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A CHARACTERISATION OF THE NORMAL DISTRIBUTION

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SUMMARY. The normal law is characterised through the local independence of certain statistics.

In this note we prove a characterisation of the normal distribution through the local independence of certain statistics. A similar result has been proved earlier by Parthasarathy (1976). Our result is the following :

Theorem : *Let X and Y be independent and identically distributed real-valued random variables with density f. Suppose the conditional densities of $X+Y$ given $X-Y = t$ exist and are equal for all $t \in E$, where E is a Borel set with $\lambda(E) > 0$. (λ denotes Lebesgue measure on the real line $\mathcal{R}$). Then f must be a normal density.*

To prove this theorem we need a few lemmas.

Lemma 1 : *Let E be a Borel subset of $\mathcal{R}$ with $\lambda(E) > 0$. Then there exist $y_n \in E$, $n = 0, 1, 2, \ldots$ such that the y_n's are distinct, each y_n is an accumulation point of E, and $y_n \to y_0$.*

Proof : There exists a compact set $E_1 \subseteq E$ with $\lambda(E_1) > 0$. By the Bolzano-Weierstrass property there exists an accumulation point $y_1 \in E_1$. Let $F_1 = \{y :$ there exist rationals r_1, r_2 not both zero such that $r_1y_1+r_2y = 0\}$. As F_1 is countable, there exists a compact set $E_2 \subseteq E_1/F_1$ with $\lambda(E_2) > 0$. Let $y_2 \in E_2$ be an accumulation point. Proceeding thus we get a sequence $y_1, y_2, \ldots$, such that the y_n's are distinct accumulation points of E_1. As E_1 is compact, $\{y_n\}$ has a convergent subsequence which may again be denoted by $\{y_n\}$. Take $y_0 = \lim y_n$. This completes the proof.

Lemma 2 : *Let $f(x) \geqslant 0$ a.e. on $\mathcal{R}$, with $\int_{\mathcal{R}} f(x)dx = 1$. Let $\alpha(x) \geqslant 0$ a.e. on $\mathcal{R}$ and > 0 on a set of positive Lebesgue measure. Let E be a Borel subset of $\mathcal{R}$ with $\lambda(E) > 0$ and let $\beta(y) > 0$ for all $y \in E$. Suppose that, for every $y \in E$, the relation*

$$f(x+y) \cdot f(x-y) = \alpha(x)\beta(y) \qquad \ldots \quad (1)$$

holds for almost all x (i.e., for all $x \notin$ some N_y^* with $\lambda(N_y^*) = 0$). Then α is continuous on the complement of a null subset of $\mathcal{R}$.

Remarks : Consider the example : $f(x) = 1$ for $0 < x < 1$ and zero otherwise; $E = [1, 2]$. Then

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(i) if $\beta = 0$ on E, (1) holds for arbitrary α for all x, so that the desired conclusion on α can be made in general only if β is positive on a set of positive Lebesgue measure.

(ii) if $\alpha = 0$ a.e. on $\mathcal{R}$, then (1) holds for any β on E. We shall therefore assume in what follows that $\alpha > 0$ on a set of positive Lebesgue measure.

Proof: Let $\xi = \sqrt{\alpha}$, $\eta = \sqrt{\beta}$ and $\zeta = \sqrt{f}$, so that $\zeta \in L^2(\mathcal{R})$. Then, by the Cauchy-Schwarz inequality, $\zeta(\cdot+y)\zeta(\cdot-y) \in L^1(\mathcal{R})$ for every fixed $y \in \mathcal{R}$. A Fubini argument then shows that, for some set N with $\lambda(N) = 0$, (1) holds for all $x \in N^c$ and for $y \in E / N_x$ for some set N_x with $\lambda(N_x) = 0$. We claim that at least on the set N^c, α is continuous. Let then $x_0 \in N^c$ and $\{x_n\}$ be a sequence of members of N^c converging to x_0. Then (1) holds for all $y \in E \setminus \bigcup_j N_{x_j}$ and we have

$$|\xi(x_n)-\xi(x_0)| \int_E \eta(y)dy = \left| \int_E \{\zeta(x_n+y)\zeta(x_n-y)-\zeta(x_0+y)\zeta(x_0-y)\}dy \right|$$

$$\leqslant \int_{\mathcal{R}} |\zeta(x_n+y)\zeta(x_n-y)-\zeta(x_0+y)\zeta(x_0-y)|\,dy$$

$$\leqslant \int_{\mathcal{R}} \zeta(x_n+y)\,|\zeta(x_n-y)-\zeta(x_0-y)|\,dy$$

$$+\int_{\mathcal{R}} \zeta(x_0-y)\,|\zeta(x_n+y)-\zeta(x_0+y)|\,dy$$

$$\leqslant 2\|\zeta\|_2\left(\int |\zeta(u+x_n-x_0)-\zeta(u)|^2du\right)^{1/2} \to 0 \text{ as } n\to\infty,$$

since $x_n \to x_0$.

Remark: Note that the same argument shows that

$$\left.\begin{array}{l} \xi(x_n)-\xi(x_n') \to 0 \text{ as } n\to\infty, \\ x_n - x_n' \to 0 \text{ and } x_n, x_n' \in N^c. \end{array}\right\} \quad \ldots \quad (2)$$

if

Lemma 3: *Let f, α, β, E and the null set N be as above and let the sequence $\{y_n\}$ be as in Lemma 1. If $[a, b]$ is a compact interval such that*

$$\inf\{\alpha(x) : x \in N^c \cap [a, b]\} > 0,$$

then $g = \log f$ is defined and equal to a quadratic polynomial on each of the intervals $(a\pm y_0, b\pm y_0)$.

Proof: The preceding remark and $\inf \alpha > 0$ over $N^c \cap [a, b]$ imply that, for some $\delta > 0$, $\inf \alpha > 0$ over $N^c \cap [a-2\delta, b+2\delta]$ as well. Let $y \in E$ be fixed. Then (1) holds for $x \in S_y = [a-2\delta, b+2\delta] / (N \cup N_y^*)$. Note that $\log f(x\pm y)$ are defined on S_y and if g denotes $\log f$, we have, for $x \in S_y$

$$g(x+y)+g(x-y) = \log \alpha(x)+\log \beta(y) = A(x)+B(y), \text{ say.}$$

Let h be a smooth function on $\mathcal{R}$ vanishing along with its derivatives of all orders on the complement of the open interval $(-\delta, \delta)$. For any real function φ defined on $[a-2\delta, b+2\delta]$, let $\bar{\varphi}$ be defined on $[a-\delta, b+\delta]$ according to

$$\bar{\varphi}(x) = \int_{-\delta}^{\delta} \varphi(x+t)h(t)dt.$$

Then we have

$$\bar{g}(x+y)+\bar{g}(x-y) = \bar{A}(x)+B(y) \int_{-\delta}^{\delta} h(t)dt,$$

for $x \in [a-\delta, b+\delta]$; therefore

$$\bar{g}'(x+y)+\bar{g}'(x-y) = \bar{A}'(x),$$

for $x \in (a-\delta, b+\delta)$. (Recall that $y \in E$ is kept fixed).

Let now $z_0 \in E$ be an accumulation point of E, so that there exists a non-constant sequence $\{z_n\}$ of members of E converging to z_0. We may assume that $|z_n-z_0| < \delta$ for all n. Then

$$\bar{g}'(x+z_n)+\bar{g}'(x-z_n) = \bar{A}'(x) = \bar{g}'(x+z_0)+\bar{g}'(x-z_0),$$

for all $x \in (a, b)$.

Note that $\bar{g}$ is defined on either of the intervals $[a\pm z_0-\delta, b\pm z_0+\delta]$. Hence it follows that $\bar{g}''(x+z_0) = \bar{g}''(x-z_0)$ for every accumulation point z_0 of E and for all $x \in (a, b)$.

Let us now take the sequence $\{y_n\}$ as in Lemma 1. Then, for any $x \in (a+y_0, b+y_0)$, if $x_n = x-2y_n+2y_0$, then $x_n \to x$ and so belongs to $(a+y_0, b+y_0)$ for all sufficiently large n. Then

$$\bar{g}''(x_n) = \bar{g}''(x_n-2y_0) = \bar{g}''(x-2y_n) = \bar{g}''(x).$$

so that $\bar{g}'''(x) = 0$ for all $x \in (a+y_0, b+y_0)$. Thus $\bar{g}$ is a quadratic polynomial on that interval, and similarly on the interval $(a-y_0, b-y_0)$ as well. Since this is true whatever be the smooth h of the kind described, it follows that g is itself a quadratic polynomial on $(a\pm y_0, b\pm y_0)$.

Lemma 4 : *Under the same hypotheses as in Lemma 2, f must be of the form $\exp Q$, where Q is a quadratic polynomaal, throughout $\mathcal{R}$.*

Proof : Since $\alpha(x_0) > 0$ for some $x_0 \in N^c$ ($\alpha > 0$ on a set of positive measure), let

$$a = \inf\{x : \inf \alpha > 0 \text{ over } N^c \cap [x, x_0]\}.$$

$$b = \sup\{x : \inf \alpha > 0 \text{ over } N^c \cap [x_0, x]\}.$$

We claim that $a = -\infty$, $b = +\infty$. Suppose not; for definiteness, let $a > -\infty$ if possible.

Let $\gamma = \inf\{\alpha(x) : x \in N^c \cap (a, x_0]\}$. We claim that $\gamma = 0$. Suppose not and that $\gamma > 0$. It then follows from the definition of a that, for every positive integer n.

$$\inf \left\{\alpha(x) : x \in N^c \cap \left[a-\frac{1}{n}, a\right]\right\} = 0$$

so that there exists a sequence $\{u_n\}$ of members of N^c such that $u_n \uparrow a$ and $\alpha(u_n) < \frac{1}{n}$; on the other hand, for any sequence $\{v_n\}$ of members of N^c such that

$v_n \downarrow a$, $\alpha(v_n) \geqslant \gamma > 0$, so that $\alpha(v_n)-\alpha(u_n) \geqslant \frac{1}{2}\gamma$ for all large n, though $v_n-u_n \to 0$, which contradicts relation (2). Hence $\gamma = 0$. Let then $\{t_n\}$ be a sequence of members of $N^c \bigcap (a, x_0]$ such that $\alpha(t_n) \to (\gamma =)\, 0$. We claim that $t_n \to a$; for let $\{t_{n_k}\}$ be any convergent subsequence of $\{t_n\}$ and let t_0 be its limit; then the possibility that $t_0 > a$ is ruled out by the definition of $\underline{a}$; hence $t_0 = a$. Thus every convergent subsequence of $\{t_n\}$ converges to $\underline{a}$, or, $t_n \to a$, while $\alpha(t_n) \to 0$.

Since $t_n \in N^c$, equation (1) holds with $x = t_n$ and $y \notin N_{t_n}$. By applying Lemma 1 to $E \setminus UN_{t_n} = E^*$ (say), we may take an accumulation point y_0 of E^* which is itself the limit of a sequence $\{y_n\}$ of accumulation point of E^*. We may then appeal to Lemma 3 to conclude that f is the form $\exp Q\pm$, where $Q\pm$ is a quadratic polynomial, on each of the sets

$$\left(a+\frac{1}{k} \pm y_0,\ x_0 \pm y_0\right) \text{ for every } k = 1, 2, \ldots$$

(It is easily seen that $Q\pm$ is independent of k). It follows then that f is of the form $\exp Q\pm$ on the sets $(a \pm y_0, x_0 \pm y_0)$. Now

$$f(t_n+y_0)f(t_n-y_0) = \alpha(t_n)/\beta(y_0), \text{ for all } n$$

$$\alpha(t_n) \to 0, \text{ as } n \to \infty$$

is then in contradiction with the facts that

$$f(t_n+y_0) = \exp Q^+(t_n+y_0) \to \exp Q^+(a+y_0),$$

$$f(t_n-y_0) = \exp Q^-(t_n-y_0) \to \exp Q^-(a-y_0).$$

Hence $a = -\infty$; and similarly $b = +\infty$. Consequently $\inf\{\alpha(x) : x \in N^c \bigcap I\} > 0$ for any compact set I. It follows from Lemma 3 that f is of the form $\exp Q$, throughout $\mathcal{R}$.

Proof of the theorem : Let $U = \dfrac{X+Y}{2}$ and $V = \dfrac{X-Y}{2}$. Then the joint density of U and V is $2f(u+v)\cdot f(u-v)$. Let $g(v) = \int f(u+v)f(u-v)du$. Let $x_0 \in E$ be fixed. Then by our hypothesis we get

$$\frac{f\left(u+\frac{x_0}{2}\right)\cdot f\left(u-\frac{x_0}{2}\right)}{g\left(\frac{x_0}{2}\right)} = \frac{f\left(u+\frac{x}{2}\right)\cdot f\left(u-\frac{x}{2}\right)}{g\left(\frac{x}{2}\right)}, \text{ a.e.u,}$$

for all x in E. Therefore

$$f\left(u+\frac{x}{2}\right)\cdot f\left(u-\frac{x}{2}\right) = \left[\frac{f\left(u+\frac{x_0}{2}\right)\cdot f\left(u-\frac{x_0}{2}\right)}{g\left(\frac{x_0}{2}\right)}\right]\cdot g\left(\frac{x}{2}\right),$$

a.e.u. for all x in E. Now the theorem follows from Lemma 4.

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Reference

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