ON THE CATEGORY OF ERGODIC MEASURES

BY

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Introduction

If (X, S, μ) is a finite measure space and G the group of all one-to-one measure-preserving transformations, then two interesting topologies can be assigned to G which make it a topological group. In certain dynamical problems it is of interest to know whether a particular transformation is ergodic or not. Even though this problem has not been solved till now, the existence of a large class of ergodic transformations has been shown by the determination of their category in G. In particular, when the measure space is nonatomic, Halmos [1] proved that the set of weakly mixing transformations is a dense G_δ in G under the weak topology. Similar results were proved by Oxtoby and Ulam [2]. Rokhlin [3] proved that under the same weak topology in G, the set of strongly mixing transformations is a set of the first category.

In problems of information-theoretic interest, we have a measurable space $(X, \mathcal{S})$ and a one-to-one both ways measurable map T of X onto itself. Here, it is of interest to know whether there are a lot of ergodic measures in the space of invariant measures. In order to study this problem, we take X to be a topological space, $\mathcal{S}$ the Borel σ-field, and T a homeomorphism of X onto itself. Then several topologies can be assigned to the space of invariant probability measures. Taking X to be a complete and separable metric space and assigning the weak topology to the space of invariant probability measures, we show that the set of ergodic measures is a G_δ. When X is a countable product of complete and separable metric spaces and T is the shift transformation, we show that the ergodic measures form a dense G_δ under the same topology. Examples are given to show that the ergodic measures need not be dense in the general case. In the case of the shift transformation we have proved that the set of strongly mixing measures is a set of the first category.

1. Preliminaries

Let $(X, \mathcal{S})$ be any measurable space, and T a one-to-one both ways measurable map of X onto itself. Whenever the space X is a topological space, we take $\mathcal{S}$ to be the Borel σ-field, and T a homeomorphism of X onto itself. By a measure, we always mean a probability measure. We denote by $\mathfrak{M}$, $\mathfrak{M}_e$, and $\mathfrak{M}_s$ the space of all invariant, ergodic, and strongly mixing measures, respectively. For these definitions we refer to [1].

A point $x \in X$ will be called periodic if for some integer k, $T^k x = x$. A measure $\mu \in \mathfrak{M}$ is periodic if for some integer k, $\mu(A \cap T^k A) = \mu(A)$ for all

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sets $A \in \mathcal{S}$. We shall denote by $\mathcal{P}$ and $\mathcal{P}_e$ the class of all invariant periodic measures and the class of all ergodic periodic measures, respectively.

When X is a topological space, we assign the weak topology to $\mathfrak{M}$ by means of the following convergence: A net $\{\mu_\alpha\}$ in $\mathfrak{M}$ converges to μ if and only if $\int f\,d\mu_\alpha \to \int f\,d\mu$ for every bounded continuous function defined on X. When X is a separable metric space, the weak topology of $\mathfrak{M}$ becomes separable and metric. If further X is complete, then $\mathfrak{M}$ is also complete [4]. Sets of the type

$$[\mu : \mu(G_i) > \mu_0(G_i) - a_i, \quad i = 1, 2, \cdots, k],$$

$$[\mu : \mu(C_i) < \mu_0(C_i) + a_i, \quad i = 1, 2, \cdots, k], \qquad a_i \geqq 0,$$

where $G_1, G_2, \cdots, G_k$ are open sets in X, $C_1, C_2, \cdots, C_k$ are closed sets in X, μ_0 is any fixed measure in $\mathfrak{M}$, and μ denotes any general invariant measure, form a neighbourhood system in $\mathfrak{M}$ at μ_0.

Another important fact which we shall make use of is the following result due to Varadarajan [5]:

THEOREM 1.1. *If X is a separable metric space, then there exists an equivalent metric d such that the space $U_d(X)$ of functions uniformly continuous with respect to d is separable in the uniform topology.*

2. Topological nature of ergodic measures in a separable metric space

In this section we shall prove the following theorem.

THEOREM 2.1. *If X is a separable metric space and T is a homeomorphism of X onto itself, then $\mathfrak{M}_e$ is a G_δ in $\mathfrak{M}$ under the weak topology.*

Proof. It is clear that the class of all Borel sets S with the property $S = TS$ form a σ-field $\mathfrak{J}$. Let $C(X)$ be the space of all real-valued bounded continuous functions defined on X. For any fixed measure μ and any $f \in C(X)$, we denote by $E_\mu(f \mid \mathfrak{J})$ the conditional expectation of $f(x)$ given the σ-field $\mathfrak{J}$. It is easy to see that μ is ergodic if and only if, for every $f \in C(X)$, the following equation holds:

$$V(f, \mu) = \int [E_\mu(f \mid \mathfrak{J})]^2\,d\mu - \left(\int f\,d\mu\right)^2 = 0. \tag{2.1}$$

It is enough if condition (2.1) is satisfied for every bounded uniformly continuous function. This is because of the fact that any bounded continuous f is a pointwise limit of a uniformly bounded sequence of uniformly continuous functions and the conditional dominated convergence theorem is applicable (cf. Doob [6], p. 23). By making use of Theorem 1.1, we can take the space $U(X)$ of bounded uniformly continuous functions to be separable in the uniform topology. We take a dense sequence $f_1(x), f_2(x), \cdots$ in $U(X)$. Thus in order that an invariant measure μ be ergodic, it is necessary and sufficient that

$$V(f_k, \mu) = 0, \qquad k = 1, 2, \cdots. \tag{2.2}$$

Let

$$V_n(f_k, \mu) = \int \left[\frac{f_k(x) + \cdots + f_k(T^{n-1}x)}{n}\right]^2 d\mu - \left(\int f\, d\mu\right)^2. \tag{2.3}$$

From the mean ergodic theorem it follows that

$$V(f_k, \mu) = \lim_{n\to\infty} V_n(f_k, \mu) = \liminf_{n\to\infty} V_n(f_k, \mu). \tag{2.4}$$

For each fixed k and n, $V_n(f_k, \mu)$ is a continuous functional in $\mathfrak{M}$ under the weak topology. From (2.2), (2.3), and (2.4), it follows that

$$\mathfrak{M}_e = \bigcap_{k=1}^{\infty} \bigcap_{r=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} [\mu : V_n(f_k, \mu) < 1/r]. \tag{2.5}$$

The continuity of $V_n(f_k, \mu)$ implies that the set $[\mu : V_n(f_k, \mu) < 1/r]$ is open in the weak topology. This together with (2.5) implies that $\mathfrak{M}_e$ is a G_δ.

3. Measures invariant under the shift transformation in a product space

Let (M, S) be a separable metric space, and let $(X, \mathcal{S})$ be the bilateral product of a countable number of copies of (M, S). X can be written as

$$X = \prod_{-\infty}^{+\infty} M_i, \qquad M_i = M \qquad (i = \cdots, -1, 0, 1, \cdots),$$

and

$$\mathcal{S} = \prod_{-\infty}^{+\infty} S_i, \qquad S_i = S \qquad (i = \cdots, -1, 0, 1, \cdots).$$

Any point $x \in X$ can be represented by

$$x = (\cdots, x_{-1}, x_0, x_1, \cdots), \qquad x_i \in M_i.$$

We introduce the shift operator T by means of the following definition:

$$Tx = y = (\cdots, y_{-1}, y_0, y_1, \cdots), \qquad y_i = x_{i-1} \quad (i = \cdots, -1, 0, 1, \cdots).$$

T is obviously a one-to-one both ways measurable map of X into itself. In the space $\mathfrak{M}$ of measures invariant under T we introduce a topology $\mathfrak{J}$ by means of the following convergence: A sequence of measures $\mu_n \in \mathfrak{M}$ converges to μ if and only if $\mu_n(A) \to \mu(A)$ as $n \to \infty$ for each finite-dimensional measurable subset A.

THEOREM 3.1. *Under the topology $\mathfrak{J}$ in $\mathfrak{M}$ the set $\mathfrak{M}_e$ is everywhere dense in $\mathfrak{M}$.*

Proof. Let μ be any measure in $\mathfrak{M}$, and μ_n^r, the restriction of μ to the σ-field

$$\mathfrak{C}_r^n = \prod_{i=r(2n+1)-n}^{r(2n+1)+n} S_i,$$

and ν_n, the product measure given by

$$\nu_n = \prod_{r=-\infty}^{+\infty} \mu_n^r$$

which is defined on $\prod_{r=-\infty}^{+\infty} \mathfrak{C}_r^n = \mathcal{S}$. Then ν_n is defined on $\mathcal{S}$ and is invariant under the transformation T^{2n+1} which is also one-to-one and both ways measurable. It is easy to verify that ν_n is ergodic under T^{2n+1}. Now we write for any set $A \in \mathcal{S}$,

$$\mu_n(A) = \frac{\nu_n(T^{-n}A) + \nu_n(T^{-n+1}A) + \cdots + \nu_n(A) + \cdots + \nu_n(T^{n}A)}{2n+1}. \tag{3.1}$$

From the invariance of ν_n under T^{2n+1}, the invariance of μ_n under T follows immediately. Let now A be any set in $\mathcal{S}$ which is invariant under T, i.e., $A = TA$. Then $\mu_n(A) = \nu_n(A)$. Since $A = T^{2n+1}A$ and ν_n is ergodic under T^{2n+1}, it follows that $\mu_n(A) = 0$ or 1, i.e., $\mu_n(A)$ is ergodic under T and hence belongs to $\mathfrak{M}_e$. We shall now prove that μ_n converges to μ under the topology $\mathfrak{I}$. Let

$$\mathfrak{C}_k = \prod_{i=-k}^{k} S_i, \qquad k = 1, 2, \cdots,$$

be the σ-field which is the $(2k + 1)$-fold product of S. $\mathfrak{C}_k$ can be considered as a sub-σ-field of $\mathcal{S}$. From the construction of ν_n, it is clear that ν_n agrees with μ on $\mathfrak{C}_n$. Let now $A \in \mathfrak{C}_k$. Then $T^{-n+k}A$, $T^{-n+k+1}A, \cdots, T^{n-k}A$ belong to $\mathfrak{C}_n$. Thus

$$\nu_n(T^r A) = \mu(A) \qquad \text{for} \quad -n + k \leqq r \leqq n - k. \tag{3.2}$$

We have from (3.1) and (3.2)

$$| \mu_n(A) - \mu(A) | = \left| \frac{\nu_n(T^{-n}A) + \cdots + \nu_n(T^{n}A)}{2n+1} - \mu(A) \right| \leqq \frac{4k}{2n+1}.$$

Thus $\mu_n(A) \to \mu(A)$ as $n \to \infty$ for every $A \in \mathfrak{C}_k$. Since this is true for each fixed k, $\mu_n \to \mu$ in the topology $\mathfrak{I}$. This completes the proof.

The following theorem is an immediate corollary of Theorems 2.1 and 3.1.

THEOREM 3.2. *If* $X = \prod_{i=-\infty}^{+\infty} M_i$, $M_i = M$ $(i = \cdots, -1, 0, 1, \cdots)$ *where M is a complete and separable metric space and T is the shift transformation in X, then $\mathfrak{M}_e$ is a dense G_δ in $\mathfrak{M}$ under the weak topology, and hence $\mathfrak{M} - \mathfrak{M}_e$ is of the first category.*

Proof. This is an immediate corollary of Theorems 2.1, 3.1, and the facts that $\mathfrak{M}$ is a complete and separable metric space under the weak topology and convergence under $\mathfrak{I}$ implies weak convergence.

Remarks. A disposition towards the method adopted in proving Theorem 3.1 may already be found in the works of I. P. Tsaregradsky [7] and A. Feinstein [8] in a different context. A result less general than Theorem 3.1 has been proved recently by M. Nisio [9].

If in Theorem 3.2, M is a compact metric space, then the space of all totally finite invariant signed measures becomes a locally convex topological vector space under the weak topology, and $\mathfrak{M}$ becomes a compact convex set with $\mathfrak{M}_e$ as the set of extreme points. From Theorem 3.2, it follows that $\mathfrak{M}_e$ is a dense G_δ in $\mathfrak{M}$. This is one of the many examples which show that in the infinite dimensional case the structure of extreme points is peculiar when compared to the finite dimensional case.

Theorem 3.2 states that in the space X with the shift operator, in some sense the ergodic measures represent the general case. But Theorem 3.2 is not true in the general case when X is any complete separable metric space, and T any homeomorphism of X onto itself. Examples are given at the end of the paper.

By making use of the same methods as in the proof of Theorem 3.1 we shall now prove the following.

THEOREM 3.3. *If* $X = \prod [illegible] M [illegible]$, *where* M *is a complete separable metric space and* T *is the shift transformation, then the set of periodic measures is dense in the set of all ergodic measures under the weak topology.*

Proof. Under the conditions of the theorem, (X, T) is a Borel system [illegible]. Thus by a result of Krylov, Bogoliouboff and Oxtoby [illegible] it follows that for any ergodic measure μ there exists a point $p \in X$ such that the sequence of measures

$$\frac{\mu_{T^{-n}p} + \mu_{T^{-n+1}p} + \cdots + \mu_{T^{n}p}}{2n+1}$$

converges weakly to μ, μ_x being the degenerate measure with mass one at the point x. We shall now approximate μ by means of periodic measures. The point p can be represented by

$$p = [illegible] \quad x_i \in M, \quad i = \cdots, -1, 0, 1, \cdots$$

We write

$$p_n = [illegible]$$

$$[illegible] \quad \text{for } k = \cdots, -1, 0, 1, \cdots, \quad -n \leq i \leq n.$$

Then p_n is a periodic point of period $2n + 1$. We consider

$$\nu_n = \frac{\mu_{T^{-n}p_n} + \mu_{T^{-n+1}p_n} + \cdots + \mu_{T^{n}p_n}}{2n+1}.$$

Since $T^{2n+1}p_n = p_n$, ν_n is a periodic measure. Proceeding exactly as in the proof of Theorem 3.1 it is not difficult to show that for every finite dimensional Borel set A, $\mu_n(A) - \nu_n(A) \to 0$. This completes the proof of Theorem 3.3.

THEOREM 3.4. *When* X *and* T *are the same as in Theorem 3.3, the set* $\mathfrak{M}_s$ *of strongly mixing measures is of the first category in* $\mathfrak{M}$ *under the weak topology.*

Let $0 < \varepsilon < \frac{1}{4}$, $\eta < \frac{1}{2}\varepsilon^2$, δ any rational number $0 < \delta < \eta/2$, r any rational number in $0 \leq r \leq 1$, F_1 and F_2 two disjoint closed sets, and G any open set such that $G \supset F_1$. We write

$$\begin{aligned} &\mathcal{E}(F_1, F_2, G, \varepsilon, r, \delta, n) \\ (3.3)\quad &= \bigcap_{k=n}^{\infty} [\mu : \mu(F_1) \geqq \varepsilon, \mu(F_2) \geqq \varepsilon, \mu(G \cap T^k G) \leqq r + \delta, \\ &\qquad r \leqq \mu^2(F_1) + \eta], \end{aligned}$$

where μ denotes any general invariant probability measure. Since F_1 and F_2 are closed and G is open, by the remarks made in Section 1, the set (3.3) is closed under the weak topology. Let

$$(3.4)\quad \mathcal{E}(F_1, F_2, G, \varepsilon) = \bigcup_{0 \leqq r \leqq 1} \bigcup_{0<\delta<\eta/2} \bigcup_{n=1}^{\infty} \mathcal{E}(F_1, F_2, G, \varepsilon, r, \delta, n).$$

It is not difficult to verify that

$$(3.5)\quad \begin{aligned} \mathcal{E}(F_1, F_2, G, \varepsilon) = \bigcup_{0 \leqq r \leqq 1} \bigcup_{0<\delta<\eta/2} [\mu : \mu(F_1) \geqq \varepsilon, \mu(F_2) \geqq \varepsilon, \\ \limsup_{k\to\infty} \mu(G \cap T^k G) \leqq r + \delta, r \leqq \mu^2(F_1) + \eta]. \end{aligned}$$

Let G_n be a sequence of open sets descending to F_1. Since, for a strongly mixing measure, $\lim_{k\to\infty} \mu(G \cap T^k G) = \mu^2(G)$, it is clear that all strongly mixing measures with the property $\mu(F_1) \geqq \varepsilon$, $\mu(F_2) \geqq \varepsilon$ belong to the set

$$(3.6)\quad \bigcup_{G_n \downarrow F_1} \mathcal{E}(F_1, F_2, G_n, \varepsilon).$$

We shall now show that the set (3.6) is of the first category. From (3.3), (3.4), and (3.6), it is clear that the set (3.6) is a countable union of the closed sets $\mathcal{E}(F_1, F_2, G, \varepsilon, r, \delta, n)$. It is enough to show that these closed sets are nowhere dense or their complements are everywhere dense.

Let P_k be the set of all periodic measures of period k and $P_n^* = \bigcup_{k>n} P_k$. Since by Theorem 3.3 periodic ergodic measures are dense in $\mathfrak{M}_e$, it follows that the set of periodic invariant measures $\mathcal{P}$ is dense in $\mathfrak{M}$. Thus P_n^* is everywhere dense in $\mathfrak{M}$. We shall complete the proof by showing that

$$(3.7)\quad P_n^* \subset \mathfrak{M} - \mathcal{E}(F_1, F_2, G, \varepsilon, r, \delta, n).$$

The inclusion relation (3.7) is satisfied if

$$(3.8)\quad \begin{aligned} P_k \subset \mathfrak{M} - [\mu : \mu(F_1) \geqq \varepsilon, \mu(F_2) \geqq \varepsilon, \mu(G \cap T^k G) \leqq r + \delta, \\ r \leqq \mu^2(F_1) + \eta]. \end{aligned}$$

Let now μ_0 be any periodic measure of period k. If either one of the inequalities $\mu(F_1) \geqq \varepsilon$, $\mu(F_2) \geqq \varepsilon$ is violated, then we are through. Otherwise, since F_1 and F_2 are disjoint, we have

$$\varepsilon \leqq \mu_0(F_1) \leqq 1 - \varepsilon, \qquad \mu_0(G \cap T^k G) = \mu_0(G).$$

Since $0 < \delta < \eta/2$, it is enough to prove that

$$(3.9)\quad \mu_0(G) \geqq \mu_0^2(F_1) + 3\eta/2.$$

Since $0 < \varepsilon < \frac{1}{2}$, $\varepsilon \leqq \mu_0(F) \leqq 1 - \varepsilon$, $G \supset F_1$, and the function

$$x - x^2 \geqq \varepsilon(1 - \varepsilon)$$

in $\varepsilon \leq x \leq 1 - \varepsilon, 0 < \varepsilon < \frac{1}{2}$, we have

$$\mu_0(G) - \mu_0^2(F_1) \geq \mu_0(F_1) - \mu_0^2(F_1) \geq \varepsilon(1 - \varepsilon) > \varepsilon^2 = 3\eta/2.$$

Thus we have proved (3.9).

Let now $S(F_1, F_2, \varepsilon)$ denote the class of all strongly mixing measures with the property

$$\mu(F_1) \geq \varepsilon, \qquad \mu(F_2) \geq \varepsilon.$$

We have proved that $S(F_1, F_2, \varepsilon)$ is of the first category. Now we take a dense sequence of points and consider all closed spheres of rational radii with centers at these points. We denote this class of sets by $\mathfrak{A}$. Then $\mathfrak{A}$ is a countable class. It is clear that the set of all nondegenerate strongly mixing measures is the same as

$$\bigcup_{\substack{F_1, F_2 \in \mathfrak{A} \\ F_1 \cap F_2 = \emptyset}} \; \bigcup_{\substack{0 < \varepsilon < 1/2 \\ \varepsilon \text{ rational}}} S(F_1, F_2, \varepsilon).$$

Since the set of degenerate strongly mixing measures is of the first category and any other strongly mixing measure is nonatomic, we have completed the proof.

4. Remarks and examples

We shall now give some examples to show that Theorem 3.2 need not be true in general.

(1) Let X_0 be a compact group with at least one periodic element, and let the transformation T_0 be the translation of X by a periodic element. Then the ergodic probability measures form a closed set under the weak topology.

(2) Let (X_0, T_0) be as above, and let (X_1, T_1) be the product space with the shift transformation. Let $X = X_0 \times X_1$ and $T = T_0 \times T_1$ be defined in the obvious manner. If X_1 is a complete separable metric space, then the set of ergodic measures is neither closed nor dense.

But in the above examples it is easily seen that there does not exist a dense orbit. However, in the example discussed by Oxtoby,[1] there exists a dense orbit, and nevertheless the ergodic measures form a closed set. Thus it would be very interesting to get a characterisation of all those homeomorphisms of a complete separable metric space for which the density theorem is true.

Now we shall make some remarks concerning Theorem 3.4. The first category theorem holds good as soon as X is a complete separable metric space and the class $\mathcal{P}_e$ of periodic ergodic measures is dense in $\mathfrak{M}_e$. Thus arises the problem of obtaining necessary and sufficient conditions on the homeomorphism T so that the periodic measures may be dense. This is true, for example, in the case when the system (X, T) is L-stable [10]. We shall now get a necessary condition in the following:

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[1] J. C. Oxtoby, *Stepanoff flows on the torus*, Proc. Amer. Math. Soc., vol. 4 (1953), pp. 982–987.

THEOREM 4.1. *If X is a complete separable metric space and the periodic ergodic measures are dense in the set of ergodic measures under the weak topology, then the complement of the closure of periodic points has measure zero for every invariant measure.*

In order to prove this theorem we require the following:

LEMMA 4.1. *If X is a complete separable metric space and μ is an ergodic measure with period k, then there exists a point $x_0 \in X$ such that $T^k x_0 = x_0$ and $\mu(x_0) = \mu(Tx_0) = \cdots = \mu(T^{k-1}x_0) = 1/k$.*

Proof. It is clear from the results of Krylov-Bogolioubov and Oxtoby [10] that an ergodic measure is either purely atomic or purely nonatomic. If it is purely atomic, we are through. In the nonatomic case the measure space becomes a Lebesgue space, and from two lemmas proved in [1] (cf. pages 70 and 71) we arrive at a contradiction without any difficulty. This completes the proof of the lemma.

Turning to the proof of Theorem 4.1, we suppose that P is the set of all periodic points, $\bar{P}$ its closure, and $G = X - \bar{P}$. Then G is an open subset of X. We shall now show that, for every ergodic measure μ, $\mu(G) = 0$. Then an application of the results of Krylov-Bogolioubov and Oxtoby will complete the proof.

Let, if possible, $\mu(G) > 0$ for some ergodic measure μ. Since by hypothesis $\mathcal{P}_e$ is dense in $\mathfrak{M}_e$, there exists a sequence $\mu_n \in \mathcal{P}_e$ such that μ_n converges weakly to μ. Since G is open, $\liminf_{n\to\infty} \mu_n(G) \geqq \mu(G) > 0$. Thus there exists an n such that $\mu_n(G) > 0$. By Lemma 4.1 there exists a point x_0 such that $T^{a_n}x_0 = x_0$ (a_n being the period of μ_n) and $\mu_n(x_0) = 1/a_n$. From the fact that $\mu_n(G) > 0$, it immediately follows that

$$(\chi_G(x_0) + \chi_G(Tx_0) + \cdots + \chi_G(T^{a_n-1}x_0))/a_n > 0,$$

where χ_G is the characteristic function of G. Thus for some r, $T^r x_0 \in G$. Since $T^r x_0$ is a periodic point, we arrive at a contradiction. This completes the proof.

The converse of Theorem 4.1 is still an open problem. It is true, for example, in the case when the system (X, T) is L-stable.

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