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ASYMPTOTIC THEORY OF ESTIMATION IN NON-LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS FOR THE MULTIPARAMETER CASE

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SUMMARY. Strong consistency and asymptotic normality of the m.l.e. for multi-dimensional parameters in non-linear stochastic differential equations is proved, using Kolmogorov type inequalities from the theory of diffusion processes.

1. Introduction

Statistical analysis of diffusion processes has received considerable attention recently. Dorogovchev (1976) has studied weak consistency of least square estimates for parameters of diffusion processes which are solutions to non-linear stochastic differential equations. Asymptotic normality and asymptotic efficiency of these estimators is investigated in Prakasa Rao (1979). See also Ibragimov and Khasminskii (1975), Kutoyants (1977), Lanska (1979) and Prakasa Rao (1980, 1981) for further work. A survey of the recent literature is given in Basawa and Prakasa Rao (1980).

In a recent paper, Prakasa Rao and Rubin (1981) studied limiting properties of a process related to a l.s.e. and discussed the asymptotic properties (strong consistency and asymptotic normality) of the m.l.e. derived from the limiting process. Their study is based on Fourier analytic methods.

In the present paper we generalize the results of Prakasa Rao and Rubin (1981) to multi-dimensional parameters, but using a different approach. Instead of the Fourier analytic methods used in Prakasa Rao and Rubin (1981) for deriving bounds for certain probabilities, we use Kolmogorov type inequalities from the theory of diffusion processes.

2. The model

Let $\{X(t) : t \geq 0\}$ be a real-valued, stationary, ergodic process satisfying the stochastic differential equation

$$dX(t) = f(\theta_0, X(t))dt + d\xi(t),\ X(0) = X_0,\ t \geq 0$$

where $\{\xi(t) : t \geq 0\}$ is a standard Wiener process and $E(X_0^2) < \infty$. (See Friedman, 1975; and Stroock and Varadhan, 1979 in this connection).

$f(\theta, x)$ is a real-valued function on $\Omega \times \mathcal{R}$, where $\Omega = \{\theta \in \mathcal{R}^d : \|\theta\| \leqslant 1\}$, $d \geqslant 1$ (finite) and $\theta_0 \in \Omega^0$ (the interior of Ω) is the unknown "true-value".

We shall assume the following conditions on f. Not all of them will be used always.

(A1) $f(\theta, x)$ is continuous on $\Omega \times \mathcal{R}$

(A2) (i) $|f(\theta, x)| \leqslant L(\theta)(1+|x|) \forall \theta \in \Omega, \forall x \in \mathcal{R}$
where $\sup \{L(\theta) : \theta \in \Omega\} < \infty$

(ii) $|f(\theta, x)-f(\theta, y)| \leqslant L(\theta)|x-y| \ \forall \theta \in \Omega, \ \forall x, y \in \mathcal{R}$

(iii) $|f(\theta, x)-f(\phi, x)| \leqslant J(x)|\theta-\phi| \forall \theta, \phi \in \Omega, \forall x \in \mathcal{R}$
where (a) $J(\cdot)$ is continuous and

(b) $E(|J(X(0))|^{d+\alpha_0}) 0 < \infty$ for some $d+\alpha_0 \geqslant 2$.

(A3) $I(\theta) = E(f(\theta, X(0))-f(\theta_0, X(0)))^2 > 0 \ \forall \theta \neq \theta_0$

(A4) The partial derivatives $f_\theta^{(i)}$ of f w.r.t. θ_i (where $\theta' = (\theta_1, ..., \theta_d)$) exists $\forall i = 1, 2, ..., d$.

Denote by $f_\theta^{(i)}(\theta^*, x)$ the derivative evaluated at θ^*.

(A5) $|f_\theta^{(i)}(\theta, x)-f_\theta^{(i)}(\phi, x)| \leqslant c(x)|\theta-\phi|^\alpha \forall \theta, \phi \in \Omega, \forall x \in \mathcal{R}$
and $E(|c(X(0))|^{d+\alpha_1}) < \infty$ for some $d+\alpha_1 \geqslant 2$, $i = 1, ..., d$

(A6) $E(f_\theta^{(i)}(\theta_0, X(0)))^2 < \infty \quad \forall i = 1, 2, ..., d$.

(A7) $\sum_{i=1}^{d} |f_\theta^{(i)}(\theta, x)| \leqslant M(\theta)(1+|x|) \forall \theta$ in a neighbourhood V_{θ_0} of θ_0
and $\{\sup M(\theta) : \theta \in V_{\theta_0}\} < \infty$.

3. A L.S.E. AND A PROCESS RELATED TO IT

Assume that the process X is observed over $[0, T]$ at the time points $t_k, k = 0, 1, ..., n$ with $0 = t_0 < t_1 < ... < t_n = T$. Form the "Sum of Squares" $Q_n^T(\theta)$ as

$$Q_n^T(\theta) = \sum_{k=0}^{n-1} \frac{[X(t_{k+1})-X(t_k)-f(\theta, X(t_k))\Delta t_k]^2}{\Delta t_k}$$

where $\Delta t_k = t_{k+1}-t_k$, $k = 0, 1, ..., (n-1)$.

An estimator $\hat{\theta}_{n,T}$ which minimizes $Q_n^T(\theta)$ is a least squares estimator. Even if such estimator exists, it is not consistent in general.

Result 3.1 : Fix $T > 0$. Suppose $\Delta_n \to 0$ as $n \to \infty$

$$(\Delta_n = \max_{0 \leqslant k \leqslant n-1} \Delta t_k).$$

Then under (A2), as $n \to \infty$,

$$Q_n^T(\theta) - Q_n^T(\theta_0) \to R_T(\theta) \text{ in probability}$$

where

$$R_T(\theta) = \int_0^T v^2(\theta, X(t))dt - 2\int_0^T v(\theta, X(t))d\xi(t)$$

and

$$v(\theta, x) = f(\theta, x) - f(\theta_0, x).$$

Proof: This is proved in Prakasa Rao and Rubin (1981).

Let

$$Z_T^*(\theta) = \int_0^T v(\theta, X(t))d\xi(t)$$

$$Z_T(\theta) = \frac{1}{\sqrt{T}} Z_T^*(\theta)$$

$$I_T(\theta) = \int_0^T v^2(\theta, X(t))dt.$$

Remark: Under (A2), w.l.g. we can assume to have chosen a version of $\{X(t) : t \geqslant 0\}$ so that $\theta \to Z_t(\theta) : 0 \leqslant t \leqslant T$ is a function from Ω into $\mathcal{C}[0, T]$ and is a.s. a continuous function of θ. This is because of Lemma 6.1 and the fact that $Z_t(\theta_0) \equiv 0$.

We now prove the weak convergence of $\{Z_T(\theta) : \theta \in \Omega\}$.

Theorem 3.1 : *Under* (A2), $\{Z_T(\theta) : \theta \in \Omega\}$ *as a process in* θ, *converges weakly (as* $T \to \infty$*) to a mean zero Gaussian process with covariance function* $R(\theta_1, \theta_2) = E(v(\theta_1, X(0))v(\theta_2, X(0)))$.

Proof: The process $\{X(t) : t \geqslant 0\}$ is stationary and ergodic. Hence by the C.L.T. for stochastic integrals (see Basawa and Prakasa Rao, 1980) there is no problem of convergence of finite dimensional distributions. Thus it suffices to prove tightness.

By Lemma 6.1

$$E\,|\,Z_T(\theta) - Z_T(\phi)\,|^{d+\epsilon_0} \leqslant c_{d+\epsilon_0}\,|\theta-\phi|^{d+\epsilon_0}\,E\,|\,J(X(0))\,|^{d+\epsilon_0}$$

which immediately gives tightness.

4. Strong consistency of the m.l.e

Denote by $\hat{\theta}_T$, the m.l.e. of θ when the process X is observed over $[0, T]$. (See Gikhman and Skorohod (1972) for the existence etc. of the relevant Radon-Nikodym derivatives). It can be shown that $\hat{\theta}_T$ is the same as $\tilde{\theta}_T$ which minimizes $R_T(\theta)$ over $\theta \in \Omega$ (see Gikhman and Skorohod, 1981 and Prakasa Rao and Rubin, 1981 for details).

We prove the consistency of the m.l.e. $\hat{\theta}_T$ through a series of Lemmas.

Lemma 4.1 : *Under* (A2),

$$P\left\{\sup_{\theta} \sup_{0 \leq t \leq T} |Z_t^*(\theta)| > c_1 \lambda^{1/d+\alpha_0}\right\} < c_2 \frac{T^{\frac{d+\alpha_0}{2}}}{\lambda}$$

Proof: Now

$$|v(\theta, x) - v(\phi, x)| \leq J(x)|\theta - \phi|.$$

Apply Lemma 6.1 (with $g(\theta, x) = v(\theta, x)$), Lemma 6.3 (with $r = d+\alpha_0$) and Cor. 6.2 in succession to get the result.

Lemma 4.2 : *Under* (A2), *for any* $\gamma > 1/d+\alpha_0$, *there exists* $H > 0$ *such that*

$$\limsup_{T \to \infty} \sup_{\theta} \frac{|Z_T^*(\theta)|}{T^{1/2}(\log T)^{\gamma}} < H \text{ a.s.}$$

Proof: The proof is similar to Lemma 4.2 of Prakasa Rao and Rubin (1981). Define

$$A_n = \left\{\sup_{2^{n-1} < t < 2^n} \sup_{\theta} |Z_t^*(\theta)| > H' 2^{n/2} . n^{\gamma}\right\}, n \geq 1.$$

Then by using stationarity of $\{X(t) : t \geq 0\}$,

$$P(A_n) = P\left\{\sup_{0 < t < 2^{n-1}} \sup_{\theta} |Z_t^*(\theta)| > H' 2^{n/2} n^{\gamma}\right\}$$

$$< \frac{c_2'(2^{n-1})^{\frac{d+\alpha_0}{2}}}{(H' 2^{n/2} n^{\gamma})^{d+\alpha_0}} \quad \text{by Lemma 4.1.}$$

By the choice of γ, $\sum_{n=1}^{\infty} P(A_n) < \infty$. The lemma now follows by applying Borel-Cantelli lemma.

Lemma 4.3 : *Under* (A2) *and* (A3),

$$\inf_{|\theta-\theta_0|>\delta} \frac{I_T(\theta)}{T} \xrightarrow{a.s.} \lambda \text{ as } T \to \infty$$

for some $\lambda > 0$, *depending on* δ.

Proof: Same as in Prakasa Rao and Rubin (1981).

Theorem 4.1 : *Under* (A2) *and* (A3),

$$\hat{\theta}_T \to \theta_0 \text{ a.s. as } T \to \infty.$$

Proof: Combine Lemma 4.2 and 4.3 as in Prakasa Rao and Rubin (1981).

5. Asymptotic normality of the M.L.E.

In this section we assume all the conditions (A1)–(A7). Since $\hat{\theta}_T$ is consistent by Theorem 4.1, $\hat{\theta}_T \in V_{\theta_0}$ with probability one for large T.

Around a neighbourhood of θ_0, we have

$$f(\theta, x) = f(\theta_0, x) + (\theta-\theta_0)'\Delta f_\theta(\theta^*, x)$$

where $|\theta^*-\theta_0| \leq |\theta-\theta_0|$ and

$$\Delta f_\theta(\theta^*, x) = \begin{pmatrix} f_\theta^{(1)}(\theta^*, x) \\ \cdot \\ \cdot \\ \cdot \\ f_\theta^{(d)}(\theta^*, x) \end{pmatrix}$$

Lemma 5.1 : *For any* $A_T > 0$,

$$\sup_{|\psi| \leq A_T} \left| I_T(\theta) - T^{-1}\int_0^T [\psi'\Delta f_\theta(\theta_0, X(t))]^2\, dt \right| \leq M_1 A_T^{3+\epsilon}\, T^{-1-\epsilon} \text{ a.s}$$

for some constant $M_1 > 0$; *where* $\theta = \theta_0 + \dfrac{\psi}{\sqrt{T}}$.

Proof:

$$I_T(\theta) = \int_0^T [f(\theta, X(t)) - f(\theta_0, X(t))]^2\, dt$$

$$= \int_0^T [(\theta-\theta_0)'\nabla f_\theta(\theta_0, X(t))]^2\, dt$$

$$+ \int_0^T \{[(\theta-\theta_0)'\nabla f_\theta(\theta^*, X(t))]^2 dt - [(\theta-\theta_0')\nabla f_\theta(\theta_0, X(t))]^2\}\, dt.$$

Now integrand in the 2nd expression

$$= (\theta-\theta_0)'(\nabla f_\theta(\theta^*, X) + \nabla f_\theta(\theta_0, X))(\theta-\theta_0)'(\nabla f_\theta(\theta^*, X) + \nabla f_\theta(\theta_0, X)).$$

Hence using (A5) and (A7),

$$|I_T(\theta)-\int_0^T[(\theta-\theta_0)'\nabla f_\theta(\theta_0, X(t)]^2\,dt| \leqslant 2M|\theta-\theta_0|^{2+\alpha}\int_0^T c(X(t)).(1+|X(t)|)\,dt.$$

Let

$$\psi = \sqrt{T}(\theta-\theta_0).$$

By using the ergodic theorem (note that $E[c(X(0))(1+|X(0)|)] < \infty$) we have the result.

Lemma 5.2 : *Let*

$$v_T(\psi, x) = f(\theta_0+\psi T^{-\frac{1}{2}}, x)-f(\theta_0, x)-\psi' T^{-\frac{1}{2}}\nabla f_\theta(\theta_0, x).$$

Then

$$P\left\{\sup_{|\psi|\leqslant A_T}\left|\int_0^T v_T(\psi, X(t))d\xi(t)\right| > c_1\lambda^{1/d+\epsilon_0}\right\} < \frac{c_2(A_T T^{-\frac{1}{2}})^{\alpha(d+\epsilon_0)}}{\lambda}.$$

Proof :

$$v_T(\psi, x)-v_T(\psi_1, x) = (\psi-\psi_1)'\nabla v_T(\zeta, x)$$

where ζ lies between ψ and ψ_1.

Hence, if $|\psi|$, $|\psi_1| \leqslant A_T$, we have by using (A5),

$$|v_T(\psi, x)-v_T(\psi_1, x)| \leqslant |\psi-\psi_1|(A_T T^{-\frac{1}{2}})^\alpha c(x).$$

Now the result follows as in Lemma 4.1.

Theorem 5.1 : *Assume*

$$I(\theta_0) = ((E[f_\theta^{(i)}(\theta_0, X(0))f_\theta^{(j)}(\theta_0, X(0))])) \quad i, j = 1, \ldots, d$$

is non-singular.

Under (A1)–(A7),

$$\sqrt{T}(\theta_T-\theta_0) \xrightarrow{\mathcal{L}} N(0, I^{-1}(\theta_0)).$$

Proof : By the ergodic theorem,

$$\frac{1}{T}\int_0^T \nabla f_\theta(\theta_0, X(t))(\nabla f_\theta(\theta_0, X(t)))'dt \to I(\theta_0) \text{ a.s. as } T\to\infty$$

and by the C.L.T. for stochastic integrals,

$$\frac{1}{\sqrt{T}}\int_0^T \nabla f_\theta(\theta_0, X(t))d\xi(t) \xrightarrow{\mathcal{L}} N(0, I(\theta_0)) \text{ as } T\to\infty.$$

Choose $A_T = \log T$ in Lemma 5.1 and 5.2, and let $T \to \infty$, we then have that the asymptotic distribution of $\hat{\theta}_T$ which minimizes $R_T(\theta)$ is same as $\hat{\psi}$, where $\hat{\psi}$ is that which minimizes $\psi' I(\theta_0)\psi - 2\psi' Z$, and Z is normal with mean zero and variance-covariance matrix $I(\theta_0)$.

But

$$\hat{\psi} = I^{-1}(\theta_0)\psi \sim N(0, I^{-1}(\theta_0)).$$

Hence it follows that $\sqrt{T}(\theta_T - \theta_0) \xrightarrow{\mathcal{L}} N(0, I^{-1}(\theta_0))$.

Remark: One might have tried to extend the results of Prakasa Rao and Rubin (1981) by generalizing the Fourier analysis results to many dimensions. For example, a generalization of Lemma 1 of Appendix in Prakasa Rao and Rubin (1981) was conveyed to the author by Professor G. J. Babu of Indian Statistical Institute.

Appendix

Let $\{X(t) : t \geq 0\}$ and $\{\xi(t) : t \geq 0\}$ be as in Section 2.

Lemma 6.1 : *Let g be a function on $\Omega \times \mathcal{R}$ such that*

$$|g(\theta, x) - g(\phi, x)| \leq J(x)|\theta - \phi| \qquad \dots (\bullet)$$

Suppose

$$Y_T(\theta) = \int_0^T g(\theta, X(t))d\xi(t), \; T \geq 0$$

is defined as a stochastic integral.

Then if $d + \alpha_0 \geq 2$,

$$E\Big(\sup_{0 \leq t \leq T} |Y_t(\theta) - Y_t(\phi)|^{d+\alpha_0}\Big) \leq c_{d+\alpha_0}|\theta - \phi|^{d+\alpha_0} T^{\frac{d+\alpha_0}{2}} E|J(X(0))|^{d+\alpha_0}$$

Proof :

$$E\Big(\sup_{0 \leq t \leq T} |Y_t(\theta) - Y_t(\phi)|^{d+\alpha_0}\Big) \leq c_{d+\alpha_0} E\Big[\int_0^T |g(\theta, X(t)) - g(\phi, X(t))^2| dt\Big]^{\frac{d+\alpha_0}{2}}$$

(See Prakasa Rao and Rubin, 1981)

$$\leq c_{d+\alpha_0} T^{\frac{d+\alpha_0}{2}-1} E\int_0^T |g(\theta, X(t) - g(\phi, X(t))|^{d+\alpha_0} dt$$

(By Hölder's Inequality)

$$\leq c_{d+\alpha_0} T^{\frac{d+\alpha_0}{2}} |\theta - \phi|^{d+\alpha_0} E|J(X(0))|^{d+\alpha_0}.$$

(By $(\bullet)$ and stationarity of $\{X(t) : t \geq 0\}$)

Lemma 6.2 : *Suppose p, ψ are continuous functions on* $[0, \infty)$

$$p(0) = \psi(0) = 0 \quad and \quad \lim_{t \to \infty} \psi(t) = \infty$$

$f : \mathcal{R}^d \to L$ *(where L is a normed linear space) is a function which is strongly continuous on* $\overline{B(a, \rho)}$, $\rho > 0$ where $B(a, \rho)$ denotes the ball of radius ρ around $a \in \mathcal{R}^d$. $\overline{B(a, \rho)}$ denotes its closure.

Suppose further that

$$\int_{B(a,\rho)} \int_{B(a,\rho)} \psi \left(\frac{\|f(\theta)-f(\phi)\|}{p(|\theta-\phi|)} \right) d\theta \, d\phi \leqslant B.$$

Then $\forall \theta, \phi \in B(a, \rho)$,

$$\|f(\theta)-f(\phi)\| \leqslant B \int_0^{2|\theta-\phi|} \psi^{-1} \left(\frac{4^{d+1} B}{\gamma_1^2 u^{2d}} \right) p(du)$$

where

$$\gamma_1 = \inf_{\theta \in B(0,1)} \sup_{1 < \rho \leqslant 2} \frac{|B(\theta, \rho) \cap B(0, 1)|}{\rho^d},$$

and $|A|$ denotes the Lebesgue measure of A.

Proof: For proof see Stroock and Varadhan, 1979, p. 60 or Stroock, , p. 7).

Corollary 6.1 : *In particular if $\psi(x) = x^r$, $p(x) = x^{\gamma/r}$, $r > 0$, $\gamma > 2d$ in Lemma 6.2, then*

$$\|f(\theta)-f(\phi)\| \leqslant c(r, \gamma, d) |\theta-\phi|^{\frac{\gamma-2d}{r}} B^{1/r}.$$

Lemma 6.3 : *Suppose $\{Y(\theta) : \theta \in \mathcal{R}^d\}$ is a class of random variables taking values in a normed linear space.*

Suppose (a) $\forall w, \theta \to Y(\theta, w)$ *is continuous on* $\overline{B(a, \rho)}$

(b) $E\|Y(\theta)-Y(\phi)\|^r \leqslant c|\theta-\phi|^{d+\alpha} \forall \theta, \phi \in B(a, \rho)$.

Then $\forall \gamma \in (2d, 2d+\alpha)$ *and* $\lambda > 0$

$$P \left\{ \sup_{\theta, \phi \in B(a,\rho)} \frac{\|Y(\theta)-Y(\phi)\|}{|\theta-\phi|^\beta} \geqslant c(r, \gamma, d) \lambda^{1/r} \right\} \leqslant \frac{cA}{\lambda}$$

where

$$\beta = \frac{\gamma-2d}{r}, \quad A = \int_{B(a,\rho)} \int_{B(a,\rho)} |\theta-\phi|^{d+\alpha-\gamma} d\theta \, d\phi.$$

Proof: By (b)

$$E\left[\int_{B(a,\rho)}\int_{B(a,\rho)}\left(\frac{\|Y(\theta)-Y(\phi)\|}{|\theta-\phi|^{\gamma/r}}\right)^r d\theta d\phi\right] \leqslant cA.$$

Hence

$$P\left\{\int_{B(a,\rho)}\int_{B(a,\rho)}\left(\frac{\|Y(\theta)-Y(\phi)\|}{|\theta-\phi|^{\gamma/r}}\right)^r d\theta d\phi > \lambda\right\} \leqslant \frac{cA}{\lambda}$$

and whenever

$$\int_{B(a,\rho)}\int_{B(a,\rho)}\left(\frac{\|Y(\theta)-Y(\phi)\|}{|\theta-\phi|^{\gamma/r}}\right)^r d\theta d\phi \leqslant \lambda,$$

we have by Corollary 6.1,

$$\|Y(\theta)-Y(\phi)\| \leqslant c(r,\gamma,d)\,|\theta-\phi|^{\beta}\,\lambda^{1/r} \;\forall\; \theta,\phi \in B(a,\rho).$$

Hence the lemma follows.

Corollary 6.2: *Suppose in additions to conditions of Lemma* 6.3, *there exists a* $\theta_0 \in B(a,\rho)$ *such that* $Y(\theta_0) = 0$.

Then

$$P\left\{\sup_{\theta\in\overline{B(a,\rho)}} \|Y(\theta)\| > (2\rho)^{\beta} c(r,\gamma,d)\lambda^{1/r}\right\} \leqslant \frac{cA}{\lambda}.$$

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