Sankhyā : The Indian Journal of Statistics
1983, Volume 45, Series A, Pt. 2 pp. 150–160.

THE BERNSTEIN-VON MISES THEOREM FOR A CERTAIN CLASS OF DIFFUSION PROCESSES

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SUMMARY. The Bernstein-von Mises theorem, showing that the posterior density, when properly normalized, converges to a normal density, is proved for a certain class of diffusion processes arising as solutions to non-linear stochastic differential equations. As an application the m.l.e. and Bayes estimators for smooth loss functions and smooth priors, turn out to be asymptotically normal. The parameter space is assumed to be a compact subset of $\mathcal{R}^k$.

1. Introduction

In this paper, we prove that under certain regularity conditions, the Bernstein-von Mises theorem holds for a class of diffusion processes arising as solutions to stochastic differential equations. For the linear case, this result was proved by Prakasa Rao (1980). For other results in this direction see Borwanker, Kallianpur and Prakasa Rao (1971) and Hipp and Michel (1976).

As a consequence of the main theorem, we obtain that the m.l.e., and Bayes estimators for smooth loss functions and smooth priors are asymptotically normal. These problems have also been studied by others, using different approaches. See Basawa and Prakasa Rao (1980) for a list of references. Recently Prakasa Rao and Rubin (1981) have proved the asymptotic normality of m.l.e. (in the one dimensional case) by using Fourier analytic methods. For higher dimensional parameter space, this was proved by Basu (1983), using Kolmogorov type inequalities from the theory of diffusion processes.

The intermediate results which led to the main theorem are as in Borwanker *et al.* (1971). The technique used is similar in spirit to what is used in Basu (1983).

Let $(X(t) : t \geq 0)$ be a real-valued, stationary ergodic process satisfying the stochastic differential equation,

$$dX(t) = f(\theta_0, X(t))dt + d\xi(t), \; X(0) = X_0 \quad E(X_0^2) < \infty, \; t \geq 0$$

and $(\xi(t) : t \geq 0)$ is a standard Wiener process.

The conditions under which such a solution exists can be found in Khasminskii (1960). See also Gikhman and Skorokhod (1972).

$f(\theta, x)$ is a real-valued function on $\Omega \times \mathcal{R}$, where $\Omega = \{\theta \in \mathcal{R}^d : \|\theta\| \leq 1\}$, $d \geq 1$ (finite) and $\theta_0 \in \Omega^0$ (the interior of Ω) is the unknown "true-value".

The following conditions will be assumed on f. Let $L(\theta)$ be a function of θ such that $\sup \{L(\theta) : \theta \in \Omega\} < \infty$

A1 (i) f is continuous on $\Omega \times \mathcal{R}$.

(ii) $|f(\theta, x)| \leq L(\theta)(1+|x|) \forall \theta \in \Omega, \forall x \in \mathcal{R}$.

(iii) $|f(\theta, x)-f(\theta, y)| \leq L(\theta)|x-y| \forall \theta \in \Omega, \forall x, y \in \mathcal{R}$.

(iv) $|f(\theta, x)-f(\phi, x)| \leq J(x)|\theta-\phi| \forall \theta, \phi \in \Omega, \forall x \in \mathcal{R}$.

where $J(\cdot)$ is continuous and $E(|J(X(0))|^{d+\alpha_0}) < \infty$ for some $d+\alpha_0 \geq 2$.

(v) $I(\theta) = E[(f(\theta, X(0))-f(\theta_0, X(0))^2] > 0 \forall \theta \neq \theta_0$.

A2 (i) The partial derivatives $f_\theta^{(i)}$ of f w.r.t. θ_i (where $\theta' = (\theta_1, \ldots, \theta_d)$) exist $\forall i = 1, 2, \ldots, d$.

Denote by $f_\theta^{(i)}(\theta^*, x)$ the derivative evaluated at (θ^*, x).

(ii) $|f_\theta^{(i)}(\theta, x)-f_\theta^{(i)}(\phi, x)| < J(x)|\theta-\phi| \forall \theta, \phi \in \Omega, \forall x \in \mathcal{R}$ $\forall i = 1, 2, \ldots, d$ and J is as in A1(iv).

(iii) $|f_\theta^{(i)}(\theta, x)| \leq L(\theta)(1+|x|) \forall \theta \in \Omega, \quad x \in \mathcal{R}, i = 1, 2, \ldots, d$.

A3 The partial derivative $\dfrac{\partial^2}{\partial\theta_i \partial\theta_j} f(\theta, x) \forall i, j = 1, \ldots, d$ exists and are continuous. Moreover, they satisfy A2(ii) and (iii).

Under the given condition A1(i), $(X(t) : t \geq 0)$ is a.s. continuous. Let μ_θ^T denote the measure induced by $(X(t) : 0 \leq t \leq T)$ on $\mathcal{C}[0, T]$, when θ is the true value. (Here $\mathcal{C}[0, T]$ is the space of all real-valued continuous functions on $[0, T]$, and is endowed with the supnorm topology).

It is well known that under given conditions on f,

$$L_T(\theta) = \frac{d\mu_\theta^T}{d\mu_{\theta_0}^T}(X(t) : 0 \leq t \leq T)$$

$$= \exp\Big(\int_0^T \Big[f(\theta, X(s)-f(\theta_0, X(s)\Big]d\xi(s)$$

$$-\frac{1}{2}\int_0^T [f(\theta, X(s))-f(\theta_0, X(s))]^2 ds \Big)$$

is the Radon-Nikodym derivative of μ_θ^T w.r.t. $\mu_{\theta_0}^T$. (See Gikhman and Skorokhod, 1972 for details.)

Assume that there exists an estimator θ_T which minimizes $L_T(\theta)$ over $\theta \in \Omega$. Then θ_T is the m.l.e. when the process $X(t)$ is observed over $[0, T]$. It has been shown in Basu (1983) that under A1, θ_T is strongly consistent.

Suppose now that Λ is a prior probability on $(\Omega, \mathcal{B})$, where $\mathcal{B}$ is the σ-algebra of Borel subsets of Ω. Assume that Λ has a density $\lambda(\cdot)$ w.r.t. the Lebesgue measure and the density is continuous and positive in an open neighbourhood of θ_0.

The posterior density of θ given $(X(t) : 0 \leqslant t \leqslant T)$ is

$p(\theta/X(t) : 0 \leqslant t \leqslant T)$

$$= \frac{d\mu_\theta^T}{d\mu_{\theta_0}^T}(X(t) : 0 \leqslant t \leqslant T)\lambda(\theta) \Big/ \int_\Omega \frac{d\mu_\theta^T}{d\mu_{\theta_0}^T}(X(t) : 0 \leqslant t \leqslant T)\lambda(\theta)d\theta.$$

Let $t = \sqrt{T}(\theta - \theta_T)$. Then the posterior density of $\sqrt{T}(\theta - \theta_T)$ is

$$p^*(t/X(t) : 0 \leqslant t \leqslant T) = \frac{1}{\sqrt{T}}\, p\Big(\theta_T + \frac{t}{\sqrt{T}} /X(t) : 0 \leqslant t \leqslant T\Big).$$

Before we prove any result, we would like to make the following important remark.

Remark 1 : Under the given conditions A1–A3 on f, all the stochastic integrals occurring henceforth can be defined path-wise. Further it is possible to differentiate (w.r.t. θ_i's) within the stochastic integral (indeed path-wise outside a fixed null set of our basic probability space.) See Karandikar (1983) for details.

For the rest of the paper, we shall assume that $d = 1$. With proper modifications, every argument in this special case goes through for higher dimensions. We will continue writing d in general.

We shall assume that solution of $\frac{\partial}{\partial \theta} \log L_T(\theta) = 0$ gives m.l.e. θ_T. Thus $\frac{\partial}{\partial \theta} \log L_T(\theta)\big|_{\theta=\theta_T} = 0$, and hence by Remark 1, outside a fixed null set (which henceforth we drop out of consideration), we have the following crucial equality.

$$\int_0^T f'(\theta_T, X(s))d\xi(s) = \int_0^T f'(\theta_T, X(s))[f(\theta_T, X(s)) - f(\theta_0, X(s))]ds. \quad \ldots \quad (1)$$

Let

$$\gamma_T(t) = \frac{d\mu^T_{\theta_T + \frac{t}{\sqrt{T}}} \Big/ d\mu^T_{\theta_0}(X(t) : 0 \leqslant t \leqslant T)}{d\mu^T_{\theta_T}/d\mu^T_{\theta_0}\,(X(t) : 0 \leqslant t \leqslant T)},$$

$$C_T = \int_{-\infty}^{\infty} \gamma_T(t)\lambda\Big(\theta_T + \frac{t}{\sqrt{T}}\Big)dt,$$

$$I_T(\theta) = \int_0^T [f(\theta, X(s)) - f(\theta_0, X(s))]^2 ds,$$

$$\beta = E[(f'(\theta_0, X(0)))^2]. \quad \text{Assume } \beta > 0.$$

Clearly

$$p^*(t/X(t) : 0 \leqslant t \leqslant T) = C_T^{-1}\gamma_T(t)\lambda\Big(\theta_T + \frac{t}{\sqrt{T}}\Big).$$

Lemma 1 : *Under* A1, *for every* $\delta > 0$

$$\lim_{T\to\infty} \inf_{|\theta-\theta_0| \geqslant \delta} \frac{I_T(\theta)}{T} = \lambda(\delta)\ (> 0)\ a.s.$$

Proof : This is proved in Prakasa Rao and Rubin (1981).

Lemma 2 : *Under* A3,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f''(\theta_0, X(s))d\xi(s) = 0\ a.s.$$

(f'' is second derivative w.r.t. θ).

Proof : $g(t) = \int_0^t f''(\theta_0, X(s))d\xi(s)$ is a continuous martingale. Hence by the martingale inequality, and stationary of $(X(t) : t \geqslant 0)$

$$P\Big\{\sup_{0\leqslant t\leqslant T} |g(t)| > \lambda\Big\} \leqslant \frac{T}{\lambda^2} E[(f''(\theta_0, X(0)))^2]$$

$$= \frac{T}{\lambda^2}\beta_1, \text{ say.}$$

Let

$$A_n = \Big\{\sup_{2^{n-1} \leqslant t \leqslant 2^n} \Big|\frac{1}{t}g(t)\Big| \geqslant 2^{-n/4}\Big\}.$$

Then

$$P(A_n) \leqslant P\left\{\sup_{2^{n-1} \leqslant t \leqslant 2^n} |g(t)| > 2^{n-1}.2^{-n/4}\right\}$$

$$\leqslant P\left\{\sup_{0 \leqslant t \leqslant 2^n} |g(t)| > 2^{n-1}.2^{-n/4}\right\}$$

$$\leqslant \frac{2^n}{(2^{n-1}.2^{-n/4})^2\, \beta_1}$$

Thus $\Sigma P(A_n) < \infty$ and the lemma follows from Borel-Cantelli lemma.

Let

$$V(\theta, x) = f''(\theta, x) - f''(\theta_0, x)$$

$$X(t, \theta) = \int_0^t V(\theta, X(s))d\xi(s).$$

Note that under A3, $(X(t, \theta) : 0 \leqslant t \leqslant T)$ as a function of θ, from Ω into $\mathcal{C}[0, T]$ is a.s. continuous for all $T > 0$ (See Basu, 1983 or Karandikar, 1983).

Lemma 3 :

(i) $$P\left\{\sup_{\theta} \sup_{0 \leqslant t \leqslant T} |X(t, \theta)| > C_1 \lambda^{1/d+\alpha_0}\right\} \leqslant C_2 \frac{T^{(d+\alpha_0)/2}}{\lambda}$$

where C_1, C_2 are positive finite constants independent of T.

(ii) *For very $\gamma > 1/(d+\alpha_0)$ there exists an H such that*

$$\lim_{T \to \infty} \sup_{\theta} \frac{|X(T, \theta)|}{T^{1/2}(\log T)^{\gamma}} \leqslant H \text{ a.s.}$$

(iii) $$\lim_{T \to \infty} \sup_{\theta} \frac{|X(T, \theta)|}{T} = 0 \text{ a.s.}$$

(iv) $$\lim_{T \to \infty} \sup_{\theta} \frac{1}{T} \int_0^T f''(\theta, X(s))d\xi(s) = 0 \text{ a.s.}$$

Proof : (i) and (ii) are proved in Basu (1983); (iii) is immediate from (ii); (iv) follows from (iii) and Lemma 2,

Lemma 4 : *Under* A1, A2 *and* A3

(a) *for each fixed t,*

$$\lim_{T \to \infty} \log \gamma_T(t) = -\frac{1}{2} \beta t^2 \text{ a.s.}$$

(b) *For every* ε, $0 < \varepsilon < \beta$, *there exists* δ_0 *and* T_0 *such that*

$$\gamma_T(t) \leq exp\left(-\frac{1}{2}\, t^2(\beta-\varepsilon)\right)$$

for $|t| \leq \delta_0 T^{1/2}$ *and* $T \geq T_0$ *a.s.*

(c) *For every* $\delta > 0$, *there exists a positive* ε *and* T_0 *such that*

$$\sup_{|t| > \delta T^{1/2}} \gamma_T(t) \leq exp\left(-\frac{1}{4}\, T\varepsilon\right) \text{ for } T \geq T_0 \text{ a.s.}$$

Proof :

$$\log\, \gamma_T(t) = \int_0^T \left[f(\theta_T+\frac{t}{\sqrt{T}}, X(s)-f(\theta_T, X(s))\right]d\xi(s)$$

$$-\frac{1}{2}\int_0^T \left[f\left(\theta_T+\frac{t}{\sqrt{T}}, X(s)\right)-f(\theta_0, X(s))\right]^2 ds$$

$$+\frac{1}{2}\int_0^T [f(\theta_T, X(s))-f(\theta_0, X(s))]^2 ds.$$

Applying mean-value theorem and then the likelihood equation (1), it easily follows that

$$\log\, \gamma_T(t) = I_1+I_2+I_3+I_4$$

where

$$I_1 = \frac{-t^2}{2T}\int_0^T f'^2(\theta_0, X(s))ds$$

$$I_2 = \frac{t^2}{2T}\int_0^T [f'^2(\theta_0, X(s))-f'^2(\theta_T^{**}, X(s))]ds$$

$$I_3 = \frac{t^2}{2T}\int_0^T f''(\theta_T^*, X(s))d\xi(s)$$

$$I_4 = -\int_0^T [f(\theta_T, X(s))-f(\theta_0, X(s))]$$

$$\left[f\left(\theta_T+\frac{t}{\sqrt{T}}, X(s)\right)-f(\theta_T, X(s))-\frac{t}{\sqrt{T}}f'(\theta_T, X(s))\right]ds$$

where $\max(|\theta_T^*-\theta_T|, |\theta_T^{**}-\theta_T|) \leq \dfrac{|t|}{\sqrt{T}}$.

(a) By Ergodic theorem

$$I_1 \to \frac{-t^2}{2}\beta \text{ a.s. as } T \to \infty.$$

Also by Ergodic theorem $\exists$ a constant M and T_0 such that

$$\frac{1}{T}\int_0^T J(X(s))(1+|X(s)|)ds \leqslant M \text{ a.s. for } T \geqslant T_0. \quad \ldots \quad (2)$$

Under assumption A_2, (ii) and (iii), consistency of θ_T and (2) it follows that $I_3 \to 0$ a.s. as $T \to \infty$.

By mean value theorem

$$|I_4| \leqslant C_0|\theta_T-\theta_0| \frac{t^2}{T}\int_0^T J(X(s))(1+|X(s)|)ds$$

and hence $I_4 \to 0$ a.s. as $T \to \infty$.

By Lemma 3, $I_3 \to 0$ a.s. as $T \to \infty$. Hence (a) is proved.

(b) Fix $\varepsilon_1 > 0$. Clearly there exists a T_1 such that

$$\forall\, T \geqslant T_1 - \frac{1}{2}\frac{t^2}{T}\int_0^T f'^2(\theta_0, X(s))ds \leqslant -\frac{1}{2}t^2(\beta-\varepsilon_1) \text{ a.s.} \quad \ldots \quad (A)$$

By Lemma 3 there exists a T_2 such that $\forall\, T \geqslant T_2$

$$\sup_\theta \frac{1}{T}\int_0^T f''(\theta, X(s))d\xi(s) \leqslant \varepsilon_1/2 \quad \text{a.s.} \quad \ldots \quad (B)$$

$$|I_2| \leqslant \frac{t^2}{2T}|\theta_T^{**}-\theta_0| \int_0^T J(X(s)(1+|X(s)|)ds$$

$$\leqslant \frac{t^2}{2T}\left(\frac{|t|}{\sqrt{T}}+|\theta_T-\theta_0|\right)\int_0^T J(X(s))(1+|X(s)|)ds$$

$$\leqslant \frac{t^2}{2T}(\delta_0+|\theta_T-\theta_0|)\int_0^T J(X(s))(1+|X(s)|)ds \quad \text{if } \frac{|t|}{\sqrt{T}} \leqslant \delta_0.$$

Using (2) and choosing δ_0 suitably and using consistency of θ_T, it follows that there exists a δ_0 and T_3 such that

$$\frac{|t|}{\sqrt{T}} \leqslant \delta_0 \text{ and } T \geqslant T_3 \text{ implies } I_2 \leqslant \frac{t^2}{2T}\varepsilon_1 \text{ a.s.} \quad \ldots \quad (C)$$

Similarly using mean-value theorem and arguing as above, there exists a T_4 and δ_1 such that

$$\frac{|t|}{\sqrt{T}} < \delta_1 \text{ and } T > T_4 \text{ implies } I_4 < \frac{t^2}{2T} \varepsilon_1 \text{ a.s.} \qquad \dots \text{ (D)}$$

Combining the estimates (A)–(D), (b) follows.

$$\text{(c) } \log \frac{\gamma_T(t)}{T} = \frac{1}{T} \int_0^T \left[f\left(\theta_T + \frac{t}{\sqrt{T}}, X(s)\right) - f(\theta_T, X(s)) \right] d\xi(s)$$

$$-\frac{1}{2} \frac{1}{T} \int_0^T \left[f\left(\theta_T + \frac{t}{\sqrt{T}}, X(s)\right) - f(\theta_0, X(s)) \right]^2 ds$$

$$+\frac{1}{2} \frac{1}{T} \int_0^T [f(\theta_T, X(s)) - f(\theta_0, X(s))]^2 ds$$

$$= A_1(t, T) + A_2(t, T) + A_3(T), \text{ say.}$$

Note that A_3 does not involve t and by arguments given earlier $A_3(T) \to 0$ a.s. as $T \to \infty$

$$\sup_t |A_1(t, T)| \leq 2 \sup_\theta \frac{1}{T} \left| \int_0^T f(\theta, X(s)) d\xi(s) \right| \to 0 \text{ a.s.}$$

Finally, by strong consistency, there exists a T_0 such that for all $T > T_0$, $|\theta - \theta_T| < \delta$ a.s.

Hence if $\frac{|t|}{\sqrt{T}} > \delta$ and $T > T_0$, we have

$$\left| \theta_T + \frac{t}{\sqrt{T}} - \theta_0 \right| > \delta/2.$$

Thus,

$$A_2 < -\frac{1}{2} \inf_{|\theta - \theta_0| > \delta/2} \frac{I_T(\theta)}{T} \to -\frac{1}{2} \lambda(\delta/2) \text{ a.s.}$$

Putting these estimates of A_1, A_2 and A_3, (c) is proved.

Let K be a non-negative measurable function such that

($K1$) There exists a number ε, $0 < \varepsilon < \beta$, for which

$$\int_{-\infty}^{\infty} K(t) \exp\left(-(\beta - \varepsilon) \frac{t^2}{2}\right) dt < \infty.$$

($K2$) For every $h > 0$ and every $\delta > 0$

$$e^{-T\delta} \int_{|t| > h} K(T^{1/2} h) \lambda(\theta_T + t) dt \to 0 \text{ a.s. as } T \to \infty.$$

Lemma 5 : *Under* A1, A2 *and* A3

(a) *There exists a* $\delta_0 > 0$ *such that*

$$\lim_{T\to\infty} \int_{|t| < \delta_0 T^{1/2}} K(t) \left| \gamma_T(t)\lambda\left(\theta_T + \frac{t}{\sqrt{T}}\right) - \lambda(\theta_0) exp\left(-\frac{1}{2}\beta t^2\right)\right| dt = 0 \ a.s.$$

(b) *For every* $\delta > 0$

$$\lim_{T\to\infty} \int_{|t| > \delta T^{1/2}} K(t) \left| \gamma_T(t)\lambda\left(\theta_T + \frac{t}{\sqrt{T}}\right) - \lambda(\theta_0) exp\left(-\frac{1}{2}\beta t^2\right)\right| dt = 0 \ a.s.$$

Theorem 1 : *Under* A1, A2 *and* A3

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} K(t) \left| p^*(t \ X(t) : 0 \leq t \leq T) - (\beta/2\pi)^{\frac{1}{2}} exp\left(-\frac{1}{2}\beta t^2\right)\right| dt = 0 \ a.s.$$

We omit the proofs of Lemma 5 and Theorem 1, since the arguments are already available in Borwanker *et al.* (1971) and Prakasa Rao (1981).

Corollary : *If further* $\int_{-\infty}^{\infty} |\theta|^m \lambda(\theta) d\theta < \infty$ *for some m, then*

$$\lim_{T\to\infty} \int_{-\infty}^{\infty} |t|^m \left| p^*(t \ X(t) : 0 \leq t \leq T) - (\beta/2\pi)^{\frac{1}{2}} exp\left(-\frac{1}{2}\beta t^2\right)\right| dt = 0 \ a.s.$$

Remark : The case $m = 0$ gives the classical Bernstein-von Mises theorem.

2. Bayes estimation

Suppose $l(\theta, \phi)$ is a loss function defined on $\Omega \times \Omega$. Assume that $l(\theta, \phi) = l(|\theta - \phi|) \geq 0$ and $l(t)$ is nondecreasing. Suppose R is a non-negative function and K and G are functions such that

(B1) $R(T) l\left(\frac{t}{\sqrt{T}}\right) \leq G(t)$ for all $T \geq 0$,

(B2) $R(T) l\left(\frac{t}{\sqrt{T}}\right) \to K(t)$ uniformly on bounded intervals of t as $T \to \infty$,

(B3) $\int_{-\infty}^{\infty} K(t+m) e^{-\frac{1}{2}\beta t^2} dt$ has a strict minimum at $m = 0$,

(B4) G satisfies $K(1)$ and $K(2)$.

A regular Bayes estimate $\tilde{\theta}_T$ based on $(X(t) : 0 \leq t \leq T)$ is that which minimizes

$$B_T(\psi) = \int_{\Omega} l(\theta, \psi) p(\theta | X(t) : 0 \leq t \leq T) d\theta.$$

Assume that such an estimator exists.

Theorem 2 : *Under* A1–A3 *and* B1–B4, *we have*

(i) $\sqrt{T}(\theta_T-\hat{\theta}_T) \to 0$ *a.s. as* $T \to \infty$,

(ii) $$\lim_{T\to\infty} R(T)B_T(\theta_T) = \lim_{T\to\infty} R(T)B_T(\hat{\theta}_T)$$

$$= (\beta/2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} K(t)e^{-\frac{1}{2}\beta t^2}\, dt.$$

Proof : The proof can be found in Borwanker *et al.* (1971).

The following is not difficult to check.

Remark : Under A1–A3 and B1–B4

(i) $\hat{\theta}_T \to \theta_0$ a.s. as $T \to \infty$

(ii) $\sqrt{T}(\hat{\theta}_T-\theta_0) \xrightarrow{\mathcal{L}} N(0, \beta^{-1})$ as $T \to \infty$.

Hence Bayes estimators are asymptotically normal and asymptotically efficient.

Remark (1) : We could have made the weaker assumption that f with its relevant derivatives are for each fixed x, Lipschitz in θ of order α, $0 < \alpha < 1$, provided we made an appropriate stronger moment condition on J.

(2) Similarly in A_2(ii) and A_3(iii) the dominating function could be taken arbitrarily instead of the specific $1+|x|$, say $g(x)$, provided we assume $Eg^2(X(0)) < \infty$.

Acknowledgement. The author is grateful to Professor J. K. Ghosh for his help.

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Paper received : March, 1982.