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ASYMPTOTIC EXPANSIONS OF PERTURBED CHI-SQUARE VARIABLES

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SUMMARY. The present paper extends the Theorem 1(b) of Chandra and Ghosh (1979) on perturbed chi-squares. It is known that such theorems are useful for getting valid expansions for test statistics whose null distributions are central chi-squares.

1. Introduction and main theorem

Let $\phi_V(x)$ and $\Phi_V(x)$ be respectively the density and distribution functions of the normal distribution on R^k with mean zero and dispersion matrix V. *In this paper, we shall assume that V is nonsingular.* Also s will stand for an integer $s \geqslant 4$. The central chi-square distribution with p degrees of freedom will be denoted by χ^2_p. For any multiindex of nonnegative integers $(a(1), \ldots, a(k))$, $|a|$ stands for the sum $a(1)+\ldots+a(k)$.

Consider a sequence of k-dimensional random vectors $\{U_n : n \geqslant 1\}$ defined on some probability space $(\Omega, \mathcal{A}, P)$ such that

$$n^{1/2}(U_n-\mu) \text{ converges weakly to } \Phi_V \quad \ldots \quad (1.1)$$

for some $\mu \in R^k$. *Let g_n be a Borel measurable real-valued function on R^k.* Put

$$U_n = g_n(n^{1/2}(U_n-\mu)), \quad n \geqslant 1. \quad \ldots \quad (1.2)$$

We are interested in obtaining the Edgeworth expansion of U_n when its asymptotic distribution is χ^2_p, $2 \leqslant p \leqslant k$. This problem was treated by Chandra and Ghosh (1979) under a set of assumptions on g_n and $n^{1/2}(U_n-\mu)$; see their Theorem 1(b). Here we relax these assumptions and give a complete proof of this theorem. In the 1979 paper, the proof of Theorem 1(b), though highly nontrivial, was only very briefly sketched; see, in this connection, page 21 of the 1979 paper and the remark following the statement of the theorem of the present paper.

In case

$$g_n(u) = l^T u + u^T L u + o(1)$$

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as $n \to \infty$, where L is a non-null positive semi-definite matrix of rank p, then in order that the asymptotic distribution of W_n be χ^2_p, it is necessary and sufficient that

$$l = 0, \ LVL = L. \qquad \ldots \quad (1.3)$$

Using 1c.3(ii) of Rao (1973), *we can and do, assume*, without loss of generality, that

$$V = I, \ u^T L u = \|u^1\|^2, \qquad \ldots \quad (1.4)$$

where *I is the $k \times k$ identity matrix, and for $u = (u^{(1)}, \ldots, u^{(k)})$, u^1 and u^2 stand for $(u^{(1)}, \ldots, u^{(p)})$ and $(u^{(p+1)}, \ldots, u^{(k)})$ respectively.* Let T be the polar transformation which sends u^1 to $(r, \theta^{(1)}, \ldots, \theta^{(p-1)})$ and keep u^2 unchanged :

$$\begin{aligned} u^{(1)} &= r \cos \theta^{(1)} \ \ldots \ \cos \theta^{(p-2)} \cos \theta^{(p-1)} \\ u^{(2)} &= r \cos \theta^{(1)} \ \ldots \ \cos \theta^{(p-2)} \sin \theta^{(p-1)} \\ &\ \vdots \\ u^{(p-1)} &= r \cos \theta^{(1)} \sin \theta^{(2)} \\ u^{(p)} &= r \sin \theta^{(1)}, \end{aligned} \qquad \ldots \quad (1.5)$$

where $0 < r < \infty$ and (θ, u^2) belongs to the set

$$A = \left\{ (\theta, u^2) \,|\, -\frac{\pi}{2} < \theta^{(i)} < \frac{\pi}{2}, \ 1 \leqslant i \leqslant p-2, \ 0 < \theta^{(p-1)} < 2\pi, \ u^2 \in R^{k-p} \right\}. \qquad \ldots \quad (1.6)$$

Here $\theta = (\theta^{(1)}, \ldots, \theta^{(p-1)})$. The Jacobian of the transformation T is $r^{p-1}J(\theta)$ where

$$J(\theta) = (\cos \theta^{(1)})^{p-2} (\cos \theta^{(2)})^{p-3} \ \ldots \ \cos \theta^{(p-2)}. \qquad \ldots \quad (1.7)$$

Let $\quad x^{(i)} = \cos \theta^{(i)}, \ y^{(i)} = \sin \theta^{(i)} \ \ i = 1, 2, \ldots, p-2,$

$$x = (x^{(1)}, \ldots, x^{(p-2)}), \ y = (y^{(1)}, \ldots, y^{(p-2)}).$$

By $R^*(r, x, y, u^2)$, we shall mean an expression of the form

$$r^{\alpha} \left\{ \prod_1^{p-2} (x^{(i)})^{a(i)} \right\} \left\{ \prod_1^{p-2} (y^{(i)})^{b(i)} \right\} \left\{ \prod_{p+1}^{k} (u^{(i)})^{c(i)} \right\}, \qquad \ldots \quad (1.8)$$

where α, $a(i)$, $b(i)$ and $c(i)$ are nonnegative integers. By $R(r, x, y, u^2)$, with or without suffixes, we shall mean a finite sum of constant multiples of terms of the form $R^*(r, x, y, u^2)$.

We say that $R^*(r, x, y, u^2)$ is odd if at least one of the nonnegative integers

$$b(1), \ldots, b(p-1), \ a(p-1), \ c(p+1), \ldots, c(k), \qquad \ldots \quad (1.9)$$

is odd. Then it is easy to see that

$$\int_A \exp\left\{ -\frac{1}{2} \|u^2\|^2 \right\} R^*(r, x, y, u^2) J(\theta) d\theta du^2 = 0 \text{ if } R^*(r, x, y, u^2) \text{ is odd}, \qquad \ldots \quad (1.10)$$

for each $r > 0$; here $\|u^2\|^2$ is the usual Euclidean norm of u^2. *We say that $R(r, x, y, u^2)$ is odd if every $R^*(r, x, y, u^2)$ appearing in it is odd.*

A function $f(u)$ is said to be a *generalized polynomial in u* if $f(Tu)$ is a polynomial in r, x, y and u^2, i.e., if $f(Tu)$ can be written as some $R(r, x, y, u^2)$.

We say that $n^{1/2}(U_n - \mu)$ satisfies condition (A_1) if there exist generalized polynomials $\{f_j(u) : j = 0, 1, \ldots, s-3\}$, $f_0 \equiv 1$, such that

$$\sup_{B_k \in \mathcal{C}_1} | P(n^{1/2}(U_n - \mu) \in B_k) - \int_{B_k} \xi_{s-1,n}(u) du = o(n^{-(s-3)/2}) \quad \ldots \quad (1.11)$$

for every family $\mathcal{C}_1$ of Borel subsets B_k of R^k satisfying

$$\sup_{B_k \in \mathcal{C}_1} \int_{(\partial B_k)^\epsilon} \phi_V(u) du = o((-\log \epsilon)^{-(s-3)/2}). \quad \ldots \quad (1.12)$$

Here
$$\xi_{s-1,n}(u) = \left\{ \sum_0^{s-3} n^{-j/2} f_j(u) \right\} \phi_V(u). \quad \ldots \quad (1.13)$$

See, in this connection, Corollary 20.3 of Bhattacharya and Ranga Rao (1975) and Theorem 3 of Bhattacharya and Ghosh (1978); see also Theorem 1.5 of Bhattacharya (1977).

Let
$$g_n(u) = \sum_1^{s-3} n^{-j/2} Q_j(u) + o(n^{-(s-3)/2}) \quad \ldots \quad (1.14)$$

where
$$u \in M_n = \{u \,|\, \|u\|^2 < dn^{-1} \log n\} \text{ for some } d > 0 \quad \ldots \quad (1.15)$$

$$Q_1(u) = u\tau Lu$$

and other Q_j are generalized polynomials in u. Put

$$h_{s-1,n} = \sum_1^{s-3} n^{-j/2} Q_j$$

$$W_n' = h_{s-1,n}(n^{1/2}(U_n - \mu)).$$

We say that g_n satisfies condition (A_2) if

$$h_{s-1,n}(Tu) = r^2 \left\{ \sum_0^{s-3} n^{-j/2} R_{1j}(r, x, y, u^2) \right\}, \quad \ldots \quad (1.16)$$

for some $R_{1j}(r, x, y, u^2)$. We say that g_n satisfies condition (A_3) if for $u \in M_n$

$$g_n(Tu) = r^2 \left\{ \sum_0^{s-3} n^{-j/2} R_{1j}(r, x, y, u^2) + O(n^{-1} \log n)^{(s-2)/2} \right\}, \; R_{10} \equiv 1, \quad \ldots \quad (1.7)$$

and
$$\frac{\partial}{\partial r} g_n(Tu) = 2r \left\{ \sum_0^{s-3} n^{-j/2} R_{2j}(r, x, y, u^2) + O(n^{-1} \log n)^{(s-2)/2} \right\}. \quad \ldots \quad (1.18)$$

We say that $R_j(r, x, y, u^2)_{j \geq 0}$ enjoy the odd-even property if the degree of each

$$R^*(r, x, y, u^2)/r^\alpha$$

appearing in $R_j(r, x, y, u^2)$ is odd or even according as j is odd or even.

We say that $R^*(r, x, y, u^2)$ satisfies the property of modulo 2 if

$$\alpha = |a|+|b| \ \text{modulo } 2;$$

here $a = (a(1), \ldots, a(p-2))$ etc. We say that $R(r, x, y, u^2)$ satisfies the property of modulo 2 if every monomial $R^*(r, x, y, u^2)$ appearing it satisfies the property of modulo 2.

The lemma below shows that "the property of modulo 2" is preserved under usual analytic operations.

Lemma 1.1 : *Let*

$$R_i \equiv R_i(r, x, y, u^2) = \sum_{1}^{s-3} n^{-j/2} R_{ij}(r, x, y, u^2), \ 1 \leqslant i \leqslant k_1.$$

and f be a real analytic function defined in an neighbourhood of the origin. Write

$$f(R_1, \ldots, R_{k_1}) = \sum_{0}^{s-3} n^{-j/2} R_{0j}(r, x, y, u^2) + o(n^{-(s-3)/2}).$$

If $\{R_{ij}(r, x, y, u^2), \ 1 \leqslant j \leqslant s-3\}$, satisfy the property of modulo 2 for each $i = 1, \ldots, k_1$, then so do $\{R_{0j}(r, x, y, u^2), \ 1 \leqslant j \leqslant s-3\}$.

Proof : Since f can be expanded in a Taylor's series and each term can be considered separately, it suffices to prove the lemma for the special case

$$f(u_1, \ldots, u_{k_1}) = u_1 u_2 \ldots u_{k_1}.$$

Fix a j, $1 \leqslant j \leqslant s-3$ and fix a monomial R^*_{0j} of $R_{0j}(r, x, y, u^2)$. This monomial is obtained by multiplying some monomials, $R^*_{ij_i}(r, x, y, u^2)$, of R_{ij_i}.

Let the powers of r in these monomials be respectively α^* and α_{j_i}. Then

$$\alpha^* = \sum_{i=1}^{s-3} \alpha_{j_i}. \qquad \ldots \ (1.19)$$

Let the powers of $x^{(l)}$, $y^{(l)}$ of these monomials be respectively denoted by $(a^*(l), b^*(l))$ and $(a_{j_i}(l), b_{j_i}(l))$, $l = 1, \ldots, p-2$. Then

$$\left.\begin{aligned} a^*(l) &= \sum_{i=1}^{s-3} a_{j_i}(l) \\ b^*(l) &= \sum_{i=1}^{s-3} b_{j_i}(l). \end{aligned}\right\} \ l = 1, \ldots, p-2. \qquad \ldots \ (1.20)$$

Now since $\{R_{ij}(r, x, y, u^2)\}$ enjoy the property of modulo 2, for each $i = 1, \ldots, s-3$ one has

$$\alpha_{j_i} = \sum_{l=1}^{p-2} a_{j_i}(l) + \sum_{l=1}^{p-2} b_{j_i}(l) \mod 2.$$

By (1.20), one can then write

$$\sum_{i=1}^{s-3} \alpha_{j_i} = \sum_{l=1}^{p-2} a^*(l) + \sum_{l=1}^{p-2} b^*(l) \mod 2.$$

Equation (1.19) now implies that

$$\alpha^* = \sum_{l=1}^{p-2} a^*(l) + \sum_{l=1}^{p-2} b^*(l) \mod 2.$$

This shows that R^*_{0j} satisfies the property of modulo 2. This completes the proof of the lemma.

The next lemma shows that "the odd-even property" is also preserved under analytic operations.

Lemma 1.2 : *Let*

$$R_i \equiv R_i(r, x, y, u^2) = \sum_{1}^{s-3} n^{-j/2} R_{ij}(r, x, y, u^2), \ 1 \leqslant i \leqslant k_1.$$

and let f be real analytic function defined in a neighbourhood of the origin. Write

$$f(R_1, \ldots, R_{k_1}) = \sum_{0}^{s-3} n^{-j/2} R_{0j}(r, x, y, u^2) + o(n^{-(s-3)}/2).$$

If $\{R_i : 1 \leqslant j \leqslant s-3\}$ enjoy the odd-even property for each $i = 1, \ldots, k_1$ then so do $\{R_{0j}(r, x, y, u^2) : 1 \leqslant j \leqslant s-3\}$.

Proof : As in Lemma 1.1, it suffices to prove the lemma when

$$f(u_1, \ldots, u_{k_1}) = u_1 u_2, \ldots, u_{k_1}.$$

To this end, fix a j, $1 \leqslant j \leqslant s-3$ and note that any monomial of $R_{0j} \equiv R_{0j}(r, x, y, u^2)$ is obtained by multiplying some monomials (of degree $k(j_i)$, say) of $R_{ij_i}(r, x, y, u^2)$, $0 \leqslant i \leqslant s-3$, where

$$j_0 + j_1 + \ldots + j_{s-3} = j; \qquad \ldots \quad (1.21)$$

also the degree, r say, of this monomial of R_{0j} will be given by

$$r = k(j_0) + \ldots + k(j_{s-3}). \qquad \ldots \quad (1.22)$$

The proof will be complete once we have established that whenever $j_0, j_1, \ldots, j_{s-3}$ are nonnegative integers satisfying (1.21) and whenever

$k(j_0), \ldots, k(j_{s-3})$ are nonnegative integers satisfying (1.23) below, r given by (1.22) will be odd or even according as j is odd or even.

For each $i = 0, 1, \ldots, s-3$, $k(j_i)$ is odd or even according as j_i is odd or even. ... (1.23)

Put
$$A = \{j_i \mid j_i \text{ is odd}, \quad 0 \leqslant i \leqslant s-3\}$$
$$B = \{j_i \mid j_i \text{ is even}, \quad 0 \leqslant j \leqslant s-3\}.$$

Then since $\sum_{i \in B} j_i$ is always even, one gets in view of (1.21),

the number of elements of A is odd or even according as j is odd or even. ... (1.24)

Since for each $i \in B$, $k(j_i)$ is even (by (1.23)), $\sum_{i \in B} k(j_i)$ is always even. On the other hand if $i \in A$, $k(j_i)$ is odd (by (1.23)) and so it follows from (1.24) that $\sum_{i \in A} k(j_i)$ is odd or even according as j is odd or even. Equation (1.22) then implies that r is odd or even according as j is odd or even. This completes the proof of the lemma.

It follows from that above lemma that the polynomials appearing in the Edgeworth expansions of normalised sum of i.i.d. random variables enjoy the odd-even property; see, in this connection, Lemma 7.1 of Bhattacharya and Ranga Rao (1975) and Theorems 2, 3 of Bhattacharya and Ghosh (1978).

Theorem : *Assume the above set up. Let* $n^{1/2}(U_n - \mu)$ *and* g_n *satisfy conditions* (A_1) *and* (A_2) *respectively. Let*

$$P_\theta(M_n^c) = o(n^{-(s-3)/2}). \qquad \ldots (1.25)$$

Then (a) *there exist nonnegative integers* $k_1, \ldots, k_{s-3}$ *and constants* a_{ij} *such that*

$$P(W_n' \in B) = \sum_0^{s-3} n^{-j/2} \left\{ \sum_{i=0}^{k_j} a_{ij} \int_B \chi^2(z; p+i)dz \right\} + o(n^{-(s-3)/2}), \qquad \ldots (1.26)$$

uniformly over all Borel subset B *of* $\mathcal{C}_2$ *satisfying*

$$\sup_{B \in \mathcal{C}_2} \int_{(\partial B)^\epsilon} \chi^2(z; p)dz = O(\epsilon), \quad \epsilon \to 0.$$

Here $\chi^2(z; p+i)$ *is the density at* z *of* χ^2_{p+i}.

(b) *The conclusion of* (a) *holds with* W_n' *replaced by* W_n *with the same choice of* $\{k_j\}$ *and* $\{a_{ij}\}$.

(c) *Let*

$$f_j(u) = R_{3j}(r, x, y, u^2), \ 1 \leqslant j \leqslant s-3 \quad (\text{cf. } (1.13)). \qquad \ldots (1.28)$$

If $\{R_{i,j}(r, x, y, u^2) : j \geq 0\}$, $i = 1, 3$, *both enjoy the odd-even property, then*

$$a_{ij} = 0 \text{ for all } i, \quad 0 < j < k_j, j \text{ odd}.$$

(d) *If* $R_{i,j}(r, x, y, u^2) : j \geq 0$, $i = 1, 3$ *both enjoy the property of modulo* 2, *then*

$$a_{ij} = 0 \text{ for all } j = 1, \ldots, s-3, \; i \text{ odd}.$$

Remark : The parts (c) and (d) implies that the asymptotic expansion of W_n will be in powers of n^{-1} (instead of in powers of $n^{-1/2}$), and that the coefficient of n^{-r} $(r \geq 1)$ will be a finite linear combination of chi-squares with degrees of freedom p, $p+2$, $p+4$ etc. (instead of with degrees of freedom p, $p+1$, $p+2$ etc.). It is, also, easy to verify that Theorem 1(b) of Chandra and Ghosh (1979) is a special case of the above theorem.

2. Proof of theorem

To prove the theorem, we need two technical lemmas.

Lemma 2.1 : *Let* g_n *satisfy condition* (A_2) *and let* (1.13) *holds with generalised polynomials* $f_j(u)$. *Then*

$$\int_{g_n^{-1}(B)} \xi_{s-1,n}(u)du = \sum_{j=0}^{s-3} n^{-j/2} \left\{ \sum_{i=1}^{k_j} a_{ij} \int_B \chi^2(z; p+i)dz \right\} + o(n^{-(s-3)/2}). \quad \ldots \quad (2.1)$$

uniformly over all Borel subsets of R^1.

Proof : Put

$$B_n = g_n^{-1}(B) \cap M_n \quad \ldots \quad (2.2)$$

and apply the transformation T given by (1.5); we then get in view of (1.25) and (1.28)

$$\int_{g_n^{-1}(B)} \xi_{s-1,n}(u)\, du$$

$$= \int_{T(B_n)} \widetilde{R}(r, x, y, u^2) \sum_{0}^{s-3} n^{-j/2} R_{2j}(r, x, y, u^2) dr d\theta du^2 + o(n^{-(s-3)/2}), \quad \ldots \quad (2.3)$$

where $\widetilde{R}(r, x, y, u^2) = (2\pi)^{-k/2} r^{p-1} J(\theta) \exp\{-\tfrac{1}{2}(r^2 + \|u^2\|^2)\}$. ... (2.4)

Henceforth we shall adopt the convention that

$$R_{ij}(r, x, y, u^2) = 1 \quad \text{if } j = 0, \; i \geq 1.$$

We apply next the transformation T' which sends (r, θ, u^2) to (r', θ, u^2) where

$$r' = g_n(T^{-1}(r, \theta, u^2))^{1/2} (r, \theta, u^2) \in T(R^k). \quad \ldots \quad (2.5)$$

In the following

$$u \in M_n \text{ and } (r, \theta, u^2) \in T(M_n).$$

In view of condition (A_3)

$$(r')^2 = r^2 \left\{ \sum_0^{s-3} n^{-j/2} R_{1j}(r, x, y, u^2) + O(n^{-1} \log n)^{(s-2)/2} \right\}$$

implying that

$$r' = r \left\{ \sum_0^{s-3} n^{-j/2} R_{4j}(r, x, y, u^2) + O(n^{-1} \log n)^{(s-2)/2} \right\}.$$

[Here and in the following, we have used the fact that any real analytic function, defined in a neighbourhood of the origin, of $\sum_0^{s-3} n^{-j/2} R_{ij}(r, x, y, u^2)$ can again be expressed in the form

$$\sum_0^{s-3} n^{-j/2} R_{ij}(r, x, y, u^2) + o(n^{-(s-2)/2+\epsilon})$$

uniformly in $(r, \theta, u^2) \epsilon T(M_n)$ for any $\epsilon > 0$].

One can then show that

$$r = r' \left\{ \sum_0^{s-3} n^{-j/2} R_{5j}(r', x, y, u^2) + o(n^{-(s-2)/2+\epsilon}) \right\} \qquad \dots \quad (2.6)$$

uniformly in $(r, \theta, u^2) \epsilon T(M_n)$.

[To verify (2.6), let $r_0 = r'$ and define inductively $r_{i,n}$ as follows

$$r_{i+1,n} = r' - r_{i,n} \sum_1^{s-3} n^{-j/2} R_{4j}(r_{i,n}, x, y, u^2) \quad 0 \leqslant i \leqslant s-4.$$

One then verifies that

$$r' - r_{i,n} \sum_1^{s-3} n^{-j/2} R_{4j}(r_{i,n}, x, y, u^2) = r + o(n^{-(s-2)/2+\epsilon}). \quad 0 \leqslant i \leqslant s-4, \epsilon > 0,$$

uniformly in $(r, \theta, u^2) \epsilon T(M_n)$, and that $r_{s-3,n}$ can be expressed, *after neglecting terms of order* $o(n^{-(s-3)/2})$, in the form

$$r' + r' \sum_1^{s-3} R_{5j}(r', x, y, u^2).$$

Plainly, Equation (2.6) holds.]

In view of (1.18), (2.5) and (2.6) we get

$$\frac{\partial r}{\partial r'} = \sum_0^{s-3} n^{-j/2} R_{6j}(r', x, y, u^2) + o(n^{-(s-3)/2}).$$

Thus (2.3) can be written as

$$\int_{\sigma_n^{-1}(B)} \xi_{s-1,n}(u)\,du$$

$$= \int_{T'T(B_n)} \widetilde{R}(r', x, y, u^2)\left\{\sum_{0}^{s-3} n^{-j/2}R_{\eta j}(r', x, y, u^2)\right\}dr'd\theta du^2 + o(n^{-(s-3)/2}). \quad \ldots \quad (2.7)$$

Now $$T'T(B_n) = T'T(B_0) \cap T'T(M_n)$$

where $$B_0 = \{(r', \theta, u^2) \in T'T(R^k) \mid (r')^2 \in B\}. \quad \ldots \quad (2.8)$$

Using Lemma 3.2, page 183 of Edwards (1973), one can show that $T'T(M_n)$ contains the set

$$\{(r', \theta, u^2) \mid (r')^2 < d \log n - \frac{1}{2} \log 2, \|u^2\|^2 < d \log n\}. \quad \ldots \quad (2.9)$$

Consequently, Equation (2.7) can be written as

$$\int_{\sigma_n^{-1}(B)} \xi_{s-1,n}(u)du$$

$$= \int_{B_0} \widetilde{R}(r', x, y, u^2)\left\{\sum_{0}^{s-3} n^{-j/2}R_{\eta j}(r', x, y, u^2)\right\}dr'd\theta du + o(n^{-(s-3)/2}). \quad \ldots \quad (2.10)$$

One new integrates with respect to (θ, u^2) and obtains (2.1). This completes the proof of Lemma 2.1.

Lemma 2.2 :

(a) $\int_{E_n(B)} \phi(u)du \leq 2 \int_B \chi^2(z; p)dz + o(n^{-(s-3)/2})$.

where $E_n(B) = \{u \mid h_{s-1,n}(u) \in B\}$

and the $o(n^{-(s-3)/2})$ *term does not depend on* B.

(b) $\int_{(\sigma(E_n(B))^\epsilon} \phi(u)du < \int_{(\partial B)^\epsilon((2s-3)\log n)^{1/2}} \chi^2(z; p)dz + o(n^{-(s-3)/2}),\ 0<\epsilon<1,$

where the $o(n^{-(s-3)/2})$ *term does not depend on* B *or* ϵ.

Proof : (a) The first part of the proof is based on the arguments similar to those given in (2.3) through (2.10) of the proof of Lemma 2.1, with the exception that r' is to be replaced by

$$r' = (h_{s-1,n}(Tu))^{1/2}.$$

One then integrates with respect to (θ, u^2) and repeatedly uses the following estimate

$$\int_{(t \log n)^{1/2}} r^j \exp\left(-\frac{1}{2}r^2\right) dr = O((\log n)^{j-1}n^{-t/2}), \quad t > 0,\ j \geq 0.$$

(b) For notational convenience, put

$$B(\epsilon) = (\partial B)^\epsilon \text{ and } A_n = \partial(E_n(B)).$$

Note that for any $n \geqslant 1$,

$$A_n \subset E_n(\partial B).$$

Get an $n_0 \geqslant 1$ such that the Euclidean norm of the gradient of $h_{s-1,n}(u)$, restricted to the sphere of radius one around M_{n_0}, is less than

$$((2s-3) \log n)^{1/2} = \epsilon_n', \text{ say}$$

Then if $n \geqslant n_0$, $0 < \epsilon < 1$, one has

$$(A_n)^c \cap M_n \subset E_n(B(\epsilon_n)) \quad M_n.$$

An appeal to the part (a) (with B replaced by $B(\epsilon_n'))$ now establishes the part (b).

Proof of theorem: (a) The proof of Part (a) is immediate from (1.25) and Lemma 2.1 (with g_n replaced by $h_{s-1,n}$) and from the fact that $n^{1/2}(U_n - \mu)$ satisfies Condition (A_1).

(b) It suffices to note that (i) for $h > 0$

$$\{W_n' \in (B)^{-\delta n}\} \cap M_n \subset \{W_n \in B\} \cap M_n \subset \{W_n \in (B)^{\delta n}\} \cap M_n$$

where $$\delta_n = hn^{-(s-3)/2}$$

and $$(B)^{-\delta n} = U\{x \mid S(x; \delta_n) \subset B\};$$

Here $S(x; \delta_n)$ is the sphere of radius δ_n with centre x.

(ii) the relation

$$(B)^{\delta n} - (B)^{-\delta n} = (\partial B)^{\delta n}$$

which is true in any metric space, every open sphere of whch is connected.

The proofs of (c) and (d) now follows from Lemma 1.2 and (1.1) respectively.

3. Counter-examples

Example 1: Let $\{Z_n\}_{n \geq 1}$ be i.i.d. two-dimensional vectors and let $Z_1^{(1)}$ and $Z_1^{(2)}$ be independent $N(0, 1)$. Let

$$W_n = nH(\bar{Z}_n), \quad \bar{Z}_n = n^{-1} \sum_1^n Z_i$$

where $$H(z) = \|z\|^2 + (z^{(1)})^5 \|z\|^{-1}.$$

Here Theorem 1(b) of Chandra and Ghosh (1979) is not applicable. As W_n has the same distribution as that of

$$X^2 + Y^2 + n^{-1} X^5 (X^2 + Y^2)^{-1/2}$$

where X, Y are i.i.d. $N(0, 1)$, it follows that our Theorem is applicable.

Example 2 : This example points out that in the part (c), one needs to bother about the odd-even property.

Let $\{Z_n\}_{n\geq 1}$ be i.i.d. $N(0, 1)$ on R^1 and put

$$W_n = n(\bar{Z}_n)^2(1+n^{-1/2}).$$

Then $$P(W_n \leqslant u) = \int_0^u \chi^2(z;1)\left(1+\frac{1}{2}(z^2-1)n^{-1/2}\right)dz+o(n^{-1/2})$$

so that the coefficient of $n^{-1/2}$ does not vanish.

Example 3 : This example points out that in the part (d), one needs to consider for property of modulo 2.

Let Z_i be as in Example 1. Put

$$H(z) = \|z\|^2+(z^{(1)})^2\|z\|^{-2}.$$

Then the Edgeworth exansion up to $o(n^{-1})$ holds uniformly over all Borel subset of R^1_+ but the coefficient of n^{-1} is a finite linear combination of χ^2_2, χ^2_1 and χ^2_3.

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References

Bhattacharya, R. N. (1977) : Refinements of the multivariate central limit theorems and applications. *Ann. Prob.*, **5**, 1-27.

Bhattacharya, R. N. and Ghosh, J. K. (1978) : On the validity of the formal Edgeworth expansions. *Ann Statist.*, **6**, 434-451.

Bhattacharya, R. N. and Ranga Rao, R. (1976) : *Normal Approximations and Asymptotic Expansions*, Wiley, New York.

Chandra, T. K. (1980) : Asymptotic expansions and deficiency. Doctoral Thesis, Indian Statistical Institute, Calcutta.

Chandra, T. K. and Ghosh, J. K. (1979) : Valid asymptotic expansions for the likelihood ratio statistic and other perturbed chi-square variables. *Sankhyā*, **41**, Ser. A, 22-47.

Edwards, C. H. Jr. (1973) : *Advanced Calculus of Several Variables*, Academic Press, New York.

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