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ON THE CENTRAL LIMIT THEOREM FOR DIFFUSIONS WITH ALMOST PERIODIC COEFFICIENTS

By RABI N. BHATTACHARYA
Indiana University
and
S. RAMASUBRAMANIAN
Indian Statistical Institute

SUMMARY. We consider a class of n-dimensional elliptic generators having almost periodic coefficients depending on finitely many rationally independent frequencies in each coordinate. A strong law of large numbers and a functional central limit theorem are proved for such diffusions.

1. Introduction

In this article we study asymptotic behaviour of diffusions on R^n whose drift and diffusion coefficients are *almost periodic* depending on M_j rationally independent frequencies $\omega_r^{(j)}$, $1 \leqslant r \leqslant M_j$, in the jth coordinate $(1 \leqslant j \leqslant n)$.

In the case of a diffusion whose generator is in the self-adjoint divergence form and whose coefficients come from a random field, a novel functional central limit theorem was obtained by Papanicolaou and Varadhan (1979) under the general condition that the random field is stationary and ergodic. Kozlov (1979), (1980) contain similar results ; but the regularity arguments in Kozlov (1979) appear to have a gap. However Kozlov (1979) contains some significant ideas which we have made use of. While Kozlov's approach is purely analytical, ours is primarily probabilistic. We also mention the work of Papanicolaou and Pironeau (1981), in which the diffusion matrix is the identity and the drift vector is a mean-zero divergence free stationary ergodic random field. In all these articles the large scale mean is zero. The point of departure in the present article is the consideration of drift velocities whose *large scale mean need not be zero*. Part of the motivation for looking at this comes from the problem of modeling solute dispersion in an aquifer (Bhattacharya *et al.*, 1987 ; Gelhar and Axness, 1983 ; Winter *et al.*, 1984) and analyzing the limiting dispersion as a function of the large scale velocity.

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It may be noted that for arbitrary strictly elliptic generators with *periodic* coefficients the pathwise central limit theorem holds (see Bensoussan, Lions and Papanicolaou (1978), Bhattacharya (1985), and the remark on p. 846 in Papanicolaou and Varadhan (1979)).

2. Preliminaries and the law of large numbers

It will be assumed throughout that $b_k(.)$, $a_{kk'}(.)$ are real-valued functions on R^n of the form

$$b_k(x) = \sum_m b_k^{(m)} \exp\left\{i \sum_{j=1}^{n} x_j \sum_{r=1}^{M_j} m_r^{(j)} \omega_r^{(j)}\right\} \quad (1 \leqslant k \leqslant n),$$

$$a_{kk'}(x) = \sum_m a_{kk'}^{(m)} \exp\left\{i \sum_{j=1}^{n} x_j \sum_{r=1}^{M_j} m_r^{(j)} \omega_r^{(j)}\right\} \quad (1 \leqslant k, k' \leqslant n). \quad \ldots \quad (2.1)$$

Here $M_1, M_2, \ldots, M_n$ are fixed positive integers ; for each $j (1 \leqslant j \leqslant n)$ one has a given set of M_j *rationally independent* (i.e., independent over the field of rationals) positive numbers $\omega_r^{(j)}$, $1 \leqslant r \leqslant M_j$; the sums in (2.1) are over a *finite* set of integer vectors $m = (m_r^{(j)} : 1 \leqslant r \leqslant M_j,\ 1 \leqslant j \leqslant n) \in Z^M$ where

$$M = M_1 + M_2 + \ldots + M_n. \quad \ldots \quad (2.2)$$

The coefficients $b_k^{(m)}$, $a_{kk'}^{(m)}$ are complex constants. For each $x \in R^n$ the $n \times n$ matrix $a(x) \doteq ((a_{kk'}(x)))$ is symmetric and positive definite and

$$\lambda_0 \doteq \inf_{x \in R^n} \text{(smallest eigenvalue of } a(x)) > 0. \quad \ldots \quad (2.3)$$

In order to avoid ending up with the periodic case it will be assumed that $M > n$.

For each $c = (c_r^{(j)} : 2 \leqslant r \leqslant M_j,\ 1 \leqslant j \leqslant n) \in R^{M-n}$ denote by H_c the n-dimensional hyperplane in R^M given by

$$H_c = \{y = (y_r^{(j)} : 1 \leqslant r \leqslant M_j,\ 1 \leqslant j \leqslant n) : y_r^{(j)} = y_1^{(j)} + c_r^{(j)},\ \ 2 \leqslant r \leqslant M_j\}. \quad \ldots \quad (2.4)$$

We shall adopt the following convention throughout : if $M_j = 1$, then terms involving subscripts $r \geqslant 2$ and superscripts j will be omitted.

Let Q denote the following discrete subgroup of R^{M-n} :

$$Q = \{m_r^{(j)}(2\pi/\omega_r^{(j)}) + m_1^{(j)}(2\pi/\omega_1^{(j)}) : 2 \leqslant r \leqslant M_j,\ 1 \leqslant j \leqslant n) : m \in Z^M\}). \quad \ldots \quad (2.5)$$

Write $f \in Trig\ (\omega)$ if f is a finite sum of the form

$$f(x) = \sum_m f^{(m)} \exp\left\{i \sum_{j=1}^{n} x_j \sum_{r=1}^{M_j} m_r^{(j)} \omega_r^{(j)}\right\}, \quad \ldots \quad (2.6)$$

where $f^{(m)}$ are complex numbers.

A complex-valued function $h(y)$ on R^M will be said to be *periodic* $(2\pi/\omega)$ if it is periodic with period $2\pi/\omega_r^{(j)}$ in the coordinate $y_r^{(j)}(1 \leq r \leq M_j, 1 \leqslant j \leqslant n)$.

If $f \in Trig\ (\omega)$ is given by (2.6) define

$$\hat{f}(y) = \sum_m f^{(m)} \exp \left\{ i \sum_{j=1}^{n} \sum_{r=1}^{M_j} m_r^{(j)} \omega_r^{(j)} y_r^{(j)} \right\}. \quad \dots \quad (2.7)$$

Then $\hat{f}$ is periodic $(2\pi/\omega)$, and f may be identified with the restriction of $\hat{f}$ to the hyperplane H_0.

Lemma 2.1 : *Q is dense in R^{M-n}.*

Proof : It is sufficient to prove that if $\omega_1, \omega_2, \dots, \omega_k$ are rationally independent positive numbers then $\{(q_1\omega_1^{-1}+q_2\omega_2^{-1},\ q_1\omega_1^{-1}+q_3\omega_3^{-1}, \dots,\ q_1\omega_1^{-1}+q_k\omega_k^{-1}) : q_1, q_2, \dots, q_k \epsilon Z\}$ is dense in R^{k-1}. Take $\omega_1 = 1$ without essential loss of generality. It is clear that

$$q_1 \,(\text{mod}\ \omega_j^{-1}) = \omega_j^{-1}\,(q_1\omega_j\,(\text{mod}\ 1)),\ j = 2, \dots, k. \quad \dots \quad (2.8)$$

Now, by Kronecker's theorem (Hardy and Wright (1959), p. 382), $\{(q_1\omega_2\,(\text{mod}\ 1),\ q_1\omega_3\,(\text{mod}\ 1), \dots, q_1\omega_k\,(\text{mod}\ 1)) : q_1 \in Z\}$ is dense in $[0, 1]^{k-1}$. Therefore, by (2.8), $\{(q_1\,(\text{mod}\ \omega_2^{-1}),\ q_1\,(\text{mod}\ \omega_3^{-1}), \dots, q_1\,(\text{mod}\ \omega_k^{-1})) : q_1 \in Z\}$ is dense in $[0, \omega_2^{-1}] \times \dots \times [0, \omega_k^{-1}]$. Consequently, $D \doteq \{q_1+q_2\omega_2^{-1},\ q_1+q_3\omega_3^{-1}, \dots, q_1+q_k\,\omega_k^{-1}) : q_1, q_2, \dots, q_k \in Z\}$ is dense in $[0, \omega_2^{-1}] \times \dots \times]0, \omega_k^{-1}] + (q_2'\omega_2^{-1}, \dots, q_k'\,\omega_k^{-1})$ for every choice of integers $q_2', \dots, q_k'$. Hence D is dense in R^{k-1}.

Henceforth $\mathcal{J}$ will denote the M-dimensional *torus* $\prod_{j=1}^{n} \prod_{r=1}^{M_j} [0,\ 2\pi/\omega_r^{(j)})$ $\{\dot{y} = y \in R^M\}$ where

$$\dot{y} \equiv \zeta(y)) \doteq (y_r^{(j)}\,(\text{mod}\ 2\pi/\omega_r^{(j)}) : 1 \leqslant r \leqslant M_j,\ 1 \leq j \leqslant n). \quad \dots \quad (2.9)$$

Let $\hat{a}_{kk'}(.)$, $\hat{b}_k(\cdot)$ be defined on R^M by (2.7). Since $a_{kk'}(.)$ may be viewed as the restriction of $\hat{a}_{kk'}(.)$ on H_0 and since $\zeta(H_0)$ is dense in $\mathcal{J}$ (Hardy and Wright, 1959, Theorem 444, p. 382), it follows by the periodicity and continuity of $\hat{a}_{kk'}(.)$ on R^M and by (2.3) that the smallest eigenvalue of $\hat{a}(y) \doteq ((\hat{a}_{kk'}(y)))$ is bounded away from zero :

$$\inf_{y \in R^M}\ (\text{smallest eigenvalue of } \hat{a}(y)) = \lambda_0 > 0. \quad \dots \quad (2.10)$$

Let $\hat{\sigma}(y) \doteq ((\hat{\sigma}_{kk'}(y)))$ denote the $n \times n$ symmetric positive definite square root of $\hat{a}(y)$.

Let $(\Omega, \mathcal{F}, p)$ be a probability space on which is defined an n-dimensional standard Brownian motion $B(t) = (B_1(t), B_2(t), \ldots, B_n(t))$, $t \geqslant 0$, which is adapted to a right continuous increasing family of P-complete sigmafields $\mathcal{J}_t$, $t \geqslant 0$.

Let $Y(t) = (Y_r^{(k)}(t) : 1 \leqslant r \leqslant M_k, 1 \leqslant k \leqslant n)$, $t \geqslant 0$, be the continuous nonaticipative solution to Itô's stochastic differential equations

$$dY_r^{(k)}(t) = \hat{b}_k(Y(t))dt + \sum_{k'=1}^{n} \hat{\sigma}_{kk'}(Y(t))dB_{k'}(t),$$

$$(1 \leqslant r \leqslant M_k, 1 \leqslant k \leqslant n), \qquad \ldots \quad (2.11)$$

subject to some initial condition $Y(0) = Z$, where Z is an M dimensional random vector independent of $B(t)$, $t \geqslant 0$.

For all $c = (c_r^{(j)} : 2 \leqslant r \leqslant M_j, 1 \leqslant j \leqslant n)$ define the functions (on R^n)

$$b_{k,c}(x) = \sum_m b_{k,c}^{(m)} \exp\left\{ i \sum_{j=1}^{n} x_j \sum_{j=1}^{M_j} m_r^{(j)} \omega_r^{(j)} \right\},$$

$$a_{kk',c}(x) = \sum_m a_{kk',c}^{(m)} \exp\left\{ i \sum_{j=1}^{n} x_j \sum_{r=1}^{M_j} m_r^{(j)} \omega_r^{(j)} \right\} \qquad \ldots \quad (2.12)$$

where

$$b_{k,c}^{(m)} = b_k^{(m)} \exp\left\{ i \sum_{j=1}^{n} \sum_{r=2}^{M_j} c_r^{(j)} m_r^{(j)} \omega_r^{(j)} \right\},$$

$$a_{kk',c}^{(m)} = a_{kk'}^{(m)} \exp\left\{ i \sum_{j=1}^{n} \sum_{r=2}^{M_j} c_r^{(j)} m_r^{(j)} \omega_r^{(j)} \right\}. \qquad \ldots \quad (2.13)$$

Note that $dY_r^{(k)}(t) - dY_1^{(k)}(t) = 0$ for $2 \leqslant r \leqslant M_k$. Hence

$$Y_r^{(k)}(t) = Y_1^{(k)}(t) + (Y_r^{(k)}(0) - Y_1^{(k)}(0)), \; t \geqslant 0, \qquad \ldots \quad (2.14)$$

with probability one.

From (2.11)-(2.14) and (2.4) the following lemma is immediate. Write

$$\partial_k = \sum_{r=1}^{M_k} \partial / \partial y_r^{(k)} \; (1 \leqslant k \leqslant n). \qquad \ldots \quad (2.15)$$

Lemma 2.2 : (i) *$Y(t)$, $t \geqslant 0$, is a singular diffusion on R^M generated, in the sense of Itô, by*

$$\widetilde{L} \doteq \frac{1}{2} \sum_{k=1}^{n} \partial_k \left[\sum_{k'=1}^{n} a_{kk'}(y) \partial_{k'} \right] + \sum_{k=1}^{n} \hat{b}_k^{*}(y) \partial_k \qquad \ldots \quad (2.16)$$

where $\hat{b}_k^{*}$ is defined on R^M by (2.7) from the function (on R^n)

$$b_k^{*}(x) \doteq b_k(x) - \sum_{k'=1}^{n} (\partial / \partial x_{k'}) \, a_{kk'}(x), \; (1 \leqslant k \leqslant n). \qquad \ldots \quad (2.17)$$

(ii) *If $Y(0) \in H_c$ (with probability one), then $(Y_1^{(1)}(t), Y_1^{(2)}(t), \ldots, Y_1^{(n)}(t))$, $t > 0$, is a nonsingular diffusion on R^n with drift coefficients $b_{k,c}(.)$ and diffusion coefficients $a_{kk'}, c(\cdot)$, and its generator may be expressed as*

$$L_c \doteq \frac{1}{2} \sum_{k=1}^{n} \partial/\partial x_k \left[\sum_{k'=1}^{n} a_{kk',c}(x)\partial/\partial x_{k'} \right]$$

$$+ \sum_{k=1}^{n} b_{k,c}^{*}(x)\partial/\partial x_k, \qquad \ldots \quad (2.18)$$

where $b_{k,c}^{*}(x) = b_{k,c}(x) - \sum_{k'=1}^{n} (\partial/\partial x_{k'}) \quad a_{kk',c}(x)$, $b_{k,0}^{*} = b_k^{*}$. In particular, if $c = 0$ then this n-dimensional diffusion has drift coefficients $b_k(.)$ and diffusion coefficients $a_{kk'}(.)$ $(1 \leqslant k, \ k' \leqslant n)$.

Note that $\dot{Y}(t) \equiv \zeta(Y(t))$, $t > 0$, is a *Markov process with state space $\mathcal{J}$*, since $\hat{b}_k(\cdot)$, $\hat{a}_{kk'}(\cdot)$ are periodic $(2\pi/\omega)$.

Lemma 2.3 : *Assume div $b^{*}(x) \doteq \sum_{k=1}^{n} (\partial/\partial x_k)b_k^{*}(x) \equiv 0$. Then* (i) *the Lebesgue measure on R^M is invariant for $Y(t)$, $t > 0$, and* (ii) *the normalized Lebesgue measure $\pi(dz)$ on $\mathcal{J}$ is an invariant probability for the Markov process $\dot{Y}(t)$, $t > 0$.*

Proof : (i) In view of the assumption div $b^{*} = 0$ the formal adjoint L_c^{*} of L_c (and $\widetilde{L}^{*}$ of $\widetilde{L}$) annihilates constant functions. One may then check that the n-dimensional Lebesgue measure is invariant for the diffusion with generator L_c. Integrating first along H_c for a fixed c and then over a set of c values the result is proved. The precise change of variables involved is given by (2.21) below.

(ii) Let $p(t\,; y, B)$, $\dot{p}(t\,; y, C)$ denote the *transition probabilities* of the processes $Y(t), t \geqslant 0$, and $\dot{Y}(t), t \geqslant 0$, respectively. For all Borel sets C of $\mathcal{J}$ one has

$$\pi(C) = \int_{R^M} p(t\,; y, C)\, dy$$

$$= \sum_{m \in Z^M} \int_{\mathcal{J} + \left(m_1^{(1)} 2\pi/\omega_1^{(1)}, \ldots, m_{M_n}^{(n)} 2\pi/\omega_{M_n}^{(n)}\right)} p(t; y, C)\, dy$$

$$= \sum_{m \in Z^M} \int_{\mathcal{J}} p\left(t\,; y - \left(m_1^{(1)} 2\pi/\omega_1^{(1)}, \ldots, m_{M_n}^{(n)} 2\pi/\omega_{M_n}^{(n)}\right), C\right) dy$$

$$= \int_{\mathcal{J}} \sum_{m} p\left(t\,; y, C + \left(m_1^{(1)} 2\pi/\omega_1^{(1)}, \ldots, m_{M_n}^{(n)} 2\pi/\omega_{M_n}^{(n)}\right)\right) dy$$

$$= \int_{\mathcal{J}} \dot{p}\,(t\,; \dot{y}, C)\, \pi(dy).$$

Let $\Gamma = C([0, \infty) : \mathcal{J})$ be the set of all continuous functions on $[0, \infty)$ into $\mathcal{J}$. Let P^y denote the distribution of $\dot{Y}(t)$, $t \geqslant 0$, (i.e., a probability measure on the Borel sigmafield of Γ) when $Y(0) \equiv y$. Clearly $P^y = P^{\dot{y}}$. Let P^π denote the corresponding distribution when $Y(0)$ has distribution π (the normalized Lebesgue measure on $\mathcal{J}$). Then $P^\pi(F) = \int_{\mathcal{J}} P^y(F)\pi(dy)$ for all Borel subsets F of Γ.

Lemma 2.4 : *Let B be a Borel subset of $\mathcal{J}$ such that, for some $t > 0$,*

$$1_B(\gamma(0)) = 1_B(\gamma(t)) \text{ for almost all (w.r.t. } P^\pi)\gamma\epsilon\Gamma, \quad \ldots \quad (2.19)$$

where $1_B(y)$ is the indicator function of the set B. Then there exists a Borel subset C of R^{M-n} such that $\pi(B\Delta\zeta(\hat{B})) = 0$ where

$$\hat{B} = \bigcup_{c\epsilon C} H_c, \quad \ldots \quad (2.20)$$

ζ is the map $y \to \dot{y}$ (see (2.9)) and Δ denotes symmetric difference.

Proof : Let φ, ψ be linear maps on R^M (into R^{M-n}, R^n, respectively) defined by

$$\varphi(y) = (y_r^{(k)} - y_1^{(k)} : 2 \leqslant r \leqslant M_k,\ 1 \leqslant k \leqslant n),$$

$$\psi(y) = (\varphi(y), y_1^{(1)}, y_1^{(2)}, \ldots, y_1^{(n)}). \quad \ldots \quad (2.21)$$

Then ψ is nonsingular with Jacobian determinant one. Let μ_r denote Lebesgue measure on R^r. For $c\epsilon R^{M-n}$, $z\epsilon R^n$, the transition probability $q(t\,;\,(c, z), D) \doteq P^\pi(\{\psi(\gamma(t)) \,\epsilon\, D\} \,|\, \psi(\gamma(0)) = (c, z))$ may be expressed as

$$q(t\,;(c, z), D) = \int_{D_{c\cdot}} f_c(t\,; z, z')\,\mu_n(dz'), \quad \ldots \quad (2.22)$$

where $D_{c\cdot} = \{z'\epsilon R^n : (c, z')\epsilon D\}$, and $f_c(t\,;\, z, z')$ is the strictly positive continuous density (w.r.t μ_n) of the transition probability of the n-dimensional diffusion generated by L_c (see (2.18)). On taking conditional expectation given $\gamma(0)$ in (2.19) one has $1_B(\dot{y}) = \dot{p}(t\,;\,\dot{y}, B)$ a.s. π, i.e.,

$$1_{\zeta^{-1}(B)}(y) = p(t\,;\, y, \zeta^{-1}(B)) \text{ a.e. } \mu_M. \quad \ldots \quad (2.23)$$

Writing $F = \psi(\zeta^{-1}(B))$ and using (2.22)), one may express (2.23) as

$$1_F((c, z)) = \int_{F_{c\cdot}} f_c(t\,; z, z')\,\mu_n(dz') \text{ a.e. } \mu_M, \quad \ldots \quad (2.24)$$

i.e., there exists a μ_M-null set J such that (2.24) holds for all $(c, z) \notin J$. Hence

$$\mu_n(R^n \setminus F_{c\cdot}) = 0 \quad \ldots \quad (2.25)$$

for almost all (w.r.t. μ_{M-n})c in

$$C = \{c \in R^{M-n} : \mu_n(F_c) > 0\}. \quad \ldots \quad (2.26)$$

It follows from (2.25), (2.26) and Fubini's theorem that

$$\mu_M((C \times R^n)\Delta F) = 0. \quad \ldots \quad (2.27)$$

Let $\hat{B}$ be as in (2.20), with C as in (2.26). Then

$$\hat{B} = \psi^{-1}(C \times R^n), \quad \ldots \quad (2.28)$$

and (2.27) implies

$$\mu_M(\hat{B}\Delta\zeta^{-1}(B)) = 0,$$

and, therefore,

$$\pi(\zeta(\hat{B})\Delta B) = 0. \quad \ldots \quad (2.29)$$

The main result of this section is the following.

Theorem 2.5 : *Suppose* div $b^*(x) \equiv 0$. (i) *If* $Y(0)$ *has distribution* π, *then* $\dot{Y}(t)$, $t \geqslant 0$, *is a stationary ergodic Markov process on* $\mathcal{T}$. (ii) *Let* $X(t\,;\,c)$ *denote an n-dimensional diffusion with drift coefficients* $b_{k,c}(\cdot)$ *and diffusion coefficients* $a_{kk',c}(\cdot)$. *Then for all* $c \in R^{M-n}$ *outside a set of zero* ($M-n$ *dimensional*) *Lebesgue measure,*

$$\lim_{t\to\infty} \frac{X(t\,;\,c)}{t} = \bar{b} \doteq (b_1^{(0)}, b_2^{(0)}, \ldots, b_n^{(0)}) \text{ a.s.}, \quad \ldots \quad (2.30)$$

whatever the initial distribution of $X(t\,;\,c)$.

Proof : (i) Suppose $Y(0)$ has distribution π. Then $\dot{Y}(t)$, $t \geqslant 0$, is a stationary process with distribution P^{π}. Let F be a shift-invariant Borel set of $\Gamma = C([0, \infty) : \mathcal{T})$. There exists a Borel set B of $\mathcal{T}$ such that (Doob, (1953), p. 460)

$$P^{\pi}(F\Delta\{\gamma(t) \in B\}) = 0 \text{ for all } t \geqslant 0. \quad \ldots \quad (2.31)$$

In particular, $P^{\pi}(\{\gamma(0) \in B\}\Delta\{\gamma(t) \in B\}) = 0$ for all $t > 0$, i.e., (2.19) holds. Hence, by Lemma 2.4, there exists a Borel set $C \subset R^{M-n}$ such that $\pi(B\Delta\zeta(\hat{B})) = 0$ with $\hat{B}$ given by (2.20). Let $G = C+Q = \{c+q : c \in C,\ q \in Q\}$, where Q is the set (2.5). Since $\zeta(H_c) = \zeta(H_{c'})$ if $c-c' \in Q$ one has $\zeta(\hat{B}) = \zeta(U_{c\in G}\ H_c)$ We need to prove $P^{\pi}(F) = 0$ or 1, i.e.,

$$\pi(\zeta(\hat{B})) = 0 \text{ or } 1. \quad \ldots \quad (2.32)$$

Suppose that (2.32) *is not true*, so that $0 < \pi(\zeta(\hat{B})) < 1$. Then

$$\mu_{M-n}(G) > 0,\ \mu_{M-n}(R^{M-n}\setminus G) > 0. \quad \ldots \quad (2.33)$$

But G is invariant under translation by elements of Q which is *dense* in R^{M-n} (Lemma 2.1). If (2.33) holds, then one may find two compact sets $K_1 \subset G$, $K_2 \subset R^{M-n} \setminus G$ both with *positive* μ_{M-n}-measure ; but the convolution $1_{K_1} * 1_{K_2}$ vanishes on the dense set Q ; this convolution is continuous (indeed its Fourier transform is integrable), so that $1_{K_1} * 1_{K_2} \equiv 0$, which is false. Hence (2.33) is false, and (2.32) is true.

(ii) Let $Y(0)$ have distribution π. Then, by the ergodic theorem applied to the time integral, and the maximal inquality applied to the stochastic integral in (2.11), one has a.s. (P),

$$\lim_{t \to \infty} \frac{Y_r^{(k)}(t)}{t} = b_k^{(0)} \; (1 \leqslant r \leqslant M_k,\ 1 \leqslant k \leqslant n). \qquad \ldots \quad (2.34)$$

Let $\bar{P}^y$ be the distribution of $(Y_1^{(1)}(t), \ldots, Y_1^{(n)}(t))$, $t \geqslant 0$, (on $C([0, \infty) : R^n)$) when $Y(0) \equiv y$. Note that $\bar{P}^y$ is the distribution of $X(t\,;\,c)$, $t \geqslant 0$, if $y \in H_c$ and $X(0\,;\,c) \equiv \bar{y} \doteq (y_1^{(1)}, \ldots, y_1^{(n)})$. Now (2.34) implies that $\mu_M(R^M \setminus B) = 0$, where $B = \{y \in R^M : g(y) = 1\}$, $g(y) \doteq P(\{(2.34)$ holds $\mid Y(0) = y\})$. Since $X(t\,;\,c)$, $t \geqslant 0$, is a nonsingular n-dimensional diffusion, $g(y)$ is continuous on H_c ; also, by the maximum principle, $g(y) \equiv 1$ on H_c if $B \cap H_c \neq \phi$ (see, e.g., Bhattacharya (1978), Lemma 2.3). It follows that $B = \bigcup_{c \in C} H_c$ with C a Borel subset of R^{M-n} such that $\mu_{M-n}(R^{M-n} \setminus C) = 0$.

3. The central limit theorem

We continue to use the notation of Section 2.

Let $\mathcal{L}^2(\mathcal{T})$ denote the usual Hilbert space of (equivalence classes) of real-valued functions square integrable with respect to the normalized Lebesgue measure π on $\mathcal{T}$. The inner product on $\mathcal{L}^2(\mathcal{T})$ will be denoted by $<, >$, and norm by $\|\cdot\|_0$. Let O_N be the subspace

$$O_N = \left\{\varphi \in \mathcal{L}^2(\mathcal{T}) : \varphi(y) = \sum_{0 \neq |m| \leqslant N} \varphi^{(m)} \exp\left\{ i \sum_{j=1}^{n} \sum_{r=1}^{M_j} y_r^{(j)} m_r^{(j)} \omega_r^{(j)} \right\}\right\} \ldots \quad (3.1)$$

where $|m| = \sum_{j,r} |m_r^{(j)}|$. We shall use $\bar{O}_N$ to denote *projection onto* O_N.

Recall the singular differential operator $\tilde{L}$ on R^M defined by (2.16)

Lemma 3.1 : *Suppose div* $b^* = 0$. *Then for each* $N \geqslant 1$, $\bar{O}_N \tilde{L}$ *is a* $1-1$ *map on* O_N *onto* O_N.

Proof: Clearly, $\overline{O}_N \widetilde{L}\varphi \in O_N$ for each $\varphi \in O_N$. Now div $b^*(x) = 0$ implies $\sum_{k=1}^{n} \partial_k \hat{b}_k^*(y) \equiv 0$, so that for every $\varphi \in O_N$

$$\int_{\mathcal{J}} \left[\sum_{k=1}^{n} \hat{b}_k^*(y)\, \partial_k \varphi(y) \right] \varphi(y)\, \pi(dy)$$

$$= -\frac{1}{2} \int_{\mathcal{J}} \left[\sum_{k=1}^{n} \partial_k \hat{b}_k^*(y) \right] \varphi^2(y)\, \pi(dy) = 0. \qquad \ldots (3.2)$$

By (2.10), (3.2) and the self-adjointness of $\overline{O}_N$, one has for every $\varphi \in O_N$, $\varphi \neq 0$,

$$< \overline{O}_N \widetilde{L}\varphi, \varphi > = < \widetilde{L}\varphi, \varphi > = -\frac{1}{2} \int_{\mathcal{J}} \left[\sum_{k,k'=1}^{n} \hat{a}_{kk'}(y)\, \partial_k \varphi(y)\, \partial_{k'} \varphi(y) \right] \pi(dy)$$

$$\leqslant -\frac{\lambda_0}{2} \int_{\mathcal{J}} \sum_{k=1}^{n} (\partial_k \varphi(y))^2\, \pi(dy) \qquad \ldots (3.3)$$

$$= -\frac{\lambda_0}{2} \sum_{k=1}^{n} \sum_{0 \neq |m| \leqslant N} |\varphi^{(m)}|^2 \left[\sum_{r=1}^{M_k} m_r^{(k)} \omega_r^{(k)} \right]^2 < 0.$$

For $\sum_{r=1}^{M_k} m_r^{(k)} \omega_r^{(k)}$ is nonzero for each k and each $m \neq 0$.

Hence $\overline{O}_N \widetilde{L}$ is $1-1$ on O_N into O_N. Since O_N is finite dimensional, $\overline{O}_N \widetilde{L}$ is $1-1$ on O_N *onto* O_N.

For infinitely differentiable periodic $(2\pi/\omega)$ functions φ on R^M define

$$\|\varphi\|_s = \left[\sum_{|\alpha| \leqslant s} \int_{\mathcal{J}} |\partial_1^{\alpha_1} \partial_2^{\alpha_2} \ldots \partial_n^{\alpha_n} \varphi(y)|^2 \pi(dy) \right]^{1/2} (s = 0, 1, 2, \ldots), \quad \ldots (3.4)$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index and $|\alpha| = \alpha_1 + \ldots + \alpha_n$.

Lemma 3.2: *Suppose div $b^*(x) \equiv 0$. Let $f \in Trig\ (\omega)$ with $f^{(0)} = 0$. Let $\hat{f}$ be given by (2.7), the sum being over m satisfying $|m| \leqslant N_0$. Then for every $N \geqslant N_0$ there exists a unique $\hat{u}_N \in O_N$ such that $\overline{O}_N \widetilde{L}\hat{u}_N = \hat{f}$, and for all $s = 0, 1, 2, \ldots$, one has*

$$\sum_{k=1}^{n} \|\partial_k \hat{u}_N\|_s^2 \leqslant c(s), \qquad \ldots (3.5)$$

where $c(s)$ does not depend on N.

Proof: Since $\hat{f} \in O_N$ for all $N \geqslant N_0$ one has, by Lemma 3.1, a unique $\hat{u}_N \in O_N$ such that $\overline{O}_N \widetilde{L}\hat{u}_N = \hat{f}$ for $N \geqslant N_0$. One then has (as in Kozlov (1979), p. 487)

$$|<\widetilde{L}\hat{u}_N, \hat{u}_N>| = |<\bar{O}_N\widetilde{L}\hat{u}_N, \hat{u}_N>| = |<\hat{f}, \hat{u}_N>| = |\sum_{m\neq 0} f^{(m)}u_N^{(m)}|$$

$$= \left|\sum_{m\neq 0} \frac{1}{\delta} f^{(m)} \left[\sum_{r=1}^{M_k} m_r^{(k)}\omega_r^{(k)}\right]^{-1} \cdot \delta\left[\sum_{r=1}^{M_k} m_r^{(k)}\omega_r^{(k)}\right] u_N^{(m)}\right|$$

$$\leqslant \frac{1}{2}\sum_{1<|m|<N_0} \frac{1}{\delta^2}|f^{(m)}|^2 \left|\sum_{r=1}^{M_k} m_r^{(k)}\omega_r^{(k)}\right|^{-2}$$

$$+\frac{1}{2}\delta^2 \sum_{1\leqslant|m|\leqslant N} |u_N^{(m)}|^2 \left[\sum_{r=1}^{M_k} m_r^{(k)}\omega_r^{(k)}\right]^2$$

$$= c_k(\delta, f) + \frac{1}{2}\delta^2 \|\partial_k \hat{u}_N\|_0^2. \quad \ldots \quad (3.6)$$

Also, from the calculations in (3.3),

$$|<\widetilde{L}\hat{u}_N, \hat{u}_N>| \geqslant \frac{\lambda_0}{2}\sum_{k=1}^{n} \|\partial_k\hat{u}_N\|_0^2. \quad \ldots \quad (3.7)$$

From (3.6), (3.7) one obtains

$$\sum_{k=1}^{n} \|\partial_k\hat{u}_N\|_0^2 \leqslant c(0), \quad \ldots \quad (3.8)$$

proving (3.5) for $s = 0$.

In order to prove (3.5) for $s > 0$, introduce the differential operator

$$\widetilde{D}_s = \left[\sum_{k=1}^{n} \partial_k^2\right]^s (s = 0, 1, 2, \ldots). \quad \ldots \quad (3.9)$$

On integration by parts one has

$$<\widetilde{D}_s\widetilde{L}\partial_k\hat{u}_N, \partial_k\hat{u}_N> = -\frac{1}{2}\sum_{j=1}^{n} <\widetilde{D}_s\sum_{j'=1}^{n} \hat{a}_{jj'}(.)\partial_{j'}\partial_k\hat{u}_N, \partial_j\partial_k\hat{u}_N>$$

$$+\sum_{j=1}^{n} <\widetilde{D}_s(\hat{b}_j^*(.)\partial_j\partial_k\hat{u}_N), \partial_k\hat{u}_N>$$

$$= -\sum_{j,j',k_1,\ldots,k_s=1}^{n} (1/2)(-1)^s \times$$

$$\times <\partial_{k_1}\ldots\partial_{k_s}(\hat{a}_{jj'}(.)\partial_{j'}\partial_k\hat{u}_N), \partial_{k_1}\ldots\partial_{k_s}\partial_j\partial_k\hat{u}_N>$$

$$+(-1)^s\sum_{j,k_1,\ldots,k_s=1}^{n} <\partial_{k_1}\ldots\partial_{k_s}(\hat{b}_j^*(.)\partial_j\partial_k\hat{u}_N), \partial_{k_1}\ldots\partial_{k_s}\partial_k\hat{u}_N>.$$

$$\ldots \quad (3.10)$$

Using Leibniz rule for differentiation of products one gets, from (3.10) and and (2.10),

$$|<\widetilde{D}_s\widetilde{L}\partial_k\hat{u}_N,\ \partial_k u_N>| \geqslant \frac{1}{2}\sum_{k_1,\ldots,k_s=1}^{n} < \sum_{j,j'=1}^{n} \partial_{jj'}(.)\partial_{k_1}\ldots\partial_{k_s}\partial_j,\ \partial_k\hat{u}_N,$$

$$\partial_{k_1}\ldots\partial_{k_s}\partial_j\partial_k\hat{u}_N> -c_1(s)\|\partial_k\hat{u}_N\|_s^2 - c_2(s)\|\partial_k\hat{u}_N\|_{s+1}\|\partial_k\hat{u}_N\|_s$$

$$\geqslant c_3(s)\|\partial_k\hat{u}_N\|_{s+1}^2 - c_1(s)\|\partial_k\hat{u}_N\|_s^2 - c_2(s)\|\partial_k\hat{u}_N\|_s\|\partial_k\hat{u}_1\|_{s+1}. \quad \ldots \quad (3.11)$$

We shall now prove (3.5) by induction on s. Suppose it holds for $s \leqslant s_0$. Then

$$|<\widetilde{D}_{s_0}\widetilde{L}\partial_k\hat{u}_N,\ \partial_k\hat{u}_N>| \leqslant |\ \widetilde{D}_{s_0}\partial_k\widetilde{L}\hat{u}_N,\ \partial_k\hat{u}_N>|$$

$$+\frac{1}{2}\left|<\widetilde{D}_{s_0}\sum_{j,j'=1}^{n}\partial_j\{(\partial_k\, a_{jj'}(.))\partial_j,\ \hat{u}_N),\ \partial_k\,\hat{u}_N>\right|$$

$$+\left|<\widetilde{D}_{s_0}\sum_{j=1}^{n}(\partial_k\,\hat{b}_j^*(.))\partial_j\,\hat{u}_N,\ \partial_k\,\hat{u}_N>\right|. \quad \ldots \quad (3.12)$$

Since $\partial_k\hat{u}_N = \overline{O}_N\partial_k\hat{u}_N$, and $\widetilde{D}_{s_0}\,\partial_k$ commutes with $\overline{O}_N$, one has

$$\left|<\widetilde{D}_{s_0}\,\partial_k\widetilde{L}\,\hat{u}_N,\ \partial_k\hat{u}_N>\right| = \left|<\widetilde{D}_{s_0}\partial_k\bar{O}_N\widetilde{L}\,\hat{u}_N,\ \partial_k\hat{u}_N>\right|$$

$$= \left|<\widetilde{D}_{s_0}\,\partial_k\hat{f},\ \partial_k\hat{u}_N>\right| \leqslant c_4(s_0)\|\partial_k\hat{u}_N\|_0 \leqslant c_5(s_0), \quad \ldots \quad (3.13)$$

by (3.8). Also, the differential operator $\widetilde{D}_{s_0}$ is of order $2s_0$ and on expressing it as a sum of products of two differential operators each of order s_0, and integrating by parts one gets

$$\frac{1}{2}\left|<\widetilde{D}_{s_0}\sum_{j,j'=1}^{n}\partial_j\{(\partial_k a_{jj'}(.))\partial_{j'}\,\hat{u}_N\},\ \partial_k\hat{u}_n>\right|$$

$$\leqslant c_6(s_0)\left[\sum_{j=1}^{n}\|\partial_j\hat{u}_N\|_{s_0+1}\right]\|\partial_k\,\hat{u}_N\|_{s_0} \leqslant c_7(s_0)\left[\sum_{j=1}^{n}\|\partial_j\,\hat{u}_N\|_{s_0+1}\right]. \quad \ldots \quad (3.14)$$

One similarly obtains

$$\left|<\widetilde{D}_{s_0}\sum_{j=1}^{n}(\partial_k\hat{b}_j^*(.))\partial_j\,\hat{u}_N,\ \partial_k\,\hat{u}_N>\right|$$

$$\leqslant c_8(s_0)\left[\sum_{j=1}^{n}\|\partial_j\,\hat{u}_N\|_{s_0}\right]\|\partial_k\,\hat{u}_N\|_{s_0} \leqslant c_9(s_0). \quad \ldots \quad (3.15)$$

Using (3.13)–(3.15) in (3.12) one gets

$$\sum_{k=1}^{n} |< \widetilde{D}_{s_0} \widetilde{L} \partial_k \hat{u}_N, \partial_k u_N >|$$

$$\leqslant c_{10}(s_0)+c_{11}(s_0) \left[\sum_{j=1}^{n} \|\partial_j \hat{u}_N\|_{s_0+1} \right]. \quad \ldots \quad (3.16)$$

On the other hand, (3.11) and the induction hypothesis yield

$$\sum_{k=1}^{n} |< \widetilde{D}_{s_0} \widetilde{L} \partial_k \hat{u}_N, \partial_k \hat{u}_N >| \geqslant c_2(s) \sum_{k=1}^{n} \|\partial_k \hat{u}_N\|^2_{s_0+1}$$

$$-c_{12}(s_0)-c_{13}(s_0) \left[\sum_{k=1}^{n} \|\partial_k \hat{u}_N\|_{s_0+1} \right]. \quad \ldots \quad (3.17)$$

From (3.16), (3.17) one easily obtains

$$\sum_{j=1}^{n} \|\partial_j \hat{u}_N\|_{s_0+1} \leqslant c_{14}(s_0). \quad \ldots \quad (3.18)$$

In the proof of Theorem 3.4 we apply Lemma 3.2 (as well as Lemma 3.3 below) with $\hat{f} = \hat{b}_k - b_k^{(0)}$, and $N_0 = \Sigma \Sigma m_r^{(j)}$ in the representation (2.1).

For the next lemma we shall need the following hypothesis (see Kozlov, 1979, p. 489) concerning $\omega_r^{(k)}$).

Condition (C). There exists a positive integer s_0 and a positive number δ such that

$$\left| \sum_{r=1}^{M_k} m_r^{(k)} \omega_r^{(k)} \right| \geqslant \delta \left[\sum_{r=1}^{M_k} \left| m_r^{(k)} \right| \right]^{-s_0} (k = 1, 2, \ldots, n), \quad \ldots \quad (3.19)$$

for all $m = (m_r^{(k)}: 1 \leqslant r \leqslant M_k, 1 \leqslant k \leqslant n) \in Z^M$ $(m \neq 0)$.

It may be noted that outside a set of Lebesgue measure (M-dimensional) zero, all M-tuples $(\omega_r^{(k)}) : 1 \leqslant r \leqslant M_k,\ 1 \leqslant k \leqslant n)$ satisfy (3.19) if $\delta > 0$ and s_0 is sufficiently large. (Sprindzuk, 1979, Theorem 12, p. 33).

It is easy to check (see Kozlov, 1979, p. 492) that condition (C) implies

$$\|\hat{u}_N\|^2_{s-s_0} \leqslant c_{15}(s) \sum_{j=1}^{n} \|\partial_j \hat{u}_N\|^2_s \quad (s \geqslant s_0). \quad \ldots \quad (3.20)$$

Now let $\dot{T}_t$, $t \geqslant 0$, denote the *semigroup of* transition operators on $\mathcal{L}^2(\mathcal{T})$ defined by

$$(\dot{T}_t f)(y) = E(f(\dot{Y}(t)) \mid \dot{Y}(0) = y) = \int_{\mathcal{T}} f(z)\dot{p}(t\,; y, dz). \quad \ldots \quad (3.21)$$

It is simple to check that this is a contraction semigroup. Let $\mathcal{D}_{\tilde{A}}$ denote the set of all f in $\mathcal{L}^2(\mathcal{J})$ such that the following limit exists in $\mathcal{L}^2$:

$$\tilde{A}f \doteq \lim_{t \downarrow 0} \frac{\dot{T}_t f - f}{t}. \qquad \dots \quad (3.22)$$

The operator $\tilde{A}$ is the *infinitesimal generator* of the semigroup and $\mathcal{D}_{\tilde{A}}$ its *domain*. Let $\mathcal{R}_{\tilde{A}}$ denote the *range* of $\tilde{A}$.

Lemma 3.3 : *Suppose div* $b^* = 0$ *and condition (C) holds. Let* $f \in Trig\ (\omega)$ *be such that* $f^{(0)} = 0$, *where* f *is represented as in* (2.6). *Let* $\hat{f}$ *be defined by* (2.7). *Then there exists* $\hat{g} \in \mathcal{D}_{\tilde{A}}$ *such that* $\tilde{A}\hat{g} = \hat{f}$, *and there exist* $\hat{g}_N \in O_N (N = 1, 2, \dots)$ *such that* $\hat{g}_N \rightarrow \hat{g}$ *and* $\tilde{A}\hat{g}_N \rightarrow \tilde{A}\hat{g} = \hat{f}$ *in* $\mathcal{L}^2$*-norm, as* $N \rightarrow \infty$.

Proof: Let $\hat{u}_N$ be the unique solution of $\bar{O}_N \tilde{L} \hat{u}_N = \hat{f}$, for $N \geqslant N_0$. By Lemma 3.2, and (3.20),

$$\sup_{n \geqslant N_0} \|\hat{u}_N\|_s^2 < \infty \ (s = 1, 2, \dots). \qquad \dots \quad (3.23)$$

Now it is easy to check using Ito's lemma and path continuity of $\dot{Y}(s)$ that all infinitely differeniable functions which are periodic $(2\pi/\omega)$, regarded as elements of $\mathcal{L}^2(\mathcal{J})$, belong to $\mathcal{D}_{\tilde{A}}$, and $\tilde{A} = \tilde{L}$ when restricted to this class of functions. Hence $\hat{u}_N \in \mathcal{D}_{\tilde{A}}$, and (3.23) implies that $\hat{u}_N$ and $\tilde{A}\hat{u}_N \equiv \tilde{L}\hat{u}_N$, $N \geqslant N_0$, are norm-bounded. Therefore, there exists a subsequence N' of the integers such that $\hat{u}_{N'}$ converges *weakly* to $\hat{g}$, say, and $\tilde{A}\hat{u}_{N'}$ converges *weakly* to $\hat{h}$, say. Thus $(\hat{g}, \hat{h})$ belongs to the *weak closure* of the graph of $\tilde{A}$ restricted to $0 = \bigcup_{N=1}^{\infty} 0_N$. Since (i) $O \subset \mathcal{D}_{\tilde{A}}$, (ii) the graph of $\tilde{A}$ is closed, and (iii) the weak closure of the graph of $\tilde{A}$ restricted to O equals its *strong closure* (Yoshida, 1966, Theorem 11, p. 125), it follows that $(\hat{g}, \hat{h})$ belongs to the graph of $\tilde{A}$, i.e., $\hat{g} \in \mathcal{D}_{\tilde{A}}$ and $\tilde{A}\hat{g} = \hat{h}$. Also for all $u \in 0$ one has

$$< \hat{h}, u > = \lim_{N' \rightarrow \infty} < \tilde{A}\hat{u}_{N'}, u >$$

$$= \lim_{N' \rightarrow \infty} < \bar{O}_{N'} \tilde{A}\hat{u}_{N'}, u > = < \hat{f}, u >. \qquad \dots \quad (3.24)$$

Since O is dense in $1^{\perp}$, $\hat{h} \in 1^{\perp}$ (since $R_{\hat{A}} \subset 1^{\perp}$; Bhattacharya, (1982, Relation (2.6)) and $f \in 1^{\perp}$, it follows that $\hat{h} = f$.

Finally, again using the fact that the weak closure of the restriction of the graph of $\tilde{A}$ to O equals its strong closure, the second assertion follows.

Theorem 3.4 : *Suppose div $b^{*} = 0$ and condition (C) holds. Define*

$$X_{\epsilon}(t\,;\,c) = \epsilon(X(t/\epsilon^{2}\,;\,c)) - \frac{t}{\epsilon}\,\bar{b}, \quad \ldots \quad (3.25)$$

where $X(t\,;\,c)$ is the n-dimensional diffusion generated by L_c in (2.18), starting at an arbitrary initial state in R^n. For all $c \in R^{M-n}$ outside a set of $(M-n)$-dimensional Lebesgue measure zero, $X_{\epsilon}(t\,;\,c)$, $t \geqslant 0$, converges weakly as $\epsilon \downarrow 0$ to a Brownian motion with zero drift and dispersion matix

$$\int_{\mathcal{T}} (\partial\hat{u}(y)-I)\hat{a}(y)(\partial\hat{u}(y)-I)'\pi(dy), \quad \ldots \quad (3.26)$$

where $\hat{u}(y) = (\hat{u}_1(y), \ldots, \hat{u}_n(y))$ is the unique solution of $\tilde{A}\hat{u}_k = \hat{b}_k - b_k^{(0)}$ $(1 \leqslant k \leqslant n)$ in $1^{\perp}$, and $\partial\hat{u}$ is the $n\times n$ matrix $((\partial_k\hat{u}_{k'}))$.

Proof: By the second part of Lemma 3.3 there exists, for each $j(1 \leqslant j \leqslant n)$, $\hat{u}_{j,N} \in O_N (N = 1, 2, \ldots)$ such that, as $N \to \infty$

$$\|\hat{u}_{j,N} - \hat{u}_j\|_0 \to 0, \; \|\tilde{A}\hat{u}_{j,N} - (\hat{b}_j - b_j^{(0)})\|_0 \to 0. \quad \ldots \quad (3.27)$$

Since (see (3.3))

$$\|\partial_k\hat{u}_{j,N} - \partial_k\hat{u}_{j,N'}\|_0^2$$

$$\leqslant \frac{2}{\lambda_0} < -\tilde{A}(\hat{u}_{j,N} - \hat{u}_{j,N'}),\, \hat{u}_{j,N} - \hat{u}_{j,N'} >, \quad \ldots \quad (3.28)$$

it follows from (3.27) that $\partial_k\hat{u}_j \in \mathcal{L}^2(\mathcal{T})$ and

$$\|\partial_k\hat{u}_{j,N} - \partial_k\hat{u}_j\|_0 \to 0 \text{ as } N \to \infty. \quad \ldots \quad (3.29)$$

Now let $Y(t)$, $t \geqslant 0$, be the continuous nonanticipative solution of (2.11) with $Y(0) = y$. Then writing

$$W_k(t) \doteq Y_1^{(k)}(t) - Y_1^{(k)}(0) - tb_k^{(0)},$$

$$W(t) \doteq (W_1(t), \ldots, W_n(t))', \quad \ldots \quad (3.30)$$

one has

$$W(t) = \int_0^t (\hat{b}(Y(s)) - \bar{b})ds + \int_0^t \hat{\sigma}(Y(s))dB(s). \quad \ldots \quad (3.31)$$

By Ito's lemma,

$$\hat{u}_{j,N}(Y(t))-\hat{u}_{j,N}(Y(0))$$

$$= \int_0^t \tilde{A}\hat{u}_{j,N}(Y(s))ds+ \int_0^t \partial\hat{u}_{j,N}(Y(s)).\,\hat{\sigma}(y(s))dB(s),\ (1 \leqslant j \leqslant n). \quad \ldots \quad (3.32)$$

In view of (3.27), (3.29) one has the representation (see Ikeda and Watanabe, 1981, Chapter II)

$$\hat{u}_j(Y(t))-\hat{u}_j(Y(0))$$

$$= \int_0^t (\hat{b}_j(Y(s))-b_j^{(0)})ds+ \int_0^t \partial\hat{u}_j(Y(s))\hat{\sigma}(Y(s))dB(s) \ \ a.s. \ \ (t \geqslant 0). \quad \ldots \quad (3.33)$$

From (3.31), (3.33) one has

$$W(t) = \hat{u}(Y(t))-\hat{u}(Y(0))$$

$$- \int_0^t (\partial\hat{u}(Y(s))-I)\hat{\sigma}(Y(s))dB(s) \ a.s. \ (t \geqslant 0). \quad \ldots \quad (3.34)$$

The quadratic variation of the martingale $W_\epsilon(t\,;\,c) \doteq X_\epsilon(t\,;\,c)-\epsilon\hat{u}(Y(t/\epsilon^2))+\epsilon\hat{u}(Y(0))$ is given by

$$Z_\epsilon(t) = \epsilon \int_0^{t/\epsilon^2} (\partial\hat{u}(Y(s))-I)\hat{a}(Y(s))\,(\partial\hat{u}(Y(s))-I)'ds. \quad \ldots \quad (3.35)$$

Since each element of the integrand is a stationary ergodic stchastic process (when $Y(0)$ has distribution π) having a finite expection, by the ergodic theorem one has a.s.

$$\lim_{\epsilon \downarrow 0} Z_\epsilon(1) = \int_{\mathcal{J}} (\partial\hat{u}(Y)-I)\hat{a}(y)\,(\partial\hat{u}(y)-I)'\pi(dy). \quad \ldots \quad (3.36)$$

It follows that (3.36) holds with $Y(0) = y_0$ for all $y_0 \in \mathcal{J}$ outside a set of null π-measure. Let $\varphi(y_0)$ denote the probability that (3.36) holds with $Y(0) = y_0$ Since the event that (3.36) holds is shift-invariant, $\varphi(y_0)$ is $\tilde{L}$-harmonic, and its restriction to H_c is L_c-harmonic (see (2.16), (2.18)). Thus if $\varphi(y_0) = 1$ for some $y_0 \in H_c$, then $\varphi(y) = 1$ for *all* $y \in H_c$, by the maximum principle for strictly elliptic operators. Therefore, for all c outside a set $\mathcal{N}$ of zero $(M-n)$-dimensional Lebesgue measure, if $y_0 \in H_c$ then (3.36) holds with initial state y_0. It now follows from (3.34)–(3.36) that with $y_0 \in H_c\,(c \notin \mathcal{N})$, $W_\epsilon(t\,;\,c)$ converges weakly to the desired Brownian motion (one may show this, e.g., by expressing $\theta.W_\epsilon(t\,;\,c)$ as a time changed one-dimensional Brownian motion, for each $\theta \in R^n$)). Finally, $\epsilon\hat{u}(Y(t/\epsilon^2))-\epsilon\hat{u}(Y(0))$ converges to zero uniformly on compact time intervals, with probability one (See Bhattacharya, 1982, p. 189) when the initial distribution is π. Again this implies that $\epsilon\hat{u}(Y(t/\epsilon^2))-\epsilon\hat{u}(Y(0))$ converges to zero uniformly on compact time intervals, with probability one when the initial state lies on H_c, for c lying outside a set of zero $(M-n)$-dimensional Lebesgue measure.

Remark 1 : One may relax the assumption that the sums in (2.1) be over a *finite* set of integer vectors m. The proof of Theorem 2.5 goes over if one assumes

$$\sum_m |b_k^{(m)}| \left[\sum_{j=1}^n \left| \sum_{j=1}^{M_j} m_r^{(j)}\omega_r^{(j)} \right| \right] < \infty \; (1 \leqslant k \leqslant n),$$

$$\sum_m |a_{kk'}^{(m)}| \left[\sum_{j=1}^n \left| \sum_{r=1}^{M_j} m_r^{(j)}\omega_r^{(j)} \right| \right] < \infty \; (1 \leqslant k, k' \leqslant n). \qquad \ldots \quad (3.37)$$

Theorem 3.4 goes over if the 'finite sum' assumption is replaced by (3.37) and (see (3.6))

$$\sum_m |b_k^{(m)}|^2 \left[\left| \sum_{r=1}^{M_j} m_r^{(j)}\omega_r^{(j)} \right| \right]^{-2} < \infty \; (1 \leqslant k \leqslant n, \; 1 \leqslant j \leqslant n). \qquad \ldots \quad (3.38)$$

In view of condition (C), (3.38) may be replaced by the condition

$$\sum_m |b_k^{(m)}|^2 \left[\sum_{r=1}^{M_j} |m_r^{(j)}| \right]^{2\nu_0} < \infty \; (1 \leqslant k \leqslant n, \; 1 \leqslant j \leqslant n). \qquad \ldots \quad (3.39)$$

Remark 2. With each $y \in \mathcal{J}$ one may associate the set of drift and diffusion coefficients $b_{k,c}(\cdot+z)$, $a_{kk'c}(\cdot+z)$, where $c = (c_r^{(k)} = \bar{y}_r^{(k)}-\bar{y}_1^{(k)} : 2 \leqslant r \leqslant M_k,\; 1 \leqslant k \leqslant n) \in R^{M-n}$ and $z = (z_k = \bar{y}_1^{(k)} : 1 \leqslant k \leqslant n) \in R^n$. When $\bar{y}$ is chosen at random with distribution π, one obtains a *random* field indexed by $x \in R^n : x \to \{(b_{k,c}(x+z))_{1 \leqslant k \leqslant n}, (a_{kk',c}(x+z))_{1 \leqslant k,k' \leqslant n}\}$. This random field is *stationary* (w.r.t. translation on R^n) and *ergodic* (See Papanicolaou and Varadhan, 1979). The proof of Theorem 3.4 shows that when the drift and diffusion coefficients arise in this random manner (i.e., as a realization of this random field) and the corresponding stochastic differential equation is solved with a Brownian motion $B(t)$ independent of this random field (i.e., independent of $\bar{y} \in \mathcal{J}$), then the solution $X(t)$, say, is asymptotically Gaussian: $\epsilon X(t/\epsilon^2)-\frac{t}{\epsilon}\,\bar{b}$, $t \geqslant 0$, converges in distribution to an n-dimensional Brownian motion with zero drift and dispersion matrix (3.26).

Remark 3 : Kozlov (1979) derives estimates such as (3.5) in the self-adjoint case, and infers the smoothness of solutions. Since these estimates concern differentiation in only n directions in an M-dimensional space, the validity of such an inference is doubtful.

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