Econometrica, Vol. 32, No. 3 (July, 1964)

OPTIMAL SAVINGS IN A TWO-SECTOR MODEL OF GROWTH[1]

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In the recent literature on mathematical models of economic growth, attention has been devoted mainly to the existence and stability of competitive equilibria. These models are based on a rather crucial but simple savings assumption: that savings form a constant proportion of income both being evaluated in terms of numéraire. In this paper, the savings decision is treated as a derived decision, i.e., as an implication of the more basic behavior of utility maximization over time. Using a simple two-sector, two-commodity, two-factor model, optimal growth paths corresponding to the maximization of the sum of the discounted future stream of consumption per worker are worked out. Savings behaviour and asymptotic properties of these optimal paths for varying positive discount rates are also discussed.

1. INTRODUCTION AND SUMMARY

IN THE RECENT literature on mathematical models of economic growth, attention has been devoted mainly to the existence and stability of competitive equilibria [2, 7, 8]. An important feature of these models is a rather simple savings function: in [2] savings form a constant proportion of income, both being evaluated in terms of a numéraire; in [7] savings equal non-wage income; and in [8] the ratio of savings to income is a function of per capita income and the instantaneous rate of interest.

The approach adopted in this paper is Ramseyan [4] in that the savings decision is a derived decision, i.e., it is an implication of the more basic behavior of utility maximization over time. We shall therefore derive the savings implications of maximizing a specific intertemporal social utility index: the utility of a growth path is the sum of the discounted future stream of consumption per worker, the discount rate being positive.[3]

It is shown in Section 2 that for the two-sector, two-factor closed economy considered, given any initial stock of capital, there exists a growth path that maximizes the utility index, provided the sector producing consumer goods is assumed to be more capital intensive than that producing capital goods.[4] The optimal paths

[1] Research undertaken by the Cowles Foundation for Research in Economics under Task NR 047-006 for the Office of Naval Research.

[2] I thank Professor Tjalling Koopmans, Drs. Mordecai Kurz, Emmanuel Drandakis, and Arnold Heertje for their valuable comments on an earlier draft. The referee's comments were also helpful. The present version was completed while I visited the Institute of Economic Growth at Delhi. I remain responsible for any errors remaining in the paper.

[3] We shall not discuss the appropriateness of the impatience implied by a positive discount rate for national planning. See Koopmans [1] for the logical connection between impatience and the existence of a continuous ordinal utility index over time.

[4] One can dispense with this assumption if the production functions in the two sectors are of the Cobb-Douglas type. Discussion of this special case has been omitted to shorten the paper. See also Solow [5]. (Added in proof) Uzawa has recently generalized our results by dropping this assumption.

corresponding to *different* initial stocks of capital but to the *same* discount rate converge to the *same* asymptotic balanced growth path. Savings, evaluated in terms of the efficiency prices associated with the optimal path, do *not*, in general, form a constant proportion of income. However, the savings-income ratio approaches a constant asymptotically. This asymptotic savings ratio is shown to exceed the asymptotic share of non-wage income in total income, for all positive discount rates. Further, it is a decreasing function of the discount rate. As the discount rate tends to zero from above, the asymptotic savings ratio tends to the asymptotic share of non-wage income. Section 3 is devoted to an alternative interpretation of this case.

2. THE MODEL AND THE RESULTS

There are two sectors in the economy. Sector 1 produces a homogeneous capital good and Sector 2 produces a homogeneous consumer good. There are two homogeneous factors of production, labor and capital, the latter being physically the same as the output of Sector 1. The production function of Sector i ($i=1,2$) is denoted by $F^i[K_i, L_i]$ where K_i is the capital input, and L_i is the rate of labor input. The entire output of the consumer good at any point in time is instantaneously consumed, and the output of capital good is added to the capital stock. Capital, once created, lasts forever.[5] The existing stock of capital can be divided between the sectors in any desired fashion. The labor force is assumed to grow exponentially at a positive rate θ.

The following assumptions are made about the production functions:[6]

(A) F^i is homogeneous of degree 1, twice continuously differentiable, and concave. Let $\delta_i=K_i/L_i$. Then $F^i[K_i, L_i]=L_iF^i[\delta_i, 1]$. Define $F^i[\delta_i, 1]=f^i(\delta_i)$. Let $f^i_1(\delta_i)$ be the derivative of $f^i(\delta_i)$ with respect to δ_i, and $f^i_{11}(\delta_i)$ be the derivative of $f^i_1(\delta_i)$ with respect to δ_i.

$$\text{(B.1)}\qquad f^i_1(\delta_i)\qquad \begin{cases}\to\infty & \text{as} \quad \delta_i\to 0\\ >0 & \text{for} \quad \delta_i>0\\ \to 0 & \text{as} \quad \delta_i\to\infty\end{cases}\qquad (i=1,2)\,.$$

$$\text{(B.2)}\qquad \frac{f^i(\delta_i)-\delta_i f^i_1(\delta_i)}{f^i_1(\delta_i)}\begin{cases}\to 0 & \text{as} \quad \delta_i\to 0\\ >0 & \text{for} \quad \delta_i>0\\ \to\infty & \text{as} \quad \delta_i\to\infty\end{cases}\qquad (i=1,2)\,.$$

(C) The consumer goods sector is more capital intensive than the capital goods sector. This means that if $\delta_1(\omega)$ and $\delta_2(\omega)$ are the unique solutions of

[5] One can introduce an exponential decay assumption about capital, but this complication does not lead to any additional insights.

[6] Our assumptions are the same as in Uzawa [8].

$$\frac{f^1(\delta_1)-\delta_1 f_1^1(\delta_1)}{f_1^1(\delta_1)} = \omega = \frac{f^2(\delta_2)-\delta_2 f_1^2(\delta_2)}{f_1^2(\delta_2)} \quad \text{for any} \quad \omega>0\,,$$

then $\delta_1(\omega)<\delta_2(\omega)$.

It can be seen that assumptions (B) imply nonnegative marginal products for capital and labor in the production of either good. Concavity of F^i implies that $f_{11}^i(\delta_i)<0$. In interpreting (C) let us first note that: if ω is the wage to rent ratio, then $\delta_i(\omega)$ represents the unit cost minimizing the capital-labor ratio in Sector i. Assumption (C) means that for any positive ω, the unit cost minimizing the capital-labor ratio in the capital goods sector is less than that in the consumer goods sector.

Let $\overline{K}(0)$ be the initial stock of capital and $K(t)$ be the stock of capital at time t. Let $C(t)$ be the rate of output of consumer goods at t. Without loss of generality we can assume the initial labor force to be equal to 1. Then the labor force at t is $e^{\theta t}$, treating time as a continuous variable. Let $c(t)\equiv e^{-\theta t}C(t)$ be the consumption per worker at t. Let $L_i(t)$, $K_i(t)$ be the labor and capital inputs devoted to the production of the ith good at t ($i=1, 2$). Let $I(t)$ be the rate of output of capital goods. Let $\dot{K}(t)$ be the rate of change of capital stock at t. Then the following relations hold for all $t\geqslant 0$.

(1) $\dot{K}(t) \leqslant I(t) \equiv F^1[K_1(t), L_1(t)]\,,$

(2) $K(t) \equiv \overline{K(0)} + \int_0^t \dot{K}(u)\,du\,,$

(3) $c(t) \equiv e^{-\theta t}C(t) \equiv e^{-\theta t}F^2[K_2(t), L_2(t)]\,,$

(4) $L_1(t)+L_2(t) \leqslant e^{\theta t}\,,$

(5) $K_1(t)+K_2(t) \leqslant K(t)\,,$

(6) $L_1(t), L_2(t), K_1(t), K_2(t), \dot{K}(t), \overline{K(0)} \geqslant 0\,.$

Inequalities (1) state that the rate of change of capital stock at any t cannot exceed the rate of output of capital goods. Equation (2) is a definitional relationship. Equation (3) states that consumption per worker is identically equal to the ratio of output of consumer goods to the labor force. Inequality (4) states that employment in the two sectors together cannot exceed the available labor at any t. Inequality (5) is a similar restraint on capital inputs. Inequalities (6) are nonnegativity restraints imposed to make economic sense.

Let the discount rate be $\rho>0$. Then the utility index is

(7) $\int_0^\infty e^{-\rho t}c(t)dt\,.$

Our problem is to maximize (7) subject to (1) through (6). Introducing the variables

$\delta(t) \equiv e^{-\theta t}K(t)$, $\overline{\delta(0)} \equiv \overline{K(0)}$, $l_1(t) \equiv e^{-\theta t}L_1(t)$, $l_2(t) \equiv e^{-\theta t}L_2(t)$, we can state this problem as follows.

Maximize

$$(7) \qquad \int_0^\infty e^{-\rho t} c(t)\,dt \qquad \rho > 0$$

subject to

$$(8) \qquad c(t) = l_2(t) f^2[\delta_2(t)]\,,$$

$$(9) \qquad 0 \leqslant \dot{\delta} + \theta\delta \leqslant l_1(t) f^1[\delta_1(t)]\,,$$

$$(10) \qquad l_1(t)\delta_1(t) + l_2(t)\delta_2(t) \leqslant \delta(t) \equiv \overline{\delta(0)} + \int_0^t \dot{\delta}(u)\,du\,,$$

$$(11) \qquad l_1(t) + l_2(t) \leqslant 1\,,$$

$$(12) \qquad c(t),\ \delta(t),\ \delta_1(t),\ \delta_2(t),\ l_1(t),\ l_2(t) \geqslant 0, \qquad \overline{\delta(0)} \text{ given}.$$

It is shown in the Appendix that the solution to this problem falls under three cases, depending on the value of $\overline{\delta(0)}$. In order to distinguish these cases let us define the two constants $\hat{\delta}_1$ and $\hat{\delta}_2$ by

$$(13) \qquad f_1^1(\hat{\delta}_1) \qquad = [\rho + \theta] \equiv \varepsilon\,,$$

$$(14) \qquad \frac{f^2\{\hat{\delta}_2\}}{f_1^2\{\hat{\delta}_2\}} - \hat{\delta}_2 = \frac{f^1(\hat{\delta}_1)}{f_1^1(\hat{\delta}_1)} - \hat{\delta}_1\,.$$

Given (A) and (B), there exist unique $\hat{\delta}_1$, $\hat{\delta}_2$ which satisfy (13) and (14). Given (C) it follows that $\hat{\delta}_2 > \hat{\delta}_1$. The meaning of (13) is that the capital-labor ratio $\hat{\delta}_1$ makes the own rate of interest on capital equal the sum of the discount rate and the rate of growth of population. Equation (14) ensures that given $\hat{\delta}_1$, the capital-labor ratio in Sector 2 is chosen in such a way as to equate the ratios of the marginal physical product of labor to that of capital in each of the two sectors.

Case I. $\hat{\delta}_2 \geqslant \overline{\delta(0)} \geqslant \hat{\delta}_1$.

The solution is

$$(15) \qquad \left.\begin{aligned} &\delta_1(t) = \hat{\delta}_1, \quad \delta_2(t) = \hat{\delta}_2\,, \\ &\delta(t) = \delta(\infty) + [\overline{\delta(0)} - \delta(\infty)]e^{-xt}\,, \\ &l_1(t) = \frac{\hat{\delta}_2 - \delta(t)}{\hat{\delta}_2 - \hat{\delta}_1}, \quad l_2(t) = 1 - l_1(t)\,, \end{aligned}\right\} \; t \geqslant 0$$

where

$$x = \theta + \frac{f^1(\hat{\delta}_1)}{\hat{\delta}_2 - \hat{\delta}_1}, \qquad \delta(\infty) = \frac{1}{x}\left(\frac{\hat{\delta}_2 f^1(\hat{\delta}_1)}{\hat{\delta}_2 - \hat{\delta}_1}\right).$$

Clearly, $x > 0$, and $\hat{\delta}_2 > \delta(\infty) > \hat{\delta}_1$.

Case II. $\overline{\delta(0)} > \hat{\delta}_2$.
The solution is

$$(16)\qquad \begin{aligned} &l_1(t)=0,\ l_2(t)=1, \\ &\delta(t)=\overline{\delta(0)}e^{-\theta t}=\delta_2(t), \end{aligned} \qquad 0\leq t\leq \bar{t};$$

$$\delta_1(t)=\hat{\delta}_1, \quad \delta_2(t)=\hat{\delta}_2$$

$$(17)\qquad \delta(t)=\hat{\delta}(\infty)+[\hat{\delta}_2-\hat{\delta}(\infty)]e^{-x(t-\bar{t})} \qquad t>\bar{t}$$

$$l_1(t)=\frac{\hat{\delta}_2-\delta(t)}{\hat{\delta}_2-\hat{\delta}_1}, \quad l_2(t)=1-l_1(t).$$

where

$$(18)\qquad \overline{\delta(0)}e^{-\theta\bar{t}}=\hat{\delta}_2 .$$

Case III. $\hat{\delta}_1 > \delta(0)$.
Let $\tilde{\delta}(t)$ be the solution of the differential equation $\dot{\delta}+\theta\delta=f^1(\delta)$ with the initial condition $\tilde{\delta}(0)=\overline{\delta(0)}$. Then the optimal solution is

$$(19)\qquad \left.\begin{aligned} &l_1(t)=1, \quad l_2(t)=0, \\ &\delta(t)=\tilde{\delta}(t)=\delta_1(t), \end{aligned}\right\} \ 0\leq t\leq \underline{t}$$

$$\delta_1(t)=\hat{\delta}_1, \quad \delta_2(t)=\hat{\delta}_2,$$

$$(20)\qquad \left.\begin{aligned} &\delta(t)=\hat{\delta}(\infty)+[\hat{\delta}_1-\hat{\delta}(\infty)]e^{-x(t-\underline{t})} \\ &l_1(t)=\frac{\hat{\delta}_2-\delta(t)}{\hat{\delta}_2-\hat{\delta}_1}, \quad l_2(t)=1-l_1(t), \end{aligned}\right\} \ t>\underline{t},$$

where

$$(21)\qquad \tilde{\delta}(\underline{t})=\hat{\delta}_1 .$$

Before discussing the properties specific to each of the above three cases, it is worthwhile to draw attention to the property common to all three. That is, as $t\to\infty$, the solution in all three cases approaches the *same* path. This path is given by the conditions:

$$(22)\qquad \delta_1(t)=\hat{\delta}_1, \quad \delta_2(t)=\hat{\delta}_2, \quad \delta(t)=\hat{\delta}(\infty),$$

$$(23)\qquad l_1(t)=\frac{\hat{\delta}_2-\hat{\delta}(\infty)}{\hat{\delta}_2-\hat{\delta}_1}, \quad l_2(t)=1-l_1(t),$$

$$(24)\qquad c(t)=\hat{c}=l_2(t)f^2(\hat{\delta}_2)=\frac{\hat{\delta}(\infty)-\hat{\delta}_1}{\hat{\delta}_2-\hat{\delta}_1}f^2(\hat{\delta}_2).$$

This asymptotic path is a balanced growth path, since all the ratio variables δ_1, δ_2, δ, l_1, and l_2 remain constant over time. An economy moving along this path looks exactly the same over time except for a scale factor $e^{\theta t}$.

Coming to the specific aspects of the three cases, we observe that in Case I, both capital goods and consumer goods are produced at all t. If $\overline{\delta(0)} > \delta_2$ as in Case II, initially there is "too much" capital relative to labor, and the economy optimally adjusts to this capital surplus by producing only consumer goods up to a certain point t in time. On the other hand, if $\delta_1 > \overline{\delta(0)}$ as in Case III, there is initially too much labor relative to capital and the economy optimally adjusts to this labor surplus by producing only capital up to a certain point t of time.[7] Except for the differences discussed in this paragraph, the solutions for the three cases are essentially the same.

Income and Savings Along the Optimal Path

Income and savings are value concepts, and we need a set of prices to convert physical magnitudes such as output of consumer goods and of capital goods into values. For this purpose we shall use the efficiency prices associated with our optimal solutions. They have the following interpretation: Given these prices and the assumption that producers choose inputs and outputs so as to maximize profits, using the appropriate rate of discount where necessary, the resulting growth paths will be the same as our optimal solutions. In other words, the optimal solutions can be realized through profit maximization, given these prices.

Let us use the following notation for prices (a price here means the value at time zero of a unit of a good or service becoming available at time t):

$p(t)$: price per unit of consumer good at t,

$q(t)$: price per unit of capital good at t,

$r(t)$: rate of rental per unit of capital stock at t,

$w(t)$: wage rate at t.

In discussing the prices and other value variables, we shall confine our attention to Case I, since the other two cases reduce to this case after a finite interval of time. It is shown in the Appendix that the efficiency prices associated with the optimal solution for this case are:

$$(25) \qquad p(t) = e^{-\varepsilon t}, \quad q(t) = \frac{1}{\varepsilon} f_1^2(\delta_2) e^{-\varepsilon t},$$

$$(26) \qquad r(t) = f_1^2(\delta_2) e^{-\varepsilon t}, \quad w(t) = [f^2(\delta_2) - \delta_2 f_1^2(\delta_2)] e^{-\varepsilon t},$$

where $\varepsilon = \rho + \theta$.

An interesting feature of this price system is that the relative prices do not change over time. Further, the prices given by (25) and (26) are also the efficiency prices associated with the balanced growth path which the optimal solutions in all three cases approach asymptotically. Thus, if the initial conditions conform to Case I,

[7] One can easily introduce a floor on consumption per worker as an additional constraint to avoid this unpleasant aspect. See Uzawa [9].

the efficiency prices associated with the optimal path are the same for all t as those associated with the asymptotic balanced growth path.

Let $Y(t)$ be income and $S(t)$ be savings at time t. It is clear that

$$Y(t)=w(t)e^{\theta t}+r(t)\,K(t)=e^{\theta t}\{w(t)+r(t)\,\delta(t)\}\,, \tag{27}$$

$$S(t)=q(t)\{e^{\theta t}l_1(t)f^1(\delta_1)\}\,. \tag{28}$$

Equation (27) states that total income equals wage income plus rental on capital stock. Equation (28) states that savings equal the value of the output of the capital goods sector. Let us denote by $s(t)$ the ratio of savings to income. Using (27) and (28) we can state that the

$$\text{saving ratio} \equiv s(t) = \frac{S(t)}{Y(t)} = \frac{q(t)\,l_1(t)f^1(\delta_1)}{w(t)+r(t)\delta(t)}$$

$$= \frac{l_1(t)f^1(\delta_1)}{\left[\left\{\dfrac{f^2(\delta_2)}{f^2_1(\delta_2)} - \delta_2\right\} + \delta(t)\right]\varepsilon}\,. \tag{29}$$

We know from (15) that $l_1(t)$ and $\delta(t)$ are not, in general, constant over time. *Hence $s(t)$ is not, in general, constant over time.* However, since $l_1(t)$ and $\delta(t)$ converge as $t\to\infty$, $s(t)$ also converges as $t\to\infty$. Let us define $s(\infty)=\lim_{t\to\infty}s(t)$. Then

$$s(\infty) = \frac{\left\{\lim\limits_{t\to\infty} l_1(t)\right\}f^1(\delta_1)}{\left[\left\{\dfrac{f^2(\delta_1)}{f^2_1(\delta_1)} - \delta_2\right\} + \delta(\infty)\right]\varepsilon}$$

$$= \frac{[\delta_2-\delta(\infty)]f^1(\delta_1)}{[\delta_2-\delta_1]\left[\left\{\dfrac{f^2(\delta_2)}{f^2_1(\delta_2)} - \delta_2\right\} + \delta(\infty)\right]\varepsilon}\,.$$

Using (14) we can rewrite $s(\infty)$ as follows:

$$s(\infty) = \left\{\frac{\delta_2-\delta(\infty)}{\delta_2-\delta_1}\right\}\left\{\frac{f^1(\delta_1)}{f^1(\delta_1)+\varepsilon(\delta(\infty)-\delta_1)}\right\}\,. \tag{30}$$

It can be shown that $s(\infty)$ decreases as the discount rate increases. This is as it should be since a larger ρ implies that future consumption is worth relatively less compared to the present. Let us compare $s(\infty)$ with the asymptotic share $\pi(\infty)$ of non-wage income to total income. Let $\pi(t)$ be the share of capital in income at t.

Then

$$\pi(t) = \frac{r(t)\,\delta(t)}{w(t)+r(t)\,\delta(t)} = \frac{\delta(t)}{\left[\dfrac{f^1(\hat{\delta}_1)}{\varepsilon} - \hat{\delta}_1 + \delta(t)\right]},$$

$$\text{(31)} \qquad \pi(\infty) = \lim_{t\to\infty} \pi(t) = \frac{\varepsilon\hat{\delta}(\infty)}{[f^1(\hat{\delta}_1)+\varepsilon(\hat{\delta}(\infty)-\hat{\delta}_1)]}.$$

Comparing (30) with (31), we find that $\pi(\infty)-s(\infty)$ has the same sign as $\varepsilon\hat{\delta}(\infty)-[(\hat{\delta}_2-\hat{\delta}(\infty))/(\hat{\delta}_2-\hat{\delta}_1)]\,f^1(\hat{\delta}_1)$. Using the definition of $\hat{\delta}(\infty)$ we can simplify this expression to the following:

$$\varepsilon\hat{\delta}(\infty) - \frac{\hat{\delta}_2-\hat{\delta}(\infty)}{\hat{\delta}_2-\hat{\delta}_1}\, f^1(\hat{\delta}_1) = \frac{\hat{\delta}_2 f^1(\hat{\delta}_1)[\varepsilon-\theta]}{\theta(\hat{\delta}_2-\hat{\delta}_1)+f^1(\hat{\delta}_1)}.$$

But $\varepsilon=\rho+\theta$ and $\rho>0$. Hence, *for all positive discount rates the asymptotic share of non-wage income exceeds the asymptotic share of savings.* It can also be seen that as $\rho\to 0$, $\pi(\infty)\to s(\infty)$. In other words, as the discount rate approaches zero the asymptotic share of capital in income approaches the asymptotic savings ratio. We shall return to this case in Section 3.

It can be easily shown from (29) that the savings ratio is a nondecreasing (nonincreasing) function of time if $\hat{\delta}(0) \geqslant (\leqslant)\, \hat{\delta}(\infty)$.

3. BEHAVIOUR OF ASYMPTOTIC BALANCED GROWTH PATHS FOR VARYING DISCOUNT RATES

We saw earlier that the optimal paths corresponding to any given positive discount rate ρ and to varying initial stocks of capital approached asymptotically the same balanced growth path given by (22) through (24). Let us now compare these balanced growth paths for various nonnegative values of ρ. First we note the obvious result that along each of these balanced growth paths, consumption per worker is a constant $\hat{C}(\rho)$ (given by (24)). *It can be shown that $\hat{C}(\rho)$ is a decreasing function of ρ and attains its maximum at $\rho=0$.* Secondly, savings and non-wage income form constant proportions of total income along each of these paths. Further, as we said earlier, *for positive discount rates, the savings ratio along an asymptotic balanced growth path exceeds the ratio of non-wage income, and these two ratios become equal as the discount rate approaches zero.*

The result that along the balanced growth path which yields the maximum consumption per worker, the savings ratio equals the ratio of non-wage income has also been obtained, under somewhat more restrictive assumptions than ours, by Kurz [2] and Phelps [3].[8] The technological nature of this result becomes apparent if we go back to equations (13) and (14), which yielded optimal capital-labor ratios for the two sectors, given the discount rate ρ.

If we let ρ tend to zero, these equations become

$$f_1^1[\hat{\delta}_1]=\theta\,, \tag{32}$$

$$\frac{f^1[\hat{\delta}_1]}{f_1^1[\hat{\delta}_1]}-\hat{\delta}_1=\frac{f^2[\hat{\delta}_2]}{f_1^2[\hat{\delta}_2]}-\hat{\delta}_2\,. \tag{33}$$

We know that θ is the rate of growth of the labor force, and it is *also* the rate of growth of the output of capital goods and consumer goods along a balanced growth path; $f_1^1[\hat{\delta}_1]$ is the marginal physical product of capital in the production of capital goods when capital-labor ratio $\hat{\delta}_1$ is used. It is, therefore, the own rate of interest on capital goods. Thus (32) states that if we are to maximize the indefinitely sustainable level of consumption per worker, we have to choose that capital-labor ratio in Sector 1 which will make the own rate of interest on capital equal the rate of growth of the system. The production function in Sector 1 and the rate of growth of the labor force are enough to determine this capital-labor ratio.

An alternative interpretation of the above result is possible. Suppose one wished to sustain indefinitely one unit of consumption per worker, using capital-labor ratios of δ_1 and δ_2 for all $t\geqslant 0$. Given that the labor force is growing at the rate θ, the direct labor input needed to produce one unit of consumer good per worker at time is

$$\eta_2=e^{\theta t}\cdot\frac{1}{f^2(\delta_2)}\,.$$

For the capital-labor ratios δ_1 and δ_2 to be maintained indefinitely, the rate of output of capital goods required at time t is $\theta[\eta_1\delta_1+\eta_2\delta_2]$, where η_1 is the employment in the capital goods sector. On the other hand, given η_1 and δ_1, the actual rate of output of new capital goods is $\eta_1 f^1(\delta_1)$. Equating the required output to actual output we have

$$\eta_1\delta_1+\eta_2\delta_2=\frac{1}{\theta}\eta_1 f^1(\delta_1)\,.$$

Hence we can solve for η_1 and get

$$\eta_1=\frac{\theta\delta_2 e^{\theta t}}{f^2(\delta_2)[f^1(\delta_1)-\theta\delta_1]}\,.$$

[8] After this paper was written, a further discussion of this result (by various authors) was published in the *Review of Economic Studies*, June, 1962.

We can call η_1 the indirect labor input needed at time t to sustain one unit of consumption per worker. Adding the direct and indirect labor inputs we have

$$(\eta_1+\eta_2) = \frac{e^{\theta t}}{f^2(\delta_2)}\left[\frac{\theta\delta_2}{f^1(\delta_1)-\theta\delta_1}+1\right].$$

Since labor is the only primary factor in this economy, the maximum sustainable rate of consumption per worker is achieved along a path which minimizes $\eta_1+\eta_2$, or equivalently minimizes $e^{-\theta t}[\eta_1+\eta_2]$. Minimization of $e^{-\theta t}[\eta_1+\eta_2]$ leads again to equations (32) and (33) for the solution of optimal δ_1 and δ_2.

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APPENDIX

We shall now establish some of the results stated in Sections 2 and 3. It may be recalled that the problem was to maximize

(A.1) $$\int_0^\infty e^{-\rho t}c(t)\,dt \qquad \rho>0$$

subject to

(A.2) $$c(t)\equiv l_2(t)f^2[\delta_2(t)],$$

(A.3) $$0\leqslant \dot{\delta}(t)+\theta\,\delta(t)\leqslant l_1(t)f^1[\delta_1(t)],$$

(A.4) $$l_1(t)\delta_1(t)+l_2(t)\delta_2(t)\leqslant\delta(t)\equiv\overline{\delta(0)}+\int_0^\infty \dot{\delta}(u)du,$$

(A.5) $$l_1(t)+l_2(t)\leqslant 1,$$

(A.6) $$c(t),\ \delta(t),\ \delta_1(t),\ \delta_2(t),\ l_1(t),\ l_2(t)\geqslant 0 \qquad \overline{\delta(0)}\text{ given}.$$

DEFINITION 1: Any set of time paths $c(t)$, $\delta(t)$, $\delta_1(t)$, $\delta_2(t)$, $l_1(t)$, and $l_2(t)$ which satisfy (A.1) through (A.6) is called a *feasible solution.*

DEFINITION 2: An *optimal solution* is a feasible solution which maximizes (A.1) among all feasible solutions.

The following lemma is useful in proving the optimality of the solutions given in Section 2.

LEMMA: *If a feasible solution* $\hat{c}(t)$, $\hat{\delta}(t)$, $\hat{\delta}_1(t)$, $\hat{\delta}_2(t)$, $\hat{l}_1(t)$, $\hat{l}_2(t)$ *and a set of auxiliary functions* $q(t)$, $w(t)$, $r(t)$ *exist such that for all* $t \geqslant 0$ *and for all feasible solutions,* $c(t)$, $\delta(t)$, $\delta_1(t)$, $\delta_2(t)$, $l_1(t)$, $l_2(t)$, *the following hold, then the solution with a caret,* ^, *on the variables is optimal:*

(a) $$q(t) \geqslant 0, \quad w(t) \geqslant 0, \quad r(t) \geqslant 0,$$

(b) $$q(t) = \int_t^\infty r(u)\,du,$$

(c.1) $$e^{-\varepsilon t}\hat{c}(t) = \hat{l}_2(t)[w(t) + r(t)\hat{\delta}_2(t)]$$
(c.2) $$e^{-\varepsilon t}c(t) \leqslant l_2(t)[w(t) + r(t)\delta_2(t)]$$
where $\varepsilon = \rho + \theta$,

(d.1) $$q(t)[d\hat{\delta}/dt + \theta\hat{\delta}] = \hat{l}_1(t)[w(t) + r(t)\hat{\delta}_1(t)],$$

(d.2) $$q(t)[\dot{\delta} + \theta\delta] \leqslant l_1(t)[w(t) + r(t)\delta(t)],$$

(e) $$w(t) = 0 \quad \text{if} \quad \hat{l}_1(t) + \hat{l}_2(t) < 1,$$

(f) $$r(t) = 0 \quad \text{if} \quad \hat{l}_1(t)\hat{\delta}_1(t) + \hat{l}_2(t)\hat{\delta}_2(t) < \hat{\delta}(t),$$

(g) $$\lim_{t\to\infty}\{e^{\theta t}q(t)\hat{\delta}(t)\} = 0.$$

PROOF: From (e), (f), and (A.5) it follows that

(A.7) $$w(t) = w(t)[\hat{l}_1(t) + \hat{l}_2(t)],$$

(A.8) $$r(t)\hat{\delta}(t) = r(t)[\hat{l}_1(t)\hat{\delta}_1(t) + \hat{l}_2(t)\hat{\delta}_2(t)].$$

Using (c.1), (d.1), (A.7), and (A.8) we have

(A.9) $$e^{-\varepsilon t}\hat{c}(t) + q(t)[d\hat{\delta}/dt + \theta\hat{\delta}] = w(t) + r(t)\hat{\delta}(t).$$

From (b) we know that $\dot{q}(t) = -r(t)$. Hence

(A.10) $$w(t) = e^{-\varepsilon t}\hat{c}(t) + e^{-\theta t}\frac{d}{dt}[e^{\theta t}q(t)\hat{\delta}(t)].$$

It follows from (a), (A.6), (c.2), and (d.2) that

$$e^{-\varepsilon t}c(t) + q(t)[\dot{\delta} + \theta\delta] \leqslant w(t)[l_1(t) + l_2(t)] + r(t)[l_1(t)\delta_1(t) + l_2(t)\delta_2(t)] \leqslant w(t) + r(t)\delta(t).$$

Hence

$$\text{(A.11)} \quad e^{-\varepsilon t}c(t) \leqslant w(t) - e^{-\theta t}\frac{d}{dt}[e^{\theta t}q(t)\,\delta(t)]\,.$$

Using (A.10) in (A.11), we have

$$\text{(A.12)} \quad e^{-\varepsilon t}c(t) - e^{-\varepsilon t}\hat{c}(t) \leqslant e^{-\theta t}\left\{\frac{d}{dt}(e^{\theta t}q(t)\hat{\delta}(t)) - \frac{d}{dt}(e^{\theta}q(t)\delta(t))\right\}.$$

Since $\varepsilon = \rho + \theta$, we can rewrite (A.12) as

$$\text{(A.12)} \quad e^{-\rho t}c(t) - e^{-\rho t}\hat{c}(t) \leqslant \frac{d}{dt}(e^{\theta t}q(t)\,\hat{\delta}(t)) - \frac{d}{dt}(e^{\theta t}q(t)\delta(t))\,.$$

Multiplying both sides of (A.12) by -1, reversing the inequality sign, integrating both sides over $[0, T]$, and utilizing the condition that $\hat{\delta}(0) = \delta(0) = \bar{\delta}(0)$, we have

$$\int_0^T e^{-\rho t}\hat{c}(t)dt - \int_0^T e^{-\rho t}c(t)\,dt \geqslant e^{\theta T}q(T)\delta(T) - e^{\theta T}q(T)\hat{\delta}(T)$$
$$\geqslant -e^{\theta T}q(T)\hat{\delta}(T)\,.$$

Letting $T \to \infty$ and utilizing (g), the optimality of the solution with ^ on the variables is established.

We can provide an interpretation for the auxiliary functions. Let us define $p(t) = e^{-\varepsilon t}$. Then $p(t)$ can be interpreted as the value at time zero of a unit of the consumer good produced at time t; $q(t)$ is the value at time zero of a unit of the capital good produced at t; and $w(t)$ and $r(t)$ are the values at time zero of the wage rate and rental rate at time t. With these price interpretations, the meaning of the conditions (a) through (g) of the lemma become clear: (a) states that the prices are nonnegative. (b) is the intertemporal efficiency condition, that the value of a unit of capital at t is the sum of the rentals on it from t on. (c.1) and (c.2) state that, given the prices $p(t)$, $q(t)$, $w(t)$, and $r(t)$, the profit from consumer goods production is zero under the solution with a ^ on the variables, and nonpositive under any other feasible solution. In other words, the solution with a ^ is profit maximizing for the production of consumer goods, at all t, given the prices $p(t)$, $w(t)$, $r(t)$. (d.1) and (d.2) are similar profit maximizing conditions on the production of capital goods. (e) and (f) state that if a resource (labor or capital) is not fully used, its price is zero. Condition (g) is a boundary condition, arising mainly because of the infinite time horizon postulated in our problem.

Although we were able to prove our Lemma without using the properties (B.1) and (B.2) on $f^1[\delta]$ and $f^2[\delta]$, having hypothesized the existence of the auxiliary functions which satisfied conditions (a) through (g), we may not, in any given problem, be able to find such auxiliary functions without (B.1) and (B.2). In fact, in what follows we shall be using (B.1) and (B.2) extensively, besides assumption (C).

We are now in a position to establish the optimality of the solutions given in

Section 2. We shall confine our attention to Cases I and III, leaving the proof for Case II to the reader. We distinguished the three cases by first defining two constants $\hat{\delta}_1$, $\hat{\delta}_2$ as follows:

$$f_1^1[\hat{\delta}_1] = s\,, \quad \frac{f^1[\hat{\delta}_1]}{f_1^1[\hat{\delta}_1]} - \hat{\delta}_1 = \frac{f^2[\hat{\delta}_2]}{f_1^2[\hat{\delta}_2]} - \hat{\delta}_2\,.$$

The existence and uniqueness of $\hat{\delta}_1$, $\hat{\delta}_2$ follow immediately from assumptions (B.1) and (B.2). We recall that these imply that $f_1^i(\delta)$ is a *decreasing function* of δ, with $f_1^i(0)=\infty$ and $f_1^i(\infty)=0$. Also $f^i[\delta]/f_1^i[\delta]-\delta$ is an increasing function of δ, approaching θ as $\delta\to 0$ and approaching ∞ as $\delta\to\infty$. Hence δ_1 and δ_2 exist and are unique. Further, our capital intensity assumption (C) implies that $\hat{\delta}_2>\hat{\delta}_1$.

Case I. $\hat{\delta}_2\geqslant\overline{\delta(0)}\geqslant\hat{\delta}_1$.

The optimal solution and the associated auxiliary functions have already been given by equations (15), (25), and (26). Given that $\hat{\delta}_2>\hat{\delta}_1>0$, it is clear that $\hat{\delta}_2>\delta(\infty)>\hat{\delta}_1>0$ and that $x>0$. Hence $\delta(t)$ is monotonic in t, and since $\hat{\delta}_2\geqslant\overline{\delta(0)}\geqslant\hat{\delta}_1$, it follows that $\hat{\delta}_2\geqslant\delta(t)\geqslant\hat{\delta}_1$ for all $t\geqslant 0$. This means that $l_1(t)$, $l_2(t)\geqslant 0$ for t. It is easy to verify that the stated solution is feasible.

In order to prove the optimality we apply our Lemma. Conditions (a), (b), (c.1), (d.1), (e), (f), and (g) are easily seen to hold. From the definition of $w(t)$ and $r(t)$, we note that their ratio, $(w(t))/(r(t))$ is a constant over time, and further:

$$\text{(A.13)} \quad \frac{w(t)}{r(t)} = \frac{f^2[\hat{\delta}_2]}{f_1^2[\hat{\delta}_1]} - \hat{\delta}_2 = \frac{f^1[\hat{\delta}_1]}{f_1^1[\hat{\delta}_1]} - \hat{\delta}_1\,.$$

In view of our constant returns to scale assumption on the two production functions, $f^i[\delta]/f_1^i[\delta]-\delta$ is the ratio of the marginal physical product of labor to that of capital in sector i, if the capital-labor ratio δ is used. Now $w(t)/r(t)$ is the ratio of wages to rents. Hence, given our assumptions (B.1) and (B.2), (A.13) means that $\hat{\delta}_1$ and $\hat{\delta}_2$ are the unique unit cost minimizing capital-labor ratios in the two sectors. Now, the unit cost of production in sector i if the capital-labor ratio $\delta_i(t)$ is used is $(w(t)+r(t)\,\delta_i(t))/f^i[\delta_i(t)]$. Hence cost minimization at the capital-labor ratios $\hat{\delta}_1$ and $\hat{\delta}_2$ implies that for all $\delta_1(t)$, $\delta_2(t)$,

$$\text{(A.14)} \quad \frac{w(t)+r(t)\delta_1(t)}{f^1[\delta_1(t)]} \geqslant \frac{w(t)+r(t)\hat{\delta}_1}{f^1[\hat{\delta}_1]} = q(t)\,,$$

$$\text{(A.15)} \quad \frac{w(t)+r(t)\delta_2(t)}{f^2[\delta_2(t)]} \geqslant \frac{w(t)+r(t)\hat{\delta}_2}{f^2[\hat{\delta}_2]} = e^{-st}\,.$$

These two inequalities imply that for any feasible solution

$$l_1(t)[w(t)+r(t)\,\delta_1(t)]\geqslant q(t)l_1(t)f^1[\delta_1(t)]\geqslant q(t)[\dot{\delta}+\theta\delta]\,,$$

$$l_2(t)\,[w(t)+r(t)\delta_2(t)]\geqslant l_2(t)f^2[\delta_2(t)]e^{-\varepsilon t}=e^{-\varepsilon t}c(t)\,.$$

Hence (c.2) and (d.2) are seen to hold, and our solution is optimal.

Case III. $\bar{\delta}_1>\overline{\delta(0)}$.

The optimal solution for this case, given by equations (19), (20), and (21) of the text is seen to depend on the solution $\bar{\delta}(t)$ of the following differential equation.

(A.16) $\dot{\delta}+\theta\delta=f^1(\delta)\,,$

the initial condition being $\bar{\delta}(0)=\overline{\delta(0)}$. Given assumptions (B.1) and (B.2), it is clear that there exist unique δ^* and δ^{**} such that:

(a) $f^1[\delta]-\theta\delta$ attains its unique maximum in $[0,\infty]$ at $\delta=\delta^*$. Clearly, $f_1^1[\delta^*]=\theta$.

(b) $f^1[\delta]-\theta\,\delta\begin{cases}>0 & \text{for } 0<\delta<\delta^{**}\\ =0 & \text{for } \delta=\delta^{**}\\ <0 & \text{for } \delta>\delta^{**}\end{cases}$.

(c) $\delta^* < \delta^{**}$.

These imply that whatever be the initial value $\delta(0)$, there exists a unique solution $\bar{\delta}(t)$ to (A.16) such that if $\delta(0)<\delta^{**}$, $\bar{\delta}(t)$ is an increasing function of t, and $\lim_{t\to\infty}\bar{\delta}(t)=\delta^{**}$ where $f^1(\delta^{**})=\theta\delta^{**}$. Now, $f_1^1[\bar{\delta}_1]=\varepsilon=\rho+\theta>\theta$, since $\rho>0$. This means that $\bar{\delta}_1<\delta^*$, since $f_1^1(\delta)$ is a decreasing function of δ. Hence $\overline{\delta(0)}<\bar{\delta}_1<\delta^*<\delta^{**}$. Also there exists a $\bar{t}$ such that $\bar{\delta}(\bar{t})=\bar{\delta}_1$.

The auxiliary functions associated with the solution given by (19), (20), and (21) are

$$p(t)=e^{-\varepsilon t}\,,$$

$$w(t)=\Big[f^1[\delta_1(t)]-\delta_1(t)f_1^1[\delta_1(t)]\Big]q(t)\,,$$

$$r(t)=f_1^1[\delta_1(t)]q(t) \qquad \text{for } 0\leqslant \bar{t}\leqslant \bar{t}$$

where $q(t)$ is the solution of $\dot{q}(t)=-f_1^1[\delta_1(t)\,q(t)]$, with the condition $q(\bar{t})=(1/\varepsilon)f_1^2[\bar{\delta}_2]e^{-\varepsilon\bar{t}}$.

The functions $p(t)$, $q(t)$, $w(t)$, and $r(t)$ have the same values as in Case I for $t>\bar{t}$.

The feasibility of this solution can be verified easily. For $t>\bar{t}$, this solution is essentially the same as in Case I, and hence the conditions (a) through (g) of the Lemma continue to hold. For any t in $0\leqslant t\leqslant \bar{t}$, conditions (a), (b), (c.1), (d.1), (e), and (f) of the Lemma are seen to hold. Condition (g), being an asymptotic property, is not relevant for this initial period. In order to establish the optimality of our solution, we need to verify (c.2) and (d.2).

We note from the definition of $q(t)$, $w(t)$, and $r(t)$ that the following hold for any t in $[0, \underline{t}]$:

$$\frac{w(t)}{r(t)} = \frac{f^1[\delta_1(t)]}{f_1^1[\delta_1(t)]} - \delta_1(t)\,,$$

$$q(t) = \frac{r(t)}{f_1^1[\delta_1(t)]}\,.$$

The first of these means that $\delta_1(t)$ is that capital-labor ratio which equates the ratio of wages over rents to the ratio of the marginal physical product of labor to that of capital in the capital goods industry. In other words $\delta_1(t)$ minimizes the unit cost of production of capital goods. The second equation means that the minimal unit cost just equals the price per unit of capital goods. Hence these two conditions together imply that for any other capital-labor ratio, profits will be nonpositive. This verifies (d.2).

In order to verify (c.2), we shall show that, given $w(t)$, $r(t)$, the minimal unit cost $\tilde{p}(t)$ of producing consumer goods exceeds its price $p(t)$, and hence it is unprofitable to produce any consumer goods. Let $\tilde{\delta}_2(t)$ be that capital-labor ratio which minimizes the unit cost of production, given $w(t)$ and $r(t)$. Then

$$\frac{f^1[\delta_1(t)]}{f_1^1[\delta_1(t)]} - \delta_1(t) = \frac{w(t)}{r(t)} = \frac{f^2[\tilde{\delta}_2(t)]}{f_1^2[\tilde{\delta}_2(t)]} - \tilde{\delta}_2(t),\ \ \tilde{p}(t) = \frac{r(t)}{f_1^2[\tilde{\delta}_2(t)]}\,.$$

We wish to show that $\tilde{p}(t) > p(t) = e^{-\varepsilon t}$ for $0 \leqslant t \leqslant \underline{t}$.

Consider $\delta_1(t)$ at any t in $[0, \underline{t}]$. We know from the choice of $\delta_1(t)$ that $\delta_1(t) < \mathring{\delta}_1$, and $\delta_1(t) > 0$. Since $f_1^1[\delta]$ is a decreasing function of δ we have:

$$f_1^1[\delta_1(t)] > f_1^1[\mathring{\delta}_1] = \varepsilon\,.$$

By definition,

$$\dot{q}(t) = -f_1^1[\delta_1(t)]q(t) < -\varepsilon q(t)\,,$$

$$\frac{\dot{q}(t)}{q(t)} < -\varepsilon\,,$$

$$\frac{d}{dt}\log q(t) < -\varepsilon\,.$$

Integrating both sides over $[t, \underline{t}]$ we have

$$\log q(\underline{t}) - \log q(t) < -\varepsilon[\underline{t} - t]\,,$$

or

$$\frac{q(\underline{t})}{q(t)} < e^{-\varepsilon[\underline{t}-t]}$$

$$q(t) > q(\underline{t})e^{\varepsilon[\underline{t}-t]} = \frac{f_1^2[\mathring{\delta}_2]}{\varepsilon}e^{-\varepsilon t}\,.$$

Now,

$$\bar{p}(t)-e^{-\alpha t} = \frac{r(t)}{f_1^2[\bar\delta_2(t)]} - e^{-\alpha t} = q(t)\frac{f_1^1[\delta_1(t)]}{f_1^2[\bar\delta_2(t)]} - e^{-\alpha t}$$

$$> e^{-\alpha t}\left\{\frac{f_1^2[\bar\delta_2]}{\varepsilon}\,\frac{f_1^1[\delta_1(t)]}{f_1^2[\bar\delta_2(t)]} - 1\right\}.$$

Let

$$\phi(t) = \frac{f_1^1[\delta_1(t)]}{f_1^2[\bar\delta_2(t)]} - \frac{\varepsilon}{f_1^2[\bar\delta_2]} = \frac{f_1^1[\delta_1(t)]}{f_1^2[\bar\delta_2(t)]} - \frac{f_1^1[\delta_1]}{f_1^2[\bar\delta_2]}.$$

Clearly $\phi(\underline{t})=0$. We shall show that $\phi(t)>0$ for $t<\underline{t}$ by showing that $\dot\phi(t)<0$ for t in $[0, \underline{t}]$:

$$\dot\phi(t) = \frac{f_{11}^1[\delta_1(t)]\dot\delta_1}{f_1^2[\bar\delta_2(t)]} - \frac{f_1^1[\delta_1(t)]}{\{f_1^2[\delta_2(t)]\}^2}\cdot f_{11}^2[\bar\delta_2(t)](d\bar\delta_2/dt).$$

We know that

$$\frac{w(t)}{r(t)} = \frac{f^2[\bar\delta_2(t)]}{f_1^2[\bar\delta_2(t)]} - \bar\delta_2(t) = \frac{f^1[\delta_1(t)]}{f_1^1[\delta_1(t)]} - \delta_1(t).$$

Differentiating with respect to t, we obtain

$$-\frac{f^2[\bar\delta_2(t)]}{f_1^2[\bar\delta_2(t)]^2}f_{11}^2[\bar\delta_2(t)](d\bar\delta_2/dt) = \frac{f^1[\delta_1(t)]}{[f_1^1[\delta_2(t)]]^2}\cdot f_{11}^1[\delta_1(t)]\dot\delta_1.$$

Substituting in the expression for $\dot\phi(t)$ and simplifying, we have

$$\dot\phi(t) = \frac{f_{11}^1[\delta_1(t)]\dot\delta_1}{f^2\delta_2(t)}\left[\frac{f^2[\bar\delta_2(t)]}{f_1^2[\bar\delta_2(t)]} - \frac{f^1[\delta_1(t)]}{f_1^1[\delta_1(t)]}\right],$$

$$= \frac{f_{11}^1[\delta_1(t)]\dot\delta_1}{f^2[\bar\delta_2(t)]}[\bar\delta_2(t)-\delta_1(t)].$$

By our capital intensity hypothesis, $\delta_1(t)<\bar\delta_2(t)$. We know also that $f_{11}^1[\delta_1(t)]<0$; $\dot\delta_1>0$; $f^2[\bar\delta_2(t)]>0$. Hence, $\dot\phi(t)<0$ for $0\leqslant t\leqslant\underline{t}$. This implies $\phi(t) > 0$ for $0 \leqslant t \leqslant \underline{t}$, which in turn implies that $\bar{p}(t)-e^{-\alpha t}>0$ for $0\leqslant t\leqslant\underline{t}$.

This completes the proof of the optimality of our solution for Case III.