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EFFECT OF NON-NORMALITY ON FOUR TEST CRITERIA FOR TESTING EQUALITY OF TWO COVARIANCE MATRICES

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SUMMARY. Pillai and Jayachandran (1968) studied the power function of four test criteria for testing the equality of two covariance matrices. For actual computation purposes they considered the case of covariance matrices of dimension two, this being necessary as the evaluation of Zonal Polynomial for higher dimension and the labour of calculation needed for simplifying the infinite series involved is quite prohibitive. Here we extend their study in that it also allows for departure from the null hypothesis even in the form of the parent distribution—which takes an extra term as shown in (1) in the text. The extra term included corresponds to second order term in Edgeworth expansion in the multivariate case. Exact tabulation of power under violations of null hypothesis as indicated is done for $p = 2$. A few sample tables are given at the end.

1. Introduction

In this paper we have studied the four test criteria used in Multivariate Analysis for tests of equality of two covariance matrices from the stand point of stability of percentage point and the power comparison when the distribution departs from usual Multivariate Normal distribution by a term which corresponds to second term in Edgeworth expansion in standardized form. This study has been done in the special case when the dimension of the covariance matrix is two. This restriction is due to unavailability of general computing technique for expression involving Zonal Polynomial.

Pillai and Jayachandran (1968) has studied the performance of these criteria for tests of hypothesis $\Sigma_1 = \Sigma_2$ against one sided alternatives where Σ_1 and Σ_2 are covariance matrices of two normal distributions of dimension two. These tests are based on Roy's largest root test, Lawley Hotelling's trace statistics, Pillai's and Wilk's criteria. The motivation behind the present problem is to study the performance of four test criteria further—under departure from the Normality assumption in the basic populations.

2. Derivation of the distribution of roots

Let $S_1\,(p\times p)$ and $S_2(p\times p)$ be two covariance matrices which are jointly distributed as

$$f(S_1, S_2)dS_1dS_2 = \exp(-1/2 \operatorname{tr} \Sigma_1^{-1}S_1)\,|S_1|^{(n_1-p-1)/2}$$

$$\exp(-1/2 \operatorname{tr} \Sigma_2^{-1}S_2)\,|S_2|^{(n_2-p-1)/2}(P_0+P_1C_1(\Sigma_1^{-1}S_1))dS_1dS_2 \quad \ldots \quad (1)$$

where P_0, P_1 are suitably chosen constant subject to total probability content and other relevant conditions so that (1) represents a density.

If $P_0 = 0$ we get the case studied by Pillai and Jayachandran (1968). Let us now make the following transformations :

$$A_1 = \Sigma_1^{-1/2} S_1 \Sigma_1^{-1/2} \quad \text{and} \quad A_2 = \Sigma_1^{-1/2} S_2 \Sigma_1^{-1/2}.$$

Then $J(S_1, S_2; A_1, A_2) = |\Sigma_1|^{(p+1)}$ as given in Deemer and Olkin (1951).

Under the above transformation (7) reduces to

$$f(A_1, A_2) dA_1 dA_2 = \exp(-1/2 \operatorname{tr}(A_1 + \Sigma_1^{1/2} \Sigma_2^{-1} \Sigma_1^{1/2} A_2))$$

$$|\Sigma_1|^{(n_1+n_2)/2} |A_1|^{(n_1-p-1)/2} |A_2|^{(n_2-p-1)/2} (P_0 + P_1 C_1(A_1)) dA_1 dA_2. \quad \ldots \quad (2)$$

Again use the following transformations :

$$A_2^{-1/2} A_1 A_2^{-1/2} = R.$$

Then $$J(A_1, A_2;\ A_2, R) = |A_2|^{(p+1)/2}$$

Thus under the transformation (2) reduces to

$$f(A_2, R) dA_2 dR = \exp(-1/2 \operatorname{tr}(R + \Sigma_1 \ \Sigma_2^{-1} \ \Sigma_1) A_2)$$

$$|\Sigma_1|^{(n_1+n_2)/2} |R|^{(n_1-p-1)/2} |A_2|^{(n_1+n_2-p-1)/2} (P_0 + P_1 C_1(A_2 R)) dA dR. \quad \ldots \quad (3)$$

Now integrating out A_2 we get from (3)

$$f(R) dR = |\Sigma_1|^{(n_1+n_2)/2} |R|^{(n_1-p-2)/2} |R + \Sigma_1 \ \Sigma_2^{-1} \Sigma_1)|^{-(n_1+n_2)/2}$$

$$(P_0 \Gamma_p((n_1+n_2)/2) + P_1 \Gamma_p((n_1+n_2)/2, 1) c_1(R((R + \Sigma_1 \ \Sigma_2^{-1} \ \Sigma_1)/2)^{-1})) dR$$

which reduces to

$$f(R) dR = |\Sigma_1|^{(n_1+n_2)/2} |R|^{-(n_2+p+1)/2} 2p(n_1+n_2)/2$$

$$(P_0 \Gamma_p((n_1+n_2)/2) + 2P_1 \Gamma_p((n_1+n_2)/2)$$

$$C_1(I + \Sigma_1 \ \Sigma_2^{-1} \ \Sigma_1 R^{-1})^{-1}) \,|I + \Sigma_1 \ \Sigma_2^{-1} \Sigma_1 R^{-1}|^{-(n_1+n_2)/2} dR. \quad \ldots \quad (4)$$

Let us now put

$$\Sigma_1^{1/2}\Sigma_2^{-1}\Sigma_1^{1/2} = \Delta_1\Lambda\Delta_1 \text{ and } R = \Delta F\Delta$$

where Δ and Δ_1 are suitably chosen orthogonal matrices and Λ and F are the diagonal matrices. Under the above transformation we get from (4)

$$f(H,F)dFdH = |\Sigma_1|^{(n_1+n_2)/2}2^{p(n_1+n_2)/2}|F|^{-(n_2+p+1)/2}$$
$$|I+HHF^{-1}|^{-(n+n)/2}\frac{n^{p^2/2}}{p^{p/2}}\prod_{i<j}(f_i-f_j)$$
$$(P_0\Gamma_p((n+n)/2)+2P_1\Gamma_p((n_1+n)/2.1)C_1(I+HH'F^{-1})^{-1})dHdF \quad \ldots \quad (5)$$

where $H = \Delta' A_1$ and thus $H'H = HH' = I$.

Now simplifying the expression (5) we get

$$f(H,F)dFdH = |\Sigma_1|^{(n_1+n_2)/2}2^{p(n_1+n_2)/2}|\Lambda|^{-(n_1+n_2)/2}$$
$$|F|^{(n-p-1)/2}|I+FH\Lambda^{-1}H'|^{-(n_1+n_2)/2}\prod_{i<j}(f_i-f_j)(P_0\Gamma_p((n_1+n_2)/2)$$
$$+2P_1\Gamma_p((n_1+n_2)/2,1)\ C_1(I+HH'F^{-1})^{-1})dFdH. \quad \ldots \quad (6)$$

3. Calculation of constant

Considering the total probability content we have to find P_0 and P_1 such that

$$\int_{S_1>0}\int_{S_2>0} f(S_1,S_2)dS_1dS_2 = 1 \quad \ldots \quad (7)$$

where $f(S_1, S_2)$ is given by (1).

To this end we note

$$\int_{S_1>0}\int_{S_2>0} \exp(-1/2\operatorname{tr}\Sigma_1^{-1}S_1)|S_1|^{(n_1-p-1)/2}$$
$$\exp(-1/2\operatorname{tr}\Sigma_2^{-1}S_2)|S_2|^{(n_2-p-1)/2}dS_1dS_2$$
$$=\int_{S_1>0}\exp(-1/2\operatorname{tr}\Sigma_1^{-1}S_1)|S_1|^{(n_1-p-1)/2}dS_1$$
$$\int_{S_2>0}\exp(-1/2\operatorname{tr}\Sigma_2^{-1}S_2)|S_2|^{(n_2-p-1)/2}dS_2$$
$$=2^{p(n_1+n_2)/2}\pi^{p(p-1)/4}$$
$$|\Sigma_1|^{(n_1/2)}|\Sigma_2|^{(n_2/2)}\prod_{i=1}^{p}\Gamma(n_1-i+1)/2\prod_{i=1}^{p}\Gamma(n_2-i+1)/2)$$

and

$$\int_{S_1>0}\int_{S_2>2} \exp(-1/2 \operatorname{tr} \Sigma_1^{-1}S_1)|S_1|^{(n_1-p-1)/2} C_1(\Sigma_1^{-1}S_1)$$

$$\exp(-1/2 \operatorname{tr} \Sigma_2^{-1}S_2)|S_2|^{(n_2-p-1)/2} dS_1 dS_2$$

$$= 2^{p(n_1+n_2)/2+1} p\pi^{p(p-1)/2}|\Sigma_1|^{(n_1/2)}|\Sigma_2|^{(n_2/2)}$$

$$\prod_{i=1}^{p} \Gamma((n_1-i+1)/2+k_i) \,.\, \prod_{i=1}^{p} \Gamma((n_2-i+1)/2)$$

where $k_1 \geqslant k_0 \geqslant k_p \geqslant 0$ and $k:(k_1, \ldots, k_0)$ is a partition of 1.

Now we use two different type of normalisation of P_0 and P_1.

Case 1 : Let us put

$$P_0 = a_0(2^{p(n_1+n_2)/2} \pi^{p(p-1)/2} \prod_{i=1}^{p} \Gamma((n_1-i+1)/2)$$

$$\prod_{i=1}^{p} \Gamma((n_2-i+1)/2)|\Sigma_1|^{(n_1/2)}|\Sigma_2|^{(n_1/2)-1}$$

and

$$P_1 = a_1(2^{p(n_1+n_2)/2+1} \pi^{p(p-1)/2} \prod_{i=1}^{p} \Gamma((n_2-i+1)/2)$$

$$\prod_{i=1}^{p} \Gamma((n_1-i+1)/2+k_i))-1.$$

With this substitution we obviously get

$$a_0+a_1 = 1.$$

As for further simplification we put $p = 2$ and get

$$P_0 = a_0(2^{(n_1+n_2)} \pi|\Sigma_1|^{(n_1/2)}|\Sigma_2|^{(n_2/2)}$$

$$\Gamma(n_1/2)\,\Gamma(n_2/2)\,\Gamma((n_1-1)/2)\,\Gamma(n_2/-1)/2))^{-1}$$

and

$$P_1 = a_1(2^{(n_1+n_2+1)} n_1\pi|\Sigma_1|^{(n_1/2)}|\Sigma_2|^{(n_2/2)}.$$

$$\Gamma((n_1-1)/2)\,\Gamma((n_2-1)/2)\,\Gamma(n_1/2)\,\Gamma(n_2/2))^{-1}.$$

With this simplification and under the assumption that $p = 2$ then (6) reduces to

$$f(H, F)dFdH = |\Lambda|^{-(n_1/2)} |F|^{(n_1-3)/2} (f_1-f_2)$$

$$|I+FH\Lambda^{-1}H'|^{-(n_1+n_2)/2} \left(\Gamma \frac{n_1}{2}\ \Gamma \frac{n_2}{2}\ \Gamma \frac{n_1-1}{2}\ \Gamma \frac{n_2-1}{2}\right)^{-1}$$

$$(a_0\Gamma_2((n_1+n_2)/2)+(a_1/n_1)\ \Gamma_2((n_1+n_2)/2,\ 1)\ C_1(I+H\Lambda H'F^{-1})^{-1})dHdF. \qquad \ldots \quad (8)$$

$$F(H, F)dFdH = |\Lambda|^{-(n_1/2)} |F|^{(n_1-3)/2} (f_1-f_2)$$

$$(\Gamma(n_1/2)\Gamma(n_2/2)\Gamma((n_1-1)/2)\Gamma(n_2-1)/2))^{-1}$$

$$|I+FH\Lambda^{-1}H'|^{-(n_1+n_2)/2} (a_0\Gamma_2((n_1+n_2)/2)$$

$$+(a_1/n_1)\Gamma_2((n+n)/2,\ 1)(1+|F|\ |\Lambda|^{-1}$$

$$|I+FH\Lambda^{-1}H|^{-1}-|I+FH\Lambda^{-1}H'|^{-1}))dFdH. \qquad \ldots \quad (9)$$

Now after some simplification and integrating H out over 0(2)—Orthogonal Group of order two we get from (9)

$$\begin{aligned} f(F)dF = (&G(n_1, n_2)(a_0+a_1(n_1+n_2)/2n_1) \\ &+a_1(n_1-1)/(2n_1(n_1+n_2-1))G(n_1+2, n_2) \\ &-a_1n_2(n_2-1)/(2n_1(n_1+n_2-1))G(n_1, n_2+2))dF. \end{aligned}$$

Where $G(n_1, n_2)$ stands for the p.d.f. of characteristic roots of $S_1S_2^{-1}$ where S_1 is distributed $W(n_2, 2, \Sigma_2)$ and S_2 as $W(n_2, 2, \Sigma_2)$ and $S_1(2\times 2)$, $S_2(2\times 2)$ are independent Wishart matrices.

Case 2 : Here we apply a different type of normalisation of the constants. For this end we proceed as follows. Assuming $p = 2$ we get

$$\int_{S_1>0}\int_{S_2>0} f(S_1, S_2)dS\,dS = (2^{(n_1+n_2)}\pi|\Sigma_1|^{(n_1/2)}$$

$$|\Sigma_2|^{(n_2/2)}\ \Gamma(n_1/2)\ \Gamma(n_2/2)\ \Gamma((n_2-1)/2)$$

$$\Gamma((n_2-1)/2))(P_0+2n_1P_1) = Z \text{ (say)}.$$

Hence to have the total probability content to be unity we take in place of P_0 and P_1, P_0/Z and P_1/Z respectively, with this substitution we get from (6)

$$f(H,F)dFdH = |\Lambda|^{-(n_1/2)}|F|^{-(n_1/2)}(f_1-f_2)$$
$$(\Gamma(n_1/2)\,\Gamma(n_2/2)\,\Gamma((n_1-1)/2)\,\Gamma((n_2-1)/2))^{-1}$$
$$|I+FH\Lambda^{-1}H|^{-(n_1+n_2)/2}(P_0(P_0+2n_1p_1)^{-1}$$
$$\Gamma_2((n_1+n_2)/2)+2P_1\Gamma_2((n_1+n_2)/2,1)$$
$$(p_0+2n_1P_1)^{-1}C_1(I+H\Lambda H'F^{-1})^{-1})dFdH. \quad \ldots \quad (10)$$

Simplifying $C_1(I+H\Lambda H'F^{-1})^{-1}$ for $p = 2$ in (10) we get on further reduction and on integrating H out over $0(2)$

$$f(F)dF = (G(n_1,n_2)(a+b(n_1+n_2)/2n_1)+b(n_1-1)/(2n_1(n_1+n_2-1))G(n_1+2,n)$$
$$-bn_2(n_2-1)/(2n_1(n_1+n_2-1))G(n_1,n_2+2))dF$$

where

$$a = P_0(P_0+2n_1P_1)^{-1} \quad \ldots \quad (11)$$
$$b = 2n_1P_1(P_0+2n_1P_1)^{-1}$$

and

$$a+b = 1.$$

From (17) we see that

$$P_0/P_1 = 2n_1(1-a)^{-1}a. \quad \ldots \quad (12)$$

4. Definitions of statistics

Let $X(p\times n_1)$ and $Y(p\times n_2)$, $p \leqslant n_i$, $i = 1, 2$ be matrix variates with columns of X independently distributed as $N(0,\Sigma_1)$ and those of Y independently distributed as $N(0, \Sigma_2)$.

Let $c \leqslant c_1 \ldots < c_p < \infty$ be the characteristic roots of the equation

$$|XX'-cYY'| = 0$$

and let $0 < \lambda_1 \leqslant \ldots \leqslant \lambda_p < \infty$ be the characteristic roots of

$$|\Sigma_1-\lambda\Sigma_2| = 0.$$

Let $C = \operatorname{diag}(c_i)$, $\Lambda = \operatorname{diag}(\lambda_i)$. To test the null hypothesis

$H_0 : \Lambda = I$ against $H_1 : \lambda_i \geqslant 1$, $(i = 1, \ldots, p)$,

$\sum_{i=1}^{p} \lambda_i > p$ the following criteria has been suggested :

(1) Roy's (1945) largest root c_p or equivalently $c_p/(1+c_p)$.

(2) Lawley and Hotelling's (1951) trace statistics $u^{(p)} = \sum_{i=1}^{p} c_i$.

(3) Pillai's (1955) criteria $V^{(p)} = \sum_{i=1}^{p} c_i/(1+c_i)$.

(4) Wilk's criterion (1932) : $W^{(p)} = \prod_{i=1}^{p} (1+c_i)^{-1}$.

5. Calculation of power of different tests

In Pillai and Jayachandran (1968) exact noncentral distribution of criteria $U^{(2)}$, $V^{(2)}$; $W^{(2)}$ are derived in connection with the above hypothesis under the normality assumption of two parent populations. A comparative study of those statistics are made from the power function point of view. The case of largest root have been studied by Pillai and Al Ani in an unpublished report.

Now as shown above under the non-normality set up of the parent population under study we get the non-central distributions of the roots as linear functions of distributions of roots studied by Pillai and Jayachandran (1968). We used the simplification technique used by them.

6. Remarks on the tabulated values

We presented a few sample tables for power of different tests under model (I). Further tables are available with the author. Using (12) we note that for fixed α, P_0 and P_1 lie on a line and as such we get corresponding values of P_0 and P_1. Thus the calculation under model (II) can be had from tabulated values.

The two models though look alike in form are basically different. In model (I) while normalizing we have not separately considered the contribution of the second part of the distribution i.e. the term by which the distribution differs from normality assumptions. In model (II) we have avoided the situation.

We now make the following observations :

(1) For the two non zero characteristic roots of the matrix $\Sigma_1 \Sigma_2^{-1}$ power of the largest root stays below those of other three tests for small to moderate deviations of the roots from null hypothetical value.

(2) Departures in normality assumption affects the percentage points least in the largest root case and as such it is least sensitive in detecting such deviations—followed by $U^{(2)}$, $W^{(2)}$ and $V^{(2)}$ in that order. Thus Pillai's $V^{(2)}$ is quite useful in such cases.

(3) In case of two non zero population roots of the covariance matrices and for small values of n_1 and n_2 largest root is least efficient in detecting small deviations from null hypothesis followed by $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$ in order of increasing sensitiveness. However, when in addition departure from normality is present the role of $U^{(2)}$ and $V^{(2)}$ are interchanged.

(4) For large deviation in two roots of the covariance matrices from null hypothetical value $L^{(2)}$ has a decided advantage in that it detects the departure most effectively. Of the other two statistics whose power are tabulated $V^{(2)}$ is least sensitive towards this type of departure.

(5) In a nutshell, when we are interested in studying the stability of the percentage points under departure from normality assumption $V^{(2)}$ is most useful. Also for two non null population roots small departures from normality is reflected by $V^{(2)}$ only less predominantly than $W^{(2)}$. However, for large deviations $L^{(2)}$ is the best statistic to apply.

(6) One interesting point indicated is the monotonicity of power functions for all the four test criteria under departure from normality assumption. This lends justifiability to the normality assumption in the null hypothetical case in such study. In case the assumption is volated it is quite likely to be detected by usual test.

Lastly we must comment on the limitations of this kind of exact study. Due to lack of convergence of the infinite series involved and sometimes due to excessive machine time needed some values of tables are not tabulated. This happened specially for calculations involving large values of n_1 and n_2 and in relation to power calculation of $V^{(2)}$.

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POWERS OF $U(2)$, $V(2)$, $W(2)$ AND $L(2)$ TESTS FOR TESTING $\lambda_1 = 1, \lambda_2 = 1$ AGAINST SIMPLE ALTERNATIVES AND VIOLATIONS OF NORMALITY ASSUMPTIONS $\alpha = .05$

POWERS FOR VIOLATIONS OF NORMALITY ASSUMPTIONS

$\lambda_1 = 1.0, \lambda_2 = 1.0$

$n_1 = 3$				$n_2 = 13$
a_1	$U(2)$	$W(2)$	$V(2)$	$L(2)$
0.00000	.0500	.0500	.0500	.0500
0.00010	.0500	.0500	.0500	.0500
0.00500	.0503	.0503	.0503	.0502
0.01000	.0505	.0505	.0506	.0505
0.10000	.0553	.0555	.0556	.0549
0.20000	.0606	.0609	.0612	.0598
0.30000	.0558	.0664	.0668	.0648
0.40000	.0711	.0719	.0723	.0697
0.50000	.0764	.0774	.0779	.0746
0.70000	.0870	.0883	.0891	.0845
0.90000	.0975	.0992	.1003	.0943

$n_1 = 5$				$n_2 = 13$
a_1	$U(2)$	$W(2)$	$V(2)$	$L(2)$
0.00000	.0500	.0500	.0500	.0500
0.00010	.0500	.0500	.0500	.0500
0.00500	.0502	.0502	.0502	.0502
0.01000	.0504	.0504	.0504	.0503
0.10000	.0536	.0538	.0539	.0532
0.20000	.0571	.0576	.0578	.0564
0.30000	.0607	.0632	.0637	.0596
0.40000	.0642	.0751	.0656	.0628
0.50000	.0678	.0689	.0690	.0660
0.70000	.0749	.0765	.0774	.0724
0.90000	.0820	.0840	.0852	.0789

POWERS FOR VIOLATIONS OF NORMALITY ASSUMPTIONS

$\lambda_1 = 1.0, \lambda_2 = 1.0$

$n_1 = 3$				$n_2 = 33$
a_1	$U(2)$	$W(2)$	$V(2)$	$L(2)$
0.00000	.0500	.0500	.0500	.0500
0.00010	.0500	.0500	.0500	.0500
0.00500	.0503	.0503	.0503	.0503
0.01000	.0507	.0507	.0507	.0506
0.10000	.0566	.0567	.0567	.0562
0.20000	.0633	.0634	.0634	.0624
0.30000	.0699	.0700	.0701	.0686
0.40000	.0765	.0767	.0768	.0749
0.50000	.0832	.0834	.0835	.0811
0.70000	.0964	.0967	.0969	.0935
0.90000	.1097	.1101	.1102	.1059

$n_1 = 5$				$n_2 = 33$
a_1	$U(2)$	$W(2)$	$V(2)$	$L(2)$
0.00000	.0500	.0500	.0500	.0500
0.00010	.0500	.0500	.0500	.0500
0.00500	.0502	.0502	.0502	.0502
0.01000	.0505	.0505	.0505	.0504
0.10000	.0547	.0548	.0548	.0543
0.20000	.0594	.0595	.0596	.0585
0.30000	.0641	.0643	.0644	.0628
0.40000	.0688	.0691	.0692	.0670
0.50000	.0735	.0738	.0740	.0713
0.70000	.0830	.0833	.0835	.0798
0.90000	.0927	.0929	.0931	.0884

POWERS FOR VIOLATIONS OF NORMALITY ASSUMPTIONS

$\lambda_1 = 1.0, \lambda_2 = 1.0$

$n_1 = 3$				$n_2 = 83$
a_1	$U(2)$	$W(2)$	$V(2)$	$L(2)$
0.00000	.0500	.0500	.0500	.0500
0.00010	.0500	.0500	.0500	.0500
0.00500	.0504	.0504	.0504	.0503
0.01000	.0507	.0507	.0507	.0507
0.10000	.0572	.0572	.0573	.0568
0.20000	.0645	.0645	.0645	.0637
0.30000	.0717	.0718	.0718	.0705
0.40000	.0790	.0790	.0790	.0773
0.50000	.0862	.0863	.0862	.0842
0.70000	.1007	.1008	.1008	.0978
0.90000	.1152	.1153	.1153	.1115

$n_1 = 5$				$n_2 = 83$
a_1	$U(2)$	$W(2)$	$V(2)$	$L(2)$
0.00000	.0500	.0500	.0500	.0500
0.00010	.0500	.0500	.0500	.0500
0.00500	.0503	.0503	.0503	.0502
0.01000	.0505	.0505	.0505	.0505
0.10000	.0553	.0553	.0553	.0548
0.20000	.0605	.0606	.0606	.0596
0.30000	.0658	.0658	.0658	.0644
0.40000	.0711	.0711	.0711	.0693
0.50000	.0763	.0764	.0764	.0741
0.70000	.0869	.0869	.0870	.0837
0.00000	.0974	.0975	.0975	.0934

POWERS FOR VIOLATIONS OF NORMALITY ASSUMPTIONS

$\lambda_1 = 1.0, \lambda_2 = 1.0$

$n_1 = 7$				$n_2 = 83$
a_1	$U(2)$	$W(2)$	$V(2)$	$L(2)$
0.00000	.0500	.0500	.0500	.0500
0.00010	.0500	.0500	.0500	.0500
0.00500	.0502	.0502	.0502	.0502
0.01000	.0504	.0504	.0504	.0504
0.10000	.0543	.0543	.0543	.0538
0.20000	.0585	.0586	.0586	.0577
0.30000	.0628	.0628	.0629	.0615
0.40000	.0671	.0671	.0671	.0653
0.50000	.0713	.0714	.0714	.0691
0.70000	.0799	.0800	.0800	.0768
0.90000	.0884	.0885	.0886	.0844

$n_1 = 13$				$n_2 = 83$
a_1	$U(2)$	$W(2)$	$V(2)$	$L(2)$
0.00000	.0500	.0500	.0500	.0500
0.00010	.0500	.0500	.0500	.0500
0.00500	.0501	.0501	.0501	.0501
0.01000	.0503	.0503	.0503	.0502
0.10000	.0529	.0529	.0529	.0525
0.20000	.0558	.0558	.0558	.0550
0.30000	.0586	.0587	.0587	.0575
0.40000	.0615	.0616	.0616	.0600
0.50000	.0644	.0645	.0645	.0625
0.70000	.0702	.0703	.0703	.0675
0.00000	.0759	.0761	.0761	.0724

VIOLATIONS OF GENERAL TYPE

$\lambda_1 = 1.0, \lambda_2 = 2.0$

$n_1 = 3$				$n_2 = 33$
a_1	$U(2)$	$W(2)$	$V(2)$	$L(2)$
0.00000	.2024	.2018		.2026
0.00010	.2024	.2018		.2025
0.00500	.2033	.2026		.2033
0.01000	.2041	.2034		.2041
0.10000	.2190	.2184		.2180
0.20000	.2356	.2350		.2335
0.30000	.2522	.2517		.2489
0.40000	.2688	.2683		.2644
0.50000	.2854	.2850		.2799
0.70000	.3185	.3182		.3108
0.90000	.3517	.3515		.3417

$n_1 = 5$				$n_2 = 33$
a_2	$U(2)$	$W(2)$	$V(2)$	$L(2)$
0.00000	.2547	.2511		.2566
0.00010	.2547	.2511		.2566
0.00500	.2554	.2518		.2572
0.01000	.2561	.2525		.2579
0.10000	.2686	.2650		.2691
0.20000	.2825	.2790		.2816
0.30000	.2963	.2929		.2941
0.40000	.3102	.3068		.3066
0.50000	.3241	.3208		.3191
0.70000	.3518	.3486		.3441
0.90000	.3795	.3765		.3691

VIOLATIONS OF GENERAL TYPE

$\lambda_1 = 1.25 \; \lambda_2 = 1.25$

$n_1 = 3$				$n = 33$
a_1	$U(2)$	$W(2)$	$V(2)$	$L(2)$
0.00000	.1105	.1109	.1110	.1059
0.00001	.1105	.1109	.1110	.1059
0.00500	.1111	.1115	.1116	.1064
0.01000	.1117	.1121	.1122	.1070
0.05000	.1163	.1168	.1169	.1112
0.10000	.1221	.1226	.1227	.1165
0.20000	.1337	.1343	.1344	.1271
0.30000	.1453	.1459	.1461	.1378
0.40000	.1569	.1576	.1578	.1484
0.50000	.1685	.1693	.1695	.1590
0.70000	.1917	.1926	.1929	.1803
0.90000	.2149	.2160	.2162	.2015

$n_1 = 5$				$n = 33$
a_1	$U(2)$	$W(2)$	$V(2)$	$L(2)$
0.00000	.1263	.1274	.1277	.1173
0.00001	.1263	.1274	.1277	.1173
0.00500	.1268	.1278	.1282	.1177
0.01000	.1272	.1283	.1286	.1181
0.05000	.1309	.1320	.1323	.1213
0.10000	.1355	.1366	.1370	.1254
0.20000	.1446	.1451	.1463	.1332
0.30000	.1538	.1552	.1556	.1412
0.40000	.1629	.1644	.1649	.1491
0.50000	.1721	.1737	.1742	.1571
0.70000	.1904	.1922	.1928	.1730
0.90000	.2087	.2108	.2114	.1889

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