously that

$$V^{(s)}_{\eta_{r,n},n} = O(n^{-\delta_1}) \text{ uniformly in } |r| \leqslant r_n,\ \ 0 \leqslant s \leqslant s_n. \quad \ldots \quad (3.25)$$

By similar techniques as in (3.21), we get after putting $h = n^{\delta_1}$ that for fixed r and s,

$$P\{U^{(s)}_{\eta_{r,n},n} > g_n\} \leqslant \exp[-K_1 n^{\delta_1-\delta}(\log n)^k(1+O(n^{\delta-\delta_1}(\log n)^{-k}))]. \quad \ldots \quad (3.26)$$

From (3.24)—(3.26), we obtain (by use of the Bonferroni inequality) that for large n, the left hand side of (3.20) is bounded above by $K_1 n^{-\delta}(\log n)^k$ with probability greater than or equal to

$$1-2(s_n+1)2r_n\exp\{-K_1 n^{\delta_1-\delta}(\log n)^k(1+o(1))\} \quad \ldots \quad (3.27)$$

since $s_n = [n^{\frac{1}{2}+\delta_1}]$, $r_n = n^{\delta_1}$ and $0<\delta<\delta_1$, K_1 can always be so selected that for $n \geqslant n^*$, (3.27) is bounded from below by $1-4\exp\{-h(\log n)^k\} \geqslant 1-4n^{-h}$ as $k>1$. This completes the proof of (3.13)

To prove (3.14), we first observe that by the same technique as in above

$$\sup_{0<t<1}\sup_{b\in I^*_n}|L^*_n(t;b)-L^*_n(t;0)| \leqslant I^*_{n1}+I^*_{n2}+I^*_{n3}, \quad \ldots \quad (3.28)$$

where

$$I^*_{n1} = \max_{0\le s\le s_n} \max_{|r|\le r_n} |L^*_n(\xi_{s,n};\eta_{r,n})-L^*_n(\xi_{s,n};0)|$$

$$I^*_{n2} = \max_{0\le s\le s_{n-1}} [L^*_n(\xi_{s+1,n};0)-L^*_n(\xi_{s,n};0)]$$

$$I^*_{n3} = \max_{0\le s\le s_n}\cdot\max_{|r|\le r_n} |W^{(s)}_{r,n}|,$$

and $W^{(s)}_{r,n} = (n+1)^{-\frac{1}{2}} \sum_{i\in S_2^{(n)}} [u(\xi_{s,n}+d_{ni};(\xi_{s,n};\eta_{r,n})-Y_i)-u(\xi_{s,n}+d_{ni}(\xi_{s,n};\eta_{r+1,n})-Y_i)]$.

Step 1 : We first show that for a given $h>0$, K_1, K_2 and n^* can be so chosen that $P\{I^*_{n1} > K_1 n^{-\delta}(\log n)^k\} \le K_2 n^{-h}$ for $n \ge n^*$. To do so, we use (3.9) and obtain that for every $g_n > 0$, $h_n > 0$

$$\log P\{L^*_n(\xi_{s,n};\eta_{r,n})-L^*_n(\xi_{s,n};0) > g_n\}$$

$$\le -(n+1)^{\frac{1}{2}}h_n g_n+\{\sum_{i\in S_1^{(n)}}+\sum_{i\in S_2^{(n)}}\}\log E\{\exp[h_n\{u(\xi_{s,n}+d_{ni}(\xi_{s,n};\eta_{r,n})-Y_i)-u(\xi_{s,n}-Y_i)\}]\}$$

$$= -(n+1)^{\frac{1}{2}}h_n g_n+\sum_{i\in S_1^{(n)}}[\log 1+(e^{h_n}-1)d_{ni}(\xi_{s,n};\eta_{r,n})]$$

$$+\sum_{i\in S_2^{(n)}}\log[1-(1-e^{-h_n})(-d_{ni}(\xi_{s,n};\eta_{r,n}))]. \qquad \dots \quad (3.29)$$

On using the inequality that for $|x|<1$, $\log(1\pm|x|)\le\pm|x|$, and on taking $h_n = n^{-\delta_1}$, $g_n = K_1 n^{-\delta}(\log n)^k$, $\left(0<\delta<\delta_1<\frac{1}{4}\right)$, we get for adequately large n that,

$$\log P\{L^*_n(\xi_{s,n};\eta_{r,n})-L^*_n(\xi_{s,n};0) > K_1 n^{-\delta}(\log n)^k\}$$

$$\le -K_1 n^{\frac{1}{2}-\delta-\delta_1}(\log n)^k+n^{-\delta_1}\Big\{\sum_{i=1}^{n} d_{ni}(\xi_{s,n};\eta_{r,n})\Big\}$$

$$+O(n^{-2\delta_1})\Big\{\sum_{i=1}^{n} |d_{ni}(\xi_{s,n};\eta_{r,n})|\Big\}$$

$$= -K_1 n^{\frac{1}{2}-\delta-\delta_1}(\log n)^k+n^{-\delta_1}[n\{O(n^{-1})(\log n)^{2k}\}]$$

$$+O(n^{-2\delta_1}).O(n^{\frac{1}{2}}(\log n)^k), \text{ by (2.10), (3.4) and (3.7).} \qquad \dots \quad (3.30)$$

Since $\delta<\delta_1$, if we let $\delta_1 \le \frac{1}{4}$, the right hand side of (3.30) is

$$-K_1 n^{\frac{1}{2}-\delta-\delta_1}(\log n)^k[1+O(n^{-\frac{1}{2}+\delta}(\log n)^k)+O(n^{-(\delta_1-\delta)})], \qquad \dots \quad (3.31)$$

and hence, for n sufficiently large,

$$P\{L^*_n(\xi_{s,n};\eta_{r,n})-L^*_n(\xi_{s,n};0) > K_1 n^{-\delta}(\log n)^k\} \exp\{-K_1' n^{\frac{1}{2}-\delta-\delta_1}(\log n)^k\},\ 0<K_1'<\infty. \qquad \dots \quad (3.32)$$

and the same bound holds for the left hand tail. Hence, by the Bonferroni inequality, for large n,

$$P\{\max_{0\leqslant s\leqslant s_n}\max_{|r|\leqslant r_n}|L_n^*(\xi_{s,n};\eta_{r,n})-L_n^*(\xi_{s,n};0)|>K_1n^{-\delta}(\log n)^k\}$$

$$<4r_n(s_n+1)\exp\{-K_1'n^{\frac{1}{2}-\delta-\delta_1}(\log n)^k\} \quad \ldots \quad (3.33)$$

$$<4n^{-h} \text{ for } k>1 \text{ and } n>n^*,$$

where $h(>0)$ is any fixed number.

Step 2 : We next show that for any fixed $h(>0)$, there exist a $K_2>0$ and an n^*, such that for $n>n^*$, $P\{I_{n2}^*\leqslant K_2n^{-\delta}(\log n)^k\}>1-n^{-h}$. Since $L_n^*(\xi_{s+1,n};0)-L_n^*(\xi_{s,n};0)=(n+1)^{-1}\sum_{i=1}^{n}[u(\xi_{s+1,n}-Y_i)-u(\xi_{s,n}-Y_i)]$ involves i.i.d. (0, 1) valued random variables, proceeding as in (3.21)–(3.24) and using the Bonferroni inequality we obtain

$$P\{\max_{0\leqslant s\leqslant s_n-1}L_n^*(\xi_{s+1,n};0)-L_n^*(\xi_{s,n};0)>K_2n^{-\delta}(\log n)^k\}$$

$$<s_n\exp[-K_2n^{\frac{1}{2}-\delta-\delta_1}(\log n)^k\{1+O(n^{\delta-\delta_1})\}]$$

$$<n^{-h} \text{ for any fixed } h>0 \text{ by proper choice of } K_2, \text{ (since } k>1). \quad \ldots \quad (3.34)$$

Step 3 : We show that for every $h>0$, there exist a $K_3>0$ and an n^*, such that for $n>n^*$, $P\{I_{n3}^*\leqslant K_3n^{-\delta}(\log n)^k\}>1-4n^{-h}$. To do so, we note that on proceeding as before

$$P\{|W_{r,n}^{(s)}|>K_3n^{-\delta}(\log n)^k\}\leqslant 2\exp[-K_3n^{\frac{1}{2}-\delta-\delta_1}(\log n)^k[1+O(n^{\delta-\delta_1})]], \quad \ldots \quad (3.35)$$

uniformly in $|r|\leqslant r_n$ and $0\leqslant s\leqslant s_n$. As before, on using the Bonferroni inequality, we get

$$P\{\max_{0\leqslant s\leqslant s_n}\max_{|r|\leqslant r_n}|W_{r,n}^{(s)}|>K_3n^{-\delta}(\log n)^k\}$$

$$\leqslant 4r_n(s_n+1)\exp\{-K_3n^{\frac{1}{2}-\delta-\delta_1}(\log n)^k(1+o(1))\}$$

$$\leqslant 4n^{-h}, \text{ for any fixed } h(>0), \text{ as } k>1. \quad \ldots \quad (3.36)$$

(3.14) then follows from steps 1, 2, 3 Q.E.D.

Theorem 3.1 will be utilized to prove the following basic result which strengthens (3.1) to a.s. convergence

Theorem 3.2 : *Under the assumptions of Section 2, for every $s(>0)$, there exist positive constants $(c_s^{(1)}, c_s^{(2)})$ and a sample size n_s, such that for $\beta=0$ and all $n\geqslant n_s$*

$$P\{\sup_{b\in I_n^*}|W(X_n,b)|>c_s^{(1)}n^{-\delta}(\log n)^{k+1}\}<c_s^{(2)}n^{-s}.$$

Hence, $\sup_{|b|<c|}|W(X_n,b)|\to 0$ *a.s., as* $n\to\infty$.

Proof : Note that if we let $J_n\left(\frac{i}{n+1}\right) = EJ(U_{ni})$, $1 \leqslant i \leqslant n$ [cf. (2.6)], we have

$$\left|\sum_{1}^{n} c_{ni}^{*}\left[J_n\left(\frac{R_{ni}}{n+1}\right) - J\left(\frac{R_{ni}}{n+1}\right)\right]\right| \leqslant \max_{1\leqslant i\leqslant n} |c_{ni}^{*}| \sum_{i=1}^{n} |J_n(i/n+1)) - J(i/(n+1))|$$

$$= O(n^{-1})\,.\,O(n^{1-\delta}) = O(n^{-\delta}), \qquad \ldots \quad (3.37)$$

by (2.10) and Theorem 3.6 of Puri and Sen (1971)(viz , pp. 408-413)). Hence, for our purpose, it suffices to work with $J_n\left(\frac{i}{n+1}\right) = J\left(\frac{i}{n+1}\right)$, $1 \leqslant i \leqslant n$. If we define $R_{ni}^{(b)} = \sum_{j=1}^{n} u(X_i - bc_{ni}^{*} - [X_j - bc_{nj}^{*}])$, $1 \leqslant i \leqslant n$, $-\infty < b < \infty$, we have

$$T(X_n) - T(X_n - bc_n^{*}) = \int_{-\infty}^{\infty} J[H_n(x, 0)]dS_n^{*}(x; 0) - \int_{-\infty}^{\infty} J[H_n(x; b)]dS_n^{*}(x; b)$$

$$= I_{n1}(b) + I_{n2}(b), \qquad \ldots \quad (3.38)$$

where

$$I_{n1}(b) = \int_{-\infty}^{\infty} J[H_n(x; 0)]d[S_n^{*}(x; 0) - S_n^{*}(x; b)], \qquad \ldots \quad (3.39)$$

$$I_{n2}(b) = \int_{-\infty}^{\infty} \{J[H_n(x; 0)] - J[H_n(x; b)]\}dS_n^{*}(x; b). \qquad \ldots \quad (3.40)$$

We first show that for every $s > 0$, there exist positive K_1, K_2 and n^{*}, such that for $n \geqslant n^{*}$, $P\{\sup_{b\in I_n^{*}} |I_{n2}(b)| \leqslant K_1 n^{-\delta} (\log n)^{k+1}\} \geqslant 1 - K_2 n^{-s}$. In (2.1), without any loss of generality, we may let $\beta_0 = 0$ and assume that $0 < F(0) < 1$. Select $x_n^{(1)}$ and $x_n^{(2)}$ such that

$$F(x_n^{(1)}) = 1 - F(x_n^{(2)}) = 4K_1 n^{-(1+\delta)} (\log n)^k, \qquad \ldots \quad (3.41)$$

where K_1 is the same as in (3.14). Then for n sufficiently large, say for $n \geqslant n_0$. We have $x_n^{(1)} < 0 < x_n^{(2)}$, and

$$I_{n2}(b) = \int_{-\infty}^{x_n^{(1)}} + \int_{x_n^{(1)}}^{0} + \int_{0}^{x_n^{(2)}} + \int_{x_n^{(2)}}^{\infty} \{J[H_n(x; 0)] - J[H_n(x; b)]\}dS_n^{*}(x, b)$$

$$= I_{n2}^{(1)}(b) + I_{n2}^{(2)}(b) + I_{n2}^{(3)}(b) + I_{n2}^{(4)}(b), \quad \text{say}. \qquad \ldots \quad (3.42)$$

Since, by (3.2), $|S_n^{*}(x; b)| \leqslant \{(n+1) \max_{1\leqslant i\leqslant n} |c_{ni}^{*}|\}H_n(x; b)$, we obtain by some standard computations that

$$|I_{n2}^{(3)}(b)| \leqslant (n+1) \max_{1\leqslant i\leqslant n} |c_{ni}^{*}| \int_{0}^{x_n^{(2)}} |H_n(x; 0) - H_n(x; b)||J'[H_n(x; b, \theta_n)]|\,dH_n(x, b), \qquad \ldots \quad (3.43)$$

where $H_n(x; b, \theta_n) = \theta_n H_n(x; 0) + (1-\theta_n)H_n(x; b)$, $0 < \theta_n < 1$. On using (3.12) and (3.14), we obtain for $n \geqslant n^{*}$ and any fixed $s(>0)$,

$$P\{\sup_{-\infty<x<\infty} \sup_{b\in I_n^{*}} |H_n(x; b) - H_n(x; 0)| > c_s^{(1)} n^{-\frac{1}{2}-\delta} (\log n)^k\} < c_s^{(2)} n^{-s}, \qquad \ldots \quad (3.44)$$

where $c_s^{(1)}$ and $c_s^{(2)}$ are suitable positive constants Hence, on using (3.44), (2.9) and (2.10), we obtain that for all $b \epsilon I_n^*$,

$$|I_{n2}^{(3)}(b)| \leqslant [c_s^{(1)} n^{-s} (\log n)^k] \int_0^{x_n^{(2)}} \{H_n(x; b, \theta_n)[1-H_n(x; b, \theta_n)]\}^{-1} dH_n(x, b), \quad \ldots \quad (3.45)$$

with probability $\geqslant 1-c_s^{(2)} n^{-s}$. Let us define $\bar{F}_n(s; b) = (n+1)^{-1} \sum_{i=1}^{n} F(x+bc_{ni}^*)$, so that $\bar{F}_n(x; 0) = (n/(n+1))F(x)$. Note that by (2.10)-(2.13)

$$\sup_{b \epsilon I_n^*} |\bar{F}_n(x; b)-\bar{F}_n(x; 0)| = O(n^{-1}(\log n)^{2k}). \quad \ldots \quad (3.46)$$

Since $H_n(x; 0) = (n+1)^{-1} \sum_{i=1}^{n} u(x-X_i)$ involves average of i.i.d. (bounded valued) r.v.'s, using the results of Hoeffding (1963), we obtain that for large n,

$$P\{|H_n(x_n^{(2)}; 0)-\bar{F}_n(x_n^{(2)}; 0)| > K_1 n^{-3/4-\delta/2}(\log n)^k\} \leqslant K_2 n^{-s}, \quad \ldots \quad (3.47)$$

where K_1 and K_2 are the same as in (3.14). Using then (3.12), (3.14), (3.41), it follows that for all $x \leqslant x_n^{(2)}$,

$$1-H_n(x; 0) \geqslant (1/3)[1-H_n(x; b)], \ \forall \ b \epsilon I_n^*, \quad \ldots \quad (3.48)$$

with probability $\geqslant 1-c_s^{(2)} n^{-s}$. Again, by using the same results of Hoeffding (1963), we have $P\{|L_n^*(F(0); 0)-\sqrt{n}F(0)| > K_1\sqrt{\log n}\} \leqslant K_2 n^{-s}$, for every $s(>0)$, and hence, by (3.12), (3.14) and the fact that $F(0) > 0$, we obtain that for all $x \geqslant 0$, as $n \to \infty$,

$$H_n(x; b) \geqslant F(0)-K_1 n^{-\frac{1}{2}} (\log n)^k[1+O(n^{-\delta})] \geqslant g_0 > 0, \ \forall \ b \epsilon I_n^* \quad \ldots \quad (3.49)$$

with probability $\geqslant 1-c_s^{(2)} n^{-s}$. From the definition of $H_n(x; b, \theta_n)$ and (3.48)-(3.49), it follows that for all $0 \leqslant x \leqslant x_n^{(2)}$;

$$H_n(x; b, \theta_n)[1-H_n(x; b, \theta_n)] \geqslant C[1-H_n(x; b)], \quad \ldots \quad (3.50)$$

(where $C > 0$), with probability $\geqslant 1-c_s^{(2)} n^{-s}$. From (3.45) and (3.50), for large n, with probability $\geqslant 1-c_s^{(2)} n^{-s}$,

$$\sup_{b \epsilon I_n^*} |I_{n2}^{(3)}(b)| \leqslant O(n^{-\delta} (\log n)^k)\left\{\sup_{b \epsilon I_n^*} \int_0^{x_n^{(2)}} \{1-H_n(x; b)\}^{-1} dH_n(x; b)\right\}$$

$$\leqslant O(n^{-\delta} (\log n)^k)\left\{\sup_{b \epsilon I_n^*} \int_{-\infty}^{\infty} \{1-H_n(x; b)\}^{-1} dH_n(x; b)\right\}$$

$$= O(n^{-\delta} (\log n)^k)\left\{\sup_{b \epsilon I_n^*} (n+1)^{-1} \sum_{i=1}^{n} \left(1-\frac{i}{n+1}\right)^{-1}\right\}$$

$$= O(n^{-\delta} (\log n)^k)\left\{1+\frac{1}{2}+\ldots+\frac{1}{n}\right\} = O(n^{-\delta} (\log n)^{k+1}). \quad \ldots \quad (3.51)$$

It follows similarly that for large n, with probability $\geqslant 1-c_s^{(2)}n^{-s}$, $\sup_{b\epsilon I_n^*} |I_{n2}^{(3)}(b)| = O(n^{-\delta}(\log n)^{k+1})$. Again

$$|I_{n2}^{(4)}(b)| \leqslant \{(n+1) \max_{1\leqslant i\leqslant n} |c_{ni}^*|\}\left\{\int_{x_n^{(2)}}^{\infty} |J[H_n(x;b)]|\,dH_n(x;b)+\int_{x_n^{(2)}}^{\infty}|J[H_n(x;0)]|\,dH_n(x;b)\right\}$$

$$= I_{n2}^{(4)}(b)_1+I_{n2}^{(4)}(b)_2, \text{ say.} \quad \ldots \quad (3.52)$$

On using (2.9), (2.10) (3.41), (3.47) and (3.48), we obtain that for large n, with probability $\geqslant 1-c_s^{(2)}n^{-s}$,

$$\sup_{b\epsilon I_n^*} |I_{n2}^{(4)}(b)_1| \leqslant O(n^{\frac{1}{2}})\left\{\sup_{b\epsilon I_n^*} \int_{H_n(x_n^{(2)};b)}^{1} \{-\log u(1-u)\}\,du\right\}$$

$$\leqslant O(n^{\frac{1}{2}})\left\{K \int_{1-O(n^{-\frac{1}{2}-\delta}(\log n)^k)} [-\log(1-u)]\,du\right\}$$

$$= O(n^{\frac{1}{2}})\,.\,O(n^{-\frac{1}{2}-\delta}(\log n)^{k+1}) = O(n^{-\delta}(\log n)^{k+1}). \quad \ldots \quad (3.53)$$

Also, $I_{n2}^{(4)}(b)_2 = O(n^{1/2})\int_{x_n^{(2)}}^{\infty}|J[H_n(x;0)]|\,dH_n(x;0)+O(n^{1/2})\int_{x_n^{(2)}}^{\infty}|J[H_n(x;0)]|\,d[H_n(x;b)-H_n(x;0)]$. By (3.53), the first term is $O(n^{-\delta}(\log n)^{k+1})$; with probability $\geqslant 1-c_s^{(2)}n^{-s}$ for large n (note that it does not depend on $b\epsilon I_n^*$), while integrating by parts and using the same technique as in the earlier cases, it follows that for large n the second term is also $O(n^{-\delta}(\log n)^{k+1})$ for all $b\epsilon I_n^*$, with probability $\geqslant 1-c_s^{(2)}n^{-s}$. Thus, as $n\to\infty$,

$$\sup_{b\epsilon I_n^*} I_{n2}^{(4)}(b) = O(n^{-\delta}(\log n)^{k+1}) \text{ with probability} \geqslant 1-c_s^{(2)}n^{-s}. \quad \ldots \quad (3.54)$$

Similarly, it follows that for large n, with probability $\geqslant 1-c_s^{(2)}n^{-s}$, $\sup_{b\epsilon I_n^*}|I_{n2}^{(1)}(b)| = O(n^{-\delta}(\log n)^{k+1})$. Consequently, for large n,

$$\sup_{b\epsilon I_n^*}|I_{n2}(b)| = O(n^{-\delta}(\log n)^{k+1}), \text{ with probability} \geqslant 1-c_s^{(2)}n^{-s}. \quad \ldots \quad (3.55)$$

Next we show that for every $s>0$, $\sup_{b\epsilon I_n^*}|I_{n1}(b)-bB(F)| \leqslant c_s^{(1)}n^{-\delta}(\log n)^{k+1}$ with probability $\geqslant 1-c_s^{(2)}n^{-s}$, where $c_s^{(1)}$ and $c_s^{(2)}$ are positive constants, and n is taken to be large. In (3.39), we now write $I_{n1}(b) = I_{n1}^{(1)}(b)+I_{n1}^{(2)}(b)$, where

$$I_{n1}^{(1)}(b) = \int_{-\infty}^{\infty} J[F(x)]d[S_n^*(x;0)-S_n^*(x;b)], \quad \ldots \quad (3.56)$$

$$I_{n1}^{(2)}(b) = \int_{-\infty}^{\infty} \{J[H_n(x;0)]-J[F(x)]\}d[S_n^*(x;0)-S_n^*(x;b)]. \quad \ldots \quad (3.57)$$

By (2.10), (3.2) and (3.57), we have

$$\sup_{b\epsilon I_n^*}|I_{n1}^{(2)}(b)| = \sup_{b\epsilon I_n^*}\left|\sum_{i=1}^{n} c_{ni}^*\{J[H_n(X_i;0)]-J[F(X_i)]\}\{u(-X_i)-u(bc_{ni}^*-X_i)\}\right|$$

$$\leqslant \{\max_{1\leqslant i\leqslant n}|c_{ni}^*|\}\{\sup_{b\epsilon I_n^*}|\ J[H_n(Cn^{-1/2}x;0)]-J[F(Cn^{-\frac{1}{2}}x)]|\}(n+1).$$

$$\{|H_n(Cn^{-\frac{1}{2}}(\log n)^k;0)-H_n(Cn^{-\frac{1}{2}}(\log n)^k;0)|\}, \quad \ldots \quad (3.58)$$

where $C(>0)$ is finite. On using the well-known result that

$$P\{\sup_{-\infty<x<\infty} n^{1/2}|H_n(x;0)-F(x)| > C_1\sqrt{\log n}\} \leqslant C_2 n^{-s}, \qquad \ldots \quad (3.59)$$

where C_1, C_2 are finite (positive) constants, and that in the neighbourhood of 0 (where $0<F(0)<1$), $|J'(F(x))|$ is bounded [by (2.9)], and further that $F(Cn^{-1/2}(\log n)^k)-F(-Cn^{-1/2}(\log n)^k) = O(n^{-1/2}(\log n)^k)$, we obtain from (3.58) that

$$\sup_{b\epsilon I_n^*}|I_{n}^{(2)}(b)| \leqslant [0(n^{-1/2})][O(n^{-1/2}(\log n))][O(n)][O(n^{-1/2}(\log n)^k]$$

$$= O(n^{-1/2}(\log n)^{k+1/2}), \text{ with probability} \geqslant 1-c_s^{(2)}(n^{-s}). \qquad \ldots \quad (3.60)$$

Further, on denoting by $c_n^* = \max_{1\leqslant i\leqslant n} |c_{ni}^*|$, we have

$$I_n^{(1)}(b) = \int_{X_{(1)}-|b|c_n^*}^{X_{(n)}+|b|c_n^*} J[F(x)]d[S_n^*(x;0)-S_n^*(x;b)]$$

$$= \int_{X_{(1)}-|b|c_n^*}^{X_{(n)}+|b|c_n^*} [S_n^*(x;b)-S_n^*(x;0)]J'[F(x)]dF(x), \qquad \ldots \quad (3.61)$$

where $X_{(1)} = \min_{1\leqslant i\leqslant n} X_i$ and $X_{(n)} = \max_{1\leqslant i\leqslant n} X_i$. Hence, by (2.22) and (3.61),

$$I_n^{(1)}(b)-bB(F) = \int_{X_{(1)}-|b|c_n^*}^{X_{(n)}+|b|c_n^*} [S_n^*(x;b)-S_n^*(x;0)-bf(x)]J'[F(x)]dF(x)$$

$$-\Big\{\int_{-\infty}^{X_{(1)}-|b|c_n^*} + \int_{X_{(n)}+|b|c_n^*}^{\infty} bf(x)J'[F(x)]dF(x)\Big\}. \qquad \ldots \quad (3.62)$$

Now, $P\{F(X_{(1)} \geqslant cn^{-1/2}\} = [1-cn^{-1/2}]^n \leqslant O(n^{-s})$, for every (fixed) $s(>0)$, and similarly, $P\{1-F(X_{(n)}) \geqslant cn^{-1/2}\} \leqslant O(n^{-s})$. Hence, by (2.13) for large n, the last term on the right hand side of (3.62) is bounded (for all $b\epsilon I_n^*$) above by $O(n^{-1/2}(\log n)^{k+1})$, with probability $\geqslant 1-c_s^{(2)}(n^{-s})$. Also, by (2.9), (3.11) and (3.13), the first term on the right hand side of (3.62) is bounded (for all $b\epsilon I_n^*$) above by

$$\{K_1n^{-\delta}(\log n)^k[1+O(n^{-1/2})]\}K \int_{X_{(1)}-Cn^{-1/2}(\log n)^k}^{X_{(n)}+Cn^{-1/2}(\log n)^k} \{F(x)(1-F(x))\}^{-1}dF(x)$$

$$\leqslant O(n^{-\delta}(\log n)^k)\{-\log F(X_{(1)}-Cn^{-1/2}(\log n)^k)-\log\{1-F(X_{(n)}+Cn^{-1/2}(\log n)^k)\}\},$$
$$\ldots \quad (3.63)$$

with probability $\geqslant 1-c_s^{(2)}(n^{-s})$, where $c>0$. In the same way as in after (3.62), it can be shown that $F(X_{(1)}-Cn^{-\frac{1}{2}}(\log n)^k) \geqslant$ const (n^{-k}) with probability $\geqslant 1-c_s^{(2)}n^{-s}$, where s is some fixed positive number, and a similar statement holds for $1-F(X_{(n)}+Cn^{-\frac{1}{2}}(\log n)^k))$. Hence, with probability $\geqslant 1-c_s^{(2)}n^{-s}$, for large n the right hand side of (3.63) is bounded above by

$$O(n^{-\delta}(\log n)^k) \,.\, O(\log n) = O(n^{-\delta}(\log n)^{k+1}). \qquad \ldots \quad (3.64)$$

The proof of the theorem is then completed by (3.55), (3.62), (3.63), (3.64) and the definition of $W(X_n, b)$, given at the beginning of this section. Q.E.D.

Let $\mathcal{F}_n$ denote the σ-field generated by $R_n = (R_{n1}, \ldots, R_{nn})$; note that $\mathcal{F}_n$ is $\uparrow$ in n. Also, let $T_n^* = C_n T_n$ and assume that $H_0 : \beta = 0$ holds. Then, Theorem 3.3 holds; the result is of fundamental use in proving the "uniform continuity in probability" (to be explained in Section 5) of $T(X_n)$ with respect to C_n^{-1}.

Theorem 3.3 : *If* $J_n(u) = EJ(U_{ni})$, $(i-1)/n < u \leqslant i/n$, $1 \leqslant i \leqslant n$, *then under* $H_0 : \beta = 0$, $\{T_n, \mathcal{F}_n\}$ *forms a martingale sequence.*

The proof of the theorem is given in Sen and Ghosh (1972), and hence, is omitted.

Note that the above martingale property does not necessarily hold when we let $J_n(i/(n+1)) = J(i/(n+1))$, $1 \leqslant i \leqslant n$ [unless $J(u) = u : 0 < u < 1$]. Also, when $\beta \neq 0$, so that given the ranks of $X_1, \ldots, X_n$, X_{n+1} does not have a constant probability $(1/(n+1))$ of having the rank $i(1 \leqslant i \leqslant n+1)$, the above martingale property does not hold, in general.

4. The proof of the main theorem

The proof is accomplished in several steps. First, we prove the following lemma.

Lemma 4.1 : *For* $\forall s(>0)$, $\exists$ *positive constants* $K_s^{(1)}$, $K_s^{(2)}$ *and a sample size* n_s, *such that for* $n \geqslant n_s$,

$$P\{[C_n(\hat\beta_{U,n}-\beta)-A\tau_{\alpha/2}/B(F)] > K_s^{(1)}(\log n)^2\} < K_s^{(2)}n^{-s}, \qquad \ldots (4.1)$$

$$P\{[C_n(\hat\beta_{L,n}-\beta)+A\tau_{\alpha/2}/B(F)] < -K_s^{(1)}(\log n)^2\} < K_s^{(2)}n^{-s}. \qquad \ldots (4.2)$$

Proof : We shall only consider the proof of (4.1) as the proof of (4.2) follows on the same line. By virtue of the translation-invariance of the estimates $\hat\beta_{U,n}$ and $\hat\beta_{L,n}$, we may, without any loss of generality, assume that $\beta = 0$. Then, by (2.7), we have

$$\begin{aligned} &P\{C_n\hat\beta_{U,n}-A\tau_{\alpha/2}/B(F) > K_s^{(1)}(\log n)^2\} \\ &= P\{T(X_n-C_n^{-1}[A\tau_{\alpha/2}/B(F)+K_s^{(1)}(\log n)^2]c_n) > -T_n^{(1)} \mid \beta = 0\} \\ &= P_{\beta=0}\{\hat T_n-\tilde T_n > -T_n^{(1)}-\tilde T_n\}, \end{aligned} \qquad \ldots (4.3)$$

where we write $\tilde T_n = T(X_n-b_0c_n^*)$, $b_0 = A\tau_{\alpha/2}/B(F)$ and $\hat T_n = T(X_n-[b_0+K_s^{(1)}(\log n)^2]c_n^*)$. By Theorem 3.2, with probability $\geqslant 1-c_s^{(2)}(n^{-s})$, for large n,

$$-\hat T_n-\tilde T_n = -K_s^{(1)}(\log n)^2B(F)+O(n^{-\delta}(\log n)^3), \quad \delta > 0. \qquad \ldots (4.4)$$

Also, using the fact that $\lim_{n\to\infty} T_n^{(1)} = -A\tau_{\alpha/2}$ and writing $S_n(x; b_0) = \sum_{i=1}^{n} c_{ni}^*F(x-b_0c_{ni}^*)$, we obtain by some standard computations that

$$\int_{-\infty}^{\infty} J[F(x)]dS_n(x; b_0) = -b_0B(F)+O(n^{-1/2}) = T_n^{(1)}+O(n^{-1/2}). \qquad \ldots (4.5)$$

Hence, it suffices to show that for large n, with probability $\geqslant 1-c_s^{(2)}(n^{-s})$,

$$\left| \int_{-\infty}^{\infty} J[F(x)]dS_n(x; b_0)-\tilde T_n \right| < K_s^{(1)}(\log n)^{3/2}. \qquad \ldots (4.6)$$

The left hand side of (4.6) can be written as

$$|\int_{-\infty}^{\infty}\{J[F(x)]-J[H_n(x;b_0)]\}dS_n(x;b_0)+\int_{-\infty}^{\infty}J[H_n(x;b_0)]dS_n[(x;b_0)-S_n^*(x;b_0)]|$$

$$\leqslant |\int_{-\infty}^{\infty}\{J[F(x)]-J[H_n(x;b_0)]\}dS_n(x;b_0)|$$

$$+|\int_{-\infty}^{\infty}[S_n^*(x;b_0)-S_n(x;b_0)]J'[H_n(x;b_0)]\,dH_n(x;b_0)|. \quad \ldots \quad (4.7)$$

It can be shown by using Theorem 2 of Hoeffding (1963) that for large n $\sup|S_n^*(x;b_0)-S_n(x;b)| = O((\log n)^{1/2})$, with probability $\geqslant 1-c_s^{(2)}(n^{-s})$. Also, $\int_{-\infty}^{\infty}|J'[H_n(x;b_0)]\,|dH_n(x;b_0) \leqslant (n+1)^{-1}\sum_{i=1}^{n} K(i(n+1-i)/(n+1)^2)^{-1} = O(\log n)$ [by (2.9)]. Hence, the second term on the right hand side of (4.7) is $O((\log n)^{3/2})$ with probability $\geqslant 1-c_s^{(2)}(n^{-s})$, for large n. Finally, precisely on the same line as in (3.42)—(3.55), it follows that for large n, the first term on the right hand side of (4.7) is also $O((\log n)^{3/2})$, with probability $\geqslant 1-c_s^{(3)}(n^{-s})$. Hence the proof is complete.

A direct consequence of Theorem 3.2 and Lemma 4.1 is the following.

Lemma 4.2 : *For every s (>0), there exist positive constants ($K_s^{(1)}, K_s^{(2)}$) and a sample size n_s, such that for $n \geqslant n_s$,*

$$P\{|C_n(\hat\beta_{U,n}-\hat\beta_{L,n})-2A\tau_{\alpha/2}/B(F)| \geqslant K_s^{(1)}n^{-\frac14}(\log n)^2\} \leqslant K_s^{(2)}n^{-s}. \quad \ldots \quad (4.8)$$

Also, we have the following lemma whose proof follows along the lines of Sen (1969, Section 3).

Lemma 4.3 : *For every real x ($-\infty < x < \infty$),*

$$\lim_{n\to\infty} P\{C_n(\hat\beta_{U,n}-\beta)B(F)/A-\tau_{\alpha/2} \leqslant x\} = (2\pi)^{-1/2}\int_{-\infty}^{x} exp\left(-\frac{1}{2}t^2\right)dt. \quad \ldots \quad (4.9)$$

Finally, for the "uniform continuity in probability" (for definition, see Anscombe (1952)) of $\{\hat\beta_{U,n}\}$ with respect to $\{C_n^{-1}\}$, we have the following :

Lemma 4.4 : *For every positive ε and η, there exists a $\delta (< 0)$, such that as $n\to\infty$*

$$P\left\{\sup_{|n'-n|<\delta n}|C_n(\hat\beta_{U,n'}-\hat\beta_{U,n})| > \eta\right\} < \varepsilon, \quad \ldots \quad (4.10)$$

and a similar statement holds for $\{\hat\beta_{L,n}\}$.

Proof : By Theorem 3.2, Lemma 4.1 (where we let $s = 1+h$, $h > 0$), and (2.17), we have, with probability $\geqslant 1-K(n^{-h})$, K being a constant depending on h, for all $|n'-n| < \delta n$,

$$\begin{aligned} C_n(\hat\beta_{U,n'}-\hat\beta_{U,n}) &= C_{n'}(\hat\beta_{U,n'}-\beta)[C_n/C_{n'}]-C_n(\hat\beta_{U,n}-\beta) \\ &= -[C_n/C_{n'}][1/B(F)][T(X_{n'}-\hat\beta_{U,n'}c_{n'})-T(X_{n'}-\beta c_{n'})+o(1)] \\ &\quad +[1/B(F)][T(X_n-\hat\beta_{U,n}c_n)-T(X_n-\beta c_n)+o(1)] \\ &= -[A/B(F)]\{\tau_{\alpha/2}[1-C_n/C_{n'}]\}+(C_n/C_{n'})[T(X_{n'}-\beta c_{n'}) \\ &\quad -T(X_n-\beta c_n)]/B(F)-T(X_n-\beta c_n)[1-C_n/C_{n'}]/B(F)+o(1). \end{aligned} \quad \ldots \quad (4.11)$$

By (2.12), $\sup_{|n'-n|<\delta n} |1-C_n/C_{n'}| < \delta'$, where $\delta'(>0)$ depends on δ, and can be made arbitrarily small when δ is made so. Also, the asymptotic normality of $T(X_n-\beta c_n)/A$ (with zero mean and unit variance) implies that $|T(X_n-\beta c_n)|$ is bounded, in probability, as $n\to\infty$. Hence, it suffices to show that as $n\to\infty$

$$P\{\sup_{|n'-n|<\delta n} |T(X_n-\beta c_n)-T(X_{n'}-\beta c_{n'})| > \eta\} < \varepsilon. \quad \ldots \quad (4.12)$$

Since $|T(X_n-\beta c_n)-T(X_{n'}-\beta c_{n'})| \leqslant |(C_{n'}-C_n)/C_{n'}|\,|T_n|+C_{n'}^{-1}|T^*_{n'}-T^*_n|$, where T^*_n is defined in Section 4), it is sufficient to show that

$$P_{\beta=0}\{\sup_{|n'-n|<\delta n} |T^*_{n'}-T^*_n| > \eta C_n\} < \varepsilon, \quad \ldots \quad (4.13)$$

and in view of the martingale property of T^*_n, see Theorem 3.3, from the Kolmogorov inequality (cf. Loève (1963), p. 386) for martingales, we get,

$$P_{\beta=0}\{\sup_{|n'-n|<\delta n} |T^*_{n'}-T^*_n| > \eta C_n\} \leqslant (\eta C_n)^{-2}E(T^{*2}_{n+[\delta n]}-T^{*2}_{n-[\delta n]})$$

$$= (\eta C_n)^{-2}[C^2_{n+[\delta n]}A^2_{n+[\delta n]}-C^2_{n-[\delta n]}A^2_{n-[\delta n]}].$$

The rest of the proof follows from (2.12) and the fact that $A^2_n = A^2+o(1)$, for large n.

We now return to the proof of Theorem 2.1. By (2.17), Lemma, 4.2 and the definition of $N(d)$, it follows that for all $d > 0$, $N(d)$ is finite a.s. and is $\downarrow$ in d. We also note that

$$[EN(d)] = \sum_{n=0}^{\infty} nP\{N(d) = n\} = \sum_{n\geqslant 0} P\{N(d) > n\}. \quad \ldots \quad (4.14)$$

Hence, in order to show that $E[N(d)] < \infty$, it suffices to show that for large n,

$$P\{N(d) > n\} = O(n^{-1-\eta}), \text{ where } \eta > 0. \quad \ldots \quad (4.15)$$

By definition, the left hand side of (4.15) is equal to

$$P\{\hat\beta_{U,k}-\hat\beta_{L,k} > 2d \text{ for all } k \leqslant n\} \leqslant P\{\hat\beta_{U,n}-\hat\beta_{L,n} > 2d\}$$
$$= P\{C_n(\hat\beta_{U,n}-\hat\beta_{L,n}) > 2dC_n\} = O(n^{-s}), \quad \ldots \quad (4.16)$$

(where we can let $s > 1$), as by (4.8), $C_n(\hat\beta_{U,n}-\hat\beta_{L,n}) = 2A\tau_{\alpha/2}/B(F)+O(n^{-\frac{1}{4}}(\log\ n)^2)$, whereas by (2.11), $2dC_n \geqslant 2dC_0 n^{1/2}$. Hence, for every $d > 0$, $E[N(d)] < \infty$. Using the fact that $\hat\beta_{U,n}-\hat\beta_{L,n} > 0$ a.s. for each n, we have $\lim_{d\to 0} N(d) = \infty$ and finally by the monotone convergence theorem $\lim_{d\to 0} E[N(d)] = \infty$.

Now (2.19) follows directly from (4.8), (2.12) and (2.22); (2.20) also follows from Theorem 1 of Anscombe (1952) after using our Lemmas 4.3 and 4.4. To prove (2.21), we let for each $d(>0)$,

$$n_i(d) = \left[Q^{-1}(\nu(d))(1+(-1)^i\varepsilon)+\frac{1+(-1)^i}{2}\right], \quad \ldots \quad (4.17)$$

$[x]$ being the large integer contained in x, and write,

$$E[N(d)]/Q^{-1}(\nu(d)) = [Q^{-1}(\nu(d))]^{-1}\{\Sigma_1+\Sigma_2+\Sigma_3 nP\{N(d) = n\}\},$$

where the summations Σ_1, Σ_2 and Σ_3 extend over $n < n_1(d)$, $n_1(d) \leqslant n \leqslant n_2(d)$ and $n > n_2(d)$, respectively. Since $Q(\circ)$ is $\uparrow$, $\lim_{d\to\infty} \nu(d) = \infty$ and (2.19) holds, for every $\varepsilon > 0$, there exists a $d_\varepsilon(>0)$ such that for all $0 < d \leqslant d_\varepsilon$, $P\{n_1(d) \leqslant N(d) \leqslant n_2(d)\} \geqslant P\{|N(d)/Q^{-1}(\nu(d))-1| < \varepsilon\} \geqslant 1-\eta$, where $\eta(>0)$ is (preassigned) small number. Hence, for $d \leqslant d_\varepsilon$,

$$\{Q^{-1}(\nu(d))\}^{-1}\Sigma_1 nP\{N(d) = n\} \leqslant (1-\varepsilon)P\{N(d) < n_1(d)\} \leqslant \eta(1-\varepsilon). \quad \ldots \quad (4.18)$$

Also,

$$\{Q^{-1}(\nu(d))\}^{-1}\Sigma_3 nP\{N(d) = n\} = \{Q^{-1}(\nu(d))\}^{-1}(n_2(d)+1)P\{N(d) > n_2(d)\}+\Sigma_3 P\{N(d) > n\}], \quad \ldots \quad (4.19)$$

where by (4.15)-(4.16), we obtain by letting $s > 1$, $(n_2(d)+1)\,P\{N(d) > n_2(d)\} < K_2[n_2(d)]^{-s+1}$ and $\Sigma_3 P\{N(d) > n\} = O([n_2(d)]^{-s+1})$, both of which converge to zero as $d \to 0$. Finally,

$$|\{Q^{-1}(\nu(d))\}^{-1}\Sigma_2 nP\{N(d) = n\}-1| \leqslant \varepsilon\Sigma_2 P\{N(d) = n\} \leqslant \varepsilon+\eta. \quad \ldots \quad (4.20)$$

The proof of (2.21) now follows from (4.18)-(4.20).

Remark : Using (4.12), the classical Wald-Wolfowtz-Noether-Hájek theorem (cf. Hájek and Sidák, 1967, p. 160) on the asymptotic normality of $T_n(\beta)$ for fixed but large n, and (2.19), it readily follows from Theorem 1 (p. 601) of Anscombe (1952) that $T_{N(d)}(\beta)$ has asymptotically (as $d \to 0$) a normal distribution with zero mean and variance A^2 defined in (2.23). Also, on using Theorem 3.2, it follows that for every $b\epsilon I_{N(d)}$, $T_{N(d)}(\beta+b)+bB(F)$ has asymptotically the same normal distribution. These results give simple proofs of the asymptotic normality of regression rank statistics based on random sample sizes both in the null ($b = 0$) and the non-null ($b \neq 0$) situations. By using Theorem 2 of Mogyorodi (1965), the results trivially extend to any stopping variable N_r, indexed by a sequence $\{r\}$ and a sequence of positive integers $\{n_r\}$, such that $n_r \to \infty$ and $\frac{N_r}{n_r} \xrightarrow{p} \lambda$, as $r \to \infty$, where λ is a positive random variable defined on the same probability space as of the X_i.

5. Asymptotic relative efficiency

For any two procedures A and B for determining (sequentially) bounded length confidence intervals for β (with the same bound $2d$), let $P_A(d)$ and $P_B(d)$ be the coverage probabilities and $N_A(d)$ and $N_B(d)$ the stopping variables corresponding to the respective procedures. Then, the ARE of the procedure A with respect to the procedure B is given by

$$e_{A,B} = \lim_{d\to 0} [EN_B(d)/EN_A(d)], \quad \ldots \quad (5.1)$$

provided $\lim_{d\to 0} P_A(d) = \lim_{d\to 0} P_B(d)$ and either of the limits exists.

Gleser (1965) considered the case when $n^{-1}C_n^2 \to C > 0$ as $n \to \infty$. However, one can easily extend his results when (2.10) and (2.11) hold. Thus, if G denotes his procedure, Theorem 2.1 also holds with the change that $\nu(d)$ has to be replaced by

$\nu_G(d) = \sigma^2\tau_{\frac{1}{2}\alpha}^2/d^2$, σ^2 being the variance of the distribution of $F(x)$ in (2.1). Writing R for the proposed procedure and using (2.21), we have now,

$$e_{R,G} = \lim_{d\to 0} \{Q^{-1}(\nu_G(d))/Q^{-1}(\nu(d))\}. \qquad \dots \quad (5.2)$$

By definition, $\nu_G(d)/\nu(d) = \sigma^2B^2(F)/A^2$ is independent of d. Write $e = s(e^*)$. Then, $e^* = s^{-1}(e)$ is monotonic in e with $e^* = 1$ when $e = 1$. Write $\phi_d = \nu_G^*(d)/\nu^*(d)$, where $\nu_G^*(d) = Q^{-1}(\nu_G(d))$, $\nu^*(d) = Q^{-1}(\nu(d))$. Both $\nu_G^*(d)$ and $\nu^*(d)$ tend to ∞ as $d \to 0$. Thus, assuming $\lim_{d\to 0} \phi(d)$ exists,

$$s(e^*) = e = \nu_G(d)/\nu(d) = \lim_{d\to 0} \{\nu_G(d)/\nu(d)\}$$

$$= \lim_{d\to 0} \{Q(\phi_d\nu^*(d))/Q(\nu^*(d))\}. \qquad \dots \quad (5.3)$$

Using (2.12) and proving by contradiction, we have,

$$\lim_{d\to 0} \phi(d) = e^* = s^{-1}(e) = s^{-1}(\sigma^2B^2(F)/A^2). \qquad \dots \quad (5.4)$$

The expression $\sigma^2B^2(F)/A^2$ is the Pitman-efficiency of a general rank order test with respect to Student's t test. In the particular case, when $J(u) = \phi^{-1}(u)$, ϕ being the distribution function (d.f.) of a standard normal variable, (normal score) it is well-known (see e.g. Puri and Sen, 1971, p. 118) that $\sigma^2B^2(F)/A^2 \geqslant 1$ for all d.f. F with a density f and a finite second moment, equality being attained when and only when F is normal $(0, \sigma^2)$ d.f. From monotonicity, it follows now from (5.4) that in this case $e^* \geqslant 1$, equality being attained if and only if F is normal. Also, when $J(u) = u$ (Wilcoxon score), $e^* = s^{-1}\left(12\sigma^2\left(\int_{-\infty}^{\infty} f^2(x)dx\right)^2\right) \geqslant s^{-1}$ (0.864) (cf. Hájek and Sidák (1967, p. 280)). In the case of equispaced regression line, $t_i = a+i$, $i = 1, 2, \dots$, $C_n^2 = n(n^2-1)/12$, $e^* = \left\{12\sigma^2\left(\int_{-\infty}^{\infty} f^2(x)dx\right)^2\right\}^{1/3}$. For normal F, this reduces to $(0.955)^{1/3} \approx .985$, while the infimum is given by $(0.864)^{1/3} \approx .953$.

In the special case when c_i is either 0 of 1, β is the difference in the location parameters of the two distributions $F(x-\alpha')$ and $F(x-\alpha'-\beta)$. This is the classical two-sample problem. If at the n-th stage, m_n of the c_i are 1 and rest zero, $C_n^2 = m_n(n-m_n)/4 \leqslant n/4$, for all $n \geqslant 1$. Looking at the definition of C_n^2, (2.19) and (2.21), we may observe that an optimum choice of m_n is $\left[\frac{1}{2}n\right]$, the integral part of $\frac{1}{2}n$. Thus, among all designs for obtaining a bounded length confidence interval for β, in this problem, and optimum design (which minimizes $EN(d)$ for small d) consists in taking every alternate observation for the two distributions. Here, $n^{-1}C_n^2 \to \frac{1}{4}$ and the ARE reduces to $\sigma^2B^2(F)/A^2$ various bounds for which have been discussed earlier,

REFERENCES

ALBERT, A. (1966): Fixed size confidence ellipsoids for linear regression parameters. *Ann. Math. Statist.*, 37, 1602-1630.

ANSCOMBE, F. J. (1952): Large sample theory of sequential estimation. *Proc. Camb. Phil. Soc.*, 48, 51-58.

CHOW, Y. S. and ROBBINS, H. (1965): On the asymptotic theory of fixed width sequential confidence intervals for the mean. *Ann. Math. Statist.*, 36, 456-462.

GLESER, L. J. (1965): On the asymptotic theory of fixed-size sequential confidence bounds for linear regression parameter. *Ann. Math. Statist.*, 36, 463-467.

HÁJEK, J. (1962): Asymptotically most powerful rank-order tests. *Ann. Math. Statist.*, 33, 1124-1147.

——— (1968): Asymptotic normality of simple linear rank statistics under alternatives. *Ann. Math. Statist.*, 39, 325-346.

HÁJEK, J. and ŠIDÁK, Z. (1967): *Theory of Rank Tests*, Academic Press, New York.

HOEFFDING, W. (1963): Probability inequalities for sums of bounded random variables. *J. Amer. Stat. Assoc.*, 58, 13-29.

JUREČKOVÁ, J. (1969): Asymptotic linearity of a rank statistic in regression parameter. *Ann. Math., Statist.*, 40, 1889-1900.

LOÈVE, M. (1963): *Probability Theory* (3rd Edition), Van Norstrand, Princeton.

MOGYORODI, J. (1965): Limit distributions for sequences of random variables with random indices. Fourth *Prague Confer. Infer Theor. Statist. Dec. fn. Random proc.*, 463-470.

PURI, M. L. and SEN, P. K. (1971): *Nonparametric Methods in Multivariate Analysis*. John Wiley, New York.

SEN, P. K. (1969): On a class of rank order tests for the parallelism of several regression lines. *Ann. Math. Statist.*, 40, 1668-1683.

SEN, P. K. and GHOSH, M. (1971): On bounded length confidence intervals based on one-sample rank order statistics. *Ann. Math. Statist.*, 42, 189-203.

——— (1972): On strong convergence of regression rank statistics. *Sankhyā, Ser. A.* (to appear).

Paper received : April, 1970.

Revised : November, 1971.