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CHARACTERIZATION OF BIJECTIVE AND BIMEASURABLE TRANSFORMATIONS FOR BIVARIATE NORMAL VARIATES

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SUMMARY. Ghosh's result (1969) in the case of univariate normal distribution based on two parametric points is extended to bivariate normal distribution based on three parametric points establishing Khatri's conjecture (1986) for bivariate situation. This result was established by Khatri (1986) under some restrictions on covariance matrices. The general multivariate situation is under consideration.

1. Introduction

Let x be a p-vector variable and $g(x)$ be a bijective and bimeasurable transformation of x. When $p = 1$, Ghosh (1969) proved that if $x \sim N(\mu_1, \sigma_i^2)$ implies $g(x) \sim N(\eta_i, \psi_i)$ for $i = 1, 2$, and $\mu_1 \neq \mu_2$, then $g(x)$ is essentially linear in x. For the p-variate normal distribution, Khatri (1986) established that if $x \sim N(\mu_i, \Sigma_i)$ implies $g(x) \sim N(\eta_i, V_i)$ for $i = 0, 1, \ldots, p$, where $\Sigma_i = \Sigma_0$ for $i = 0, 1, \ldots, r\ (> 0)$ and $\Sigma_i = \Sigma_p$ for $i = r+1, \ldots, p$ with $\Sigma_0 \neq \Sigma_p$, and

$$[\Sigma_1^{-1}(\mu_0 - \mu_1), \Sigma_2^{-1}(\mu_0 - \mu_2), \ldots, \Sigma_p^{-1}(\mu_0 - \mu_p)]$$

is nonsingular, then $g(x)$ is essentially linear in x. Further, he conjectured that this is true even without the conditions on Σ_i's as in Khatri (1986). This conjecture is established here for $p = 2$ and this can be mentioned as

Theorem 1 : *Let $x \sim N(\mu_i, \Sigma_i)$ imply $g(x) \sim N(\eta_i, V_i)$ for $i = 0, 1, 2$ and $p = 2$. Assume that*

$$(\Sigma_1^{-1}(\mu_1 - \mu_0), \Sigma_2^{-1}(\mu_2 - \mu_0))$$

is nonsingular. Then $g(x)$ is essentially linear in x. Here $\Sigma_1, \Sigma_2, \Sigma_0, V_1, V_2, V_0$ are all 2×2 positive definite matrices and $\Sigma_1, \Sigma_2, \Sigma_0$ may be all distinct.

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Further, the following result is established :

Theorem 2 : *Let* $x \sim N(0, \Sigma_i)$ *imply* $g(x) \sim N(\eta_i, V_i)$ *for* $i = 0, 1, 2, 3$, *where* $p = 2$ *and* Σ_i *and* V_i $(i = 0, 1, 2, 3)$ *are positive definite. Let*

$$A_i = \Sigma_0^{\frac{1}{2}} \Sigma_i^{-1} \Sigma_0^{\frac{1}{2}} - I = \begin{pmatrix} a_{1i} & a_{2i} \\ a_{2i} & a_{3i} \end{pmatrix}$$

for $i = 1, 2, 3$, *and assume that*

$$\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

is nonsingular. Then $\eta_i = \eta_0$ *for* $i = 1, 2, 3$, *and there exists a non-singular matrix* P *such that*

$$(g(x)-\eta_0)(g(x)-\eta_0)' = Pxx'P'.$$

2. Some lemmas and proofs of main theorems

We shall first prove some results necessary for the proof of the main theorem mentioned in section 1.

Lemma 1 : *Let* A_i *and* B_i $(i = 1, 2, \ldots, k(>1))$ *be* 2×2 *symmetric matrices and*

$$\left| I_2 + \sum_{i=1}^{k} v_i A_i \right| = \left| I_2 + \sum_{i=1}^{k} v_i B_i \right|,$$

for all real $v_1, v_2, \ldots, v_k$. *Then there exists an orthogonal matrix* P *such that* $A_i = PB_iP'$ *for* $i = 1, 2, \ldots, k$.

Proof : The given equation implies that for each i, $|I+v_iA_i| = |I+v_iB_i|$ for every real v_i and this implies that A_i and B_i have the same eigenvalues. Hence, if $A = \alpha I$ for some α, then $B_i = \alpha I$. Then any orthogonal matrix will be suitable for the result. From this point, we shall assume that none of the A_i's is proportional to I. Without loss of generality, assume that

$$A_1 = B_1 = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \; h_1 \neq h_2.$$

Taking $A_j = \begin{pmatrix} a_{1j} & a_{2j} \\ a_{2j} & a_{3j} \end{pmatrix}$ and $B_j = \begin{pmatrix} b_{1j} & b_{2j} \\ b_{2j} & b_{3j} \end{pmatrix}$ for $j = 2, 3, \ldots, k$, the given equation is equivalent to

$$\Big(1+h_1v_1+\sum_{j=2}^{k} v_ja_{1j}\Big)\Big(1+h_2v_1+\sum_{j=2}^{k} v_ja_{3j}\Big)-\Big(\sum_{j=2}^{k} v_ja_{2j}\Big)^2 =$$

$$\Big(1+h_1v_1+\sum_{j=2}^{k} v_jb_{1j}\Big)\Big(1+h_2v_1+\sum_{j=2}^{k} v_jb_{3j}\Big)-\Big(\sum_{j=2}^{k} v_jb_{2j}\Big)^2,$$

for all $v_1, v_2, \ldots, v_k$. Equating the coefficients of v_i's, v_i^2's, v_iv_j's, we get

$a_{1j}+a_{3j} = b_{1j}+b_{3j}$, $h_1a_{3j}+h_2a_{1j} = h_1b_{3j}+h_2b_{1j}$ for $j = 2, 3, \ldots, k$;

$a_{1j}a_{3j}-a_{2j}^2 = b_{1j}b_{3j}-b_{2j}^2$ for $j = 2, 3, \ldots, k$;

$a_{1i}\,a_{3j}+a_{1j}a_{3i}-2a_{2i}a_{2j} = b_{1i}b_{3j}+b_{1j}b_{3i}-2b_{2i}b_{2j}$

for all $i \neq j$, $i, j = 2, 3, \ldots, k$.

From the above equalities, we get

$$a_{1j} = b_{1j},\ a_{3j} = b_{3j},\ a_{2j}^2 = b_{2j}^2,\ a_{2i}a_{2j} = b_{2i}b_{2j},$$

for all $i \neq j$, $i, j = 2, 3, \ldots, k$. This shows that

$$A_j = VB_jV \text{ for } j = 1, 2, \ldots, k,$$

where $V = I$, or $V = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ or $V = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$ or $V = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$.

This proves the lemma.

Remark 1 : Suppose $|I_p+\sum_{i=1}^{k} v_iA_i| = |I_p+\sum_{i=1}^{k} v_iB_i|$ for all real $v_1, v_2, \ldots, v_k$, where $A_1, \ldots, A_k$, $B_1, \ldots, B_k$ are $p\times p$ symmetric matrices and $p \geqslant 3$. Consider for example,

$$A_1 = B_1 = \begin{pmatrix}h_1 & o & o\\ o & h_2 & o\\ o & o & h_3\end{pmatrix},\ A_2 = \begin{pmatrix}a & x & o\\ x & a & z\\ o & z & a\end{pmatrix}, B_2 = \begin{pmatrix}a & o & y\\ o & a & w\\ y & w & a\end{pmatrix},$$

where $h_1 > h_2 > h_3 > 0$. Note that A_2 and B_2 will be positive definite if $a > o$, $a^2 > x^2+z^2$ and $a^2 > y^2+w^2$. It may be seen that

$$|I_3+v_1A_1+v_2A_2| = |I_3+v_1B_1+v_2B_2|$$

holds for all real v_1, v_2 provided

$$x^2 = (h_1-h_2)y^2/(h_1-h_3),\ z^2 = ((h_1-h_3)w^2+(h_2-h_3)y^2)/(h_1-h_3).$$

Considering non-zero choices of x, z, y, w, satisfying the above conditions, it follows that Lemma 1 does not hold for $p \geqslant 3$ and $k \geqslant 2$.

Lemma 2 : *Let A_1, A_2, B_1, B_2 be nonsingular 2×2 symmetric matrices on the real space and μ_1, μ_2, ν_1, ν_2 be 2×1 vectors such that*

$$\Big(\sum_{i=1}^{2} v_iA_i\mu_i\Big)'\Big(I+\sum_{i=1}^{2} v_iA_i\Big)^{-1}\Big(\sum_{i=1}^{2} v_iA_i\mu_i\Big)$$

$$=\Big(\sum_{i=1}^{2} v_iB_i\nu_i\Big)'\Big(I+\sum_{i=1}^{2} v_iB_i\Big)^{-1}\Big(\sum_{i=1}^{2} v_iB_i\nu_i\Big)$$

and

$$|I+\sum_{i=1}^{2} v_iA_i|=|I+\sum_{i=1}^{2} v_iB_i|$$

for all real v_1, v_2. Then there exists an orthogonal matrix P such that $\mu_i = P\nu_i$ and $A_i = PB_iP'$ for $i = 1, 2$, provided $(A_1\mu_1, A_2\mu_2) = \Delta$ is nonsingular.

Proof: By Lemma 1, there exists an orthogonal matrix Q such that $A_i = QB_iQ'$ for $i = 1, 2$, or $(I+v_1A_1+v_2A_2) = Q(I+v_1B_1+v_2B_2)Q'$, for all real v_1, v_2. Then, defining

$$\alpha_i = A\,\mu_i,\ \beta_i = QB_i\nu_i\ (i = 1, 2),$$

the first equation yields

$$\begin{vmatrix} 1 & v_1\alpha_1'+v_2\alpha_2' \\ v_1\alpha_1+v_2\alpha_2 & I+\sum_{i=1}^{2} v_iA_i \end{vmatrix}\begin{vmatrix} 1 & v_1\beta_1'+v_2\beta_2' \\ v_1\beta_1+v_2\beta_2 & I+\sum_{i=1}^{2} v_iA_i \end{vmatrix}$$

for all real v_1, v_2. Hence for all real v_1, v_2,

$$\Big|I+\sum_{i=1}^{2} v_iA_i-(v_1\alpha_1+v_2\alpha_2)(v_1\alpha_1+v_2\alpha_2)'\Big|$$

$$=\Big|I+\sum_{i=1}^{2} v_iA_i-(v_1\beta_1+v_2\beta_2)(v_1\beta_1+v_2\beta_2)'\Big|.$$

Now, directly equating the coefficients of the powers of v_i's, we get the following equations :

$$\alpha_1'\alpha_1 = \beta_1'\beta_1,\ \alpha_2'\alpha_2 = \beta_2'\beta_2,\ \alpha_1'\alpha_2 = \beta_1'\beta_2, \qquad \dots\ (2.1)$$

$$\alpha_1'A_1^{-1}\alpha_1 = \beta_1'A_1^{-1}\beta_1, \qquad \dots\ (2.2a)$$

$$\alpha_2'A_2^{-1}\alpha_2 = \beta_2'A_2^{-1}\beta_2, \qquad \dots\ (2.2b)$$

$$2|A_1|\alpha_1'A_1^{-1}\alpha_2+|A_2|\alpha_1'A_2^{-1}\alpha_1 = 2|A_1|\beta_1'A_1^{-1}\beta_2+|A_2|\beta_1'A_2^{-1}\beta_1, \qquad \dots\ (2.2c)$$

$$2|A_2|\alpha_2'A_2^{-1}\alpha_1+|A_1|\alpha_2'A_1^{-1}\alpha_2 = 2|A_2|\beta_2'A_2^{-1}\beta_1+|A_1|\beta_2'A_1^{-1}\beta_2. \qquad \dots\ (2.2d)$$

The solution of (2.1) shows that

$$(\alpha_1, \alpha_2) = R(\beta_1, \beta_2), \quad \ldots \quad (2.3)$$

where R is an orthogonal matrix and (β_1, β_2) is nonsingular. Using this in (2.2) and defining $C_i = |A_i|\,(RA_i^{-1}R' - A_i^{-1})$ for $i = 1, 2$, we get

$$\alpha_i' C_i \alpha_i = 0 \ (i = 1, 2),\ 2\alpha_1' C_1 \alpha_2 + \alpha_1' C_2 \alpha_1 = 0,$$
$$2\alpha_2' C_2 \alpha_1 + \alpha_2' C_1 \alpha_2 = 0,\ \text{tr}\, C_i = 0 \ (i = 1, 2). \quad \ldots \quad (2.4)$$

Let us write

$$\alpha_1 = \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix},\ \alpha_2 = \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix},\ C_1 = \begin{pmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{pmatrix},\ C_2 = \begin{pmatrix} y_1 & y_2 \\ y_2 & -y_1 \end{pmatrix}.$$

Notice that given α_1 and α_2, (2.4) represents a system of linear equations in x_1, x_2, y_1, y_2, and can be written as

$$Td = 0, \quad \ldots \quad (2.4a)$$

where $d = (x_1, x_2, y_1, y_2)'$, and

$$T = \begin{bmatrix} a_{11}^2 - a_{12}^2 & 2a_{11}a_{12} & 0 & 0 \\ 0 & 0 & a_{21}^2 - a_{22}^2 & 2a_{21}a_{22} \\ a_1^* & a_2^* & a_{11}^2 - a_{12}^2 & 2a_{11}a_{12} \\ a_{21}^2 - a_{22}^2 & 2a_{21}a_{22} & a_1^* & a_2^* \end{bmatrix},$$

with $a_1^* = 2(a_{11}a_{21} - a_{12}a_{22})$, $a_2^* = 2(a_{12}a_{21} + a_{11}a_{22})$. Note that by routine explicit calculation, $|T| = 4(a_{11}a_{22} - a_{12}a_{21})^4 > 0$. Hence, (2.4a) shows that $d = 0$, i.e., $C_1 = C_2 = 0$. Therefore, we have

$$QB_iQ' = A_i = RA_iR' \text{ and } A_i\mu_i = RQB_i\nu_i \ (i = 1, 2).$$

From these, we get $\mu_i = P\nu_i$ and $A_i = PB_iP'$ with $P = RQ$ and $A_i = RA_iR'$ for $i = 1, 2$, where P, R and Q are orthogonal matrices. This proves Lemma 2.

Remark 2 : Is Lemma 2 true for $p \geqslant 3$? In view of Remark 1, it appears that Lemma 2 will not be true for $p \geqslant 3$ and $k \geqslant 3$.

Proof of Theorem 1 : Since $g(x)$ is bijective and bimeasurable function of x, using Theorem 1 of Khatri (1986), we have

$$(x - \mu_0)'(\Sigma_i^{-1} - \Sigma_0^{-1})(x - \mu_0) + 2(\mu_0 - \mu_i)'\Sigma_i^{-1}(x - \mu_0)$$
$$= (g(x) - \eta_0)'(V_i^{-1} - V_0^{-1})(g(x) - \eta_0) + 2(\eta_0 - \eta_i)' V_i^{-1}(g(x) - \eta_0), \quad \ldots \quad (2.5)$$

for all real x and for $i = 1, 2$,

$$\left| I - t\sum_{i=1}^{2} v_i(\Sigma_i^{-1} - \Sigma_0^{-1})\,\Sigma_0 \right| = \left| I - t\sum_{i=1}^{2} v_i(V_i^{-1} - V_0^{-1})V_0 \right| \quad \ldots \quad (2.6a)$$

and

$$\left(\sum_{i=1}^{2} v_i \, \Sigma_0^{-1}(\mu_i-\mu_0)\right)'\left(\Sigma_0^{-1}-t \sum_{i=1}^{2} v_i \, (\Sigma_i^{-1}-\Sigma_0^{-1})\right)^{-1}\left(\sum_{i=1}^{2} \; v_i \, \Sigma_i^{-1}(\mu_i-\mu_0)\right)$$

$$=\left(\sum_{i=1}^{2} v_i V_i^{-1}(\eta_i-\eta_0)\right)'\left(V_0^{-1}-t \sum_{i=1}^{2} v_i(V_i^{-1}- V_0^{-1})\right)^{-1}\left(\sum_{i=1}^{2} v_i V_i^{-1}(\eta_i-\eta_0)\right) \quad \ldots \quad (2.6b)$$

for all real t, v_1, v_2. Taking

$$A_i = \Sigma_0^{\frac{1}{2}} \Sigma_i^{-1}\Sigma_0^{\frac{1}{2}}, B_i = V_0^{\frac{1}{2}}V_i^{-1}V_0^{\frac{1}{2}}, \; \mu_{(i)} = \Sigma_0^{-\frac{1}{2}}(\mu_0-\mu_i), \; \eta_{(i)} = V_0^{-\frac{1}{2}}(\eta_0-\eta_i),$$

for $i = 1$, 2, and defining $x_0 = \Sigma_0^{-\frac{1}{2}}(x-\mu_0)$, $y_0 = V_0^{-\frac{1}{2}}(g(x)-\eta_0)$, we can rewrite (2.5) and (2.6) as

$$x_0'(A_i-I)x_0+2\mu_{(i)}'A \, x_0 = y_0'(B_i-I)y_0+2\eta_{(i)}B_i y_0 \; (i = 1, 2), \quad \ldots \quad (2.7)$$

and

$$\left| I+ \sum_{i=1}^{2} v_i A_i \right| = \left| I+ \sum_{i=1}^{2} v_i B_i \right|, \quad \ldots \quad (2.8a)$$

for all real v_1, v_2,

$$\left(\sum_{i=1}^{2} v_i A_i \mu_{(i)}\right)' \left(I+ \sum_{i=1}^{2} v_i A_i\right)^{-1} \left(\sum_{i=1}^{2} v_i A_i \mu_{(i)}\right)$$

$$= \left(\sum_{i=1}^{2} v_i B_i \eta_{(i)}\right)' \left(I+\sum_{i=1}^{2} v_i B_i \right)^{-1}\left(\sum_{i=1}^{2} v_i B_i \eta_{(i)}\right), \quad \ldots \quad (2.8b)$$

for all real v_1, v_2. By Lemma 2, (2.8) shows that there exists an orthogonal matrix P such that

$$\mu_{(i)} = P\eta_{(i)}, \; A_i = PB_iP' \; (i = 1, \; 2), \quad \ldots \quad (2.9)$$

when $p = 2$. Using these in (2.7), we have

$$(x_0-Py)' \, (2\Delta+H) = 0 \quad \ldots \quad (2.10)$$

for all real x, where

$$2\Delta+H = [2A_1\mu_{(1)}+(A_1-I)(x_0+Py_0), \; 2A_2\mu_{(2)}+(A_2-I)(x_0+P y_0)].$$

Then, arguing as done by Ghosh (1969) or Khatri (1986), we get

$$Py_0 = x_0 \text{ or } g(x) = \eta_0+ V_0^{\frac{1}{2}}P'\Sigma_0^{-\frac{1}{2}}(x-\mu_0),$$

for almost all real x, which proves Theorem 1.

Proof of Theorem 2 : Since $g(x)$ is bijective and bimeasurable function of x, by Theorem 1 of Khatri (1986),

$$x'(\Sigma_i^{-1}-\Sigma_0^{-1})x = (g(x)-\eta_0)' \, (V_i^{-1}- V_0^{-1}) \, (g(x)-\eta_0) \; (i = 1, 2, 3), \quad \ldots \quad (2.11)$$

for all real x, and for all real v_1, v_2, v_3,

$$\left| I+\sum_{i=1}^{3} v_i A_i \right| = \left| I+\sum_{i=1}^{3} v_i B_i \right| \text{ and } \eta_i = \eta_0 \ (i = 1, 2, 3), \qquad \ldots \quad (2.12)$$

where $A_i = \Sigma_0^{1/2}\Sigma_i^{-1}\Sigma_0^{1/2}-I$, $B_i = V_0^{1/2}V_i^{-1}V_0^{1/2}-I$ $(i = 1, 2, 3,)$. Using Lemma 1 in (2.12), there exists an orthogonal matrix P such that

$$A_i = PB_iP' \ (i = 1, 2, 3).$$

Using this in (2.11), we have

$$(\Sigma_0^{-1/2}x)' A_i(\Sigma_0^{-1/2}x) = (PV_0^{-1/2}(g(x)-\eta_0))'A_i(PV_0^{-1/2}(g(x)-\eta_0)), \qquad \ldots \quad (2.13)$$

for all real x and $i = 1, 2, 3$. Taking

$$(\Sigma_0^{-1/2}x)(\Sigma_0^{-1/2}x)'-(PV_0^{-1/2}(g(x)-\eta_0))(PV_0^{-1/2}(g(x)-\eta_0))' = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3 \end{pmatrix},$$

$$A_i = \begin{pmatrix} a_{1i} & a_{2i} \\ a_{2i} & a_{3i} \end{pmatrix} \ (i = 1, 2, 3),\ \delta' = (x_1, x_2, x_3)$$

and

$$T = \begin{pmatrix} a_{11} & 2a_{21} & a_{31} \\ a_{12} & 2a_{22} & a_{32} \\ a_{13} & 2a_{23} & a_{33} \end{pmatrix}.$$

(2.13) gives $T\delta = 0$ and hence $\delta = 0$, or

$$(g(x)-\eta_0)(g(x)-\eta_0)' = Axx'A',$$

with $A = V_0^{1/2}P'\Sigma_0^{-1/2}$, for all real x. This proves Theorem 2.

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